Critical dimension of Spectral Triples

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Abstract

It is open the possibility of imposing requisites to the quantisation of Spectral Triples in such a way that a critical dimension D=26 appears.

From [1] it is known that commutative spectral triples contain the Einstein Hilbert action, which is extracted by using the Wodziski residue over $\displaystyle{\not}D^{-2}|{\not}D|^n$, being ${\not}D$ a Dirac operator.

The theorem was initially enunciated [1] with a complicated proportionality factor,

$$c_n = \frac{n-2}{12} \frac{1}{(4\pi)^{n/2}} \frac{1}{\Gamma(n/2+1)} 2^{[n/2]}$$

and initial proofs where given independently by Kastler, and by Kalau and Walze. There it was already noticed that the normalisation of the residue was somehow arbitrary. Actually, the factor in the previous expression contains three elements:

-a volume Ω_n of the n-dimensional sphere

-the dimension $2^{[n/2]}$ of the fiber of the Dirac operator.

-an extant term $\frac{n-2}{24}$ (!!!)

Further development of the theory has driven to include the two first elements in the normalisation of the residue, so that the definition coincides with the generic integral over a non commutative manifold. Besides, modern

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1

proofs of the theorem get the extra factor in a very independent way: during a expansion of the spectral density kernel, it comes [5, eq. 8.12]from a coefficient $\frac{n-2}{2}(\frac{1}{6}r(x) - c(x))$ when one applies Lichnerowicz formula, $c = \frac{1}{4}r$.

Thus the right normalised form of the fundamental theorem for commutative spectral triples has evolved to show explicitly the extra factor. For instance already [4, pg. 33] enounces

$$S(\slashed{D}) = -\frac{n-2}{24} \int_M s\sqrt{\det g}\, d^n x$$

(the minus sign still there is due to Euclidean Gravity, which is the usual formulation in the context of non commutative geometry) The change in normalisation can be traced back to [2, Th. 11.2], in a book which is to NCG theory as Polchinski's is to string theory.

Coming naturally from the mathematical development, it seems that nobody has raised any issue about this extant factor. But it is evident that if we require it to be equal to one, we are imposing space time dimension n=26, an amusing coincidence. Now, is there some situation where the extant factor can be required to be unity? Surely some different ways can be found to impose this requeriment, as it happened during the evolution of string theory.

Our first suggestion is that a perturbative quantisation of the action given by the normalised trace of the spectral triple will accumulate powers of this term, so it will coincide with perturbative quantisation of gravity only when the factor is 1. Thus we claim that a perturbative quantisation of commutative spectral triples gives gravity only if the dimension of the triple is n=26.

At the moment we have not argument to tell that quantisation in other number of dimensions is inconsistent. This is slightly different from the proofs in string theory, where consistency implies gravity and at the same time consistency requires 26 dimensions.

Nature has given us, up to today, sort of 24 degrees of freedom: 12 elementary fermions and 12 gauge bosons-. Bosonic strings have, of course, 24 transverse directions, but no fermions. Heterotic strings have a narrow miss trying to score this target, but model builders opt towards a not straightforward implementation of the Standard Model spectrum. Having another theory with criticality at D=26 could guide us towards the M[issing] theory. Let us point out that in NCG the Higgses are very coordinate-like, so perhaps

they can be counted jointly with space-time coordinates in order to partner the fermions.

The intriguing point in taking a NCG approach is that we are not using any tool from string theory. Thus we confront two alternatives: either the existence of a critical dimension D=26 is a phenomena independent of string theory, or commutative spectral triples can be obtained as a limit of bosonic strings. This second possibility is very interesting because it reopens the door of directly studying [bosonic?] strings in the context of noncommutative spectral triples. Formulation of spectral triples associated to string interactions had already been done in the past, before the flood of papers on non commutativity. Check [3] for an instance.

In is tempting to speculate if and how could the almost commutative limit of a concrete string theory give not only the justification of the critical factor, but also the 0-dim part of an almost commutative spectral triple, ie the Connes-Lott model.

References

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