On the section of a cone

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Abstract

A problem from Democritus is used to illustrate the building and use of infinitesimal covectors.

The Friday before Passover I was forced to make some bureaucratic consultations in our Ministery of Defence. So I landed Tuesday in our desolated airport at Zaragoza and, two days after, I took the train to Madrid, hoping to get lodgement in the house of Miss Ana Leal in the folkish town of *La Latina*.

Such happenings use to be intellectually exciting, and this one was not exception. Miss Leal suggested extending some hours the visit in order to be able to attend a lecture of Agustin Garcia Calvo in Lavapies. This Agustin is a well-known classical linguist, and a kind of anarchist philosopher, who likes to teach in the Greek stile, and we had already enjoyed his *tertulias* in the old institution of the *Ateneo de Madrid*. This one was supposed to be a more technical lecture, addressed to secondary school teachers.

Indeed, it was a very amenable lecture. According to my notebook, he made a good point of the use of language for political control, opposing the vocabulary against grammar, the former owned by the power to construct the reality, the latter unconsciently managed by the people driving a "raison en marche". Being practising mathematician, one can easily to feel this confrontation; gramatician placeholders, names, adjectives, etc, are not very different of our variables and constants, and our whole fight is to leave all the weight over proofs, over our grammar, avoiding to get any conclusions from the loose vocabulary of definitions. I meditated on these parallelisms while hearing the linguist' admonitions.

Then, Agustin centred in the lecture main theme, the teaching of philosophy in secondary education, and somehow malevously suggested three examples to be proposed to students. The Lewis Carroll approach to Zeno paradoxes, the Zeno paradox itself (beautifully glossed as *no se vive mientras se besa, no se besa mientras se vive*, one does not live while kissing, one does not kiss while living) and, for my surprise, the dilemma of the cone from Democritus!

Heath, following Plutarch, enunciates it asking what happens if we cut a cone using a plane parallel and very close to the basis. Is the resulting circle equal to the one of the basis? Our XXth century presocratic philosopher prefers a more intrincate set up; let me to go to my notes and remember this. Take a cone and cut it through a plane, which for simplicity we can still take parallel to the basis. Now, look at the resulting figures, a smaller cone with basis B and a conical trunk with top B'. The question is, are the circles B and B' equal or unequal?

In any case, if they are not equal, it results that some discontinuity happens in the complete, joined, cone, and the generating line should present jumps, small steps. But if, on the contrary, both surfaces are not unequal, their fusion should build a cylinder, no a cone.

To solve this paradox, we can negate the possibility of the described action. We can claim that the cone is a real figure, and then it is not proofed that it is a mathematical cone, and its generating line could really be irregular. But the mathematical problem still exists, and we can ask about the ideal cone[1].

Again, in this setup, we can negate the starting point and claim that the cut is really the intersection of the plane with the cone. There are no two surfaces to be compared.

But then the paradox can be got again by using a Gedankenexperiment. Instead of Plutarch quote, let me get the same music, if not the notes, from the last point of Agustin: Imagine we cut a carrot, or a turnip with a cutter, so we can not deny we have two surfaces. In principle the cutter retires some slice of matter from the carrot, slice thickness being related to the one of the cutter. Imagine we make the cutter thinner and thinner, so no mass is moved out of the carrot when we cut it. Now imagine the same operation on the mathematical setup, we have two surfaces and the paradox again.

And from here we are in our own[2].

The result of the progressive thinning of the cutter has been a pair of planes becoming progressively nearer. This is to be noted: the operative problem involves no a plane, but two planes approaching one to other.

This figure, a pair of parallel planes, is known to mathematics following Shouten and Golab as being a covector (in a tridimensional space). Technically, it is specified by giving a unit segment (axial vector) perpendicular to the plane, and then a modulus measuring the separation of both planes and a support point where the first of the planes lies[3].

[1] We can claim that "we can see it with the eye of the mind; and we know, by force of demonstration, that it cannot be otherwise", as Democritus himself claimed for the tangent of the circle.

