The finite Dirac operator as a boundary condition on an interval

(Dated: Draft, 3 November 2026)

Every finite Dirac operator D_F with the signs of KO-dimension six is exactly the lowest spectral window of the pure derivative on an interval of length l, for $l\|D_F\| < \pi/2$, with no mass term. The boundary data are a grading at each wall, and $D_F = (2l)^{-1}\Gamma_0 \log(\Gamma_1\Gamma_0)$, so the singular values of D_F are principal angles over l and the masses are quantized momenta. Preserving the domain restricts the observables at a wall to the commutant of its grading. Order one holds in this parent for any finite data with order zero. The compression of an observable is its average under e^{ixD_F}, $|x| \le l$, which tends to the finite representation with rate l^2. Aligned walls compress a bulk prealgebra stable under the grading to its even part, with the grading inherited. Even scalar fluctuations are pure gauge, and on a moving window they reproduce the quadratic term of Chamseddine-Connes-van Suijlekom. Odd ones change the angles; for $M_2(\mathbb{H})$ their field space is S^4. In a Yang-Mills parent with a wall connector the W and Z masses are exact, with the constants 2 and 16/5 at small angle. Unfolded, the parent is $-i\,d/dx + D_F$ on a circle, and at leading order in small masses, for a vector of constant profile in the transported frame, the sum rule of the spectral action becomes $M_W^2 = \frac{3-d}{24}\sum N_f m_f^2$, with d the spectral dimension of the circle at the cutoff, so that m_t/M_W goes from $\sqrt{8/3}$ at $d = 0$ to 2 at $d = 1$. All finite data are inputs and the parent algebra is matrix-valued, so the selection of the finite algebra is untouched.

I. INTRODUCTION

In an almost-commutative geometry the masses of the fermions are the entries of a finite Dirac operator D_F, and in a Kaluza-Klein reduction a mass can be a momentum in the internal direction, quantized by the boundary conditions. We relate the two for an interval. The Connes-Lott mass is the lowest such momentum, $\pi/(2l)$, and the known boundary formula $\tan(m_n L) = M_D L$ of Higgsless models [1] gives the interpolation for one flavor. Here we carry this through for the whole finite operator, with its real structure and grading, and we ask what the parent says about the algebra, order one, and fluctuations.

The finite data remain inputs, so the existence of the window is close to a reparametrization. The content lies in the existence of a consistent parent, and in the form it gives to the wall algebra, the compression and the fluctuations.

II. THE WINDOW

Let $H_F = H_+ \oplus H_-$ be finite-dimensional, $\Gamma_F = \mathrm{diag}(1,-1)$, and $c : H_+ \to H_-$ antiunitary, with $J_F(a,b) = (c^{-1}b, ca)$. Let D_F have blocks M and M^*, with $M = cM^*c$, which is $J_F D_F = D_F J_F$. These are the signs $(+,+,-)$. For $l\|D_F\| < \pi/2$, define $X : H_- \to H_+$ as the block of $\tan(lD_F)$, so that X^* is the other block.

On $L^2([0,l];\mathbb{C}^2)\otimes(H_+\oplus H_-)$, write ψ with components $(Q_L, Q_R; U_L, U_R)$, and let $D_X = E\,d/dx$, where $(E\psi)_L = \psi_R$ and $(E\psi)_R = -\psi_L$, with domain

$$\begin{aligned} &Q_R(0) = U_L(0) = 0, \\ &Q_R(l) = XU_R(l), \quad U_L(l) = -X^*Q_L(l). \end{aligned} \tag{1}$$

Let $\hat\Gamma$ be $+1$ on ψ_L and -1 on ψ_R, and let $\hat J\psi$ have components $(c^{-1}U_R, -c^{-1}U_L; -cQ_R, cQ_L)$.

Theorem 1. *D_X is self-adjoint with compact resolvent, with signs $(+,+,-)$. Let $R(x) = \exp(x\Gamma_F D_F)$, and let $W : H_F \to L^2$ send (a,b) to the function with L-components $l^{-1/2}R(x)(a,0)$ and R-components $l^{-1/2}R(x)(0,b)$. Then W is an isometry, and*

$$D_X W = W D_F, \quad \hat J W = W J_F, \quad \hat\Gamma W = W\Gamma_F. \tag{2}$$

With $\theta_k = l\sigma_k$ for the singular values σ_k of M, the signed spectrum is $(n\pi \pm \theta_k)/l$, $n \in \mathbb{Z}$, so $\operatorname{ran} W$ is the spectral subspace of $|D_X| \le \|D_F\|$, and the next level is at least $(\pi - l\|D_F\|)/l$.

