Finite spectral windows with an opening tower in seven-dimensional parents

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The statement that a Kaluza-Klein dimension opens as the W and Z become massless needs a family, since along a general path to $M_W = M_Z = 0$ no tower opens. We give families with the hierarchy $m \ll E_\varepsilon \ll \Delta \to 0$ in which a moving low-energy window keeps a fixed finite number of states and the count at fixed positive energy diverges. On an interval this follows from exact formulas, and in a five-dimensional toy the vector poles follow the same hierarchy. On the disc bundle of $\Lambda^2_- T^*S^4$, a connected seven-dimensional commutative parent with boundary $\mathbb{CP}^3$, we treat a fermion window: for a chosen class of metrics, a local Dirac operator with a Callias potential has exactly eight light states, one copy of the pseudoreal **4** of $Sp(2)$ near each end, with exponentially small mixing, and real functions compress to the two-sheet finite maps generating $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$. Along a family with logarithmically growing potential and expanding metric, the eight states stay separated at the moving scale, and the count at fixed energy has seven-dimensional Weyl growth. The proofs rest on exact sums of squares given by $Sp(2)$-equivariance. On the disc bundle no calculation identifies the tunnelling scale with M_W or M_Z: the vector poles, the gauge couplings, the connector of the real bimodule, the chirality of the physical spectrum and any dynamics selecting the family remain open.

I. INTRODUCTION

In the towerless reading of almost-commutative geometry proposed in [1], a finite spectral triple is the low-energy window of a parent with an internal geometry, and the labels 9 and 11 are limits of an even interior at finite Fermi constant, where the physical point would lie. In that proposal, the limit towards 11 is the one in which the W and Z become massless and a dimension opens. This note asks what that statement can mean.

It needs a path. On the interval of [2], the vector masses obey $\sin^2(M_W l) = \frac{1}{2}\sin^2 2\theta$, so $\theta \to 0$ at fixed l makes them vanish with the tower spacing fixed. The criterion we use is a hierarchy along a family indexed by $\varepsilon \to 0$,

$$m_\varepsilon \ll E_\varepsilon \ll \Delta_\varepsilon \to 0, \tag{1}$$

with m_ε the light scale and Δ_ε the first discarded level. Then a window at the moving scale E_ε holds a fixed number of states. The hierarchy alone does not make the count at fixed $E > 0$ diverge, so we require that as a separate property, and prove it for each family below. For such a family the two observations have different limits, which reconciles a finite description near the weak scale with a tower at fixed energy.

The results of Sections II and III, and the formulation of (1), follow an audit of this project (see the Provenance). The new content is Section IV and what rests on it: a window obtained from a local operator on a connected seven-dimensional parent, rather than prescribed.

II. THE INTERVAL

Let $(H_F, D_*, J_F, \Gamma_F)$ have the signs of KO-dimension six, with $r = \dim H_F$ and an algebra A_F acting through π, as in [2]. For $0 < \alpha < 1$ and $l_* > 0$, let $D_F = \varepsilon D_*$ and $l_\varepsilon = l_* \varepsilon^{-\alpha}$, and let D_ε be the derivative on the interval with the boundary conditions that realize D_F exactly [2]. Its signed spectrum is $(n\pi \pm \theta_k)/l$, $n \in \mathbb{Z}$, with $\theta_k = l\sigma_k$ and σ_k the singular values of the block of D_F.

Theorem 1. *For small ε, an isometry W_ε gives $W_\varepsilon^* D_\varepsilon W_\varepsilon = \varepsilon D_*$, with J_F and Γ_F exact, and*

$$\begin{aligned} \Delta_\varepsilon &= \frac{\pi}{l_*}\varepsilon^\alpha - \varepsilon\|D_*\|, \\ \|Q_\varepsilon(a) - \pi(a)\| &\le \tfrac{2}{3} l_*^2 \|D_*\|^2 \varepsilon^{2(1-\alpha)} \|a\|. \end{aligned} \tag{2}$$

For $\alpha < \beta < 1$, $E_* > 0$ and $E_\varepsilon = E_* \varepsilon^\beta$, the number of levels with $|D_\varepsilon| \le E_\varepsilon$ is r. For fixed $E > 0$ it is asymptotic to $(2r l_* E/\pi)\, \varepsilon^{-\alpha}$.

