Rhythmos, Diathige, Trope

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Abstract

It is argued that properties of Democritus atoms parallel those of volume forms in differential geometry. This kind of atoms has not "size" of finite magnitude.

Aristotle, in Metaphysics I, 4, 985b [1, 67A6] comments Democritus' characterization of atoms, which consists in the three words above in the header of this note. His translation is, respectively, *skhema, taxis, thesis*. Roughly, Democritus wording could translate to structure, contact and direction, while Aristotle translation stands for figure, order and position.

In any case, it should puzzle to commentators the absence of a property standing for "size". It seems than most people, starting already in the antiquity, ascribes to Aristotle "figure" this property. And from here starts a centuries wide debate about how can atoms be physically indivisible if they have a finite extension and, then, they are mathematically divisible. I suggest that such finite size was not present in the original theory, and the difficulties of commentators to access (or to understand) the atomists books induced them to imagine such property.

The mistake comes from other notices of Aristotle about atoms. Indeed, in Generation and Corruption I 8, 326a, and in On Heaven IV, 2, 309a (see [1, 68A60]) it is claimed that the weight of an atom is proportional to its magnitude or size.

It is probable that even commentators in the Ancient Age had difficulties to access a key text of this "weight": the letter of Archimedes to Erathostenes, popularly known as "The Method", whose only (partly) extant copy was found at the very end of the XIXth century. In "The Method", Archimedes explains that some results on the volume of planar and solid figures can be obtained by assigning a weight to the infinitesimal slices which compose the figure, and then adding the weights using some mechanical method, for instance counterweighting every slice via lever rule. Archimedes rejects the mathematical validity of proofs coming from this method, but he suggest it is a good hint to guess a result that can be then proofed using acceptable methods, such as Eudoxus exhaustion.

Moreover, Archimedes starts his letter giving us a reference that no other author had preserved: that Eudoxo found the proof of the relationship between volume of cone and cylinder, but that Democritus was the first one enunciating this formula, without an acceptable proof.

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This notice is so recent that it is not contained in [1], I believe. On other hand, [1, 68B155] gives us a later comment of Plutarch where Democritus studies the surfaces got by slicing a cone with a plane parallel to the basis. In the quote, we are informed of that Democritus was unable to answer one of the two alternatives there presented: either the surfaces were unequal, and the cone was then a kind of ziggurat of microscopical steps, or both surfaces were equal, and its addition should build a cylinder instead. The point of Democritus being unable to decide for the first option has been interpreted by commentators as marking a definite difference between physical figures and mathematical ones, the first ones having finite size atoms, the later being continuously divisible. This is not so clear under our interpretation.

The difference between slices in the cone could be one of the points considered by Epicurus to claim [1, 68A43] that "atoms can not have any magnitude...but there is between them certain differences of magnitude". In any case, Epicurus comes after Platon and Aristotle[1], so the confusion related to magnitude (megethos) was already present.

Lastly, and jumping a bunch of centuries, we should remark the insight of XVIIth century geometers, starting with Cavalieri group -and perhaps also Viviani and Barrow[2] should be named here-. On some unknown basis, they decide to call the infinitesimal components of a figure "atoms", and then they proceed to build the modern integral calculus as a sum of atoms.

Independently of if Viviani or others got access to antiquity work, it can be postulated that the name comes obvious if we think that every geometrical object with magnitude can be divided. (Differential Geometric) atoms have not extension in the usual sense, so the division postulate does not apply to them. They have not got extension (size), so they are atomic.

To understand the properties of Democritean atoms, it is useful to try to solve the decomposition of a cone in slices and its addition. Every planar slice has the area of a circle, and it is to be put in a determinate ordering. This kind of slice has not magnitude in the direction of the symmetry axis, but still has in a finite area. But the same technique can be applied on the area, then on the resulting lines. The resulting elements will have not magnitude, but in order to be able to add them we need to preserve its ordering and spatial orientation, as well as a weight.

Thus we have the three properties claimed.

Archimedes calculus, in the method, uses division into planar or linear slices, driven by the symmetry of the problems he is focusing. So we have there explicit examples of rhythmos and diathige, both needed to make the addition of the slices. Ordering or diathige is also fundamental for the rigorous proof, as the convergence of volume is proofed, from Eudoxus[3] perhaps, by substracting two ordered series of inscribed and circumscribed (for the cone problem) finite slices,

[1] In [2], it is made a good case to suggest that atomism is used politically by Epicurus, to undermine the political thesis of Platon and his school. Involvement of atomists could be earlier, as some biographic comments of Democritus point out [1, 68A14,68A2,68A16]. One wonders if this confrontation could be real and to have some influence in the transmission of atomism, as the popular legend [1, 68A1] says.

[2] The interested reader can find some speculation about them in [3], where it is pointed that Barrow actually traveled to the sites where the manuscript could be, and he states to be looking for religious readings similar to the ones shared in the extant copy.

[3] For the area of each circle, an unrigorous exhaustion is ascribed to Antiphon. This sophist could share with atomists the concept of weight, see [1, 87B42]

then taking the infinite limit and getting the cancellation from the ordering in the substraction.

So, do atoms have a size? By now, it should be clear that weight is related to rhythmos, unrelated to size. Most classical arguments fall apart under this observation.

As for mathematical atoms, if they are different of physical ones, it can surely be said that they have not size, as we know that they are the indivisible slices reached in the limit process of integration. We can think of them either as the infinitesimal volume form or as the ordered product of differential one forms which build that volume form.

The idea of one forms could be read into the old theory. In modern geometry, covectors (one forms) have not size, even infinitesimal. The "size property" is carried by the vector fields, and integration is done by acting a form over a current.

In a similar manner, I should suggest that "size" is not a property of the atom, but of the vacuum where atoms are. If they are some difference between mathematical atoms and physical ones, it is that in the first ones the vacuum is exhausted by infinite planes, while in the second cases something, perhaps related to the possibility of interaction, change or movement, obstructs the possibility of completely filling the vacuum.

Which could be the origin of the obstruction is still unclear to me. From a modern point of view, we known (with constructions as deformed calculus, or tangent groupoids, to name two) that the way to get movement and to avoid the limit at the same time is to introduce a quanta h giving us some control of the filling but still driving to classical results (via Erhenfest theorem or classical or semi-classical limits). Such arguments are far from the possibilities of the ancients. On other hand, it is clear from the variation of areas in the cone that movement, or change in length or position, can be defined without recoursing to the vacuum to cure Zeno' objections. I could be driven to believe that vacuum is needed to explain change as transformation of "molecules" or to provide playground for interaction -in atomic theory it is sometimes done via "idols" (eidolon), which should be a kind of photons if atoms were a kind of fermions-. But I could also be driven to believe on some unexpected objection coming from the development of the ones from Zeno.

As final remark, let me notice that perhaps the funniest coincidence of all is a notational one, which can serve as reminder note: We modernly use the same letter, ω, as convention for "differential form" and for "frequency".

References

[1] Diels and W. Kranz, *Die Fragmente der Vorsokratiker*, Berlin 1934-1937, 1956

[2] B. Farrington, *Science and Politics in the Ancient World*, London 1946.

[3] A. Rivero, *Dream of a Christmas lecture*, filed in xxx.lanl.gov