Revisiting the Tangent Groupoid

A. Rivero*

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Abstract

The setup of a quantization in a Tangent Groupoid structure is reviewed from the point of view of the action of Butcher Group. In this sense, quantization is implied by renormalization.

1

Consider a particle in a manifold M. Its movement can be described with a groupoid scructure in $M \times M \times R$:

$$(x, y, t)(y, z, t') = (x, z, t + t')$$

Alain Connes discovered other groupoid structure,

$$(x, y, \epsilon)(y, z, \epsilon) = (x, z, \epsilon)$$

defining the Tangent Groupoid of M. This groupoid contains TM, defined as $(x, X)(x, Y) = (x, X + Y)$ in the boundary $\epsilon \to 0$. When considering the functions over the groupoid, continuity translates to Weyl quantization rule. This was claimed by Connes in a lecture at Les Houches and verifyed later by some teams, ours between them [3, 4].

It remained to understand why the natural groupoid was substituted for the Tangent Groupoid. The author pursued some hints [9, 10] related to the scaling between both groupoids: the result $(x, y, 2t)$ in the first one must rescale to (x, y, ϵ) in the latter, somehow going back from a 2ϵ spacing to the original ϵ scale. While this points to a renormalization group scheme, it was unclear at that time how RG was to be formulated in this context.

Recently it was found that Renormalization Group can be coded in a Hopf algebra whose simplest example is the algebra of rooted trees. Elaborated uses of this algebra have been done by some people: Kreimer, Connes, Brouder, Wulkenhaar, Frabetti, and others[1] . The use of rooted trees to label, and to

*Dep. Informatica, Univ Carlos III Madrid, Spain

[1] Reader is encouraged to do extensive search on the subject in http://www.arxiv.org/, where most of the matter is scattered across hep-th, math-ph and math.QA. Students are specially warned to keep a critical eye when reading unreferred research (just as this preprint, for instance): there is always some gaps to be filled and some misunderstandings to be solved only by long hours of study.

order, terms in solutions of ODEs was known from Cayley, and it had reentered the field in 1972, when Butcher [2] fully implemented a suggestion of Merton to label Runge-Kutta methods; then it appeared that it was possible to define a composition of methods [2].

Very much as decimation proceeded, this composition proceeds by halving scales. Given a RK method going from x to y in a step ϵ, we compose it with another one starting from y and going to z. Linearity let us to consider both methods as a single one, from x to z.

In [6], Connes and Kreimer comment about how existence of labeling of RK methods by rooted trees can be seen as the appropiate generalization of the solution of ODEs. They point again to the Tangent Groupoid as the appropiate context to study the phenomena. So it is appropiate to revisit the groupoid and see if the new math available can be used to justify the existence of a quantization principle.

2

Poincare suggested a method to codify the information about the behavior of a function $f(t)$ as $t \to \infty$. The most inmediate information is the limit of the function in the limit point. Call it a_0. Then we can consider

$$\lim_{t\to\infty} t(f(t) - a_0)$$

and if it exists, call it a_1. And so on. It can happen that all the limits exist,

$$a_n = \lim_{t\to\infty} t^n[f(t) - a_0 - a_1 t - ... - a_{n-1}t^{-n+1}]$$

. Then we say we have an *asymtotic expansion*. For instance [5], the integral $e^t \int_t^\infty s^{-1}e^{-s}ds$ has an a.e. $\sum_{n=0}(-1)^{n+1}n!t^{-n-1}$.

Note that the asymtotic series could to be divergent for all x. Because of this, physics folklore defines a.s. just as "series which are not convergent". This is a way to say that a.e. can be operated in the same way that convergent series.

Most important, different functions can have the same asymptotic expansion. This can be easyly seen by considering the function $e^{-\alpha t}$, which has a null a.e. $a_i = 0 \forall i$

In this paper we are interested on the kind of a.s. that appear when evaluating any integral. Suppose we divide the integral in N segments and then we proceed to sum the fragments. Obviously we can resume the method by giving an asymptotic Al expansion on N. Lets say

$$\int r(x)dx \simeq \sum^N \frac{L}{N} r(\frac{L}{N} i) \sim a_0 + a_1\frac{1}{N} + a_2\frac{1}{N^2} + \dots$$

[2] The next monography from Hairer et al [8] is expected to contain a full review of the uses of trees in numerical analysis, including the Hopf algebra formulation. Meanwhile, see [1]. For an introduction to numerics, the book [7] is a handy tool

Of course the first term, a_0, will be the value of the integral if the integral exists.

