

Renormalization in 1-D Quantum Mechanics: contact interactions

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Abstract

We have commemorated the 20th anniversary of the Wilson-Kogut review [14] by building a toy model of the W-K RG in one dimensional Quantum Mechanics... With it, we show (well, sort of) that the RG flow in the set of 1-dimensional finite range S matrices fulfilling $S_{-k}^\dagger = QS_kQ$ defines the known four parametric set of zero-range interactions.

1 Introduction

It is presented here a gadget model of the Wilson-Kogut [14] renormalization group implemented in a Quantum Mechanical problem, a bit following the mood of [7]. But our scheme is complex enough to be a good introduction before to go to full QFT-oriented reviews, as the recent one from Ball-Thorpe [2].

Examples of the renormalization group in QM have been built using the traditional beta function setup (by example, see [6]) and, recently, the path integral formalism [12]. To get a non trivial W-K flow we work with QM on R^1 . This one-dimensional setup is richer (and more complicated) than typical "tridimensional" problems in $R^3/O(3)$, which are usually reduced to problems in the one dimensional half line. By working with the full real line we are forced to calculate in matrix form, which make the problem more illuminating in the long way. This can be seen, by example, by comparing Newton [11] vs. Fadeev [5] solutions of the 1-D inverse scattering problem.

Our scheme moves close to the standard studies of contact interactions: self-adjoint extensions [1], series of hamiltonians [3], regularizations [10, 8] etc. So it can illuminate some recent conflicts in the literature, such as the status of the controversial δ' interaction (which, btw, would be scale-invariant in one dimension or at least to present characteristics close to the $1/x^2$ studies from [7]).

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This preprint represents work in course. Rigorization of convergence issues in the perturbative analysis is in process. Effort has been done to implement major features of the renormalization group, but some interesting points, as C-functions or correlation lengths are not implemented yet.

Plan of this paper is as follows: In section 1 we make some introductory remarks and the plan of the paper is presented. Section 2 defines the interaction we are going to study and sketch some needed formulae. In section 3 the wilson-kogut RG is built and fixed points are calculated. Section 4 show some examples of trajectories got directly from known solutions, for comparison. Section 5 completes the topological analysis of the RG flow calculating the stable and unstable directions at fixed points. Sections 6 and 7 sketch some examples showing how the mechanics of of regularized potentials and renormalized couplings. Both sections are mainly didactical and only needed points are detailed. We conclude in section 8 with some specific remarks about contact interactions.

2 The cut-off interaction

A localized interaction with cut-off a will correspond to an interaction which is free out of the interval $(-a, a)$, but can have any form in the interior of this interval. So we work only with data external to $(-a, a)$. Equivalently, it can be said that the cutoff "hides" or "averages" any characteristic of the interaction in distances lower than the cutoff, see figure 1.

We can characterize such interactions either by their scattering matrix,

$$S_k = \begin{pmatrix} T^r & R^l \\ R^r & T^l \end{pmatrix} \quad (1)$$

or by some matrix specifying the boundary conditions in $-a, a$. An useful one, given its dimensional and scaling properties [3], is $M \equiv \begin{pmatrix} \alpha + \rho & -\rho e^{i\theta} \\ -\rho e^{-i\theta} & \beta + \rho \end{pmatrix}$

$$\begin{pmatrix} -\psi'(-a) \\ \psi'(a) \end{pmatrix} = M_k^a \begin{pmatrix} \psi(-a) \\ \psi(a) \end{pmatrix} \quad (2)$$

Where the parameters in M are reals, but can become indeterminates or infinity for some interactions. In such case, we could use other formulations [13, 9], closer to the standard formalism of self adjoint extensions.

Note that, in principle -and forgetting some of inverse scattering theory-, different hamiltonians could be localized in the same interval with equal conditions at the boundary; thus the a -cutoff in some sense hides data about the interaction to distance less than $2a$.

The interaction being free out of this interval, the asymptotic solution of the Schroedinger equation must remain valid over all this zone $\{R - (-a, a)\}$.

We can relax this condition to be only "with range less than a"

Thus we can use the explicit definition of the S -Matrix to connect the boundary conditions at both sides of the interval. As we will need it in our examples, lets sketch the formulae. For each eigenvalue k , we can chose two independent solutions u_1, u_2 of the Schroedinger eq. fulfilling:

$$u_1(x) = \begin{cases} e^{ikx} T^l & x > a \\ e^{ikx} + e^{-ikx} R^l & x < -a \end{cases} \quad (3)$$

$$u_2(x) = \begin{cases} e^{-ikx} + e^{ikx} R^r & x > a \\ e^{-ikx} T^r & x < -a \end{cases} \quad (4)$$

$$u_2(x) = \begin{cases} e^{-ikx} + e^{ikx} R^r & x > a \\ e^{-ikx} T^r & x < -a \end{cases} \quad (5)$$

$$u_2(x) = \begin{cases} e^{-ikx} + e^{ikx} R^r & x > a \\ e^{-ikx} T^r & x < -a \end{cases} \quad (6)$$

and evaluate them at $-a$ and a to solve for the matrix M_k^a ; and reciprocally for S_k . We get the following relationships.

