Supplemental Material: Finite spectral triples as towerless intervals

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Part S23 (The heavy-W prescription with every fermion mass fixed is incompatible with the CCM relation, and a heavy-W limit forces m_max -> infinity) 366

Part S24 (If the up-type Dirac mass of one generation is the only diverging Dirac mass, the CCM relation gives m_t/M_W -> sqrt(8/3)) 367

Part S25 (With a boundary Yukawa matrix X between a doublet-type and a singlet-type field on the interval, D_X is self-adjoint, its kernel is ker X^dagger + ker X, and ind D_X = N_Q - N_U for every X; equivariantly the index is [rho_Q] - [rho_U]) 368

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Part S27 (The light eigenvalues of the boundary-Yukawa interval are the fixed points lambda = sigma_k(A^{1/2} X B^{1/2}), with two-sided bounds, and in the towerless limit the retained block is the Yukawa block Phi_Q Y_hat Phi_U) 370

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Part S29 (All zero modes of the boundary-Yukawa interval lift only if the residual theory is vectorlike, so without nu_R a massless left neutrino is forced for every X preserving electromagnetism) 371

Part S30 (The boundary-Yukawa interval has the exact secular equation det(lambda^2 - A(lambda) X B(lambda) X^dagger) = 0 below the first poles, and for one massless flavour its spectrum (n pi +- arctan|X|)/l interpolates from types (a) + (b) to two copies of type (c)) 372

Part S31 (With the three-generation CCM trace and one massive colour triplet of mass m = pi/(2l), the spectral-action W mass is M_W = sqrt(3/8) m = sqrt(3/8) M_KK, with no interval-overlap correction) 373

Part S32 (The minimal two-sheet spectral action gives a massless diagonal vector and a relative vector of mass sqrt(6) m, against the Neumann-Dirichlet tower mass m = pi/(2l)) **374**

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Part S36 (The gauge-covariant lowest-mode reduction of the massless mixed interval still gives M_spec^2 = (3/5) M_KK^2) **377**

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Part S42 (Matching one vector mass fixes neither the relative gauge couplings nor the scalar kinetic coefficients transverse to the gauge orbit) **382**

Part S43 (Limits in the propinquity of commutative quantum compact metric spaces are commutative, and the number of connected components does not increase in the limit) **383**

Part S44 (A sequence of metric spectral triples converges to (C, C^N, D_F) in the spectral propinquity iff the diameter tends to 0, the gap tends to infinity and the light-block error tends to 0, so the finite limits of connected commutative triples are exactly the triples (C, C^N, D_F)) **384**

Part S45 (A metric triple on M_2(C) whose Dirac operator is the traceless projection commutes with every inner conjugation, so Proposition 15.10 needs order one) **385**

Part S46 (A metric spectral triple with a finite spectral block of gap g lies within 1/g^2 + max(delta/2, Delta) + Delta + sqrt(Delta) of (C, C^N, D_F) in the spectral propinquity, with no anticommutation hypothesis) **385**

Part S47 (For a symmetric spectral block the grading and the real structure of the parent descend to the block with their signs, and the tunnel of the gap criterion controls the

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Part S48 (In a finite real triple with order zero and order one, a unitary u of the algebra whose inner automorphism u J u J^{-1} commutes with D commutes with D itself, so a metric such triple has no nontrivial unbroken inner gauge transformation) 387

Part S49 (On the doubled parent H_0 (x) conj(H_0) with the graded flip J the compressed data satisfy order zero and order one for every generated algebra, noncommutative or not, and these two conditions are formal consequences of the tensor square) 388

Part S50 (Compressing functions on a connected commutative space to a fixed spectral window gives a noncommutative finite algebra with no limit taken: M_2(C) + M_2(C) at the second level of the chiral interval and M_4(C) at level one on CP^3) 389

Part S51 (The product of the truncated coarse triple with the Clifford point (C, C^2, 0, sigma_1 K, sigma_3) has the KO-6 signs, order zero, order one and dim H = 2 k^2, and fails finite orientability) 390

Part S52 (A grading inherited by compression from an even parent commutes with all compressed functions, so the grading Gamma_5 of the twistor-level-one data cannot come from the stated even parent) 390

Part S53 (With an inherited pointwise real structure J_0 a J_0^{-1} = a^*, compressed data satisfy order zero if and only if the generated algebra is commutative) 391

Part S54 (On the doubled parent of twisted CP^3 scaled by 1/l, compression and generation give the constant finite data A_F = M_4(C), dim H_F = 16, D_F = 0, while the whole triple converges to (C, C^16, 0)) 392

Part S55 (At twistor level one on two sheets, real functions on S^4 and on CP^3 generate M_2(H) + M_4(C) on E + E^* of dimension 32 with J(x, y) = (y^*, x^*), the algebra, bimodule and real structure of Chamseddine-Connes, as abstract finite data) 392

Part S56 (A two-sheet coarse parent gives the KO-6 signs with order zero and order one, and its truncation for equal dimensions and full algebras is the Chamseddine-Connes pair M_k(C) + M_k(C) with J(x, y) = (y^*, x^*)) 393

Part S57 (The unitaries of the quark sector commuting with the colour actions, the grading, J_F and D_F form G(M) = {(V_L, V_R) : V_L M = M V_R}, with G(0) = U(N_f) x U(N_f)) 394

Part S58 (At finite volume and fixed background the regulated condensate tends to zero as m -> 0 apart from the zero-mode term, and the Banks-Casher form pi rho_0(0) needs a gauge measure, infinite volume, an order of limits and uniform control) 395

Part S59 (On a subalgebra commuting with D_F and with the involution, inner fluctuations have no scalar part, so the Yukawa block of the reduced triple is an explicit constant mass) 396

Part S60 (A spacetime-dependent reduction of the fixed operator gives no kinetic term for Sigma: the spectral action of D is unchanged, and a kinetic term needs a new operator with m != 0, whose coefficient is N_c f(0) m^2/(4 pi^2)) 397

Part S61 (The fermion action of M x F on its physical subspace equals (1/2) <jj psi, (D_M + gamma_5 d) psi> on an unconstrained spinor, and a paired Dirac sector gives one Dirac fermion of explicit mass for either sign s) 398

Part S62 (The admissible involutions of each sign on the colour-only quark sector are the Sigma in U(N_f) with Sigma^* M Hermitian, exactly the stationary points of Re tr(M Sigma^*), and at M = 0 they form the homogeneous space (U(N_f) x U(N_f))/U(N_f)_V) 399

Part S63 (As m_t -> infinity at fixed cutoff the broken vectors keep a finite nonzero kinetic form and a vanishing mass, so heavy fermions do not select the centralizer by mass decoupling) 400

Part S64 (For V_0 = m sigma_3 and T = sigma_1 the Taylor kinetic coefficient of the exact quadratic spectral action is negative for m^2 > Lambda^2, while the full form has a unique positive pole with M^2 ~ 4 m^2) 400

Part S65 (The quadratic spectral action of a transverse vector about a constant finite background has an exact kernel with no expansion in m/Lambda, gauge invariant through the Goldstone orbit, whose small-mass limit gives the mass formula (3/2) Q^{-1/2} K Q^{-1/2} and the CCM relation) 401

Part S66 (With one common Dirac mass m for every species and no Majorana masses, the W and Z pole masses diverge at fixed cutoff, 2x < z_W < 2x + 2 and 16x/5 < z_Z < 16x/5 + 2, with M_W^2/M_Z^2 -> 5/8 at both ends and no light fermions) 402

Part S67 (The top-only Taylor result extends to nonnegative compactly supported cutoffs, and with several heavy doublets the Taylor W mass tends to zero whenever a complete massless doublet is retained) 403

Part S68 (In the top-only heat-cutoff family, m_t, M_W and M_Z diverge together only jointly with Lambda -> infinity, and the limit m_t/M_W -> sqrt(8/3) needs m_t/Lambda -> 0) 404

Part S69 (In the top-only limit the vanishing vector masses do not restore SU(2) x U(1) on the light space: the compressed generators on b_L fail the SU(2) relations, and the limiting coefficients have K_W != K_33 and K_3Y != 0) 404

Part S70 (For cutoffs in the positive Laplace class and without an added potential or constraint, no finite c > 0 is stationary along a radial ray V = c V_* of the spectral action) 405

Part S71 (On flat R^4 the radial derivative of the spectral action is U'(c) = -(c Lambda^2/(2 pi^2)) sum_j lambda_j^2 int_{c^2 lambda_j^2/Lambda^2}^infinity f(z) dz, so non-negative cutoffs of unbounded support have no finite stationary point and compactly supported ones have a plateau) 406

Part S72 (In the three-generation Standard Model with only the top massive, the W mass at fixed cutoff tends to zero as M_W^2 ~ Lambda^4/m_t^2 for both the Taylor and the pole definitions, with M_W^2 <= Lambda^2/e) 407

Part S73 (In the top-only heat-cutoff family the photon and gluon coefficients keep the ratio 8/3 and the neutral-current angle keeps cos^2 theta_NC = 5/8 for every m/Lambda, while M_Z tends to zero exponentially and the matching angle tends to 103/157) 408

Part S74 (For the heat cutoff and doublets with one massive member, 8 kappa M_{W,Taylor}^2 = sum_f N_f m_f^2 rho(m_f^2/Lambda^2) with rho(x) = 2[1 - (1 + x) e^{-x}]/x^2, so the CCM relation is the small-mass term of a weighted sum rule) 409

Part S75 (Along a gauge orbit the modulus of the link stays fixed for the covariant projection of the parent and for the finite data with the quadratic term, and grows at second order for the fixed compression and for the finite data with linear terms only) 410

Part S76 (The inner fluctuation without order one of the truncated two-point data factorizes: m |Phi|^2 sigma_1, gauge-neutral, with the pointwise real structure, and a link m Phi^2 of charge 2, whose quadratic term keeps the modulus fixed along a gauge orbit, with the reflection real structure) 411

Part S77 (A block T: E -> E^* added to a sheet-diagonal finite Dirac operator keeps the KO-6 signs iff T = tau(T) and hT + Tg = 0, and has order one on B iff T pi_1(a) - pi_2(a) T intertwines the opposite actions; the maximal such B are the maximal A(R) over odd R with T = R + tau(R)) 412

Part S78 (Full scalar gluing f_Q(l) = f_U(l) on the one-flavour boundary-coupled interval gives a parent with signs (+, +, -), order zero and order one whose lowest window C + C, m sigma_1, sigma_1 K fails order one) 413

Part S79 (The same-chirality analogue of the Horava-Witten walls is type (a) at both ends with opposite localization masses, and unbroken E_8 x E_8 allows no bilinear connector between (248, 1) and (1, 248)) 414

Part S80 (Two massless interval fields valued in E and E^* with wall coupling X = t S^* and constant multipliers B = A(S) give, in the lowest window, finite data with connecting block m S, m = arctan(t)/l, the KO-6 signs, order zero and order one) 414

Part S81 (In unchanged separate windows, scalar gluing on closed sets with empty interior leaves the compressed image unchanged, so marking points on the twistor windows selects no further algebra) 415

Part S82 (A rank-one connecting block T x = kappa K x K with K = v w^* has intertwining algebra C + H + M_3(C), of real dimension 24, the unique maximal-dimensional admissible algebra up to automorphism) 416

Part S83 (If the antisquarks of three generations are symmetric diquarks of the light quarks, their counts force (n_u, n_d) = (2, 3), with the top out, and leave three antitriplets of charge +4/3, half of a chiral-multiplet pairing and anomalous as superpartners) 417

Part S84 (For the five quarks other than the top, 30 = (5, 3) + (5-bar, 3-bar), and the adjoint of SO(30) contains in (15, 3-bar) and (24, 1) + (1, 1) exactly the charged scalar-partner counts of three generations, plus three triplets of charge +4/3 and one neutral state) 418

Part S85 (With only an up-type Dirac mass the finite heat trace is 84 + 12 e^{-x}, and the 96 states of H_F split as 84 + 12 in two ways, by the top and by the neutrinos) 419

Part S86 (In the exact quadratic spectral action, among single charged species only an up-type quark can be removed while keeping both cos^2 theta_NC = 5/8 and the photon/gluon ratio 8/3) 420

Part S87 (The compression of a constant observable to the interval window is the average Q_l(a) = sinc(l ad_{D_F})(pi(a)) of e^{ixD_F} pi(a) e^{-ixD_F} over abs(x) <= l, so compressed observables converge to pi(A_F) at rate l^2 while D_F, J_F and Gamma_F stay exact) 421

Part S88 (With a fixed light block M_0 and a heavy block Theta/l, the limit l -> 0 keeps

the light masses explicit with couplings vanishing as m l, and sends W and Z to pi/(2l) and the top to Theta/l, with m_t/M_W = 2 Theta/pi < 1 and M_W = M_Z) 422

Part S89 (Every finite (H_F, D_F, J_F, Gamma_F) of signs (+, +, -) with l ||D_F|| < pi/2 is exactly the lowest spectral window of the pure derivative on an interval of length l, with the wall coupling X = tan(l D_F)) 423

Part S90 (The scalar inner fluctuations of the interval parent are pure gauge under its allowed endpoint gauge group and act on the moving window by the gauge orbit of D_F, which reproduces on that orbit the quadratic term of Chamseddine-Connes-van Suijlekom) 424

Part S91 (On one interval with fixed wall coupling, no bulk mass and even scalar inner fluctuations on the moving window, the singular values of the finite Dirac operator are fixed, so this parent has no radial Higgs mode) 425

Part S92 (On the interval parent a smooth A_F-valued function preserves the domain iff its wall value lies in the commutant C_F of D_F, and the parent is then a real even spectral triple with order zero and order one for any finite data with order zero) 426

Part S93 (With a bulk prealgebra P and a grading at each wall, a P-valued function preserves the domain iff it commutes with the grading at each wall, and aligned walls compress P to its even part, with the grading inherited from the parent operator, when Gamma_0 P Gamma_0 = P) 427

Part S94 (With P = M_2(H) + M_4(C), a connector at one wall and an assumed five-dimensional Yang-Mills action of one coupling, the lowest W and Z masses are sin^2(M_W l) = sin^2(2 theta)/2 and sin^2(M_Z l) = (4/5) sin^2(2 theta), with constants 2 and 16/5 at small theta and M_W^2/M_Z^2 -> 5/8 at both ends) 428

Part S95 (A self-adjoint boundary condition for the derivative on an interval is a grading at each wall, and the finite Dirac operator is D_F = (1/2l) Gamma_0 log(Gamma_1 Gamma_0), with singular values the principal angles between the two gradings divided by l) 429

Part S96 (In the flat massless family with this gauge content and frequency-one fermions, the one-loop potential has its unique global minimum at theta = pi/2 for every n_f > 0 and decreases strictly for n_f >= 10.8, with no interior minimum on the grids computed below that) 430

Part S97 (A scalar connection along the interval moves the second wall grading along its unitary orbit, and its odd part changes the principal angles, which supplies a radial mode; for M_2(H) the field space is S^4 and the Higgs modulus is half the geodesic distance between the points chosen by the two walls) 431

Part S98 (For the heat cutoff and a universal angle theta, the spectral action of an interval channel is a Jacobi theta function, strictly decreasing on [0, pi/2] and stationary at theta = pi/2, the Connes-Lott value m = pi/(2l)) 432

Part S99 (Local wall gradings satisfying a principal-symbol condition give a self-adjoint elliptic realization preserved by every smooth scalar function, while the spectral walls are preserved only by scalar functions constant on CP^3 at each wall) 433

Part S100 (On the unchanged level-one window of Example 16.9, the compression of a local multiplication grading G is a grading exactly when G commutes with the projection onto the window, and it is then +1 or -1) 434

Part S101 (For the flat massless one-loop determinant of lemma 23, theta = pi/2 is the unique global minimum for every n_f > 0; the potential decreases strictly for n_f >= 54/5, and for 0 < n_f < 54/5 its only interior stationary point is a strict maximum) 435

Part S102 (On CP^3 x I, nonlocal spectral walls that prescribe two gradings on the level-one kernel C^4 give exactly the two-gradings window of lemma 23, with every other mode above Delta_K/r, so the higher K-modes decouple as the radius shrinks) 436

Part S103 (The compression from the scalar cylinder to the finite window is Phi_l(f) = l^-1 int_0^l E_0(R(x)* T_f(x) R(x)) dx, independent of the radius, and with aligned walls it generates R + R on the S^4 factor, against H + H when M_2(H) is generated between the two stages) 437

Part S104 (For the heat cutoff at leading order in small masses, a vector of constant transported profile on the interval has M^2 = ((3 - d_eff)/2) ||[D_F, T]||^2/||T||^2, with d_eff the spectral dimension of the unfolded circle at the cutoff, so that M_W^2 = ((3 - d_eff)/24) sum N_f m_f^2 and m_t/M_W runs from sqrt(8/3) to 2) 438

Part S105 (Unfolded, the interval parent is -i d/dx (x) 1 + 1 (x) D_F on a circle of length 2l, so its tower is D_F shifted by the momenta n pi/l, D_F is a constant connection, and an even generator constant in the transported frame becomes 1 (x) T) 439

Part S106 (In the flat family with pure frequency-1 and frequency-2 channels at one loop, an interior global minimum exists for n_2 >= n_c = 2.8855..., at theta = pi/4 with M_KK = 4 M_W when the frequency-2 channels act alone, and in (pi/4, pi/2) when 0 < n_1 < n_* = (4/3)(4 n_2 - 54/5)) 440

Part S107 (In the step-threshold model of the flat parent, differences of the inverse gauge couplings do not depend on a common bulk-fermion term, and the Sp(2) left-right difference alpha_R^-1 - alpha_2^-1 runs only logarithmically, with no power of the tower variable S) 441

Part S108 (In the one-loop step-threshold model of the flat parent, without frequency-2 multiplets, the three gauge couplings meet at mu_* = 2.21 x 10^14 GeV with M_KK = 3.15 x 10^13 GeV, the only solution with S >= 1) 442

Part S109 (In the displayed charge-2/3 channel of the (10, 6) or the (5, 15), a pure frequency-2 channel carries an exact massless coloured exotic, and a wall mass of any finite positive size leaves a coloured state of charge 2/3 with 0 < m < M_W) 443

Part S110 (As Lambda -> infinity on the reduced action, the axial anomaly of a traceless flavour generator has only the electromagnetic term, with coefficient 2 N_c tr(T^3 Q^2) = 1 for two flavours, while at fixed cutoff a heavy top weights it as (4 e^{-x} - 1)/3 for the heat cutoff) 444

Part S111 (On the reduced operator D_M(A) + gamma_5 d with masses commuting with the gauge field, the regulated axial trace is sum_f tau_f f(m_f^2/Lambda^2) ind_f at every cutoff, and the regulated mass insertion equals it) 445

Part S112 (For two massless flavours and a reduction Sigma : S^4 -> SU(2) in the nontrivial class of pi_4(SU(2)), the spectral action is unchanged and the Weyl Pfaffian of N_c doublets, transported along connections from 0 to the pure gauge Sigma d Sigma^*, has the relative sign (-1)^{N_c}) 446

Part S113 (For the pull-back connection, the leading wall operator on the representation 4 has a one-dimensional kernel whose compression has fibre part -w(rho), w(rho) =

2(rho^2 + 2)^2/(9 + 2(rho^2 + 2)^2), which reaches the level-one value only as rho -> infinity, while the line-preserving collar connection gives the level-one value at every radius) 447

Part S114 (Conjecture: for large mass the doubled operator on the disc bundle has an isolated window of eight states, one copy of the 4 of Sp(2) at the zero section and one at the boundary, and its compression maps tend to the two endpoint maps) 448

Part S115 (Unital completely positive maps that converge in norm can generate M_4(C) for every positive parameter and M_2(H) at the limit, so generated algebras need not converge when compression maps do) 449

Part S116 (The normal model Q = sigma . d_y + y . J with spin 3/2 has ker Q = 0 and ker Q^* = 3 + 1 with a gap, and over S^4 this gives exactly four core states, one copy of the 4 of Sp(2), the twistor spinors as sections of S) 450

Part S117 (On the disc bundle X of Lambda^2_- over S^4, whose smooth functions have independent values on the zero section S^4 and on the boundary CP^3, prescribed rank-eight windows at the two ends compress real functions to maps that generate M_2(H) + M_4(C) at every localization scale) 451

Part S118 (On the disc bundle with a connection metric, the doubled Callias-type operator built from T_M = D_X^E + i Phi_M, with a local chirality condition at the boundary, is self-adjoint, elliptic and odd with compact resolvent, every smooth scalar function preserves its domain, and no condition enters at the zero section) 452

Part S119 (On the eight-state window, the compression of a C^1 function on the disc bundle converges to the pair of endpoint maps (Q_B(a|*{S^4}), Q_Y(a|*{CP^3})) with error O(M^{-1/2}), and real functions generate M_2(H) + M_4(C) under the limit map) 453

Part S120 (The singular value lambda_M that links the two ends of the eight-state window satisfies the exact identity lambda (<u, chi u> - <v, chi v>) = <v, c(d chi) u> and is exponentially small, |lambda_M| <= C exp(-(1 - eta)(M/2) int_e^{l-a} f dr)) 454

Part S121 (With the pull-back connection throughout, Conjecture 28.6 holds at the Fubini-Study wall radius, where the kernel condition K(1) holds and the wall mode has fibre part -2/3, and at another radius it holds under the kernel condition K(rho)) 455

Part S122 (With the line-preserving collar connection and the Fubini-Study metric at the wall, and flat fibres near the zero section, the doubled operator on the disc bundle has, for each fixed 0 < delta < sqrt(3) and mass above a threshold depending on delta, exactly eight eigenvalues in (-delta, delta), one copy of the 4 of Sp(2) at each end, which is Conjecture 28.6 in case (b)) 456

Part S123 (On the disc bundle with mass M_eps proportional to log(1/eps) and dilation R_eps = eps^{-alpha}, the operator keeps exactly eight eigenvalues, of size at most C eps, below delta eps^alpha, its compression converges to the endpoint maps, and its count at fixed energy has seven-dimensional Weyl growth C_X E^7 eps^{-7 alpha}) 457

Part S124 (With bulk mass mu_eps = mu_* eps^alpha and length c_eps/mu_eps, c_eps = (1 - alpha) log(1/eps) + c_0, the light pair of the interval has mass m_eps ~ 2 mu_* e^{-c_0} eps, the gap is ~ mu_* eps^alpha, the localization width over the length tends to zero, and the opposite-wall coefficient is exactly (m/(mu + kappa))^2) 459

Part S125 (For the core copy of the 4 in the disc-bundle window, every bounded real observable compresses to within 2 eta_t of a quaternionic operator and products of

k of them to within 2 k eta_t, and the defect eta_t is O(M^{-1/2}) in general and exponentially small in M when the quaternionic structure of E is parallel) 460

Part S126 (On the interval with D_F = eps D_* and length l_* eps^{-alpha}, the finite data stay exact in a window of r = dim H_F states below E_* eps^beta, while the tower gap closes as eps^alpha and the count at fixed energy grows as eps^{-alpha}) 461

Part S127 (A scale-free potential with G(theta) = theta^4 near theta = 0 is stationary along every family m = eps, l = eps^{-alpha} with 0 < alpha < 1, so a flat direction allows a joint path without selecting it) 462

Part S128 (The chiral interval of length l is unitarily the antiperiodic circle of radius l/pi, with the grading as the reflection x -> -x, the walls as its two fixed points and the scalar functions as the reflection-even functions) 463

Part S129 (Under the hypotheses of Theorem 9 of lemma 24, the angle-dependent one-loop potential, relative to theta = 0, has its minimum at theta = pi/2 with value -93 zeta(5) n_f/(512 pi^2 l^4), and near theta = 0 it is (3 zeta(3)/(16 pi^2))(54/5 - n_f) theta^2/l^4) 464

Part S130 (For a scale-free potential V = l^{-4} G(theta), a free angle gives a runaway of l at every angle with G != 0, and an angle tied to a mass, theta = l m, has stationary lengths l = theta_*/m at isolated roots of theta G' = 4 G, so M_W l is the same at every m) 465

Part S131 (Abelian fluxes on CP^2 x CP^1 along hypercharge and B - L, with one line bundle per Standard Model field and the canonical spin^c shift, give no generation, with or without nu^c, and adding any real third charge does not change this) 466

Part S132 (With fluxed charges F (universal), chi (the grading) and B - L, the pairings P^(1) = c_0 chi + c_1 (B - L) and P^(2) = (3F + chi)/2 give one generation with nu^c as the exact index content on CP^2 x CP^1 for (c_0, c_1) = (-9/4, -3/4) or (-3/2, -3), and without chi or without B - L there is no solution) 467

Part S133 (The circle bundle Y_{a,b} over CP^2 x CP^1 with Euler class a h_1 + b h_2, a != 0, has H^2 = Z + Z/gcd(a, b), b_3 = 0 and H^4 of order a^2, and is spin exactly when a is odd and b is even; for (3, 2), H^2 = Z, H^4 = Z/9, and the divisor classes of U and V go to 2 and -3) 468

Part S134 (Among the circle quotients Y_{a,b} of S^5 x S^3 with coprime weights only (3, 2) has a Calabi-Yau cone, and a free circle quotient of a moment level in a complex representation of dimension n has dimension 2n - 3, which is 7 only for n = 5: for SU(5) with hypercharge, the 5 and the 5-bar) 469

Part S135 (The hypercharge quotient Y_{3,2} = (mu_Y^{-1}(0) cap S^9)/U(1)_Y of the 5-bar is a smooth simply connected seven-manifold, the circle bundle of O(3, 2) over CP^2 x CP^1, with a Calabi-Yau cone and effective group (U(3) x U(2))/U(1)_Y; its identification with M^{111} is conditional on the recalled fibration) 470

Part S136 (Every monomial in U, V and every state of Lambda^*(5-bar) satisfies 2t + 3d + 6Y = 0 mod 6, the U(1)_Y-neutral monomials have zero triality and integer weak isospin, and no tensor Kaluza-Klein mode on Y_{3,2}, on an invariant background, is a colour triplet or a weak doublet) 471

Part S137 (Among blocks of 12 states made of whole Dirac species, exactly one up-type quark and the three neutrinos together keep both spectral-action values cos^2 theta_NC

= 5/8 and kappa_em/kappa_s = 8/3 on removal) 472

Part S138 (Two commuting gradings of order three on e8, by SU(9) and by the family SU(3), place the left-handed 10 and H_u of a generation in the C_3 sector and (d^c, L), N and (D^c, H_d), nu^c in the C_6 and gravity sectors according to a sign t, so that the 96 states split as 45 + 45 + 6 for t = +1/2 and 33 + 33 + 30 for t = -1/2) 473

Part S139 (Under colour F1 = 10 (3 + 3-bar) + 24 (1) and F2 = 12 (3 + 3-bar) + 12 (1), so no colour-commuting unitary exchanges the two families of 84, and for every homomorphism SU(3) -> SU(9) the exterior cube Lambda^3 C^9 has the colour content of neither family) 474

Part S140 (In the aligned three-family embedding in e8 with its two gradings, no complex automorphism preserving or inverting the family grading maps the top block onto the neutrino block, and the Chevalley involution maps each block to its conjugate) 475

Part S141 (If the cubic couplings come from the structure constants of e8 and both gradings are symmetries, the up-type Yukawa couplings have all three fields in the C_3 sector, and the down-type, charged-lepton and Dirac-neutrino couplings take one field from each sector) 476

Part S142 (With four families, M = -P, where P exchanges the colour and family coordinates and the two weak coordinates, is an involution in W(E8) that preserves SU(2)_L and the 128 roots and exchanges the up-type quark of family 0 with the neutrinos of families 1, 2 and 3) 477

Part S143 (In a four-family finite geometry with KO-6 signs, the flip of colour and family through a unitary identification is a unitary involution commuting with gamma and J that conjugates the top Dirac block into the Dirac blocks of the neutrinos of families 1, 2 and 3, so the two heavy limits have conjugate light spaces of 116 states) 478

Part S144 (With four families, no isometry of R^8 maps the 44 roots of the fourth family with the top block onto the 44 roots of the fourth family with the neutrino block: up to family permutations their degree multisets are {10^16, 12^12, 13^16} and {10^24, 13^16, 16^4}) 479

Part S145 (The colour SU(3) of the root table lies in no Spin(9) in SO(16) with 128 -> 44 + 84, so the split 44 + 84 of the four-family spinor is not colour-covariant and the fit 44 = 32 + 12 is numerical) 479

Part S146 (In the aligned three-family embedding in the roots of E8, the two families of 84 have equal numbers of pairs at each inner product, and no isometry of R^8 maps F1 onto F2, so no element of W(E8) does and no isometry of H_F maps the top block onto the neutrino block) 480

Part S147 (For none of the 192 assignments of the up-type quarks, down-type quarks, charged leptons and neutrinos to three of the four family slots, aligned or misaligned, does an isometry of R^8 map S - T onto S - N) 481

Part S148 (The so(9) fixed by the Chevalley involution on sl(9) is conjugate in E8 to the Spin(9) of the four-family count, with k = so(9) + 84 and p = 44 + 84, the 44 inside the gravity sector, and both are SO(9) inside E8) 482

Part S149 (On each eigenspace of d^2 in H_+, of dimension n, the unitaries commuting with the antiunitary j of an odd factorization, j^2 = s, form O(n) for s = +1 and USp(n) for s = -1, in which case n is even) 483

Part S150 (In the parent of lemma 22, k channels at a common angle theta give the signed eigenvalue theta/l multiplicity k for 0 < theta < pi/2 and 2k at the ends, and the lowest eigenvalue of abs(D) multiplicity 2k in the interior, 2k at theta = 0 and 4k at theta = pi/2) 484

Part S151 (The two del Pezzo surfaces of degree eight have different lattices orthogonal to the canonical class: P^1 x P^1 has a class of square -2 there and dP_1 has none, so the empty far wall with E_1 cannot be assigned to either surface indifferently) 485

Part S152 (Given the type I' slope rule and sixteen D8-branes at x_k, the coupling function is z(x) = z(0) + c sum_k min(x/2, x_k - x/2), so z(L) = z(0) - c (8L - sum_k x_k), and infinite coupling at the far wall fixes z(0) by the sum of the brane positions) 485

Part S153 (Under U(15), the adjoint of SO(30) is Lambda^2 V + Lambda^2 V-bar + V (x) V-bar and that of USp(30) is S^2 V + S^2 V-bar + V (x) V-bar, and the colour antitriplets of Lambda^2 V and S^2 V carry the charge counts (6, 6, 3) and (3, 6, 1)) 486

Part S154 (For an antiunitary J exchanging V and V-bar with J^2 = eps and a Hermitian X with J X = eps' X J, the particle-antiparticle block obeys B = eps eps' B^T, so in the labels 9, 10 and 11 a J-compatible operator has no nonzero antisymmetric block within one multiplet) 487

Part S155 (Under SU(16) x U(1) the complex adjoint of SO(32) is 120 + 120-bar + 255 + 1, and under SU(5) x SU(3) the 105 in the 120 is (15, 3-bar) + (10, 6), whose antitriplet has the charge counts (6, 6, 3)) 488

Part S156 (The Wilson-line angle of a type I' D8-brane at x_k is theta_k = (pi/2) W(x_k)/W(L) with W(x) = int_0^x z^{-1/3} dy, so the linear dictionary x_k = (2L/pi) theta_k holds as a function of every position exactly for constant z, and asymptotically for large R) 489

Part S157 (The angle-independent one-loop coefficient of the flat interval is 8 pi^2 a_0 = (3 zeta(5)/16)(4 n_D' - 3 (n_i - (15/16) n_h) - n_s - 5 n_g), positive for the corpus content, so at v = 0 the interval runs to large length) 490

Part S158 (In the flat one-loop family with a_0 restored and the angle tied to a mass, top-only content has G > 0 on [0, pi/2] and no stationary point in the window, while a universal mass gives one stable stationary angle, theta_* = 1.009 for three generations and 1.031 for one) 491

Part S159 (For a potential with a regular joint expansion in theta and alpha'/l^2 around the flat family, the type I' dilaton gradient supplies no second scale for Proposition 1.5: its relative variation is bounded by 6/R^2 along an opening path, and it can matter only if a_0 = 0, and then only with b = 0 when a_1 = 0) 492

Part S160 (In the chamber 15 + 1 with the top's brane on the far wall, the D0 states massless there at infinite coupling satisfy M_D0 R = R^2 - R_*^2, so W bosons of this kind are massless at the string-scale radius and grow heavier as R grows, and a neutral Z cannot be of this kind) 493

Part S161 (On a type I' segment with N != 8 branes to its left the proper length is linear in the inverse coupling, so the classical profile of the chamber 15 + 1 has no region that is both strongly coupled and resolved at the formal local scale g^{1/3} l_s) 494

Part S162 (With fifteen D8-branes at x = 0, one at x_t and infinite coupling at the far wall, theta_t = (pi/2) 8 (1 - rho)/(8 - rho) and R_*^2 = 14 (8 - rho^2)/(8 - rho)^2

with rho = (z(x_t)/z(0))^{2/3}, so infinite coupling sits at the smallest radius of the chamber, at the string scale) 495

Part S163 (If the tower of the SO(32) Wilson-line channel is identified with the type I' winding tower, the length l of the interval of H and the proper length L_p of the type I' interval satisfy l L_p = pi^2 alpha' at constant dilaton, and l -> infinity at fixed heterotic coupling shrinks the type I' interval and ends at ten-dimensional SO(32)) 496

Part S164 (With the real algebra M_30(R) on the massless light block and wall algebras containing 1, the self-adjoint one-forms of the interval parent are exactly E a_x with a_x an so(30)-valued function free at the walls, whose block between particles and antiparticles lies in Lambda^2 V) 497

Part S165 (Under a five-dimensional Yang-Mills action the massless four-dimensional scalars from the interval component of an so(30) connection form the orthogonal complement of k_0 + k_l, so lemma 22's parent has none, wall algebras M_15(C) give exactly Lambda^2 V + Lambda^2 V-bar, and a generator has a massless vector or a massless scalar, never both) 498

Part S166 (For s = +1 the skew-Hermitian operators on the 30-dimensional massless block of the five light quarks commuting with j form so(30) = u(15) + Lambda^2 V, with u(15) the commutant of quark number, and for s = -1 they form usp(30) = u(15) + S^2 V; the colour antitriplets carry the counts (6, 6, 3) and (3, 6, 1)) 499

Part S167 (The interval component of an so(30) connection compresses to the finite window through its commutator with the masses, B_ij -> -((1 - cos(l (m_i - m_j)))/(l (m_i - m_j))) B_ij, which maps the (S^2 5, 3-bar) of Lambda^2 V onto the (Lambda^2 5, 3-bar) of S^2 V, with counts (3, 6, 1), and vanishes for a universal mass) 500

Part S168 (As SU(3) x SU(2) x U(1)_Y representations, Lambda^1 + Lambda^3 + Lambda^5 of the 5-bar is one generation with nu^c, the 16, and Lambda^0 + Lambda^2 + Lambda^4 is its conjugate, with the charge X = 5 - 2k on Lambda^k) 501

Part S169 (On the brane coordinates (U, V) with Betti charges (2, -3), the point U = 0, V = (v, 0) is fixed by SU(3) and by T_3 + Y and moved by T_3 - Y, and the Fayet-Iliopoulos quotient at zeta != 0 has residual Z_3 or Z_2 stabilizers along CP^1 or CP^2, so it is a partial orbifold resolution) 502

Part S170 (AdS5 x CP^2 x S^2 with G_4 = f_1 vol(CP^2) + f_2 J wedge vol(S^2) solves the equations of eleven-dimensional supergravity, with Ricci components -(f_1^2 + 2 f_2^2)/6, f_1^2/3 + f_2^2/6 and 2 f_2^2/3 - f_1^2/6, and round factors exist exactly when abs(f_1) < 2 abs(f_2)) 503

Part S171 (For every homomorphism SU(3) -> SO(9), Lambda^3 R^9 (x) C and Lambda^6 R^9 (x) C have the colour content of neither family of 84, and on Y_{3,2} no tensor Kaluza-Klein mode of C_3 or of the metric is a colour triplet, so the coloured states of the 84 are not modes of C_3 there) 504

Part S172 (The signed spectrum {(n pi +- theta)/l} of one channel of the interval parent depends only on theta modulo theta -> -theta and theta -> theta + pi, with fundamental domain [0, pi/2], whose ends are the periodic spectrum with a double zero mode and the antiperiodic spectrum with every value doubled) 505

Part S173 (The window of the doubled torus with flux p carries the degenerate representation a on C^p and 0 on conj(C^p), and for p = 3 no unital M_3(C) representation on C^3 + conj(C^3) satisfies order zero with a unital opposite action) 506

Part S174 (On a flat torus with a line bundle of degree p, the doubled Dirac operator with J(psi, chi) = (C chi, -C psi) has the signs of KO-dimension six, a kernel C^p + conj(C^p) below Landau levels of order (p/A)^{1/2}, and compressions of the functions generating M_p(C), so the window carries the unital finite datum M_p(C) + C) 507

Part S175 (For the SO(32) Wilson line W_a = diag(1_30, R(2 pi a)), a = theta/pi in [0, 1/2], the centralizer is SO(32) at a = 0, SO(30) x SO(2) for 0 < a < 1/2 and S(O(30) x O(2)) at a = 1/2, where the (30)_{+-1} tower is antiperiodic, and the one-loop minimum of the flat family is a = 1/2) 508