On other hand, we are more interested in the integral of a vector field,

$$\oint_0^\lambda f(t) d\vec{r}(t) = \vec{r}(\lambda) - \vec{r}(0)$$

3

The unit in Butcher group is the trivial method x, x, ϵ. The inverse of a method x, y, ϵ is the method y, x, ϵ that takes the inverse path across the vector field.

Besides holding an infinite number of integration points, Picard method has some interesting scaling properties [2, Th 7.1].

We can anticipate that Picard methods are going to be a line of renormalized methods. The RG will switch from the absurdity of a infinite sequence of infinitesimal one-step RK evolutions to the absurdity of a infinite number of intermediate integration points. The former is controlled by a cut-off ϵ, the later is controlled by an arbitrary scale $\hbar$

...

4

A perturbative series can be renormalized following an standard procedure. First regularize the divergences using a cutoff $epsilon$. Then define a projection method R which gives, for each term, a corresponging one still containing the same divergence. Substract both terms and there you are.

A teacher at Vietri gave us the following example: consider divergent terms such as $\frac{\phi(\epsilon)}{\epsilon}, \frac{\phi(2\epsilon)}{\epsilon^2}, \ldots$ You need to find counterterms $\lambda/\epsilon, \ldots$ that still contain the divergent part, and that can be adjusted to get a finite result. In this simple case the adjustment gives,

$$\lim_{\epsilon \to 0} \frac{\phi(\epsilon) - \phi(0)}{\epsilon}, \lim_{\epsilon \to 0} \frac{\phi(2\epsilon) - 2\phi(\epsilon)}{\epsilon^2}, \ldots$$

ie, $f'(0), f''(0), \ldots$. In some sense, usual derivatives are an example of perturbative renormalization.

Real life is more strange. One must refer to an arbitrary scale h to define the finite part while removing the cutoff. And, being arbitrary, it is possible to move to other level h' without changing the physics. This implies a group structure called the Renormalization Group.

Kreimer found that the methodology can be more accurately described in terms of Hoft algebras. The renormalization scheme R will twist the antipode of the algebra, and the twisted antipode acts in the bare series to bring it back to the renormalized one at some scale h.

Now, in our setup it is also convenient to recall the decimation picture of RG, in the line of Kadanoff, Wilson y Kogut. Imagine a lattice where the physical quantity is quantized to only two values for each point. Now, grop the points in boxes, lets say with 3^d points in each. This can be seen as a new physical system with 2×3^d degrees of freedom in each point, and we can built a hamiltonian to get the same physics that the former system. But notice that the scale has been moved down a factor three.

In this way the general RG acts: the regularization parameter ϵ generates a system for each value of ϵ, but the RG steps can always be used to refer it back to a constant scale h, and the limit is taken along this family of systems.

Notice that the coproduct in RK makes the role of a RG step. The new method gets more parameters, in the same way that a hamiltonian gets more terms when doing the decimation. Picard methods are the perfect actions of the RG

When searching for the RG fixed points, the cutoff scale is the only one needed to remove physical units. This is because physicist have a constant h relating units in position space and momentum space, and a constant c relating space and time units.

In a purely mathematical setting the role of units is not explicit. It could be assimilated to a metric, but we prefer to see a unit as a way of labeling different sets of euclidean coordinates. [3].

Scales on the Tangent Groupoid Alejandro RiveroRochester, Kent (UK). email rivero@wigner.unizar.es

Abstract

Tangent groupoid is reviewed in the light of Connes and Kreimer recent exposal of renormalization group formalism. N points functions, block transformations and cotangent groupoid are new concepts to be fitted.

5 motivation

Besides its flirt with M(atrix) theory, Non Commutative Geometry got recently a fortunate[4] impulse with the discovery of a Hopf algebra underlying the process of renormalization diagrammatics [**?**]. Moreover, Kreimer use of rooted trees let Brouder [1] to point a relation with the Butcher group of Runge-Kutta numerical integration methods, with links which go back until the works of Cayley [**?**].