$$M_k = -ik \begin{pmatrix} \frac{1 - e^{2iak} R^l + e^{2iak} R^r - e^{4iak} R^l R^r + e^{4iak} T^l T^r}{1 + e^{2iak} R^l + e^{2iak} R^r + e^{4iak} R^l R^r - e^{4iak} T^l T^r} & \frac{2T^r}{1 + e^{2iak} R^l + e^{2iak} R^r + e^{4iak} R^l R^r - e^{4iak} T^l T^r} \\ \frac{2T^l}{1 + e^{2iak} R^l + e^{2iak} R^r + e^{4iak} R^l R^r - e^{4iak} T^l T^r} & \frac{1 + e^{2iak} R^l - e^{2iak} R^r - e^{4iak} R^l R^r + e^{4iak} T^l T^r}{1 + e^{2iak} R^l + e^{2iak} R^r + e^{4iak} R^l R^r - e^{4iak} T^l T^r} \end{pmatrix} \quad (7)$$

$$S_k = -e^{-2iak} \begin{pmatrix} \frac{2e^{i\theta} ik\rho}{\alpha\beta - i\alpha k - i\beta k - k^2 + \alpha\rho + \beta\rho - 2ik\rho} & \frac{\alpha\beta - i\alpha k + i\beta k + k^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha k - i\beta k - k^2 + \alpha\rho + \beta\rho - 2ik\rho} \\ \frac{\alpha\beta + i\alpha k - i\beta k + k^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha k - i\beta k - k^2 + \alpha\rho + \beta\rho - 2ik\rho} & \frac{2e^{-i\theta} ik\rho}{\alpha\beta - i\alpha k - i\beta k - k^2 + \alpha\rho + \beta\rho - 2ik\rho} \end{pmatrix} \quad (8)$$

There are no problem going from one description to the other, as here the only role of both S_k and M_k here is to select a pair of eigenfunctions. ¹

To be fully "Wilson-Kogut compliant" and draw the renormalization flow in the space of fixed cut-off theories, it is need to work with the adimensional matrices \tilde{S}_k, \tilde{M}_k :

$$S_k \equiv \tilde{S}_{ak} \quad M_k \equiv \frac{1}{a} \tilde{M}_{ak} \quad (9)$$

In this form, the relationship becomes:

$$\tilde{M}_k = -i\tilde{k} \begin{pmatrix} \frac{1 - e^{2i\tilde{k}} R^l + e^{2i\tilde{k}} R^r - e^{4i\tilde{k}} R^l R^r + e^{4i\tilde{k}} T^l T^r}{1 + e^{2i\tilde{k}} R^l + e^{2i\tilde{k}} R^r + e^{4i\tilde{k}} R^l R^r - e^{4i\tilde{k}} T^l T^r} & \frac{2T^r}{1 + e^{2i\tilde{k}} R^l + e^{2i\tilde{k}} R^r + e^{4i\tilde{k}} R^l R^r - e^{4i\tilde{k}} T^l T^r} \\ \frac{2T^l}{1 + e^{2i\tilde{k}} R^l + e^{2i\tilde{k}} R^r + e^{4i\tilde{k}} R^l R^r - e^{4i\tilde{k}} T^l T^r} & \frac{1 + e^{2i\tilde{k}} R^l - e^{2i\tilde{k}} R^r - e^{4i\tilde{k}} R^l R^r + e^{4i\tilde{k}} T^l T^r}{1 + e^{2i\tilde{k}} R^l + e^{2i\tilde{k}} R^r + e^{4i\tilde{k}} R^l R^r - e^{4i\tilde{k}} T^l T^r} \end{pmatrix} \quad (10)$$

$$\tilde{S}_k = -e^{-2i\tilde{k}} \begin{pmatrix} \frac{2e^{i\theta} i\tilde{k}\rho}{\alpha\beta - i\alpha\tilde{k} - i\beta\tilde{k} - \tilde{k}^2 + \alpha\rho + \beta\rho - 2i\tilde{k}\rho} & \frac{\alpha\beta - i\alpha\tilde{k} + i\beta\tilde{k} + \tilde{k}^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha\tilde{k} - i\beta\tilde{k} - \tilde{k}^2 + \alpha\rho + \beta\rho - 2i\tilde{k}\rho} \\ \frac{\alpha\beta + i\alpha\tilde{k} - i\beta\tilde{k} + \tilde{k}^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha\tilde{k} - i\beta\tilde{k} - \tilde{k}^2 + \alpha\rho + \beta\rho - 2i\tilde{k}\rho} & \frac{2e^{-i\theta} i\tilde{k}\rho}{\alpha\beta - i\alpha\tilde{k} - i\beta\tilde{k} - \tilde{k}^2 + \alpha\rho + \beta\rho - 2i\tilde{k}\rho} \end{pmatrix} \quad (11)$$

where now the (k -dependent) matrix terms $\alpha\beta\rho\theta$ refer to the adim matrix \tilde{M}_k .

¹BTW, We can check that making $a \rightarrow 0$ and requesting S_k to be unitary and complete, the set of admissible solutions coincide with the result got in [3], as S_k unitary iff M_k hermitian, and $H^{in} = H^{out} = L^2(R)$ would imply $M_k = \text{cte}$ when $a = 0$.