Part S176 (The Pati-Salam finite data are orientable, with gamma = rho(p_L' - p_R') rho(1_4)^o - rho(1_4) rho(p_L' - p_R')^o, while the Standard Model data and the light block of colour plus electromagnetism admit no orientation cycle) 509

Part S177 (Along the chain Pati-Salam, Standard Model, colour plus electromagnetism, the skew trace forms of the finite data have ranks 2, 2 and 0, so the trace-form Poincaré duality fails at every step, at the last because every isotypic range of the light block has graded dimension zero) 510

Part S178 (The left-action traces of the abelian generator of C and of the common colour phase are (4i, 12i) on the 32 states, (2i, 12i) with the Majorana pair removed, (-2i, 6i) with the up-type sector removed and (-4i, 6i) on the light block, and only the inherited unimodularity condition gives the photon its charges) 511

Part S179 (The Sp(2)-orbits of the pair of marked lines (Cw, Cv) of the connector are classified by a = abs(h(w, v)) and b = abs(omega(w, v)), with a^2 + b^2 <= 1, and the orbits of the connector by a, b, abs(kappa) and the phase of kappa alpha^2 when alpha = h(w, v) is nonzero) 512

Part S180 (With one copy of the 4 of Sp(2) in each degree of the end grading, the off-diagonal part of the window operator is a scalar lambda_M I_4, invariant under Sp(2), so the equivariant window operator carries no Higgs representation; G-breaking off-diagonal blocks do) 513

Part S181 (For a marked complex line l in the quaternionic sheet W = H^2 with quaternionic line L over p in S^4, T_L S^4 = (2, 2) = Hom_H(L, L^perp) and T_l CP^3 = (2)_{+-1} + (1)_{+-2} under Sp(1)_a x U(1)_b: the Higgs doublet of hypercharge 1/2 along the base and a charged singlet of hypercharge 1 along the twistor fibre) 514

Part S182 (Changing the coefficient connection by a horizontal alpha changes the Berry connection on a constant-rank kernel bundle by the fibrewise compression P alpha P, exactly, and every connection on the kernel bundle is a Berry connection of such a change exactly when the compression Gamma(F_b, u(E)) -> u(K_b) is surjective at every point, that is, when the products s_i s_j^* of a kernel frame are linearly independent) 515

Part S183 (Scalar changes of the coefficient connection attain every connection on the kernel bundle exactly when the functions <s_i, s_j> are independent on the fibre, which holds for H^0(CP^3, O(1)) and fails for functions pulled back from S^4, whose compressed image has real dimension six) 516

Part S184 (On the neutral block of the seesaw family the light mass is 2 t^{2 alpha} abs(y)^2 / (t + sqrt(t^2 + 4 t^{2 alpha} abs(y)^2)), zero for alpha < 1/2, abs(y)^2 at alpha = 1/2 and divergent above) 517

Part S185 (Along the seesaw scaling the algebra retained by bounded commutators is C + M_3(C) for any growing Dirac block, so under that criterion a finite seesaw mass and

the Standard Model algebra belong to two different limits) 517

Part S186 (The 18-state orientable KO-6 bimodule of the corpus is the two-sheet construction of lemma 16 over the two orbit types of the Weyl crossed product) 518

Part S187 (A Yukawa block X = q + i q' on a doublet, with q and q' quaternions, gives the two members the masses m_+-^2 = |q|^2 + |q'|^2 +- 2 |Im(conj(q) q')|: equal exactly when X is quaternionic up to a complex factor, as every block compressed from a function on S^4 is; one massless member exactly when X has rank one with the two halves of equal norm; and when split, the masses fix a maximal torus of SU(2)_R, the hypercharge circle of the connector exactly when its line is an eigenline) 520

Part S188 (On the quaternionic sheet W = H^2 with grading axis p, the Hermitian blocks odd for the grading form the complex bidoublet Hom_C(L_p, L_p^perp) = (2, 2) (x) C of Sp(1)_a x Sp(1)_b; the quaternionic half is T_p S^4, the odd part of the compressed coordinates of S^4, and the anti-quaternionic half lies outside M_2(H), inside the compression of the functions on CP^3) 521

Part S189 (In the eight-state window of lemma 29 every cross block between the core and the wall states of a bounded multiplication operator is bounded by the exponential that bounds the tunnelling mass, the cross block of a coordinate of S^4 is c_M Gamma_a, and a window operator with block lambda + mu n . Gamma has the eigenvalues +-(lambda^2 + |mu|^2 +- 2 lambda Re mu)^{1/2}, each of multiplicity two) 522

Part S190 (On the Berger five-sphere the Killing-form coupling ratio of the central circle with charge c on the coordinates to the SU(3) is e^2/g_s^2 = (3 + eta^2)/(24 eta^2 c^2); with the spinor charge unit c = 2/3 it is 3(eta^2 + 3)/(32 eta^2), equal to Connes' matching value 3/8 at the round point) 523

Part S191 (The lift of the Weyl group S_3 x Z_2 to the spinors of T^6 is (Dic_3 x Z_4)/Z_2; the spinorial part of R[Dic_3] at the orbit type (a, a, a) is C + H, complexified at the fixed point by the central lift of the inversion; the two-sheet bimodule of lemma 45 with its first sheet there carries the real Standard Model algebra C + H + M_3(C)) 524

Part S192 (The spinor fibre of the four-torus with its Clifford action and chirality is the quaternionic sheet of Example 16.9 with the chirality as the grading axis; the lifted Weyl stabilizer generates C + H with C the Cartan of H_R spanned by the lifted transposition, which is the hypercharge circle; the hypercharge-invariant Yukawa blocks are the diagonal ones, and the whole fibre carries one quark generation) 526

Part S193 (The compression of 5 w'^* B w'/w'^* w' on CP^3 is B + tr(B): the grading and the projectors on the marked lines are compressions of height functions, and the top block is the anticommutator of a compressed base coordinate and a compressed height function of the wall sheet) 527

Part S194 (The rank-one Yukawa block with the eigenline condition is the Clifford action of a Higgs vector times the projector on the connector's line: one point of the twistor fibre carries the Majorana block and the top's right-handed states) 528

Part S195 (The centralizer in A_SM of the top-only Dirac operator is C + M_3(C) with the summand C embedded in H along the top's left-handed line: the algebra of the label 9, the retained algebra of lemma 44 and the corpus row rs_F1) 529

Part S196 (A sheet-diagonal Yukawa block split along projections of the second sheet keeps order one exactly when the projections commute with the second-sheet algebra: uni-

versal on the Pati-Salam M_4(C), with the lepton Yukawas equal to the quark ones, and separable exactly on the connector's algebra C + M_3(C), with the lepton line the connector's line) 530

Part S197 (The torus Dirac operator with a Wilson line and a torsion commutes with the lift of a transposition exactly when both data lie in its mirror plane; then the window block commutes with the lifted transposition on both chiralities, the condition [D_F, C_F] = 0 of Chamseddine, Connes and Marcolli, and these are all the invariant blocks; nonzero data along a mirror break S_3 to that transposition, and no nonzero data are invariant under S_3) 531

Part S198 (For a Wilson line and a torsion in a mirror plane the doublet has equal masses exactly when the torsion is parallel to the Wilson line, masses |A| +- |eta| exactly when perpendicular, and a massless member exactly when eta = +- J A, the quarter turn in the complex structure of the elliptic curve; the up and down Yukawas are A + J_w eta and A - J_w eta in the complex structure J_w = +-J of the Majorana line; the finite data at the mirror stratum are the quark sector of the label 9 with explicit masses) 532

Part S199 (On the flat four-torus with a constant connection and a constant three-form the Dirac operator is c(k + A) + i c(eta) Gamma on each momentum sector; the constant spinors are its lowest window for |A| + |eta| < pi/l, and there it is the Yukawa block c(A) + i c(eta) Gamma, with the Wilson line as the custodial half and the three-form as the splitting half, of masses m_+-^2 = |A|^2 + |eta|^2 +- 2 |A wedge eta|) 533

Part S200 (The lepton sub-bimodule of the two sheets has eight states and contains the connector; the connector's order-one condition reduces C + H + C' + M_3(C) to the Standard Model algebra, identifying the two copies of C, exactly when the Majorana line is a weight line of the lifted transposition; the trace-form pairing of the lepton generation over C + H is nondegenerate) 534

Part S201 (A Majorana spinor of nine-dimensional Minkowski space is, under Spin(1, 3) x Spin(5), the complex space (2, 1) (x) 4: four left-handed Weyl spinors in the spinor fibre of the five-dimensional internal space, the sheet W = H^2, which under K_p is the left doublet and the right-handed pair, one lepton generation with nu_R; on its self-conjugate sector the masses are the torsion |eta| for both members, the sectors of nonzero momentum pairing into a generation and a mirror, and a Majorana mass for nu_R alone is wall data in the scope of constant bulk data) 535

Part S202 (Three lepton generations with nu_R (24 states) and the complexified adjoint of the flavour SU(5) of the five light quarks are isomorphic as representations of U(1)_em, with multiplicities 12, 6, 6; the adjoint is (3, 1)_0 + (1, 8)_0 + (1, 1)_0 + (2, 3-bar)_{+1} + (2-bar, 3)_{-1} under SU(2)_u x SU(3)_d x U(1)_em, and the charged states count (up flavours) x (down flavours) as the charged leptons count (chiralities) x (generations)) 536

Part S203 (Orientability is lost under a heavy centralizer while first order survives) 537

Part 03

Part A. Statement.

For a constant finite operator commuting with the gauge connection, the Yang–Mills term is

$$\boxed{S_{\mathrm{YM}} = -\frac{1}{24\pi^2}\int_M \sum_j f(m_j^2/\Lambda^2)\ \mathrm{tr}_{H_j}(F_{\mu\nu}F^{\mu\nu})\,dV.}$$

Here H_j is an eigenspace of D_F^2. The trace includes its entire finite multiplicity; the spin trace has already been taken.

Hypotheses.

Use Hermitian Euclidean gamma matrices, $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$, and skew-Hermitian gauge curvature

$$F_{\mu\nu} = [\nabla_\mu + A_\mu, \nabla_\nu + A_\nu]_{\text{gauge}}.$$

The spectral action is the literal trace on $L^2(M,S)\otimes H_F$, without a subsequent chirality or reality projection.

To distinguish the Laplace measure from the mass ratios, write

$$f(z) = \int_{(0,\infty)} e^{-sz}\, d\nu(s), \qquad 0 < \nu((0,\infty)) < \infty, \qquad \int (s^{-2} + s^2)\, d\nu(s) < \infty.$$

This class includes $f(z) = e^{-z}$. In particular, $f(0) < \infty$, $f(\infty) = 0$, and the spectral action is trace class.

Assume the finite-sector decomposition is fixed as Λ varies, A is smooth and independent of Λ, and

$$\partial_\mu D_F = 0, \qquad [A_\mu, D_F] = 0.$$

Proof.

Put $\mathcal{D} = D_{M,A}$, suppressing identity factors. Chirality satisfies

$$\{\mathcal{D}, \gamma_5\} = 0, \qquad \gamma_5^2 = 1.$$

Consequently,

$$(\mathcal{D} + \gamma_5 D_F)^2 = \mathcal{D}^2 + D_F^2 + \gamma_5[D_F, \mathcal{D}] = \mathcal{D}^2 + D_F^2.$$

The two summands strongly commute. Since A preserves every eigenspace H_j of D_F^2,

$$\boxed{\operatorname{Tr} e^{-tD_A^2} = \sum_j e^{-tm_j^2} \operatorname{Tr} e^{-tD_{M,A_j}^2}.}$$

Let $P_j = D_{M,A_j}^2$, and use integrated heat coefficients:

$$\operatorname{Tr} e^{-tP_j} = t^{-2}a_0(P_j) + t^{-1}a_2(P_j) + a_4(P_j) + R_j(t).$$

For fixed smooth backgrounds, $|R_j(t)| \le Ct$ for all $t > 0$: the small-t estimate is the heat expansion, and the large-t estimate follows from $P_j \ge 0$.

Define

$$x_j = m_j^2/\Lambda^2, \qquad \Phi_k(x) = \int s^{-k} e^{-sx}\, d\nu(s).$$

Integration of the exact factorization gives

$$S = \sum_j \left[\Lambda^4 \Phi_2(x_j) a_0(P_j) + \Lambda^2 \Phi_1(x_j) a_2(P_j) + f(x_j) a_4(P_j)\right] + O(\Lambda^{-2}).$$

The remainder is bounded by

$$C\Lambda^{-2} \int s\, d\nu(s),$$

uniformly for all $x_j \ge 0$. It is also uniform over compact smooth families of connections. Thus fixed mass ratios require no expansion in m_j/Λ.

It remains to compute the gauge part of a_4. With $\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, the gauge-dependent endomorphism in the convention $P = -(\nabla^2 + E)$ is

$$E_F = \tfrac{1}{2}\gamma^{\mu\nu} F_{\mu\nu}.$$

The relevant terms are

$$a_4(P)\big|_{F^2} = (4\pi)^{-2} \int_M \operatorname{tr}_{S,F} \left(\frac{1}{12} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} E_F^2 \right) dV.$$

Using

$$\operatorname{tr}_S 1 = 4, \qquad \operatorname{tr}_S(\gamma^{\mu\nu}\gamma^{\rho\sigma}) = 4(g^{\mu\sigma}g^{\nu\rho} - g^{\mu\rho}g^{\nu\sigma}),$$

the two contributions are $+\frac{1}{3}\operatorname{tr}_F F^2$ and $-\operatorname{tr}_F F^2$. Hence

$$a_4(P)\big|_{F^2} = -\frac{1}{24\pi^2} \int_M \operatorname{tr}_F F^2 \, dV.$$

This proves the statement.

For $F_{\mu\nu} = F^a_{\mu\nu} T_a$, with skew-Hermitian T_a, the coefficient matrix is therefore

$$-\frac{1}{24\pi^2} \sum_j f(x_j) \operatorname{tr}_{H_j}(T_a T_b).$$

It is positive semidefinite. The normalization is per full four-component spin Dirac operator, with the finite representation traced explicitly.

The three regimes are:

Heavy mass ratio	Weight	Consequence for the kinetic term
$m_h/\Lambda \to 0$	$f(0)$	Full inherited finite trace
$m_h/\Lambda \to c \in (0,\infty)$	$f(c^2)$	Partial suppression
$m_h/\Lambda \to \infty$	0	Trace over the light sectors

Absolute divergence $m_h \to \infty$ does not distinguish these regimes.

Part B. Statement.

Write

$$\ell = f(0), \qquad h = f(\mu^2).$$

For N generations with exactly one heavy up-type quark, the coefficients on the full finite space are

$$\boxed{\begin{aligned} \kappa_{\text{em}}^{(N)} &= \frac{1}{24\pi^2} \frac{16}{3} [(2N-1)\ell + h], \\ \kappa_s^{(N)} &= \frac{1}{24\pi^2} 2[(2N-1)\ell + h]. \end{aligned}}$$

Here $S_{\text{YM}} = \int (\kappa_{\text{em}} F_{\text{em}}^2 + \kappa_s \sum_a G_a^2)\, dV$; there is no $1/4$ in this definition.

Thus

$$\boxed{\frac{\kappa_{\text{em}}^{(N)}(\mu)}{\kappa_{\text{em}}^{(N)}(0)} = \frac{\kappa_s^{(N)}(\mu)}{\kappa_s^{(N)}(0)} = \frac{(2N-1)f(0) + f(\mu^2)}{2N f(0)}.}$$

Hypotheses.

Use $Q = T_3 + Y$ and the following particle states:

Multiplet	Number of states	Y	T_3	Q
(u_L, d_L)	6	$1/6$	$+1/2, -1/2$	$2/3, -1/3$
u_R	3	$2/3$	0	$2/3$
d_R	3	$-1/3$	0	$-1/3$
(ν_L, e_L)	2	$-1/2$	$+1/2, -1/2$	$0, -1$
ν_R	1	0	0	0
e_R	1	-1	0	-1

Charge conjugates double this to the standard 32-dimensional finite space. For color, use Hermitian generators with

$$\mathrm{tr}_{\mathbf{3}}(t_a t_b) = \tfrac{1}{2}\delta_{ab}.$$

All three colors of the selected up-type quark have the same Dirac mass. Photon and gluon generators commute with these masses.

Proof.

On the particle space, both chiralities contribute

	u	d	e
$\mathrm{tr}\, Q^2$	8/3	2/3	2
color index	1	1	0.

Therefore, for one generation,

$$K_{\mathrm{em}}^{\mathrm{particle}} = \frac{8}{3}(h + \ell), \qquad K_s^{\mathrm{particle}} = h + \ell.$$

Charge conjugation doubles both expressions. Each additional light generation adds $32\ell/3$ and 4ℓ, respectively, on the full finite space. Part A now gives the stated coefficients.

In particular:

Heavy sector	Both kinetic-coefficient ratios	$\mu \to \infty$
Up quark in one generation	$\dfrac{f(0) + f(\mu^2)}{2f(0)}$	1/2
Top only in three generations	$\dfrac{5f(0) + f(\mu^2)}{6f(0)}$	5/6

The common suppression implies

$$\boxed{\kappa_{\mathrm{em}}/\kappa_s = 8/3}$$

throughout this particular family. The absolute kinetic form changes, while its photon-to-gluon ratio remains unchanged.

The same charge table gives, on the particle space,

$$K_W = 3 \cdot \tfrac{1}{2} + \tfrac{1}{2} = 2,$$

$$K_Y = 6(1/6)^2 + 3(2/3)^2 + 3(1/3)^2 + 2(1/2)^2 + 1 = \frac{10}{3},$$

reproducing the conventions in section 14.3.

Part C. Statement.

For broken generators, a mass-dependent "F^2 coefficient" requires a prescription for separating gauge and Higgs derivative terms. I use the **integrated quadratic transverse kinetic coefficient**: expand about zero gauge field with D_F held constant, remove the gauge-field mass term, and extract the coefficient of $q^2|a_T(q)|^2$. Since

$$F_{\mu\nu}(q)^* F_{\mu\nu}(q) = 2q^2|a_T(q)|^2,$$

this specifies the quadratic F^2 coefficient, including all Higgs contributions to it.

For a vectorial doublet with nonnegative masses m_1, m_2, put

$$x_i = m_i^2/\Lambda^2, \quad z(u) = (1-u)x_1 + ux_2, \quad b(u) = u(1-u).$$

The normalized weight is

$$\boxed{w_f(x_1,x_2) = \frac{3}{2}\int_0^1 \left[(1-2b)f(z) + b(\sqrt{x_1}-\sqrt{x_2})^2 f'(z)\right] du.}$$

It is symmetric and satisfies $w_f(x,x) = f(x)$.

For the heat cutoff,

$$\boxed{w_{\exp}(x_1,x_2) = \frac{3}{2}\int_0^1 e^{-z(u)}\left[1-(2+(\sqrt{x_1}-\sqrt{x_2})^2)b(u)\right] du.}$$

This is generally more complicated than a divided difference of an antiderivative.

Hypotheses and scope.

Compute the local, gravity-independent coefficient in flat coordinates, with fixed m_i/Λ and $q/\Lambda \to 0$. The result concerns the quadratic action; longitudinal derivative terms and the full covariant Higgs-operator expansion remain distinct.

For an odd finite operator, the vectorial model can be realized as

$$D_F = \begin{pmatrix} 0 & M \\ M & 0 \end{pmatrix}, \qquad M = \mathrm{diag}(m_1, m_2), \qquad T = \begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}.$$

The Standard Model instead has a left doublet and right singlets. That representation changes the answer, as calculated below.

This distinction is necessary: prescribing D_F^2 alone does not prescribe $[T, D_F]$. Moreover,

$$[\nabla_\mu, \nabla_\nu]D_F = [F_{\mu\nu}, D_F],$$

so a noncommuting curvature cannot coexist with a covariantly constant finite operator. A separate "pure curvature" coefficient obtained by discarding Higgs derivative operators would require an additional operator-basis convention.

Proof by Duhamel expansion.

Without the commuting assumption,

$$\boxed{D_A^2 = D_{M,A}^2 + D_F^2 + i\gamma^\mu\gamma_5[A_\mu, D_F].}$$

The last term is essential at fixed D_F/Λ.

First take $f(z) = e^{-z}$, put $\tau = \Lambda^{-2}$, and diagonalize

$$D_F = \mathrm{diag}(\lambda_i).$$

Use $A_\mu = i\, a_\mu T$, with T Hermitian. The free squared operator is

$$P_0 = -\partial^2 + D_F^2.$$

The linear vertex between internal states j and i, carrying momentum q, is

$$(V_1)_{ij} = T_{ij}\left[(2p+q)\cdot a + \gamma^{\mu\nu} q_\mu a_\nu + (\lambda_i - \lambda_j)\gamma^\mu \gamma_5 a_\mu\right].$$

The quadratic vertex has no external momentum and therefore makes no contribution to the q^2 coefficient. The second-order heat trace is

$$\frac{\tau^2}{2}\int_0^1 \mathrm{Tr}\,\left(e^{-(1-u)\tau P_0} V_1 e^{-u\tau P_0} V_1\right) du,$$

together with the quadratic-vertex term. Completing the Gaussian square gives the factor

$$(4\pi\tau)^{-2} e^{-\tau[(1-u)\lambda_i^2 + u\lambda_j^2 + bq^2]}.$$

For $q\cdot a = 0$, the Gaussian-averaged spin trace of the vertex product is

$$4\left[\frac{2}{\tau} + (\lambda_i - \lambda_j)^2 + q^2\right]|a|^2.$$

Expanding to order q^2, and writing $r_i = \lambda_i/\Lambda$, yields

$$\frac{q^2|a|^2}{8\pi^2}\sum_{ij}|T_{ij}|^2 \int_0^1 e^{-[(1-u)r_i^2 + u r_j^2]}\left[1 - 2b - b(r_i - r_j)^2\right] du.$$

Comparison with $F^2 = 2q^2|a|^2$ and Part A's normalization proves the heat-cutoff formula.

Integration over the Laplace measure replaces

$$e^{-z} \mapsto f(z), \qquad -se^{-sz} \mapsto f'(z),$$

giving the stated general formula. The moment assumptions justify these operations. The small-momentum expansion is uniform for mass ratios in compact sets.

For a vectorial positive-mass pair, the two relevant signed-mass branches both have $(r_i - r_j)^2 = (\sqrt{x_1} - \sqrt{x_2})^2$. At equal masses, that term vanishes and

$$\frac{3}{2}\int_0^1 (1 - 2u(1-u))\, du = 1,$$

which recovers Part A.

The divided difference does occur as

$$\int_0^1 f(z(u))\, du = \frac{\mathcal{F}(x_1) - \mathcal{F}(x_2)}{x_1 - x_2}, \qquad \mathcal{F}' = f.$$

The orbital vertices and finite-operator commutator supply the additional terms in w_f.

Standard Model doublet.

For the left-handed weak generator,

$$T = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \qquad [T, D_F] = \begin{pmatrix} 0 & tM \\ -Mt & 0 \end{pmatrix}.$$

For an off-diagonal t, summing these matrix elements replaces

$$(\sqrt{x_1} - \sqrt{x_2})^2 \quad \text{by} \quad x_1 + x_2.$$

Thus

$$w_L(x_1, x_2) = \frac{3}{2} \int_0^1 e^{-z(u)}[1 - (2 + x_1 + x_2)b(u)]\, du.$$

In particular,

$$w_L(x, x) = e^{-x}(1 - x/2).$$

Equal masses do not make a left-only weak generator commute with D_F, because its right partners are weak singlets. The conjectured diagonal identity therefore applies to the vectorial model, but not to this chiral representation.

For a heavy up quark and massless down quark, both models give

$$\boxed{w(x) := w_L(x, 0) = 3 \int_0^1 u^2 e^{-xu}\, du = \frac{3[2 - (x^2 + 2x + 2)e^{-x}]}{x^3}, \qquad x = \mu^2.}$$

The value at zero is defined continuously. In particular,

$$w(x) = 1 - \frac{3}{4}x + \frac{3}{10}x^2 + O(x^3), \qquad w(x) \sim \frac{6}{x^3} \quad (x \to \infty).$$

The divided-difference candidate $(1 - e^{-x})/x$ already disagrees at first order.

Weak-angle consequences.

Use particle traces below; charge conjugation doubles the entire kinetic matrix. Let $N = 1$ or 3, with one heavy up-type quark, $x = \mu^2$, and $r = e^{-x}$. The charged and neutral transverse coefficients are

$$\begin{aligned} K_W &= 2N - \frac{3}{2} + \frac{3}{2}w(x), \\ K_{33} &= 2N - \frac{3}{4} + \left(\frac{3}{4} - \frac{3x}{8}\right) r, \\ K_{YY} &= \frac{10N}{3} - \frac{17}{12} + \left(\frac{17}{12} - \frac{3x}{8}\right) r, \\ K_{3Y} &= -\frac{1}{4} + \left(\frac{1}{4} + \frac{3x}{8}\right) r. \end{aligned}$$

The neutral action contains $K_{33}F_3^2 + 2K_{3Y}F_3F_Y + K_{YY}F_Y^2$.

For completeness, a Dirac species of mass ratio squared x, with neutral charges q_L^a, q_R^a, contributes

$$e^{-x}\left[q_L^a q_L^b + q_R^a q_R^b - \frac{x}{2}(q_L^a - q_R^a)(q_L^b - q_R^b)\right].$$

Applying this formula to the three up colors proves the neutral entries above. The commutator term is responsible for the terms proportional to xr.

At $x = 0$,

$$K_W = K_{33} = 2N, \qquad K_{YY} = 10N/3, \qquad K_{3Y} = 0,$$

so the usual matching angle is 5/8.

For $x > 0$, generally $K_W \neq K_{33}$ and $K_{3Y} \neq 0$. Consequently, the usual electroweak relations no longer define one common weak angle. Two precise continuations are:

1. **Continue the charged-W/hypercharge matching prescription.** Define

$$\boxed{\cos^2\theta_{\text{match}}(x) := \frac{K_{YY}}{K_W + K_{YY}} = \frac{80N - 34 + (34 - 9x)e^{-x}}{128N - 70 + (34 - 9x)e^{-x} + 36w(x)}.}$$

   Then

$$\cos^2\theta_{\text{match}}(0) = \frac{5}{8}, \qquad \cos^2\theta_{\text{match}}(\infty) = \begin{cases} 23/29, & N = 1, \\ 103/157, & N = 3. \end{cases}$$

   This prescription alone does not diagonalize the neutral kinetic terms.

2. **Define the angle from the neutral current.** When the neutral mass term has the Standard Model direction $(W_3 - B)^2$, the photon direction is $(1, 1)$. Its kinetic-orthogonal massive direction is proportional to

$$(K_{YY} + K_{3Y},\ -K_{33} - K_{3Y}).$$

Writing its current as $T_3 - \sin^2\theta_{\mathrm{NC}} Q$ gives

$$\cos^2\theta_{\mathrm{NC}} = \frac{K_{YY} + K_{3Y}}{K_{33} + 2K_{3Y} + K_{YY}}.$$

Here

$$K_{33} + K_{3Y} = 2N - 1 + r, \qquad K_{YY} + K_{3Y} = \frac{5}{3}(2N - 1 + r),$$

and therefore

$$\boxed{\cos^2\theta_{\mathrm{NC}}(x) = 5/8.}$$

This equality is special to the chosen heavy-up family. It does not restore a common $SU(2)$ kinetic coefficient or the usual relations among charged and neutral couplings.

As a consistency check,

$$K_{33} + 2K_{3Y} + K_{YY} = \frac{8}{3}(2N - 1 + e^{-x}),$$

exactly reproducing Part B's photon coefficient on the particle space.

Verdict on corpus claim.

Item 1 states: "The centralizer and compressed resolvent do not determine the surviving gauge kinetic terms." The calculation confirms that qualification.

- **The full inherited kinetic form survives** the joint limit $m_h \to \infty$, $\Lambda \to \infty$, $m_h/\Lambda \to 0$.
- **Finite nonzero ratios produce threshold weights.** When $m_h/\Lambda \to \infty$, the residual kinetic form is the light-sector trace. For the specified up-quark removals, photon and gluon coefficients retain their ratio but acquire the common reductions $1/2$ or $5/6$.
-

The corpus's $5/8$ is a valid electroweak matching condition when the full module is light relative to the matching cutoff, or when that condition is imposed before taking the heavy limit. The broken action does not retain one universal weak angle. Its neutral-current angle happens to remain $5/8$ in this family, whereas the continued charged-W/hypercharge matching ratio changes as above. At the strict $SU(3)\times U(1)_{\mathrm{em}}$ endpoint, a weak angle requires reference to such an earlier matching prescription.

Independent checks (Claude, 2 Nov 2026)

- Part A: (1/12) tr_S 1 + (1/2)(1/4) tr_S(gamma^{mu nu} gamma^{rho sigma}) F F gives (1/3 - 1) tr F^2 = -(2/3) tr F^2. Dividing by 16 pi^2 gives -1/(24 pi^2), the standard spectral-action Yang-Mills coefficient.
- Part B: the per-generation traces tr Q^2 = 8/3 (u), 2/3 (d), 2 (e) and color index 1, 1, 0 were rechecked by hand.
- Part C: w_L(x, 0) = 3 int_0^1 u^2 e^{-xu} du = 3[2 - (x^2+2x+2) e^{-x}]/x^3 agrees by quadrature to 1e-10 at x = 0, 0.3, 1, 4, 12. The limits cos^2 theta_match(infinity) = (80N-34)/(128N-70) give 23/29 (N=1) and 103/157 (N=3). The identity (K_YY + K_3Y) = (5/3)(K_33 + K_3Y), hence cos^2 theta_NC = 5/8 for every x, was verified algebraically.

Part 05

The geometric identification exists: the minimal grading-preserving spectral truncation of a suitable chiral interval is exactly Connes–Lott, with

$$|m| = \frac{\pi}{2l}, \qquad d_{\text{trunc}} = \frac{2l}{\pi}.$$

The truncated endpoint states are smeared states on the original interval. Moreover, this interval has no zero mode, which limits the physical interpretation of both extremes.

Part 1. Deconstructed chain

Statement. There is a natural family with opposite endpoint chiralities, an exact Connes–Lott member at $N = 2$, and endpoint distance converging to l. The answer depends on the representation of the site algebra; the familiar lattice-distance anomaly is not universal. (Parts 1 and 2 were refereed by Claude Opus 5.5 on 5 November 2026, report in historical/reviews/2026-11-05_opus_check_05_21_34.md: correct as written, every formula recomputed, with the scope of the three convergence statements made explicit at the end of the Statement of Part 2.)

Hypotheses. We consider complex even spectral triples, without imposing additional real-structure or KO-dimension axioms. Let $N \geq 2$, $n = N - 1$, and

$$A_N = \mathbb{C}^N, \qquad H_N = \text{span}\{L_1, \dots, L_{N-1}, R_2, \dots, R_N\}.$$

Thus interior sites carry both components, site 1 only L, and site N only R. Define

$$\pi_N(a) = \text{diag}(a_1, \dots, a_{N-1}, a_2, \dots, a_N), \qquad \Gamma_N = \text{diag}(I_n, -I_n).$$

If S_n is the lower shift, set

$$B_N = h_N(I_n - S_n), \qquad D_N = \begin{pmatrix} 0 & B_N \\ B_N^* & 0 \end{pmatrix}.$$

These are forward/backward differences:

$$(D_N\psi)_{L_i} = h_N(R_{i+1} - R_i), \qquad (D_N\psi)_{R_j} = h_N(L_{j-1} - L_j),$$

with the missing components interpreted as

$$R_1 = 0, \qquad L_N = 0.$$

There are nearest-neighbour couplings and onsite L-R terms at interior sites.

For compatibility with the spectral truncation in Part 3, choose

$$\theta_N = \frac{\pi}{4N - 2}, \qquad h_N = \frac{\pi}{4l \sin\theta_N} \sim \frac{N}{l}. \tag{1}$$

This fixes the smallest positive eigenvalue to $\pi/(2l)$ at every N.

Proof.

The representation is faithful, $[\Gamma_N, \pi_N(a)] = 0$, and $\Gamma_N D_N = -D_N\Gamma_N$. At $N = 2$,

$$A_2 = \mathbb{C}^2, \quad H_2 = \mathbb{C}^2, \quad D_2 = \begin{pmatrix} 0 & h_2 \\ h_2 & 0 \end{pmatrix}, \quad h_2 = \frac{\pi}{2l}.$$

This is exactly the stated two-point triple; a diagonal unitary supplies any desired phase of m.

Ordering the basis as

$$L_1, R_2, L_2, R_3, \dots, L_{N-1}, R_N$$

makes D_N a path adjacency matrix with alternating signs. A diagonal unitary removes those signs. The usual sine eigenvectors of a path therefore give

$$\operatorname{spec}(D_N) = \left\{ 2h_N \cos\frac{j\pi}{2N-1} : 1 \le j \le 2N-2 \right\} = \{\pm\lambda_{N,r} : 1 \le r \le N-1\},$$

where

$$\boxed{\lambda_{N,r} = 2h_N \sin((2r-1)\theta_N).} \tag{2}$$

Consequently,

$$\boxed{K_N(t) = \operatorname{Tr} e^{-tD_N^2} = 2\sum_{r=1}^{N-1} e^{-4th_N^2 \sin^2((2r-1)\theta_N)}.} \tag{3}$$

For a real site function a, the onsite terms commute with $\pi_N(a)$, while

$$B_N\pi_R(a) - \pi_L(a)B_N = h_N \operatorname{diag}(a_2 - a_1, \dots, a_N - a_{N-1}).$$

Hence

$$\|[D_N, \pi_N(a)]\| = h_N \max_i |a_{i+1} - a_i|.$$

The triangle inequality bounds $|a_j - a_i|$ by $|j-i|/h_N$; the function $a_i = (i-1)/h_N$ attains this bound. Thus

$$\boxed{d_N(i,j) = \frac{|i-j|}{h_N}.} \tag{4}$$

In particular,

$$\boxed{d_N(1,N) = \frac{4l(N-1)}{\pi}\sin\frac{\pi}{4N-2} \longrightarrow l.} \tag{5}$$

One can instead choose the geometric normalization

$$h_N = \frac{N-1}{l}. \tag{6}$$

Then $d_N(1,N) = l$ exactly for every N, with the same continuum limit. However, its $N = 2$ mass is $1/l$, rather than the continuum lowest eigenvalue $\pi/(2l)$. Equations (1) and (6) distinguish spectral calibration from metric calibration at finite resolution.

Two other natural choices clarify the literature:

- **One-component staggered path.** Take $A = \mathbb{C}^M$, $H = \mathbb{C}^M$, alternating grading, and $D = h$ times path adjacency. Opposite endpoint chiralities require even M. Its spectrum is $2h\cos(j\pi/(M+1))$. For $k = |i-j|$,

$$\boxed{d(i,j) = \begin{cases} (k+1)/(2h), & k \text{ odd}, \\ \sqrt{k(k+2)}/(2h), & k \text{ even}. \end{cases}} \tag{7}$$

  In particular $d(1,M) = M/(2h)$. Thus $h = M/(2l)$ gives endpoint distance l, whereas $h = M/l$ gives $l/2$.

  Here is a direct proof of (7). Restrict the commutator to the connecting subpath and extend the function constantly outside it. After removing phases, its norm is that of weighted adjacency, with dimensionless edge weights $w_i = h|\Delta a_i|$. If its two bipartition sizes are p, q, the off-diagonal block C satisfies

$$\sum_i w_i = \mathbf{1}_p^T C \mathbf{1}_q \le \sqrt{pq}\,\|C\|.$$

For $p = q$, alternating weights $1, 0, 1, \dots, 1$ attain the bound. For $p = q + 1$, the weights

$$\frac{(q, 1, q-1, 2, \dots, 1, q)}{\sqrt{q(q+1)}}$$

attain it: the normalized constant vectors on the two bipartitions are singular vectors with singular value 1, and positivity gives the norm 1. This proves (7).

These are the nonadditive distances studied by Bimonte–Lizzi–Sparano, §5 and given explicitly by Atzmon, Eq. (30). Reduction to the connecting path also agrees with Iochum–Krajewski–Martinetti, Proposition 4.

- **Vertex-edge incidence Dirac.** Let $\partial : \mathbb{C}^N \to \mathbb{C}^{N-1}$ be the path difference matrix and use

$$D_{\text{inc}} = h \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix}.$$

Its spectrum and heat trace are

$$\{0\} \cup \left\{ \pm 2h \sin \frac{k\pi}{2N} : 1 \le k < N \right\},$$

$$K_{\text{inc}}(t) = 1 + 2 \sum_{k=1}^{N-1} e^{-4th^2 \sin^2(k\pi/(2N))}.$$

These follow by substituting the cosine eigenvectors into the path Laplacian $\partial^*\partial$. Representing a site function on each edge by its value at the tail gives distance $|i-j|/h$. But H has dimension $2N-1$, has a zero mode, and both endpoint vertex components have the same grading. At $N = 2$, its spectrum is $0, \pm\sqrt{2}h$, so it is not Connes–Lott.

Thus the quoted integer deconstruction tower belongs to the incidence construction. Opposite chiral boundary conditions produce the shifted tower (2).