In this context, and reviewing issues of operator expnsion, Connes andn Kreimer [**?**] draw paralells with compactifications of the Fulton-MacPherson [**?**] typer. Theses structures, that study diagonals in the configuration space of n particles, happen to be stratified by rooted trees too. The appropiate instrument here, it is suggested, should be a generalization of the tangent groupoid, extended "from the two point case to the full set-up of the configurations of n points". Such instrument is expected to have scales available for any strata.

Of course calculus, either codified in the usual mode or over the tangent groupoid, is easy to recast in renormalization group language, mostly for teaching purposes. Such examples have been sometimes given by Connes for the

[3] So, a given point can be at 0.1 cm or 1.0 mm or 0.01 meters: the value of the coordinate depends on the unit

[4] No mathematically surprising, but really lucky given the low number of researches involved in the area

"classical" renormalization group (see eq. **??**) and we ourselves included a "lattice" wilsonian example as appendix of [?] a collection of imaginery for the groupoid. Partly this work is being done as a kind of expansion of the previous one, thus we expect be forgiven by keeping some lighter tones. For instance, the following section.

6 Poetical motivation

Differential calculus is the mathematical toolt built to confront two dual[5] problems. The first one comes from Zeno, who argued [?] that an moving arrow can no be described only telling where it is, nor can be described only telling where it is going to be. The second one comes from Democritus who, facing the problem of integration of revolving figures, whondered how to distinghuish infinitesimal slices coming from different figures (eg a cilynder and a cone). For, the two circles liming a infinitesimal slice are expected to be equal, but then its composition should build always a cylinder.

Eventually, mathematicians got to address both problems[6]. To solve the first, we note both points, and then a limit method to build from this kind of vector a covariant one which becomes the instantaneus velocity of the arrow. To solve the second, we codify both planes with an axial vector, and again a limit method is used to built a contravariant vector over each point; this assignment becomes a differential form.

How such limits are done? Take the movement problem. The first obvious answer is to describe it by giving both the starting and ending points, and some measure of the time interval for the arrow to go from one to another. A natural composition rule seems to be

$$(x, y, t)(y, z, t) \to (x, z, 2t)$$

But this rule does not drive to define a linear velocity space attached to every instant of the trajectory. For that,

$$(x, y, \epsilon)(y, z, \epsilon) \to (x, z, \epsilon)$$

proves more adequate: in the limit $y \to x, z \to x, \epsilon \to 0$ the composition rule defines addition on the tangent space $T_x M$, ie

$$(X, x) + (Y, x) = (Z, x)$$

with $X = \lim_{\epsilon \to cero} \frac{y-x}{\epsilon}$ and so on (see [?] for the rigourous construction a generic manifold M).

[5] or at least, connected via a Wick rotation, just make time a height and speed an area

[6] Democritus himself got to integrate the cone, and a method was devised by Archimedes, who noticed the need of a proper limit by integrating slices both over and under the figure. The first problem showed itself harder, and while Archimedes got some progress defining the tangent line to some curves, we do not know if he or his sucessors made some advance, as most work was lost or deleted. It took the simultaneus genious of Newton and Leibnitz to figure out a generic solution

The object $M \times M \times [0,1) \bigcup TM$ with the previous operation (**??**) was defined by Connes [**?**] in a way such that it can be studied with the basic tool of NCG, the algebra of continous functions over the object, which he denominated Tangent Groupoid.

This constuction was done in order to study objects far more sophisticated that a manifold: foliations, quotient spaces, etc. But even in the simplest case a surprise was patiently waiting us: It was noticed, by Connes himself, that this aproach involved a quantization method relating operator kernels with functions over T^*M.

This link was examined with detail in [**?**, **?**], but no deeper examination was done. Some paralell threads, coming from q-deformations and discrete calculus [**?**, **?**] seem to stop at the same depth.

7 objects

7.1 tangent groupoid

Def: such that, for a chart $M \ni U \to R^n$, the sucesion $\frac{c^\mu(x_n)-c^\mu(y_n)}{\epsilon_n}$ converges to $X^\mu \in T_xM$

it is possible actually to define T_xM as the points at infinite of a compactification given by this kind of series. this avoids us the nuance of duplicated information.