This is a direct consequence of the exact formulas (Supplemental Material, part 30): $l_\varepsilon \|D_F\| \to 0$, and $\varepsilon \ll \varepsilon^\beta \ll \varepsilon^\alpha$. The finite data remain inputs, the wall data vary by hand, and a fixed Majorana block does not scale as εD_*, so an application to an electroweak Dirac block needs a reducing sector.

In the five-dimensional Yang-Mills toy of [2], with universal mass $m_\varepsilon = \varepsilon m_*$ and $\theta_\varepsilon = l_\varepsilon m_\varepsilon \to 0$, the light poles are $M_W = l^{-1}\arcsin(\sin 2\theta/\sqrt{2})$ and $M_Z = l^{-1}\arcsin(2\sin 2\theta/\sqrt{5})$, so

$$M_W^2 \simeq 2m_\varepsilon^2, \qquad M_Z^2 \simeq \tfrac{16}{5} m_\varepsilon^2, \tag{3}$$

and the vector poles follow (1). The small-mass coefficients of the finite spectral action, in the family with both members of each doublet at one Dirac mass and no Majorana mass, are 3 and 24/5. The leading Taylor quotient of [2], $M^2 = \frac{1}{2}(3 - d_{\text{eff}}(s))\, \|[D_F, T]\|^2/\|T\|^2$ with $s = l\Lambda$, connects the two pairs at the level of this coefficient: $d_{\text{eff}} = 0$ gives the finite values, and $d_{\text{eff}} = 1$ the interval values. It does not identify the spectral-action pole problem with the Yang-Mills toy. Along $\Lambda = E_\varepsilon$ we have $s \to 0$, and at fixed Λ we have $s \to \infty$; a joint use needs a remainder uniform in s. Varying the cutoff of the action is an additional prescription.

A family closer to the geometry below uses localization instead of boundary angles. With a bulk mass $\mu_\varepsilon = \mu_* \varepsilon^\alpha$ and a new length $l_\varepsilon^{\text{loc}} = c_\varepsilon/\mu_\varepsilon$, where $c_\varepsilon = (1 -$

$\alpha)\log(1/\varepsilon)+c_0$, the light pair has $m=\mu\operatorname{sech}\kappa l$ with $\kappa l=c\tanh\kappa l$, so that

$$m_\varepsilon\sim 2\mu_* e^{-c_0}\varepsilon,\qquad \Delta_\varepsilon\sim\mu_*\varepsilon^\alpha,\qquad \frac{1/\kappa}{l_\varepsilon^{\mathrm{loc}}}\to 0. \tag{4}$$

The widths and the length both grow, with a shrinking ratio. Writing the profile as $e^{-\kappa x}-re^{\kappa x}$, the opposite-wall coefficient is exactly $r=(m/(\mu+\kappa))^2$, quadratic in the light mass.

III. A CYLINDER WITH A PRESCRIBED WINDOW

Let A be the Dolbeault-Dirac operator of $\mathbb{CP}^3$ twisted by $\mathcal{O}(1)$, with kernel $\mathbb{C}^4$ and first nonzero level Δ_K, and let $\mathcal{D}_r=E\partial_x+ZA/r$ on $I_l\times\mathbb{CP}^3$, with the spectral walls of Supplemental Material, part 24, which prescribe finite data on this $\mathbb{C}^4$, with gradings of type $(2,2)$, so here $r=4$. Taking $r_\varepsilon=r_*\varepsilon^{-\gamma}$ and the scaling of Theorem 1, the next level is at least $\min\{(\Delta_K/r_*)\varepsilon^\gamma,\,(\pi/l_*)\varepsilon^\alpha-\varepsilon\|D_*\|\}$, so a cutoff $E_*\varepsilon^\beta$ with $\max(\alpha,\gamma)<\beta<1$ isolates the window, and the count at fixed energy grows as $l_\varepsilon r_\varepsilon^6E^7$, by Dirichlet and Neumann bracketing. For $\gamma=\alpha$ all seven directions expand homothetically. So (1) is compatible with seven internal dimensions, for $4+7=11$. The window here is imposed by nonlocal walls.