3 The RG transformation, a la Wilson Kogut. Fixed Points

As usual, we take the space $\{\tilde{S}\}$ of all the a_0 -cutoff interactions, in dimensionless form. Each interaction can be given by a unitary $S(k)$, which by standard scattering theory (see e.g. [11]) will fulfill

$$S_{-k}^\dagger = QS_kQ \quad (12)$$

where $Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and \dagger is the hermitian conjugate. We could in addition restrict ourselves to interactions invariant under time reversal. In such case, we would add the condition $S_{-k} = S_k^*$, which implies $T^r = T^l$ and the know reciprocity theorem

$$\hat{S}_k = QS_kQ \quad (13)$$

Now, we define that two theories are in the same line of the Renormalization Group flow if there are a pair of scales $\{a, e^t a\}$ such that when we apply them to its respective theories, we get the same physics (see figure 2), ie the same S -matrix in physical dimensions. Equivalently, \tilde{S}_k^t will be the result of applying a RG transformation to $\tilde{S}_{\tilde{k}}$ iff $\tilde{S}_{ak} = \tilde{S}_{ae^t k}^t$, this is,

$$\tilde{S}_k^t = T^t[\tilde{S}_{\tilde{k}}] = \tilde{S}_{e^{-t}\tilde{k}} \quad (14)$$

So the fixed points will be constant \tilde{S} matrices. This is, the subset of $U(2)$ fulfilling property (12), namely:

$$\{I_{\theta,\phi} \equiv \begin{pmatrix} e^{i\theta} \cos \phi & -\sin \phi \\ \sin \phi & e^{-i\theta} \cos \phi \end{pmatrix}\} \cup \{\pm Q\} \quad (15)$$

If we want to study only T-invariant potentials, we must add condition (13), and the set of fixed points reduces further to

$$\left\{ \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix} \right\} \cup \{\pm Q\} \quad (16)$$

Where the continuous circle of fixed points can be interpreted corresponding to Kurasov δ' [9]. The rest of contact interactions is not scale-invariant, so we need to study the flow near the fixed points to find them, as relevant parameters.

Note that there are additional assumptions on the analytic properties of S , but we dont need to impose them to determine the fixed points, so they will be commented when needed in section 5. At this point, note simply that the characterization of space of interactions is supposed to be restricted to potentials with range smaller than the cut-off.

4 The RG flow, as seen from the QM solution

Before entering in perturbative theory, it is good to give an idea about what results we expect, to more easily follow the argument. There are a four-parametric family of self-adjoint extensions to the free hamiltonian in $R - \{0\}$. In this section we calculate some subfamilies of scattering matrices for known extensions and show its form near a fixed point.

Following the standard theory([1, 13, 3]), lets take contact-interaction given by the constant matrix at a cutoff 0.

$$M_k^0 = \begin{pmatrix} \alpha + \rho & -\rho e^{i\theta} \\ -\rho e^{-i\theta} & \beta + \rho \end{pmatrix} = cte. \quad (17)$$

Its scattering matrix is:

$$S_k = - \begin{pmatrix} \frac{2e^{i\theta} ik\rho}{\alpha\beta - i\alpha k - i\beta k - k^2 + \alpha\rho + \beta\rho - 2ik\rho} & \frac{\alpha\beta - i\alpha k + i\beta k + k^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha k - i\beta k - k^2 + \alpha\rho + \beta\rho - 2ik\rho} \\ \frac{\alpha\beta + i\alpha k - i\beta k + k^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha k - i\beta k - k^2 + \alpha\rho + \beta\rho - 2ik\rho} & \frac{2e^{-i\theta} ik\rho}{\alpha\beta - i\alpha k - i\beta k - k^2 + \alpha\rho + \beta\rho - 2ik\rho} \end{pmatrix} \quad (18)$$

which, using the lenght a to remove dimensions, corresponds to a line

$$\tilde{S}_{k,a} = - \begin{pmatrix} \frac{2e^{i\theta} i\frac{k}{a}\rho}{\alpha\beta - i\alpha\frac{k}{a} - i\beta\frac{k}{a} - (\frac{k}{a})^2 + \alpha\rho + \beta\rho - 2i\frac{k}{a}\rho} & \frac{\alpha\beta - i\alpha\frac{k}{a} + i\beta\frac{k}{a} + (\frac{k}{a})^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha\frac{k}{a} - i\beta\frac{k}{a} - (\frac{k}{a})^2 + \alpha\rho + \beta\rho - 2i\frac{k}{a}\rho} \\ \frac{\alpha\beta + i\alpha\frac{k}{a} - i\beta\frac{k}{a} + (\frac{k}{a})^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha\frac{k}{a} - i\beta\frac{k}{a} - (\frac{k}{a})^2 + \alpha\rho + \beta\rho - 2i\frac{k}{a}\rho} & \frac{2e^{-i\theta} i\frac{k}{a}\rho}{\alpha\beta - i\alpha\frac{k}{a} - i\beta\frac{k}{a} - (\frac{k}{a})^2 + \alpha\rho + \beta\rho - 2i\frac{k}{a}\rho} \end{pmatrix} \quad (19)$$

of renormalized interactions. By construction, the RG transformation (??) can be compensated by a change in the "spacing" (or cutoff) a .

As explained in figure 3, we expect solution lines to be end-pointed by fixed points. Specifically, we see that:

a) For $\rho = 0$; α, β finite, which correspond to two separate half-lines, the RG flow goes from $\tilde{S}_k = Q$ to $\tilde{S}_k = -Q$. We get this result in general for any ρ, α, β finite and different of zero.

b) For ρ infinite, α, β finite, which for $\theta = 0$ is the traditional δ -interaction, we get the flow going

$$\text{from } \tilde{S}_k = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \text{ to } \tilde{S}_k = -Q \quad (20)$$

In particular, we see that the fixed point governing the δ is the Identity.

c) For ρ finite but $\alpha = \beta = 0$ (which when $\theta = 0$ is the so-called (by [1, 3]) δ' -interaction) the flow travels along

$$\tilde{S}_k = Q \rightarrow \tilde{S}_k = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \quad (21)$$

So all the interactions of this kind are to be governed either by transparent interactions or by the totally reflective one. These observations are summarized in figure 4.