Status: PROVED

Part 2. Continuum limit

Statement. The chain above converges to an interval Dirac operator in three precise senses: ordered low eigenvalues, heat traces locally uniformly for $t > 0$, and pure-state metric spaces in Gromov–Hausdorff distance. It also has a controlled one-dimensional heat window. Here spectral convergence means that each eigenvalue lambda_{N,r} increases to (2r-1) m_1 at fixed r, so that the counting function on [-E, E] converges for every E outside spec(D_l); the Hilbert spaces differ and no resolvent convergence is asserted. The heat traces converge uniformly on [eps, infinity) for every eps > 0. The Gromov-Hausdorff statement concerns the pure-state spaces with their Connes metrics, at distance at most l/(N-1) (referee, 5 November 2026).

Hypotheses. On $H_l = L^2([0, l], \mathbb{C}^2)$, define

$$D_l = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix}, \qquad \operatorname{Dom} D_l = \{(L, R) \in H^1 : R(0) = 0,\ L(l) = 0\}, \tag{8}$$

with scalar multiplication by $C^\infty([0, l])$ and grading $\Gamma = \operatorname{diag}(1, -1)$.

These are opposite local chiral conditions. Reflection parity is Γ at 0 and $-\Gamma$ at l; their composition gives antiperiodicity on the doubled interval.

Proof.

Integration by parts gives the boundary form

$$[\overline{L}\,\widetilde{R} - \overline{R}\,\widetilde{L}]_0^l.$$

The conditions in (8) form a maximal isotropic boundary subspace, so D_l is self-adjoint. Its resolvent is compact by the compact embedding $H^1 \hookrightarrow L^2$.

Solving $D_l\psi = \lambda\psi$ gives

$$\psi_\lambda(x) = c(\cos\lambda x, \sin\lambda x), \qquad \cos(\lambda l) = 0.$$

Therefore

$$\boxed{\operatorname{spec}(D_l) = \left\{ \pm\frac{(r-\frac{1}{2})\pi}{l} : r = 1, 2, ... \right\}.} \tag{9}$$

The kernel is zero. Squaring gives mixed Neumann–Dirichlet conditions on L, and Dirichlet–Neumann conditions on R.

For scalar f,

$$\|[D_l, f]\| = \|f'\|_\infty, \qquad d_l(x, y) = |x - y|. \tag{10}$$

Under normalization (1), with $m_1 = \pi/(2l)$,

$$\lambda_{N,r} = m_1 \frac{\sin((2r-1)\theta_N)}{\sin\theta_N} \longrightarrow (2r-1)m_1. \tag{11}$$

This proves low-eigenvalue convergence. Since

$$\sin y \geq \frac{2y}{\pi} \quad (0 \leq y \leq \pi/2),$$

we also have

$$\lambda_{N,r} \geq \frac{2r-1}{l}. \tag{12}$$

A summable Gaussian majorant now proves, uniformly on every $t \in [\varepsilon, T]$,

$$K_N(t) \longrightarrow K_l(t) = 2\sum_{r=1}^{\infty} e^{-t((r-\frac{1}{2})\pi/l)^2}. \tag{13}$$

Poisson summation gives

$$\boxed{K_l(t) = \frac{l}{\sqrt{\pi t}}\left(1 + 2\sum_{q=1}^{\infty}(-1)^q e^{-q^2l^2/t}\right).} \tag{14}$$

In particular, $K_l(t) \sim l/\sqrt{\pi t}$ as $t/l^2 \to 0$.

The finite-chain heat window has the following rigorous meaning. For every sequence $t_N > 0$ satisfying

$$\frac{t_N}{l^2} \to 0, \qquad \frac{N^2 t_N}{l^2} \to \infty, \tag{15}$$

one has

$$\boxed{K_N(t_N) \sim \frac{l}{\sqrt{\pi t_N}}, \qquad -2t_N\frac{K_N'(t_N)}{K_N(t_N)} \to 1.} \tag{16}$$

To prove this, put $u_r = \pi\sqrt{t_N}(r - \frac{1}{2})/l$. On bounded u_r-intervals, the sine argument in (11) tends uniformly to zero, so $\sqrt{t_N}\lambda_{N,r}/u_r \to 1$. The mesh tends to zero and its upper endpoint tends to infinity. Bound (12) controls the tails. Gaussian Riemann sums, and then the same sums weighted by $t_N\lambda_{N,r}^2$, prove both assertions.

Finally, identify site i with $x_i = (i-1)l/(N-1)$. Equation (4) gives

$$d_N(i, j) = \rho_N|x_i - x_j|, \qquad \rho_N = \frac{N-1}{lh_N} = 1 + O(N^{-1}).$$

The metric distortion and the grid mesh are both $O(l/N)$, proving Gromov–Hausdorff convergence to $[0, l]$.

The endpoint chirality here concerns allowed boundary components. It does not supply localized chiral zero modes.

Status: PROVED

Part 3. Minimal spectral truncation

Statement. The minimal nonzero grading-preserving spectral truncation of (8) is unitarily equivalent to Connes–Lott, with

$$\boxed{|m| = m_1 = \frac{\pi}{2l}, \qquad c = \frac{\pi}{2}.}$$

Its natural truncated endpoint states have distance $1/m_1 = 2l/\pi$. Literal endpoint evaluations do not descend through compression.

Hypotheses. Let

$$P = \mathbf{1}_{[0,E]}(|D_l|), \qquad m_1 \le E < 3m_1,$$

and give $S = P\,C([0, l])\,P$ its inherited operator-system structure. This is the compression construction described in Connes–van Suijlekom, §1.

Proof.

An orthonormal chiral basis of PH_l is

$$u_L(x) = \sqrt{\frac{2}{l}} \begin{pmatrix} \cos(\pi x/(2l)) \\ 0 \end{pmatrix}, \qquad u_R(x) = \sqrt{\frac{2}{l}} \begin{pmatrix} 0 \\ \sin(\pi x/(2l)) \end{pmatrix}.$$

In this basis,

$$PD_lP = \begin{pmatrix} 0 & m_1 \\ m_1 & 0 \end{pmatrix}, \qquad P\Gamma P = \mathrm{diag}(1, -1). \tag{17}$$

Compression of multiplication is

$$Q(f) = \begin{pmatrix} \dfrac{2}{l}\displaystyle\int_0^l f(x)\cos^2\frac{\pi x}{2l}\,dx & 0 \\ 0 & \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin^2\frac{\pi x}{2l}\,dx \end{pmatrix}. \tag{18}$$

Since

$$Q(1) = I, \qquad Q(\cos(\pi x/l)) = \tfrac{1}{2}\,\mathrm{diag}(1, -1),$$

its range is exactly the full diagonal algebra:

$$\boxed{S = \mathbb{C} \oplus \mathbb{C}.} \tag{19}$$

Here the compressed operator system happens to be an algebra, although Q is not multiplicative.

Evaluation on the finite-dimensional space PH_l is bounded. At 0, only u_L has a nonzero boundary value; at l, only u_R does. The normalized evaluation vectors therefore define

$$\omega_0(\mathrm{diag}(a, b)) = a, \qquad \omega_l(\mathrm{diag}(a, b)) = b.$$

Using (17),

$$\|[PD_lP, \mathrm{diag}(a, b)]\| = m_1|a - b|,$$

and hence

$$\boxed{d_{\mathrm{trunc}}(\omega_0, \omega_l) = \frac{1}{m_1} = \frac{2l}{\pi} < l.} \tag{20}$$

Minimality follows because there is no zero mode and the grading exchanges the eigenspaces at $+m_1$ and $-m_1$. A nonzero grading-preserving spectral subspace must retain both.

There is an essential distinction between these states and literal evaluations. For

$$g(x) = \cos(3\pi x/l),$$

equation (18) gives $Q(g) = 0$, while $g(0) = 1$ and $g(l) = -1$. Thus neither endpoint evaluation factors through Q.

The pullbacks of the truncated endpoint states are instead the probability measures

$$d\mu_L = \frac{2}{l}\cos^2\frac{\pi x}{2l}\,dx, \qquad d\mu_R = \frac{2}{l}\sin^2\frac{\pi x}{2l}\,dx.$$

Their distance using the **original** continuum seminorm is

$$d_{D_l}(\mu_L, \mu_R) = \int_0^l \left| \int_0^x (d\mu_L - d\mu_R) \right| dx = \frac{4l}{\pi^2}. \tag{21}$$

The equality follows by integration by parts and is attained by $f = -x$. Equations (10), (20), and (21) concern three different state/metric comparisons.

Status: PROVED

Part 4. The two limits

Amendment (Terra 6.1, 3 Nov 2026; report). The statements of this part are correct for the stated bare chain, with one precision: the condition N/l -> infinity gives uniform convergence of the dispersion on bounded energy windows and of normalized heat traces and counting densities, and does not by itself give convergence of the unnormalized difference of heat traces (for N = floor(l^(5/4)) it diverges). Escapes for the empty light sector: a tuned bulk mass or a scaled boundary coupling.

Statement. At fixed finite rank, $m \to 0$ produces infinitely separated finite geometry, with heat dimension zero. A resolved interval requires a growing tower. As $m \to \infty$, the metric collapses, but the light Hilbert space is zero.

Hypotheses. Identify

$$m = m_1 = \frac{\pi}{2l},$$

use normalization (1), and hold physical energies or heat times fixed unless stated otherwise.

Proof.

For fixed N, $D_N = mT_N$ with T_N independent of m. Therefore, as $m \to 0$,

$$D_N \to 0, \qquad K_N(t) \to 2(N-1), \qquad d_N(i,j) \to \infty \quad (i \neq j).$$

For the minimal truncation,

$$K_2(t) = 2e^{-tm^2}.$$

Every finite N has ultraviolet heat dimension zero. Making its finitely many modes light does not change that conclusion.

For the full interval, a fixed energy E is in the resolved one-dimensional regime when

$$\frac{1}{l} \ll E.$$

For its finite-chain approximation, one additionally needs

$$\boxed{\frac{1}{l} \ll E \ll \frac{N}{l}.} \tag{22}$$

Thus, as $l \to \infty$, an exact continuum limit throughout a fixed positive energy window follows if $N/l \to \infty$. Equivalently, as $m \to 0$, one needs $Nm \to \infty$.

The continuum counting function and heat density make the emergence explicit:

$$\#\{|\lambda| \le E\} = 2\left\lfloor \frac{lE}{\pi} + \frac{1}{2} \right\rfloor, \qquad \frac{K_l(t)}{l} \longrightarrow \frac{1}{\sqrt{\pi t}} \quad (l \to \infty). \tag{23}$$

Based at an endpoint, the metric limit is a half-line; the opposite endpoint recedes to infinity.

The order of limits matters even for the normalized heat trace:

$$\lim_{l\to\infty}\lim_{N\to\infty} \frac{K_N(t)}{l} = \frac{1}{\sqrt{\pi t}}, \qquad \lim_{N\to\infty}\lim_{l\to\infty} \frac{K_N(t)}{l} = 0. \tag{24}$$

Likewise, taking $t \to 0$ at fixed N gives dimension zero, whereas taking $N \to \infty$ first gives the interval's dimension one.

As $m \to \infty$, all distances tend to zero. At fixed N, the algebra elements with uniformly bounded commutators reduce to constants. But every eigenvalue has magnitude at least m, so

$$\|(D_N - i)^{-1}\| \le \frac{1}{\sqrt{m^2+1}} \longrightarrow 0. \tag{25}$$

The same holds for D_l, after identifying the varying intervals by dilation. Its heat trace tends to zero at every fixed $t > 0$.

Thus the metric limit is a point, while the embedded resolvent limit has

$$H_{\text{light}} = \ker T_N = 0.$$

There is no finite operator limit on the original nonzero Hilbert space, and no surviving fermion pair at fixed energy.

Finally, the two-state spectral truncation exists for

$$m_1 \le E < 3m_1.$$

A genuine spectral cutoff $E \ll 1/l$ retains no modes. The ratio between the first two positive levels is always 3, so the minimal pair has no parametrically large separation from the next level.

Status: PROVED

Verdict on hypothesis H (theorem T1)

Supported for T1's geometric assertion. Connes–Lott is exactly the minimal nonzero grading-preserving spectral compression of the opposite-chirality interval above, with

$$l = \frac{\pi}{2|m|}.$$

The chain recovers its spectrum, heat window, and metric.

The precise replacement for the stronger interpretation of H is:

> The two sheets are truncated endpoint evaluation states, whose pullbacks are smeared over the interval. Refinement restores the interval. Decompactification produces a one-dimensional heat regime only when the number of retained modes grows sufficiently rapidly. Collapse identifies the metric points but removes all fermionic modes of this particular interval at fixed energy.

Accordingly, a nonempty two-state sector at $E \ll 1/l$, or surviving light fermions after collapse, is disproved by this model. The construction supplies no KO-6 or Standard Model identification.

Independent numerical check (Claude, 2 Nov 2026)

For l = 1 and N = 2, 3, 5, 9 we checked the following. The spectrum of D_N agrees with +/- 2 h_N sin((2r-1) theta_N) to 1e-14. The lowest positive eigenvalue equals pi/(2l) for every N. For random real site functions, ||[D_N, pi_N(a)]|| equals h_N max abs(a_{i+1} - a_i), which is the identity behind d_N(i,j) = abs(i-j)/h_N. The endpoint distance d_N(1,N) = 0.6366, 0.7869, 0.8844, 0.9398 tends to l. For the minimal truncation, Q(cos(3 pi x/l)) = 0, and the smeared-state distance 0.405290 matches 4l/pi^2. The truncated distance is 2l/pi = 0.6366, identical to d_2(1,2).

Literature (../LITERATURE.md, section A). The Kaluza-Klein reading of the two-point space is in Connes-Lott (sections I and III: "two copies of Z separated by a distance L"); the two points as points of a fifth dimension are in Viet-Wali 1996; the two-point model as an approximation of a manifold, to be related to dimensional reduction, is in Dimakis-Muller-Hoissen 1994 (section III, Example 4, and section VII). Connes-van Suijlekom (section 1) note that compressions P A P are operator systems, of which the chiral interval is an exception. The truncation theorem with |m| = pi/(2l) was not found in print.

Part 06

The proposed uniqueness is false: **the same KO-6 triple can admit both reductions**. Moreover, under the stated definition, the strict heavy-limit triple is a doubling of KO-7, although it is not a doubling of KO-5. The distinction between the full real Hilbert space and its physical particle subspace is essential.

1. Lemma — Product convention and necessary correction of the "Clifford point."

Under ordinary spectral-triple conventions, a graded triple on `C^2` is not itself an odd KO-1 or KO-7 triple. The appropriate interpretation is an odd point

$$P_k = (\mathbb{C}, \mathbb{C}, 0, K), \qquad k = 1 \text{ or } 7,$$

together with the auxiliary `C^2` introduced by the odd-times-odd product. For real algebras, use the point algebra `R`, so that the product algebra remains `A`.

I use the following products, with the standard antilinear tensor product of real structures.

For even × even,

$$D = D_1 \otimes 1 + \gamma_1 \otimes D_2, \qquad \gamma = \gamma_1 \otimes \gamma_2,$$
$$J = J_1 \otimes (J_2 \gamma_2^b), \qquad b = (1 - \epsilon_1'')/2.$$

For even × odd, use the same formula for `D`, no grading, and

$$J = (J_1 \gamma_1^a) \otimes J_2, \qquad (-1)^a = \epsilon_1'' \epsilon_2'.$$

Odd × even is obtained by transporting the even-first construction through the tensor flip.

For odd × odd,

$$H = H_1 \otimes H_2 \otimes \mathbb{C}^2, \quad D = D_1 \otimes 1 \otimes \sigma_1 + 1 \otimes D_2 \otimes \sigma_2, \quad \gamma = 1 \otimes 1 \otimes \sigma_3,$$

$$J = J_1 \otimes J_2 \otimes (BK),$$

where

$(\epsilon_1', \epsilon_2')$	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
B	σ_1	1	σ_3	$-i\sigma_2$.

This agrees, up to harmless phases, with Dąbrowski–Dossena, equation (25), Table 6.

Proof. In the odd-odd case,

$$B\sigma_1 = \epsilon'_1\sigma_1 B, \qquad B\sigma_2 = -\epsilon'_2\sigma_2 B.$$

The second minus sign compensates for complex conjugation of `sigma_2`. Consequently `JD=DJ`; computing `J^2` and `J gamma` gives the signs of the sum degree. The grading twists in the other two formulas follow by the same calculation. Order zero and first order pass to these products.

Thus the intended doubling has Hilbert space `H_0 ⊗ C^2`, with the displayed grading on the auxiliary factor. All subsequent statements use this corrected interpretation and the zero-Dirac point. ■

2. Theorem — Exact doubling criterion in KO-dimension six.

Let `(A,H,D,J,gamma)` be a finite KO-6 triple. For `s=±1`, the following are equivalent:

1. It is a product `T_n × P_k`, where

$$(s, n, k) = (+1, 7, 7) \quad \text{or} \quad (-1, 5, 1).$$

2. There is a complex-linear unitary `U` satisfying

$$U^2 = 1, \quad U\gamma = -\gamma U, \quad [U, a] = [U, a^o] = [U, D] = 0, \quad UJ = sJU,$$

   where `a^o=Ja^\*J^{-1}`.

Write `H_L=H_{\gamma=+1}`, `H_R=H_{\gamma=-1}`, and

$$D = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}.$$

Equivalently, there is an `A`-bimodule unitary `V:H_L→H_R` such that

$$V^*M = (V^*M)^*, \qquad (V^*C)^2 = s, \quad C = J|_{H_L} : H_L \longrightarrow H_R.$$

Here `C` and `V^\*C` are antilinear.

Proof. Given `U`, identify

$$H_L \oplus H_L \longrightarrow H, \qquad (x, y) \longmapsto x + Uy.$$

Put

$$H_0 = H_L, \qquad d = (UD)|_{H_L}, \qquad j = (UJ)|_{H_L}.$$

The assumptions give

$$d = d^*, \qquad j^2 = s, \qquad jd = s\,dj.$$

Hence `(j^2,jd/dj)` is `(+,+)` for KO-7 and `(-,-)` for KO-5. In these coordinates,

$$a = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$J(x, y) = (sjy, jx).$$

These are precisely the products in Result 1: `B=σ_1` for `(7,7)` and `B=-iσ_2` for `(5,1)`.

The descended opposite action is

$$ja_0^*j^{-1} = a^o|_{H_L}.$$

Since `[d,a_0]=U[D,a]|_{H_L}`, order zero and first order descend. Conversely, the product has `U=1⊗σ_1`, with all the required properties.

Finally,

$$U = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}.$$

Its commutation with `D` is equivalent to self-adjointness of `V^\*M`, while `UJ=sJU` is equivalent to `(V^\*C)^2=s`.

If one prefers an anticommuting convention, replacing `U` by `i gamma U` exchanges `[U,D]=0` with `{U,D}=0` and preserves `s`. ■

In particular, ordinary equality of left and right module multiplicities is only part of the criterion; compatibility with `D` and `J` must also be checked.

3. Counterexample — J does not select a unique reduction; electrodynamics admits both.

Consider the electrodynamics triple in the basis `(e_L,e_R,bar e_L,bar e_R)`. After a phase change compatible with `A,J,gamma`, its matrices are

$$\rho(a,b) = \mathrm{diag}(a,a,b,b), \qquad \gamma = \mathrm{diag}(1,-1,-1,1),$$

$$J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} K, \qquad D = m \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad m \geq 0.$$

These are the input data of van den Dungen–van Suijlekom, §4.1, equations (4.1)–(4.2).

Define

$$U_+ = \mathrm{diag}(\sigma_1, \sigma_1), \qquad U_- = \mathrm{diag}(\sigma_1, -\sigma_1).$$

Proof. Both involutions reverse the grading and commute with the algebra, opposite algebra, and `D`. Direct multiplication gives

$$U_+ J = J U_+, \qquad U_- J = -J U_-.$$

Thus the same triple, with the same `J`, admits both factorizations.

On `H_0=span(e_L,bar e_R)`, the two descended triples have

$$d_+ = mI_2, \qquad j_+ = \sigma_1 K$$

and

$$d_- = \mathrm{diag}(m,-m), \qquad j_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K.$$

They have KO-degrees seven and five, respectively. This also holds at `m=0`. ■

4. Lemma — Exact neutral-sector criterion, including Majorana masses.

Suppose a KO-6 sector has

$$H = H_+ \oplus H_-, \qquad \dim H_+ = \dim H_- = r,$$

and both algebra actions are the same scalar character throughout this sector. Then:

- a KO-7 reduction always exists;
- a KO-5 reduction exists exactly when **every singular value of `M`, including zero, has even multiplicity**.

Proof. Choose an orthonormal basis of `H_+` and its `J`-image as the basis of `H_-`. Then

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} K.$$

The equation `JD=DJ` says `M^T=M`. Takagi factorization, implemented by a basis change preserving this form of `J`, reduces `M` to

$$\Sigma = \mathrm{diag}(\mu_1, \ldots, \mu_r), \qquad \mu_i \geq 0.$$

Taking `V=I` satisfies Result 2 with `s=+1`.

For `s=-1`, the descended antiunitary `j` squares to `-1` and commutes with `d^2=D^2|_{H_+}`. Every eigenspace of `d^2` must therefore have even complex dimension. These dimensions are exactly the singular-value multiplicities.

Conversely, pair equal entries of `Sigma` and choose `V=σ_2` on each pair. Then `V` is a skew-symmetric Hermitian unitary commuting with `Sigma`; it satisfies the criterion with `s=-1`. ■

A single neutral Weyl state and its conjugate therefore **are** a KO-7 doubling, even without a Dirac partner. They cannot be a KO-5 doubling. The same statement holds for a single Majorana pair, whether its mass vanishes or not.

5. Proposition — The strict one-generation heavy limit is a KO-7 doubling, and KO-5 is obstructed.

Here is an explicit convention for the Standard Model representation. Set

$$F = \mathbb{C} \oplus \mathbb{C}^3, \qquad H_p = (\mathbb{C}^2 \otimes F)_L \oplus (\mathbb{C}^2 \otimes F)_R, \qquad H = H_p \oplus H_{\bar{p}}.$$

For `a=(lambda,q,m)`, put

$$Q_\lambda = \mathrm{diag}(\lambda, \bar{\lambda}), \qquad B_{\lambda,m} = \mathrm{diag}(\lambda, m),$$
$$\rho_p(a) = (q \otimes I_4) \oplus (Q_\lambda \otimes I_4),$$
$$\rho_{\bar{p}}(a) = (I_2 \otimes B_{\lambda,m}) \oplus (I_2 \otimes B_{\lambda,m}).$$

Use

$$J(p, \bar{p}) = (K\bar{p}, Kp), \qquad \gamma = \mathrm{diag}(I_8, -I_8, -I_8, I_8).$$

On particles, the Dirac block is

$$S_p = \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix},$$

where `Y` has entries `y_nu,y_e,y_u I_3,y_d I_3` on the corresponding species. The antiparticle block is `bar S_p`; the additional Majorana block connects `nu_R` with `J nu_R`. This realizes the bimodule and mass-block structure in CCM, Proposition 2.4 and Theorem 2.21.

Assume precisely the strict projection described in canonical §3: remove `nu_R,J nu_R` and the complete up-type Dirac sector, and restrict to

$$B = \{(\lambda, Q_\lambda, m)\} \simeq \mathbb{C} \oplus M_3(\mathbb{C}).$$

The surviving space consists of `d_L,d_R,e_L,e_R,nu_L` and their conjugates, with `D_light=PSP`.

Proof. The surviving down-quark and electron masses pair identical `B`-bimodules. After making their masses nonnegative, each sector has both involutions from Result 3, with the identity on color.

On

$$N = \mathrm{span}(\nu_L, J\nu_L),$$

both algebra actions are the scalar character `lambda`, and `PSP|_N=0`. Thus

$$\gamma_N = \sigma_3, \qquad J_N = \sigma_1 K, \qquad U_N = \sigma_1$$

gives a KO-7 reduction. Combining this with `U_+` on the charged sectors gives a global KO-7 reduction.

A KO-5 reduction is impossible: the full space has dimension eighteen, so its descended space would have dimension nine, which cannot carry an antiunitary squaring to `-1`. More specifically, the neutral bimodule is inequivalent to the charged bimodules, so any permitted `U` preserves it; Result 4 already obstructs KO-5 there. ■

Thus the anticipated claim "the neutral sector prevents any doubling" is false under the definition in Question 1.

6. Proposition — Modifications and the distinction from physical vectorlikeness.

For the reduced algebra `B` above, the following conclusions hold.

Surviving neutral content	Odd reductions of the full light triple
No neutral sector	Both KO-5 and KO-7
`nu_L,nu_R` and conjugates, pure Dirac mass	Both KO-5 and KO-7
Only `nu_L,J nu_L`	KO-7 only
A single isolated Majorana pair	KO-7 only
General neutral Dirac–Majorana block	Always KO-7; KO-5 exactly under Result 4

Proof. The charged sectors admit both reductions by Result 3. Pure Dirac neutrinos have the same four-state construction, so adding or retaining `nu_R` restores both possibilities, including when its Dirac mass is zero.

For one generation with Dirac mass `m` and right Majorana mass `R`, the neutral symmetric matrix can be written

$$M_N = \begin{pmatrix} 0 & m \\ m & R \end{pmatrix}.$$

Phase changes make `m,R` nonnegative. Its singular values are

$$\mu_\pm = \frac{\sqrt{|R|^2 + 4|m|^2} \pm |R|}{2}.$$

They coincide exactly when `R=0`. Hence any nonzero right Majorana mass obstructs KO-5 in this one-generation neutral sector, but never KO-7. Treating that sector separately leaves a charged KO-5 reduction; it does not turn the complete direct sum into a KO-5 reduction. ■

There is an additional physically meaningful condition absent from Question 1. Let `Q` be `+1` on particles and `-1` on antiparticles.

No-go theorem under the extra hypothesis `[U,Q]=0`. An unpaired `nu_L` prevents doubling.

Proof. Such a `U` must map particle states of positive grading to particle states of negative grading within the same bimodule. No particle `nu_R` exists. The neutral involution used in Result 5 instead maps `nu_L` to its conjugate and violates `[U,Q]=0`. ■

This is why physical chirality and doubling of the complete real triple are different tests.

7. Proposition — Three generations and arbitrary flavor mixing.

Assume the reduced algebra acts trivially on generation indices.

If all three right-neutrino Majorana pairs are removed and only the top Dirac sector is made heavy, the light space has

$$96 - 6 - 12 = 78$$

complex dimensions. If all three up-type sectors are removed, it has dimension fifty-four. In either case, with three surviving left neutrinos and `D_light=PSP`, the light triple admits KO-7 and excludes KO-5.

Dropping the remaining neutral sector, or retaining three right neutrinos with purely Dirac neutrino masses, permits both reductions.

Proof. Each surviving charged species has equal left/right flavor multiplicities. Singular-value decomposition of its mass matrix uses bimodule unitaries; conjugate transformations on antiparticles preserve `J`. Consequently every charged sector reduces to copies of Result 3, irrespective of mass rank or flavor mixing.

The three unpaired neutral states give `r=3`, `M_N=0` in Result 4, excluding KO-5 and admitting KO-7. Equivalently, the proposed descended spaces have odd dimensions thirty-nine and twenty-seven.

With three right neutrinos retained, the general neutral matrix is

$$M_N = \begin{pmatrix} 0 & Y_\nu^T \\ Y_\nu & R \end{pmatrix}, \qquad R^T = R.$$

For R=0, every singular value of Y_nu appears twice, so both reductions exist. For general R, KO-7 always exists, while KO-5 exists exactly when every singular value of this six-by-six matrix has even multiplicity. Thus nonzero Majorana masses are not an unconditional obstruction in several generations; the exact obstruction is the stated multiplicity condition. ■

8. No-go theorem — The massless Standard Model triple is not a doubling.

Let the algebra remain the full

$$A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

with the representation of Result 5, for g≥1 generations. Then no involution satisfying Result 2 exists, for any admissible D. In particular this applies at v=0, when only the Majorana block remains.

Proof. Consider the central projection

$$e_{\mathbb{H}} = (0, 1_{\mathbb{H}}, 0).$$

Its represented range lies entirely in the particle left-handed sector:

$$\dim \rho(e_{\mathbb{H}})H_+ = 8g, \qquad \dim \rho(e_{\mathbb{H}})H_- = 0.$$

An algebra-linear grading-reversing unitary would identify these spaces, which is impossible.

In physical representation-ring notation,

$$V_L = (3,2)_{1/6} \oplus (1,2)_{-1/2},$$

$$V_R = (3,1)_{2/3} \oplus (3,1)_{-1/3} \oplus (1,1)_{-1} \oplus (1,1)_0.$$

Thus I_G=[V_L]-[V_R]≠0. For the complete real graded Hilbert space, the relevant gauge character is

$$[H_+] - [H_-] = I_G - \overline{I_G},$$

not merely I_G. It too is nonzero: the coefficient of (3,2)_{1/6} is g. Either this full-character calculation or the central-projection argument proves the obstruction. ■

Consequence for the 10 -> 9 claim

9. Proposition — What dimension reduction is actually established.

The heavy projection itself preserves KO-degree six, as canonical §3 explicitly states. Hence its spacetime product still has degree 4+6=10 mod 8.

A subsequent choice of a KO-5 reduction removes a degree-one Clifford factor and gives the formal representative 4+5=9. Such a reduction exists for the charged truncation or the pure-Dirac completion, but **both also admit KO-7 reductions**. Their fixed J does not select nine.

For the strict light spectrum with one or three unpaired left neutrinos, **KO-5 is obstructed; only KO-7 is available among the proposed reductions**. This gives the formal representative eleven. If particle preservation is additionally required, that strict light spectrum admits neither reduction.

Proof. These are Results 2–7. For completeness, the spacetime real structures are

$$J_{4+5} = (J_M \gamma_M) \otimes j_5, \qquad J_{4+7} = J_M \otimes j_7.$$

Thus even the odd reductions require different product conventions. No finite-dimensional operation here changes heat dimension: a finite internal factor remains spectrally zero-dimensional. Consequently the claimed loss of one dimension is, at most, a chosen Clifford-factor removal—not a proved metric collapse, and not a consequence of vectorlike charged matter alone. ■ —

Independent numerical check (Claude, 2 Nov 2026)

We used the electrodynamics data of Result 3, with m = 0.7, J = [[0, I_2],[I_2, 0]] K and gamma = diag(1,-1,-1,1). The triple has J^2 = 1, JD = DJ and J gamma = -gamma J, so its KO-dimension is 6. Both U_+ = diag(sigma_1, sigma_1) and U_- = diag(sigma_1, -sigma_1) satisfy U^2 = 1 and U gamma = -gamma U. Both commute with random algebra elements, their opposites and D; and U_+ J = J U_+, U_- J = -J U_-. On H_0 = span(e_L, bar e_R), U_+ descends to j^2 = +1 with jd = dj (KO-7) and U_- to j^2 = -1 with jd = -dj (KO-5). For No-go 8 we also rechecked by hand the dimension count dim rho(e_H) H_+ = 8g, dim rho(e_H) H_- = 0 in the representation of Result 5.

Terminology. Here "doubling" means the factorization of an even triple as an odd triple times a Clifford point (the doubling of an odd triple in Dabrowski-Dossena's product construction). It is unrelated to the "fermion doubling" of the noncommutative Standard Model (Lizzi-Mangano-Miele-Sparano). Prior work on products and on J versus J gamma: ../LITERATURE.md, section D; the factorization criterion and the 9/11 results were not found in print.

Referee note

Terra 6.1 (3 Nov 2026; report) checked Result 3, Lemma 4, Proposition 5, Result 7, No-go theorem 8 and the no-go under [U, Q] = 0: all correct. One amendment to Result 7: "arbitrary flavor mixing" means arbitrary matrices within the displayed species blocks, with the heavy projection chosen onto the specified top Dirac sector; mixing between inequivalent species is outside the displayed Dirac operator. Escapes: for KO-5, neutral-sector pairing or a symmetry enforcing even multiplicities; for No-go 8, a change of algebra or representation, the even description, and boundary or defect realizations of chirality; for the no-go under [U, Q] = 0, a matching particle bimodule of opposite grading.

Part 07

Does the chiral interval select 9 or 11?

Questions 1-3 of prompts/07. The Astra job stopped on the codex usage limit before producing output, so these are Claude Opus 5.5's proofs, checked by hand. Astra job 08 confirmed Lemmas 1-3 under the pointwise reality hypothesis of Lemma 1 and answered Question 4: see lemma 08 and the amendments below.

Let I_l = [0, l], H = L^2(I_l, C^2) with psi = (L, R), D = E d/dx with E = [[0, 1], [-1, 0]], Dom D = {psi in H^1 : R(0) = 0, L(l) = 0}, Gamma = diag(1, -1) (job 05). D and Gamma are real.

Lemma 1 (real structures on the chiral interval). Let J be antiunitary with J f J^{-1} = conj(f) for f in C^infty(I_l), J Dom D = Dom D, J^2 = eps, JD = eps' DJ, J Gamma = eps'' Gamma J. Then J = W K with W = diag(alpha, beta) constant, abs(alpha) = abs(beta) = 1, and

```
eps = +1,   eps'' = +1,   alpha = eps' beta,   eps' in {+1, -1}.
```

Both values of eps' occur, and they are exchanged by J -> J Gamma. The chiral boundary conditions exclude every real structure anticommuting with Gamma; the bare line allows such ones, e.g. W = E with signs (-1, +1, -1).

Proof. Antilinearity together with J f = conj(f) J gives J = W K for a unitary matrix function W(x). Since E is real, D J psi = E W' conj(psi) + E W conj(psi'), while J D psi = W E conj(psi'). Hence eps' D J = J D forces E W' = 0, so W is constant, and W E = eps' E W. Preservation of R(0) = 0 for arbitrary L(0) forces W_21 = 0, and preservation of L(l) = 0 forces W_12 = 0. So W is diagonal and unitary. Then W conj(W) = 1 gives eps = +1, and W commutes with Gamma, so eps'' = +1. Finally diag(alpha, beta) E = [[0, alpha], [-beta, 0]] and E diag(alpha, beta) = [[0, beta], [-alpha, 0]], so alpha = eps' beta. Replacing J by J Gamma keeps eps = eps'' = +1 and flips eps'. ■

With Connes' even convention (eps' = +1) the chiral interval therefore has KO-dimension 0 mod 8, although its metric dimension is 1.

Lemma 2 (compression). Every J of Lemma 1 commutes with P = 1_{[0,E]}(abs(D)). On the truncation span(u_L, u_R) of job 05 it induces J_2 = diag(alpha, beta) K, with D_2 = m_1 sigma_1, Gamma_2 = sigma_3 and the signs (+1, eps', +1).

Proof. JD = eps' DJ gives J D^2 = D^2 J, so J commutes with abs(D) and with its spectral projections. The basis functions u_L = sqrt(2/l)(cos(pi x/2l), 0) and u_R = sqrt(2/l)(0, sin(pi x/2l)) are real, so J u_L = alpha u_L and J u_R = beta u_R. ■

Lemma 3 (the reflection fixes s = eps'). Let (sigma psi)(x) = a (R(l - x), L(l - x)). Then sigma preserves Dom D, anticommutes with Gamma and commutes with constants; [sigma, D] = 0 holds exactly when the phases on the two components agree, and sigma^2 = 1 forces a = +1 or -1. On the truncation, sigma = a sigma_1. For every J of Lemma 1,

```
sigma J = s J sigma   with   s = beta/alpha = eps'.
```

Proof. The swap with x -> l - x exchanges the conditions R(0) = 0 and L(l) = 0. With general phases a, b on the two components, (D sigma psi)_L = -b L'(l - x) and (sigma D psi)_L = -a L'(l - x), so commutation forces a = b. One checks sigma u_L = u_R and sigma u_R = u_L. Then sigma J psi = a (beta conj R(l - x), alpha conj L(l - x)) and J sigma psi = a (alpha conj R(l - x), beta conj L(l - x)), which gives s = beta/alpha = eps'. ■

Consequence. As the interval collapses, only constants keep bounded commutators, and the reflection becomes a canonical L <-> R involution. Its sign against J is s = eps', so the interval fixes s exactly as far as the real structure on it is fixed, and no further. The 9 versus 11 ambiguity of job 06 is the same as the standard ambiguity J versus J gamma for even real spectral triples (Dabrowski-Dossena). This matches job 06, Proposition 9: the two odd reductions use J_M gamma_M (for 4 + 5) and J_M (for 4 + 7) on spacetime. Under Connes' convention eps' = +1 the reflection gives s = +1, i.e. KO-7, formally 11.

Conjecture (Question 4; false as stated, see amendment 4). For F = F_0 x (collapsing chiral interval) with the product conventions of job 06, Lemma 1, the induced U = 1 (x) sigma satisfies UJ = sJU with s = eps' (interval) times a sign determined by the product grading twist. In particular, which of 9 and 11 is selected is fixed by choosing which spacetime real structure (J_M or J_M gamma_5) represents physical charge conjugation. A KO-0 interval does not change the KO-degree of a product, so the "Clifford point" of job 06 must come from the odd x odd product structure rather than from the interval alone; the open point is how the two descriptions match. Attempted by: Claude Opus 5.5 (Questions 1-3 above). The gpt-6-astra job was cut off by the usage limit before attempting it.

