Explicit masses and the chiral limit in odd reductions of almost-commutative geometries

(Dated: Draft, 3 November 2026)

A finite KO-6 triple with a grading-reversing involution factors as an odd triple times a point. We show that on the physical subspace the fermion action of the product with spacetime is exactly the action of the odd triple on an unconstrained spinor, with the Yukawa block becoming the constant matrix d in $D_M + \gamma_5 d$. A paired sector gives one Dirac action of explicit mass for either sign of the involution, up to a change of Grassmann variables, and on a subalgebra commuting with D_F and the involution no scalar fluctuation remains on the kept mass blocks. At zero quark mass the symmetry of the colour-only quark sector is $U(N_f) \times U(N_f)$. The admissible involutions are the $\Sigma \in U(N_f)$ with $\Sigma^* M$ Hermitian, which are the stationary points of the mass term, and at $M = 0$ they form a homogeneous space of the chiral group. All of this is kinematic. A spacetime-dependent reduction of the fixed operator leaves the spectral action unchanged, so that a pion kinetic term needs another operator or a quantum effective action, and the condensate needs a measure on gauge fields, infinite volume and an order of limits.

I. SETTING

The noncommutative Standard Model has a finite factor of KO-dimension 6 [1–3]. Reference [4] treats products of real triples. A companion paper [1] studies when a KO-6 triple factors as an odd triple times a Clifford point, with formal KO-labels $4 + 5 = 9$ and $4 + 7 = 11$. Here we ask how the Yukawa coupling reads in the odd triple, and what the data say about the chiral limit. The labels are formal: the metric dimension of the product is 4, and the label 9 exists only for sectors with an involution of sign $s = -1$. Charged paired sectors have one. The strict light spectrum with one or three unpaired neutrinos has none, since its H_0 has odd dimension.

Let M be a compact Riemannian spin 4-manifold with (D_M, J_M, γ_5), $J_M^2 = -1$, $J_M D_M = D_M J_M$, $J_M \gamma_5 = \gamma_5 J_M$, $\gamma_5 D_M = -D_M \gamma_5$. Let $F = (A_F, H_F, D_F, J_F, \gamma_F)$ be finite with KO-6 signs. On $L^2(S) \otimes H_F$ put $D = D_M \otimes 1 + \gamma_5 \otimes D_F$, $J = J_M \otimes J_F$, $\Gamma = \gamma_5 \otimes \gamma_F$. The fermion action is $\frac{1}{2}\langle J\xi, D\xi\rangle$ on Grassmann ξ in $\{\Gamma = +1\}$.

Let U be a unitary involution with $U\gamma_F = -\gamma_F U$, $[U, a] = [U, a^\circ] = [U, D_F] = 0$ and $UJ_F = sJ_F U$. On $H_0 = \{\gamma_F = +1\}$ put $d = UD_F$ and $j = UJ_F$, so that $j^2 = s$ and $jd = sdj$.

II. THE REDUCED ACTION

Theorem 1. *Define $R(\phi \otimes x) = \phi_+ \otimes x + \phi_- \otimes Ux$ on $L^2(S) \otimes H_0$. Then R is unitary onto $\{\Gamma = +1\}$, and*

$$\langle JR\psi, DR\psi'\rangle = \langle \mathcal{J}\psi, \mathcal{D}\psi'\rangle, \quad \mathcal{D} = D_M \otimes 1 + \gamma_5 \otimes d, \quad (1)$$

with $\mathcal{J} = J_M \otimes j$ for $s = +1$ and $\mathcal{J} = \gamma_5 J_M \otimes j$ for $s = -1$. So the action is $\frac{1}{2}\langle \mathcal{J}\psi, \mathcal{D}\psi\rangle$ on an unconstrained spinor ψ. Moreover $\mathcal{D}^2 = D_M^2 + d^2$, and $\operatorname{Tr} f(D^2/\Lambda^2) = 2 \operatorname{Tr} f(\mathcal{D}^2/\Lambda^2)$ when $f(D^2/\Lambda^2)$ is trace class.