/Identity/... but would we say the trivial fixed point?

It is instructive to look the interactions in the form $S = S^0 + \Delta S$ near a fixed point (around an endpoint, if we prefer to ignore RG terminology in this section). We get for "+Q"

$$\begin{aligned}\Delta S_a &= \tilde{S}_{\tilde{k},a} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \\ &= - \begin{pmatrix} \frac{2e^{i\theta} i \frac{\tilde{k}}{a} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} & 2 \frac{\alpha\beta - i\alpha \frac{\tilde{k}}{a} + \alpha\rho + \beta\rho - i \frac{\tilde{k}}{a} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} \\ 2 \frac{\alpha\beta - i\beta \frac{\tilde{k}}{a} + \alpha\rho + \beta\rho - i \frac{\tilde{k}}{a} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} & \frac{2e^{-i\theta} i \frac{\tilde{k}}{a} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} \end{pmatrix}\end{aligned}\quad (22)$$

for $-Q$:

$$\begin{aligned}\Delta S_a &= \tilde{S}_{\tilde{k},a} - \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \\ &= - \begin{pmatrix} \frac{2e^{i\theta} i \frac{\tilde{k}}{a} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} & 2 \frac{+i\beta \frac{\tilde{k}}{a} + (\frac{\tilde{k}}{a})^2 + i \frac{\tilde{k}}{a} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} \\ 2 \frac{+i\alpha \frac{\tilde{k}}{a} + (\frac{\tilde{k}}{a})^2 + i \frac{\tilde{k}}{a} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} & \frac{2e^{-i\theta} i \frac{\tilde{k}}{a} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} \end{pmatrix}\end{aligned}\quad (23)$$

and for each $I_{\theta,0}$:

$$\begin{aligned}\Delta S_a &= \tilde{S}_{\tilde{k},a} - \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = \\ &= - \begin{pmatrix} \frac{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} e^{i\theta} & \frac{\alpha\beta - i\alpha \frac{\tilde{k}}{a} + i\beta \frac{\tilde{k}}{a} + (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} \\ \frac{\alpha\beta + i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} + (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} & \frac{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} e^{-i\theta} \end{pmatrix}\end{aligned}\quad (24)$$

With this, we can see outgoing and ingoing trajectories near a fixed point:

-Lines starting from +Q with $\rho = 0$

(Notar que ΔS tambien va con determinante uno)

$$\begin{aligned}\Delta S_a &= - \begin{pmatrix} 0 & 2 \frac{\alpha\beta - i\alpha \frac{\tilde{k}}{a}}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2} \\ 2 \frac{\alpha\beta - i\beta \frac{\tilde{k}}{a}}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2} & 0 \end{pmatrix} = - \begin{pmatrix} 0 & 2 \frac{\alpha}{\alpha - i \frac{\tilde{k}}{a}} \\ 2 \frac{\beta}{\beta - i \frac{\tilde{k}}{a}} & 0 \end{pmatrix} \quad (25) \\ &= -2 \begin{pmatrix} 0 & \frac{1}{1 - \frac{i}{\alpha} \frac{\tilde{k}}{a}} \\ \frac{1}{1 - \frac{i}{\beta} \frac{\tilde{k}}{a}} & 0 \end{pmatrix} \approx_{a \ll 1} \begin{pmatrix} 0 & -i \frac{2\alpha a}{\tilde{k}} \\ -i \frac{2\beta a}{\tilde{k}} & 0 \end{pmatrix}\end{aligned}\quad (26)$$

-Starting from "+Q" with $\alpha = \beta = 0$

$$\Delta S_a = \tilde{S}_{\tilde{k},a} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} \frac{2e^{i\theta} i \rho}{-\frac{\tilde{k}}{a} - 2i\rho} & 2 \frac{-i\rho}{-\frac{\tilde{k}}{a} - 2i\rho} \\ 2 \frac{-i\rho}{-\frac{\tilde{k}}{a} - 2i\rho} & \frac{2e^{-i\theta} i \rho}{-\frac{\tilde{k}}{a} - 2i\rho} \end{pmatrix} = \quad (27)$$

$$= - \begin{pmatrix} -e^{i\theta} & 1 \\ 1 & -e^{-i\theta} \end{pmatrix} \frac{1}{1 - \frac{i}{2\rho a} \frac{\tilde{k}}{a}} \approx -i \begin{pmatrix} -e^{i\theta} & 1 \\ 1 & -e^{-i\theta} \end{pmatrix} \frac{2\rho a}{\tilde{k}} \quad (28)$$