[2] I intend to hide some complex or distracting comments under the carpet of the footnotes. The reader could prefer to avoid them in a first reading

[3] The specification of the axial vector changes peculiarly when we make a change of coordinates of the system, fitting the usual definition of covectors. And the space of covectors (n-1 segments figures as specified) is dual to the space of vectors (1 dimensional oriented segments). An equivalence relationship can be added to get a space of free covectors, but here this step is not needed. We can say that two covectors can be added when the final plane of one coincides with the starting plane of the other, then fusing it to make a grosser cutter. To be honest, this restriction is stronger than the usual for "free" covectors, and in fact it reflect that we are interested in a slightly looser structure, which we could call q-covectors (the q making

Now, lets to make the cutter slimmer and slimmer. Then the modulus of our covectors goes to zero, but we still have two planes.

This is in fact the resolution of the paradox. We distinguish between a cylinder and a cone because we have more information: the continuity should be claimed between one surface of the pair and the next one of the following "slice". The difference between a cone and a cylinder resides in the internal structure of the pair. When the pair becomes so-to-say infinitesimal, both planes of the slice live over the same points of the space. All the infinitesimal slices over the same area can be added without taking care of the fusion condition above, and then get additional structure[4].

Of course, it must be seen that there is something in the limit of approaching surfaces, i.e., we must to give sense to this limit and proof there are really something over a single cut. In our modern XXth century we could jump directly to use scaling transformations in the spirit of Wilson-Kogut. But perhaps it is better to start from Archimedian methods, which the reader can enjoy in the interesting book of T.L. Heath. For instance, we can see how the volume of a conoid can be extracted by using our bifacial knife. And, as we ignore the full detail of Greek methods[5], it could be perhaps forgiven if we avoid refilling the discussion with Leibnitzian meat, trying instead to keep the spicy flavour of our local cooking.

Imagine again the cone, divided into slices of finite size, lets say using our finite cutter. Each slice can be fitted between two cylindrical ones, a smaller one, with takes as basis the small circle of the conic slice, and a greater one, taking as basis the big circle.

By joining the cylinders we have two circular "ziggurats", a small one inscribed inside the cone, and a greater one circumscribed on it. We can consider then the calculation of a quantity such all the volume of these whole figures from the corresponding quantity of the pieces.

The difference between the circumscribed and the inscribed figure amounts only to the greater slice of the circumscribed one. This is because each cylinder of the inscribed one is equal to the previous one of the circumscribed one. Here we see that the importance of the correct pasting condition: it must be between the circle of one slice and the immediate of the following one. Only in this

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reference to a scale of the thickness of the cutter, and -indirectly- to the deformed differential calculus of Majid)

[4] Mathematicians call this space of infinitesimal covectors over a point "cotangent space", the whole set being the "cotangent bundle". An application selecting one covector over each point is called a "differential form".

[5] Heath quotes Wallis regretting that "nearly all the ancients so hid from posterity their method of Analysis (though it is clear that they had one) that more modern mathematicians found it easier to invent a new Analysis than to seek out the old". Indeed, the lack of texts is surprising, all a branch of reason cleared white as the recycled folia where "The Method" was found in 1909, *palimpsesta sunt, scriptura antiqua (litteris minusculis s. X) aqua tantum diluta plerumque oculis intentis discipi potest (de foll 1..., 119-122 tamen desperandum mihi erat)*, a inmense cleaning which justifies Wallis' paranoia. But again, even Newton kept secret his own method, until that Leibnitz developments forced him to show it.

manner the subtraction keeps control, all the difference being the volume of only one slice. Thus when the cutter thickness goes to zero, so goes the difference between figures, and their volume converges to the volume of the cone.

Lets examine this convergence with more detail. It involves two operations: to increase the number of slices, to decrease the width of the slices. Both operations related, of course, because the product is the height of the cone. But here we do not see the structure of the limiting objects, so it is possible yet to hold some doubts about the process. An alternative approach is the averaging method of Wilson [6]. Two consecutive cylinders can be substituted by an unique cylinder averaging them, i.e., with a volume that is the sum of the volumes of both cilinders and a thickness equal to the union of them, thus double of the original one.