Proof. Let $\tilde{U} = 1 \otimes U$ and $rx = (x + Ux)/\sqrt{2}$. Then $D_F r = rd$, $(1 + \tilde{U})R/\sqrt{2} = 1 \otimes r$, so $\tilde{U}DR = R\mathcal{D}$. Using $J_F x = Ujx$ and $UJ_F U = sJ_F$, one finds $\tilde{U}JR = R\mathcal{J}$. Finally $\langle J\xi, D\xi'\rangle = \langle \tilde{U}J\xi, \tilde{U}D\xi'\rangle$. □

So the doubling is removed by the chirality projection, and the physical space is exactly the Hilbert space of the odd product.

Corollary 1. *For a paired sector $H_0 = \operatorname{span}(e_L, \bar{e}_R)$ with $m > 0$, $(d_+, j_+) = (mI, \sigma_1 K)$ and $(d_-, j_-) = (\operatorname{diag}(m, -m), i\sigma_2 K)$, the action is*

$$S_+ = \langle J_M \psi_2, (D_M + \gamma_5 m)\psi_1\rangle \quad (2)$$

and $S_-(\psi_1, \psi_2) = S_+(\psi_1, \gamma_5 \psi_2)$. With $R_5 = e^{i\pi\gamma_5/4}$, $R_5(D_M + \gamma_5 m)R_5 = D_M + im$. This is one Dirac fermion of explicit mass m for either sign, up to the displayed change of Grassmann variables.

An unpaired reduction has a Majorana-type action: for $\gamma_F = \sigma_3$, $J_F = \sigma_1 K$, $D_F = m\sigma_1$ and $U = \sigma_1$, H_0 is one-dimensional and the action is $\frac{1}{2}\langle J_M\psi, (D_M + \gamma_5 m)\psi\rangle$. The neutral sector of the strict light spectrum has this form with $m = 0$. The rotation R_5 is formal, and carries a Jacobian in a quantized theory.

Proposition 1. *Let $B \subset A_F$ be a unital subalgebra with $[D_F, B] = 0$, and let U commute with B and B°. Inner fluctuations by $C^\infty(M) \otimes B$ contain only Clifford one-forms, and D_A reduces to $D_M(A) + \gamma_5 \otimes d$ with d constant.*

In the Standard Model representation with $B = \{(\lambda, Q_\lambda, m)\} \cong \mathbb{C} \oplus M_3(\mathbb{C})$, u_L, u_R carry the left character λ, d_L, d_R carry $\bar{\lambda}$, each pair with one opposite colour action, and e_L, e_R pair as well, so the kept mass blocks commute with B. This holds in the strict projection and when only a vectorlike top pair is removed. The reading "photon and gluons" needs unimodularity or an inherited restriction, since $U(B)$ contains $U(1) \times U(3)$. A canonically normalized radial field couples to a fermion of fixed mass with strength m_f/v.

[1] Finite spectral triples as towerless intervals (draft).

III. THE CHIRAL LIMIT

Consider N_f flavours, $q_L, q_R \in \mathbb{C}^{N_f} \otimes \mathbb{C}^3$, with $D_F = \left(\begin{smallmatrix} 0 & M \\ M^* & 0 \end{smallmatrix}\right) \otimes 1_3$ on (q_L, q_R) and the conjugate on antiparticles.