-From $I_{\theta,0}$ with $\rho \rightarrow \infty$:

$$\Delta S_a = \tilde{S}_{\tilde{k},a} - \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} = - \begin{pmatrix} \frac{+\alpha+\beta}{\alpha+\beta-2i\frac{\tilde{k}}{a}} & \frac{+\alpha+\beta}{\alpha+\beta-2i\frac{\tilde{k}}{a}} \\ \frac{+\alpha+\beta}{\alpha+\beta-2i\frac{\tilde{k}}{a}} & \frac{+\alpha+\beta}{\alpha+\beta-2i\frac{\tilde{k}}{a}} \end{pmatrix} = \quad (29)$$

$$- \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \frac{1}{1 - \frac{2i}{(\alpha+\beta)} \frac{\tilde{k}}{a}} \approx - \begin{pmatrix} i & i \\ i & i \end{pmatrix} \frac{2(\alpha+\beta)a}{\tilde{k}} \quad (30)$$

Now, for the incoming lines we define $\underline{a} \equiv 1/a$, so $\underline{a} \ll 1$ and we get:

-Lines incoming to I_θ , $\alpha = \beta = 0$.

$$\Delta S_{\underline{a}} = - \begin{pmatrix} \frac{-\underline{a}\tilde{k}}{-\underline{a}\tilde{k}-2i\rho} & \frac{+\underline{a}\tilde{k}}{-\underline{a}\tilde{k}-2i\rho} \\ \frac{+\underline{a}\tilde{k}}{-\underline{a}\tilde{k}-2i\rho} & \frac{-\underline{a}\tilde{k}}{-\underline{a}\tilde{k}-2i\rho} \end{pmatrix} = \quad (31)$$

$$= - \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \frac{1}{1 - \frac{2\rho}{i\underline{a}\tilde{k}}} \approx - \begin{pmatrix} -i & i \\ i & -i \end{pmatrix} \frac{\underline{a}\tilde{k}}{2\rho} \quad (32)$$

-Arriving to $-Q$: (a) with $\rho = 0$

$$\Delta S_{\underline{a}} = - \begin{pmatrix} 0 & 2\frac{+i\beta\underline{a}\tilde{k}+\underline{a}\tilde{k}^2}{\alpha\beta-i\alpha\underline{a}\tilde{k}-i\beta\underline{a}\tilde{k}-\underline{a}\tilde{k}^2} \\ 2\frac{+i\alpha\underline{a}\tilde{k}+\underline{a}\tilde{k}^2}{\alpha\beta-i\alpha\underline{a}\tilde{k}-i\beta\underline{a}\tilde{k}-\underline{a}\tilde{k}^2} & 0 \end{pmatrix} = \quad (33)$$

$$= - \begin{pmatrix} 0 & 2\frac{+i\underline{a}\tilde{k}}{\alpha-i\underline{a}\tilde{k}} \\ 2\frac{+i\underline{a}\tilde{k}}{\beta-i\underline{a}\tilde{k}} & 0 \end{pmatrix} = -2 \begin{pmatrix} 0 & \frac{1}{1-\frac{\alpha}{i\underline{a}\tilde{k}}} \\ \frac{1}{1-\frac{\beta}{i\underline{a}\tilde{k}}} & 0 \end{pmatrix} \quad (34)$$

$$\approx - \begin{pmatrix} 0 & -\frac{2}{\alpha}i \\ -\frac{2}{\beta}i & 0 \end{pmatrix} \underline{a}\tilde{k} \quad (35)$$

and from lines type (b):

$$\Delta S_{\underline{a}} = - \begin{pmatrix} \frac{2e^{i\theta}i\underline{a}\tilde{k}}{\alpha+\beta-2i\underline{a}\tilde{k}} & 2\frac{i\underline{a}\tilde{k}}{\alpha+\beta-2i\underline{a}\tilde{k}} \\ 2\frac{i\underline{a}\tilde{k}}{\alpha+\beta-2i\underline{a}\tilde{k}} & \frac{2e^{-i\theta}i\underline{a}\tilde{k}}{\alpha+\beta-2i\underline{a}\tilde{k}} \end{pmatrix} = \quad (36)$$

$$= - \begin{pmatrix} -e^{i\theta} & -1 \\ -1 & -e^{-i\theta} \end{pmatrix} \frac{1}{1 - \frac{(\alpha+\beta)}{2i\underline{a}\tilde{k}}} \approx -i \begin{pmatrix} e^{i\theta} & 1 \\ 1 & e^{-i\theta} \end{pmatrix} \frac{2}{\alpha+\beta} \underline{a}\tilde{k} \quad (37)$$

(Note that the approximations here are given in a non rigorous way, simply to have a reference for the next section)

It's worth to note that the coupling constants appear clearly related to the (dimensional) constant we used to remove dimensions of k . Compare e.g. with [10, 6].

...or to give dimensions to \tilde{k}

5 Stability. Relevant et irrelevant directions

getting ourselves out of the forest...

Now, we need to develop a perturbation theory² around the fixed point directly in the S-matrix formalism. In some neighbourhood of the identity, where the exponential map is one to one, we can use the generators L_i of the $U(2)$ group, and write the perturbed system as

$$S^{\vec{a}} = S_0 e^{\vec{a}(k) \cdot \vec{L}} \quad (38)$$

with $\vec{a}(k) \in R^4$. For $\int ||\vec{a}_k|| dk$ small³, we can put it as:

$$S^{\vec{a}} - S_0 \approx \vec{a}(k) \cdot (S_0 \vec{L}) \quad (39)$$

The generators of $U(2)$ are

$$L = \left\{ \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right\} \quad (40)$$

(remember that $-iL = \{I, \sigma_x, \sigma_y, \sigma_z\}$ So, around $+Q$ we get

$$S_0^{+Q} L = \left\{ \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \right\} \quad (41)$$

And for the other fixed points we have

$$S_0^{-Q} L = -S_0^{+Q} L, S_0^{I_0, 0} L = L, S_0^{I_0, \phi} L = \dots \quad (42)$$

As our space of interactions impose restrictions to the admissible S_k , we get restrictions on $\vec{a}(k)$ depending on the fixed point. Imposing condition (12) around I_0 we get

$$a_0(-k) = -a_0(k) \quad (43)$$

$$a_1(-k) = -a_1(k) \quad (44)$$

$$a_2(-k) = a_2(k) \quad (45)$$

$$a_3(-k) = a_3(k) \quad (46)$$

Of course, the additional assumption (13) of reciprocity imposes $a_3(k) = 0$.

