Dream of a Christmas lecture

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Abstract

We recall the origins of differential calculus from a modern perspective. This lecture should be a victory song, but the pain makes it to sound more as a oath for vendetta, coming from Syracuse two milenia before.

A visitor in England, if he is bored enough, could notice that our old 20 pound notes are decorated with a portrait of Faraday imparting the first series of "Christmas lectures for young people", which began time ago, back in the XIXth century, at his suggestion. Today they have become traditional activity in the Royal Institution.

This year the generic theme of the lectures was quantum theory and the limits implied by it. The BBC uses to broadcast the full sessions during the holidays, and I decided to enjoy an evening seeing the recording. This day, the third of the series, is dedicated to the time scale of quantum phenomena. The main hall is to be occupied, of course, by the children who have come to enjoy the experimental session, and the BBC director, a senior well trained to control this audience, keeps the attention explaining how the volunteers are expected to enter and exit the scene. While he proceeds to the customary notice, that "*all the demonstrations here are done under controlled conditions and you should not try to repeat them at home*", I dream of a zoom over a first bowl with some of the bank notes, and the teacher starting the lecture.

He wears the white coat and in a rapid gesture drops a match in the bowl, and the pieces of money take fire. The camera goes from the flames to the speaker, who starts:

Money. Man made, artificial, unnatural. Real and Untrue.

And then a slide of a stock market chart:

But take a look to this graph: Why does it move with the same equations that a grain of pollen? Why does it oscillate as randomly as a quantum mechanic system?.

Indeed. It is already a popular topic that the equations used for the derivative market are related to the heat equation, and there is some research running in this address. But the point resonating in my head was a protest, formulated[20] a couple of months before by Mr. W.T. Shaw, a researcher of financial agency, Nomura:

"Money analysts get volatility and other parameters from the measured market data, and this is done by using the inverse function theorem. If a function has a derivative non zero in a point, then it is invertible at this point. But, if

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we are working out discrete calculus, if we are getting discrete data from the market, how can we claim that the derivative is non zero? Should we say that our derivative is *almost non zero*? What control do we have over the inversion process?".

Most meditations in this sense drive oneself to understand the hidings under the concept of stability of a numerical integration process.

But consider just this: discrete, almost zero, almost nonzero calculus!. It is a romantic concept by itself. Infinitesimals were at the core of the greatest priority dispute in Mathematics. On one side, at Cambridge, the second Lucasian chair, Newton. On the other side, at the political service of the elector of Mainz, the mathematical philosophy of Leibniz. And coming from the dark antiquity, old problems: *How do you get a straight line from a circle? How do you understand the area of any figure? What is speed? Is the mathematical continuum composed of indivisible "individua", mathematical "atoms without extension"?*

Really all the thinking of calculus is pushed by two paradoxes. That one of the volume and that one of the speed.

The first one comes, it is said, from Democritus. Cut a cone with a plane parallel and indefinitely near to the basis. Is the circle on the plane smaller or equal than the basis?

Other version makes the infinite more explicit. Simply cut the cone parallel to the basis. The circle in the smaller cone should be equal to the one in the top of the trunk. But this happens for every cut, lets say you make infinite cuts, always the circles will be equal. How is that different of a cylinder? You can say, well, that the shape, the area, decreases between the cuts, no in the cuts. Ok, good point. But take a slice bounded by two cuts. As we keep cutting we make the slice smaller, indefinitely thinner, until the distinction between to remove a slice and to make a cut is impossible. How can this distinction be kept? And we need to kept it in a mathematically rigorous way, if possible.

The second paradox is a more popular one, coming from the meditations of Zeno. In more than one sense, it is dual to the previous one. Take time instead of height and position instead of circular area. How can an arrow to have a speed? How can an arrow to change position if it is resting at every instant? In other version, it is say that it can not move where it is fixed, and it can not move where it is not yet. Or, as Garcia-Calvo, a linguist and translator of Greek, formulated once: "One does not kiss while he lives, one does not live when he kisses".

