Introduction to the Tangent Grupoid

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Abstract

We present some plausible definitions for the tangent grupoid of a manifold M, as well as some of the known applications of the structure. This is a kind of introductory note.

Students first meet the grupoid law when they learn about affine vectors, this is, ordered pairs of points jointly with an addition rule: $\vec{xy} + \vec{yz} = \vec{xz}$. Sometimes they are able even to notice the ambiguity of any attempt to add a "multiplication by scalars", before being driven to quotient by an equivalence relationship, thand then to arrive to the standard concept of free vector.

Perhaps the simplest mathematical object preserving the naive vector law is the tangent grupoid. It is basically the union of two old pieces of differential geometry: a well known one, the tangent bundle, living in the envolvent of the manifold, and an older but less controlled one, finite differences, which can be thought to live in the secant envolvent of the manifold.

A giant step into the analysis of this structure was given by A. Connes, namely to study the algebra of functions over the grupoid, using the tools of non commutative geometry [3].

We aim here to present this object in a form more close to standard geometry books (e.g.[2, 6]), so it becomes easier to focus the geometric significance of its use.

The paper can be read as a prequel to [9, section 6], and we strongly suggest the reader jump to that lecture, or some similar one, in order to get a good grasp of the structure and its uses.

1 Definition

Given a manifold M, and the set $A = C^1(M)$ of differentiable functions over it, the tangent bundle TM is defined as a specific subset of the dual A^*: the one of evaluations of directional derivatives. To be concrete, a vector X tangent to M on x is associated to the distribution (continuous linear functional) that evaluates at x the directional derivative along X:

$$< [x, X]|f >= \partial_X f|_x \tag{1}$$

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We expand this subset to a bigger one $G \subset A^*$ adding the distributions which give us finite differences

$$< [x, y, \epsilon] | f >= \frac{f(x) - f(y)}{\epsilon} \tag{2}$$

Even if we only had TM, we already confront the following problem: While A^* is not closed under products, its addition is yet an internal operation. We would like to preserve thus this addition in the new subset. The solution comes giving to G a grupoid structure, which restricts the pairs of elements that can be added.

We define functions range and source, from G to $M \times \mathbb{R}$:

$$\begin{array}{ll} r([x, X]) = (x, 0) & s([x, X]) = (x, 0) \\ r([x, y, \epsilon]) = (y, \epsilon) & s([x, y, \epsilon]) = (x, \epsilon) \end{array} \tag{3}$$

and the obvious immersion of $M \times \mathbb{R}$ in G,

$$e(x, \epsilon) = \begin{cases} [x, 0] & \epsilon = 0 \\ [x, x, \epsilon] & \epsilon > 0 \end{cases} \tag{4}$$

With these functions, the addition in A^* defines a grupoid structure on G. By inducing the appropriate diferentiable structure, G can be made a smooth grupoid, which is named *the tangent grupoid to a manifold M*.

While the topology can be taken directly from A^*, for constructive and operational reasons we can be interested on making explicit the pasting between the two parts of the grupoid. This can be done [3] in an atlas of the manifold, asking

$$\lim x_n = \lim y_n = x, \lim_{\epsilon_n \to 0} \frac{x_n - y_n}{\epsilon_n} = X. \tag{5}$$

but other methods can be employed depending of applications.

2 Geometric Meaning

Before going deeper into the structure, it is a good idea to test how it fits in our previous geometric visualizations. It is comfortable to check that the usual interpretations of the tangent bundle can be enhanced to host the full tangent grupoid. We explain this next.

2.1 Tangent bundle and secant (pre)sheaf

When we consider M immersed on some $\mathbb{R}^m$, there is a natural identification of TM inside the set of straight lines in $\mathbb{R}^m$, to be concrete with the ones making the tangent envelope of M. This is done by seeing them as linear applications$f : \mathbb{R} \to \mathbb{R}^m$, and then mapping each point $[x, X]$ of TM to the line tangent to M in x with velocity vector X.