IV. A WINDOW FROM A LOCAL OPERATOR

Let X be the disc bundle of $\Lambda^2_-T^*S^4$, with $G=Sp(2)$, $K=Sp(1)_a\times Sp(1)_b$, and $H=Sp(1)_a\times U(1)_b$. It has zero section S^4 and boundary $\mathbb{CP}^3$, and a smooth function has independent values at the two ends. Let $S=G\times_K(1,\mathbf{2})$, let $E=S^3(S)$ with its pull-back to X, and let J be the spin-$\frac{3}{2}$ action on it. On X take

$$T_M=D_X^E+i\Phi_M,\qquad \Phi_M=\epsilon Mf(r)\,n\cdot J, \tag{5}$$

with $\widehat{D}_M$ the doubled operator, $Z=-ic(dr)$, and the local wall condition $Zu=-u$ for T_M, $Zv=+v$ for T_M^*, which gives a self-adjoint elliptic operator (Supplemental Material, part 28, Theorem 28.9). The sign ϵ is the one that puts the kernel of the end model of T_M at the zero section. We assume the following: the metric is $dr^2+\pi^*g_{S^4}+\rho(r)^2g_{S^2}$, with S^4 of radius 1 and ρ,f smooth and positive for $r>0$, with $\rho=f=r$ on $r\le 2e$, and with $\rho=1$, $f=f_0>0$ on a product collar $l-2a\le r\le l$, which is the Fubini-Study metric on $\mathbb{CP}^3$. The connection on E is $Sp(2)$-invariant, in radial parallel gauge. It is either (a) the pull-back throughout, or (b) the pull-back near S^4 and, on the collar, the connection preserving the four line summands $\mathcal{O}(3),\mathcal{O}(1),\mathcal{O}(-1),\mathcal{O}(-3)$.

Theorem 2. *For each fixed $0<\delta<\sqrt{3}$ in case (b), and each fixed $0<\delta<\min(2,\kappa_a)$ in case (a), with $\kappa_a\ge\sqrt{5}-\frac{3}{2}$, there is M_δ such that for $M\ge M_\delta$, $\widehat{D}_M$ has exactly eight eigenvalues in $(-\delta,\delta)$. The spectral subspace is one copy of the* **4** *of $Sp(2)$ in each $\hat{\Gamma}$ degree, within $O(M^{-1/2})$ of the cut-off core model in the domain of T_M, and of the cut-off wall model in the domain of T_M^*.*

The proof (Supplemental Material, part 29) is a partition into three regions, and each estimate holds on a fixed manifold, so uniformity over the representations of $Sp(2)$ needs no separate argument.

The zero section. Sections on a tube are K-equivariant maps $G\to\Sigma_4\otimes L^2(\mathbb{R}^3;\mathbb{C}^2\otimes\mathbb{C}^4)$. Put $D_H=i\sum\gamma_ae_a$ and $\hat{Q}=\gamma_5\sigma\cdot\partial_y+\epsilon My\cdot J$. Then $T_M=T_0+R$, with $T_0=D_H+i\hat{Q}$ and $|R|\le C|y|$ the O'Neill term. The operator $\hat{Q}$ acts on the values and is K-equivariant, so it commutes with the derivatives e_a. With the Clifford signs, this gives exactly

$$T_0^*T_0=D_H^2+\hat{Q}^*\hat{Q}. \tag{6}$$

On one chirality of Σ_4 the fibre operator has kernel $(1,\mathbf{3})\oplus(1,\mathbf{1})$, and its square is at least of order M on the complement and on the other chirality. The projection onto this K-type, on both chiralities, commutes with D_H, and on its range D_H is a twisted Dirac operator on S^4, with $D_H^2=\mathrm{Cas}_G(V)+\frac{3}{2}-\mathrm{Cas}_K$ [3]. This vanishes only on the **4** and is at least 4 otherwise. Since (6) also bounds $M^2\||y|u\|^2$, the term R is a small perturbation. The stronger localization theorem of [4] has a hypothesis that fails for this mass, since $(n\cdot J)^2$ depends on n, and it is not needed here; nor is the block analysis of [5].

