On Generations

Alejandro Rivero[*]

February 22, 2002

Abstract

The well known operator ordering ambiguity could motivate the existence of generations. This possibility is explored by exploiting the relationship between ordering and discretization rules.

1   Wilson fermion as analogy

Non Commutative Geometry is not the only area knowing how to get mass terms from nowhere. Lattice theorists learnt time ago a nice trick from Wilson. Consider the forward and backward discrete derivatives, ∂_+, ∂_- as usual

$$\partial_+ f(x) = \frac{f(x+\epsilon) - f(x)}{\epsilon}, \partial_- f(x) = \frac{f(x) - f(x-\epsilon)}{\epsilon}$$

Suitable derivatives in the lattice can be implemented as any linear combination of both, $\frac{1}{a+b}(a\partial_+ + b\partial_-)$ Now, by noticing that $\partial_-(f(x+\epsilon) - f(x)) = (\partial_+ - \partial_-)f(x)$ we can rewrite

$$\frac{a\partial_+ + b\partial_-}{a+b} = \frac{1}{2}(\partial_+ + \partial_-) + \frac{a-b}{a+b}(\partial_+ - \partial_-) = \frac{1}{2}(\partial_+ + \partial_-) + \lambda\partial_-\partial_+$$

where $\lambda \equiv \epsilon\frac{a-b}{a+b}$. Here Wilson stops and we follow up: If some technical or quantum effect avoids the limit λ going to zero, then the system taking account of the ambiguity should be written $\frac{d}{dx}f \to \partial f + \lambda\partial^2 f$ And, if two derivatives are done, in order to account for the ambiguity we shall rewrite

$$m\frac{d^2}{dx^2}f \to m(\partial + \lambda_1\partial^2)(\partial + \lambda_2\partial^2)f$$

Just as a funny notation, define $\Psi_e(x) \equiv f(x), \Psi_\mu \equiv f', \Psi_\tau \equiv f''$, then:

$$m\frac{d^2 f}{dx^2} = M_0\partial^2\Psi_e + M_1\partial^2\Psi_\mu + M_2\partial^2\Psi_\tau$$

Note that M_0, M_1, M_2 contain orders of $\epsilon^0, \epsilon^1, \epsilon^2$ respectively.

---

[*]somewhere in Kent, England. Email: `rivero@wigner.unizar.es`

2 Introduction

Please keep in mind that the real game is expected to be played in the NCG space, where some additional sheets can accomodate more structure.

Proposals to get the structure of quantum elementary objects from the discretization of space or its regularization are significant in the literature. But we are not aware of any suggestion to extract from there the spectrum of elementary fermions. This would include four flavours of particles, two chiralities and three generations.

We advanced this idea last year [5], jointly with other conjectures of diverse quality (mostly slippery). As a piece of the puzzle, it was already suggested that the existence of three generations of elementary particles is the method Nature uses to parametrize the ambiguity of quantization. Here some additional toy models are proposed to explore this point.

This letter is mainly a progress report, please refer to an hypothetical [4] for context. Definite advance will come someday from the interplay of lattice theory and NCG; some attention has been put on it from the lattice side [6, 7], but the NCG counterpart seems to sleep since [8] et al.

3 From Bosons to Pseudofermions

For instance,

$$
\begin{aligned}
\frac{1}{2}m\dot{x}^2 &= \frac{1}{2}m\left(\frac{f(t+\Delta)-f(t-\Delta)}{2\Delta}\right)^2 = \\
&= \frac{1}{4}m\,\mathrm{Tr}\left(\begin{array}{cc}\left(\frac{f_+-f_-}{2\Delta}\right)^2 & \\ & \left(\frac{f_+-f_-}{2\Delta}\right)^2\end{array}\right) = -\frac{1}{4}m\,\mathrm{Tr}\left(\begin{array}{cc} & \frac{f_--f_+}{2\Delta} \\ \frac{f_+-f_-}{2\Delta} & \end{array}\right)^2 = \\
&= -\frac{1}{4}[D,A]^2
\end{aligned}
$$

where $A = \left(\begin{array}{cc} f(t+\Delta) & \\ & f(t-\Delta)\end{array}\right)$ and $D = \left(\begin{array}{cc} & \frac{m}{2\Delta} \\ \frac{m}{2\Delta} & \end{array}\right)$

Alternatively, we can define a vector $|\Psi> \equiv \left(\begin{array}{c}\psi^+ \\ \psi^-\end{array}\right)$ and to say that

$$
\frac{1}{2}m\dot{x}^2 = -\frac{1}{4}<\Psi|[D,A]^2|\Psi>
$$

It is possible also to try to start sooner, let say from a naive ladder $D_L\Phi(x) \equiv \Phi(x-L)$, to see it as an infinite matrix mechanics, and then duplicate, collect and reorder vectors to connect with the above scheme. This naive ladder is easier to link with other formulations, say Kauffman or Costella, to name a pair of interesting ones.