Structure of the proof. Takagi factorization gives $M = V|M|$ with $V = cV^*c$. Then $R(x)(a,0)$ has components $\cos(x|M|)a$ and $-V\sin(x|M|)a$, and $R(x)(0,b)$ has components $\sin(x|M|)V^*b$ and $\cos(x|M^*|)b$. The wall conditions follow from $X = \tan(l|M|)V^*$, and $D_X W = W D_F$ follows by differentiating R. Constant unitaries on Q and U reduce X to $\mathrm{diag}(\tan\theta_k)$, and each channel has the known one-flavor spectrum (Supplemental Material, part 22). □

So D_X has no mass term. A light eigenvalue is a momentum $k = \theta/l$, fixed by $\tan(kl) = \tan\theta$, a singular value of X. In the finite description this condition is lost and the value appears as free data, which offers a reading of why the finite triple needs Yukawa couplings. One matrix X encodes the Yukawa and Majorana blocks together. Also, $R^* D_X R = D_0 + E\Gamma_F D_F$, with D_0 uncoupled: after a unitary change of domain, D_F is a constant matrix connection, with relative endpoint rotation $R(l)$.

Theorem 2. *The boundary conditions are gradings: ψ_L lies in the $+1$ eigenspace of Γ_i and ψ_R in the -1 eigenspace at each wall, with $\Gamma_0 = \Gamma_F$ and Γ_1 equal to $R(l)\Gamma_0 R(l)^*$. If -1 is not an eigenvalue of $\Gamma_1\Gamma_0$, then*

$$D_F = \frac{1}{2l}\,\Gamma_0 \log(\Gamma_1\Gamma_0), \tag{3}$$

and the numbers $l\sigma_k$ are the principal angles between the $+1$ spaces. A real structure is given by an antiunitary J_1, with $J_1^2 = -1$, that anticommutes with Γ_0 and Γ_1. Then $J_F = \Gamma_0 J_1$, and J_F carries Γ_1 to $-\Gamma_0\Gamma_1\Gamma_0$.

At an angle $\pi/2$ the channel has the chiral conditions of the Connes-Lott interval, with no zero mode and lowest level $\pi/(2l)$.

III. ALGEBRA, ORDER ONE, COMPRESSION

Let a unital algebra A_F act evenly and faithfully, with order zero, and let C_F be the commutant of D_F in A_F. Order one is not assumed.

Theorem 3. *A function $f \in C^\infty([0,l], A_F)$ preserves the domain of D_X iff $f(l) \in C_F$. Let A_l be the algebra of such f. Then $[D_X, \pi(f)] = E\pi(f')$, and we have a real even spectral triple with order zero and order one.*

Structure of the proof. The wall conditions say that $\pi(f(l))$ commutes with $\tan(lD_F)$, equivalently with D_F, for $l\|D_F\| < \pi/2$. The opposite action is pointwise, so by finite order zero $E\pi(f')$ commutes with it. □

So A_F acts freely in the bulk, and the wall carries the commutant of D_F. In the Standard Model data with nonzero Yukawa blocks, this is $\{(\lambda, \lambda, m)\}$, the algebra of the photon and gluons. Finite order one does not arise as a condition on the parent.

Theorem 4. *Let $Q_l(T) = W^*TW$. For even $T(x)$,*

$$\begin{aligned} Q_l(T) &= \frac{1}{l}\int_0^l \cos(x\,\mathrm{ad}_{D_F})(T(x))\,dx, \\ Q_l(ET) &= -\frac{1}{l}\int_0^l \sin(x\,\mathrm{ad}_{D_F})(T(x))\,dx. \end{aligned} \tag{4}$$

For constant a,

$$\begin{aligned} Q_l(a) &= \frac{1}{2l}\int_{-l}^{l} e^{ixD_F}\pi(a)e^{-ixD_F}dx \\ &= \mathrm{sinc}(l\,\mathrm{ad}_{D_F})(\pi(a)), \end{aligned} \tag{5}$$

so $\|Q_l(a) - \pi(a)\| \le (l^2/6)\|[D_F,[D_F,\pi(a)]]\|$. Moreover $Q_l(A_l) = Q_l(C^\infty([0,l],A_F))$, which contains $Q_l(a)$ for every $a \in A_F$, and the order defects of the compressed data differ from the finite ones by $O(l^2)$ with explicit constants.