The bulk. For w supported away from both ends, $\|T_Mw\|^2=\|Dw\|^2+\|\Phi_Mw\|^2+\langle w,i[D,\Phi_M]w\rangle\ge\int(M^2f^2/4-CM)|w|^2$, for every representation.

The wall. On the collar in case (b), with $q=\epsilon n\cdot J$ and $m=Mf_0$,

$$\begin{aligned}\|T_Mw\|^2=\|(\partial_x+mq)w_+\|^2+\|Aw_+\|^2\\ +\|(\partial_x-mq)w_-\|^2+\|Aw_-\|^2,\end{aligned} \tag{7}$$

where A is the sum of line-twisted Dirac operators. Only the channels $Z=-1,\ q>0$ for T_M and $Z=+1,\ q>0$ for T_M^* have a stable profile, and on these the lower bound comes from A. On the Fubini-Study metric $D_k^2=C_2+\frac{3}{2}-\frac{3}{8}k^2$ on the $SU(4)$-type with Casimir C_2; the normalization passes the Kirchberg limit $R/3$ for $k=0$ [6]. So on the stable lines $\mathcal{O}(3)$ and $\mathcal{O}(1)$ the kernel is exactly one copy of the **4**, in $Z=+1$, and all other values have $D^2\ge 3$, for all representations. In case (a) the line summands mix through a bounded term with overlap $\sqrt{3}/2$ between the profiles, and a Schur complement of order $1/m$ reduces the wall to an elliptic operator on $\mathbb{CP}^3$, with fibre part $-w(\rho)$, $w(\rho)=2(\rho^2+2)^2/(9+2(\rho^2+2)^2)$, for its kernel in the limit, so $-2/3$ at the Fubini-Study radius. At other radii the theorem holds under one kernel condition.

The transition regions lie in the bulk, where the mass controls $\|u\|$ with a factor M^{-2}, so the partition costs nothing at the scale of the gap. The end models fix the expected low values, which were written down before a direct check in Python (Table I).

TABLE I. Lowest singular values of T_M on the $Sp(2)$-representation V, case (b), from a radial solve in Python at $M = 32$, $l = 3$, against the values of the core (c) and wall (w) end models. The smallest computed value on the **4** is dominated by the mesh floor at this resolution and is omitted; the observed mesh discrepancy of the wall values is about 0.005.

V	$M = 32$	limit
4	1.7323, 1.9991, 2.4497	$\sqrt{3}$ w, 2 c, $\sqrt{6}$ w
16	2.2351, 2.2409, 2.4497	$\sqrt{5}$ c, $\sqrt{5}$ w, $\sqrt{6}$ w
20	2.2409, 2.8275, 2.8286	$\sqrt{5}$ w, $\sqrt{8}$ c, $\sqrt{8}$ w
40	3.4626, 3.4642, 3.4672	$\sqrt{12}$ (three)

Proposition 3. *Let $T_M u = \lambda v$, $T_M^* v = \lambda u$ be a unit singular pair of the window. For radial χ equal to 1 near S^4 and 0 near the wall,*

$$\lambda\big(\langle u, \chi u\rangle - \langle v, \chi v\rangle\big) = \langle v, c(d\chi)u\rangle, \tag{8}$$

and for every $\eta \in (0,1)$,

$$|\lambda_M| \le C_\eta \exp\Big(-(1-\eta)\frac{M}{2}\int_e^{l-a} f\, dr\Big). \tag{9}$$

The identity is an integration by parts, and Agmon estimates from each end give the bound, with rate set by the smallest eigenvalue $\frac{1}{2}$ of $|n\cdot J|$. The summands of E on $\mathbb{CP}^3$ have indices $4, 0, 0, -4$, of total zero, as equivariant cobordism invariance requires, since their sum extends over X, although the individual lines do not ([7], preprint pp. 43-44). The window has index $\mathbf{4} - \mathbf{4} = 0$, so index theory allows $\lambda \neq 0$, and the eight states come from the estimates. The **4** is pseudoreal, and the auxiliary grading $\hat{\Gamma}$ with zero virtual index says nothing yet about the chiral spectrum of the Standard Model.