Amendments from job 08

1. Lemma 1 needs its hypothesis J f J^-1 = conj(f). Without it, J_ref = sigma_1 R_l K (R_l the reflection x -> l - x) preserves the chiral boundary conditions and has the KO-6 signs (Counterexample 08.4), so the chiral boundary conditions alone do not force KO-dimension 0.
2. Lemma 2 holds for the compressed operators. With the full compressed diagonal algebra and the inherited J_2, the two-point data fail the first-order condition: [[D_2, e], e°] = D_2 for e = diag(1, 0) (Counterexample 08.4).
3. The sign s = eps' in the Consequence above is the sign of the reflection against the interval's own J. In the standard internal-first KO-6 product the required grading twist changes it: the bare reflection 1 (x) sigma always has s = -1, formally 4 + 5 = 9, for either eps' (Theorem 08.5).
4. Question 4 (Conjecture 07.4) is false as stated: a fixed spacetime J_M and particle-number conservation admit both signs (Proposition 08.9, Counterexample 08.11, Theorem 08.12).

Part 08

Product real structures and the choice between 9 and 11

Prompt: prompts/08 (Task A: check Lemmas 07.1-07.3; Task B: Conjecture 07.4). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5: by hand (results 1-3, 4 second

part, 8, 12) and numerically in Python (results 5, 6, 7, 11 and 8; section at the end).

Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): all 12 numbered statements are correct. Two points of scope: the case t = -1 in Result 1 uses the extended convention 0_-, since the standard KO-0 signs have t = +1; and Theorem 7 concerns the scalar-algebra truncation, since the full compressed two-sheet algebra with its inherited real structure fails first order (Counterexample 4).

Outcome. Lemmas 07.1-07.3 hold under the pointwise reality hypothesis. Conjecture 07.4 is false as stated: Counterexample 11 fixes the physical D_F, J_F, Gamma_F and particle number Q and still admits admissible involutions of both signs. What holds instead: in the standard internal-first KO-6 product, the bare interval reflection always has s = -1 (Theorem 5, formally 4+5 = 9); the product is exactly T_5(m) x P_1 with d_5 = D_0 + m Gamma_0 (Theorem 7); particle preservation pairs the two signs whenever [Q, D] = 0 (Theorem 12); and the literal collapse of the massless interval leaves no light sector (No-go theorem 8).

We use the sign convention

$$J^2 = \epsilon, \qquad JD = \epsilon' DJ, \qquad J\Gamma = \epsilon'' \Gamma J.$$

Unless stated otherwise, KO-6 means the standard signs $(+,+,-)$. We require the order-zero and first-order conditions when calling data a real spectral triple; we impose no orientability or Poincare-duality assumption.

1. Lemma (07.1: classification under pointwise reality).

Let

$$H_I = L^2([0,l],\mathbb{C}^2), \quad D_I = E\partial_x, \quad E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma_I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with

$$\operatorname{Dom} D_I = \{(L,R) \in H^1 : R(0) = 0,\ L(l) = 0\}.$$

Suppose J_I is antiunitary, preserves this domain, and satisfies

$$J_I f J_I^{-1} = \overline{f}, \qquad J_I D_I = t D_I J_I, \quad t \in \{+1,-1\}.$$

Then

$$J_I = \operatorname{diag}(\alpha,\beta)K, \qquad |\alpha| = |\beta| = 1, \qquad \alpha = t\beta.$$

Consequently,

$$(\epsilon_I, \epsilon_I', \epsilon_I'') = (+,t,+).$$

Proof. The linear unitary $W = J_I K$ commutes with every scalar multiplication operator. Hence W is multiplication by a measurable unitary matrix $W(x)$.

Domain preservation, applied to compactly supported smooth sections, gives $W \in H^1_{\mathrm{loc}}$. Comparing the coefficients of derivatives in

$$WE\overline{\psi'} = tE(W'\overline{\psi} + W\overline{\psi'})$$

gives

$$WE = tEW, \qquad EW' = 0.$$

Thus W is constant. Preservation of the boundary conditions gives $W_{21} = W_{12} = 0$, because the remaining endpoint traces are arbitrary.

Writing $W = \operatorname{diag}(\alpha,\beta)$, the equation $WE = tEW$ becomes $\alpha = t\beta$. Finally,

$$W\overline{W} = 1, \qquad W\Gamma_I = \Gamma_I W.$$

These imply the asserted signs. Conversely, every displayed J_I satisfies all the hypotheses. □

Thus Lemma 07.1 is correct under its stated pointwise reality hypothesis. Up to phase, the two possibilities are K and $\Gamma_I K$. They are 0_+ and 0_- in the extended convention; Connes' even convention selects $t = +1$. See Dąbrowski–Dossena, arXiv v1, Table 1, p. 5.

2. Lemma (07.2: spectral compression).

Every J_I in Result 1 commutes with the spectral projections of $|D_I|$. On the minimal nonzero grading-preserving spectral subspace,

$$D_2 = m\sigma_1, \qquad \Gamma_2 = \sigma_3, \qquad J_2 = \mathrm{diag}(\alpha, \beta)K, \qquad m = \frac{\pi}{2l}.$$

Proof. The relation $J_I D_I = tD_I J_I$ implies that J_I preserves every eigenspace of D_I^2, hence every spectral subspace of $|D_I|$.

Solving $D_I\psi = \lambda\psi$ with the boundary conditions gives $\lambda = \pm(r - \frac{1}{2})\pi/l$. The lowest pair has the real orthonormal chiral basis

$$u_L = \sqrt{\frac{2}{l}}\begin{pmatrix}\cos(mx)\\ 0\end{pmatrix}, \qquad u_R = \sqrt{\frac{2}{l}}\begin{pmatrix}0\\ \sin(mx)\end{pmatrix}.$$

Direct differentiation gives $D_I u_L = mu_R$, $D_I u_R = mu_L$. Also $J_I u_L = \alpha u_L$, $J_I u_R = \beta u_R$. □

This proves Lemma 07.2 as a statement about the compressed operators. Result 4 below addresses the first-order condition.

3. Lemma (07.3: reflection sign).

Let $R_l\psi(x) = \psi(l - x)$. A component-exchanging reflection

$$\sigma = \begin{pmatrix}0 & a\\ b & 0\end{pmatrix} R_l, \qquad |a| = |b| = 1,$$

commutes with D_I exactly when $a = b$. Requiring additionally $\sigma^2 = 1$ gives $a = b = \pm 1$. For these reflections,

$$\sigma\Gamma_I = -\Gamma_I\sigma, \qquad \sigma J_I = tJ_I\sigma.$$

On the minimal truncation, $\sigma = \pm\sigma_1$.

Proof. Reflection exchanges the two endpoint conditions. Since $\partial_x R_l = -R_l\partial_x$, commutation with D_I requires

$$-E\begin{pmatrix}0 & a\\ b & 0\end{pmatrix} = \begin{pmatrix}0 & a\\ b & 0\end{pmatrix} E,$$

which is equivalent to $a = b$. Its square is then a^2, so $a = \pm 1$.

For real a, multiplication of the matrices representing σ and J_I gives

$$\sigma J_I = tJ_I\sigma$$

using $\alpha = t\beta$. The identities $\cos(m(l - x)) = \sin(mx)$ and $\sin(m(l - x)) = \cos(mx)$ give the compressed reflection. □

Thus all three lemmas agree with the direct calculation. Notice, however, that

$$\sigma f\sigma^{-1} = f(l - \cdot).$$

The reflection commutes with constants, but generally does not commute with the interval algebra.

4. Counterexample (two stronger interpretations of the interval lemmas fail).

First, chiral boundary conditions alone do not force KO-dimension zero. Without pointwise reality, the antiunitary

$$J_{\mathrm{ref}} = \sigma K, \qquad \sigma = \sigma_1 R_l,$$

has the standard KO-6 signs.

Second, the minimal two-point compression with its full diagonal algebra and inherited J_2 is not a real spectral triple satisfying the first-order condition when $m > 0$.

Proof. For the first assertion,

$$J_{\mathrm{ref}}^2 = 1, \qquad J_{\mathrm{ref}} D_I = D_I J_{\mathrm{ref}}, \qquad J_{\mathrm{ref}} \Gamma_I = -\Gamma_I J_{\mathrm{ref}}.$$

It implements

$$J_{\mathrm{ref}} f J_{\mathrm{ref}}^{-1} = \overline{f}(l - \cdot).$$

The opposite action is still scalar multiplication, so order zero and first order hold. This example violates precisely the pointwise reality hypothesis of Result 1.

For the second assertion, the compressed algebra is

$$A_2 = \{\mathrm{diag}(a, b)\}.$$

Indeed, compression sends 1 to 1 and $\cos(\pi x/l)$ to $\frac{1}{2}\sigma_3$. Let $e = \mathrm{diag}(1, 0)$. The inherited diagonal J_2 gives $e^o = e$, whereas

$$[[D_2, e], e^o] = D_2 \neq 0.$$

Moreover, $\sigma_1 e \sigma_1 = 1 - e$, so the reflection is not an A_2-module map. □

For the finite products below, we therefore explicitly restrict the interval algebra to scalar constants. This restriction makes the compressed interval a real triple and makes its reflection algebra-linear. It is an additional assumption about the proposed collapsed model.

5. Theorem (B1–B2: the internal-first KO-6 product).

Let

$$F_0 = (A_0, H_0, D_0, J_0, \Gamma_0)$$

be a finite real KO-6 triple:

$$J_0^2 = 1, \qquad J_0 D_0 = D_0 J_0, \qquad J_0 \Gamma_0 = -\Gamma_0 J_0.$$

Take either the interval or its scalar-algebra truncation, and use the internal-first product

$$D_F = D_0 \otimes 1 + \Gamma_0 \otimes D_I, \qquad \Gamma_F = \Gamma_0 \otimes \Gamma_I.$$

Among the grading-twisted tensor real structures

$$J_{a,b} = (J_0 \Gamma_0^a) \otimes (J_I \Gamma_I^b), \qquad a, b \in \{0, 1\},$$

the standard KO-6 conditions force

$$a = 0, \qquad t(-1)^b = -1.$$

For the induced reflection $U = 1 \otimes \sigma$,

$$\boxed{U J_F = -J_F U.}$$

Proof. Direct calculation gives

$$J_{a,b}^2 = (-1)^a, \qquad J_{a,b} \Gamma_F = -\Gamma_F J_{a,b}.$$

The two Dirac summands satisfy

$$J_{a,b}(D_0 \otimes 1) = (-1)^a (D_0 \otimes 1) J_{a,b},$$

$$J_{a,b}(\Gamma_0 \otimes D_I) = -t(-1)^b (\Gamma_0 \otimes D_I) J_{a,b}.$$

Thus $J_F^2 = 1$ requires $a = 0$, and $J_F D_F = D_F J_F$ requires $t(-1)^b = -1$.

Since $\sigma J_I = t J_I \sigma$ and $\sigma \Gamma_I = -\Gamma_I \sigma$,

$$U J_{a,b} = t(-1)^b J_{a,b} U.$$

The required product sign is therefore -1. □

Explicitly:

Interval $t = \epsilon'_I$	Required b	Product reflection sign s
$+1$	1	-1
-1	0	-1

For $t = +1$, this is exactly job 06's formula

$$J_F = J_0 \otimes (J_I \Gamma_I).$$

Changing J_I to $J_I\Gamma_I$ while adjusting the required twist leaves the resulting J_F unchanged, up to phase. The general formula $s = t(-1)^b$ is proved, but t and b are not independent choices when the product must have standard KO-6 signs. The relevant product constructions are Dąbrowski–Dossena, §4.1, equation (18) and Tables 2–3.

A KO-0 factor leaves the KO-degree unchanged. Hence F_0 must already have degree six in this ordinary even-even construction. An odd internal triple multiplied by an even interval remains odd under the standard product rules.

6. Proposition (B1–B2: the grading and the lift of reflection).

The preceding construction has three necessary distinctions.

1. The interval grading alone cannot be the total KO-6 grading for a tensor real structure of the displayed form.
2. Twisting the reflection by the internal grading can change s, but generally destroys commutation with D_F.
3. Transporting all data between equivalent product conventions preserves s.

Proof. If one sets $\Gamma_F = 1 \otimes \Gamma_I$, then every $J_{a,b}$ commutes with Γ_F, contradicting the required KO-6 relation. The product grading must be $\Gamma_0 \otimes \Gamma_I$.

For

$$U_c = \Gamma_0^c \otimes \sigma, \qquad c \in \{0, 1\},$$

we obtain

$$U_c J_F = (-1)^c \, t(-1)^b J_F U_c,$$

and

$$[U_c, D_F] = [\Gamma_0^c, D_0] \otimes \sigma.$$

Thus U_0 always commutes with D_F. For $c = 1$,

$$[U_1, D_F] = 0 \quad \Longleftrightarrow \quad D_0 = 0,$$

because $\{\Gamma_0, D_0\} = 0$. When $D_0 = 0$, both lifts are admissible and have opposite signs.

To compare product conventions, let

$$W = P_+^0 \otimes 1 + P_-^0 \otimes \Gamma_I, \qquad P_\pm^0 = \frac{1 \pm \Gamma_0}{2}.$$

Then

$$W D_F W = D_0 \otimes \Gamma_I + 1 \otimes D_I,$$

while

$$W(1 \otimes \sigma)W = \Gamma_0 \otimes \sigma.$$

Unitary transport preserves the relation $UJ = sJU$. Keeping the bare symbol $1 \otimes \sigma$ after changing convention can therefore mean choosing a different involution. □

The interval can still carry the physical L/R label. For example, if $\Gamma_0 = Q_0$ distinguishes particles from antiparticles, then

$$\Gamma_F = Q_0 \otimes \Gamma_I$$

has the usual reversal of finite grading between a particle and its conjugate.

Multiplying the total real structure by Γ_F also flips s, but changes its signs to $(-,-,-)$. It therefore leaves the standard KO-6 convention required by job 06.

7. Theorem (B1: exact reconciliation with the odd Clifford point).

For the scalar-algebra minimal truncation, choose the harmless overall phase so that

$$D_F = D_0 \otimes 1 + m\Gamma_0 \otimes \sigma_1, \quad \Gamma_F = \Gamma_0 \otimes \sigma_3, \quad J_F = J_0 \otimes (\sigma_3 K).$$

With $U = 1 \otimes \sigma_1$, this triple is exactly the odd-odd product

$$\boxed{F \cong T_5(m) \times P_1,}$$

where

$$T_5(m) = (A_0, H_0,\ d_5 = D_0 + m\Gamma_0,\ j_5 = J_0\Gamma_0),$$

and

$$P_1 = (\mathbb{C}, \mathbb{C}, 0, K).$$

For real algebras, use the scalar point algebra $\mathbb{R}$.

Proof. The operator d_5 is self-adjoint, and

$$j_5^2 = -1, \qquad j_5 d_5 = -d_5 j_5.$$

Its opposite action is unchanged, and $[d_5, a] = [D_0, a]$, so order zero and first order descend.

Let $e_\pm$ be the eigenvectors of σ_3, and define

$$R : H_0 \longrightarrow H_{F,+}, \qquad Rx = P_+^0 x \otimes e_+ + P_-^0 x \otimes e_-.$$

This is unitary onto the positive total-grading subspace. Direct calculation gives

$$R^* U D_F R = D_0 + m\Gamma_0, \qquad R^* U J_F R = J_0 \Gamma_0.$$

Under the unitary identification

$$(x, y) \longmapsto Rx + URy,$$

the operators become

$$D_F = \begin{pmatrix} 0 & d_5 \\ d_5 & 0 \end{pmatrix}, \quad \Gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_F(x, y) = (-j_5 y, j_5 x).$$

These are precisely the odd-odd product matrices with auxiliary matrix $B = -i\sigma_2$. □

More generally, any admissible involution in a standard KO-6 triple gives

$$d = (UD)|_{H_+}, \qquad j = (UJ)|_{H_+},$$

with

$$j^2 = s, \qquad jd = s\, dj.$$

Thus

$$s = -1 : \quad F \cong T_5 \times P_1, \qquad s = +1 : \quad F \cong T_7 \times P_7.$$

The auxiliary matrices are respectively $-i\sigma_2$ and σ_1, in agreement with Dąbrowski–Dossena, equation (25), Table 6, p. 12.

The even KO-0 interval is therefore not identified with an odd point. Its two-dimensional space becomes the auxiliary doubling space after reorganizing the complete product, while its mass enters d_5. The point itself has one-dimensional Hilbert space and zero Dirac operator. Because its Dirac operator is zero, the distinction between P_1 and P_7 includes a choice of KO label.

8. No-go theorem (literal collapse produces no light finite sector).

Assume the heavy-limit model uses exactly the product Dirac operator of Result 5, with $m = \pi/(2l)$. At every fixed energy cutoff Λ, its light subspace is zero for sufficiently small l.

Proof. Anticommutation of D_0 and Γ_0 gives

$$D_F^2 = D_0^2 \otimes 1 + 1 \otimes D_I^2.$$

The chiral interval and its minimal truncation satisfy

$$D_I^2 \geq \frac{\pi^2}{4l^2}.$$

Hence

$$|D_F| \geq \frac{\pi}{2l}, \qquad \mathbf{1}_{[0,\Lambda]}(|D_F|) = 0 \quad \text{if } l < \frac{\pi}{2\Lambda}.$$

The same conclusion holds for the alternative product convention by unitary equivalence. □

Consequently, the exact product supplies an algebraic model at finite l, but cannot yield a nonempty light vectorlike sector by literal collapse at fixed energy. Retaining such a sector requires additional dynamics or a modified limiting prescription.

9. Proposition (B3: spacetime grading twists do not select the finite involution).

Let the four-dimensional Euclidean spacetime triple satisfy

$$J_M^2 = -1, \qquad J_M D_M = D_M J_M, \qquad J_M \gamma_M = \gamma_M J_M.$$

For an odd reduction (d, j_s) with

$$j_s^2 = s, \qquad j_s d = s\, d j_s,$$

the job-06 even-odd convention requires

$$J_{M\times T_s} = (J_M \gamma_M^{a_s}) \otimes j_s, \qquad (-1)^{a_s} = s.$$

Therefore

$$J_{4+5} = (J_M \gamma_M) \otimes j_5, \qquad J_{4+7} = J_M \otimes j_7.$$

These formulas do not imply that choosing physical spacetime charge conjugation selects s.

Proof. For

$$D_{M\times T_s} = D_M \otimes 1 + \gamma_M \otimes d,$$

the proposed real structure has commutation signs $(-1)^{a_s}$ and s against the two summands. They must agree. Its square is $-s$, giving KO-1 for $s = -1$ and KO-3 for $s = +1$.

We can also retain the removed point and compare the complete products. Write

$$F = T_s \times P_k, \qquad J_F = j_s \otimes (B_s K),$$

where $B_- = -i\sigma_2$, $B_+ = \sigma_1$. The unitary

$$V = P_+^M \otimes 1 \otimes 1 + P_-^M \otimes 1 \otimes \sigma_1$$

identifies $(M \times T_s) \times P_k$ with $M \times F$. In particular,

$$V((J_M \gamma_M^{a_s}) \otimes j_s \otimes B_s K)V^* = J_M \otimes J_F.$$

Here the extra factor $\gamma_M^{a_s}$ cancels because $\sigma_1 B_s = s B_s \sigma_1$. □

Thus both reconstructions can use the same original J_M. The different spacetime factors in the reduced products are required by the product rules.

10. Proposition (B3: Euclidean and Lorentzian charge conjugation).

In the conventions used by Chamseddine–Connes–Marcolli and van den Dungen–van Suijlekom, the Euclidean spacetime charge-conjugation operator is J_M, with

$$(J_M^2, \epsilon'_M, \epsilon''_M) = (-, +, +).$$

The operator $J_M\gamma_5$ is its other even real-structure convention. Lorentzian charge conjugation requires changing signature and is not obtained merely by choosing between these two Euclidean operators.

Proof and source identification. CCM state, "The operator J is the charge conjugation operator," in §3.1, and use $J = J_M \otimes J_F$ in §3.2. Their calculation in §3.5.2 explicitly uses $J_M D_M = D_M J_M$. See CCM, arXiv v1, §§3.1–3.2, p. 20, and §3.5.2, p. 27.

Van den Dungen–van Suijlekom likewise identify J_M as "the charge conjugation operator on the spinors" in Example 2.1 and use $J = J_M \otimes J_F$ in Definition 2.3. Their fermionic action uses this J_M. See the published paper, pp. 435–436 and Proposition 4.3, p. 452.

For an explicit signature comparison, take Lorentzian signature $(+---)$, chiral gamma matrices, and

$$\gamma_E^0 = \gamma_L^0, \qquad \gamma_E^j = -i\gamma_L^j.$$

Up to overall phases, standard conjugations are

$$C_E = i\gamma_E^0\gamma_E^2 K, \qquad C_L = -i\gamma_L^2 K.$$

These are the conventions in D'Andrea–Kurkov–Lizzi, arXiv v2, equations (3.4), (3.13)–(3.15).

The Clifford relations give

$$C_E^2 = -1, \quad [C_E, \gamma_5] = 0, \quad [C_E, D_E] = 0, \qquad D_E = -i\gamma_E^\mu \nabla_\mu,$$

whereas

$$C_L^2 = 1, \qquad C_L\gamma_5 = -\gamma_5 C_L.$$

In the displayed common matrix basis,

$$C_L = -i\gamma_E^0 C_E.$$

Both C_E and $C_E\gamma_5$ square to -1 and preserve Euclidean chirality; neither is already C_L.

Finally, for real m, Lorentzian charge conjugation carries

$$(i\gamma_L^\mu(\partial_\mu + ieA_\mu) - m)\psi = 0$$

to the same equation with charge $-e$. Composing C_L with γ_5 instead reverses the scalar mass sign. Thus the usual massive Dirac-field identification distinguishes charge conjugation once the signature and Dirac-operator conventions are fixed. □

This identification determines the starting spacetime convention. By Result 9, it does not determine the finite involution.

11. Counterexample (fixed physical real structure permits both 9 and 11).

There is a single scalar-interval product with fixed D_F, J_F, Γ_F, Q admitting particle-preserving involutions of both signs.

Proof. Let τ_i act on an internal particle/antiparticle space and σ_i on the interval truncation. Set

$$\rho(a, b) = \mathrm{diag}(a, b) \otimes 1, \qquad D_F = m\tau_3 \otimes \sigma_1,$$

$$\Gamma_F = \tau_3 \otimes \sigma_3, \qquad J_F = (\tau_1 \otimes \sigma_3)K, \qquad Q = \tau_3 \otimes 1, \quad m > 0.$$

This is Result 5 with

$$D_0 = 0, \quad \Gamma_0 = \tau_3, \quad J_0 = \tau_1 K.$$

It has standard KO-6 signs. Its Dirac operator commutes with the algebra, so the first-order condition holds.

Define

$$U_- = 1 \otimes \sigma_1, \qquad U_+ = \tau_3 \otimes \sigma_1 = QU_-.$$

Both are grading-reversing unitary involutions commuting with D_F, the two algebra actions, and Q. Nevertheless,

$$U_- J_F = -J_F U_-, \qquad U_+ J_F = J_F U_+.$$

The unitary

$$W = \mathrm{diag}(1, \sigma_3)$$

on the particle/antiparticle blocks transforms the data into

$$D'_F = m\, 1 \otimes \sigma_1, \qquad J'_F = (\tau_1 \otimes 1)K,$$

while preserving Γ_F, Q, ρ. The transformed involutions are

$$U'_- = \mathrm{diag}(\sigma_1, -\sigma_1), \qquad U'_+ = \mathrm{diag}(\sigma_1, \sigma_1).$$

Thus this is also the electrodynamics counterexample in job 06, expressed as an interval product. Tensoring with a fixed spacetime J_M leaves both possibilities available. □

This disproves the assertion in Conjecture 07.4 that physical spacetime charge conjugation alone selects the reduction. Selecting specifically the bare reflection $1 \otimes \sigma$ is an additional restriction and gives $s = -1$ in the internal-first convention.

12. Theorem (B4: particle preservation and sign pairing).

Let Q be a self-adjoint involution equal to $+1$ on particles and -1 on antiparticles.

For an interval product with $Q = Q_0 \otimes 1$, the bare reflection automatically satisfies

$$[1 \otimes \sigma, Q] = 0.$$

More generally, assume a standard KO-6 triple satisfies

$$[Q, D] = [Q, \Gamma] = [Q, a] = [Q, a^o] = 0, \qquad QJ = -JQ.$$

If an admissible doubling involution U preserves particle number, then QU is another particle-preserving doubling involution with the opposite sign against J.

Proof. The tensor-product assertion is immediate. For the general assertion, $[Q, U] = 0$ gives

$$(QU)^2 = 1.$$

The remaining commutation and grading properties follow from those of Q and U. If $UJ = sJU$, then

$$(QU)J = sQJU = -sJ(QU).$$

Also $[QU, Q] = 0$. □

Thus particle preservation does not select a sign in a particle-number-conserving sector. The additional assumption $[Q, D] = 0$ is essential to the sign-pairing argument; Majorana terms can violate it.

Conversely, particle-preserving doubling requires the particle spaces of opposite grading to be isomorphic as bimodules: U restricts to such an isomorphism. An unpaired left-handed neutrino with no matching

right-handed particle in its bimodule therefore obstructs particle-preserving doubling. The formal interval product builds in the missing partner and cannot represent that strict light spectrum without changing its particle content.

Consequence for 9 versus 11

Lemmas 07.1–07.3 hold under pointwise reality, but their proposed physical consequence does not follow. For the internal-first standard KO-6 product, the required grading twist gives the bare interval reflection $s = -1$, hence a KO-5 reduction and formal $4+5 = 9$. Changing the interval's even real-structure convention consistently does not change this result. A fixed spacetime charge conjugation J_M, and even particle-number preservation, can coexist with both reductions; choosing the lift of L/R reflection remains additional data. Finally, literal collapse of this chiral interval leaves no light fermions, so neither 9 nor 11 has been established as the dimension of a surviving heavy-W/Z limit.

Independent numerical check (Claude, 3 Nov 2026)

Python (numpy), antiunitaries represented as J = M K, with J X J^-1 = M conj(X) M^-1, J^2 = M conj(M), and U J = s J U iff U M = s M conj(U). F_0: H_0 = C^2 (+) C^2 with Gamma_0 = diag(g, -g), D_0 = diag(d, conj(d)) (d off-diagonal, random complex entry), J_0 = block swap times K; KO-6 signs verified. Interval truncation D_I = m sigma_1 (m = 0.7), Gamma_I = sigma_3, J_I = diag(t, 1) K.

- **Theorem 5.** Of the eight choices (t, a, b), exactly (t, a, b) = (+1, 0, 1) and (-1, 0, 0) give the KO-6 signs (+, +, -); both give s = -1 for U = 1 (x) sigma_1, and U commutes with D_F.
- **Theorem 7.** With R x = P_+ x (x) e_+ + P_- x (x) e_-: R* U D_F R = D_0 + m Gamma_0 and R* U J_F R = J_0 Gamma_0, with j_5^2 = -1 and j_5 d_5 = -d_5 j_5 (KO-5 signs).
- **Proposition 6.** Gamma_0 (x) sigma_1 commutes with D_F for D_0 = 0 and fails to commute for D_0 != 0.
- **Counterexample 11.** KO-6 signs, order zero and first order hold for random diagonal algebra elements; U_- and U_+ are both admissible (involutive, grading-reversing, commuting with D_F, Q, the algebra and its opposite), with s = -1 and s = +1.
- **No-go theorem 8.** D_F^2 = D_0^2 (x) 1 + 1 (x) D_I^2 holds exactly.
- **Counterexample 4 (first order), by hand.** For e = diag(1, 0) and the inherited diagonal J_2, e° = e and [[m sigma_1, e], e] = m sigma_1.

Terminology. Here "doubling" means the factorization of an even triple as an odd triple times a Clifford point (the doubling of an odd triple in Dabrowski-Dossena's product construction). It is unrelated to the "fermion doubling" of the noncommutative Standard Model (Lizzi-Mangano-Miele-Sparano). Prior work on products and on J versus J gamma: ../LITERATURE.md, section D; the factorization criterion and the 9/11 results were not found in print.

Part 09

The chiral interval with a bulk mass: chirality, light pairs and both limits

Theorem T3 of ../STRATEGY.md, core part. Proved by Claude Opus 5.5 without Astra or Fable; checked numerically against an independent first-order discretization. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026): of 11 statements, 4 correct, 3 needed amendments, 2 had gaps (closed by the referee) and 2 were wrong as first stated (Corollary 9.10 and Lemma 9.11). All corrections are incorporated below; the referee report is appended.

Setting

Let I = [0, l], H = L^2(I; C^2) with psi = (L, R), and Gamma = diag(1, -1). For M in R let A = -d/dx + M, so that A^dagger = d/dx + M, and

```
D_M psi = (A^dagger R, A L),   i.e.   (D psi)_L = R' + M R,   (D psi)_R = -L' + M L.
```

At M = 0 this is the operator of job 05. We consider three boundary conditions:

- (a) R(0) = R(l) = 0;
- (b) L(0) = L(l) = 0;
- (c) R(0) = 0 and L(l) = 0 (the chiral interval of job 05).

Lemma 9.1. For each of (a), (b) and (c), D_M is self-adjoint on {psi in H^1 : boundary condition} and has compact resolvent.

Proof. For psi, phi in H^1 the mass terms cancel (M is real) and integration by parts gives <D psi, phi> - <psi, D phi> = [conj(R) phi_L - conj(L) phi_R]_0^l. Each condition is a Lagrangian subspace of the four-dimensional space of boundary values: two independent conditions on which the form vanishes. Distributional differentiation shows Dom D_M^* is contained in H^1, so the boundary form forces the same conditions and D_M is self-adjoint. The estimate norm(psi') <= norm(D_M psi) + abs(M) norm(psi) makes the graph norm control the H^1 norm, so the resolvent maps L^2 boundedly into H^1, which embeds compactly in L^2. ■

Chirality lives in the boundary conditions

Theorem 9.2. For every M in R and l > 0:

- (a) ker D = C (e^{Mx}, 0), so ind = dim ker_{Gamma=+1} - dim ker_{Gamma=-1} = +1;
- (b) ker D = C (0, e^{-Mx}), so ind = -1;
- (c) ker D = 0, so ind = 0.

Proof. D psi = 0 iff A L = 0 and A^dagger R = 0, i.e. L = alpha e^{Mx} and R = beta e^{-Mx}. Exponentials never vanish, so a condition at either end forces the corresponding coefficient to 0. ■

Corollary 9.3 (equivariant index). Let a compact group G act on H_G = (+)_rho V_rho (x) L^2(I; C^2), a sum of finitely many finite-dimensional representations V_rho, with boundary type tau(rho) in {a, b, c} and mass M_rho on each summand. (With infinitely many summands the operator need not be Fredholm: the referee report gives a counterexample.) Then the G-index of D in R(G) is

```
sum over tau(rho) = a of [V_rho]   minus   sum over tau(rho) = b of [V_rho],
```

independent of every M_rho and of l. Assigning (a) to the summands of V_L and (b) to those of V_R realizes I_G = [V_L] - [V_R] of research_note 7. Vectorlike pairs of type (c) contribute 0. ■

Spectrum of the mixed condition with a bulk mass

Theorem 9.4. Take condition (c). The spectrum is symmetric under lambda -> -lambda, all eigenvalues are simple, and 0 is not an eigenvalue. Write c = mu l.

1. M = -mu with mu > 0 and c > 1: there is exactly one positive eigenvalue below mu,

   ```
   lambda_0 = mu sech(kappa_0 l),   where kappa_0 in (0, mu) is the unique positive root of
    ↪  tanh(kappa l) = kappa/mu,
   ```

   and the remaining positive eigenvalues are lambda_n = sqrt(mu^2 + k_n^2), n >= 1, with k_n the unique root of tan(k l) = k/mu in (n pi/l, (n + 1/2) pi/l).

2. M = -mu and c < 1: no eigenvalue in (0, mu]. The positive eigenvalues are sqrt(mu^2 + k_n^2), n >= 0, with k_0 in (0, pi/(2l)) and k_n in (n pi/l, (n + 1/2) pi/l) roots of tan(k l) = k/mu. At c = 1 the lowest eigenvalue is exactly mu.

3. M = +mu > 0: every positive eigenvalue is sqrt(mu^2 + k_n^2) with k_n in ((n - 1/2) pi/l, n pi/l), n >= 1, a root of tan(k l) = -k/mu, so all exceed sqrt(mu^2 + (pi/2l)^2).

For M = 0 the spectrum is +/-(n + 1/2) pi/l, n >= 0 (job 05).

Proof. If D psi = lambda psi with lambda > 0, then R = A L / lambda and A^dagger A L = lambda^2 L, where A^dagger A = -d^2/dx^2 + M^2. The conditions become L(l) = 0 and L'(0) = M L(0). Conversely,

every solution of this Sturm-Liouville problem with lambda^2 > 0 yields the eigenvector (L, A L/lambda), whose R-component vanishes at 0. The Sturm-Liouville problem has simple eigenvalues, and the condition L(l) = 0 forces each eigenfunction into one of three forms:

- E = M^2 + k^2: L = sin(k(l - x)), giving tan(k l) = -k/M;
- E = M^2 - kappa^2 with 0 < kappa < abs(M): L = sinh(kappa(l - x)), giving tanh(kappa l) = -kappa/M;
- E = M^2: L = l - x, giving M l = -1.

Counting roots for M = -mu, with t = k l or t = kappa l:

- g(t) = tanh t - t/c is concave on t > 0, with g(0) = 0, g'(0) = 1 - 1/c and g -> -infinity. It has a positive root iff c > 1, and that root is unique.
- h(t) = tan t - t/c is negative on (n pi - pi/2, n pi). On (n pi, n pi + pi/2) we have h' = sec^2 t - 1/c, which changes sign at most once from - to +. Since h(n pi) <= 0 and h tends to +infinity, there is exactly one root there for n >= 1. For n = 0 there is one iff c < 1.

For M = +mu, the function tan t + t/c is strictly increasing on (n pi - pi/2, n pi), from -infinity to n pi/c > 0, so it has exactly one root there. The hyperbolic equation and M l = -1 have no solutions. Finally, s = kappa_0 l satisfies s = c tanh s, which gives lambda_0^2 = mu^2 - kappa_0^2 = mu^2 sech^2 s. ■

Remark 9.4a (the boundary identity; from the Farsi-Latrémolière assessment of 3 Nov 2026, section 4.1). Write D_M = D_0 + M sigma_1 with D_0 the massless operator. For psi = (L, R) in the domain of condition (c),

```
||D_M psi||^2 = ||D_0 psi||^2 + M^2 ||psi||^2 + M (|R(1)|^2 + |L(0)|^2),
```

since 2 Re <D_0 psi, sigma_1 psi> = [|R|^2]_0^l - [|L|^2]_0^l and the condition kills R(0) and L(l). So the anticommutator {D_0, sigma_1} vanishes in the interior and its quadratic form is the boundary term: this is exactly the hypothesis that Theorem 3.4 of Farsi-Latrémolière needs and that the interval fails, by the term that creates the wall modes (results/R1_counterexample_farsi_latremoliere.md). For M = +mu the term is nonnegative and the identity gives ||D_M psi||^2 >= ((pi/(2l))^2 + mu^2) ||psi||^2, a one-line proof of the bound in item 3; for M = -mu it is negative, which is where item 1 finds its eigenvalue below mu.

Lemma 9.5 (size and shape of the light pair). For c > 1,

```
mu sech(mu l) <= lambda_0 <= 2 mu e^{-kappa_0 l},   so   lambda_0 = 2 mu e^{-mu l} (1 + o(1))   as
↪  mu l -> infinity.
```

The pair is spanned by L_0 = sinh(kappa_0 (l - x)), concentrated at x = 0, and R_0 = A L_0 / lambda_0 = (kappa_0 cosh(kappa_0(l - x)) - mu sinh(kappa_0(l - x)))/lambda_0, which vanishes at 0 and is concentrated at x = l. Moreover ||R_0|| = ||L_0||.

Proof. The lower bound holds because sech is decreasing and kappa_0 < mu. For the upper bound, lambda_0^2 = (mu - kappa_0)(mu + kappa_0) <= 2 mu (mu - kappa_0) = 2 mu^2 (1 - tanh(kappa_0 l)) <= 4 mu^2 e^{-2 kappa_0 l}. The equality of norms follows from ||A L_0||^2 = <A^dagger A L_0, L_0> = lambda_0^2 ||L_0||^2. For the asymptotic, put s = kappa_0 l, so that c = s/tanh s and 0 < c - s = c(1 - tanh s) <= 2c e^{-2s} -> 0; hence lambda_0 = mu sech s = 2 mu e^{-c} e^{c-s}/(1 + e^{-2s}) = 2 mu e^{-c}(1 + o(1)). Finally, mu = kappa_0 coth(kappa_0 l) and lambda_0 = kappa_0/sinh(kappa_0 l) give the exact identity R_0(x) = sinh(kappa_0 x) = L_0(l - x): the two profiles are mirror images, decreasing away from their walls, with localization on the scale 1/kappa_0 as s -> infinity (and approaching multiples of l - x and x as c -> 1). ■

Theorem 9.6 (the light pair is Connes-Lott with m = lambda_0). Let P_0 project onto the eigenvalues +/- lambda_0 (c > 1), and use the basis u_L = (L_0, 0)/||L_0|| and u_R = (0, R_0)/||R_0||. Then:

- P_0 D P_0 = lambda_0 sigma_1 and P_0 Gamma P_0 = sigma_3;
- the compression of C(I) is Q(f) = diag(int f abs(u_L)^2, int f abs(u_R)^2), whose range is C + C;
- the truncated end states are at Connes distance 1/lambda_0.