Thus D_F, J_F, Γ_F are exact, and the observables converge to $\pi(A_F)$ as $l \to 0$ at fixed D_F, as maps. The generated algebra is discontinuous at $l = 0$: compressions of commuting projections can fail to commute at order l^2, in line with the remark of Connes and van Suijlekom [2] on generated algebras (Supplemental Material, part 22).

Theorem 5. *Let P be a $*$-subalgebra of $\mathrm{End}(H)$ that need not commute with Γ_0, acting pointwise and identically on both spin components. A function f valued in P preserves the domain iff $[f(0),\Gamma_0] = 0$ and $[f(l),\Gamma_1] = 0$, and $\hat\Gamma$ commutes with every such f. For aligned walls $\Gamma_1 = \Gamma_0$, $D_F = 0$, $\hat\Gamma$ restricts to Γ_0 on the window, and an ambient constant $a \in P$ has $Q(a) = E_0(a) = (a + \Gamma_0 a \Gamma_0)/2$. If $\Gamma_0 P \Gamma_0 = P$, then $E_0(P)$ is the even part of P, and it is the image of the boundary algebra.*

So a matrix-valued bulk prealgebra is compressed to its even part, with the grading inherited from the parent operator $\hat\Gamma$. For $M_2(\mathbb{H})\oplus M_4(\mathbb{C})$ this gives $\mathbb{H}\oplus\mathbb{H}\oplus M_4(\mathbb{C})$, the step of the classification [3] that a compression of functions on S^4 reaches only with an added grading. The commutative-parent question remains open.

IV. FLUCTUATIONS

Theorem 6. *Let h be smooth with anti-Hermitian values in P, with $u' = -hu$, $u(0) = 1$, $w = u(l)$, and let $D(\Gamma_0,\Gamma_1)$ be the derivative with those gradings. Then $D(\Gamma_0,\Gamma_1) + Eh$ equals $uD(\Gamma_0, w^*\Gamma_1 w)u^*$. The self-adjoint scalar one-forms are exactly the Eh. If h takes values in the even algebra, then $w^*\Gamma_1 w$ has the same angles with Γ_0; on the moving window the finite operator is $\pi(w)^* D_F \pi(w)$, and with the real fluctuation $U_w^* D_F U_w$, where $U_w = \pi(w)J_F\pi(w)J_F^{-1}$. In that case the connection is gauge equivalent to zero under the allowed endpoint gauge group.*

With $v = w^*$, this is the gauge orbit of [4], Eqs. (10)-(11), namely UDU^*, which equals $D + A_{(1)} + \tilde A_{(1)} + A_{(2)}$, quadratic term included, for finite data that fail order one, while the parent has order one and no quadratic term. The fixed compression is a different object: for constant h, $Q_l(Eh) = -(l/2)[D_F, h] + O(l^3)$, against $-l[D_F, h]$ for the first derivative of the transported operator.

No-go theorem (the radial mode). For one interval, fixed X, no bulk masses, even scalar inner fluctuations and the moving window, the singular values of the finite operator are fixed. With nonzero Yukawa data, the Higgs quaternion stays on the unit quaternions, with no radial mode (Supplemental Material, part 22). The following lie outside the hypotheses, and each needs its own construction: X as a field on the wall; bulk terms odd for the grading; bulk masses not commuting with the gauge action; more internal dimensions; an added composite degree of freedom. At the $10 \to 9$ endpoint this agrees with a frozen Higgs field.

The odd escape is explicit. For $W = \mathbb{H}^2$, the gradings with quaternionic eigenspaces form $S^4 = Sp(2)/(Sp(1)\times Sp(1))$, with $\Gamma_p = p_a\Gamma_a$. For two points at geodesic distance d, all principal angles equal $\theta = d/2$. On $\mathrm{Hom}(V,W)\oplus\mathrm{Hom}(W,V)$, the connection $K(x,y) = (\theta/l)(Fx, yF)$, with F odd in $M_2(\mathbb{H})$ and $F^2 = -1$, gives a universal mass θ/l, with $J_1 = \Gamma_0 J$ and a second grading at the point p on the first summand and its mirror image on the second. So (θ, q) are polar coordinates of a Higgs quaternion on S^4, the coset of the minimal composite Higgs model [5], whose warped construction and potentials are not those here; the analogy with gauge-Higgs unification [6] has the same limit. That S^4 is also the base of the twistor window is an observation here, with no identification. J_1 induces an opposite action twisted by the inner automorphism $\gamma_W \cdot \gamma_W$, a structural analogy

with the twisted triples of the grand symmetry, whose automorphism is different [7].