Seriously taken, the paradox throws strong doubts about the concept of instantaneous speed. Or perhaps about the whole conception of what "instantaneous" is. While Democritus asked how indefinite parts of space could add up to a volume, Zeno wonders how a movement can be decomposed to run across indefinite parts of time. A Wicked interplay.

It is interesting to notice that physicists modernly do not like to speak of classical mechanics as a limit $\hbar \to 0$, but as a cancellation of the trajectories that differ from the classical one. Perhaps this is more acceptable. Anyway, the paradoxes were closed in false by Aristotle with some deep thoughts about the infinity. Old mathematics was recasted for practical uses and, at the end, lost.

But in the late mid ages, some manuscripts were translated again. A man no far from my homeland, in the Ebro river in Spain, took over a Arabic book to be versed into Latin. It was the *Elements*, that book all you still "suffer" in the first courses of math in the primary school, do you remember? Circles, angles, triangles, and all that. And, if your teacher is good enough, the art of mathematical skepticism and proof comes with it.

Of course the main interesting thing in the mid ages is a new art, Algebra. But that is a even longer history. To us, our interest is that with the comeback of geometry, old questions were again to be formulated. *If continuous becomes, in the limit, without extension, then is such limit divisible? And if it is indivisible, atomic, how can it be?*

That automatically brings up other deterred theory to compare with. That one which postulates Nature as composed with indivisible atoms but, having somehow extension, or at least some vacuum between them.

Such speculation had begun to be resuscitated in the start of the XVII, with Galileo Galilei himself using atomic theory to justify heat, colours, smell. His disciple Vincenzo Viviani will write, time-after, that then, with the polemic of the book titled "Saggiatore", the eternal prosecution of Galileo actions and discourses began.

Mathematics was needing also such atomic objects, and in fact the first infinitesimal elements were named just that, atoms, before the modern name was accepted.

(By the way, Copernico in "De revolutionibus" explains how the atomic model, with its different scales of magnitude, inspires the astronomical world: the distance of the earth to the center of the stars sphere is said to be negligible by inspiration from the negligibility of atomic scale. It is very funny that some centuries later someone proposed the "planetary" model of atoms.)

Back to the lecture. Or to the dream. Now the laboratory has activated a sort of TV projector bringing images from the past. Italy.

Viviani. He made a good effort to recover Archimedes and other classical geometers. So it is not strange that the would-to-be first lucasian, Isaac Barrow, become involved when coming to Florence. And Barrow understood how differentiation and integration are dual operations.

Noises...

Perhaps Barrow learn of it during his Mediterranean voyage

Noises of swords and pirates sound here in the TV scene, and Barrow himself enters in the lecture room.

He is still blooding from the encounter with the pirates. Greets the speaker, cleans himself, and smiles to the children in the first row:

"We become involved in a stupid war. Europe went to war about sacraments, you know, the mystery of eucharistic miracle and all that niceties. And there we were, with individia, indivisilia, atoms... things that rule out difference between substance and accidents. You can not make a bread into a divine body if it is only atoms, they say."

Indeed, someone filed a denounce against Galileo claiming that is theory was against the dogma of transustantiation[17]. Touchy matter, good for protestant faith but not for the dogmas from Trent concilium.

For a moment he raises the head, staring to us, in the upper circle. Then he goes back to the young public: *"Yes, there was war. Protestants, Catholics, Anglicans. Dogmas and soldiers across Europe. Bad time to reject Aristotle, worse even to bring again Democritus. With Democritus comes Lucretius, with Lucretius comes Epicurus. Politically inconvenient, you know. Do the answers pay the risk?"*

He goes away. He went away to Constantinople, perhaps to read the only extant copy of the Archimedean law. *Perhaps he found the lost Method. Perhaps he lost other books when his ship was burned in Venice.*

Yes, Bourbaki says (according [1]) that Barrow was the first one proofing the duality between derivatives and integration. At least, with his discrete "almost zero" differential triangle, doubting about the risks of jumping to the limit, was closer to our modern [3] view. Three or four years ago Majid, then still in Cambridge, claimed its resurrection in the non commutative calculus $f(x)dx = dxf(x-\lambda)$. Even the formulation of fermions in the lattice, according Luscher, depends on this relationship to proof the cancellation of anomalies.