Now, we put the RG transformation in differential form.

$$T^{\delta t} S_k = S_{e^{-\delta t} k} \approx S_k + (O_k S_k) \delta t \quad (47)$$

where

$$O_k S_k = \left. \frac{\partial S_{e^{-t} k}}{\partial t} \right|_{t=0} = -S'(k) k \quad (48)$$

²The model is simple enough to be exactly solved, but we consider more didactical to remain close to Wilson-Kogut papers

³for simplicity, we are going to be a bit loose with this condition

We are looking for vectors $\vec{a}(k)$ such that

$$T^{\delta t} S^{\vec{a}(k)} \approx S^{\lambda \vec{a}(k)} \quad (49)$$

which to first-order amounts to

$$S^0 + \vec{a}(k) S^0 \vec{L} - \vec{a}'(k) S^0 \vec{L} k \delta t \approx S^0 + \lambda \vec{a}(k) S^0 \vec{L} \quad (50)$$

So we have an eigenvalue equation

$$-k \vec{a}'(k) S^0 \vec{L} \delta t = (\lambda - 1) \vec{a}(k) S^0 \vec{L} \quad (51)$$

which solves to

$$(\lambda - 1) = n \delta t \quad (52)$$

$$\vec{a}(k) = k^{-n} \vec{a}_0 \quad (53)$$

As an example, lets calculate the flow near I_0 with some detail. It could seem that the valid $\vec{a}(k)$ would be

$$\{0, 0, 1, 0\} \quad \textit{marginal} \quad \lambda = 1 \quad (54)$$

$$\{0, 0, 0, 1\} \quad \textit{marginal} \quad (55)$$

$$\{k, 0, 0, 0\} \quad \textit{irrelevant} \quad \lambda = 1 - \delta t \quad (56)$$

$$\{0, k, 0, 0\} \quad \textit{irrelevant} \quad (57)$$

$$\{1/k, 0, 0, 0\} \quad \textit{relevant} \quad \lambda = 1 + \delta t \quad (58)$$

$$\{0, 1/k, 0, 0\} \quad \textit{relevant} \quad (59)$$

Now, finite range condition when required in even potentials implies ⁴ for the fase shifts at small k

$$\text{even wave :} \quad \tan \delta_0 \approx 1/k \quad (60)$$

$$\text{odd wave :} \quad \tan \delta_1 \approx k \quad (61)$$

So, this condition for our space of interactions rules out the combinations

$$\{-k, k, 0, 0\} \quad (62)$$

$$\{-1/k, -1/k, 0, 0\} \quad (63)$$

as well as higher orders in k , and let us with four directions,

$$\{0, 0, 1, 0\} \quad \textit{marginal} \quad \lambda = 1 \quad (64)$$

$$\{0, 0, 0, 1\} \quad \textit{marginal} \quad \lambda = 1 \quad (65)$$

$$\{k, k, 0, 0\} \quad \textit{irrelevant} \quad \lambda < 1 \quad (66)$$

$$\{1/k, -1/k, 0, 0\} \quad \textit{relevant} \quad \lambda > 1 \quad (67)$$

⁴Well, I dont know of any rigourous proof at this moment...

which coincides with the known result, derived in the previous section. Here, the marginal direction (64) can be associated with the circle of kurasov' δ' family; the relevant one (67) is the usual δ , and the irrelevant parameter one (66) can be seen as coming from the fixed point $+Q$, then producing the line corresponding to Albeverio et al. so-called δ' (which happens to be not scale-invariant). Finally, lets note that direction (65) produces a family of scale invariant interactions which haven't time-reversal symmetry.

6 Regularizations and its flow

Given a series of cutoff interactions the RG mechanism, as explained in figure 3, let us to obtain. a renormalized interaction at a given scale a_0 ,

Lets see, as first example, the series of effective pseudopotentials $\{V_a\}$ proposed by Carreau [3]. Each V_a is zero out of the interval $(-a, a)$, and the M matrix at $\{-a, a\}$ always the same and independent of k . In such case, we get a series $\{M^a \equiv aM_0\}$ in the space of dimensionless cutoff interactions. The \tilde{S} matrix is:

$$\tilde{S}_{\tilde{k}} = -e^{-2i\tilde{k}} \begin{pmatrix} \frac{2e^{i\theta} i \frac{\tilde{k}}{a} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} & \frac{\alpha\beta - i\alpha \frac{\tilde{k}}{a} + i\beta \frac{\tilde{k}}{a} + (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} \\ \frac{\alpha\beta + i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} + (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} & \frac{2e^{-i\theta} i \frac{\tilde{k}}{a} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a} - i\beta \frac{\tilde{k}}{a} - (\frac{\tilde{k}}{a})^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a} \rho} \end{pmatrix} \quad (68)$$