Proof. We have D u_L = A L_0/||L_0|| = lambda_0 u_R. Pointwise orthogonality of the L and R components makes Q(f) diagonal. Q(1) = I and Q(x) = diag(_L, _R) with _L < _R, so the range is the full diagonal algebra. Finally, |||[lambda_0 sigma_1, diag(a, b)]|| = lambda_0 abs(a - b). ■

The two-sheet distance is therefore 1/(fermion mass), as in the Standard Model finite geometry; it depends on l only through lambda_0. At M = 0 (job 05) it was 2l/pi.

The two limits

Lemma 9.7. For c > 1 let s(c) be the positive root of tanh s = s/c, and set F(c) = (lambda_0 l)^2 = c^2 sech^2 s(c). Then F is strictly decreasing on (1, infinity), F(1+) = 1 and F(infinity) = 0.

Proof. Differentiating tanh s = s/c gives s'(1/c - sech^2 s) = s/c^2. Here 1/c - sech^2 s > 0, since 1/c = tanh(s)/s and sinh 2s > 2s. Using c^2 - s^2 = c^2 sech^2 s, the condition F'(c) = 2c - 2 s s' < 0 is equivalent to c^2 sech^2 s < c^3 sech^2 s, i.e. to c > 1. The limits follow from s -> 0 as c -> 1 and s/c -> 1 as c -> infinity. ■

Theorem 9.8 (decompactification, l -> infinity at fixed mu, M = -mu). We have lambda_0 <= 2 mu e^{-kappa_0 l} -> 0 with kappa_0 -> mu. The normalized profiles of L_0 near x = 0 and of R_0 near x = l converge, uniformly on compact sets of y and in L^2([0, infinity)) after extension by zero, to sqrt(2 mu) e^{-mu y}, where y is the distance to the respective wall: these are exact chiral zero modes of the two half-line operators (proof: norm(L_0)^2 = sinh(2s)/(4 kappa_0) - l/2 ~ e^{2s}/(8 kappa_0), kappa_0 -> mu, L_0(y) = R_0(l - y), and the same formula gives a uniform exponential bound). Since k_n lies in (n pi/l, (n + 1/2) pi/l), the rest of the spectrum obeys

```
#{n : lambda_n <= E} = (l/pi) sqrt(E^2 - mu^2) + O(1)   for E > mu,
```

a one-dimensional Weyl law (for the positive spectrum; both signs double the leading term). One dimension emerges above the bulk gap mu, and the chirality sits on the two walls. ■

Theorem 9.9 (collapse at fixed light mass). Fix m_f > 0. For each l in (0, 1/m_f) there is a unique mu(l) > 1/l with lambda_0 = m_f. As l -> 0:

- c(l) = mu(l) l -> infinity, with c = log(2c/(m_f l)) + o(1); equivalently c = -W_{-1}(-m_f l/2) + o(1) (Lambert W, lower branch), so c = L + log L + o(1) with L = log(2/(m_f l)): the leading scaling log(1/(m_f l))/l carries a log log correction, and it is specific to this localization model;
- the next eigenvalue exceeds sqrt(mu(l)^2 + (pi/l)^2) -> infinity;
- the light pair stays Connes-Lott with mass m_f (Theorem 9.6), while the interval collapses to a point in Gromov-Hausdorff distance. This is a statement about the compression at each l. As whole triples in the spectral propinquity, the family converges to (C, C^2, m_f sigma_1), a point carrying a massive two-dimensional Dirac block, at distance at least min(sqrt(2)/2, (1/m_f - l)/2) from the Connes-Lott triple (Corollary 15.5 of lemma 15). Both statements hold: the metric compares the whole algebra, and the compression is a different object.

Proof. Existence and uniqueness follow from Lemma 9.7 applied to F(c) = (m_f l)^2 in (0, 1). As l -> 0 we have F -> 0, so c -> infinity, and Lemma 9.5 gives m_f = 2 (c/l) e^{-c}(1 + o(1)). The bound on the next eigenvalue is Theorem 9.4.1. ■

Corollary 9.10 (collapse at fixed bulk mass). If instead mu is fixed and l -> 0, then eventually c < 1, every eigenvalue has absolute value greater than mu, and the smallest absolute value is (pi/(2l))(1 + o(1)) (the lowest root t_0 = k_0 l of tan t/t = 1/(mu l) tends to pi/2). No light fermion survives. This extends Proposition 05-4 to M != 0.

Gauge fields on the interval (scale lemma for T4)

Lemma 9.11. For -d^2/dx^2 on [0, l]:

- Neumann-Neumann conditions give eigenvalues (n pi/l)^2, n >= 0, including a massless mode;
- Neumann-Dirichlet (in either order) gives ((n - 1/2) pi/l)^2, n >= 1;
- Dirichlet-Dirichlet gives (n pi/l)^2, n >= 1.

A gauge field with a Dirichlet condition at one end and a Neumann condition at the other therefore has lightest mass pi/(2l), which equals the Connes-Lott mass of the massless chiral interval (job 05); with Dirichlet conditions at both ends the lightest mass is pi/l. ■

Remark: fermion mass as internal momentum

In this family, masses are internal momenta, with an explicit rule for their quantization. At M = 0 the Connes-Lott mass is the lowest allowed momentum (r - 1/2) pi/l, quantized by the chiral boundary conditions. With a bulk mass, lambda_n^2 = mu^2 + k_n^2 adds the bulk mass and the quantized momentum in quadrature. The light pair's mass, mu sech(kappa_0 l), is exponentially sensitive to mu l. Order-one differences of mu_i l between generations therefore give hierarchical masses of about 2 mu_i e^{-mu_i l}. This is the localization mechanism for Yukawa hierarchies (Arkani-Hamed-Schmaltz 2000; Gherghetta-Pomarol 2000; Grossman-Neubert 2000). In noncommutative-geometry terms, the finite operator D_F of the two-sheet space is the towerless remnant of internal momentum, and its texture becomes localization and overlap data.

Numerical check (Claude)

The secular equations of Theorem 9.4 were compared with the eigenvalues of an independent first-order discretization: the chain of job 05 with mass term (M/2)(R_{i+1} + R_i), N = 4000.

mu (l = 1)	secular	chain
1e-4	1.57073, 4.71237, 7.85397	1.57054, 4.71178, 7.85299
0.5	1.26828, 4.63129, 7.80591	1.26809, 4.63070, 7.80493
3	0.30255, 5.06274, 8.05194	0.30244, 5.06227, 8.05103
6	0.02975, 7.04564, 9.33873	0.02972, 7.04536, 9.33799

At mu = 6, 2 mu e^{-mu l} = 0.029745, against lambda_0 = 0.029747. Lemma 9.7 was confirmed on (1, 40] using the cancellation-free form c^2 sech^2 s. For the collapse at m_f = 1:

l	mu l	mu	lambda_0	next eigenvalue
0.5	2.234	4.47	1.000000	9.56
0.1	4.501	45.0	1.000000	59.2
0.01	7.284	728	1.000000	812
0.001	9.893	9893	1.000000	10487

Referee report (gpt-6.1-terra, effort max, 3 Nov 2026)

Prompt: prompts/terra_check_09.md. Reproduced as received; the corrections above implement it. Claude checked the two errors, the identity R_0(x) = L_0(l - x) and the asymptotic of 9.5 by hand.

We take $\ell > 0$ throughout. All checks below are analytic; we used no numerical computation.

Lemma 9.1 Verdict: Correct with amendment. The compact-resolvent argument needs the graph-norm estimate

$$\|\psi'\|_{L^2} \le \|D_M\psi\|_{L^2} + |M|\,\|\psi\|_{L^2}.$$

The boundary form in the proof is correct. Each proposed boundary subspace is maximal isotropic. To apply this to the adjoint, distributional differentiation shows that every vector in $\mathrm{Dom}\, D_M^*$ lies in H^1; the boundary form then forces it to satisfy the same boundary conditions. Thus D_M is self-adjoint.

The displayed estimate makes the graph norm control the H^1-norm. Consequently, the resolvent maps L^2 boundedly into H^1, and compactness of $H^1 \hookrightarrow L^2$ gives compact resolvent.

Theorem 9.2 Verdict: Correct.

The zero-mode equations give

$$L(x) = \alpha e^{Mx}, \qquad R(x) = \beta e^{-Mx}.$$

The boundary conditions force $\beta = 0$ in case (a), $\alpha = 0$ in case (b), and both coefficients to vanish in case (c). The grading preserves each domain and anticommutes with D_M, so the stated chiral indices follow.

Corollary 9.3 Verdict: Correct with amendment. Assume that the displayed direct sum has finitely many finite-dimensional G-representations. Under that hypothesis, Theorem 9.2 gives the stated element of $R(G)$.

Without such a hypothesis, the direct-sum operator need not be Fredholm, even if every summand has boundary type (c). For a counterexample, take $\ell = 1$, $G = S^1$, the distinct one-dimensional representations $V_j(z) = z^j$, $j \geq 2$, and $M_j = -j$. Choose a smooth χ equal to 1 on $[0, 1/2]$ and equal to 0 at 1, and put

$$f_j(x) = \chi(x) e^{-jx}.$$

Then $(f_j, 0)$ satisfies condition (c), while

$$A_j f_j = -\chi'(x) e^{-jx}, \qquad \frac{\|A_j f_j\|}{\|f_j\|} = O\big(\sqrt{j}\, e^{-j/2}\big) \longrightarrow 0.$$

After normalization, these vectors lie in distinct orthogonal representation summands. The direct sum has no kernel, but it is not bounded below and hence is not Fredholm. Its ordinary $R(G)$-valued Fredholm index is therefore undefined.

Theorem 9.4 Verdict: Correct with amendment. Specify $\mu > 0$ in the negative-mass cases and $n \geq 1$ in case 3. For an exhaustive list, add $M = 0$, where

$$\operatorname{spec} D_0 = \left\{ \pm \frac{(n + \frac{1}{2})\pi}{\ell} : n \geq 0 \right\}.$$

At $c = 1$, the threshold eigenvalue is μ, and the remaining positive roots have $n \geq 1$ in the intervals stated in case 1.

For $\lambda \neq 0$, eliminating R gives

$$-L'' + M^2 L = \lambda^2 L, \qquad L(\ell) = 0, \quad L'(0) = M L(0).$$

Conversely, every solution gives $R = AL/\lambda$. The squared operator is nonnegative, and Theorem 9.2 excludes $\lambda = 0$. Separated boundary conditions make each squared eigenvalue simple; $\Gamma D_M = -D_M \Gamma$ gives the symmetric spectrum.

For $M = -\mu$, put $c = \mu\ell$. In the subthreshold case, the equation is

$$\frac{\tanh t}{t} = \frac{1}{c}, \qquad t = \kappa\ell > 0.$$

The left side decreases strictly from 1 to 0, so there is exactly one positive root if $c > 1$, and none if $c \leq 1$. It satisfies $\kappa < \mu$.

For the oscillatory roots, let $h(t) = \tan t - t/c$. It is negative on each positive interval $(n\pi - \pi/2, n\pi)$. On $(n\pi, n\pi + \pi/2)$, its derivative increases strictly. For $n \geq 1$, it starts negative in value and tends to $+\infty$, so it has exactly one root. On $(0, \pi/2)$, it has exactly one positive root when $c < 1$, and none when $c \geq 1$. At $c = 1$, the separate linear solution $L = \ell - x$ gives $\lambda = \mu$.

For $M = +\mu$, $\tan t + t/c$ increases strictly from $-\infty$ to a positive value on each $((n - \frac{1}{2})\pi, n\pi)$, $n \geq 1$, giving exactly one root there. It has no positive root on the other half-periods, and there is no subthreshold or threshold solution. These counts establish all three stated cases.

Lemma 9.5 Verdict: Gap, closed here. The displayed proof does not justify the relative asymptotic or make "concentrated" precise.

Write $s = \kappa_0 \ell$ and $c = \mu \ell$. The root equation gives $c = s/\tanh s$. As $c \to \infty$, $s \to \infty$ and $s/c \to 1$; moreover,

$$0 < c - s = c(1 - \tanh s) \leq 2ce^{-2s} \longrightarrow 0.$$

Thus

$$\lambda_0 = \mu \operatorname{sech} s = 2\mu e^{-c} \frac{e^{c-s}}{1 + e^{-2s}} = 2\mu e^{-c}(1 + o(1)).$$

The two stated inequalities also follow directly from $s < c$ and $\operatorname{sech} s \leq 2e^{-s}$.

The root equation simplifies the right profile exactly:

$$R_0(x) = \sinh(\kappa_0 x) = L_0(\ell - x).$$

This proves the equality of norms and shows that the two profiles decrease away from their respective walls. Exponential localization on the scale $1/\kappa_0$ holds when $s \to \infty$. It is not uniform throughout $c > 1$: at fixed ℓ, as $c \downarrow 1$, the normalized profiles approach multiples of $\ell - x$ and x.

Theorem 9.6 Verdict: Correct.

The two chiral basis vectors span the $\pm\lambda_0$ eigenspaces together, and the first two matrix identities follow from

$$D_M u_L = \lambda_0 u_R, \qquad D_M u_R = \lambda_0 u_L.$$

By the reflection identity in Lemma 9.5, the normalized left density exceeds the right density on the left half of the interval, with the reverse inequality on the right half. Hence

$$\langle x \rangle_L < \ell/2 < \langle x \rangle_R.$$

Thus $Q(1)$ and $Q(x)$ span the diagonal algebra, and the commutator norm is $\lambda_0 |a - b|$. The distance between its two pure states is $1/\lambda_0$.

The pullbacks of these states through Q are the two smeared probability measures, not literal endpoint evaluations. Also, the unnumbered sentence immediately after the proof needs correction: the distance is **not determined by ℓ alone**, but it does depend on ℓ. In fact, Lemma 9.7 gives $1/\lambda_0 > \ell$ for $c > 1$.

Lemma 9.7 Verdict: Correct.

Set $q = \operatorname{sech}^2 s$. Since

$$\frac{1}{c} = \frac{\tanh s}{s} > q,$$

implicit differentiation is valid and gives

$$s' = \frac{s}{c(1 - cq)}, \qquad F'(c) = \frac{2cq(1 - c)}{1 - cq} < 0.$$

As $c \downarrow 1$, the inverse relation $c = s/\tanh s$ gives $s \to 0$, so $F \to 1$. As $c \to \infty$, $s \to \infty$ and $c/s \to 1$, so $c^2 \operatorname{sech}^2 s \to 0$.

Theorem 9.8 Verdict: Gap, closed here. The statement asserts convergence of the normalized profiles without proving it or specifying a mode of convergence. The convergence holds uniformly on compact sets of wall distance and in $L^2([0, \infty))$, after extension by zero.

Let $B_\ell = \|L_0\| = \|R_0\|$. With $s = \kappa_0 \ell$,

$$B_\ell^2 = \frac{\sinh(2s)}{4\kappa_0} - \frac{\ell}{2} \sim \frac{e^{2s}}{8\kappa_0}.$$

At fixed μ, $s/c \to 1$, so $\kappa_0 \to \mu$. For fixed $y \geq 0$,

$$\frac{L_0(y)}{B_\ell} = \frac{R_0(\ell - y)}{B_\ell} \longrightarrow \sqrt{2\mu}\, e^{-\mu y}.$$

The same formula supplies a uniform exponential bound, proving the stated L^2 convergence. The limits are exact zero modes of the two separately oriented half-line operators, with left and right chirality respectively. They are not zero modes at finite ℓ.

For $E > \mu$, set $K = \sqrt{E^2 - \mu^2}$. The root bounds

$$n\pi < k_n \ell < (n + \tfrac{1}{2})\pi$$

give

$$\#\{n \geq 1 : k_n \leq K\} = \frac{\ell K}{\pi} + O(1).$$

This is the displayed one-sided counting law. Counting both signs of the Dirac spectrum doubles its leading term.

Theorem 9.9 Verdict: Correct.

Lemma 9.7 makes F a continuous strictly decreasing map from $(1, \infty)$ onto $(0, 1)$. Since $m_f \ell \in (0, 1)$, it gives the unique $c = \mu\ell > 1$. As $\ell \to 0$, $c \to \infty$, and Lemma 9.5 yields

$$m_f \ell = 2ce^{-c}(1 + o(1)), \qquad c = \log \frac{2c}{m_f \ell} + o(1).$$

The first root above the light pair has $k_1 > \pi/\ell$, proving the stated lower bound and divergence.

The original interval metric is $|x - y|$: the mass term commutes with scalar functions, so it does not change the Dirac commutator seminorm. Therefore

$$d_{\mathrm{GH}}([0, \ell], \{\text{point}\}) = \ell/2 \longrightarrow 0.$$

The compressed two-state geometry uses a different seminorm and retains distance $1/m_f$.

Corollary 9.10 Verdict: Error. The assertion that "every eigenvalue exceeds μ" is false: the spectrum is symmetric and contains negative eigenvalues.

The corrected statement is that, for fixed bulk mass and sufficiently small ℓ, **every eigenvalue has absolute value greater than** μ, and the smallest absolute eigenvalue is

$$\min_{\lambda \in \operatorname{spec} D_M} |\lambda| = \frac{\pi}{2\ell}(1 + o(1)).$$

For $M = -\mu < 0$, the lowest root $t_0 = k_0 \ell \in (0, \pi/2)$ satisfies

$$\frac{\tan t_0}{t_0} = \frac{1}{\mu\ell} \longrightarrow \infty,$$

so $t_0 \to \pi/2$, proving the formula. For $M = 0$ it is exact; the positive-mass case follows similarly from the root just above $\pi/2$. Thus the conclusion that no light fermion survives is correct.

Lemma 9.11 Verdict: Error. The three eigenvalue lists are correct, but the gauge-field conclusion lacks a boundary condition at the other end.

For a Dirichlet condition at **both** ends, which in particular includes a Dirichlet condition at one end, the lightest mass is π/ℓ, not $\pi/(2\ell)$. The conclusion is correct for a Dirichlet condition at one end **and a Neumann condition at the other**, for the stated operator $-d^2/dx^2$.

Even if "at one end" is read as exactly one Dirichlet condition, another boundary condition need not be Neumann: $u(0) = 0$, $u'(\ell) = u(\ell)/\ell$ admits the zero mode $u(x) = x$.

Amendments and errors

- **9.1:** Add the graph-norm estimate for compact resolvent.
- **9.3:** Require a finite direct sum, or impose other hypotheses ensuring an equivariant Fredholm operator with index in $R(G)$.
- **9.4:** Specify the root indices and $\mu > 0$, and include $M = 0$ for completeness.
- **9.5:** Supply the relative-asymptotic argument and state localization precisely; the gap is closed above.
- **9.6:** Correct the adjacent unnumbered claim that the distance is unrelated to ℓ.
- **9.8:** Prove and specify the profile convergence; the gap is closed above.
- **9.10:** Replace "every eigenvalue exceeds μ" by the corresponding statement about absolute values.
- **9.11:** Require Neumann at the opposite endpoint for the claimed lightest mass.

Overall: 11 statements checked; 4 correct, 3 correct with amendment, 2 gaps closed above, and 2 errors.

Literature (../LITERATURE.md, section B). Opposite-chirality boundary modes with exponentially small mixing are in Jackiw-Rebbi 1976, Kaplan 1992 and Shamir 1993 (section 3); collapse of spectral triples to a point is treated by Farsi-Latremoliere (J. London Math. Soc., 2026). The tuned collapse mu(l) of Theorem 9.9 was not found in print.

Part 10

Toeplitz compression of the twistor coordinates on CP^3

Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026): the three identities are correct; the displayed fourth moment had an index error (pairing two barred indices), corrected below. This answers Part A of parked job 04 (research_note 4 and canonical 4, which labelled the result "Checked calculation", although no script computes it). Proved by Claude Opus 5.5 without Astra or Fable; Monte Carlo check with 2 x 10^6 samples.

Let Gamma_1, ..., Gamma_5 be Hermitian generators of Cl(5) on C^4, so that Gamma_a Gamma_b + Gamma_b Gamma_a = 2 delta_ab. For [w] in CP^3 put x_a([w]) = w* Gamma_a w / w* w, and define the level-one Berezin-Toeplitz compression

```
Q(f) = 4 int_{CP^3} f([w]) P_w dmu([w]),
```

where P_w is the rank-one projector onto w and mu is the Fubini-Study probability measure, normalized so that Q(1) = I.

Proposition 10.1.

```
Q(x_a) = Gamma_a / 5,   Q(x_a)^2 = I/25,   Q(x_a^2) = I/5.
```

In particular Q is not multiplicative.

Proof. Lift to w uniform on S^7 in C^4. For n = 4, writing w = g/abs(g) with g a standard complex Gaussian (abs(g) and g/abs(g) are independent), Wick's theorem gives

```
E[w_j w_k conj(w_i) conj(w_l)] = (delta_ij delta_kl + delta_jl delta_ki) / (n(n+1)),
```

and the sixth moment is the sum over the six matchings of unbarred with barred indices, divided by n(n+1)(n+2) = 120.

First, Q(x_a)_kl = 4 sum_ij (Gamma_a)_ij E[conj(w_i) w_j w_k conj(w_l)] = (4/20)(tr(Gamma_a) delta_kl + (Gamma_a)_kl) = Gamma_a/5, because tr Gamma_a = 0.

Second, Q(x_a^2)_kl = (4/120) sum over matchings of {j, q, k} with {i, p, l} of the corresponding contraction of (Gamma_a)_ij (Gamma_a)_pq. Matchings that contain tr Gamma_a vanish. The remaining three give tr(Gamma_a^2) delta_kl = 4 delta_kl, (Gamma_a^2)_kl = delta_kl and (Gamma_a^2)_kl = delta_kl. The total is 6 delta_kl, so Q(x_a^2) = (24/120) I = I/5.

The middle identity follows from Gamma_a^2 = I. ■

Numerical check. With Gamma_a built as Pauli tensor products (Clifford relations verified), the Monte Carlo deviations are max abs(Q(x_a) - Gamma_a/5) <= 0.0009 and max abs(Q(x_a^2) - I/5) <= 0.0006, at a sampling error of about 10^-3.

Part 11

Heavy W in the spectral-action family forces a heavy fermion

Theorem T5 of ../STRATEGY.md. Proved by Claude Opus 5.5 without Astra or Fable. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026): Proposition 11.1 correct; Corollaries 11.2 and 11.3 and the regime amendment overstated their conclusions as first written; corrected below, report appended. The result is conditional on the Chamseddine-Connes-Marcolli normalization at the matching (unification) scale.

Hypothesis (CCM relation). In the spectral action of the almost-commutative Standard Model, the gauge normalization g^2 f_0 / (2 pi^2) = 1/4 and the Higgs normalization by a = Tr(Y_nu* Y_nu + Y_e* Y_e + 3(Y_u* Y_u + Y_d* Y_d)) give, at the matching scale,

```
8 M_W^2 = sum over generations of (m_nu^2 + m_e^2 + 3 m_u^2 + 3 m_d^2),
```

where the m are Dirac masses (Chamseddine-Connes-Marcolli, Adv. Theor. Math. Phys. 11 (2007), section 5). In outline (H the unrescaled Higgs field of CCM; the normalization is the three-generation one): the Dirac masses satisfy sum N_f m_f^2 = a abs()^2, while M_W^2 = (g^2/4)(a f_0/pi^2) abs()^2 with f_0 = pi^2/(2 g^2). Majorana masses do not enter a.

Proposition 11.1. Assume the CCM relation at fixed gauge coupling, with N_g generations, and let m_max be the largest Dirac mass. Then

```
m_max / (2 sqrt 2) <= M_W <= sqrt(N_g) m_max.
```

In particular M_W -> infinity if and only if m_max -> infinity.

Proof. Every term has weight at least 1, so 8 M_W^2 >= m_max^2. Each generation has weights 1 + 1 + 3 + 3 = 8, so 8 M_W^2 <= 8 N_g m_max^2. ■

Corollary 11.2 (no-go for the fixed-mass prescription). The heavy-W prescription of canonical 14.1 and research_state 9 (v -> infinity with y_f v fixed for every fermion and the gauge couplings fixed, so that the Yukawa couplings tend to zero) is incompatible with the CCM relation: the right-hand side stays bounded while M_W -> infinity. In the spectral-action family a heavy-W/Z limit forces m_max -> infinity, with m_max >= M_W/sqrt(N_g); some species is then unbounded along a subsequence, though no single fixed species need diverge along the whole family (the referee report gives a family with m_e^2 + m_nu^2 = 8 t^2 in which each mass returns to zero infinitely often). ■

Corollary 11.3 (only the top heavy). Assume that the up-type Dirac mass of one generation (three colors) diverges and every other Dirac mass stays bounded. Then

```
m_t^2 = (8/3) M_W^2 + O(1),   so   m_t / M_W -> sqrt(8/3), approximately 1.633.
```

Top dominance is a hypothesis here: the relation alone also allows, for example, y_e = 2g with every other Yukawa zero, which gives m_e = 2 sqrt(2) M_W and m_t = 0. Under the hypothesis, the surviving SU(3) x U(1)_em theory is vectorlike and anomaly-free, and job 03 gives the residual kinetic weights. ■

Remark (interval comparison). In the massless chiral interval of job 05, the fermion truncation mass and the Neumann-Dirichlet gauge mass both equal pi/(2l) (Lemma 9.11). Under top dominance (Corollary 11.3), the CCM relation instead fixes m_t / M_W = sqrt(8/3) at matching. The interval model and the spectral action therefore assign the top and the W to the same scale with different order-one ratios. Matching them is part of T4.

Amendment (regime of validity; review of 3 Nov 2026; corrected after the referee report). The CCM relation is a relation in the truncated asymptotic (heat-kernel) action at the matching scale Lambda; small Dirac masses relative to Lambda alone do not make it exact for the full finite-cutoff action, which also

needs control of momenta and of the Majorana sector. With fixed Yukawa couplings, M_W and every mass with nonzero Yukawa diverge together as v -> infinity, with ratios fixed by all the Yukawas through the matching constraint (for example y_t = y_nu = g gives m_t/M_W = sqrt 2); Corollary 11.3's ratio needs top dominance. At fixed Lambda a heavy-W limit leaves every fixed small-mass hierarchy (m_max <= eps Lambda implies M_W <= sqrt(N_g) eps Lambda), so it needs Lambda -> infinity jointly or a calculation of the finite-cutoff action. The weights f(m^2/Lambda^2) of lemma 03, Part A, hold under its commuting finite-operator hypothesis and do not apply directly to the broken gauge and Higgs kinetic terms (Part C's w_L(x, 0) = 1 - 3x/4 + O(x^2) differs from f); near the cutoff the coefficients need their own calculation (STRATEGY T5b).

Literature (../LITERATURE.md, section E). The mass relation and the top ratio are in CCM 2007: equation (1.5), Lemma 4.6 equation (4.16) and section 5.4 equation (5.27). Corollary 11.3 restates that result; Proposition 11.1 and Corollary 11.2 apply it to the heavy-W prescription of the corpus.

Referee report on lemmas 10 and 11 (gpt-6.1-terra, effort max, 3 Nov 2026)

Prompt: prompts/terra_check_10_11.md. Reproduced as received; the corrections above (and in lemma 10) implement it. Claude checked the counterexamples (the rotating e/nu family; y_e = 2g; y_t = y_nu = g) and the corrected fourth moment by hand.

We count Proposition 10.1, Proposition 11.1, the two corollaries, and the regime-of-validity amendment as five statements. The checks below are analytic; we did not rerun the Monte Carlo calculation.

Proposition 10.1 **Verdict: Correct with amendment.** All three identities and the normalization of Q are correct. The displayed fourth-moment formula in the proof has an index error, and its subsequent contraction is consequently incorrect as written.

For w uniformly distributed on $S^7 \subset \mathbb{C}^4$, the moments are

$$\mathbb{E}\left[\prod_{r=1}^{p} w_{j_r}\overline{w_{i_r}}\right] = \frac{\displaystyle\sum_{\sigma\in S_p}\prod_{r=1}^{p}\delta_{j_r i_{\sigma(r)}}}{4\cdot 5\cdots(3+p)}.$$

Thus the fourth moment required in the proof is

$$\mathbb{E}[w_j w_k \overline{w_i}\,\overline{w_l}] = \frac{\delta_{ij}\delta_{kl}+\delta_{jl}\delta_{ki}}{20}.$$

The file instead has $\delta_{il}\delta_{kj}$ as its second term. For example, with $i = l = 1$ and $j = k = 2$, its formula gives $1/20$, whereas the expectation is zero by phase invariance. Contracting the erroneous formula with A_{ij} gives A_{lk}, not the claimed A_{kl}.

The sixth-moment statement is correct: its denominator is $4\cdot 5\cdot 6 = 120$, and its numerator is the sum over all six matchings. Also,

$$Q(1) = 4\mathbb{E}[ww^*] = I.$$

For any 4×4 matrix A, the corrected moments give

$$Q(w^*Aw) = \frac{\mathrm{tr}(A)I + A}{5}$$

and

$$Q((w^*Aw)^2) = \frac{((\operatorname{tr} A)^2 + \operatorname{tr}(A^2))I + 2\operatorname{tr}(A)A + 2A^2}{30}.$$

For $A = \Gamma_a$, conjugation by a different Clifford generator shows $\operatorname{tr}\Gamma_a = 0$, while $\Gamma_a^2 = I$ and $\operatorname{tr}(\Gamma_a^2) = 4$. Consequently,

$$Q(x_a) = \frac{\Gamma_a}{5}, \qquad Q(x_a)^2 = \frac{I}{25}, \qquad Q(x_a^2) = \frac{I}{5}.$$

Proposition 11.1 **Verdict: Correct.** For a fixed, finite N_g, the proof follows from the assumed relation. If S denotes its weighted mass sum, then

$$m_{\max}^2 \le S \le 8N_g m_{\max}^2.$$

Substituting $S = 8M_W^2$ proves both bounds and the equivalence of the two divergence statements.

The CCM relation is quoted correctly for the three-generation model. In Chamseddine–Connes–Marcolli, arXiv:hep-th/0610241v1, equations (1.5), (4.10), (4.15)–(4.17), and (5.25)–(5.26), the weights are $1, 1, 3, 3$; g is the weak coupling at matching; and the neutrino terms are **Dirac Yukawa mass parameters**, not the eigenmasses of the full seesaw matrix. The Majorana matrix does not enter the trace a, although it enters other terms in the action. CCM states that the mass relation is valid "at unification scale."

The outline preceding the proposition should identify its $\langle H \rangle$ as the **unrescaled** spectral Higgs field. CCM's canonically normalized field is $\mathbf{H} = (\sqrt{af_0}/\pi)\varphi$. Equivalently, using the standard normalized Yukawas and v,

$$m_f = \frac{y_f v}{\sqrt{2}}, \qquad M_W = \frac{gv}{2}, \qquad \sum_f N_f y_f^2 = 4g^2,$$

which gives the quoted mass relation.

There is also a scope distinction for $N_g \neq 3$. Proposition 11.1 remains true under its explicitly assumed relation, but CCM derives the coefficient 8 for three generations. If we replicate the same gauge representations N_g times and normalize their gauge kinetic term in the same way, the gauge trace scales by $N_g/3$, giving instead

$$\sum_f N_f m_f^2 = \frac{8N_g}{3} M_W^2.$$

Thus the N_g-generation version with coefficient 8 is an additional assumption beyond the cited normalization.

Corollary 11.2 **Verdict: Error.** The fixed-mass no-go and the displayed bound are correct. The subsequent assertion that a particular Dirac mass must *tend to infinity* is stronger than the relation proves for a general family.

If all finitely many Dirac masses are bounded, Proposition 11.1 bounds M_W. Moreover, it gives exactly

$$m_{\max} \ge \frac{M_W}{\sqrt{N_g}}.$$

Thus $M_W \to \infty$ forces $m_{\max} \to \infty$, and at least one fixed species is unbounded along a subsequence. It need not force one fixed species to tend to infinity along the whole family.

For a counterexample with fixed gauge coupling, let $t \to \infty$, put $M_W = t$, and take all Dirac masses except one charged-lepton mass and one neutrino Dirac mass to be zero. Set

$$\theta(t) = \frac{\pi}{4}(1 + \sin t), \quad m_e = 2\sqrt{2}\, t \cos\theta(t), \quad m_\nu = 2\sqrt{2}\, t \sin\theta(t).$$

Then $m_e^2 + m_\nu^2 = 8t^2$, so the assumed CCM relation holds. Nevertheless, each of m_e and m_ν vanishes on an unbounded sequence. The corresponding Yukawa couplings vary; the gauge coupling can remain fixed.

The description of the fixed-mass prescription should also say **gauge couplings fixed**: nonzero fixed fermion masses with $v \to \infty$ require their Yukawa couplings to tend to zero.

Corollary 11.3 **Verdict: Error.** The displayed asymptotic formula is correct under the stated "only the top heavy" hypothesis. The claim that a heavy top follows from the spectral action, without that choice or another hypothesis selecting it, is false.

Let R be the weighted sum of all the other Dirac masses squared. Under the hypothesis, $R = O(1)$, and

$$8M_W^2 = 3m_t^2 + R \quad \Longrightarrow \quad m_t^2 = \frac{8}{3}M_W^2 - \frac{1}{3}R.$$

This proves the stated $O(1)$ formula and $m_t/M_W \to \sqrt{8/3}$.

For a counterexample to the unconditional heavy-top claim, take $y_e = 2g$ and every other normalized Yukawa coupling, including y_t, equal to zero. The matching constraint is satisfied, $m_e = 2\sqrt{2}\, M_W$, and $m_t = 0$. The relation therefore permits a heavy charged lepton with no heavy top. CCM's equation (5.27) likewise obtains the top ratio by neglecting the other Yukawa couplings; its equation (5.29) considers a comparable neutrino Yukawa instead. The interval-comparison remark needs the same top-dominance qualification.

The assertion about the surviving $SU(3) \times U(1)_{\mathrm{em}}$ representation is correct for the stated Standard Model field content: each surviving charged Dirac fermion has identical left- and right-handed representations of the unbroken group. Their anomaly contributions cancel pairwise, and neutral Majorana states do not contribute to these gauge anomalies.

Amendment on the regime of validity **Verdict: Error.** The warning about a fixed cutoff is justified, but the amendment overstates both the fixed-Yukawa conclusion and the form of the threshold weights.

First, fixed Yukawa couplings do not by themselves imply a heavy top or the ratio in Corollary 11.3. A fixed nonzero y_t gives

$$\frac{m_t}{M_W} = \frac{\sqrt{2}\, y_t}{g},$$

but its value depends on the other Yukawas through the matching constraint. For example, $y_t = y_\nu = g$, with all others zero, satisfies that constraint and gives $m_t/M_W = \sqrt{2}$; both masses diverge as $v \to \infty$. If $y_t = 0$, the top does not diverge. Corollary 11.3 applies only when the other Dirac masses remain bounded.

Second, the displayed CCM relation is a relation in the **truncated asymptotic action**. Small Dirac masses relative to Λ alone do not establish it as an exact relation for the full finite-cutoff action: the expansion also needs control of the other relevant scales, including momenta and the Majorana sector, or a suitable uniform remainder estimate. CCM's own discussion places a Majorana scale of order Λ in one scenario; see its equations (5.16)–(5.21).

At fixed Λ, the amendment correctly observes that a heavy-W limit cannot remain within a fixed small-mass hierarchy. Indeed, if $m_{\max} \le \varepsilon\Lambda$, Proposition 11.1 gives $M_W \le \sqrt{N_g}\varepsilon\Lambda$. A limit within that hierarchy requires a joint family with $\Lambda \to \infty$, together with the necessary control of the remaining scales.

Finally, the simple weights $f(m^2/\Lambda^2)$ in lemma 03, Part A have a commuting finite-operator hypothesis. They cannot be assigned generally to broken gauge and Higgs kinetic terms. In particular, that same file's Part C gives, for a broken weak generator with one heavy and one massless partner and the squared-operator cutoff $f(x) = e^{-x}$,

$$w_L(x, 0) = 3 \int_0^1 u^2 e^{-xu}\, du = 1 - \frac{3}{4}x + O(x^2),$$

which differs from $f(x) = 1 - x + O(x^2)$. Near the cutoff, the appropriate coefficients and any replacement for the CCM relation require a calculation under specified hypotheses.

Amendments and errors found

- Correct the fourth-moment indices in Proposition 10.1's proof; the three proposition identities are unchanged.
- Specify the unrescaled Higgs field in the CCM outline and retain the three-generation scope of the cited normalization.
- In Corollary 11.2, assert divergence of $m_{\max}$, or add a hypothesis ensuring that one fixed species tends to infinity.
- In Corollary 11.3 and the interval remark, make top dominance a hypothesis; it is not forced by the mass relation.
- In the regime amendment, qualify fixed-Yukawa claims, distinguish the truncated action from the finite-cutoff action, and restrict the simple threshold weights to their applicable commuting setting.

Overall: 5 statements checked; 1 correct, 1 amended, 0 gaps, 3 errors.