For our additional points $[x, y, \epsilon]$ there is also an obvious subset of lines, the ones secant to M through x, y. To keep up with the topology, we must require that each $[x, y, \epsilon]$ is to be mapped to the line f going through x then y with speed v such that distance$(x, y)/|v| = \epsilon$.

→ We could rephrase this in terms of the natural parameter of the straight line

In the same manner that to each point $x \in M$ we can associate the set of lines tangent to x, we can associate to every open set $O \subset M$ the set of lines cutting through it. While the former set is a natural receptacle for T_xM, this latter one is the adequate to map the set $S_OM \equiv \{[x,y,\epsilon]|x,y \in O \subset M\}$, but the conditions to be put over the immersion for the map to be one-to-one are more stringent in the latter case.

Here we see one of the main weakness of G. While TM can be made naturally a fiber bundle with base M, we have not an unique procedure to build a fiber of elements $\{[x,y,\epsilon]\}$ over each point $z \in M$. The presheaf $O \to S_OM$ of elements going through each subset O is yet unique, but it is a bit too general. Even if we give an arbitrary procedure to choose a fiber, say $\pi([x,y,\epsilon]) = x$, we can only get a true vector structure in the limit $\epsilon \to 0$. Additionally, exotic elections for $\pi : SM \to M$ would give problems to define the exponential map.

2.2 Equivalence of curves

Other common interpretation of TM is to see each point as a class of equivalence of curves in M, which "coincide at order ϵ" in a neighborhood of the point x. This is written, given some chart φ, as:

$$[x,X] = \{f : \mathbb{R} \to M \ni f(0) = x, \lim_{\epsilon \to 0} \frac{\varphi(f(\epsilon)) - \varphi(f(0))}{\epsilon} = X\} \tag{6}$$

Our generalization is obvious: each new point $[x,y,\epsilon]$ can be seen as the class of curves passing through x and y for determinate values of its parameter. For example, if we want to be consistent[1] with the previous formula for TM, we can postulate:

$$[x,y,\epsilon] = \{f : \mathbb{R} \to M | f(0) = x, f(\epsilon) = y\} \tag{7}$$

Both spaces are pasted using simple set-theory techniques: a sequence $\{[x_n,y_n,\epsilon_n]\}$ is said to converge to $[x,X]$ if and only if $\{\epsilon_n\}$ goes to zero and

$$\emptyset \neq \bigcap_n [x_n,y_n,\epsilon_n] \subset [x,X] \tag{8}$$

Alternatively, we can even avoid to define TM and postulate it directly as the class of non empty intersections of such sequences.

2.3 Leibnitz rule

A vector of T_xM can be defined[2] also as an element $v_x \in A^*$ fulfilling Leibnitz rule,

$$v_x(fg) = f(x)v_x(g) + g(x)v_x(f) \tag{9}$$

The definition can be expanded to the rest of the grupoid by permitting "braided" Leibnitz rules:

$$v_{xy}(fg) = f(x)v_{xy}(g) + g(y)v_{xy}(f) \tag{10}$$

[1] The ambiguity in the definition of derivative must translate to an ambiguity in the value of the parameter of the curve which goes through x.

[2] In finite dimensional manifolds [2].

In this sense, distributions $\langle [x,y,\epsilon]|$ are considered as "deformed" derivations. Sometimes formula (10) is written using coordinates more or less implicitly; e.g. with a coordinate function ϕ:

$$v_{xy}(f) = \frac{f(x) - f(y)}{{}_\phi x - {}_\phi y} v_{xy}(\phi) \tag{11}$$

2.4 TM as boundary of the grupoid

The set-theory building of TM as limiting set of SM was already apparent in section 2.1, where we can observe than the set of tangent lines lies in the border of the set of secant ones.

The goal of pasting the two parts of G in a form compatible with the usual definition of derivatives brings this result as byproduct: the tangent bundle is the boundary of the secant grupoid.