V. GAUGE SECTOR AND THE TWO LIMITS

Assume a flat product and a positive five-dimensional Yang-Mills kinetic form with no boundary terms. Then A_μ at a wall lies in the wall algebra, the directions in it are Neumann and the others Dirichlet, as in Higgsless models [8].

The strict family. For even walls with nonzero Yukawa data, the unbroken directions have masses $n\pi/l$ and the broken ones $(n-\frac{1}{2})\pi/l$, so $M_W = M_Z = \pi/(2l)$, with massless photon and gluons after unimodularity. Fix a light block M_0 and a heavy block Θ/l with $0 < \Theta < \pi/2$. As $l \to 0$, the light couplings $l|M_0|$ vanish at fixed mass, the heavy mass Θ/l and M_W diverge with $m_t/M_W = 2\Theta/\pi < 1$, and the light compression converges. This realizes the $10 \to 9$ proposal at this level, with $X_q \sim (\pi/2)m_q/M_W$, in the form of $y_q = \sqrt{2}m_q/v \to 0$ at fixed mass. Identifying X_q with a four-dimensional Yukawa coupling is a hypothesis.

Theorem 7. *Take $P = M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$, with $sp(2) \oplus su(4)$ and the trace form, with one coupling, gradings at distance 2θ, and at $x = l$ a connector on a line w in the $+1$ eigenspace of Γ_p and a line v in $\mathbb{C}^4$, whose stabilizer is $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$. Then for $0 < \theta < \pi/2$ exactly nine modes are massless, the photon, with $1/e^2 = (8/3)/g^2$, and the gluons, and*

$$\begin{aligned} \sin^2(M_W l) &= \tfrac{1}{2}\sin^2 2\theta, \\ \sin^2(M_Z l) &= \tfrac{4}{5}\sin^2 2\theta. \end{aligned} \tag{6}$$

With $m = \theta/l$, as $\theta \to 0$: $M_W^2 \to 2m^2$ and $M_Z^2 \to \frac{16}{5}m^2$. As $\theta \to \pi/2$: $M_W, M_Z \to 0$ and $m \to \pi/(2l)$. In both limits $M_W^2/M_Z^2 \to 5/8$, and its minimum is $((\pi/4)/\arctan 2)^2 = 0.503$ at $\theta = \pi/4$.

Here the connector is prescribed wall data for the gauge field; as a term in the fermion boundary operator it also changes the affected fermion tower. The constants 2 and 16/5 are the Hilbert-Schmidt ratios $\|[D_F, a]\|^2/\|a\|^2$, and they are the coefficients of the lower bounds for the poles of the fully massive family in the exact quadratic spectral action (Supplemental Material, part 18), whose small-mass poles and Taylor mass differ from them by the heat coefficients. The second limit is a geometric form of heavy fermions with light vectors: at antipodal points the two walls have the same stabilizer in $sp(2)$.

Proposition 8. *For the heat cutoff and universal θ, with $s = l\Lambda$, the trace per channel is $2\sum_{n\in\mathbb{Z}} \exp(-(n\pi + \theta)^2/s^2)$, a Jacobi theta function, strictly decreasing on $[0, \pi/2]$. For $s \to 0$ at fixed $x = \theta^2/s^2$, it tends to $2e^{-x}$.*