Part 12

Boundary Yukawa matrices on the interval: from chiral to vectorlike boundary conditions, and towerless textures

Theorems T3 (remaining part) and T8 of ../STRATEGY.md. Proved by Claude Opus 5.5 without Astra or Fable. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026): of 9 statements, 4 correct, 2 needed amendments and 3 were wrong as first stated (Theorem 12.4 item 2, Theorem 12.7, Corollary 12.9); all corrections are incorporated below and the referee report is appended. The secular equation was checked against direct numerical integration of the ODE; the bounds of Theorem 12.6 and Lemma 12.8 were checked on random samples (section "Numerical check").

Setting

Let I = [0, l]. A doublet-type field Q in L^2(I; C^2) (x) C^{N_Q} and a singlet-type field U in L^2(I; C^2) (x) C^{N_U} have components (L, R), with grading Gamma = diag(1, -1) on each. As in lemma 09, the bulk operator is (D psi)_L = R' + M R, (D psi)_R = -L' + M L, with diagonal bulk masses M_Q = diag(m_i^Q) and M_U = diag(m_j^U). For a complex N_Q x N_U matrix X, let D_X be this operator on the domain of H^1 functions with

```
Q_R(0) = 0,   U_L(0) = 0,   Q_R(l) = X U_R(l),   U_L(l) = -X^dagger Q_L(l).
```

At X = 0, Q has boundary type (a) and U type (b) of lemma 09. The matrix X is a boundary Yukawa coupling: it couples the two fields only at the wall x = l. We write c_i^Q = -m_i^Q l and c_j^U = m_j^U l, so that c > 0 means that the zero mode of the field at X = 0 is localized near x = 0, away from the Yukawa wall.

Lemma 12.1. D_X is self-adjoint with compact resolvent, and Gamma D_X = -D_X Gamma.

Proof. The boundary form of lemma 9.1, summed over both fields, is [conj(Q_R) phi_{Q,L} - conj(Q_L) phi_{Q,R} + conj(U_R) phi_{U,L} - conj(U_L) phi_{U,R}] evaluated between 0 and l. At x = l the conditions leave Q_L and U_R free, which is half of the boundary values, and on them the form becomes conj(U_R) X^dagger phi_{Q,L} - conj(Q_L) X phi_{U,R} - conj(U_R) X^dagger phi_{Q,L} + conj(Q_L) X phi_{U,R} = 0. At x = 0 the conditions are those of types (a) and (b). Hence the conditions define a Lagrangian subspace, and the argument of lemma 9.1 applies. Gamma flips the signs of Q_R and U_R together, so it preserves each condition. ■

Kernel, index and symmetry breaking

Theorem 12.2. For every X,

```
ker D_X = {(e^{M_Q x} a, 0; 0, 0) : X^dagger e^{M_Q l} a = 0}  (+)  {(0, 0; 0, e^{-M_U x} b) : X
↪  e^{-M_U l} b = 0},
```

so dim ker_{Gamma=+1} = N_Q - rank X, dim ker_{Gamma=-1} = N_U - rank X, and ind D_X = N_Q - N_U.

Proof. As in Theorem 9.2, D psi = 0 forces L = e^{Mx} a and R = e^{-Mx} b on each field. The conditions at 0 remove Q_R and U_L, and the conditions at l reduce to X U_R(l) = 0 and X^dagger Q_L(l) = 0. ■

Let a compact group G act unitarily on C^{N_Q} and C^{N_U} by rho_Q and rho_U, commuting with the bulk masses. The domain of D_X is G-invariant exactly for g in the stabilizer H_X = {g : rho_Q(g) X = X rho_U(g)}.

Corollary 12.3 (equivariant index and a no-go). In R(H_X),

```
ind_{H_X} D_X = [ker X^dagger] - [ker X] = [rho_Q] - [rho_U]   (restricted to H_X),
```

because X maps (ker X)^perp isomorphically and H_X-equivariantly onto ran X = (ker X^dagger)^perp. At X = 0 we have H_X = G, and the index is the chiral class [V_L] - [V_R] of Corollary 9.3. D_X has no zero modes iff X is invertible. This requires rho_Q and rho_U to be isomorphic as H_X-representations, i.e. the residual theory must be vectorlike. For the Standard Model with H = SU(3) x U(1)_em, the quark sector satisfies this. A lepton sector without nu_R does not: a massless left-handed neutrino is forced for every X preserving electromagnetism (H contained in H_X), in agreement with the unpaired nu_L of job 06. ■

Exact secular equation

For a field of mass m and lambda > 0 let omega = (m^2 - lambda^2)^{1/2}, and read omega coth(omega l) as k cot(k l) when omega = ik and as 1/l when omega = 0. Define

```
alpha_i(lambda) = omega coth(omega l) + m_i^Q,     beta_j(lambda) = omega coth(omega l) - m_j^U,
```

A = diag(alpha_i) and B = diag(beta_j). Let W be the set of lambda > 0 with lambda^2 < m^2 + pi^2/l^2 for every bulk mass m of the system.

Theorem 12.4. For lambda in W, lambda is an eigenvalue of D_X iff det(lambda^2 - A(lambda) X B(lambda) X^dagger) = 0, and its multiplicity is dim ker(lambda^2 - A X B X^dagger). Moreover:

1. alpha_i(0) = 2m/(1 - e^{-2ml}) with m = m_i^Q, which equals f_{Q_i}^2, the value at x = l of the normalized zero mode e^{mx}; likewise beta_j(0) = 2m/(e^{2ml} - 1) = f_{U_j}^2 for e^{-mx}. In terms of c, f^2 = phi(c)^2/l with phi(c)^2 = 2c/(e^{2c} - 1) and phi(0) = 1.
2. On its own window 0 < lambda^2 < m^2 + pi^2/l^2, each alpha_i (and each beta_j) is a strictly decreasing, concave function of lambda^2, positive at 0 and tending to -infinity at the edge of that window, so it vanishes at exactly one point: the lowest positive eigenvalue Lambda_i^Q of Q_i alone with R(0) = 0 and L(l) = 0, respectively Lambda_j^U of U_j alone with L(0) = 0 and R(l) = 0 (the type-(c) problems of lemma 09). On the common window W it is still decreasing and concave but may stay positive (e.g. l = 1, M_Q = diag(0, 4): alpha_2 > 0 on all of W = (0, pi)).

Proof. With R(0) = 0, the Q-field solution is R = a sinh(omega x)/omega and L = a(cosh(omega x) + m sinh(omega x)/omega)/lambda. With L(0) = 0, the U-field solution is L = b sinh(omega x)/omega and R = b(m sinh(omega x)/omega - cosh(omega x))/lambda. On W, every S = sinh(omega l)/omega is nonzero. Inserting these into the conditions at l and eliminating b gives lambda^2 c = X B X^dagger A c for c = diag(S_Q) a. The map c -> eigenfunction is linear and injective. Finally, lambda^2 - X B X^dagger A and lambda^2 - A X B X^dagger have kernels of equal dimension for lambda != 0. For item 2, the expansion x coth x = 1 + sum_{n >= 1} 2x^2/(x^2 + n^2 pi^2) gives

```
z^{1/2} coth(z^{1/2} l) = (1/l) [1 + sum_{n >= 1} 2 z l^2 / (z l^2 + n^2 pi^2)],
```

which is strictly increasing and concave in z > -pi^2/l^2. Substituting z = m^2 - lambda^2 gives the claim. The function tends to -infinity at the edge of its own window, and alpha_i = 0 means Q_L(l) = 0. ■

Example 12.5 (one flavor, no bulk mass: the interpolation). For N_Q = N_U = 1 and m = 0 we get alpha = beta = lambda cot(lambda l), and the equation reduces to tan^2(lambda l) = abs(X)^2. With theta = arctan abs(X) in [0, pi/2], the positive spectrum is

```
theta/l,   (n pi - theta)/l,   (n pi + theta)/l   (n >= 1).
```

At X = 0 the light pair is the pair of chiral zero modes of types (a) and (b): zero then has multiplicity two and the positive eigenvalues are n pi/l, each of multiplicity two; theta = pi/2 is reached only as a limit. As abs(X) -> infinity the conditions tend to Q_L(l) = 0 and U_R(l) = 0, so each field becomes type (c), and the spectrum tends to two copies of (r - 1/2) pi/l. The light pair theta/l therefore interpolates continuously between the chiral zero modes and the Connes-Lott mass pi/(2l) of job 05. Its eigenvectors span a Gamma-graded two-dimensional space on which D acts as (theta/l) sigma_1, with the Gamma = +1 vector carried by (Q_L, U_L) and the Gamma = -1 vector by (Q_R, U_R). ■

Light spectrum and towerless limit

Let F_Q = diag(f_{Q_i}), F_U = diag(f_{U_j}) and Y_0 = F_Q X F_U. Let Lambda_Q = min_i Lambda_i^Q, Lambda_U = min_j Lambda_j^U and lambda_* = min(Lambda_Q, Lambda_U). For 0 < lambda < lambda_* the matrices A and B are positive, and we put Y(lambda) = A(lambda)^{1/2} X B(lambda)^{1/2}. Singular values sigma_k are listed in decreasing order.

Theorem 12.6.

1. lambda in (0, lambda_*) is an eigenvalue of multiplicity m iff exactly m indices k satisfy sigma_k(Y(lambda)) = lambda.
2. For each k, the equation lambda = sigma_k(Y(lambda)) has at most one solution lambda_k in (0, lambda_). *It has one whenever 0 < sigma_k(Y_0) < lambda_.*
3. Every such solution satisfies

   ```
   sigma_k(Y_0) (1 - lambda_k^2/Lambda_Q^2 - lambda_k^2/Lambda_U^2) <= lambda_k <= sigma_k(Y_0).
   ```

Proof. Each Lambda lies in the window of its own field, so (0, lambda_) *is contained in W. (1) Conjugating by A^{1/2} turns A X B X^dagger into Y Y^dagger, and dim ker(lambda^2 - Y Y^dagger) = #{k : sigma_k(Y) = lambda}. Then apply Theorem 12.4. (2) A and B decrease entrywise in lambda, and sigma_k(P Z Q) <= norm(P) norm(Q) sigma_k(Z). Hence sigma_k(Y(lambda)) is continuous and nonincreasing, so lambda - sigma_k(Y(lambda)) is strictly increasing. It is negative near 0 and at least lambda_* - sigma_k(Y_0) > 0 near lambda_*. (3) The upper bound is (2). For the lower bound, concavity places alpha_i above its chord: alpha_i(lambda) >= alpha_i(0)(1 - lambda^2/(Lambda_i^Q)^2), and likewise for beta_j. Therefore Y(lambda) = D_1 Y_0 D_2 with diagonal D_1, D_2 whose entries are at least (1 - t_Q)^{1/2} and (1 - t_U)^{1/2}, where t_Q = lambda^2/Lambda_Q^2 and t_U = lambda^2/Lambda_U^2. Hence sigma_k(Y) >= ((1 - t_Q)(1 - t_U))^{1/2} sigma_k(Y_0) >= (1 - t_Q - t_U) sigma_k(Y_0). The last step uses a^2 + ab + b^2 <= a + b when a + b <= 1; for a + b > 1 the bound is trivial. ■

Theorem 12.7 (towerless limit of the texture). Fix the localization parameters c_i^Q and c_j^U and the matrix Y_hat = X/l, and let l -> 0. Put Phi_Q = diag(phi(c_i^Q)), Phi_U = diag(phi(c_j^U)) and M_F = Phi_Q Y_hat Phi_U. For every E > 0 and all small l, the spectrum of D_X in [-E, E] consists of:

- 0 with multiplicity N_Q + N_U - 2 rank Y_hat;
- plus/minus lambda_k(l) for those k <= rank Y_hat with sigma_k(M_F) <= E (since lambda_k(l) < sigma_k(M_F) and lambda_k(l) -> sigma_k(M_F)), where

  ```
  abs(lambda_k(l) - sigma_k(M_F)) <= sigma_k(M_F)^3 l^2 (1/L_Q^2 + 1/L_U^2),
  ```

  and L_Q = l Lambda_Q and L_U = l Lambda_U depend only on the c's.

All eigenvalues outside zero and the light set {+/- lambda_k} have absolute value greater than lambda_* = min(L_Q, L_U)/l, which tends to infinity. For E > norm(M_F) the light space has dimension N_Q + N_U and, after grading-preserving unitary identifications, its compressed Dirac matrices converge to D_F below. Calling the limit a spectral triple requires specifying an algebra and its representation; with that understood, the towerless remnant is the finite even triple with H_F = C^{N_Q} (+) C^{N_U}, Gamma_F = diag(1, -1) and

```
D_F = [[0, M_F], [M_F^dagger, 0]],   M_F = Phi_Q Y_hat Phi_U,
```

up to unitaries on each chirality, which is the Yukawa block of the Chamseddine-Connes finite Dirac operator.

Proof. By item 1 of Theorem 12.4, Y_0 = (1/l) Phi_Q X Phi_U = M_F. Rescaling x -> x/l shows that l Lambda_i depends on c_i alone. Apply Theorem 12.6 with sigma_k(Y_0) fixed and lambda_* -> infinity, and use lambda_k <= sigma_k(M_F) in the error term. Theorem 12.2 counts the zero modes. ■

At fixed fermion masses, the towerless limit therefore requires X = l Y_hat -> 0: the boundary condition tends to the chiral type (a) (+) (b), while the normalized profiles at the wall grow like l^{-1/2}.

Hierarchies from localization

Lemma 12.8 (graded singular values). Let M = F X G with F = diag(f_1 <= … <= f_N) and G = diag(g_1 <= … <= g_N) positive, and X in M_N(C). For p = 1, …, N let T_p = {N-p+1, …, N}, Delta_p = abs(det X_{T_p T_p}), D_p = max over p-subsets I, J of abs(det X_{IJ}), eps_p^F = f_{N-p}/f_{N-p+1}, eps_p^G = g_{N-p}/g_{N-p+1} (both 0 for p = N), and K_p = binom(N, p) - 1. Then

```
f_{T_p} g_{T_p} Delta_p  <=  sigma_1 ... sigma_p  <=  f_{T_p} g_{T_p} [Delta_p + D_p ((1 + K_p
↪  eps_p^F)(1 + K_p eps_p^G) - 1)],
```

where f_I is the product of f_i over i in I.

Proof. By Cauchy-Binet, the p-th compound is Lambda^p M = Lambda^p F Lambda^p X Lambda^p G, with entries f_I g_J det X_{IJ}, and sigma_1 … sigma_p = norm(Lambda^p M). The lower bound is the (T_p, T_p) entry. For the upper bound, norm(A) <= sum abs(A_{IJ}). Any p-subset I != T_p trades at least one index of T_p for an index <= N - p, so f_I/f_{T_p} <= eps_p^F. Hence sum_I f_I <= f_{T_p}(1 + K_p eps_p^F), and similarly for g. ■

Corollary 12.9 (masses from localization). Assume every trailing minor Delta_p is nonzero. Put E_0 = E_N = 0 and E_p = (D_p/Delta_p)((1 + K_p eps_p^F)(1 + K_p eps_p^G) - 1) for 1 <= p < N. Then

```
1/(1 + E_{p-1}) <= sigma_p / (f_{N-p+1} g_{N-p+1} Delta_p/Delta_{p-1}) <= 1 + E_p,
```

by dividing the bounds of Lemma 12.8 for p and p - 1. Indexing generations by k = N - p + 1, m_k = f_k g_k (Delta_{N-k+1}/Delta_{N-k})(1 + O(eps)) as eps = max_p (eps_p^F + eps_p^G) -> 0, for fixed X and N (uniform statements need uniform bounds on D_p/Delta_p). Without the minor condition the formula fails: F = G = diag(t, 1) and X = sigma_1 give Delta_1 = 0 while both singular values equal t. With M_F = Phi_Q Y_hat Phi_U and Y_hat anarchic (all minors of order one), the masses follow phi(c_k^Q) phi(c_k^U). Since phi(c) = (2c)^{1/2} e^{-c} (1 + O(e^{-2c})) for large c, order-one differences

of the c's between generations give exponential hierarchies (the ratio also carries the factor sqrt(c_k^Q c_k^U/(c_{k+1}^Q c_{k+1}^U)), bounded when adjacent c's differ by bounded amounts):

```
m_k / m_{k+1} ~ e^{-(c_k^Q - c_{k+1}^Q) - (c_k^U - c_{k+1}^U)}   up to factors of order one. ■
```

Remark 12.10 (mixing; numerical, not proved here). For an up sector and a down sector sharing Phi_Q, we drew 500 anarchic complex samples with c^Q = (3, 2, 0.5). The medians of abs(V_ij)/(phi(c_i^Q)/phi(c_j^Q)) were 1.02, 0.92 and 1.09 for (ij) = (12), (23) and (13). So the left mixing follows the doublet localization alone, abs(V_ij) ~ phi(c_i^Q)/phi(c_j^Q), as in the split-fermion and warped-flavor literature (Huber-Shafi 2001; Agashe-Perez-Soni 2005; Casagrande et al. 2008). An upper bound with explicit constants, |V_ij| <= 3 K_u K_d f_i/f_j for i < j, is proved in the referee report appended below, which also notes that there is no universal lower bound; Astra 6 wrote out the cofactor step that the report names (validation, item 10).

Consequences for STRATEGY

- **T3 (remaining part).** The boundary Yukawa X deforms the chiral types (a) (+) (b) (X = 0) continuously into two copies of the vectorlike type (c) (abs(X) -> infinity). The index N_Q - N_U is constant along the way. What changes is the symmetry: X reduces G to H_X, and the zero modes lift exactly as far as the H_X-index allows (Corollary 12.3).
- **T8.** In the towerless limit, the Yukawa block of D_F is Phi_Q Y_hat Phi_U. The texture of the finite geometry is localization data (the c's) times an order-one boundary matrix, the masses are internal momenta and overlaps, and the hierarchies are exponential in the c's (Corollary 12.9).

Numerical check (Claude)

Method: roots of the secular equation are bracketed on fine grids and refined by Brent's method; the shooting check integrates the ODE from x = 0 with scipy `solve_ivp` (rtol 1e-12) and finds the zeros of the boundary-condition determinant at x = l.

- **Secular equation against ODE shooting** (scipy `solve_ivp`, rtol 1e-12, real X). For one massless flavor and X = 0.3, 1 and 5, the shooting roots agree with (n pi +/- arctan X)/l to 6 digits. For three flavors with masses (-3, -1.5, 0.2) and (2.5, 1, -0.3), l = 1 and random X, both methods give 0.024689, 0.048653, 0.534569 and 2.03374.
- **Theorem 12.6** (c^Q = (2, 1, 0), c^U = (1.5, 0.5, -0.5), X = 0.05 x Gaussian, lambda_* = 0.577). The eigenvalues in the window are 0.000576, 0.035602 and 0.061096, against sigma_k(Y_0) = 0.000576, 0.035632 and 0.061188. All three two-sided bounds hold.
- **Theorem 12.7** (c^Q = (4, 2, 0.3), c^U = (3, 1, -0.2), Gaussian Y_hat, sigma(M_F) = 0.000735, 0.072042, 0.786929). The relative errors of the three light eigenvalues are:

l	relative errors	lambda_*
1	1.5e-5, 1.1e-2, 2.5e-1	0.147
0.1	below 1e-6, 1.1e-4, 4.2e-3	1.47
0.03	below 1e-6, 1.0e-5, 3.9e-4	4.90

  The errors decrease as l^2.

- **Lemma 12.8:** no violation in 2000 random complex 3 x 3 cases with log-uniform gradings over 8 e-folds (6000 inequalities).

Referee report (gpt-6.1-terra, effort max, 3 Nov 2026)

Prompt: prompts/terra_check_12.md. Reproduced as received; the corrections above implement it. Claude checked the three counterexamples (M_Q = diag(0, 4); arctan(l)/l -> 1; F = G = diag(t, 1), X = sigma_1) by hand.

We take $\ell > 0$ and the bulk masses to be real, as in Lemma 09. All arguments below are analytic; we used no numerical computation.

Lemma 12.1 Verdict: Correct.

At $x = \ell$, the free boundary values are $Q_L(\ell)$ and $U_R(\ell)$. Substitution of the boundary conditions into the boundary form gives zero, and the allowed boundary values have half the total boundary dimension. The conditions at $x = 0$ are likewise maximal isotropic. The adjoint boundary conditions therefore coincide with the stated ones, so D_X is self-adjoint.

The graph norm controls the H^1-norm, and the embedding $H^1([0,\ell]) \hookrightarrow L^2([0,\ell])$ is compact. Thus the resolvent is compact. Finally, Γ preserves the boundary conditions and anticommutes with the differential expression.

Theorem 12.2 Verdict: Correct.

The zero-mode equations give

$$L(x) = e^{Mx}a, \qquad R(x) = e^{-Mx}b.$$

The conditions at 0 remove Q_R and U_L, and those at ℓ impose exactly the two displayed kernel conditions.

The endpoint exponentials are invertible, so the two chiral kernel dimensions are $N_Q - \operatorname{rank} X$ and $N_U - \operatorname{rank} X$. Their difference is $N_Q - N_U$.

Corollary 12.3 Verdict: Correct with amendment. The neutrino conclusion holds for every X that preserves the stated electromagnetic symmetry, meaning $H \subseteq H_X$. The phrase "for every X" needs that qualification.

The endpoint exponentials commute with the group actions. They identify the actual chiral kernels equivariantly with $\ker X^\dagger$ and $\ker X$. The orthogonal complements are equivariantly isomorphic through X, proving the displayed index identity. Both kernels vanish exactly when X is invertible, in which case X is an equivariant isomorphism.

If electromagnetic symmetry is not required, the neutrino assertion can fail. For one lepton generation with zero bulk masses, take

$$Q_L = (\nu_L, e_L), \qquad U_R = e_R, \qquad X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The left kernel is then spanned by e_L, while ν_L is coupled to e_R. This X breaks electromagnetic charge. If X intertwines the electromagnetic representations and there is no neutral right-handed lepton, the neutrino subspace lies in $\ker X^\dagger$, as claimed.

Theorem 12.4 Verdict: Error. The secular equation, multiplicity formula, endpoint values, monotonicity and concavity are correct. The assertion that *each* function has a zero **on the common window** W is false.

For example, take $\ell = 1$, $M_Q = \operatorname{diag}(0, 4)$, and $M_U = (0)$. Then $W = (0, \pi)$, but throughout W,

$$\alpha_2(\lambda) = \sqrt{16 - \lambda^2}\, \coth\sqrt{16 - \lambda^2} + 4 > 0.$$

Its individual lowest mixed-boundary eigenvalue is greater than 4, hence lies outside W. The proof's assertion that every function tends to $-\infty$ at the edge of W fails for the same reason.

The corrected zero statement uses each mass's **individual** first-pole window

$$0 < \lambda < \sqrt{m^2 + \pi^2/\ell^2}.$$

On that interval the corresponding α or β has exactly one zero, at the lowest positive eigenvalue of its mixed-boundary problem. The U problem is equivalent to the type-(c) problem after exchanging components and changing the mass sign.

For the monotonicity and concavity, write $s = \lambda^2$ and $a_n = (n\pi/\ell)^2$. Differentiating the displayed partial-fraction expansion on the individual window gives

$$\frac{d}{ds}(\omega \coth(\omega\ell)) = -\frac{2}{\ell}\sum_{n\geq 1}\frac{a_n}{(m^2 - s + a_n)^2} < 0,$$

$$\frac{d^2}{ds^2}(\omega \coth(\omega\ell)) = -\frac{4}{\ell}\sum_{n\geq 1}\frac{a_n}{(m^2 - s + a_n)^3} < 0.$$

The function is positive at $s = 0$ after the appropriate $\pm m$ is added, and tends to $-\infty$ at its own first pole. This proves the corrected zero claim.

For the secular equation, the proof's boundary variables c, d satisfy

$$\lambda c = -XBd, \qquad \lambda d = -X^\dagger Ac.$$

Eliminating d gives $\lambda^2 c = XBX^\dagger Ac$. For $\lambda \neq 0$, the kernels of $\lambda^2 I - CA$ and $\lambda^2 I - AC$, with $C = XBX^\dagger$, are isomorphic via A, with inverse C/λ^2. This also proves the stated multiplicity.

Example 12.5 Verdict: Correct with amendment. At $X = 0$, omit $\theta/\ell = 0$ from the **positive** spectrum; instead, zero has multiplicity two. The positive eigenvalues are then $n\pi/\ell$, each with multiplicity two. For finite X, $\theta < \pi/2$; the endpoint $\theta = \pi/2$ is a limit.

For all positive λ, direct substitution of the massless solutions into the boundary conditions gives

$$\sin^2(\lambda\ell) - |X|^2\cos^2(\lambda\ell) = 0.$$

For $X \neq 0$, its roots are exactly the listed values. This derivation also includes the roots at $n\pi/\ell$ when $X = 0$, even though the secular formula of Theorem 12.4 was derived only before the first pole. The negative spectrum follows by grading symmetry. The stated two-dimensional graded block and the limiting mixed-boundary spectrum follow from these roots and their eigenvectors.

Theorem 12.6 Verdict: Correct.

Although Theorem 12.4 requires the correction above, it still supplies what is needed here. The smallest individual first pole belongs to some field; that field's first zero lies before its pole. Consequently, λ_* lies before the smallest first pole, and $(0, \lambda_*) \subset W$.

On this interval A, B are positive. Conjugation reduces the secular equation to the eigenvalue equation for $Y(\lambda)Y(\lambda)^\dagger$, proving the multiplicity claim. If $\lambda_2 > \lambda_1$, then

$$Y(\lambda_2) = P\,Y(\lambda_1)\,Q$$

for diagonal contractions P, Q. Thus every singular value is continuous and nonincreasing, so it meets the strictly increasing function λ at most once. The stated condition at $\lambda = 0$ gives a crossing before λ_*.

Concavity gives, with $t_Q = \lambda^2/\Lambda_Q^2$ and $t_U = \lambda^2/\Lambda_U^2$,

$$\sigma_k(Y(\lambda)) \geq \sqrt{(1 - t_Q)(1 - t_U)}\,\sigma_k(Y_0) \geq (1 - t_Q - t_U)\sigma_k(Y_0).$$

Together with $\sigma_k(Y(\lambda)) \leq \sigma_k(Y_0)$, this proves both bounds.

Theorem 12.7 Verdict: Error. The fixed-energy spectrum cannot contain every listed light eigenvalue for every $E > 0$.

Take one massless flavor, $\widehat{Y} = 1$, and $E = 1/2$. Then $X = \ell$, and Example 12.5 gives the positive light eigenvalue

$$\lambda(\ell) = \frac{\arctan\ell}{\ell} \longrightarrow 1.$$

For sufficiently small ℓ, it is outside $[-1/2, 1/2]$, contrary to the stated list.

The corrected fixed-energy statement includes only those $\pm\lambda_k(\ell)$ with $\lambda_k(\ell) \le E$. Equivalently, for each fixed $E > 0$ and all sufficiently small ℓ, include exactly those indices for which

$$\sigma_k(M_F) \le E.$$

The equality case is included because the strict decrease of A and B gives $\lambda_k(\ell) < \sigma_k(M_F)$ for every positive light eigenvalue.

The displayed error estimate follows from Theorem 12.6:

$$0 < \sigma_k(M_F) - \lambda_k(\ell) \le \sigma_k(M_F)^3 \ell^2 \left(L_Q^{-2} + L_U^{-2}\right).$$

Zero has the stated multiplicity. Apart from zero and the **entire** light set $\{\pm\lambda_k\}$, the eigenvalues have **absolute value** greater than λ_* for small ℓ. At λ_*, the limiting matrices A, B are nonnegative and the relevant singular values are at most $\|M_F\|$, whereas $\lambda_* \to \infty$; hence equality at the boundary is excluded for small ℓ.

For $E > \|M_F\|$, the full low-energy space has dimension $N_Q + N_U$. After grading-preserving unitary identifications, its compressed Dirac matrices converge to the displayed D_F. Calling the limit a spectral triple additionally requires an algebra and its representation to be specified.

Lemma 12.8 Verdict: Correct.

The entries of $\bigwedge^p M$ are $f_I g_J \det X_{IJ}$, and its operator norm is $\sigma_1 \cdots \sigma_p$. Its (T_p, T_p) entry gives the lower bound.

For every $I \neq T_p$, at least one factor from the top p indices is replaced by one no larger than $N - p$, so $f_I / f_{T_p} \le \epsilon_p^F$; the analogous estimate holds for g. Bounding the operator norm by the sum of absolute entries, while keeping the (T_p, T_p) entry separate, gives the upper bound. For $p = N$, both bounds reduce to the determinant identity.

Corollary 12.9 Verdict: Error. The displayed mass formula is not valid under the hypotheses of Lemma 12.8: its denominator can vanish, and its leading term can be zero when the singular value is positive.

For $0 < t < 1$, take

$$F = G = \operatorname{diag}(t, 1), \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then FXG has both singular values equal to t, but $\Delta_1 = 0$ and $\Delta_2 = 1$. The formula for m_2 gives zero, and the formula for m_1 divides by zero.

A sufficient correction is to require every trailing minor Δ_p to be nonzero and to define the small parameter. More precisely, put $E_0 = E_N = 0$ and, for $1 \le p < N$,

$$E_p = \frac{D_p}{\Delta_p} \left[(1 + K_p \epsilon_p^F)(1 + K_p \epsilon_p^G) - 1\right].$$

Lemma 12.8 then gives the explicit bounds

$$\frac{1}{1 + E_{p-1}} \le \frac{\sigma_p}{f_{N-p+1} g_{N-p+1} \Delta_p / \Delta_{p-1}} \le 1 + E_p.$$

Indeed, apply the lemma to the products of the first p and first $p-1$ singular values and divide their bounds. For fixed X, N, if

$$\epsilon = \max_{p<N} (\epsilon_p^F + \epsilon_p^G) \longrightarrow 0,$$

these inequalities give the stated $1 + O(\epsilon)$ formula. Uniform estimates require uniform bounds on D_p/Δ_p.

The localization conclusion follows under quantitative order-one upper and lower bounds on the relevant minors. For large positive c, the mass ratio also contains the factor

$$\sqrt{\frac{c_k^Q c_k^U}{c_{k+1}^Q c_{k+1}^U}}.$$

It is bounded when the adjacent large c's differ by bounded amounts; without such control it cannot be suppressed uniformly as an order-one factor.

Remark 12.10 Bound. We can prove an upper bound independent of the right-hand localization profiles. It does not prove the reported sample medians or a universal lower bound.

For each sector $s = u, d$, let

$$M_s = FX_sG_s, \qquad F = \mathrm{diag}(f_1 \le f_2 \le f_3), \qquad G_s = \mathrm{diag}(g_1^{(s)} \le g_2^{(s)} \le g_3^{(s)}),$$

with all diagonal entries positive. Assume X_s is invertible and its trailing minors $\Delta_1^{(s)}, \Delta_2^{(s)}$ are nonzero. With $D_p^{(s)}$ as in Lemma 12.8, define

$$K_s = \left(1 + \frac{\sqrt{3} D_1^{(s)}}{\Delta_1^{(s)}}\right)\left(1 + \frac{\sqrt{3} D_2^{(s)}}{\Delta_2^{(s)}}\right).$$

Choose the columns of the left singular-vector matrices U_s in increasing order of singular value, and put $V = U_u^\dagger U_d$. Then, for $i < j$,

$$\boxed{|V_{ij}| \le 3 K_u K_d \frac{f_i}{f_j}.}$$

To prove this, set $H_s = M_s M_s^\dagger$, and temporarily suppress s. Write

$$a = \frac{\sqrt{3} D_1}{\Delta_1}, \qquad b = \frac{\sqrt{3} D_2}{\Delta_2}.$$

For the largest eigenvalue λ_3 of H,

$$\lambda_3 \ge f_3^2 g_3^2 \Delta_1^2, \qquad H_{ii} \le 3 D_1^2 f_i^2 g_3^2.$$

Cauchy–Schwarz therefore gives $|U_{i3}| \le a f_i / f_3$. Applying the same argument to H^{-1}, and using $(X^{-1})_{11} = \det X_{\{2,3\},\{2,3\}} / \det X$, gives $|U_{i1}| \le b f_1 / f_i$. Cofactor identities for the unitary U bound the two remaining off-diagonal entries. Together,

$$|U_{ij}| \le (1 + a)(1 + b) \min\left(\frac{f_i}{f_j}, \frac{f_j}{f_i}\right).$$

Summing the three terms in $V_{ij} = \sum_k \overline{(U_u)_{ki}} (U_d)_{kj}$ proves the boxed inequality.

There is no corresponding positive universal lower bound: identical up and down matrices have $V = I$, including when their minors are nonzero. The numerical medians also require the sampling distribution and sampled matrices, neither of which is specified in the remark.

Amendments and errors

- **12.3:** Restrict the neutrino assertion to couplings preserving electromagnetic symmetry.
- **12.4:** State the unique-zero claim on each field's individual first-pole window; it can fail on the common window.
- **12.5:** Treat zero separately at $X = 0$, and treat $\theta = \pi/2$ as a limit.
- **12.7:** Filter the light eigenvalues by E; state the gap in absolute value; specify an algebra and representation for a spectral-triple claim.

- **12.9:** Require nonzero trailing minors and define the separation parameter and its dependence. Retain the square-root localization factors unless they are uniformly bounded.

Overall: 9 statements checked: 4 correct, 2 correct with amendment, 0 gaps, and 3 errors.

Literature (../LITERATURE.md, section C). The one-flavour spectrum of Example 12.5 is equation (5.6) of Csaki-Grojean-Hubisz-Shirman-Terning, Fermions on an interval (2004); the warped multi-flavour analogue of the light spectrum and of the localization textures is in Casagrande et al. 2008 (equations 65, 87, 90, 94-97). No NCG paper deriving D_F from localization was found.

Part 13

T4: gauge fluctuations of the chiral interval

Prompt: prompts/13 (STRATEGY T4). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5 by hand: Results 2, 3, 4, 5 and 6 (the compression integrals, the counterexample B = -2a/(3 pi) with trivial Wilson line, the Neumann-Dirichlet mass, the 1 : 3 ratio of the a_4 gauge and scalar terms with the canonical normalization, and the eigenvalues 0 and 6m^2); Results 7 and 8 reproduce the standard CCM relations (8 M_W^2 = sum N_f m_f^2, M_W/M_Z = sqrt(5/8) with Q_2 = 12, Q_Y = 20).

Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): of 8 statements, 7 are correct and Proposition 8 needs one amendment, recorded there.

We prove an exact compression formula, but it does not identify the continuum gauge theory with the full Connes–Lott gauge theory. The fermion boundary conditions impose no boundary conditions on the gauge potentials, and fixed spectral compression produces only links of the form `phi = m I + i B`, with `B` Hermitian. The proposed ratio `M_W/m = sqrt(3/8)` is correct under the three-generation Standard Model CCM normalization, with one massive colour triplet and all other Dirac masses zero. It is not a consequence of the interval alone: the minimal two-sheet spectral action has a different gauge representation and gives a massive relative gauge field of mass `sqrt(6) m`.

1. Theorem (inner fluctuations and boundary conditions).

Let M be a closed Riemannian spin four-manifold. We use Hermitian Euclidean gamma matrices, $D_M = -i\gamma^\mu \nabla^S_\mu$, and

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad D = D_M \otimes 1 + \gamma_5 \otimes E\partial_x.$$

Let

$$\mathcal{H} = L^2(S_M) \otimes L^2([0,l], \mathbb{C}^2) \otimes \mathbb{C}^n,$$

with the algebra acting by the same matrix multiplication on L and R. The interval boundary conditions are

$$R(y,0) = 0, \qquad L(y,l) = 0.$$

We consider complex inner fluctuations $D + \omega$, where $\omega = \sum_j a_j[D, b_j]$ and $\omega^* = \omega$. No real structure or additional bimodule is assumed.

Then the fluctuated operators are exactly

$$\boxed{D_A = -i\gamma^\mu(\nabla^S_\mu + iA_\mu) + \gamma_5 E(\partial_x + iA_5),}$$

where $A_\mu(y,x)$ and $A_5(y,x)$ are arbitrary smooth Hermitian $n \times n$ matrices. They are connections on the trivial $U(n)$ bundle over $M \times I_l$. Restriction to traceless potentials gives the $SU(n)$ case.

The fermion conditions induce **no Dirichlet, Neumann, or endpoint-value conditions on A_μ or A_5.**

Proof. For a matrix-valued function b,

$$[D, b] = -i\gamma^\mu \partial_\mu b + \gamma_5 E \partial_x b.$$

Consequently, represented one-forms have precisely these five Clifford components. Self-adjointness makes their connection coefficients anti-Hermitian, hence iA_α with A_α Hermitian. Conversely, smooth matrix-valued one-forms are finite sums $a\,db$, using local coordinates and a partition of unity.

Every such fluctuation is a bounded self-adjoint perturbation of D, so D_A is self-adjoint on the original domain. A bounded perturbation need not map that domain into itself; requiring this would impose an additional, unjustified condition.

Every smooth $g : M \times I_l \to U(n)$ preserves the fermion boundary conditions:

$$(gR)(y, 0) = 0, \qquad (gL)(y, l) = 0.$$

The usual transformation

$$A_\alpha \longmapsto g A_\alpha g^* + i(\partial_\alpha g) g^*$$

therefore allows arbitrary endpoint values of g.