We take usually $\epsilon \in [0,1)$ when we want to see G as a manifold with boundary. In this case we will conventionally call $G^{(0)} \equiv TM$ to the tangent part, $\epsilon = 0$, and $G^{(1)} \equiv SM$ to the secant part, $0 < \epsilon < 1$.

3 A Dilatation Structure

The pasting of SM to TM uses some charting of M in order to control the rate of convergence of x_n, y_n. This is, to say the least, antieconomical, as only the scaling structure provided by the chart is used. In addition, the atlas itself is unavailable if we choose to define TM directly from sequences in SM.

Alternatively, we can control the convergence by defining a dilatation structure over SM

$$\tau_\lambda : SM \to SM \tag{12}$$

$$[x,y,\epsilon] \to [x',y',\lambda\epsilon] \tag{13}$$

with $\tau_{\lambda\lambda'} = \tau_\lambda \circ \tau_{\lambda'}$ (For a general discussion on this structure, see [10, ch.12]).

We will say that a sequence $\{[x_n,y_n,\epsilon_n]\}$ going to $[x,x,0]$ has a limit on TM ("goes to a point of TM") if the scaled sequence $\{\tau_{\varepsilon_0/\epsilon_n}[x_n,y_n,\epsilon_n]\}$ has a limit in SM. ($SM = M \times M \times \mathbb{R}^+$)

In some sense, this technique is reminiscent of the tensorial definition: "a vector is a geometrical object which transforms as a vector", but here transformations are hidden under the carpet of the dual A^*.

If we need explicitly to know the limit vector on the tangent bundle, we must do a bit more of work, as remarked in section one, or, alternatively, specify the recipe compatible with the dilatation flow. This is not surprising: we have used the flow as a substitute for the explicit chart in (6), but then we have lost a method to name explicitly X. We must then specify a recipe compatible with the "hidden" coordinate system.

For example, take the flow τ_λ to be such that

$$\tau_\lambda[x, \exp_x X, \epsilon] = [x, \exp_x \lambda X, \lambda\epsilon] \tag{14}$$

Then each point $[x,X]$ must be canonically associated to the limit point $[x, \exp_x X, \varepsilon_0]$.

To put other common example, if we choose a "mid-point rule" for scaling:

$$\tau_\lambda[\exp_z -X, \exp_z X, \epsilon] = [\exp_z -\lambda X, \exp_z \lambda X, \lambda\epsilon] \tag{15}$$

the limit point "$[x, Y]$" will be $[\exp_x -Y, \exp_x +Y, \varepsilon_0]$

We have not a general proof of existence for this process of limit. It could be suspected that, while working well for directional derivatives, it could fail for more exotic constructions, such as covariant derivatives depending on symmetry groups (from a physicist point of view, we would like geometry to be able to detect physical properties of the connection, specifically renormalizability).

4 The Tangent Grupoid of a Lie Group

When the manifold M has a Lie group structure, we can say a lot more about the tangent bundle: we know how to get each fiber T_xM from an action of the group over the fiber at the identity, T_eM. And we know that this fiber, the tangent space at the identity, can be show to be equivalent to the Lie algebra g of the group (and/or to the left invariant fields over M), and we can build A^* from the enveloping algebra of g.

Now, if we are able to get such quantity of information from the $\epsilon = 0$ part of our grupoid, it is natural to ask ourselves how this information is present in any part of it $\epsilon = \epsilon_0 > 0$. A partial answer in the language of quantum groups is provided by [5], which builds the distributions associated to the tangent grupoid as a particular first-order example of more general deformed (braided) differential calculus over M. There, a explicit selection of fiber over each $x \in M$ is needed, entering the yet developing world of deformed q-vector bundles.