Also we would note that his calculus was "renormalized" to a finite scale, as instead of considering directly $\Delta f/\Delta x$, he first scaled this relationship to a finite triangle with side $f(x)$. The freedom to choose either the triangle on $f(x)$ or the one in $f(x+\lambda)$ was lost when people start to neglect this finite scale.

Really, this is mathematical orthodoxy. Consider a series $\sin(1/n)$ and another one $1/n$, both going to zero. The quotient, then, seems to go to an indefinite 0/0, but if you scale all the series to a common denominator, call it $S_a(n)/a$, you will find that $S_a(n)$ goes to a as n increases. Wilson in the seventies made the same trick for statistical field theory (or for quantum field theory), which was at that moment crowded of problematic infinities.

There is also a infinite there in the Barrow idea, but it is a very trivial one. Just the relation between the vertical of the finite triangle and the horizontal of the small one, $\frac{f(x)}{\Delta x}$. It goes to infinity, but this divergence can be cured by subtracting another infinite quantity, $\frac{f(x+\lambda)}{\Delta x}$, so that the limit is finite[1].

Barrow died in sanctity. But in his library [13] there was no less that three copies of Lucretius "De Rerum Natura", a romam poem about atomistic Nature, already critiquized in the antiquity because in supports the Epicurean doctrine: that gods, if they exist, are not worried about the human affairs, so we must build our moral values from ourselves and our relationships with our friends and society.

In the lecture room, the slides fly one ager other. Back in the XVII, with heat, smell, colour, and other accidents, black storms blow in the air. It has been proposed that the sacred eucaristic mystery was in agreement with Aristotle, as it could be said that the substance of wine and bread was substituted by the substance of the Christ, while the accidents remained. Go tell to the Luterans.

First August 1632. The Compañia de Jesus forbids the teaching of the atomic doctrine. 22nd June 1633, Galileo recants. "Of all the days that was the one / An age of reason could have begun" [2]

In the "Saggiatore", Galileo had begun to think of physical movement of atoms as the origin of the heat. It would take three centuries for Einstein to get the Brownian key. But even that was already disentangled of pure mathematics, so it took some other half century for to discover the same equations again, now for the stock market products. The history has not finished.

It sounds not to surprise that Dimakis has related discrete calculus to the Ito calculus, the basic stochastic in the heat equation, the play of money, that Black and Scholes rediscovered. In some sense it is as if the physical world described by mathematics were dependent on mathematics only, as it it were

[1] This example was provided by Alain in Vietri at the request of the public, but it was not to be related to the hoft algebra of trees, as far as I can see

the unique answer to organise things in a localized position.

Dark clouds will block our view. *Barrow survived to his ship and crossed Germany and come home to teach Newton. But Newton himself missed something greater when, for sake of simplicity, the limit to zero was taken. In this limit, he can claim the validity of series expansion to solve any differential equation, so it is a very reasonable assumption.* Yes, but it had been more interesting to control the series expansion even without such limit.

Leibniz come to the same methods and the jump to the limit is to be the standard. Mathematical atoms, scales and discrete calculus will hide its interplay with the infinitesimal ones for some centuries. Only two years ago Mainz, in voice of Dirk Kreimer, got again the clue to generalized Taylor series. The wood was found to be composed of trees.

Vietri is a small village in the Tyrrenian sea, near Salerno, looking at the bay of Amalfi. Good fishing and intense *limoncello* liquor. About the 20th of Mars, 1998, there Alain come, to explain the way Kreimer had found a Hopf algebra structure governing perturbative renormalization. The algebra of trees was not only related to Connes Moscovici algebra, but also with the old one proposed by Cayley to control the Taylor series of the vector field differential equation.

And to close the circle, Runge-Kutta numerical integration algorithms can be classified with a hoft algebra of trees. *Today it can be said[5] that the generic solution to a differential equation is not just the function, but also some information codified in the Butcher group.* Which can be related to the physics monster of this century, the renormalization group we have mentioned before.

