Some minor examples on discrete geometry

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By assuming a minimum value for area measurement, the emergence of quantum mechanics can be easily motivated from simple consideration of gravitational forces. Here we provide some examples and extensions that can be used for pedagogical purposes.

At the same time, the role of Planck units is shown to be of some theoretical influence even at low energies.

A QUANTUM GRAVITY HAIKU

Given a particle of mass m, for which radius will a circular gravitational orbit around the particle have the property of sweeping one Planck area in exactly one Planck time?

With this question we began a thread at PhysicsForums website [1] and the user Marcus, always optimistic, praised as a "QG haiku". As a catching motivation, I argued that below that radius, it should be possible to use *Planck time beats* to divide area into regions smaller that previous. And so, a fundamental break of physics will happen at *quantum Kepler length* of the particle m.

Of course this is exaggerated, but really the question is about emerging quantum mechanics [3, 4, 5, 6]. i.e., about to accept the discretizacion of area as a first principle, then deriving from it the rules of quantum mechanics. Markopoulou and Smolin [3] have showed that QM emerges if some stochasticity principle is incorporated to the description. Here this small puzzle shows that such principle is a requeriment, as the answer is

$$R = \frac{\hbar}{mc}$$

ie, Compton radius of the particle m.

We are saved from contradiction because relativistic quantum mechanics comes to help us: no body can be located beyond its Compton radius.

In 1949, Osborne [8], in an unnoticed (except by [9]) paper, studied the possibility of measuring the curvature of a Schwartzchild solution of mass m using geodesical triangles from a test particle. He applied quantum uncertainty to the test particle and then he derived sequentially Planck mass, Compton length and Planck length as successive bounds barring the measurement of curvature. Our example reverses the path, taking Planck length as the fundamental principle.

The results of [8] show that, while we have invoked classical gravity for simplicity, general relativity also contains the same argument. Also, similar results could surely be derived from other standpoints, for example the Veneziano [7, formula 4.2] string.

DEPENDENCE ON NUMBER OF DIMENSIONS

Note that Planck length has dissapeared above, giving place to the usual QM relationship. This is a peculiarity of gravity on 3+1 dimensional space.

Consider a generic force Gmm'/r^q so that the units of G will depend on q. In general, asking $A(t_P)$ to be a multiple n of Planck Area A_P, we have

$$2n = G^{\frac{1}{2}-\frac{1}{q}} m^{\frac{1}{2}} r^{\frac{3}{2}-\frac{q}{2}} c^{\frac{3}{q}-1} \hbar^{-\frac{1}{q}}$$

And only for the usual inverse-square law, $q = 2$, we get to cancel Newton constant.

If we assume that the value of q comes from Fourier transformation of a wave propagator, then we are forced to fix space time to be 3+1.

A DEPENDENCE ON INDETERMINACY

But there is also a dimension-independent way to inverse square forces. Time before Feynman, the mathematician J.L. Singe proposed [10] to link the potential energy $V(x,t)$ to the total energy of the photons exchanged to generate V. It did not work very well, but it suggests the following argument:

Assume that the preferent wavelength for exchanged photons is of the order of the distance r between particles, and that this exchange happens under the cloak of indeterminacy principle. Then we have a momentum of order $p \sim \hbar/r$. On the other hand the photon has an energy $E = pc = \hbar c/r$ with the associated time $t \sim \hbar/E = r/c$. Thus

$$F = \frac{\Delta p}{\Delta t} \sim \hbar c \frac{1}{r^2}$$

(To justify this Δt, imagine for instance a stable circular orbit. In this situation the particle changes momentum but keeps the energy constant. Thus the photon is virtual and it can only exist during the indeterminacy time. This is the key use of virtual, off-shell, particles)

It is possible to do the same trick for massive mediators if we start from $\Delta p = \frac{1}{c}\sqrt{E^2 - m^2 c^4}$ and, ad hoc, $\Delta t = r/c$. Then one gets a short distance approximation of yukawian force.

Thus from quantum indeterminacy it seems that forces should be always inverse square. And again if we want them to come from a wave propagator we are forced to fix space time to be 3+1.

CANCELLATION, OR INDEPENDENCE

It could be worth to research the mutual cancellation of the two previous arguments. We could define gravitational force as the result of a virtual exchange in the above way. Then the units of G should be independent of space time, and we would always be able to cancel it and Newton constant to obtain purely the Compton radius.

In exchange, the Fourier transform of the potential would be a "standard wave propagator" only for three spatial dimensions.

Regretly this mechanism imposes upon us the need to invoke quantum mechanics, thus it is muddier than the first, QG only, procedure.

EMERGENCE OF QUANTUM MECHANICS

On other hand, if we have got Compton Length, can we get quantum mechanics from it? It is tricky. Compton Length is not exactly a quantum condition, but the result of pair creation, via indeterminacy principle, for extreme localisation of energy. We could think that consistency of quantum gravity implies pair creation and Zitterbewegung, but not the whole quantum mechanics.