The radial no-go of the finite action [9] concerns finite data and a free ray, so this family lies outside it: the radial direction is compact, and the tower gives a stationary point at $m = \pi/(2l)$. There, by Theorem 7, the vectors are massless, so this is not the physical point. For the one-loop potential with vector and fermion loops in the flat massless family, with fermions counted by $n_f = 2N_D$, $\theta = \pi/2$ is the unique global minimum for $n_f > 0$, the potential is strictly decreasing for $n_f \geq 54/5$, and for $0 < n_f < 54/5$ it has exactly one interior stationary point, a strict maximum, for the stated flat massless determinant with no other angle-dependent term (Supplemental Material, part 23 and Supplemental Material, part 24). In a flat tree-level comparison, neither the Wilson-line family, with $M_W^2/M_Z^2 \in [0.503, 0.625)$, nor a finite wall scalar, with $[0.625, 1)$, fits both the top ratio and the weak ratio. Pure fermion channels of frequency 2 under the Wilson line, which the bimodule does not contain, change this: for $n_2 \geq n_c = 2.8855\ldots$ they give an interior global minimum, at $\theta = \pi/4$ when they act alone, where $\pi/l = 4M_W$. In the multiplets containing both top chiralities, in the displayed channel, a state at $2M_W$ has a lighter coloured partner for every finite wall mass. In a step-threshold model without those multiplets, the left-right difference of the couplings runs only logarithmically, and at one loop the three couplings meet at $2.2 \times 10^{14}\,\mathrm{GeV}$ with $\pi/l = 3.15 \times 10^{13}\,\mathrm{GeV}$, where the flat potential has no global minimum at the required angle (Supplemental Material, part 26). These negative statements hold in those flat families. They point toward changed form factors, as in warped composite models [5], or a wall scalar with its own potential, for the physical interior, consistently with the reading of the odd labels as limits.

VI. THE SUM RULE AND THE EFFECTIVE DIMENSION

Unfolding gives a sharper statement. For $x \in [-l, l]$, put $\Psi_Q(x) = Q_L(|x|) + i\,\mathrm{sgn}(x) Q_R(|x|)$ and $\Psi_U(x) = U_R(|x|) - i\,\mathrm{sgn}(x) U_L(|x|)$. Up to the factor $1/\sqrt{2}$, this is unitary, and it carries $R^* D_X R = D_0 + E\Gamma_F D_F$ to

$$-i\frac{d}{dx} \otimes 1 + 1 \otimes D_F \tag{7}$$

on the circle of length $2l$. So the tower is D_F shifted by $k_n = n\pi/l$, $n \in \mathbb{Z}$, and an even generator T constant in this transported frame becomes $1 \otimes T$. In the original frame it is RTR^*, generally with an odd bulk component, so it needs an enlarged gauge algebra, such as the prealgebra above. The exact quadratic kernel of the spectral action [9] then applies block by block, with Gaussian convergence.

Theorem 9. *For the heat cutoff, with $s = l\Lambda$, the Taylor mass of a vector of constant profile in the transported frame is, at fixed s and for fixed nonzero T,*

$$M^2 = \frac{3-d}{2}\,\frac{\|[D_F, T]\|^2}{\|T\|^2} + O\!\left(\frac{\|D_F\|^4}{\Lambda^2}\right), \tag{8}$$

in Hilbert-Schmidt norms, where $d = d\log S/d\log s$ *and* $S(s) = \sum_{n\in\mathbb{Z}} e^{-n^2\pi^2/s^2}$. *Here* S *is the heat trace of the unshifted circle at* $t = \Lambda^{-2}$, *so* d *is its spectral dimension at the cutoff, with* $d \to 0$ *for* $s \ll 1$ *and* $d \to 1$ *for* $s \gg 1$. *For the three-generation Standard Model representation, at this order,*

$$M_W^2 = \frac{3-d}{24}\sum_f N_f m_f^2, \tag{9}$$

so for the top alone $m_t/M_W = \sqrt{8/(3-d)}$, *and* $M_W^2/M_Z^2 = 5/8$ *at all* s.

Structure of the proof. With $a_i = (k+\mu_i)^2$ and $d_{ij} = \mu_i - \mu_j$, the kernel at $q = 0$ expands to second order in μ as $d_{ij}^2 e^{-k^2}(1-\frac{2}{3}k^2)$, and at order q^2 as $\frac{2}{3}e^{-k^2}$. Terms odd in k cancel between n and $-n$. Summing over n gives the factor $\frac{3}{2} - \langle k^2\rangle$, and $\langle k^2\rangle = \frac{1}{2}d\log S/d\log s$. □