For smooth sections in $\mathrm{Dom}(D_A^2)$, the additional conditions are

$$(\partial_x + iA_5)L(y, 0) = 0, \qquad (\partial_x + iA_5)R(y, l) = 0.$$

These are conditions on the fermions, not on A_5. They follow by applying the original boundary conditions to $D_A\psi$.

An orbifold extension would add different hypotheses. Requiring A_μ to be even and A_5 odd under endpoint reflections gives $\partial_x A_\mu = 0$ and $A_5 = 0$ there. Those parity requirements do not follow from the stated fermion domain. □

2. Theorem (exact fixed-projection compression and its image).

Let

$$m = \frac{\pi}{2l}, \qquad u_L = \sqrt{\frac{2}{l}} \begin{pmatrix} \cos(mx) \\ 0 \end{pmatrix}, \qquad u_R = \sqrt{\frac{2}{l}} \begin{pmatrix} 0 \\ \sin(mx) \end{pmatrix},$$

and let P project onto their span, tensored with the spacetime and gauge factors. This is the unfluctuated lowest-pair projection of Lemma 05, Part 3.

Define

$$w_L(x) = \frac{2}{l}\cos^2(mx), \qquad w_R(x) = \frac{2}{l}\sin^2(mx), \qquad k(x) = \frac{1}{l}\sin\frac{\pi x}{l}.$$

Then, under the identification $P\mathcal{H} \simeq L^2(S_M) \otimes (\mathbb{C}^n_L \oplus \mathbb{C}^n_R)$,

$$\boxed{PD_AP = -i\gamma^\mu \left(\nabla^S_\mu + i \begin{pmatrix} A_{L,\mu} & 0 \\ 0 & A_{R,\mu} \end{pmatrix} \right) + \gamma_5 \begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix},}$$

where

$$\boxed{A_{L,\mu} = \int_0^l w_L A_\mu \, dx, \qquad A_{R,\mu} = \int_0^l w_R A_\mu \, dx,}$$

and

$$\boxed{\phi = mI_n + iB, \qquad B = \int_0^l k(x) A_5(y, x)\, dx = B^*.}$$

Here ϕ denotes the total link; its additive fluctuation is $\phi - mI_n$.

The exact image consists of arbitrary Hermitian sheet connections together with links satisfying

$$\boxed{\phi + \phi^* = 2mI_n.}$$

In particular, not every Connes–Lott fluctuation arises by this compression.

Proof. The spacetime term is diagonal in interval chirality, giving the two weighted averages. The unfluctuated interval operator satisfies

$$D_I u_R = m u_L, \qquad D_I u_L = m u_R.$$

The upper-right matrix element of iEA_5 is

$$i\frac{2}{l}\int_0^l \cos(mx)\sin(mx)A_5\,dx = iB;$$

the lower-left element is $-iB$. This proves the operator identity without approximation.

The elementary integrals

$$\int_0^l w_L\,dx = \int_0^l w_R\,dx = 1, \quad \int_0^l w_L \cos\frac{\pi x}{l}\,dx = \frac{1}{2}, \quad \int_0^l w_R \cos\frac{\pi x}{l}\,dx = -\frac{1}{2}$$

show that any prescribed A_L, A_R have the lift

$$A_\mu(y,x) = \frac{A_{L,\mu} + A_{R,\mu}}{2} + (A_{L,\mu} - A_{R,\mu})\cos\frac{\pi x}{l}.$$

Since $\int_0^l k\,dx = 2/\pi$, any Hermitian B has the lift

$$A_5(y,x) = \frac{\pi}{2}B(y).$$

Thus the stated image is exact.

By contrast, the finite algebra

$$M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$$

with $D_F = m\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ admits an arbitrary complex matrix as its total link. Indeed, writing p_L, p_R for its central projections, $p_L[D_F, p_R]$ supplies an arbitrary upper-right block after multiplication by the left algebra; adding its adjoint supplies the lower-left block. The allowed finite fluctuation $\phi = 0$, for example, has no continuum lift under the displayed compression.

For $SU(2)$, B is additionally traceless. The continuum A_5 is adjoint-valued; an electroweak Higgs doublet needs a further identification.

If an additional finite factor is included in the internal-first convention, and the gauge algebra commutes with D_0, Γ_0, its compressed internal operator is

$$D_0 \otimes 1 + \Gamma_0 \otimes \begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix}.$$

There is no extra overlap factor. If $[D_0, a] \neq 0$, additional finite scalar fluctuations occur. Moreover, the cited Theorem 5 explicitly restricts the interval algebra to constants for its real-triple assertion; it does not supply a Standard Model real structure for the full compressed algebra. □

3. No-go theorem (fixed compression is not a gauge-covariant Wilson-link construction).

Let P be the fixed projection of Result 2. Its link $\phi[A_5]$ is not a functional of the Wilson line alone, and fixed compression does not descend to continuum gauge-equivalence classes.

Proof. It suffices to consider $U(1)$ potentials independent of y. Define parallel transport by

$$W'(x) = -iA_5(x)W(x), \qquad W(0) = 1.$$

Then

$$D_{I,A} = W D_I W^*.$$

Multiplication by W preserves both fermion boundary conditions, so the complete interval spectrum is unchanged.

Take

$$A_5(x) = a \cos \frac{2\pi x}{l}.$$

Its Wilson line is $W(l) = 1$, but

$$B = \frac{a}{\pi} \int_0^\pi \sin t \cos(2t)\, dt = -\frac{2a}{3\pi}.$$

Thus

$$\phi = m - \frac{2ia}{3\pi}, \qquad |\phi| = \sqrt{m^2 + \frac{4a^2}{9\pi^2}}.$$

The zero connection and this connection are gauge equivalent even through a transformation equal to 1 at both endpoints, yet their fixed compressed operators have different internal eigenvalues.

The obstruction is that multiplication by an x-dependent gauge transformation generally fails to preserve $P\mathcal{H}$. Compression of multiplication is not multiplicative, and PgP generally fails to be unitary.

The normalization difference is already visible for constant $A_5 = a$:

$$\frac{\phi}{m} = 1 + i\frac{4l}{\pi^2}a, \qquad W(l)^* = e^{ial}.$$

Neither equality with the Wilson link nor a universal correction to the mass follows from these expressions.

A different, gauge-adapted construction is possible. Use the transported interval frames

$$u_L(x)W(x), \qquad u_R(x)W(x)W(l)^*.$$

Their vertical Dirac matrix is exactly

$$\begin{pmatrix} 0 & mW(l)^* \\ mW(l) & 0 \end{pmatrix}.$$

Its link obeys $\phi^*\phi = m^2 I$, so it also omits general radial Higgs fluctuations. If W depends on y, derivatives of these frames additionally enter the sheet connections.

Finally, compression differs from eliminating the omitted fermions. With $Q = 1 - P$, block elimination adds

$$-PD_AQ\,[Q(D_A - z)Q]^{-1}QD_AP$$

whenever the inverse exists. This follows by solving the Q-component of $(D_A - z)\psi = 0$ and substituting into the P-component. It is the distinction relevant to the effective-operator discussion in research state §6.10. □

4. Lemma (the continuum vector mass requires specified gauge boundary conditions).

Let the quadratic transverse-vector operator in the interval direction be $-\partial_x^2$, with

$$f'(0) = 0, \qquad f(l) = 0.$$

Assume no boundary kinetic terms or other modifications of this operator. Then

$$\boxed{M_{\mathrm{KK}} = \frac{\pi}{2l} = m.}$$

Proof. The Neumann condition selects $f(x) = C\cos(kx)$. The Dirichlet condition gives

$$kl = \left(r + \frac{1}{2}\right)\pi, \qquad r = 0, 1, \ldots.$$

The lowest eigenvalue is therefore m^2.

A Dirichlet condition at one endpoint alone does not determine this answer. Dirichlet conditions at both endpoints give lowest mass π/l, whereas two Neumann conditions give a zero mode. Thus the comparison uses specifically the mixed boundary problem listed in Lemma 9.11, and Result 1 does not derive that problem from the fermion boundary conditions. □

5. Proposition (general spectral-action vector-mass formula).

Let H_F be a specified finite fermion space with Hermitian represented gauge generators T_a. Let

$$\mathcal{D} = -i\gamma^\mu(\nabla^S_\mu + iA_\mu) + \gamma_5 V,$$

where $V = V^*$ is the total finite scalar matrix. Use the ordinary spectral trace, an even cutoff function with $f_0 = f(0) > 0$, and the terms through order Λ^0. Assume no additional kinetic terms.

Let V_0 be the chosen constant vacuum value. Define

$$Q_{ab} = \mathrm{Re}\,\mathrm{Tr}_{H_F}(T_aT_b), \qquad K_{ab} = \mathrm{Re}\,\mathrm{Tr}_{H_F}([T_a, V_0]^*[T_b, V_0]).$$

On the faithfully represented gauge algebra, the canonically normalized vector mass-squared matrix is

$$\boxed{\mathcal{M}^2 = \frac{3}{2}\,Q^{-1/2}KQ^{-1/2}.}$$

Consequently, relative to the mixed-boundary continuum mass,

$$\boxed{\frac{M_j^2}{M_{\mathrm{KK}}^2} = \frac{3}{2m^2}\,\lambda_j\big(Q^{-1/2}KQ^{-1/2}\big)\,.}$$

Proof. We use the heat coefficient

$$a_4 \supset \frac{1}{16\pi^2}\,\mathrm{Tr}\,\left(\frac{1}{2}\mathcal{E}^2 + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu}\right),$$

as recorded in CCM, Theorem 6.1, equation (6.10). Here $\mathcal{D}^2 = \nabla^*\nabla - \mathcal{E}$.

Write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \qquad D_\mu V = \partial_\mu V + i[A_\mu, V].$$

The relevant endomorphism terms are

$$\mathcal{E}_F = \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}, \qquad \mathcal{E}_V = i\gamma^\mu\gamma_5 D_\mu V.$$

The Clifford traces give

$$\mathrm{tr}_S(\mathcal{E}_F^2) = 2F_{\mu\nu}F^{\mu\nu}, \quad \mathrm{tr}_S(\mathcal{E}_V^2) = 4D_\mu V D^\mu V, \quad \mathrm{tr}_S(\Omega_F^2) = -4F_{\mu\nu}F^{\mu\nu}.$$

Hence the relevant Lagrangian is

$$\boxed{\mathcal{L} = c\ \mathrm{Tr}_{H_F}(F_{\mu\nu}F^{\mu\nu}) + 3c\ \mathrm{Tr}_{H_F}(D_\mu V D^\mu V), \qquad c = \frac{f_0}{24\pi^2}.}$$

At $V = V_0$, the quadratic gauge terms are

$$cQ_{ab}F^a_{\mu\nu}F^{b,\mu\nu} + 3cK_{ab}A^a_\mu A^{b,\mu}.$$

The change of variables $A_{\mathrm{can}} = 2\sqrt{c}\,Q^{1/2}A$ gives kinetic normalization $F^2_{\mathrm{can}}/4$ and mass normalization $A_{\mathrm{can}}\mathcal{M}^2A_{\mathrm{can}}/2$, proving the formula.

Thus the necessary data are the represented gauge generators, their full trace multiplicities, and the vacuum matrix V_0. The common coefficient f_0 cancels. However, the bare finite Dirac matrix need not itself be the vacuum: determining V_0 requires the scalar potential, including its cutoff moments and curvature dependence. All masses below are conditional on the stated vacuum and refer to this leading spectral-action approximation. □

6. Counterexample (the minimal two-sheet action does not give the CCM triplet ratio).

Let

$$H_F = (\mathbb{C}_L^n \oplus \mathbb{C}_R^n) \otimes \mathbb{C}^N, \qquad V_0 = m \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \otimes I_N.$$

Gauge both sheet groups, $U(1)_L \times U(1)_R$ for $n = 1$, or $SU(2)_L \times SU(2)_R$ for $n = 2$. Use the action of Result 5.

Then the diagonal gauge combination is massless and the relative combination has

$$\boxed{M_{\text{relative}} = \sqrt{6}\, m.}$$

This remains true for colour multiplicity $N = 3$.

Proof. For one normalized generator t, set

$$T_L = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \qquad T_R = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}, \qquad \kappa = N \operatorname{tr}(t^2).$$

Direct multiplication gives

$$Q = \kappa I_2, \qquad K = 2\kappa m^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Therefore

$$\mathcal{M}^2 = 3m^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

whose eigenvalues are 0 and $6m^2$. The factor N cancels between scalar and gauge normalization. For $SU(2)$, this calculation applies to each $t = \sigma_a/2$.

If only the diagonal group is gauged, every generator commutes with V_0, and all these vector masses vanish. Neither choice realizes the assumed continuum mixed-boundary breaking automatically.

The dependence on the full fermion trace can also be exhibited explicitly. In the $U(1)$ example, add S otherwise identical charged pairs with zero finite Dirac operator. They contribute to Q but not K, giving

$$M^2_{\text{relative}} = 6\frac{N}{N+S}m^2.$$

The mass and colour multiplicity of the original pair have not changed.

There is another distinction if one restricts the continuum field to its lowest mixed-boundary profile before evaluating the finite action:

$$A_\mu(y, x) = a_\mu(y)\cos(mx).$$

Result 2 then gives

$$A_{L,\mu} = \frac{8}{3\pi}a_\mu, \qquad A_{R,\mu} = \frac{4}{3\pi}a_\mu.$$

For a restriction $A_L = c_L a,\ A_R = c_R a$, Result 5 gives the one-field quadratic mass

$$M^2_{\text{restricted}} = 3m^2\frac{(c_L - c_R)^2}{c_L^2 + c_R^2}.$$

Thus this particular restriction gives

$$M^2_{\text{restricted}} = \frac{3}{5}m^2.$$

It is an additional restriction of the action, and differs from the mass eigenvector of the unrestricted two-sheet theory. It demonstrates that retaining a continuum gauge profile and fluctuating the full compressed algebra are different prescriptions. □

7. Corollary (the precise conditional CCM comparison).

Let the finite model instead have the full three-generation Standard Model representation and its usual Higgs doublet. Retain all charged fermion states in the spectral trace, including those whose Yukawa masses vanish. Identify one colour-triplet Dirac mass at the vacuum with

$$m = \frac{\pi}{2l},$$

and suppose every other Dirac mass is zero.

Then, at the matching scale,

$$\boxed{M_W^2 = \frac{3}{8}m^2, \qquad \frac{M_W}{M_{\mathrm{KK}}} = \sqrt{\frac{3}{8}}.}$$

No additional factor from the interval wavefunctions or from Wilson-line normalization multiplies this ratio.

Proof. We can recover the normalization directly from Result 5. For $T_1 = \sigma_1/2$, one Standard Model generation of particles contributes

$$\mathrm{Tr}(T_1^2) = \frac{1}{2} + 3\frac{1}{2} = 2.$$

Antiparticles double this, and three generations give

$$Q_{11} = 12.$$

For a left doublet and its right singlets with mass matrix $Y = \mathrm{diag}(m_u, m_d)$, the particle contribution to K_{11} is

$$2\,\mathrm{tr}(Y^* T_1^2 Y) = \frac{1}{2}(m_u^2 + m_d^2).$$

Including antiparticles and all multiplicities gives

$$K_{11} = \sum_f N_f m_f^2.$$

Therefore

$$M_W^2 = \frac{3}{2}\frac{K_{11}}{12} = \frac{1}{8}\sum_f N_f m_f^2.$$

This is the relation in CCM, Lemma 4.6, equation (4.16), assumed in Lemma 11. Substitution of $K_{11} = 3m^2$ proves the claimed ratio.

The normalization identity is equivalently

$$\frac{f_0 g_2^2}{2\pi^2} = \frac{1}{4}.$$

Its coefficient comes from $Q_{11} = 12$. Applying that coefficient after deleting the other charged states would no longer canonically normalize the gauge kinetic term.

The interval basis already satisfies $\|u_L\| = \|u_R\| = 1$, and $\langle u_L, D_I u_R\rangle = m$. Thus its unfluctuated fermion mass is exactly m. The factor $2/\pi$ belongs to the coupling of a constant A_5 to the fixed compressed link; it does not rescale the background mass. A change of scalar coordinates likewise cannot change a mass ratio when both kinetic and Yukawa terms are transformed consistently.

If other Dirac masses are nonzero, the exact conditional comparison becomes

$$\frac{M_W^2}{M_{\mathrm{KK}}^2} = \frac{1}{8}\left(3 + \sum_{\text{other } f} N_f \frac{m_f^2}{m^2}\right).$$

If "a coloured doublet" means two massive coloured Dirac species, both of mass m, their contribution is $6m^2$, giving $M_W/m = \sqrt{3}/2$ when all other masses vanish. □

8. Proposition (electroweak doublet and the neutral-vector ratio).

Amendment (referee). In the last paragraph of this result, the law phi -> g_L phi g_R^* holds for the link of the independently gauged finite two-sheet completion. The fixed compression of the continuum field does not obey it in general: Result 3 gives an endpoint-fixed gauge transformation that changes phi. The decomposition 1 + 3 under the diagonal SU(2) stays correct.

Let an additional electroweak assignment supply one complex Higgs doublet h, with

$$T_a = \frac{\sigma_a}{2}, \qquad Y_h = \frac{1}{2}, \qquad \langle h \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix},$$

where $Q_{\mathrm{em}} = T_3 + Y$. Let the gauge fields have canonical kinetic terms and couplings g_2, g_Y.

Then

$$\boxed{M_W^2 = \frac{g_2^2 v^2}{4}, \qquad M_Z^2 = \frac{(g_2^2 + g_Y^2) v^2}{4},}$$

and consequently

$$\boxed{\frac{M_W}{M_Z} = \cos\theta_W, \qquad \rho = \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1, \quad \tan\theta_W = \frac{g_Y}{g_2}.}$$

Proof. At the vacuum,

$$|D_\mu h|^2 = \frac{v^2}{8} \left[g_2^2 ((W_\mu^1)^2 + (W_\mu^2)^2) + (g_2 W_\mu^3 - g_Y B_\mu)^2 \right].$$

Diagonalizing the neutral quadratic form gives one massless photon and the displayed Z mass.

Under a common spectral-action coefficient, canonical normalization additionally gives

$$g_2^2 Q_2 = g_Y^2 Q_Y.$$

For the full three-generation Standard Model trace, including antiparticles,

$$Q_2 = 12, \qquad Q_Y = 20,$$

so

$$\boxed{\frac{M_W}{M_Z} = \sqrt{\frac{5}{8}}}$$

at that matching scale. These trace weights agree with the per-generation calculation in research state §6.10.

The original interval multiplication representation acts identically on both chiralities. Its matrix link transforms as

$$\phi \longmapsto g_L \phi g_R^*,$$

and under the diagonal $SU(2)$ it decomposes as $1 \oplus 3$. It therefore does not specify the electroweak doublet assignment used above. That assignment, its hypercharges, and the charged fermion trace must be supplied separately. Likewise, the continuum value of ρ requires the neutral-sector boundary conditions and kinetic normalization; the single mixed-boundary eigenvalue in Result 4 cannot determine it. □

Consequence for T4 and hypothesis H

The minimal fermion compression supports the geometric identification $m = \pi/(2l)$, but it does not establish an equivalence of gauge fluctuations, gauge boundary conditions, or bosonic actions. The ratio $\sqrt{3/8}$ is correct for the specified CCM Standard Model completion and receives no interval-overlap correction. Hypothesis H therefore requires an additional gauge-covariant reduction and a matching of representations, vacuum data, and kinetic forms before it can identify the finite vector spectrum with the interval Kaluza–Klein spectrum.

Part 14

T4b: gauge-covariant projection and conditional bosonic matching

Prompt: prompts/14 (STRATEGY T4b). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5 by hand: Theorem 5 (both mass matrices as quadratic forms), Counterexample 6 (the ratio 3/5 agrees with the restricted computation of lemma 13, Result 6: c_L = 8/(3 pi), c_R = 4/(3 pi)), Proposition 7 (t = 3 pi/4: M = 3 pi/(4l), lambda^2/m^2 = 9/2, normalization l(1/2 + 1/(3 pi)), ratio 27/26, then the intermediate value theorem) and No-go theorem 8 (M_spec^2 <= 6 norm(V_0)^2). Reading: the spectral-action vector mass matches the Neumann-Dirichlet tower only through a fermion whose mass is at the tower scale (the matching bulk mass M_* > 0 leaves no light pair, Theorem 9.4 case 3), and with bounded light fermions no matching is possible as l -> 0.

Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): of 9 statements, 5 are correct, Theorems 1 and 5 and Proposition 4 need amendments, and the global form of Theorem 3 is wrong as first written. Each is recorded at its statement.

We prove that a gapped spectral projection defines a gauge-covariant light-fermion bundle with its induced Grassmannian connection. For the mixed interval with a gauge-invariant bulk mass, its finite link has fixed singular values; it does not supply the full scalar fluctuation space of a two-sheet finite triple. At the leading four-dimensional spectral-action order, equality with a specified Kaluza–Klein vector mass is equivalent to an explicit trace-and-overlap sum rule. Covariance does not enforce that rule: the massless-bulk example gives $M_{\mathrm{spec}}^2 = \frac{3}{5} M_{\mathrm{KK}}^2$, whereas a suitable positive bulk mass gives exact equality. We also prove that bounded retained fermion masses cannot match a Neumann–Dirichlet vector mass diverging as $1/l$. Matching one vector mass does not, by itself, determine relative gauge couplings or all scalar kinetic coefficients.

1. Theorem (covariant spectral projection and the data controlling its spectrum).

Amendment (referee). The double-coset statement needs the endpoint transformations to act independently through H_0 and H_l, with extendability understood, and it is a local pointwise classification unless the global topology is specified. Periodic conditions require g(l) = g(0), and then the holonomy is classified by conjugation.

Let $I = [0, l]$, and let

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let the vertical operator be

$$D_{A_5,\mathsf{M},\mathcal{B}} = E(\partial_x + iA_5) + \begin{pmatrix} 0 & \mathsf{M} \\ \mathsf{M} & 0 \end{pmatrix},$$

where A_5 and M are Hermitian matrix-valued functions, and $\mathcal{B}$ specifies a self-adjoint boundary domain. Gauge transformations act in the same representation on both interval chiralities.

Let the operators form a smooth family in y, in the norm-resolvent sense after local identification of their domains. Assume that the selected light cluster is separated from the rest of the spectrum and contains complete eigenspaces, including any gauge degeneracy. Let $P_A(y)$ be its spectral projection.

Then

$$\boxed{P_{A^g} = G_g P_A G_g^\dagger,}$$

provided the transformation includes

$$A_5^g = gA_5g^\dagger + i(\partial_x g)g^\dagger, \qquad \mathsf{M}^g = g\mathsf{M}g^\dagger, \qquad \mathcal{B}^g = G_g\mathcal{B}.$$

For fixed $\mathsf{M}, \mathcal{B}$, this assertion applies exactly to the transformations preserving those data.

Define parallel transport by

$$\partial_x W = -iA_5 W, \qquad W(0) = 1, \qquad W_l = W(l).$$

The vertical spectrum is determined by the axial-gauge data

$$\boxed{\widetilde{\mathsf{M}}(x) = W(x)^\dagger \mathsf{M}(x) W(x), \qquad \widetilde{\mathcal{B}} = W^\dagger \mathcal{B}.}$$

Consequently:

- For a gauge-invariant bulk mass and the mixed conditions

$$R(0) = 0, \qquad L(l) = 0,$$

  the entire spectrum is independent of A_5.
- With nontrivial endpoint data and gauge-invariant bulk coefficients, the relevant holonomy information is the Wilson line relative to those endpoint data. If the endpoint groups are H_0, H_l, its gauge class is represented by

$$H_l \backslash G / H_0.$$

- A bulk matrix not commuting with parallel transport can introduce dependence on the whole transported profile $\widetilde{\mathsf{M}}(x)$, beyond W_l.

An open Wilson line is not generally gauge invariant:

$$W_l^g = g(l) W_l g(0)^\dagger.$$

It is invariant under transformations fixed at both endpoints; it transforms by conjugation when the endpoint transformations are identified. With independently unrestricted endpoints, its double-coset class is trivial.

Proof. Direct differentiation, including the boundary domain, gives

$$D_{A^g, \mathsf{M}^g, \mathcal{B}^g} = G_g D_{A, \mathsf{M}, \mathcal{B}} G_g^\dagger.$$

For a contour enclosing the selected cluster,

$$P_A = \frac{1}{2\pi i} \oint (z - D_A)^{-1} \, dz.$$

Conjugation of the resolvent proves covariance. Smoothness follows from the same formula and the assumed gap.

Multiplication by W removes the connection:

$$W^\dagger D_{A_5, \mathsf{M}, \mathcal{B}} W = D_{0, \widetilde{\mathsf{M}}, \widetilde{\mathcal{B}}}.$$

This proves the assertion about spectral data. For a commuting bulk mass and the mixed boundary conditions, both transformed data equal their unfluctuated values.

Finally, the transport equation gives $W^g(x) = g(x) W(x) g(0)^\dagger$. The endpoint transformation law and the stated quotients follow. If the gap closes, no fixed-rank smooth light bundle is asserted. □

2. Theorem (the induced connection and finite operator).

Let M_4 be a closed Riemannian spin four-manifold. Use Hermitian Euclidean gamma matrices and

$$\mathcal{D}_A = -i\gamma^\mu (\nabla_\mu^S + \partial_\mu + i A_\mu) + \gamma_5 D_{I, A_5}.$$

Assume the hypotheses of Result 1, and let

$$\mathcal{E}_y = \operatorname{Ran} P_A(y).$$

Then the induced connection and vertical endomorphism are

$$\boxed{\nabla_\mu^{\mathcal{E}} = P_A (\partial_\mu + i A_\mu) P_A, \qquad V = P_A D_{I, A_5} P_A \big|_{\mathcal{E}}.}$$

The compressed first-order operator is exactly

$$\boxed{P_A \mathcal{D}_A P_A\big|_{\mathcal{E}} = -i\gamma^\mu(\nabla^S_\mu + \nabla^{\mathcal{E}}_\mu) + \gamma_5 V.}$$

In a local orthonormal frame $U(y) : \mathbb{C}^r \to \mathcal{E}_y$,

$$\boxed{\mathcal{A}_\mu = U^\dagger A_\mu U - iU^\dagger \partial_\mu U, \qquad V_F = U^\dagger D_{I,A_5} U.}$$

The inner products here include integration over the interval. Its curvature is

$$[\nabla^{\mathcal{E}}_\mu, \nabla^{\mathcal{E}}_\nu] = P_A(iF_{\mu\nu})P_A + P_A[\nabla_\mu P_A, \nabla_\nu P_A]P_A,$$

where $\nabla_\mu P_A = \partial_\mu P_A + i[A_\mu, P_A]$.

Proof. Write a light section as $U\chi$. Applying $\partial_\mu + iA_\mu$ and multiplying by $U^\dagger$ gives the displayed connection. The spectral projection commutes with the vertical operator, giving the displayed finite endomorphism. Expanding the commutator of the projected covariant derivatives and using $P_A^2 = P_A$ gives the curvature formula.

If

$$U^g = G_g U h^\dagger,$$

then

$$\mathcal{A}^g_\mu = h\mathcal{A}_\mu h^\dagger + i(\partial_\mu h)h^\dagger, \qquad V^g_F = hV_F h^\dagger.$$

Thus the construction is gauge covariant.

This is an exact compression. It is not generally the operator obtained by eliminating the other fermions. With $P^\perp = 1 - P_A$, elimination additionally produces

$$-P_A \mathcal{D}_A P^\perp \big[P^\perp(\mathcal{D}_A - z)P^\perp\big]^{-1} P^\perp \mathcal{D}_A P_A,$$

when the inverse exists. Horizontal derivatives and $A_\mu(x)$ generally make these off-diagonal blocks nonzero. □

3. Theorem (the finite algebra and fluctuation image for the mixed interval).

Correction (referee; the global statement as first written is false). The statement that the continuum link image consists of all links with Phi^* Phi = lambda^2 I holds locally in spacetime. Globally the unitary link must admit a path from the identity in the chosen trivialization. For M_4 = S^1 x S^3, n = 1 and Phi = lambda exp(i theta), every globally smooth real A_5 gives Phi = lambda exp(i int A_5 dx), with zero winding around S^1, so this Phi is not attained. The displayed formulas, the compression range and the reducing-subalgebra criterion are correct.

Let

$$D_{I,A_5} = \begin{pmatrix} 0 & \partial_x + iA_5 + M \\ -\partial_x - iA_5 + M & 0 \end{pmatrix}, \qquad M \in \mathbb{R},$$

with $R(0) = 0$, $L(l) = 0$, acting on $\mathbb{C}^n$-valued fields. Let the parent algebra be $C^\infty(I, M_n(\mathbb{C}))$, acting identically on both interval chiralities.

Let p_L, p_R be real normalized profiles for the lowest pair:

$$(-\partial_x + M)p_L = \lambda p_R, \qquad (\partial_x + M)p_R = \lambda p_L, \qquad \|p_L\| = \|p_R\| = 1.$$

Use the endpoint-adapted light frame

$$U_L = p_L W, \qquad U_R = p_R W W_l^\dagger.$$

Then

$$\boxed{V_F = \begin{pmatrix} 0 & \lambda W_l^\dagger \\ \lambda W_l & 0 \end{pmatrix}.}$$

Writing $w_L = p_L^2,\ w_R = p_R^2$ and

$$B_\mu = W^\dagger A_\mu W - iW^\dagger \partial_\mu W,$$

the induced connections are

$$\boxed{\begin{aligned} \mathcal{A}_{L,\mu} &= \int_0^l w_L B_\mu \, dx, \\ \mathcal{A}_{R,\mu} &= W_l \left(\int_0^l w_R B_\mu \, dx \right) W_l^\dagger - iW_l \partial_\mu W_l^\dagger. \end{aligned}}$$

The range of algebra compression is

$$A_F = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$$

on

$$H_F = \mathbb{C}_L^n \oplus \mathbb{C}_R^n, \qquad \Gamma_F = \mathrm{diag}(1, -1).$$

Taking this range with its matrix product gives a complex even finite triple. Its unitary group is $U(n)_L \times U(n)_R$, and its unrestricted inner fluctuations have an arbitrary complex matrix as their total link.

The continuum construction, however, produces exactly

$$\boxed{\Phi^\dagger \Phi = \Phi \Phi^\dagger = \lambda^2 I_n}$$

for the link. With unrestricted A_μ, it produces arbitrary pairs of Hermitian sheet connections locally in spacetime. Thus its field space is the fixed-singular-value restriction of that finite triple's fluctuation space.

The inherited local gauge action is

$$h = \mathrm{diag}(g(0), g(l)).$$

Additional gauge boundary conditions restrict this endpoint group. Globally, only endpoint transformations extending across $M_4 \times I$ are inherited.

Proof. Gauge removal and the two profile equations give the formula for V_F. Applying Result 2 to U_L, U_R gives the connection formulas. Under a continuum transformation,

$$U_L^g = gU_L g(0)^\dagger, \qquad U_R^g = gU_R g(l)^\dagger,$$

which proves the endpoint action.

The two functions w_L, w_R are linearly independent: one vanishes at $x = 0$, and the other does not. Therefore the map

$$b(x) \longmapsto \left(\int w_L b, \int w_R b \right)$$

is onto $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. This proves both the algebra-range assertion and the connection-surjectivity assertion. Every unitary W_l has a local smooth path lift, proving the stated link image.

For the finite triple with

$$D_F = \lambda \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

multiplication of $p_L[D_F, p_R]$ by the left algebra generates arbitrary upper-right matrices; adjoining the adjoint generates arbitrary self-adjoint odd fluctuations. The full finite fluctuation space therefore includes radial changes absent from the continuum image.

There is a further distinction concerning algebra multiplication. For any compression $C(a) = PaP|_{PH}$,

$$C(a^\dagger a) - C(a)^\dagger C(a) = Pa^\dagger(1-P)aP.$$

Hence C is a representation on a unital $*$-subalgebra precisely when that subalgebra reduces P. In the present model, multiplication by $a(x)$ reduces the light pair precisely when

$$a(x) = W(x)bW(x)^\dagger$$

for a constant matrix b. Its restriction is a single diagonal $M_n(\mathbb{C})$, commuting with V_F, and its finite scalar one-forms vanish.

Thus the two-sheet algebra is obtained by equipping the compression range with its matrix product; the interval multiplication algebra has no multiplicative quotient onto it. No real structure, real gauge-group prescription, or Standard Model representation follows from this construction. □

4. Proposition (boundary Yukawa data determine a finite operator, but not a unique finite triple).

Amendment (referee). To obtain the entire limiting block, retain the full zero-mode-descended cluster of dimension N_Q + N_U: for two massless flavours and X = l diag(1, 2), the cluster containing only the lowest pair is gapped and has a 1 x 1 limiting block. The nonzero singular values of Y are the retained positive eigenvalues; zero singular values and unmatched chiral zero modes are included separately.

Let D_X have the boundary conditions of Lemma 12:

$$Q_R(0) = 0, \quad U_L(0) = 0, \qquad Q_R(l) = XU_R(l), \quad U_L(l) = -X^\dagger Q_L(l).$$

Let the bulk masses be M_Q, M_U, and let the gauge representations commute with these masses.

Gauge covariance requires

$$X^g = g_Q(l)Xg_U(l)^\dagger.$$

If X is fixed, the allowed endpoint transformations satisfy

$$g_Q(l)X = Xg_U(l).$$

After axial gauge fixing,

$$\widetilde{X} = W_Q(l)^\dagger X W_U(l).$$

Let P_A retain a gapped grading-invariant light cluster. Then

$$H_F = H_{F,+} \oplus H_{F,-}, \qquad V_F = \begin{pmatrix} 0 & Y \\ Y^\dagger & 0 \end{pmatrix},$$

where the singular values of Y are the retained positive vertical eigenvalues. These eigenvalues alone do not determine the represented gauge generators or their trace data.

In the towerless scaling of Theorem 12.7,

$$X = l\widehat{X}, \qquad M_Q l,\ M_U l \text{ fixed},$$

the limiting block is, up to changes of chiral orthonormal frames,

$$\boxed{Y = \Phi_Q \widehat{X} \Phi_U,} \qquad \phi(c)^2 = \frac{2c}{e^{2c}-1}, \quad \phi(0) = 1.$$

Every rectangular complex matrix Y is attainable as an operator block by choosing

$$\widehat{X} = \Phi_Q^{-1} Y \Phi_U^{-1}.$$

This does not establish that every finite algebra, real structure, gauge representation, or inner-fluctuation space is attained.

Proof. Transforming the boundary equations gives the transformation law for X. Applying $W_Q^\dagger, W_U^\dagger$ gives $\widetilde{X}$. Since the domain is grading invariant, its symmetric spectral projection commutes with the grading; the restricted operator is therefore odd and has the displayed block form.

For an explicit spectral determination, define

$$\alpha_i(\lambda) = \omega_i \coth(\omega_i l) + m_i^Q, \qquad \beta_j(\lambda) = \omega_j \coth(\omega_j l) - m_j^U, \qquad \omega^2 = m^2 - \lambda^2.$$

In the low window where the transfer denominators are nonzero, the boundary equations give

$$\det\left[\lambda^2 I - \operatorname{diag}(\alpha_i)\widetilde{X}\operatorname{diag}(\beta_j)\widetilde{X}^\dagger\right] = 0.$$

This follows by solving the constant-coefficient first-order equations and imposing the two conditions at l, as in Theorem 12.4.

At zero spectral parameter, the normalized wall values give

$$F_Q = l^{-1/2}\Phi_Q, \qquad F_U = l^{-1/2}\Phi_U.$$

Consequently

$$F_Q X F_U = \Phi_Q \widehat{X} \Phi_U.$$

The bounds of Theorem 12.6 give convergence of the retained positive eigenvalues to its singular values, with error $O(l^2)$; the remaining rescaled eigenvalues stay separated from zero. Positivity of the diagonal entries of Φ_Q, Φ_U proves attainability of every operator block.

To obtain a finite triple, one must additionally specify the algebra to be represented on the retained space. A multiplicative restriction must satisfy the reducing-subalgebra criterion in Result 3; taking the algebra generated by compressed operators is a different prescription. Once an algebra A_F is specified, its complex gauge group is the represented $U(A_F)$, and its scalar one-forms are

$$\Omega^1_{V_F}(A_F) = \operatorname{span}\{a[V_F, b] : a, b \in A_F\}.$$

Neither this algebra nor this space is determined by the light eigenvalues. □

5. Theorem (the exact quadratic matching condition at leading spectral-action order).

Amendment (referee). Both quadratic forms must be compared on one common profile space on which G^(5) and Q are positive, because a Kaluza-Klein profile can be invisible to compression. For the massless mixed pair, f = cos(2 pi x/l) satisfies Neumann conditions at both ends and has positive norm, while int w_L f = int w_R f = 0, so Q = K = 0 for this mode.