Theorem 4. *For $a \in C^1(X)$, with the norm in fixed shape coordinates and after equivariant identifications of the window with the model copies of the **4**, the compression satisfies*

$$\|Q_M(a) - (Q_B(a|_{S^4}) \oplus Q_Y(a|_{\mathbb{CP}^3}))\| \le C\|a\|_{C^1} M^{-1/2}, \tag{10}$$

with $Q_B(a) = \int a(x)(1 - x\cdot\Gamma)\, d\mu$, and with Q_Y the level-one Toeplitz map of the twistor space in case (b), and in case (a) its form $B = 2((1+w)p_z + (1-w)p_{jz})$, with $w = w(\rho)$. Real functions on X generate $M_2(\mathbb{H}) \oplus M_4(\mathbb{C})$ under the limit map.

This is a statement about compression maps. Generated algebras need not converge when compression maps do, in line with the setting of [8].

V. THE JOINT LIMIT ON THE DISC BUNDLE

Under $g \to R^2 g$, with the potential divided by R, the operator is unitarily equivalent to $\widehat{D}_M/R$. So the light mass and the gap both scale as $1/R$, with the compression maps unchanged. Let $G_0 = \int_e^{l-a} f\, dr$, $0 < \alpha < \beta < 1$, $E_* > 0$ and $\eta \in (0,1)$.

Theorem 5. *Take $M_\varepsilon = \frac{2(1-\alpha)}{(1-\eta)G_0}\log\frac{1}{\varepsilon}$, $R_\varepsilon = \varepsilon^{-\alpha}$, and $D_\varepsilon = \widehat{D}_{M_\varepsilon}/R_\varepsilon$, in the setting of Theorem 2. For small ε:*

1. *D_ε has exactly eight eigenvalues in $(-\delta\varepsilon^\alpha, \delta\varepsilon^\alpha)$, equal to $\pm m_\varepsilon$ with multiplicity four, or zero with multiplicity eight, and $m_\varepsilon \le C\varepsilon$; the next level is at most $C\log(1/\varepsilon)\,\varepsilon^\alpha$, so it tends to zero;*
2. *the number of levels with $|D_\varepsilon| \le E_*\varepsilon^\beta$ is 8;*
3. *the compression converges to the endpoint maps in shape coordinates, with error $O((\log 1/\varepsilon)^{-1/2})$;*
4. *at fixed $E > 0$, the number of levels with $|D_\varepsilon| \le E$ is asymptotic to $C_X E^7 \varepsilon^{-7\alpha}$.*

Item 4 follows from the Weyl law for $\widehat{D}_0$, since $\widehat{D}_M - \widehat{D}_0$ is bounded by cM_ε, which is small against ER_ε. So (1) holds for a fermion window that comes from the operator, with growth of a seven-dimensional internal space at fixed energy. The bound on m_ε is one-sided, since a lower bound on λ is open. The fixed shape expands, so the pointed limit at fixed energy is flat at an interior point and a half-space at a boundary point. Nothing here identifies it with the complete Bryant-Salamon manifold [9, 10], whose metric lies outside our class, or with a solution of field equations.

VI. THE QUATERNIONIC DEFECT

Proposition 28.16 of Supplemental Material, part 28 gives the test for the core block: with $B(r) = F^*F = \mathrm{diag}(b_a I_2, b_+, b_-)$, all real multipliers compress to operators commuting with the quaternionic structure j if and only if $b_+ = b_-$. Put $\eta_t = \int |b_+ - b_-|\, dr / \int \mathrm{tr}\, B\, dr$, and let $P_H(A) = \frac{1}{2}(A + jAj^{-1})$.