where now the parameters $\alpha, \beta, \rho, \theta$ are the constants of the initial matrix M . The limit $a \rightarrow 0$ is $\tilde{S}^0 = -e^{-2i\tilde{k}} \tilde{S}^{fp}$, where \tilde{S}^{fp} one of the fixed points studied in section 5. Obviously the RG transformation moves S^0 towards S^{fp} ; so when using the RG we will move near S^{fp} and the renormalized series will converge to a renormalized interaction in the relevant line. To be concrete, we begin with a cut-off a_0 , and for each a we recover the original scale by applying $T^{\log(a_0/a)}$, thus getting:

$$(T\tilde{S})_{\tilde{k}} = -e^{-2i\frac{a\tilde{k}}{a_0}} \begin{pmatrix} \frac{2e^{i\theta} i \frac{\tilde{k}}{a_0} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a_0} - i\beta \frac{\tilde{k}}{a_0} - \frac{\tilde{k}}{a_0}^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a_0} \rho} & \frac{\alpha\beta - i\alpha \frac{\tilde{k}}{a_0} + i\beta \frac{\tilde{k}}{a_0} + \frac{\tilde{k}}{a_0}^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a_0} - i\beta \frac{\tilde{k}}{a_0} - \frac{\tilde{k}}{a_0}^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a_0} \rho} \\ \frac{\alpha\beta + i\alpha \frac{\tilde{k}}{a_0} - i\beta \frac{\tilde{k}}{a_0} + \frac{\tilde{k}}{a_0}^2 + \alpha\rho + \beta\rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a_0} - i\beta \frac{\tilde{k}}{a_0} - \frac{\tilde{k}}{a_0}^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a_0} \rho} & \frac{2e^{-i\theta} i \frac{\tilde{k}}{a_0} \rho}{\alpha\beta - i\alpha \frac{\tilde{k}}{a_0} - i\beta \frac{\tilde{k}}{a_0} - \frac{\tilde{k}}{a_0}^2 + \alpha\rho + \beta\rho - 2i \frac{\tilde{k}}{a_0} \rho} \end{pmatrix} \quad (69)$$

and we see that in the limit $a \rightarrow 0$ we recover (19), as expected.

To go for a more complicated example, lets use the two-deltas regulator for δ' interaction (this would be as a core-shell regularization), $V = \frac{g}{2a}(\delta(x+a) - \delta(x-a))$

the matching conditions are:

$$\frac{2ika}{g}((A-1)e^{-ika} - (B-R^l)e^{ika}) = e^{-ika} + R^l e^{ika} = Ae^{-ika} + Be^{ika} \quad (70)$$

$$-\frac{2ika}{g}(Be^{-ika} - (A - T^l)e^{ika}) = T^l e^{ika} = Be^{-ika} + Ae^{ika} \quad (71)$$

The scattering matrix is:

$$S_k^a = \begin{pmatrix} \frac{1}{1 - (\frac{g}{4ak})^2 (e^{4ika} - 1)} & e^{-2ika} \frac{(e^{4ika} - 1)(1 - \frac{ig}{4ak})}{\frac{4ak}{ig} + \frac{ig}{4ak} (e^{4ika} - 1)} \\ \dots & \dots \end{pmatrix} \quad (72)$$

which goes to $-Q$ as $a \rightarrow 0$.

In the adim space this stuff becomes:

$$\frac{i2\tilde{k}}{g}((A - 1)e^{-i\tilde{k}} - (B - R^l)e^{i\tilde{k}}) = e^{-i\tilde{k}} + R^l e^{i\tilde{k}} = Ae^{-i\tilde{k}} + Be^{i\tilde{k}} \quad (73)$$

$$-\frac{i2\tilde{k}}{g}(Be^{-i\tilde{k}} - (A - T^l)e^{i\tilde{k}}) = T^l e^{i\tilde{k}} = Be^{-i\tilde{k}} + Ae^{i\tilde{k}}$$

$$\tilde{S}_k^a = \begin{pmatrix} \frac{1}{1 - (\frac{g}{4k})^2 (e^{4i\tilde{k}} - 1)} & \frac{(e^{2i\tilde{k}} - e^{-2i\tilde{k}})(1 - \frac{ig}{4k})}{\frac{4k}{ig} + \frac{ig}{4k} (e^{4i\tilde{k}} - 1)} \\ \dots & \dots \end{pmatrix} \quad (74)$$

and the limits are:

$$\tilde{S}_k^{a \rightarrow 0} = \tilde{S}_k^a \quad \tilde{S}_{k \rightarrow 0}^a = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad (75)$$

This shows the qualitative difference between both formalisms. In (72) we simply take the limit $a \rightarrow 0$ expecting in to be well-behaved (as it happens in this simple case). In the RG approach, we first got the limit point, and then we look for the fixed point attracting it.

Here, any limiting procedure will carry us inexorably to the Dirichlet fixed point (see Seba). If we want to get a non trivial result, we need to implement a dependence for the coupling constant. This can be seen a la Tarrach in the implicit equations; if $g(a) \rightarrow 0$ the eq (73) has a indetermination and we will need go to $g'(a)$.

