Flashes of noncommutativity

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Abstract

Noncommutativity lays hidden in the proofs of classical dynamics. Modern frameworks can be used to bring it to light: *-products, groupoids, q-deformed calculus, etc.

Flash one.

Time ago, newborn Classical Mechanics simply described how the inertial law was disturbed by the action of a force. One can consider two inertial trajectories from x to y and then from y to z, where a change is applied in a point y.
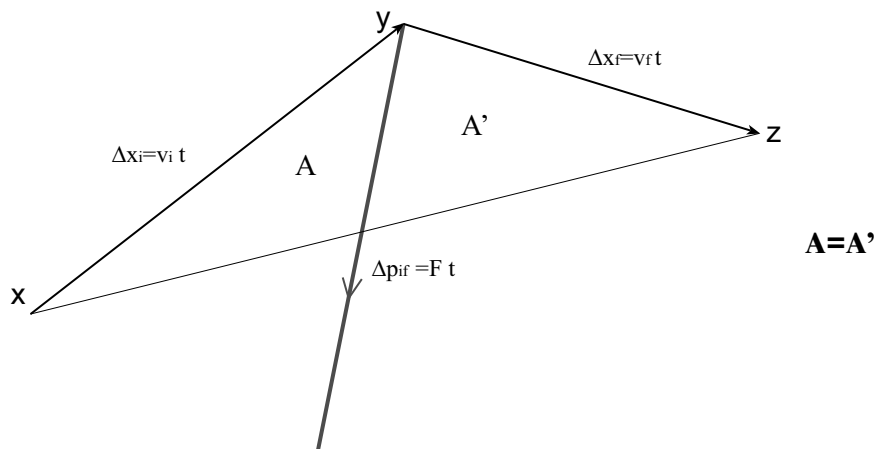
If we ask for equal-time segments, then areas A and A' are equal too. In modern language, we are showing that dynamics of a physical system is given by the rule

$$\Delta x_i \times \Delta p = \Delta x_f \times \Delta p$$

By applying this principle to central forces Newton was able to introduce time in geometry, mimicking Kepler second law. This is proposition 1 of Book I in the Principia. Historians tell us that this proposition was rebuilt at least three times while doing the built, and it was already present in the previous paper *De Motu*.

It is very troublesome to define evolution, or consistence within a trajectory, by claiming the equality of two areas, and then asking both areas to go to zero. Paraphrasing my colleague E. Forgy (from a different context), I believe the old fathers could be asking themselves: Is mechanics just a series of $0 = 0$ statements?

Two close remarks:

- It is known that path integral measure is concentrated in continuous everywhere, differentiable nowhere, trajectories. This shows how troublesome is to try to

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approach the classical path. By the way, Feynman path integral is about the limit of an equal-time discretization of trajectory, just as Newton Proposition 1.

- Itô's stochastic calculus includes a factor $\sqrt{t}$. It should be interesting, from the point of view of didactics, to look for the geometrical origin of this root.

Flash two.

Consider the natural "elementary school" groupoid over configuration space:

$$(x \ y \ u) \circ (y \ z \ v) = (x \ z \ u + v)$$

Its algebra of functions has the product

$$(AB)(x, z, t) = \int \int A(x, y, r) B(y, z, t - r) dy dr$$

Fourier transforming the t component we see this product is equivalent to:

$$(\tilde{A}\tilde{B})(x, y, \hat{t}) = \int \tilde{A}(x, y, \hat{t}) \tilde{B}(y, z, \hat{t}) dy$$

which in turn, by changing $\epsilon \sim 1/\hat{t}$, corresponds to the product defined in the $\epsilon \neq 0$ part of Connes's Tangent Groupoid. Thus the algebra of functions in the latter is a subalgebra of the one of the "elementary school" groupoid.

The $\epsilon = 0$ part of the Tangent Groupoid is defined using the product $(x, X) \circ (x, Y) = (x, X + Y)$ of elements of the Tangent Bundle. It is well known that continuity of functions in all the groupoid is just a (de)quantization condition, as $\epsilon > 0$ defines a product of operators in Hilbert Space.

