Lessons from numerical analysis

A. Rivero[*]

February 22, 2002

Abstract

It is reviewed how Renormalization Group, species doubling and CKM mixing are known to appear in numerical analysis: Butcher group, parasitic solutions and higher order methods

1 Butcher Group

In the late sixties Butcher [2] found a composition rule for Runge Kutta methods by using a multiplication of rooted trees. Recently the same multiplication has been found in the perturbative renormalization of Feynman diagrams (Kastler) and in the group of diffeomorphisms of a manifold (Connes and Moscovici). The recent findings were done under the cover of Hopf algebras, but C. Brouder[1] pointed the similarity with the previous work of Butcher.

Butcher group, as it is, has not enough capacity to hold the quantity of renormalized parameters in a QFT theory, so it is usually extended to decorated trees. Again, this is not unusual, as the treatment of methods beyond Runge Kutta (numerical geometrical integration methods, for instance) also uses labeled nodes to classify them.

An up-to-date review of the use of this structure in Numerical Analysis, including some kinds of decorated trees, will be found in the monograph [4] of the Geneva team. Unlabeled trees are well covered in the books of Butcher and Hairer and Wanner.

Composition of Runge Kutta methods implies scale change in a way very similar to Wilson's view of the renormalization group. A RK method (or two different ones) can be applied two times, from $y(x)$ to $y(x+h)$ and then, taking $y(x+h)$ as starting value, to $y(x+2h)$. But the composition is again a RK method, from $y(x)$ to $y(x+h)$, with a more complicated description, in the same way that a block-spin composition gives as a new lattice Hamiltonian with a more complicated action.

It is possible to build a method, with an infinite (continuous) set of RK parameters, giving as result the exact solution of the integration problem. This

---

[*]Dep. Informatica, Univ Carlos III Madrid, Spain

is called the Picard method, and its associated B-series can be seen to have special properties preserving its form under scaling.

The undecorated Butcher group is powerful enough to control the change of scales in a structure pertinent to quantization, the tangent groupoid. Continuous functions over this groupoid are known to be determined by Weyl quantization, and the points of the groupoid are parametrized by a variable that jumps from a role of cutoff to a role of Plank constant. This should happen by a spontaneous apparition of scale associated to the choosing of renormalization point, but the details are of the action of Butcher Group inside the groupoid are still to be studied.

2 Doubling

It was noticed by Rothe [6, p.48 ff] that the multiplicity of solutions in multisteps methods, in the particular case of the symmetric derivative method, corresponds to the fermion doubling of lattice QCD.

Numerical methods rule out the doubled solution by imposing a continuity in the solution, which it turn imposes to the parasitic solution a coefficient of order h^2. This is enough for the method to work, although it remains weakly unstable.

In field theory, for fermions, the extra solution can hide under the cover of a different representation of gamma matrices, and it can not be eliminated. So d-dimensional field theory will produce 2^d fermionic degrees of freedom when discretized.

In lattice QFT, $d = 4$, the doubling can be reduced to four species by using Susskind formulation. This is just the discrete version of Dirac-Kahler equation, a old recipe from Käler where differential forms are used to implement a Dirac equation containing four degenerate species of fermions. It is unknown if degeneracy can be broken in order to obtain the four fundamental fermions: there is a huge difference of masses and charges, specially the confinement of two fermions. In [5] it is speculated that the quark sector u, d should be seen as differential forms corresponding to angular coordinates, while the leptonic sector e, ν should be the radial part of a volume form $d\theta \wedge d\phi \wedge dr \wedge dt$. But no further work has been done in this direction.

3 Cabbibo angles

If we expand a function in powers of the cutoff,

$$f(x + nh) = f(x) + nf'(x)h + \frac{n^2}{2}f''(x)h^2 + ...$$

we can see the basis of multi-step methods: by choosing adequate combinations of $f(x), f(x - h), f(x - 2h), ...$ it is possible to approach $f'(x)$ at higher orders of the step size h. For instance, if we work with $f(x), f(x - h), f(x + h)$,

the symmetric derivative approaches the continuous one with order h^2. Besides chiral preservation, this is a motivation to work with symmetric derivatives. Forward or backward derivatives have only order h.

Now, for higher orders, the symmetric solution, or the combination of symmetric derivatives, is not optimal. So an optimal fixing of the combination must be found if we need high precision or, also, if we need to approach further derivatives $f', f'', ...$

Suppose you want to calculate $f''(x)$ by nesting some discrete derivations $(f(x + a_i + \beta_i h) - f(x + a_i))/\beta_i h$. Then the optimal combination will correspond to some adimensional relationships between parameters. If derivations are formulated as an NCG in discrete space, such relationships can be made to appear as a CKM matrix.

References

[1] C. Brouder, *Runge Kutta methods and renormalization*, hep-th/9904014

[2] J.C. Butcher, *An Algebraic Theory of Integration Methods*, Mathematics of Computation, v 29, n 117, p 79-117

[3] A. Connes et D. Kreimer, *Lessons from Quantum Field Theory*, hep-th/9904044

[4] E. Hairer et al., *Geometric Integration*, monography to appear.

[5] A. Rivero, *Some conjectures looking for a non commutative geometry*, hep-th/9804169

[6] H J Rothe, Lattice Gauge Theories, WS 1977.