Proposition 2. *The unitaries on this sector commuting with both colour actions, γ_F, J_F and D_F are the pairs $(V_L, V_R) \in U(N_f)^2$ with $V_L M = M V_R$. So $G(0) = U(N_f) \times U(N_f)$, $G(mI) = U(N_f)_V$, and $G(M) = U(1)_V^{N_f}$ for positive distinct masses. With the electric charge, $G(0) = U(1)^4$ for two differently charged flavours.*

Theorem 2. *With the colour algebra only, the admissible involutions of sign s are $\left[\begin{smallmatrix} 0 & \Sigma \\ \Sigma^* & 0 \end{smallmatrix}\right]$ on (q_L, q_R), and s times the conjugate on antiparticles, with $\Sigma \in U(N_f)$ and $\Sigma^* M$ Hermitian. This holds iff Σ is a stationary point of $-\mathrm{Re}\,\mathrm{tr}(M\Sigma^*)$. The reduced mass on $(q_L, \bar{q}_R)$ is $\Sigma M^* \oplus s\,\overline{\Sigma^* M}$, tensored with 1_3. At $M = 0$ the set is all of $U(N_f)$, a homogeneous space of $G(0)$ under $\Sigma \mapsto V_L \Sigma V_R^*$, with stabilizer $U(N_f)_V$ at $\Sigma = 1$. At $M = mI$ it is $\{\Sigma = \Sigma^*, \Sigma^2 = 1\}$; for positive distinct masses it is the 2^{N_f} sign matrices in a mass eigenbasis; for invertible M, the polar part is the unique global minimum. With the electric charge, the set is the intersection with the commutant of the charge, with $\Sigma^* M$ still Hermitian: for u, d this leaves diagonal phases at $M = 0$, and signs for positive nonzero diagonal masses.*

Proof. Commuting with both colour actions and reversing the grading gives the form, and $U J_F = s J_F U$ fixes the rest. Commuting with D_F is $\Sigma M^* = M \Sigma^*$ and $\Sigma^* M = M^* \Sigma$, which are equivalent. Along Σe^{itX} the derivative of $\mathrm{Re}\,\mathrm{tr}(M\Sigma^*)$ is $\mathrm{Im}\,\mathrm{tr}(\Sigma^* M X)$. The minimum is von Neumann's trace inequality. □

The homogeneous space at $M = 0$ has the same stabilizer as in the symmetry breaking of QCD. The mass condition says that $\Sigma^* M$ is Hermitian, and it captures all stationary points, including saddles and maxima. Vacuum alignment [5] asks which of them minimizes a physical energy (Theorems 1 and 2 there give at least a local minimum, and a global one is left as a conjecture), and these data contain no such energy. So the theorem gives the shape of the set of reductions, and a map to a quantum order parameter is missing. The anomalous singlet direction also separates the classical $U(N_f)$ from the pion manifold. Choosing another reduction of a fixed even triple changes a sign in the reduced block, and differs from a physical mass-sign change, for which the determinants at $\pm m$ differ by $(-1)^{n_0}$.

Proposition 3. *Let $U(x)$ be pointwise admissible, $c^\mu = [D_M, x^\mu]$ and $B_\mu = (U \partial_\mu U)|_{H_0}$. Then $R^{-1} \tilde{U} D R = D_M + c^\mu P_- B_\mu + \gamma_5 d(x)$, which is generally not self-adjoint. If $\Sigma(x)$ only changes coordinates for D, the spectral action does not depend on it. A kinetic term needs another operator, chosen separately: for $D_M + \gamma_5 d_\Sigma(x)$ on flat M with $M = mI$, $\Sigma = \Sigma^*$, $\Sigma^2 = 1$, the heat expansion contains, tracing both reduced blocks,*

$$\frac{N_c f(0) m^2}{4\pi^2} \int \mathrm{tr}(\partial_\mu \Sigma^* \partial^\mu \Sigma), \tag{3}$$

which vanishes at $M = 0$. This does not construct the massless pion effective action.

This points to a nonzero m with a new operator, for instance the Higgs-type inner fluctuation $\Phi = m\Sigma(x)$ of an even triple whose algebra contains flavour algebras on L and R, with accompanying flavour gauge fields, or to a quantum effective action.