Still, we can try in a antique way: the Bohr-Sommerfeld quantum condition can be formulated, at least for circles, via a Newton-Kepler principle: any bound particle sweeps a multiple of Compton Area in a unit of Compton Time. This principle does not need gravity; it works for any central force. Note that we have shifted the point of view; instead of considering the mass of the central particle, here we have a fixed force field and we consider the mass of the orbiting particle.

The usual way to get BS quantisation is to invoke the De Broglie wavelength to check for destructive interference. And then, also, a bound particle sweeps a multiple of De Broglie Area in a unit of De Broglie Time. Really if we use any speed v to define area and time, the same rule apply. While in the first example Planck Length was cancelled out, here speed simplifies and we are left only with the quantisation constant.

A historically minded reader could here enjoy the setup of the area principle in Newton [2, book 1, sect 2, Prop 1]; it is defined first for discrete areas and impulses.

UGLY DIMENSIONAL ANALYSIS

Naive dimensional analysis can be used also to justify inverse-square forces. In natural units, force has a dimension $[L]^{-2}$. In absence of masses, the only scale available is the separation r, between particles. Thus either the force will have a dimensional coupling constant (spoiling renormalizability: try naive power counting) or it must use this unique scale available, then inducing a dependence on inverse square of distance.

Also, we can use a regulator mass M in adimensional way:

$$f = \frac{K}{r^2}(1 - (Mr)^p)^q$$

and then we get a sort of approximation to short range, yukawian, forces (note q=1/2, p=2 for instance).

Even if trivial, it is perhaps worth to remark that, when we add some masses, naive dimensional analysis offers the possibility of justifying constant and inverse quartic forces. The corresponding equations, with a K still adimensional, are

$$f = Kmm', \text{ and } f = \frac{K}{mm'x^4}$$

This is of some value because a constant force appears as a limit of QCD, while inverse quartic is a way to approach Fermi theory of contact interactions. The mass in this later case is known to come from the electroweak bosons M_W and M_Z and not from the fermions involved.

SCALES OF MASS

Another user of PF, nicknamed Orion1, suggested to try the two body problem instead of the Kepler one. I am a bit slow to follow Orion1' calculations, so I have redone them, basically confirming the results. Now, it is interesting to look also to the intermediate steps, so let me play them here.

We have two bodies 1,2 circling around the center of mass, thus with a common angular velocity ω such that $\omega^2 R_i = Gm_j/R^2$. Here R is the sum of both radius. The equation is consistent with the center of mass condition

$$R_1 m_1 = R_2 m_2$$

The sum of cases 1 and 2 let us to solve for ω,

$$\omega = \sqrt{G\frac{M}{R^3}}$$

Now we impose that the area $A_i(t_P)$ must be a multiple n_i of Plank Area. This translates to

$$n_i = \frac{1}{2}\sqrt{\frac{cM}{\hbar}}\frac{R_i^2}{R^{3/2}}$$

Or, using the C.M. condition to substitute R,

$$R_i = \frac{4\hbar}{c}\frac{M^2}{m_j^3}n_i^2$$

Note now that using again this condition over the already solved radiouses, we get a condition on the multiples of area, namely $(m_1/m_2)^2 = n_2/n_1$. Or, say, $m_1^2 n_1 = m_2^2 n_2$

Now lets go for the total angular momentum $L = m_1\omega R_1^2 + m_2\omega R_2^2$. Substituting and after a little algebra we get

$$L = \frac{2\hbar}{m_P}\frac{M^{3/2}}{(m_1 m_2)^{3/2}}\frac{m_1^7 n_1^4 + m_2^7 n_2^4}{(m_1^3 n_1^2 + m_2^3 n_2^2)^{3/2}}$$

Which, using the relationship between n and m, simplifies to

$$L = \frac{2\hbar}{m_P}\frac{m_1 + m_2}{m_1 m_2}m_1^2 n_1$$

Orion' case $L = \hbar, m_1 = m_2 \equiv m$ gives us, accordingly,

$$m = m_P/4n$$

Also, if we take m_1 a lot greater than m_2, we recover the initial Compton for R_2 and also we get a total angular momentum

$$L_{m_1 \gg m_2} \approx 2n_1\hbar\frac{m_1^2}{m_P m_2} = 2n_2\hbar\frac{m_2}{m_P}$$

which shows that Planck mass keeps its role as a bound.

Last, a interesting mistake happens if we try to impose simultaneously low quantum numbers (n1 and n2 small) and big mass differences (m1 a lot greater than m2). Then we are driven to write

$$L_{m_1 \gg m_2}^{WRONG} \approx 2n_1\hbar\frac{m_1}{m_P}(\frac{m_1}{m_2})^{\frac{3}{2}}$$

that is not completely out of physical ranges, if for instance we put $m_1 = 175$ GeV and m_2 of the order of neutrino differences. The first section of this note has taught us that QFT scales can be implied by cancelling planckian scales. This last equation, even if unjustified, tell us that plankian scales can be adequate to study the span of masses in the known standard model of particles. [12]

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[11] Alejandro Rivero *Standard Model Masses and Models of Nuclei*, http://arxiv.org/abs/nucl-th/0312003
[12] for the next step in the ladder, check the plots at [11]; the masses of the standard model, in turn, could generate the extra coupling that guarantees the scale of nuclear stability.