4 From Quantization Ambiguity to generations

Consider $\dot{x}_+ = \frac{f(t+\Delta)-f(t)}{\Delta}$ and $\dot{x}_- = \frac{f(t)-f(t-\Delta)}{\Delta}$. See [2] for their relation to quantization ordering rules. For generic ordering take $\dot{x}_{(\lambda\mu)} = \lambda\dot{x}_+ + \mu\dot{x}_-$.

Apply the previous method. You will need to duplicate the vector space, and presumably to draw a more sophisticated representation A_f of functions.

For instance, let it be

$$f_\uparrow = \frac{1}{2} \begin{pmatrix} f(t+\Delta) + f(t) \\ & f(t-\Delta) + f(t) \end{pmatrix}, f_\downarrow = \frac{1}{2} \begin{pmatrix} f(t+\Delta) + f(t-\Delta) \\ & 2f(t) \end{pmatrix}$$

$M = \begin{pmatrix} m_1 & m_3 \\ m_4 & m_2 \end{pmatrix}$, and set $D = \begin{pmatrix} & M \\ M^* & \end{pmatrix}$, $A = \begin{pmatrix} f_\uparrow & \\ & f_\downarrow \end{pmatrix}$

Thus, putting $m_1 = m_2 = 2\mu/\Delta$, $m_3 = m_4 = 2\lambda/\Delta$,

$$[D, A]^2 = \begin{pmatrix} & Mf_\downarrow - f_\uparrow M \\ M^* f_\uparrow - f_\downarrow M^* & \end{pmatrix}^2 =$$

$$= \begin{pmatrix} & -\mu\frac{f(t)-f(t-\Delta)}{\Delta} & -\lambda\frac{f(t+\Delta)-f(t)}{\Delta} \\ & \lambda\frac{f(t+\Delta)-f(t)}{\Delta} & \mu\frac{f(t)-f(t-\Delta)}{\Delta} \\ \mu\frac{f(t)-f(t-\Delta)}{\Delta} & -\lambda\frac{f(t+\Delta)-f(t)}{\Delta} \\ \lambda\frac{f(t+\Delta)-f(t)}{\Delta} & -\mu\frac{f(t)-f(t-\Delta)}{\Delta} \end{pmatrix}^2 =$$

$$= -\begin{pmatrix} \mu^2\dot{x}_-^2 + \lambda^2\dot{x}_+^2 & -2\lambda\mu\dot{x}_+\dot{x}_- \\ -2\lambda\mu\dot{x}_+\dot{x}_- & \lambda^2\dot{x}_+^2 + \mu^2\dot{x}_-^2 \\ & & ... & ... \\ & & ... & ... \end{pmatrix}$$

So that ambiguity fixing at a combination λ, μ can be related via those toys to a two generations structure with mass eigenvalues proportional to $\lambda \pm \mu$. Far away from the order of magnitude of the real thing except if $\lambda \approx \mu$, which, on the other hand, is the usual symmetric guess, $\lambda = \mu$ to get Weyl.

We conjecture that if our Lagrangian had second order derivatives, we should need three generations (alternatively, it could be considered to take different ambiguity fixing in each derivation). Simultaneous fitting of first and second derivatives is also interesting, because it suggests special mass relationships (for instance, $\lambda - \mu \sim 1/\Delta$).

The A matrix in the example shows an aspect more sensate in the basis of mass eigenvectors, but we have preferred to keep it diagonal. Out of the toy model, more serious derivations must be done, for instance using $[D, A] = G[D, X] = [D, X]\tilde{G}$ with X coming from the obvious coordinate function $f(x) = x$, and then linking the so defined derivatives $G, \tilde{G}$ with the objects of Barrow-Majid non-commutative calculus, and so on.