7 Coupling constant renormalization

Lets continue with the previous example. We ask for a $g(a)$ dependence giving us a non trivial limit. The example is simple enough to directly read the answer from (72). Regretly the RG mechanism in QM is too simple [7] and it is not possible to get remarkable differences. Lets sketch the method anyway.

Equation (74) let us define a subset $\{\tilde{S}^a(g)\}$ of interactions in the space S . We need to get series $\tilde{S}^a \equiv S_{k/a}(g(a))$ such that the limit point $a \rightarrow 0$ falls in the attraction point of a non trivial fixed point. Any $g(a)$ going to zero as $a \rightarrow 0$ makes the trick, falling directly in the fixed point $I_{0,0}$. Furthermore, we want the corresponding renormalized series $T^{-\log t(a)}\tilde{S}^a$ to have a non trivial

limit. This is enforced in the usual manner, asking for no dependence of a in the limit. This is get by putting

$$t(a) = \alpha g^2(a) \quad (76)$$

and then

$$\lim_{a \rightarrow 0} T^{-\log t(a)} \tilde{S}_k^a(g(a)) = \begin{pmatrix} \frac{1}{1 - (\frac{1}{\alpha k})} & \cdots \\ \cdots & \cdots \end{pmatrix} \quad (77)$$

which is the S-matrix of the δ interaction. Of course, if we put $t(a) = a/a_0$, as given by the usual scaling, we get

$$g(a) = \frac{1}{a_0 \alpha} a^{1/2} \quad (78)$$

So we have got an alternate derivation of the known result of Seba [13].

8 Remarks

We can always get a known regularization of the δ or the δ' and look for the renormalized interaction. It can be got partial but important information simply taking the limit of the unrenormalized series and asking which fixed point is reached when applying the RG to this limit interaction. By example, lets note that even if the nonrenormalized interaction falls in the attraction domain of the two half-lines fixed point, $\tilde{S} = +Q$, it is unlikely to reach any interaction in the renormalized line if the series happens to fall in the domain of $\tilde{S} = -Q$.

In particular, if we want to reach a limit of the kind of Albeverio et al. " δ' ", we will need series of interactions in the attraction domain of the $I_{0,0}$, and with its limit in the domain of $+Q$. Such properties seem to imply that any regularization for this interaction would fullfill $S_{k \rightarrow 0} \rightarrow I$, which greatly restricts the class of candidates.

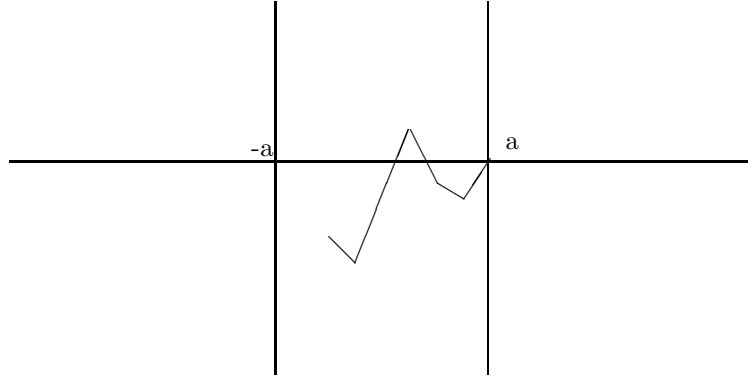
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there are difference between $Lim_k Lim_a$ and $lim_a lim_k$. If both limits commute, would we fall into the scale-invariant interaction, the fixed point?

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x

Figure 1: Any information about the interaction to distances shorter than the cutoff is hidden "under the cutoff"

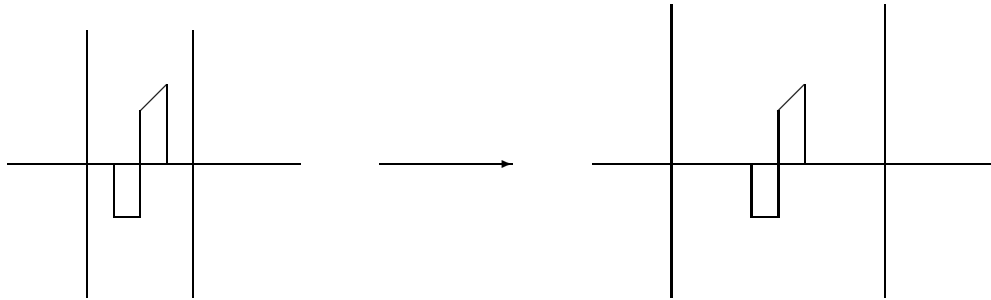


Figure 2: The renormalization group transformation joins theories with the same physics but different cut-off

Figure 3: Renormalization group scheme

Line A contains a series of unrenormalized theories with decreasing cutoff. The renormalization group transformations let us to map this to one series (line C) of theories with the same cutoff, say a_0 . Such series has as limit a point in the line B of renormalized interactions ($T^\infty\{\tilde{S}\}$). The flow corresponding to renormalized interactions is limited by fixed points (endpoints of B), but any other theory could be driven out of the space of interactions when integrated back with the renormalization group transformation (case D).

Figure 4: Expected topology of the RG flow (artist's view)

The usual δ potential corresponds to the line from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. It is unlikely to reach the " δ' line" ($\alpha = \beta = 0$) by renormalizing interactions in the domain of the $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ fixed point, so some regularizations will give us renormalized interactions in the line of the δ . Note that this drawing is somehow a projection of the ∞ -dim space, and RG trajectories doesn't cross in reality.