IV. THE CONDENSATE

Proposition 4. *Let $D_M(A)$ have eigenvalues λ_n, with n_0 zero modes, on volume V, and let $c_V(m; A) = V^{-1} \sum_n m/(\lambda_n^2 + m^2)$, for $m > 0$, regularized compatibly with the pairing $\pm\lambda$ by a subtraction whose finite remainder is $O(m)$, for example odd and analytic in m. At fixed A on compact M, $c_V = n_0/(Vm) + O(m)$ as $m \to 0$. If $\rho_m(\lambda) = \lim_V \langle V^{-1} \sum_n \delta(\lambda - \lambda_n) \rangle_m$ is equicontinuous at 0 with $\rho_m(0) \to \rho_0(0)$, with tails and subtraction controlled uniformly in m and the infinite-volume limit taken inside the integral, then $\lim_{m\to 0} \lim_V \langle c_V \rangle = \pi \rho_0(0)$, the form of the Banks-Casher relation as stated in [6], Section 2; the original gives the integral and the condition $\rho \to$ const at fixed A, without the factor π or an order of limits with the volume ([7], eqs. (49), (50)); the hypotheses here are ours, and the sign and flavour normalization of a condensate depend on convention.*

So the spectral triple supplies the operator $D_M + \gamma_5 m$, the density of each fixed operator, and the continuity of that operator and its square in m at fixed background; the regularized function itself has the zero-mode pole above. The averaged density, the regulated determinant or Pfaffian measure, the sequence of volumes and the order of limits need further input. A proof of existence of QCD with fermions would supply a measure, and would not by itself give $\rho_0(0) > 0$. With the anti-self-adjoint $\tilde{D} = -iD_M$, $\det(\tilde{D} + m) = m^{n_0} \prod_{\lambda > 0} (\lambda^2 + m^2) > 0$ with a compatible regulator, which is part of the method of [8].

V. THE ANOMALY ON THE REDUCED ACTION

On the commuting subalgebra, let $d = \mathrm{diag}(m_f)$ and let T be a Hermitian flavour matrix with eigenvalues τ_f, commuting with d and with the connection. Then $\mathcal{D}^2 = D_M^2 + d^2$.

Proposition 5. *Let $f(\mathcal{D}^2/\Lambda^2)$ be trace class, with $\gamma_5 = +1$ on S^+ and ind_f the index on the block of flavour f.*

At every Λ*,*

$$\operatorname{Tr}\gamma_5 T f(\mathcal{D}^2/\Lambda^2) = \sum_f \tau_f f(m_f^2/\Lambda^2)\operatorname{ind}_f, \qquad (4)$$

and for $m_f > 0$ *the mass insertion with the same regulator equals it. If* $\operatorname{tr} T = 0$*,* f *is continuous at* 0 *and* $f(0) = 1$*, then as* $\Lambda \to \infty$ *twice this trace tends to* $2N_c\mathrm{tr}(TQ^2)(e^2/8\pi^2)\int F \wedge F$*, with no gluonic or gravitational term. For* $T = T^3$ *on* (u, d) *the coefficient is* $2N_c\mathrm{tr}(T^3Q^2) = 1$*.*

The nonzero modes pair between chiralities, so only the zero modes survive, with their signed count, where the scalar density above has the unsigned count. The identity is a Ward identity under the common regularization; the weights at finite Λ depend on the regulator.

Proposition 6. *Let* $N_f = 2$*,* $M = 0$*, with colour background zero on the round* S^4*, and let* $\Sigma : S^4 \to SU(2)$ *generate* $\pi_4(SU(2)) = \mathbb{Z}_2$*. In independent quark and antiquark variables, the operator of the previous section has the pure-gauge connection* B *on one chirality. The Weyl Pfaffian of* N_c *doublets, transported along* $A_t = tB$*, has the relative sign* $(-1)^{N_c}$ *[9]. The spectral action does not depend on* Σ*.*

This is a relative sign for that transport and endpoint identification; the pullback of the measure through R_Σ keeps the value of the integral. It is the sign of the two-flavour discrete term $(-1)^{N_c[\sigma]}$ [10], and matching a pion field needs a sigma model and its anomaly-matching assumptions. By footnote 2 of [9], the sign is the parity of the zero modes of a five-dimensional Dirac operator on the nontrivial $SU(2)$ bundle on S^5.