We have sometimes referred to the $\epsilon > 0$ part of the tangent groupoid as the *secant groupoid*. It gives a good picture as an element $(x, y, \epsilon$ living on a line cutting[7] the manifold through points x, y. From this image and the definition you can see that each level $\epsilon = \tilde{a}$ of the groupoid is simply an affine vector space living in M, and that the limiting process makes from this affine space a free vector one over each point.

Exa:

7.2 cotangent groupoid

Def: such that, for a chart $M \ni U \to R^n$, the sucesion $\frac{\gamma_n}{c^\mu(x_n)-c^\mu(y_n)}$ converges to $X_\mu \in T^*_xM$

As in the previous case, it is possible actually to define T^*_xM as the points at infinite of a compactification given by this kind of series. Again, this avoids us the nuance of duplicated information.

name

We have made $x - y$ a covector

Ex: typical elements (x, y, a) of the tangent groupoid can be thought as slices of an integral.

[7] regretly, the traditional name for the segment joining the extremes of an arc is already overused

While for the mathematical image of the groupoid could be enough note how affine covectors are dual to the affine vector above (an approach followed in some textbooks, cite)....

While the tangent groupoid comes naturally from the effort to build derivatives, the cotangent groupoid can be easily found in the process of formalising integration. For a first idea, take an increasing function $f : [0,1] \to R$, and use a partition of scale a_0 to build the usual rectangles above and below the funtion. The convergence of the integral as a goes to zero is got from the fact that the difference between the upper set of triangles and the lower one is only given by the a rectangle in the extremes, so it goes to zero as the thickness of the rectangle[8].

You can guess then that the basic condition to get a integral working is to be able to fit one slice into the following one. This information is stored in the paralell planes of every differential segment, you can imagine one of then defining the small rectangle, the other defining the big one, and in composition done in a form such that the big rectangle of a slice must be equal to the small one of the following.

7.3 trees

Sit down under the appletree, but forget the apple falling just there and contemplate branches instead. Imagine a branch as a point which at every node, as you progress towards the leaves, splits in other points which separate ones from others. Got the idea? Now, do it backwards, points going nearer, the ones coming near faster join in an only branch which progress to join next ones. Well, you have now a pretty idea of a point at infinity in the -McPherson compactification. Note that you are not forced to join all the branchs in a single point to finish the process; this unattached trees make also points of the compactification, and make place for some permutation counting to torture students.

7.4 McPherson trees. orders

Def:

Notice we are making pure topology. Nice thing, but perhaps excesive as we have no clue to compare orders, this is, the speed which a point separates from other at, for different bifurcations happening in different branches. We can solve this by making the nodes bifurcate at different "depths": simply permit single branches to enter play. We lost the easy code of nested sets, but we win some nice algebra.

7.5 Kreimer trees. scales. Other trees similar

The algebra of rooted trees.

Loops. First order diff eq. Iterated integrals.

[8] this proof is from Archimedes, "the paraboloid", cite

Cayley.

Scales.

7.6 n-point chains

8 actions

Lets go back to the integral example and to the array of groupoid elements approximating it. Suposse you want to multiply this array of rectangles, by a function $f(x)$. A question arises: must I multiply the small rectangles, getting the big ones from the composition condition, or the contrary.

It happens that this chain is a bimodule, which can be acted in two different forms. In this instance, we see that if a is the grossor of the slices, then multiplicacion "of the upper" by $f(x)$ amount to multiplication "of the lower" by $f(x+a)$. This is generalized making the space of differential to be an bimodule over the one of funtions. In this simple example we had $f(x)dx = dxf(x+a)$, and Majid [**?**] does so. Remembering that out forms are mapped more generically to pairs (x, y), we put

$$f(x)dx = dxf(y)$$

. Of course if $x(.)$ is the coordinate function associated to dx, we have $[dx, x] = \Lambda$ yet.

define:....

For the tangent grupoid, the same can be done:

define:...

9 dualities

A definite volume form must have an orientation. This is got by asking anticonmutation to the differentials, so that $dxdy = -dydx$

Poincare duality, via the intersection product, warrants the existence of interated integration, but we can contemplate yet a problem, we need integration to commute, ie $\int dx \int dy = \int dy \int dx$.

duality and at the abstract, categorical, level?