The theory of differential calculus for q-groups is currently well established. Basically, it deforms both the algebra A of functions over M and the dual algebra of distributions A^* (to be precise, the enveloping algebra, as we have said), but keeps both in duality by using (an equivalent of) the usual relation

$$< X, f >= Xf|_0, \ f \in A, X \in g \subset A^* \quad (16)$$

Some additional structure is required, mainly as properties of the coproducts, see [8]. In the clasical, undeformed, case, these are $\Delta(X) = X \otimes 1 + 1 \otimes X$ and $\Delta(a(x)) = a(x.y)$, for $X \in A^*$ and $a \in A$ respectively.

currently, Sweedler notation is used, $X_{(1)} \otimes X_{(2)}$ and $a_{(1)} \otimes a_{(2)}$ respectively

The tangent space to the identity, L, is characterized technically through the action of the quantum double $D(A^*)$ in ker $e \subset A^*$, where e is the counit of A^*. The usual product of functions and vector fields, $a(x)X$, generalizes to the action

$$a \triangleright y =< a, y_{(1)} > y_{(2)} - 1 < a, y > \quad (17)$$

while the Lie derivation of vector fields retain its aspect as adjoint action Ad_x

$$x \triangleright y = x_{(1)} y S x_{(2)} \quad (18)$$

$X \triangleright Y = XY - YX$

In a dual manner, over A we can locate a space $L^* = \ker e'/M$, being M some $D(A)$-submodule and e' the counit of A. Such space will have a natural action of A by multiplication from the left, and an acting of A^* as coadjoint action, Ad^*, both following from the construction of the double $D(A)$. These actions are used to define the space of left-invariant 1 forms over the group, explicitely $\Omega^1 = L^* \otimes A$.

Details of actions on Ω^1 are show in [5] jointly with explicit examples for $\mathbb{R}^n$. In particular, when $n=1$ any function $c(p)$ defines an invariant q-tangent space

$$L = \text{span } \{p_n = c^{(n)}(p) - c^{(n)}(0)\} \tag{19}$$

with corresponding (braided) derivation and Leibnitz rule, which we omit for sake of brevity.

We obtain L^* taking the quotient of kerϵ', this is, functions $f \ni f(0) = 0$, by its subspace

$$\{f \ni \partial_{p_n} f|_0 = 0\} \tag{20}$$

thus keeping with the usual method given by (16).

For $c_\lambda(p) = \lambda^{-2} e^{\lambda p}$, L is one-dimensional, and its partial derivative comes out to be

$$\partial_{p_1} f = \frac{f(x+\lambda) - f(x)}{\lambda} \tag{21}$$

while for $c_{\lambda\to 0}$ we get the usual commutative, $q \to 1$, calculus, which is also our limiting process $\epsilon \to 0$. For other derivatives it would be interesting to investigate limits to other roots of the unit.

5 Applications

Connes approach [3] studies the tangent grupoid through its Gelfand-Naimark dual, i.e. the set of continuous (alt. smooth) functions over it, which form an algebra, its product being the convolution product associated to the grupoid law.

$$(a * b)(\gamma) = \sum_{\substack{\alpha, \beta \in G \\ \alpha\beta = \gamma}} a(\alpha) b(\beta) \tag{22}$$

For fixed ϵ, greater than zero, this is product of kernels of operators in $L^2(M)$ (see [9]). Thus we have is a very nice host for a K theory.

In fact the main use of the tangent grupoid has been to work its K theory as receptacle for Bott periodicity, then to formulate proofs of index theorems.

Also, it has been observed [4] that the set of continuous functions over the grupoid defines a quantization procedure. This is because a observable function over T^*M can be described with its Fourier transform in TM, the pointwise product passing then to convolution product. If we assume that the function in TM is the $\epsilon = 0$ part of a continuous function over the full grupoid, it can be seen that any fixed $\epsilon = \epsilon_0$ part of it is described as the kernel of a quantized operator associated to the original function. The ambiguity of this procedure is studied in [1]

It could be worth to note that the quantization phenomena appears also in the example of [5], where formula (19) can be seen as implementing the indeterminacy principle needed in any quantization context (If we prefer to see A as a set of fields on 0+1 dimensions, then it is most appropriate to try interpret (19) as the quantum mechanics evolution rule).

