

# Ah, that 3/8! Weinberg's angle and $Z^0$ decay width.

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## Abstract

The GUT value of Weinberg's angle is also the value that minimises the total square matrix elements of  $Z^0$  decay, independently of any GUT consideration, and thus the one that maximises the neutrino branching ratio against total width. We review the proof of this result and some related facts.

From any textbook (eg [1]), the amplitude for decay of  $Z^0$  into a fermion pair, at leading order, is

$$\Gamma(Z^0 \rightarrow f\bar{f}) = C_f(|V_f|^2 + |A_f|^2) \frac{G_F M_Z^3}{6\sqrt{2}\pi} \quad (1)$$

where  $C_f$  is a colour normalisation constant, 1 for fermions and 3 for quarks, and  $V_f$  and  $A_f$  are the vector and axial charges,

$$V_f = T_f^3 - 2Q_f \sin^2 \theta_W \quad (2)$$

$$A_f = T_f^3 \quad (3)$$

If we are interested on the decay into a set of fermions, we add the contributions to get the total square matrix element:

$$K_{\{f\}} = \sum_f C_f ((T_f^3)^2 + (T_f^3 - 2Q_f \hat{s})^2) \quad (4)$$

We want to know for which value of  $\hat{s} \equiv \sin^2 \theta_W$  will the relative coupling, and then the decay width<sup>1</sup>, to be a minimum. Thus we ask

$$0 = K'_{\{f\}}(\hat{s}) = \sum_f 2C_f(T_f^3 - 2Q_f \hat{s})(-2Q_f) = 4 \sum_f C_f(2\hat{s}Q_f^2 - T_f^3 Q_f) \quad (5)$$

and then using that  $T_f^3 Q_f = T_f^3(Y + T_f^3) = (T_f^3)^2$  for sums across an isospin multiplet, accounting colour in the whole sum, and passing it from Dirac to Weyl species we get

$$\hat{s}_{min} = \frac{\sum_f T_f^3 Q_f}{2 \sum_f Q_f^2} = \frac{\sum_f (T_f^3)^2}{\sum_f Q_f^2} \quad (6)$$

When the set of fermions is a whole generation, this last formula equals the very well known result (e.g. exercise VII.5.2 in [2]) for  $\sin^2 \theta_W$  at the GUT scale of any unification based on a simple group. It is independent of the specific fermion content of the theory except that the factor 2 cancels because right fermions live in isospin singlets, thus  $T^3 = 0$  for them.

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<sup>1</sup>Except for the kinematics if we still want  $M_Z$  to depend of  $\sin \theta_W$ .

The proof is even shorter if we make use of the argument from D. H. Perkins [6, section 9.3]. According this argument, normalisation of the coupling amplitudes in the GUT scale amounts to forbid fermion-loop mixing of  $Z^0$  and photon, i.e. to ask directly  $\sum_f C_f Q(T^3 - 2\hat{s} \cdot Q) = 0$  when the loop is run for all the fermions in a multiplet of the GUT group. Diagrammatically we can move between our formulation and Perkins's by joining the two fermion lines in the  $Z^0$  decay to form a tadpole diagram and then inserting an external photon line, the whole process resembling the setup of a Ward identity, but where now the derivative is against Weinberg's sine squared.

Another early try to get the right normalisation of couplings from considerations in the fermion-boson vertex can be found in [7].

Now lets concentrate particularly in the fermion content of a generation of the standard model, where we have  $\hat{s}_{min} = 3/8$  (and  $K_{\{u,d,\nu,e\}}(3/8) = 2.5$ ).

The charge assignments of the standard model can be imposed by hand or via the requisites of anomaly cancellation. In any case, they have an extra property when we pay attention to minimisation of  $Z^0$  decay: the value  $3/8$  also minimises separately the partial decay width towards an  $u$  (or  $c$ ) quark. And thus it minimises also the partial decay into the set  $\{d, \nu_e, e\}$  (or  $\{s, \nu_\mu, \mu\}$  or  $\{b, \nu_\tau, \tau\}$ )

This means that for the standard model,  $3/8$  is not only the value minimising decay into first and second generation; it also minimises the decay of  $Z^0$  into the third generation, even if the top quark is kinematically out of reach of the gauge meson. A posteriori, we could interpret this fact of an indication of the particular characteristics of the top quark. The same arguments could be run from Perkins requisite, but the sum of fermion loops does not underline the special role of top quark, while  $Z^0$  decay stresses it.