At $d = 0$ this is the relation $8M_W^2 = \sum N_f m_f^2$ of the asymptotic spectral action. At $d = 1$ it is $12M_W^2 = \sum N_f m_f^2$, which covariance requires on the unfolded circle: the fifth component of the Dirac operator is a constant $A_5 = D_F$, and $F_{5\mu}^2$ and $F_{\mu\nu}^2$ have one coefficient. So the factor 3/2 between the finite spectral-action coefficient and the Yang-Mills commutator quotient is, in this constant-profile calculation, the ratio of the two ends of one function, crossing at $l\Lambda \approx 2$. In terms of couplings, $y_t = 2g/\sqrt{3}$ at $d = 0$ and $y_t = \sqrt{2}g$ at $d = 1$. The endpoint $m_t = 2M_W$ is known in flat gauge-Higgs models with suitable representations [10]; we did not find the coefficient $(3-d)/24$ or the crossover, and a wider search is open. These are leading-order statements about a prescribed background: the towerless top-only expansion already has the correction $-\frac{23}{128}m_t^4/\Lambda^2$. They hold at a matching scale, for flat profiles and no running. With bulk masses for the light fields the kinetic coefficient must be recomputed. Comparison with measured values needs a specified effective theory with thresholds and running.

VII. STATUS

As mathematics, the one-flavor formula and the gauge boundary conditions are known [1, 8], compression of operator systems is treated in [2], and spectral action on spaces with boundary in [11], for a boundary of the manifold, with a local chirality projector. In our searches we did not find the finite real triple as an exact window on an internal interval, with the wall algebra and the average formula; the searches are recorded in the Supplemental Material and do not establish absence. The parent algebra is matrix-valued, so the founding question for a commutative parent remains open, as does any selection of A_F. On $\mathbb{CP}^3 \times I$, nonlocal spectral walls give the exact window for the operator and the grading, with the higher K-modes decoupling with the radius, but they reduce scalar functions at the walls to constants, while direct aligned compression of real functions on S^4 gives only $\mathbb{R}\oplus\mathbb{R}$, against $\mathbb{H}\oplus\mathbb{H}$ when the matrix algebra is generated at the intermediate stage (Supplemental Material, part 24).

PROVENANCE

This note has no listed authors. The results were proved by Claude Opus 5.5, except that the statements on the commutative cylinder and the proof for the one-loop potential are by Astra 6 (gpt-6-astra, effort xhigh). All were refereed by Terra 6.1 (gpt-6.1-terra, effort max), whose reports led to corrections, with Python (numpy, scipy) checks of the identities and the numerical statements. Details are in the Supplemental Material of [9], parts 22 to 26, with the referee reports.

[1] C. Csáki, C. Grojean, J. Hubisz, Y. Shirman, and J. Terning, Fermions on an interval: quark and lepton masses without a Higgs, Phys. Rev. D **70**, 015012 (2004).

[2] A. Connes and W. D. van Suijlekom, Spectral truncations in noncommutative geometry and operator systems, Commun. Math. Phys. **383**, 2021 (2021), 2004.14115.

[3] A. H. Chamseddine and A. Connes, Why the Standard Model, J. Geom. Phys. **58**, 38 (2008), 0706.3688.

[4] A. H. Chamseddine, A. Connes, and W. D. van Suijlekom, Inner fluctuations in noncommutative geometry without the first order condition, J. Geom. Phys. **73**, 222 (2013), 1304.7583.

[5] K. Agashe, R. Contino, and A. Pomarol, The minimal composite Higgs model, Nucl. Phys. B **719**, 165 (2005), hep-ph/0412089.

[6] Y. Hosotani, Dynamical mass generation by compact extra dimensions, Phys. Lett. B **126**, 309 (1983).

[7] F. D'Andrea, F. Lizzi, and P. Martinetti, Spectral geometry with a cut-off: topological and metric aspects, J. Geom. Phys. **82**, 18 (2014), 1305.2605.

[8] C. Csáki, C. Grojean, H. Murayama, L. Pilo, and J. Terning, Gauge theories on an interval: unitarity without a Higgs boson, Phys. Rev. D **69**, 055006 (2004), hep-ph/0305237.

[9] Finite spectral triples as towerless intervals (2026), draft, 3 November 2026, this repository.

[10] G. Cacciapaglia, C. Csáki, and S. C. Park, Fully radiative electroweak symmetry breaking, J. High Energy Phys. **03**, 099, hep-ph/0510366.

[11] A. H. Chamseddine and A. Connes, Noncommutative geometric spaces with boundary: spectral action, J. Geom.

Phys. **61**, 317 (2011), 1008.3980.