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Appendix: Toy Renormalization Group Triangle

Lets review how we use a dilatation structure on our set of distributions $\{\langle [x,y,\epsilon]|\}$. Ideally, this structure is a set of transformations $\{\tau_\lambda\}$ which are obtained by duality from other set $\{\tau^*_\lambda\}$ of "rescaling" operators on $A = C^\infty(M)$.

$$< \tau_\lambda v | \tau^*_\lambda a > = < v | a > \tag{23}$$

Such pairing can be built explicitly in simple cases, by example for $\mathbb{R}$ we can put τ_λ as in (14) and $\tau^*\lambda$ such that for some family of charts ϕ, any function $a \in A$ transforms as in a linear change of variable,

$$\tau^*_\lambda a(x) = \lambda^1 a(\phi^{-1}\lambda^{-1}\phi x). \tag{24}$$

But regretfully the pairing is not one on one, so in practice (15, 14, etc.) we start directly from a concrete $\{\tau_\lambda\}$ set.

Now, dilatations τ_λ draw a flow on the space $(M \times M \times \mathbb{R})$ that parametrizes $SM \subset A^*$, and this can be interpreted as a RG flow. Lets see how.

We put some mild restrictions in transformations τ_λ. They must form a multiplicative semigroup, $\tau_{\lambda\mu} = \tau_\lambda \tau_\mu$. They must preserve the subspace $M \times M \times \mathbb{R}$ of distributions. In order to simplify operations, they shall act multiplicatively in $\mathbb{R}$, $\tau_\lambda[x,y,\epsilon] = [x', y', \lambda\epsilon]$

The dilatation condition takes apart any pair of points x, y, and additionally moves ϵ. Then the only fixed points of the flow are $\{[x,x,0]\}$. The standard approach ask us to linearize the transformation near the fixed point, and then study its flow.

The fixed point lies in the critical surface $\{\langle [x,y,0]|, x, y \in M\}$, where distributions become indefinite. Any series going to a point $[z,z,0]$ in the critical surface draws a nonrenormalized line of "bare" distributions

$$\{[x_n, y_n, \epsilon_n]\}, \epsilon_n \to 0, y_n \to x_n \to z, n \to \infty \tag{25}$$

To fix it, we choose a "physical" scale ε_0 and use the renormalization flow transformations $\tau_{\varepsilon_0/\epsilon_n}$ to rescale the points, getting a renormalized series

$$\left\{\tau_{\frac{\varepsilon_0}{\epsilon_n}}[x_n, y_n, \epsilon_n]\right\} \tag{26}$$

which lives in the surface $[x, y, \varepsilon_0], x, y \in M$. If the bare series had a limit $[x, X]$ in the associated chart ϕ, then the renormalized series has a limit, which we can name $[x, X]_{\varepsilon_0}$.

The mathematical image is straightforward: both limits are related "until energy ε_0", which is the scale of validity of the renormalized one;

$$\langle [x,X]|a\rangle = \langle [x,X]_{\varepsilon_0}|a\rangle + o(\varepsilon_0) \tag{27}$$

Renormalized limits at different scales are joined by RG transformations, correspoding to a relevant direction coming out from the fixed point.

Other usual folklore of Renormalization Group theory can equally be translated to this context: So, formula (23) is the traditional assesment which claims physics to be independent of the scale; the arbitrarity choosing an specific R.G. transformation translates to the freedom to choose the limiting process that defines the derivative of a function, while the global character of the obtained tangent vector comes from the global properties of the fixed point and the flow near it.

The main drawback of the example is the character of the fixed point: it has not got any irrelevant direction, then any "bare" line must go directly to it in order to define a good limit. This effect can be attributed to the simple form of the derivative being defined in the process. If we want to look for examples owning irrelevant directions in the critical surface, we would try more complicated geometrical notions, such as covariant derivatives. Also, we could try Jackson q-derivatives, for q going to a complex root of unit.

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