By Fourier transform we can see that Groupoid algebra in Tangent Space is pointwise product of functions in Cotangent space. And if we choose a concrete limiting procedure then we can associate a deformed, star product to the functions of the Cotangent Space. Whose star-exponential, Fourier transformed again, can be proved to be Feynman path integral.

Flash three.

While the law of areas can be used to prove angular momentum preservation, its origin is deeper, and simpler, than our modern Noether's theorem. It comes from the combination of the law of addition of vectors and the first law of Newton, and in this way it simply expresses the equality of the projection of velocities in the plane orthogonal to the variation of momentum. Thus, force -or variation of momentum-, which describes a plane, is compulsed to be a covector. And then we can look for the potential functions whose gradient is the force.

The visualization of the plane associated to the force covector let us to extend the equal-time area law to more arbitrary time steps: now we just ask for inertia law in [the projection on] this plane. Noncommutativity is still there, hidden in the definition of variation of momentum just as something with happens "between" velocity steps (whose size does not matter anymore). Lets say, when position changes there is not such thing as a change of momentum, and when momentum changes there is not such thing as a change of position.

It could be interesting to review the introduction of a gauge field. Then it is known that the new momentum (as defined from the Lagrangian) is not anymore the canonical conjugate of position. But the point is that the new field, at least in the simplest U(1) case, generates a force orthogonal to the velocity. In some sense it is exploiting a hole in the "projected inertia law" of the previous paragraph. One could

relate this exploit to the proof of Maxwell's [homogeneous] laws from that famous -and slippery- report of Feynman to Dyson. Remember that noncommutativity is needed for the proof.

Remark.

If we were used to the derivation of the Lagrangian (and the Euler-Lagrange equations) in a deductive way from Newton dynamics, as historically was done, then neither the groupoid formalism nor Feynman path-integral approach should surprise us. It is very easy to attach a Lagrangian action to an element of groupoid and then to formulate the variation mechanism. And the extremal condition can be shown to be the one of the path integral. More on this below in the bibliography.

The real quest is the use of these formalisms for fields beyond 0+1.

Flash four.

During a quantization process it always happens that we choose an ordering of operators. Above it was the limiting procedure. It is equivalent to select one concrete star product. Also, it is known that it corresponds in the path integral formalism to a choosing of what discretization method do we apply to the Lagrangian.

The classical limit of course, does not see the discretization nor the ordering.

Now, A huge -for the standards of the family- group of theorists like to study the noncommutative differential calculus coming from the rule

$$f(x).dx = dx.f(x - \lambda),$$

that approaches the usual calculus when $\lambda \to 0$.

The previous formula is a bit formal, and it is usually supplemented by choosing whether the associated difference equations are to be considered with $x \in \mathbb{Z}$ or with $x \in \mathbb{R}$. In any case, there is still the same question that in the physical methods: While keeping with the noncommutativity rule, it could be possible to do various combinations of forward and backward derivations or to choose different values of a displacement parameter such that $\lambda f'(x) = f(\lambda + x + \mu) - f(x + \mu)$ with $\mu \to 0$ when $\lambda \to 0$.

As other possibility, a doubly twisted differential calculus can be defined via

$$D_{qr} f(x) = \frac{f(qx) - f(rx)}{(q - r)x}$$

and the standard technologies

$$D_{qr} x^n = <n> x^{n-1}, \ <n> = \frac{q^n - r^n}{q - r} = \sum_{i=0}^{n-1} q^{n-i} r^i, \ \text{etc.}$$

It is interesting because from the physics side no results are published, as far as I known, beyond $0+1$, so the q-geometry approach could contain some surprises. It should be good to be aware that quantum mechanics does not need renormalization, and renormalization is about scale-keeping in a limit process.

And even in the trivial case it can be studied an "angle" q/m depending of the position x but disappearing in the commutative limit.

Flashback.

It was not the first time Newton ran into troubles with the infinitesimal limit. Back in 1666 he noticed how his binomial was the key for an algebraic approach to the

method of calculation of tangents. The combination with Barrow theorem, giving the inverse operation, was to be enough to dominate analysis during three centuries. Probably Barrow got enthusiastic by the discovery, because Newton was asked to write a small document, which was showed to Collins.