VI. THE CHIRAL LIMIT ON AN INTERVAL

In the scalar-algebra realization on a chiral interval, $\mathcal{D} = D_M + \gamma_5(D_0 + m\Gamma_0)$. The limit $m \to 0$ has two routes. With a bulk mass $-\mu$, $\mu l > 1$ and $\tanh(\kappa_0 l) = \kappa_0/\mu$, the mass $m = \mu\,\mathrm{sech}(\kappa_0 l)$ is positive at finite parameters, with $m \sim 2\mu e^{-\mu l}$ for $\mu l \to \infty$, and at fixed μ it tends to 0 as $l \to \infty$: an exponentially small residue, with the localization and overlap mechanism of domain-wall fermions [11, 12]. With a boundary Yukawa matrix, $X = 0$ gives exact zero modes, and for flavour-degenerate bulk masses X preserves exactly $\{(V_Q, V_U) : V_Q X = X V_U\}$, so that X is a spurion on the wall and the symmetry is exact at finite l when $X = 0$. This boundary-coupled operator is a different construction from the tensor-product truncation above, and identifying them at finite l would need a proof.

PROVENANCE

This note has no listed authors. The results were proved by Claude Opus 5.5, with the proposition on spacetime-dependent Σ by Terra 6.1, and refereed by Terra 6.1, whose reports led to corrections of five items and several amendments. Python on finite models checked the reduced action for both signs, the involution conditions, the formula for the reduced mass, stationarity at sample points and the rotation R_5. The full proofs are in the project files `lemmas/17_explicit_masses_chiral_limit.md` and `lemmas/27_anomaly_reduced_action.md`.

[1] A. Connes, Noncommutative geometry and the standard model with neutrino mixing, J. High Energy Phys. **11**, 081, hep-th/0608226.

[2] J. W. Barrett, Lorentzian version of the noncommutative geometry of the standard model of particle physics, J. Math. Phys. **48**, 012303 (2007), hep-th/0608221.

[3] A. H. Chamseddine, A. Connes, and M. Marcolli, Gravity and the standard model with neutrino mixing, Adv. Theor. Math. Phys. **11**, 991 (2007), hep-th/0610241.

[4] L. Dąbrowski and G. Dossena, Product of real spectral triples, Int. J. Geom. Methods Mod. Phys. **8**, 1833 (2011), 1011.4456.

[5] R. F. Dashen, Some features of chiral symmetry breaking, Phys. Rev. D **3**, 1879 (1971).

[6] L. Giusti and M. Lüscher, Chiral symmetry breaking and the Banks-Casher relation in lattice QCD with Wilson quarks, J. High Energy Phys. **03**, 013, 0812.3638.

[7] T. Banks and A. Casher, Chiral symmetry breaking in confining theories, Nucl. Phys. B **169**, 103 (1980).

[8] C. Vafa and E. Witten, Eigenvalue inequalities for fermions in gauge theories, Commun. Math. Phys. **95**, 257 (1984).

[9] E. Witten, An SU(2) anomaly, Phys. Lett. B **117**, 324 (1982).

[10] Y. Lee, K. Ohmori, and Y. Tachikawa, Revisiting Wess-Zumino-Witten terms, SciPost Phys. **10**, 061 (2021), 2009.00033.

[11] D. B. Kaplan, A method for simulating chiral fermions on the lattice, Phys. Lett. B **288**, 342 (1992), hep-lat/9206013.

[12] Y. Shamir, Chiral fermions from lattice boundaries, Nucl. Phys. B **406**, 90 (1993), hep-lat/9303005.