Proposition 6. *For real f with $\|f\|_\infty \le 1$, $\|Q(f) - P_H Q(f)\| \le 2\eta_t$, and products of k such compressions have defect at most $2k\eta_t$. Moreover, $\eta_t = O(M^{-1/2})$. If the quaternionic structure j_E of E is parallel, as for the pull-back connection and for the real affine interpolation between it and the line-preserving connection, then for every $\eta \in (0,1)$,*

$$\eta_t \le C_\eta \exp\Big(-(1-\eta)M\int_{2e}^{l-a} f\, dr\Big). \tag{11}$$

With j_E parallel, the charge conjugation C commutes with T_M and exchanges the two wall conditions, so only the wall breaks the symmetry. On copies of the **4**, $F \mapsto CFj^{-1}$ is a real structure, which fixes the model kernel. Then $\zeta = u - Cu$ obeys $T_M\zeta = \lambda(v - Cv)$ in the interior, with vanishing projection of its cut-off to the

model kernel, and it is exponentially small at the core, sourced at the wall. A weighted Cauchy-Schwarz inequality doubles the exponent of Proposition 3, with the integral starting at $2e$. So under this hypothesis, bounded expressions in the core block lie exponentially close to $M_2(\mathbb{H})$. An unrestricted generated algebra may use coefficients of size $1/\eta_t$, and it is $M_4(\mathbb{C})$ at finite mass when $b_+ \neq b_-$ and the **5** component is nonzero. The wall block generates $M_4(\mathbb{C})$, which requires the loss of C there.

VII. RELATION TO PRIOR WORK

The mechanisms have precedents, which we distinguish from the combination studied here. Chiral localization on defects and boundaries goes back to [11, 12], and gauge symmetry breaking by interval boundary conditions to [13, 14]; a tree-level $m_t = 2M_W$ for a suitable representation is in [15]. The square identity with a zero-order cross term is in [16], in a concentrating-pair setting, and a horizontal and vertical factorization of the Dirac operator with a curvature contraction is in [17], under other hypotheses. Index statements of Callias type in families are in [18, 19], and the finite boundary window is a separate question. Kaluza-Klein modes that become massless during gauge restoration are followed in [20], with a different mechanism and geometry. Targeted searches found these and did not establish whether the eight-state theorem has appeared elsewhere; they were not exhaustive.

VIII. WHAT IS MISSING

The vector sector. On the family of Theorem 5 one needs the kinetic matrix and the mass Hessian of the vector fluctuations, then the momentum-dependent kernel, with the aim $M_W, M_Z \sim c\lambda/R$. The tunnelling eigenvalue alone does not establish this, so the words "massless electroweak limit" apply to the disc bundle only as a target.

The couplings. For a bulk Yang-Mills field with a constant profile, $g_4^{-2} = \mathrm{Vol}(X)/g_{11}^2$, so with fixed microscopic coupling a plain dilation gives $g_4 \propto R^{-7/2}$, which must be separated from restoration at finite coupling. Gauge fields from isometries have Killing overlap integrals instead, and non-Killing fields behave differently again [21].

The dynamics. A potential $a\,r^{-p} + b\,\varepsilon^2 r^q$ has its minimum at $r \propto \varepsilon^{-2/(p+q)}$, so it would select $\alpha = 2/(p+q)$ if $p + q > 2$. Nothing has derived such a potential, and adding it by hand derives nothing.

The finite structure. The window gives the graded eight-dimensional space and the limit algebra. The connector of the bimodule of complex dimension 32 with a real structure, the quaternionic grading and the chirality of the physical spectrum remain inputs. The product of two copies of the parent has heat exponent t^{-7}, so the bimodule should come from a finite-rank correspondence, to keep seven dimensions.

IX. STATUS

Theorem 1, the vector poles and the cylinder are direct consequences of refereed lemmas, with proofs in Supplemental Material, part 30. Theorems 2, 4 and 5 and Propositions 3, 6 are new here, for the stated class of metrics, with proofs in Supplemental Material, part 29 and Supplemental Material, part 30. They were read by an independent referee, GPT-6.1 Terra, whose reports found errors in case (a) and a missing hypothesis in the exponential defect bound. These are corrected, and the corrections have had no further independent reading. The referee ran no computation. Other metrics, and case (a) at other radii, stay open. Table I is a Python computation; by the rule of the project the script is not kept, and the values and the setting are recorded in Supplemental Material, part 29.