It seems that objections were raised about the method, and Newton was convinced to bury it in the coffin of unspeakable resources. At that time Barrow influence was high. He had narrowly escaped death two times while adventuring in foreign lands, trying to rebuild the methods of Archimedes, and then returning Cambridge to be awarded the (first) Lucasian chair. And Collins was also remarkable, he was the key editor in the age, even if his own math was not so impressive. So it is not surprising that Newton was refrained of speaking calculus, at least until Leibnitz happened to claim the same results.

With time -and this is a sad history- Barrow's star should decline, retired to monastic studies far from math. He evolved to write Collins that their superiors -in his Order- had completely "forbidden" him to do study in any mathematical research, and it is rumored he eventually overdoped opium until death. He was not there, then, to discuss with Newton about the Principia.

The limit of Newton's Proposition 1 was ultimately justified in empirical grounds: it fitted the known dynamics of physical bodies. It should take centuries to verify that the fit was wrong.

References

The relationship between path integral and *-products was first noticed by Pankaj Sharan [9] (I thank J. Clemente, besides general criticism, his good memory, pointing this reference).

The Tangent Groupoid is defined in Connes [3, II.5]; its relation with deformed *-products and Weyl quantization was insinuated at the end of [3, IV.D] and also in the 1995 sesion of Les Houches. It was rigorized in a first wave by Cariñena *et al.* and Landsman. The use of groupoids in mechanics, quantum and classical, has been revised by Weinstein [12, 11]. In particular the least action principle for groupoids was formulated in [11]. Dirac original paper shows that one must use Feynman formula; alternatively a derivation based in the δ' distribution is exemplified in [8]

Links between deformed differential calculus and discretization have been claimed by diverse groups during the last decade; perhaps the deepest work come from A. Dimakis, F. Müller-Hoissen and collaborators. In the math side, the laboratoire Emile Picard is doing hard work on q-differences. Also q-group work, as for instance S. Majid, is relevant. A direct line to stochastic calculus is exposed in [4]; also E. Forgy had a work published on this, but in Wilmott financial bulletin!. A recent account of the ordering problem related to path integrals can be found in [6], from where references can be backtracked.

I hold a webpage at `http://dftuz.unizar.es/~rivero/` which can be browsed for more references, specially `/research/` and `/research/ncactors.html`.

[1] M. Asorey, J.F.Cariñena and M. Paramio, *Quantum Evolution as a Parallel Transport*, J. Math. Phys. 23, v8 (1982)

[2] R. Carroll, *Discretization and Moyal Brackets*, arxiv:math.QA/0101229

[3] A. Connes, *Noncommutative Geometry*, Academic Press 1994.

[4] A. Dimakis & C. Tzanakis, J. Phys. A: Math. Gen. 33 (2000), 5267-5301, and references therein.

[5] M.V. Karasev and T.A.Osborn, *Symplectic areas, quantization, and dynamics in electromagnetic fields*, arxiv:quant-ph/0002041

[6] T. Kashiva, S. Sakoda & S.V. Zenkin, *Ordering, Symbols and Finite-Dimensional Approximations of Path Integrals*, Progr. Theor. Phys. 92(3), 669-685

[7] R. Kemmoku & S. Saito, *Phase Space Discretization and Moyal Quantization*, hep-th/9510077

[8] A. Rivero, *A short derivation of Feynman formula*, arxiv:quant-ph/9803035

[9] P. Sharan, *Star-product representation of path integrals*, Phys. Rev. D, 20(3) 414-418

[10] P. Sharan and P. Chingangbam, *Lagrangian in quantum mechanics is a connection one-form*

[11] A. Weinstein, *Lagrangian mechanics and groupoids*, Mechanics Day, Fields Institute, June 1992.

[12] A. Weinstein, *Groupoids: Unifying Internal and External Symmetry*, arxiv:math.RT/9602220

[13] C. Zachos, *A survey of star product geometry*, arxiv:hep-th/0008010