Aplying this procedure to the whole "ziggurat" we get a new figure which is no more inscribed (or circunscribed) to the cone, but has the same volume that the starting one.

Now, this method can be used to control the limit process in the following way: we choose an arbitrary scale of thickness, say for instance the one half of the height of the cone, and for each "ziggurat" in the converging series we apply the averaging until we get back to a figure composed of cylinders of the choosen thickness.

This new series of figures[7] is composed of finite objects, each one having equal number of cylinders and cylinder thickness being the same in every figure. Then the limit process of this series is not affected by the two infinities, in thickness and in number of cylinders, that were incrusted into the previous series. Even if we do not believe in the infinitesimal slices, we should have not problem admiting the regularized slices built at given, but arbitrary, scale.

Readers could note that a "ziggurat" composed of cylinders of equal radius should be invariant under the Wilson transformation, only the external, nominal, scale of thickness changes. A deeper examination would show that the existence of this set of invariant figures is the key for the convergence of the whole process[8]. In some sense, this invariant shape is the amplification, to a finite scale, of the infinitesimal cylinders of the first convergence process. We can choose the arbitrary reference scale as near to zero as we wish, so its limit zero[9] can be interpreted as the home of such "differentials". In fact the existence of

[6] The interested reader can see some examples in the article published by Wilson himself in 1979 in the Scientific American

[7] The new series could be called "renormalized", if we call the original "bare"

[8] By iteration of the transformation, we finish over some invariant figure, and the slice we are slimming is also similar to the invariant figure. If the transformation were given by a continous group, we should see trajectories on the space the figures, approaching the trajectory of invariant cylinders, and the renormalized series would be a line cutting across trajectories and converging to a point in the invariant trajectory. In our case, with discrete transformations, we can still hit into the invariant trajectory by choosing the length (for instance, using as fundamental length the heigth of the cone, instead of one half as above).

[9] Which we could be tempted to call "classical limit" instead of the usual "continous limit"

limit for our finite series depends of the existence of the line of invariant figures, and the existence of this line relates to the existence of a fixed point, from which the line starts. Such fixed point can be linked the above suggested zero limit. All the pieces of the puzzle fit together.

Note that equal that we can cut the cone, we can also cut its axis, the only difference being that the former lives in three dimensions, the latter in one, then the covectors over this latter are specified by pairs of points instead of planes. Following conventions, we can call dV to the infinitesimal slice over the cone, and dz to the slice over the axis. It is also usual to write $dV = \frac{\partial V}{\partial z} dz = A(z) dz$ but of course such writing must also be justified.

Another stroke could be draw if we choose to imagine the cone as developing in the time, i.e., the axis being some kind of temporal direction. Thus each cut is a circle which grows in the time, and the connection with Zeno paradox becomes evident [10]

And further developments could be done, for instance connecting the scaling procedure to the ones currently in use in theoretical physics, or to work out the q-covectors composition rule aiming to a cotangent groupoid similar to the tangent groupoid raised by Monsieur Alain Connes. The chain of reasonments is enough tight to rule out impossible relations and, as an old friend used to say, when impossible is ruled out, the only remaining thing is the answer. In this footing, we could follow up trying to build the arguments in four dimensional spaces and within field theory, from where we have already taken part of our terminology.

For sure that readers can imagine a lot of additional quests.

So, which our conclusion is? Well, we have seen how the discussion about such an old problem becomes an argument for the teaching of modern mathematics and physics. Perhaps this is the whole point of this note, although surely it was not the one of Garcia Calvo when spelling this old tale to the philosophical audience. But again, mathematics works in its own pace, independently of our own intentions, just as sometimes science gets to be taught independently of the intention of educational programs. Well, this one was probably the very point of Agustin, ours is only to give voice to the math through ourselves. He says *dejarse hablar*.

[10] Sketching a parallelism with modern physics terminology, we could say that Democritus paradox is the "Wick-rotated" version of this from Zeno. The axis should be the "imaginary time", and volume and area should perhaps correspond to position and velocity.