Can we control the inversion of the Taylor series using trees? Then these doubts about the inverse function theorem in stock markets could be sorted out. Will us be able to expand in more than one variable? Still ignorabimus.

Worse, it is progressively clear that this kind of pre-Newtonian calculus are a natural receptacle for quantum mechanics. Even the stock market Ito equations are sometimes honoured as "Feynman-Kac-Ito" formula, so marking its link with the quantum world. The difference comes from the format of the time variable in both worlds. One should think that time is more subtle than the intuitive "dot" that Newton put in the fluxion equations.

Perhaps we are now, then, simply correcting a flaw made three hundred years ago. A flaw that Nature pointed to us, when it was clear the failure of classical mechanics in the short scale.

But how did come to exist the conditions to such failure? Why did geometry need to be reborn in the XVIIth century? Why did the mathematicians so little information, so that the mistake has a high probability[2] to happen?

If calculus, or "indivisilia", were linked to atomism already in the old age, it could be a sort of explanation. Archimedes explain that Democritus was the first finding "without mathematical proof" the volume of the cone. And with Democritean science there was political problems already in Roman times:

A leftist scholar, Farrington[12], claims that political stability was thought to reside in some platonic tricks, lies, proposed in "Republic": a solid set of unskeptic faith going up the pyramid until the divine celestial gods. Epicurus is seen as a fighter for freedom, putting at risk social stability. Against Plato ideals,

[2] And, by the way, Why there was not the slightest notion of probability in the old mathematics texts, so they were unable even to consider it?

Epicurus casts in his help the Ionian learning, including Democritean mathematics and physics. According Farrington, if government aspires to platonic republic, it must control or suppress such kind of mathematics and physics.

No surprise, if this is true, that the man who understood the floating bodies and the centers of gravity, who stated the foundations and integration, the process of mechanical discovery and mathematical rigor, who was fervently translated by Viviani and Barrow, was killed and dismissed. To be buried without a name, it could have been Archimedes' own wish. But to get his books left out of the copy process for centuries until there were extinct, that is a different thing.

"Only a Greek copy of the Floating Bodies extant, found at Constantinople. See here the palimpsest, the math almost cleared, a orthodox liturgy, perhaps St John Chrisostom, wrote above instead."

"Let me to pass the pages, and here you have, the only known version of Archimedes letter about the Mechanical Method. Read only by three persons, perhaps four, since it was deleted in the Xth century. Was this reading the goal of Barrow in orient?"

And even then, is it the same? It has been altered, the last occasion in this century, when someone painted four evangelists over the Method.

Hmm. Last Connes report [7] quotes the Floating Bodies principle, doesn't it?. More and more associations. Stop!

And recall.

Had our research been different if we had been fully aware of the indivisilia problems, if we had tried hard for rigour? Perhaps. Only in the XVI, rescued Apollonius and Archimedes, the new mathematics re-taken the old issue. And, as we have seen, in a dark atmosphere. Enough to confuse them and go into classical mechanics instead of deformed mechanics. Instead of quantum mechanics.

The matter of copernicanism has been usually presented as a political issue. Brecht made a brilliant sketch of it while staying in Copenhagen with some friends, physicists which become themselves caught in the dark side of our own century. We suspect that the matter of atomism also has suffered because of this, and now it appears that Differential Geometry itself has run across a world of troubles since the assesination of his founder in Sicily two milenia ago. The truth has been blocked again and again by the status quo, by the "real world" preferring tales of stable knowledge to inquisitive minds learning to crawl across, and with, the doubt.

If the goal of the Christmas lectures is to move young people to start a career in science, here is our statement: it is for the honour of human spirit. It is because understanding, reading the book of nature, we calm our mind. Call it ataraxia, athambia, or simply tranquility.

But we have been mistaken, wronged, delayed. The world has tricked, outraged, raped us. *When we have been wronged, should we not to revenge?* Then our main motivation is here: when reality is a lie, the song of science must be a song of vengeance. A man in Syracuse has been killed, all our milenia-old family has been dishonoured. Every mother, every child, every man in Sicily knows then the word. Vendetta.

Go to your blackboards, my children, and sing the song. Just to clear any trace of pain in the soul.

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