5 From Pseudofermions to infinitesimal

Use Δ to regularize the Lagrangian. Our differential elements are then replaced by finite differential slices.

A discrete differential "in" a point is an affine covector instead of a free one, which is the usual limit case. In $0+1$ it is given by two points, that become two parallel planes in physical space. The famous[3] "wine dance" argument from Feynman applies: rotation of one point is topologically equal to exchange.

Our pseudo fermions in the previous section would absorb into themselves this property, for the discrete summation to proceed without topological consid-

erations. Ideally, the rule should be applied for each of the four differentials[1] in 3+1 space in order to get four fundamental particle fields. Vectorial gauge fields and anti-particles must be added following recipes in [1], in order to establish dualities between homology and cohomology, thus guaranteeing the usual properties of integral calculus. This should also require some relationship between masses of different particles, hopefully.

An additional clue should come from the uses of Grassman variables (for instance, in superspace formalism). Reminder that such variables were suggested in the XIXth century as a way to formalize infinitesimal calculus. A Grassman variable could be presented as a very small interval such that its square was orders of magnitude smaller, thus zero in approximative calculations. We could expect that our fermions had some map into Grassman variables when the continuous limit is taken [2].

Other attemtp to link differentials and fermions is the kahler-dirac equation.

6 From There to the Limit?

Field Theory is about finding the extremal of a functional $F[f]$. To accomplish it, we regularize the functional introducing two scales, two units $a, \tilde{a}$ of position and momentum respectively. Then the limit is taken to remove the scale. If $a\tilde{a} \to 0$, it is called classical field theory. If $a\tilde{a} \to h$, being h a finite constant, it is called quantum field theory.

Point is, references to continuous and discrete worlds seem to forgot that the former is built from the latter. The proof of existence of classical objects involves a regularization jointly with a jump to the limit. It has been so since Archimedes age. But from Wilson we know that such proofs are equivalent to the existence of fixed points in the space of regularizations, and that the argument can be developed by referring anything to an scale h_0. Just as the limit of a series $a_n/b_n \to 0/0$ is got by referring all its elements to a finite constant denominator.

The metaphysical justification of our toys would be to come someday to claim that QFT is the correct proof of existence of classical field theory (well, it should be closer to the proof of existence of a variational calculus). And that, by some reason, Nature does not like to make the jump into this limit.

Postlude

Even if you has not liked anyone of the previous, I am sure you will like this: *Copenhagen*, yet playing at the Duchess theater, up the Strand.

I was prevented about the script, but the author has made a deep research, more careful than the usual newspaper comments about the meetings between

---

[1] If you feel uncomfort identifying differentials and particles, try instead to say "low energy modes of an 1-brane" (sorry, the temptation was too strng, at least I have put it on a footnote. But please remember why they were called "p"-things, the p-branes). To identify particle fields with special choosing of coordinate fields seems too far-fetched, but it is appealing to think of confined fields as "angular" coordinates, without length units. So we have mixed feeling about it.

[2] This is touchy, as those variables in classical mechanics are only a trick to formalize variational calculus (we simply look for solutions $F[f + \theta] = F[f]$); their role could be different here

Bohr and Heisenberg. Better, the main plot has a lot of deviations you will enjoy, for instance when Bohr remembers his doubts about the spin, or when he asks for simplicity in the dialogue, because they "must be able to explain everything to Margaritha"

References

[1] A. Connes, Gravity coupled with matter and foundation of non commutative geometry, *Comm. Math. Phys.*, **182** (1996)

[2] J.S. Dowker, Path Integrals and Ordering Rules, *J. Math. Phys.* **17** (1976)

[3] Roy R. Gould, *Am. J. Phys.*, **63**, n. 2 (1995)

[4] A. Rivero, *Report on a plan to derive the standard model*, to be filed in hep-th.

[5] A. Rivero, *Some conjectures looking for a NCG theory*, hep-th/9804169

[6] T. Fujiwara, H. Suzuki and K. Wu, *Non-commutative Differential Calculus and the Axial Anomaly in Abelian Lattice Gauge Theories*, hep-lat/9906015

[7] M. Luescher, *Chiral gauge theories on the lattice with exact gauge invariance*, hep-lat/9909150

[8] M. Goekeler and T. Schueker, *Does noncommutative geometry encompass lattice gauge theory?*, hep-th/9805077