10 transforms

Fourier: is it from X or from x?

11 results

tensor.

strata.

renorm and first degree. Non renorm, greater than first degree diff eq? Reducible to first?

11.1 Scales triangle

wilson...

p

12 physics

Velocity space relates to momentum space via m, the inertial mass.

The map **??** is fixed through the CKM mixing matrix.

Position space relates to momentum space via h, Plank constant

Position space relates to velocity space via (exponentiation map) a scaling parameter with units of time.

Again, time relates to space via c, light speed.

Is c also the only solution, via contractions and all that, to make compatible fourier transform and manifolds? Its imposition of locality could be just the needed tool. (is that general relativity?)

Variations can be controlled through a sugerence attibuted to Grassman: incorporate the requeriment $\Delta x)^2 << \Delta x$ in a rigourous manned by using a set of variables θ_i such that $\theta_i^2 = 0$, and $\theta_i \theta_j = -\theta_j \theta_i$, so we have a "small" volume element with definite orientation.

It is a fortunate coincidence that Dirac sea requires, to be filled, the Pauli exclusion principle, and this in turn ask field operators not to be able to create two particles with the same spin and momenta. This is, the one particle creation operatir $a_\mu(k)$ must be such that $a_\mu(k)a_\mu(k) = 0$ (for a textbook on "primitive" quantum fielt theory, we suggest [**?**]).

Renormalization substraction were (accoring [**?**] suggested by Kramer and developed in [**?**], in the case if mass renormalization.

13 further comments

-It sounds pedantic, but the Tangent Groupoid formulation let us pinpoint a *mistake* of classical mechanics. It happens that both differential and integral calculus are defined bytaking the limit of a scale, which goes to zero in the later, to infinity in the former. This is rigourous, and calculations are correct while we keep each one in its field. But if both calculus are mixed, then an indeterminacy $\infty 0$ has been hidden under the carpet! Classical Mechanics chooses to fix that indeterminacy, the product of units $\tilde{a}a$ to zero, and this is, to say the least, arbitrary. Indeed, it happens that Nature, with more mathematical sense of the origin of its equations, make this limit go to a finite constant value; such fact should be taken as an axiom of mechanics if we were to formalize it[9]

[9] think, for instance, in Hilbert 6th problem

-Configuration space has two roles, one as target as trajectories to be selected by some differential equation, other as domail for functions which are to be minimized according some funtional. It is a ironical aside that the goal of the later was to cure mechanics from action-to-distance, from nolocality. But we are supossed to cope with both roles now.

-We would like to involve also a sense of urgency. From the point of pure mathematic, we only have some issues that have been patiently waiting for us for two milennia, so they will surely to wait some decades more. But from physics, if this quest is going to have some relevance in particles, it seems better to have results while the experimentalist are yet providing fresh data, does it?.

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Queremos entender el principio de minima accion cuando todo lo que contamos para ello es con el grupoide tangente. Sabemos como funcionan las cosas en $\epsilon = 0$, es la macanica lagrangiana. Pero dado que a los vectores de TM llegamos a traves de un proceso de limite en $M \times M \times (0,1)$, nos interesa formular todo antes del propio limite.

Nuestro primer problema: Construir una trajectoria $\phi(t), \dot{\phi}(t)$ en el contexto del grupoide tangente.

Como plantearlo? Con promedios y block-spin. Vamos alla

Tomamos una funcion $\phi(t)$, continua y "derivable". La exploramos a una escala ϵ, esto es, tomando muestras

$$\phi_n^\epsilon \equiv \phi(n\epsilon),\ n \in Z$$

(aqui tenemos distintas alternativas de muestreo: promediar, coger el maximo, etc. Para cada una debemos de dar una interpretacion compatible en los elementos del grupoide).

Esta discretizacion ϕ_n tiene una derivada $\dot\phi$ que es el conjunto de elementos del grupoide del tipo

$$\dot\phi_n \equiv (\phi_n, \phi_n + 1, \epsilon)$$

Tenemos con ello una "trayectoria" $\phi^\epsilon, \dot\phi^\epsilon$ en cada capa del grupoide.