PROVENANCE

The formulation of the joint limit and the interval and cylinder statements come from an audit of the repository by GPT-6 Astra, kept in its `reviews` folder. The proofs of Sections IV and V and of the defect bound are by Claude Opus 5.5, using results of GPT-6 Astra on the operator and its two ends, with referee reports by GPT-6.1 Terra on the earlier lemmas. Claude Sonnet 5.5 rechecked the audit in Python. The two referee reports on lemmas 29 and 30 and this note are in the repository, and referee status is recorded in its ledger.

[1] Finite spectral triples as towerless intervals (2026), draft, 3 November 2026, this repository.

[2] The finite Dirac operator as a boundary condition on an interval (2026), draft, 3 November 2026, this repository.

[3] R. Parthasarathy, Dirac operator and the discrete series, Ann. of Math. **96**, 1 (1972).

[4] M. Maridakis, A localization theorem for Dirac operators, (2021), arXiv:2110.11654.

[5] J.-M. Bismut and G. Lebeau, Complex immersions and Quillen metrics, Publ. Math. IHES **74**, 1 (1991).

[6] K.-D. Kirchberg, An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature, Ann. Global Anal. Geom. **4**, 291 (1986).

[7] E. Witten, Fermion quantum numbers in Kaluza-Klein theory, in *Shelter Island II: Proceedings of the 1983 Shelter Island Conference on Quantum Field Theory and the Fundamental Problems of Physics* (MIT Press, 1985) p.

227, preprint, Princeton, October 1983.
[8] A. Connes and W. D. van Suijlekom, Spectral truncations in noncommutative geometry and operator systems, Commun. Math. Phys. **383**, 2021 (2021), 2004.14115.
[9] R. L. Bryant and S. M. Salamon, On the construction of some complete metrics with exceptional holonomy, Duke Math. J. **58**, 829 (1989).
[10] S. Karigiannis and J. D. Lotay, Bryant-Salamon G_2 manifolds and coassociative fibrations, (2020), arXiv:2002.06444.
[11] D. B. Kaplan, A method for simulating chiral fermions on the lattice, Phys. Lett. B **288**, 342 (1992), hep-lat/9206013.
[12] Y. Shamir, Chiral fermions from lattice boundaries, Nucl. Phys. B **406**, 90 (1993), hep-lat/9303005.
[13] C. Csáki, C. Grojean, J. Hubisz, Y. Shirman, and J. Terning, Fermions on an interval: quark and lepton masses without a Higgs, Phys. Rev. D **70**, 015012 (2004).
[14] C. Csáki, C. Grojean, H. Murayama, L. Pilo, and J. Terning, Gauge theories on an interval: unitarity without a Higgs boson, Phys. Rev. D **69**, 055006 (2004), hep-ph/0305237.
[15] G. Cacciapaglia, C. Csáki, and S. C. Park, Fully radiative electroweak symmetry breaking, J. High Energy Phys. **03**, 099, hep-ph/0510366.
[16] M. Maridakis, Spinor pairs and the concentration principle for Dirac operators, (2015), arXiv:1510.07004.
[17] J. Kaad and W. D. van Suijlekom, Factorization of Dirac operators on almost-regular fibrations of spin^c manifolds, (2017), arXiv:1710.03182.
[18] C. Callias, Axial anomalies and index theorems on open spaces, Commun. Math. Phys. **62**, 213 (1978).
[19] C. Kottke, An index theorem of Callias type for pseudodifferential operators, (2009), arXiv:0909.5661.
[20] S. Iso, N. Kitazawa, and T. Suyama, Gauge symmetry restoration by Higgs condensation in flux compactifications on coset spaces, (2021), arXiv:2111.06508.
[21] J. Baptista, Chiral interactions of fermions and massive gauge fields in Kaluza-Klein models, J. High Energy Phys. **05**, 008, 2506.09126.