Up to here the main comment, or result<sup>2</sup>, of this note: that the GUT formula for Weinberg angle at unification scale is also got without GUT, by minimising  $Z_0$  decay. The following few paragraphs are random musings distilled from the above:

- A consequence of the derivation here presented is that a model can get into GUT angle by asking for some minimisation requisite, without looking for a GUT group. Spectral actions of Connes-Chamseddine could be a good candidate for this, as would some other approaches from non commutative geometry [9, 10]. And I wonder if Ibañez string-inspired approaches to Weinberg angle are also a consequence of hidden minimisation.

- We have an alternate, more physical if you wish, way to state the Hierarchy problem. Instead of asking "Why the unification scale is so high compared with the electroweak SSB scale", now we can ask "Why the  $Z_0$  decay (the squared matrix elements  $K$ ) should reach a minimum at the GUT scale".

- If we contemplate  $K_{\{u,d,\nu,e\}}$  we can wonder for the value of this coupling at the experimental scale of decay, ie when  $\hat{s}$  is about 0.232. In GUT theories there is an scale available from which the value of the angle descends via renormalisation flow [3, 4]; here we haven't such scale to start with. But a related unexplained fact is that

$$K_{\{u,d,\nu,e\}}(0.231948\dots) = \exp(1) = \sum_0^{\infty} \frac{1}{n!} \quad (7)$$

We haven't the slightest idea of why the transcendent number  $e$  could have a reason to appear here. The minimum,  $K = 2.5$ , is a member of the simple series expansion of  $e$ , up to three terms. But on the other hand the values  $K = 1$  and  $K = 2$ , which we could get by using the lower terms, need of a complex  $\hat{s}$ .

Related to this, it is perhaps worth to mention that the experimental  $Z^0$  width, which we can calculate by summing the three families, has another intriguing empir-

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<sup>2</sup>The whole note is motivated because I have been unable to find this remark in standard textbooks; I'd thank any information about previous statements of it

ical accident: it scales straightforwardly, via the cube of the mass, from neutral pion width[8], so that currently the scaling down from  $Z^0$  to  $\pi^0$  is a precise "prediction" of the mean life of the latter particle.

- Also for the standard model assignment of charges, and in terms of the decay to a whole family, we have the relationships  $K_{\{d,e\}} = \frac{1}{2}K$ ,  $K_\nu = \frac{1}{2}$ ,  $K_u = \frac{1}{2}K - \frac{1}{2}$ . This implies that for the above mentioned  $K = 1$ , the decay probability into upper quarks vanishes.

- The value of  $K\nu$  does not depend of  $\sin^2 \theta_W$ . Then we can use its corresponding decay width to establish branching ratios, getting rid of  $G_F$  and  $M_{Z^0}^3$ . So, we can state for instance that at tree order, the value of  $\sin^2 \theta_W$  that maximises the branching ratio of neutrinos in  $Z^0$  decay is the same value that  $\sin^2 \theta_W$  has at the Grand Unification Scale. And so on. This is a traditional trick, used by instance in [11], where regrettably Wilczek et al. only plot ratios dependence of  $\sin^2 \theta_W$  in the quark sector, then getting a near miss of the formulation here presented.

- Just for analytic commodity we can solve the equation for  $\hat{s}$  in terms of the decay to a whole family. We have

$$\hat{s} = \frac{3}{8} \left( 1 - \sqrt{\frac{2}{3}} \sqrt{K - \frac{5}{2}} \right) \quad (8)$$

where some care must be taken about the lack of analyticity of  $||^2$  and its conversion to  $()^2$ . Actually the possible values of  $\hat{s}$  for a given  $K$  form an hyperbola in the complex plane, that becomes degenerate when  $K = 5/2$ . For values  $K < 5/2$  the hyperbola does not touch the real axis and the above equation gives the position of its vertex. Incidentally, for  $K=1$  such vertex is at a distance  $3/8$  of the real axis and the hyperbola is symmetric to the one for  $K=4$ .

- Finally, let me note that another common apparition of the factor  $3/8$  is in perturbative expansions of electromagnetism, and that the use of the GUT to correct  $\alpha_{EM}$  has been vindicated in some exponential adjustments between Planck and electron scales, eg by Laurent Nottale. I strongly doubt that the development here can be connected to these ones, albeit a corner should be left to accidental technicalities from  $\sin^4, \cos^4$  expansions.

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