Nuestro objetivo inicial es saber cuando dos series de este tipo definen, en el limite $\epsilon \to 0$, un par de funcion y derivada viviendo en TM. La condicion para obtener un punto en TM la empezamos a discutir en un apendice hace tiempo, una version sofisticada de la definicion de limite de una serie de fracciones cuyo numerador y denominador van a cero.

(Intermedio: basicamente, la historia es que una serie

$$\{\frac{f_1}{\epsilon_1}, \frac{f_2}{\epsilon_2}, \frac{f_3}{\epsilon_3}...\}$$

que tiende a $0/0$ es compensada multiplicando numerador y denominador por la serie

$$\{\frac{\epsilon_0}{\epsilon_1}, \frac{\epsilon_0}{\epsilon_2}, \frac{\epsilon_0}{\epsilon_3}...\}$$

De forma que la serie resultante si que tiene un limite sensato como fraccion, simplemente $\frac{\epsilon_0 f}{\epsilon_0}$. En aquel caso, traspasabamos de un limite que debia caer en TM a un limite mas sencillo simplemente en $M \times M \times \{\epsilon_0\}$)

Aqui se trata no de un punto sino de una trayectoria completa, y esto se reflejara en una condicion adicional, posiblemente el cumplir un escalado

$$\phi_n^{2\epsilon} = \frac{\phi_n^\epsilon + \phi_{n+1}^\epsilon}{2}$$

o algo asi.

Segunda parte, tenemos que entender como se construyen funcionales $L[\phi]$, que viene a ser entender integracion a con una escala dada.

Tendremos que oftener una familia L^ϵ, y al cambio $\phi^\epsilon \to \phi^{n\epsilon}$ le correspondera un cambio $L^\epsilon \to L^{n\epsilon}$ que nos garantize que en cierto limite obtendremos la accion clasica. El cambio disminuye el numero de observaciones discretas en las que evalua la accion, digamos que el grupoide se va haciendo mas "basto" segun la escale.

Tenemos que conseguir utilizando reescalados que la existencia de un limite clasico en $\epsilon \to 0$ se corresponda con la existencia de otro de $L_0^\epsilon[\phi^{\epsilon_0, n}]$ en la loncha ϵ_0 del grupoide.

NUESTRA INTUICION ES QUE LA CONDICION DE MINIMIZAR LA ACCION QUE ESTE LIMITE REPRESENTA ES LA LEY DE MINIMA ACCION DE MECANICA CUANTICA. La ϵ_0 seria la autentica constante de plank, y no las "running ϵ_n" que aparecen en el proceso de construccion del grupoide. Naturalmente, ϵ_0 es arbitraria en el sentido de que debe determinarse experimentalmente. Esto es lo que ocurre con cualquier escala que aparece inducida por el proceso de renormalizacion, no es raro. Y coincide con lo que sabemos de la cte de Plank.

Tercero, como implementamos la condicion de minimo de este funcional? Seguramente tendremos que seguir el reastro de lo que ya hizo Feymann. As un aside, recordemos que *formalmente* la condicion de minimo en el formalismo de feymann es un invariante bajo reescalados de la cte de plank,

$$< \frac{\delta L}{\delta \phi} >= \int e^{\frac{iL[\phi]}{h}} \frac{\delta L}{\delta \phi} d\phi = \int e^{\frac{iL}{h}} dL = \delta(1/h)$$

Esto es, $< \frac{\delta L}{\delta \phi} >$ es cero para cualquier valor de h, no solo para el limite clasico.

En algun sitio antes de ir a este tercer paso hemos de reencontrarnos con la formulacion de path integral, ya que cuando mas afinamos ϵ, mas cerca estamos de reproducir el conjunto de caminos que la path integral pesa. El problema mas raro al que nos enfrentamos por en medio es la existencia de los leyes "naturales" de composicion en el grupoide: por un lado, la que usamos en la construccion del limite de connes,

$$(x, y, \epsilon)(y, z, \epsilon) = (x, z, \epsilon)$$

Por otro, la natural de considerar ϵ como un parametro temporal, dando el tiempo en ir de x a y:

$$(x, y, \epsilon)(y, z, \epsilon) = (x, z, 2\epsilon)$$

Es curioso que las dos estan relacionadas por una dilatacion en la variable epsilon.