Let a five-dimensional Yang–Mills theory have no boundary kinetic or mass terms. Let $\langle\ ,\ \rangle_5$ be its positive invariant gauge form, including its coupling constants. Expand the retained transverse vectors about $A_5 = 0$ as

$$A_\mu(y, x) = \sum_a a_\mu^a(y) F_a(x),$$

where $F_a(x)$ are fixed Hermitian Lie-algebra-valued profiles satisfying the chosen gauge boundary conditions. Define

$$G^{(5)}_{ab} = \int_0^l \langle F_a, F_b\rangle_5\, dx, \qquad H^{(5)}_{ab} = \int_0^l \langle F_a', F_b'\rangle_5\, dx.$$

Then

$$\mathcal{L}^{(2)}_{\mathrm{KK}} = \frac{1}{4} G^{(5)}_{ab} f^a_{\mu\nu} f^{b,\mu\nu} + \frac{1}{2} H^{(5)}_{ab} a^a_\mu a^{b,\mu},$$

and

$$\boxed{\mathcal{M}^2_{\mathrm{KK}} = (G^{(5)})^{-1/2} H^{(5)} (G^{(5)})^{-1/2}.}$$

Let U_0 be the light frame at this background, and set

$$T_a = U_0^\dagger F_a U_0, \qquad V_0 = U_0^\dagger D_I U_0.$$

For the literal spectral trace, retain the gauge and scalar derivative terms through a_4, assume $f_0 > 0$, and add no independent kinetic terms. Define

$$Q_{ab} = \operatorname{Re}\operatorname{Tr}(T_a T_b), \qquad K_{ab} = \operatorname{Re}\operatorname{Tr}([T_a, V_0]^\dagger [T_b, V_0]).$$

On the subspace where $Q > 0$,

$$\boxed{\mathcal{M}^2_{\text{spec}} = \frac{3}{2} Q^{-1/2} K Q^{-1/2}.}$$

Thus equality of the vector mass spectra is equivalent to equality of the generalized spectra

$$\boxed{\operatorname{spec}\left(\tfrac{3}{2}K, Q\right) = \operatorname{spec}\left(H^{(5)}, G^{(5)}\right).}$$

For a retained sector whose KK mass matrix is $\kappa^2 I$, this is equivalent to the sum rule

$$\boxed{K = \frac{2}{3}\kappa^2 Q.}$$

Equality of the quadratic actions in the same field coordinates is stronger:

$$\boxed{G^{(5)} = 4\eta Q, \qquad H^{(5)} = 6\eta K, \qquad \eta = \frac{f_0}{24\pi^2}.}$$

For a chiral finite block

$$V_0 = \begin{pmatrix} 0 & Y \\ Y^\dagger & 0 \end{pmatrix}, \qquad T_a = \operatorname{diag}(t_{L,a}, t_{R,a}),$$

put

$$C_a = t_{L,a} Y - Y t_{R,a}.$$

The trace data are explicitly

$$\boxed{Q_{ab} = \operatorname{Re}\operatorname{Tr}(t_{L,a} t_{L,b} + t_{R,a} t_{R,b}), \qquad K_{ab} = 2\operatorname{Re}\operatorname{Tr}(C_a^\dagger C_b).}$$

Here Y and the $t_{L/R,a}$ must both be computed from the interval eigenfunctions and the retained gauge profiles.

Proof. Expanding the five-dimensional action gives the two integral forms $G^{(5)}, H^{(5)}$. Canonical normalization by $(G^{(5)})^{1/2}$ gives its mass matrix.

For

$$\mathcal{D} = -i\gamma^\mu(\nabla^S_\mu + i\mathcal{A}_\mu) + \gamma_5 V,$$

the relevant heat-coefficient traces are

$$\operatorname{tr}_S(\mathcal{E}_F^2) = 2F_{\mu\nu}F^{\mu\nu}, \quad \operatorname{tr}_S(\mathcal{E}_V^2) = 4D_\mu V D^\mu V, \quad \operatorname{tr}_S(\Omega_F^2) = -4F_{\mu\nu}F^{\mu\nu}.$$

Their coefficients $1/2, 1/2, 1/12$ give

$$\mathcal{L}_{\text{spec}} = \eta \operatorname{Tr} F_{\mu\nu}F^{\mu\nu} + 3\eta \operatorname{Tr} D_\mu V D^\mu V.$$

At V_0, this becomes

$$\eta Q_{ab} f^a_{\mu\nu} f^{b,\mu\nu} + 3\eta K_{ab} a^a_\mu a^{b,\mu}.$$

The normalization $a_{\text{can}} = 2\sqrt{\eta}\, Q^{1/2} a$ proves the coefficient $3/2$, agreeing with Proposition 13.5.

Equality with a scalar matrix $\kappa^2 I$ is equivalent to

$$Q^{-1/2} K Q^{-1/2} = \frac{2}{3}\kappa^2 I.$$

The block formulas follow by multiplying

$$[T_a, V_0] = \begin{pmatrix} 0 & C_a \\ -C_a^\dagger & 0 \end{pmatrix}.$$

This proves all the matching conditions.

For mixed fermion pairs labelled by r, let their masses be λ_r, and let a single gauge profile f have overlaps

$$c_{L,r} = \int |p_{L,r}|^2 f, \qquad c_{R,r} = \int |p_{R,r}|^2 f.$$

If $\tau_r = \mathrm{tr}_r(t^2)$ includes every multiplicity, the one-vector condition becomes

$$\boxed{3\sum_r \tau_r \lambda_r^2 (c_{L,r} - c_{R,r})^2 = \kappa^2 \sum_r \tau_r (c_{L,r}^2 + c_{R,r}^2).}$$

This is the trace-and-overlap sum rule for that gauge reduction.

If instead both sheet connections are fluctuated independently, with the same representation on each pair, their relative mass is

$$M^2_{\text{relative}} = 6\frac{\sum_r \tau_r \lambda_r^2}{\sum_r \tau_r}.$$

Equating this single eigenvalue to κ^2 requires

$$6\sum_r \tau_r \lambda_r^2 = \kappa^2 \sum_r \tau_r.$$

That comparison has an additional massless diagonal vector and is not the same restriction as retaining one interval gauge profile.

All these spectral-action equalities concern the stated a_4 approximation and chosen vacuum. For a full finite scalar fluctuation space, identifying V_0 with the compressed bare operator additionally requires the scalar potential to select that vacuum. At masses comparable to the cutoff, Lemma 03, Part C shows why weighting Q alone does not give the broken-generator answer.

Finally, a nonabelian truncation to nonconstant lowest profiles need not be closed under gauge transformations: $[ft_a, ft_b] = f^2[t_a, t_b]$ generally leaves the retained span. The theorem compares quadratic transverse actions; it does not assert equality of full interacting theories. □

6. Counterexample (the covariant lowest-mode reduction still gives 3/5).

Let the gauge group be $U(1)$, and impose

$$\partial_x A_\mu(0) = 0, \qquad A_\mu(l) = 0,$$

with complementary conditions

$$A_5(0) = 0, \qquad \partial_x A_5(l) = 0.$$

Retain

$$A_\mu = a_\mu(y)\cos(\kappa x), \qquad A_5 = b(y)\sin(\kappa x), \qquad \kappa = \frac{\pi}{2l}.$$

Let one charge-one fermion have the mixed boundary conditions and scalar bulk mass M. Let λ be its lowest positive vertical eigenvalue, and define

$$c_L = \int_0^l p_L^2 \cos(\kappa x)\, dx, \qquad c_R = \int_0^l p_R^2 \cos(\kappa x)\, dx.$$

The gauge-covariant light projection gives

$$\boxed{M^2_{\text{spec}} = 3\lambda^2 \frac{(c_L - c_R)^2}{c_L^2 + c_R^2}, \qquad M^2_{\text{KK}} = \kappa^2.}$$

For $M = 0$,

$$\boxed{M^2_{\text{spec}} = \frac{3}{5} M^2_{\text{KK}}.}$$

Proof. The vector boundary problem selects $f(x) = \cos(\kappa x)$. Direct integration gives

$$\frac{\int (f')^2}{\int f^2} = \kappa^2,$$

which is the Neumann–Dirichlet result of Lemma 9.11.

The retained abelian gauge transformations have parameter

$$\epsilon(y, x) = \alpha(y) \cos(\kappa x).$$

With the conventions of Result 1,

$$a_\mu \mapsto a_\mu - \partial_\mu \alpha, \qquad b \mapsto b + \kappa\alpha.$$

Thus this vector-scalar restriction is gauge closed.

Parallel transport is

$$W(x) = \exp\left[-\frac{ib}{\kappa}(1 - \cos \kappa x)\right].$$

Result 3 gives

$$\Phi = \lambda e^{ib/\kappa},$$

and

$$\mathcal{A}_L = c_L a - \frac{1 - c_L}{\kappa}\, db, \qquad \mathcal{A}_R = c_R a + \frac{c_R}{\kappa}\, db.$$

Consequently,

$$D\Phi = i\Phi(c_L - c_R)\left(a + \frac{db}{\kappa}\right), \qquad F_L = c_L\, da, \quad F_R = c_R\, da.$$

The quadratic spectral action is therefore

$$\eta(c_L^2 + c_R^2)(da)^2 + 6\eta\lambda^2(c_L - c_R)^2 \left(a + \frac{db}{\kappa}\right)^2.$$

Canonical vector normalization proves the mass formula. In particular, the Berry terms have been included; the calculation does not use a fixed projection.

At $M = 0$,

$$\lambda = \kappa, \qquad p_L = \sqrt{\frac{2}{l}} \cos(\kappa x), \qquad p_R = \sqrt{\frac{2}{l}} \sin(\kappa x).$$

Elementary integration gives

$$c_L = \frac{8}{3\pi}, \qquad c_R = \frac{4}{3\pi}.$$

Hence

$$\frac{M^2_{\text{spec}}}{M^2_{\text{KK}}} = 3\frac{(8-4)^2}{8^2 + 4^2} = \frac{3}{5}.$$

The matrices $\mathrm{diag}(c_L, c_R)$ describe the coefficient of the retained vector in the induced connection; they are not the endpoint $U(1)$ charge assignment. The explicit transformation formulas above establish covariance. □

7. Proposition (exact matching is possible after imposing an additional bulk-mass condition).

Keep the gauge and fermion boundary conditions of Result 6. At fixed l, let the fermion bulk mass vary over

$$0 \le M \le \frac{3\pi}{4l}.$$

Then at least one value

$$\boxed{0 < M_* < \frac{3\pi}{4l}}$$

satisfies

$$M_{\mathrm{spec}}(M_*) = M_{\mathrm{KK}}.$$

An exact defining condition is

$$\boxed{3\lambda(M_*)^2(c_L(M_*) - c_R(M_*))^2 = \frac{\pi^2}{4l^2}(c_L(M_*)^2 + c_R(M_*)^2).}$$

No uniqueness is asserted.

Proof. For $M \ge 0$, write

$$t = kl \in [\pi/2, \pi), \qquad M = -k\cot t, \qquad \lambda = \frac{k}{\sin t}.$$

The lowest chiral profiles are proportional to

$$p_L(x) \propto \sin(k(l-x)), \qquad p_R(x) \propto \sin(kx).$$

The relation $M = -k\cot(kl)$ follows by imposing $p_L'(0) = Mp_L(0)$; substitution in the first-order equation gives the displayed p_R and λ.

At

$$t = \frac{3\pi}{4}, \qquad M = \frac{3\pi}{4l},$$

we have

$$\lambda^2 = \frac{9\pi^2}{8l^2}.$$

The common squared normalization integral is

$$\int_0^l \sin^2(kx)\,dx = l\left(\frac{1}{2} + \frac{1}{3\pi}\right).$$

Integrating against $\cos(\pi x/(2l))$ gives

$$c_L = \frac{9}{3\pi + 2}, \qquad c_R = \frac{6}{3\pi + 2}.$$

Therefore

$$\frac{M_{\mathrm{spec}}^2}{M_{\mathrm{KK}}^2} = 3\frac{9}{2}\frac{(9-6)^2}{9^2 + 6^2} = \frac{27}{26} > 1.$$

At $M = 0$, Result 6 gives $3/5 < 1$. The lowest eigenvalue is simple, and the displayed root and normalized profiles depend continuously on M. The intermediate value theorem proves the existence of M_*. Thus universal impossibility of mass matching is false, while automatic matching is also false. □

8. No-go theorem (bounded light fermions cannot match the collapsing Neumann–Dirichlet tower).

Let a family of induced finite operators satisfy

$$\|V_0(l)\| \le C < \infty.$$

Use the leading spectral action of Result 5, with no additional vector-mass or kinetic terms. The retained gauge couplings may depend on l, and any finite multiplicities may be included.

Then every canonically normalized spectral-action vector mass satisfies

$$\boxed{M^2_{\mathrm{spec}} \le 6C^2.}$$

Consequently it cannot equal

$$M^2_{\mathrm{KK}} = \frac{\pi^2}{4l^2}$$

for arbitrarily small l.

This applies to the fixed-light-mass collapse of Theorem 9.9 and to the bounded-texture limit of Result 4.

Proof. For every Hermitian retained coupling matrix T,

$$\|[T, V_0]\|_{\mathrm{HS}} \le \|TV_0\|_{\mathrm{HS}} + \|V_0T\|_{\mathrm{HS}} \le 2C\|T\|_{\mathrm{HS}}.$$

Hence

$$K \le 4C^2Q$$

as quadratic forms. Result 5 gives

$$\mathcal{M}^2_{\mathrm{spec}} \le 6C^2 I.$$

The same inequality holds after restricting to any gauge-profile subspace. Multiplicities cannot evade it because they enter both Hilbert–Schmidt norms.

The Neumann–Dirichlet mass diverges as l^{-1}, proving the contradiction. More quantitatively, matching any such vector requires

$$\boxed{\|V_0(l)\| \ge \frac{\pi}{2\sqrt{6}\,l}.}$$

Thus some retained fermion scale must grow at least as $1/l$, unless the bosonic action or the stated approximation is changed. □

9. Proposition (gauge-factor and scalar normalizations require additional conditions).

Let the finite gauge kinetic form be diagonal on two specified factors, with fixed generator normalizations,

$$Q|_{\mathfrak{g}_s} = q_s I, \qquad s = 1, 2.$$

Let their five-dimensional kinetic terms have independent couplings $g_{5,s}$, and let

$$J_s = \int_0^l f_s(x)^2\, dx.$$

Then the leading finite spectral action predicts

$$\frac{1}{g^2_{4,s}} = 4\eta q_s, \qquad \frac{g^2_{4,1}}{g^2_{4,2}} = \frac{q_2}{q_1},$$

whereas the KK action gives

$$\frac{1}{g^2_{4,s}} = \frac{J_s}{g^2_{5,s}}.$$

Vector-mass matching does not imply equality of these relative normalizations.

For scalar coordinates φ^i, let the induced connection at vanishing ambient A_μ be

$$\mathcal{A}_\mu = \mathcal{B}_i(\varphi)\partial_\mu\varphi^i,$$

and define

$$H_i = \partial_i V + i[\mathcal{B}_i, V].$$

The spectral-action scalar metric, in the convention $\frac{1}{2}Z_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j$, is

$$\boxed{Z_{ij}^{\mathrm{spec}} = 6\eta\ \mathrm{Re}\,\mathrm{Tr}(H_iH_j).}$$

Mass matching constrains this metric on the corresponding gauge-orbit directions. It does not generally determine the metric on additional physical scalar directions.

Proof. Canonical normalization of $\eta q_s F_s^2$ gives the finite coupling formula. Integrating the five-dimensional kinetic term gives the KK formula. Rescaling $g_{5,s}^{-2}$ multiplies both $G_s^{(5)}$ and $H_s^{(5)}$ by the same factor and leaves every KK mass unchanged. Therefore mass matching cannot determine the independent ratio $g_{5,1}/g_{5,2}$. A common simple parent group with a specified invariant form could remove this freedom, but that is additional geometric input.

For the scalar metric,

$$D_\mu V = H_i\partial_\mu\varphi^i.$$

Substitution into $3\eta\,\mathrm{Tr}(D_\mu V D^\mu V)$ proves the formula.

The eaten scalar in Result 6 illustrates the constrained directions precisely. Put

$$G = \frac{l}{2g_5^2}, \qquad q = c_L^2 + c_R^2, \qquad d = c_L - c_R.$$

Its KK action is

$$\frac{G}{4}(da)^2 + \frac{G}{2}(db + \kappa a)^2.$$

If both the gauge kinetic coefficient and the vector mass match,

$$4\eta q = G, \qquad 3\lambda^2 d^2 = \kappa^2 q,$$

then the spectral scalar coefficient automatically satisfies

$$Z_b^{\mathrm{spec}} = \frac{12\eta\lambda^2 d^2}{\kappa^2} = G.$$

Thus no separate tuning is needed for this eaten scalar once those two equalities hold.

A scalar direction transverse to the gauge orbit is not tested by that mass equation. In the commuting-bulk mixed model, A_5 supplies only the unitary link of Result 3, so there is no radial Higgs kinetic term to match. If boundary Yukawa matrices are promoted to independent scalar fields, their continuum dynamics, including any boundary kinetic form, must be specified separately. □

Consequence for T4b and hypothesis H

Gauge-covariant projection repairs the covariance defect and supplies an exact light-bundle connection, but it does not establish equality with the full finite-triple fluctuation space or with the interacting KK theory. At fixed length, vector-mass equality is possible precisely when the appropriate trace-and-overlap condition holds; covariance supplies no selection principle for it or for the remaining kinetic normalizations. Hypothesis H therefore requires additional geometric relations, and its collapse regime cannot reproduce a Neumann–Dirichlet vector of mass $\pi/(2l)$ from a leading finite spectral action whose retained fermion masses all remain bounded.

Part 15

Collapse in the spectral propinquity: a gap criterion and limits of commutative spaces

Lemmas R2 and R3 of the Farsi-Latrémolière assessment, written out and extended. Proved by Claude Opus 5.5 without Astra or Fable, from the definitions in the LaTeX source of arXiv:2404.00240v3, Section 2. The sketches came from the reading subagent; the lower bound in the sketch of R2 omitted the cap sqrt(2)/2 of the metric and is corrected here. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): Theorem 15.2 item 2 had a gap, closed by the referee; Remark 15.4 omitted the cap; Proposition 15.9 needed amendments; the T0 section first contained two invalid arguments (on unbroken symmetry and on Morita equivalence). All corrections are incorporated below. Proposition 15.10 and Counterexample 15.11 are the referee's.

Scope. Everything here concerns one metric, the spectral propinquity, applied to the whole algebra of the parent triple. The section "What this says about T0" states what follows for the founding problem and what does not; lemma 16 treats truncations, where other answers hold.

Conventions

These are the definitions of Farsi-Latrémolière, Section 2.

- A *quantum compact metric space* (A, L) is a unital C*-algebra A with a seminorm L on a dense subspace of sa(A) such that ker L = R 1; the metric mk_L(phi, psi) = sup{abs(phi(a) - psi(a)) : L(a) <= 1} metrizes the weak* topology of the state space S(A); L satisfies the Leibniz inequality for the Jordan product a o b = (ab + ba)/2 and the Lie product {a, b} = (ab - ba)/(2i); and {L <= 1} is closed.
- A *quantum isometry* is a *-epimorphism pi with L_B(b) = inf{L_A(a) : pi(a) = b}. Its dual map is an isometry of state spaces.
- A *tunnel* is a quantum compact metric space (E, L_E) with quantum isometries pi_1, pi_2 onto the two spaces. Its *extent* chi is the larger of the Hausdorff distances, in mk_{L_E}, from S(E) to pi_1^*S(A_1) and to pi_2^*S(A_2). The *propinquity* Lambda is the infimum of extents.
- A *metric spectral triple* T = (A, H, D) is a spectral triple for which L_T(a) = norm([D, a]) is an L-seminorm. In particular ker L_T = R 1. We write DN_T(xi) = norm(xi) + norm(D xi) and diam T for the diameter of S(A) in mk_{L_T}.
- A *metrical tunnel* between T_1 and T_2 consists of a C*-correspondence P from E to a unital C*-algebra B, with L_E and L_B as above and a norm DN_P on a dense subspace of P such that: the unit ball of DN_P is compact in P; DN_P dominates the module norm; DN_P(e Theta) <= (norm(e) + L_E(e)) DN_P(Theta) (*modular Leibniz*); L_B applied to the real and to the imaginary part of <Theta, Theta'>*B is at most 2 DN_P(Theta) DN_P(Theta')* (inner Leibniz*; this is the condition of Latrémolière, arXiv:1811.10843, in the definition of a metrical C*-correspondence, whereas the display in Farsi-Latrémolière applies L to the non-self-adjoint inner product and names L_A). It carries maps (Pi_j, pi_j, theta_j) onto (H_j, A_j, C), compatible with the actions and inner products, with DN*{T_j}(xi) = inf{DN_P(Theta) : Pi_j Theta = xi}. Its extent is the larger of the extents of (E, pi_1, pi_2) and (B, theta_1, theta_2).
- The *pseudo-states* of T are the functionals xi -> <omega, xi> with DN_T(omega) <= 1. The *separation* of two operator families (a_t), (b_t) is the Hausdorff distance between {(phi a_t Pi_1)_t} and {(psi b_t Pi_2)_t}, phi and psi ranging over pseudo-states, in the metric sup_t sup{abs(F_t(Theta) - G_t(Theta)) : DN_P(Theta) <= 1}. The *dispersion* is the larger of extent and separation.
- The *spectral propinquity* Lambda_spec(T_1, T_2) is the infimum of sqrt(2)/2 and of those eps > 0 for which some metrical tunnel has dispersion < eps for the families exp(i t D_1), exp(i t D_2), 0 <= t <= 1/eps.

Three elementary consequences.

(a) If tau is a tunnel of extent chi, then abs(diam_1 - diam_2) <= 2 chi. Take phi, phi' realizing diam_1; their pullbacks lie within chi of pullbacks of states of A_2, and the dual maps are isometries.

(b) Lambda_spec(T_1, T_2) >= min(sqrt(2)/2, Lambda(A_1, A_2)), because the dispersion dominates the extent of the top tunnel. So convergence in Lambda_spec implies convergence in Lambda.

(c) Combining: Lambda_spec(T_1, T_2) >= min(sqrt(2)/2, abs(diam T_1 - diam T_2)/2).

The cap is dimensionful: Lambda_spec has the units of a length and is cut off at sqrt(2)/2 in the chosen unit.

(d) If a = a* in (A, L), and z is the midpoint of the convex hull of Sp(a), then norm(a - z) <= (diam/2) L(a). The endpoints of the hull are values of states on a.

Comparison along a tunnel

Lemma 15.1. Let tau = (E, L_E, pi_1, pi_2) be a tunnel of extent chi, and let e = e* with L_E(e) finite. For {j, k} = {1, 2},

```
norm(pi_j e) <= norm(pi_k e) + chi L_E(e),     norm(e) <= norm(pi_j e) + chi L_E(e).
```

For a real polynomial p = sum p_m x^m, the element p(e) has finite L_E, with L_E(p(e)) <= L_E(e) sum_{m >= 1} abs(p_m) m norm(e)^{m-1}.

Proof. The norm of a self-adjoint element is the supremum of abs(state) on it. A state phi of A_j pulls back to S(E), and lies within chi of the pullback of a state psi of A_k, so abs(phi(pi_j e)) <= abs(psi(pi_k e)) + chi L_E(e). The second inequality is the same argument starting from a state of E. For powers, e^m = e o e^{m-1}, and induction with the Jordan-Leibniz inequality gives L_E(e^m) <= m norm(e)^{m-1} L_E(e); L_E(1) = 0. ■

Limits of commutative spaces (Lemma R3)

Theorem 15.2. Let (A_n, L_n) -> (A, L) in the propinquity.

1. If every A_n is commutative, then A is commutative.
2. If every A_n = C(X_n) with X_n having at most k connected components, then the spectrum of A has at most k connected components. In particular, a finite-dimensional limit is C^j with j <= k, and for k = 1 it is C.

Both statements hold for limits in Lambda_spec, by (b).

Proof. Let tau_n be tunnels of extent chi_n -> 0.

(1) Let a_1, a_2 in sa(A) have finite L. Choose lifts d_i in E_n with pi_A(d_i) = a_i and L_{E_n}(d_i) <= L(a_i) + 1. Lemma 15.1 bounds norm(d_i) uniformly. The Lie product e = {d_1, d_2} satisfies pi_n(e) = 0 and pi_A(e) = {a_1, a_2}, and L_{E_n}(e) <= norm(d_1) L(d_2) + norm(d_2) L(d_1) is uniformly bounded. So norm({a_1, a_2}) <= chi_n L_{E_n}(e) -> 0. Elements of finite L are dense in sa(A), so A is commutative.

(2) By (1), A = C(X). Suppose X has at least k + 1 components. Since components and quasicomponents agree in a compact Hausdorff space, X splits into nonempty clopen sets X_0, ..., X_k. Let a in sa(A) have finite L and norm(a - sum_j j 1_{X_j}) < 1/8, so that Sp(a) meets each [j - 1/8, j + 1/8] and lies in their union. Take lifts d_n as above, and b_n = pi_n(d_n), a real function on X_n. All spectra lie in a fixed compact interval I for large n, by Lemma 15.1.

Let g : I -> [0, 1] be continuous, 0 on the union of the [j - 1/8, j + 1/8] and 1 outside the union of the open intervals (j - 3/8, j + 3/8), and let p be a polynomial with abs(p - g) < 1/8 on I. Then norm(p(a)) <= 1/8, so Lemma 15.1 gives norm(p(b_n)) <= 1/4 for large n, because L_{E_n}(p(d_n)) is uniformly bounded. Hence the range of b_n lies in the union of the disjoint intervals (j - 3/8, j + 3/8), and each component of X_n maps into one of them. So b_n reaches at most k of them, and along a subsequence it misses one fixed j_*. Let h : I -> [0, 1] be continuous, 1 on [j_* - 1/8, j_* + 1/8] and 0 outside radius 3/8 around j_*, and let q be a polynomial with abs(q - h) < 1/8 on I. Then norm(q(a)) >= 7/8 and norm(q(b_n)) <= 1/8, contradicting Lemma 15.1 for large n. ■

The theorem is one-directional: noncommutative spaces may converge to commutative ones, as matrix algebras do to the sphere (Rieffel; Latrémolière) and spectral truncations of tori do to the tori in the Gromov-Hausdorff sense on state spaces (Leimbach-van Suijlekom). We did not find Theorem 15.2 stated in the sources we searched (arXiv:1302.4058, 1311.0104, 2404.00240, 2603.19128); its proof is elementary and it may be folklore.

The gap criterion (Lemma R2)

Theorem 15.3. Let T = (A, H, D) be a metric spectral triple with delta = diam T. Let Sigma be a finite subset of R, P = 1_Sigma(D) of rank N >= 1 and Q = 1 - P, and let g > 0 be such that abs(lambda) >= g for every lambda in Sp(D) outside Sigma. Let D_F be a Hermitian N x N matrix, W : PH -> C^N a unitary, extended by 0 on QH, and Delta = norm(W D W* - D_F). For T_F = (C, C^N, D_F),

```
min(sqrt(2)/2, delta/2) <= Lambda_spec(T, T_F) <= 1/g^2 + max(delta/2, Delta) + Delta +
↪ sqrt(Delta).
```

For Delta = 0 the upper bound reads delta/2 + 1/g^2. No anticommutation hypothesis enters.

Proof. The lower bound is (c), since diam T_F = 0. For the upper bound fix rho > 0 with rho >= max(delta/2, Delta), and put s = rho + 1/(2 g^2), c = 1/(1 + Delta), D_B = W D W*.

Top tunnel. On E = A (+) C let L_E(a, z) = max{L_T(a), norm(a - z)/rho}. Then ker L_E = R (1, 1). The Leibniz inequality for the Jordan product follows from a o b - z w = (a - z) o b + z (b - w), and for the Lie product from {a, b} = {a - z, b}. The set {L_E <= 1, mu(a) = 0}, for a fixed state mu of A, is contained in a product of compact sets, since abs(z) <= norm(a) + rho and norm(a) <= delta there, so L_E is an L-seminorm by Rieffel's criterion, with the closed ball following from lower semicontinuity. The projection onto A is a quantum isometry by (d), as rho >= delta/2; the projection onto C is one because L_E(z 1, z) = 0. A state of E has the form (a, z) -> t mu(a) + (1 - t) z. On L_E(a, z) <= 1 it differs from (a, z) -> mu(a) by (1 - t) abs(z - mu(a)) <= norm(a - z) <= rho, and from (a, z) -> z by t abs(mu(a) - z) <= rho. So the extent is at most rho.

Base tunnel. On B = C (+) C let L_B(x, y) = abs(x - y)/s. The two projections are quantum isometries onto (C, 0), and the extent is at most s.

Module. Let P_mod = H (+) C^N, a right Hilbert B-module with <(xi, eta), (xi', eta')> = (<xi, xi'>, <eta, eta'>) and left action (a, z)(xi, eta) = (a xi, z eta). On Dom D (+) C^N set

```
DN_P(xi, eta) = max{DN_T(xi), DN_F(eta), norm(W xi - eta)/rho}.
```

The unit ball is closed in a product of compact sets, because D has compact resolvent. The norm is dominated.

Modular Leibniz. DN_T(a xi) <= (norm(a) + L_T(a)) DN_T(xi) and DN_F(z eta) = abs(z) DN_F(eta). Further

```
norm(W a xi - z eta) <= norm(W (a - z) xi) + abs(z) norm(W xi - eta) <= norm(a - z) norm(xi) +
↪ abs(z) rho DN_P(xi, eta),
```

and dividing by rho gives at most (L_E(a, z) + norm(a, z)) DN_P(xi, eta). No commutation of P with A is needed, because the limit algebra acts by scalars.

Inner Leibniz. On Q H the spectrum has abs(lambda) >= g, so norm(Q xi) <= norm(D xi)/g <= DN_T(xi)/g. Then <xi, xi'> = <W xi, W xi'> + <Q xi, Q xi'> and

```
abs(<W xi, W xi'> - <eta, eta'>) <= norm(W xi - eta) norm(W xi') + norm(eta) norm(W xi' - eta') <=
↪ 2 rho DN_P DN_P',
```

so abs(<xi, xi'> - <eta, eta'>) <= (2 rho + 1/g^2) DN_P DN_P' = 2 s DN_P DN_P'. This complex estimate implies the condition on real and imaginary parts.

Quotient conditions. Given xi, take eta = c W xi. Since P reduces D and D_F W = W D - (D_B - D_F) W on Dom D, we get DN_F(c W xi) <= c (norm(W xi) + norm(W D xi) + Delta norm(W xi)) <= DN_T(xi), and norm(W xi - c W xi) <= Delta norm(xi) <= rho DN_T(xi). So DN_P(xi, c W xi) = DN_T(xi). Given

eta, take xi = c W* eta. Then DN_T(xi) = c (norm(eta) + norm(D_B eta)) <= DN_F(eta) and norm(W xi - eta) <= Delta norm(eta), so DN_P(c W* eta, eta) = DN_F(eta).

Separation. Let DN_T(omega) <= 1 and put omega' = c W omega, so that DN_F(omega') <= 1. For DN_P(xi, eta) <= 1 and real t, since P commutes with exp(i t D),

```
<omega, e^{itD} xi> - <omega', e^{itD_F} eta> = <Q omega, e^{itD} Q xi> + <W omega, (e^{itD_B} -
↪  e^{itD_F}) W xi> + <W omega, e^{itD_F} (W xi - eta)> + (1 - c) <W omega, e^{itD_F} eta>,
```

and the four terms are bounded by 1/g^2, abs(t) Delta, rho and Delta. Conversely, let DN_F(omega') <= 1 and put omega = c W* omega', so that DN_T(omega) <= 1 and Q omega = 0; the same decomposition without the first term gives at most abs(t) Delta + rho + Delta.

Conclusion. Put K = 1/g^2 + rho + Delta. The extent is at most s <= K and the separation on [0, 1/eps] is at most K + Delta/eps. If eps > K + sqrt(Delta), then Delta/eps < sqrt(Delta) (or Delta = 0), so the dispersion is below eps. Hence Lambda_spec <= K + sqrt(Delta), and we let rho decrease to max(delta/2, Delta). ■

Remark 15.4 (rescaling). For T_eps = (A, H, D/eps) with 0 in Sp(D), Sigma = {0} and D_F = 0, the theorem gives min(sqrt(2)/2, eps delta_1/2) <= Lambda_spec <= eps delta_1/2 + eps^2/g_1^2, where delta_1 = diam T_1 and g_1 is the gap of D. This is Corollary 4.1 of Farsi-Latrémolière with a sharp leading rate for small eps. Counterexample R1 of the assessment violates the gap hypothesis, since the eigenvalue x stays bounded outside Sigma.

Corollary 15.5 (the tuned interval, Theorem 9.9). Let T_l = (C([0, l]), L^2([0, l]; C^2), D_{-mu(l)}) with condition (c) and mu(l) as in Theorem 9.9, c = mu(l) l, and let P project onto +/- m_f. For l small,

```
l/2 <= Lambda_spec(T_l, (C, C^2, m_f sigma_1)) <= l/2 + l^2/(pi^2 + c^2).
```

For the Connes-Lott triple CL(m_f) = (C + C, C^2, m_f sigma_1) of diameter 1/m_f,

```
Lambda_spec(T_l, CL(m_f)) >= min(sqrt(2)/2, (1/m_f - l)/2).
```

Proof. Multiplication by a Lipschitz function preserves condition (c), and the mass term commutes with functions, so L(f) = norm(f')_infinity and diam T_l = l. Theorem 9.6 gives W with W D W* = m_f sigma_1, so Delta = 0, and Theorem 9.4 gives g >= sqrt(mu^2 + (pi/l)^2). The second bound is (c). ■

So the whole triple converges to a point carrying a two-dimensional massive Dirac block, while the compression at each l is the Connes-Lott triple (Theorem 9.6). The two statements concern different objects, and both hold; see "What this says about T0".

Corollary 15.6 (towerless Yukawa block, Theorem 12.7). Fix the c's and Y_hat of Theorem 12.7, with T_{X,l} = (C([0, l]), L^2 (x) (C^{N_Q} (+) C^{N_U}), D_X). Then T_{X,l} is a metric spectral triple of diameter l, and

```
Lambda_spec(T_{X,l}, (C, C^{N_Q + N_U}, D_F)) = O(l),     D_F = [[0, M_F], [M_F^dagger, 0]],   M_F
↪  = Phi_Q Y_hat Phi_U.
```

Proof. Multiplication by f preserves the conditions at l because f takes one value there, and norm([D_X, f]) = norm(f')_infinity. Theorem 12.7 gives g >= min(L_Q, L_U)/l. Mapping eigenvectors of +/- lambda_k to the singular vectors of M_F gives W with Delta = max_k abs(lambda_k - sigma_k(M_F)) = O(l^2). Theorem 15.3 applies, with sqrt(Delta) = O(l). ■

This answers the remark in Theorem 12.7 that a limit triple needs a choice of algebra: in this metric the algebra of the limit is C.

The converse

Theorem 15.7. Let T_n = (A_n, H_n, D_n) be metric spectral triples with Lambda_spec(T_n, T_F) -> 0, T_F = (C, C^N, D_F). Then diam T_n -> 0; for every R > norm(D_F), D_n has exactly N eigenvalues in [-R, R] for large n, so that the gap g_n to the remaining spectrum tends to infinity (with g_n infinite if

no spectrum remains); and there are unitaries W_n from the light space onto C^N with norm(W_n D_n W_n* - D_F) -> 0. With Theorem 15.3,

```
Lambda_spec(T_n, T_F) -> 0   if and only if   diam T_n -> 0, g_n -> infinity, Delta_n -> 0.
```

Proof. The diameter statement is (c). Let tau_n have dispersion < eps_n on [0, 1/eps_n], with eps_n -> 0.

Inner estimate. The characters theta_1, theta_2 are states of the base algebra B_n, and S(C) is one point, so mk_{L_B}(theta_1, theta_2) < eps_n. Applying this to the real and imaginary parts of <Theta, Theta'> and using inner Leibniz,

```
abs(<Pi_1 Theta, Pi_1 Theta'> - <Pi_2 Theta, Pi_2 Theta'>) <= 4 eps_n DN_P(Theta) DN_P(Theta').
```

At most N eigenvalues. Suppose D_n has N + 1 orthonormal eigenvectors xi_i with eigenvalues in [-R, R], so DN(xi_i) <= 1 + R. Lift them to Theta_i with DN_P(Theta_i) <= 2(1 + R), and put eta_i = Pi_2 Theta_i in C^N. The Gram matrix G of the eta_i has rank at most N, so norm(1 - G) >= 1, while the inner estimate gives norm(1 - G) <= 16 (N + 1)(1 + R)^2 eps_n. This fails for large n.

Concentration. Let e be a unit eigenvector of D_F with eigenvalue lambda, C = 1 + norm(D_F), and lift e to Theta with DN_P(Theta) <= 2C and xi = Pi_1 Theta. Apply the separation to the pseudo-state omega = xi/(2C): there is omega' with DN_F(omega') <= 1 and, for t in [0, 1/eps_n], abs(<omega, e^{itD_n} xi> - e^{it lambda} <omega', e>) <= 2C eps_n. Comparing with t = 0 and multiplying by 2C,

```
abs(<xi, e^{itD_n} xi> - e^{it lambda} norm(xi)^2) <= 8 C^2 eps_n.
```

Let nu be the spectral measure of D_n at xi. Taking real parts and averaging over t in [0, 1/eps_n] gives the integral of 1 - sinc((x - lambda)/eps_n) d nu(x) at most 8 C^2 eps_n. Since 1 - sinc(u) >= 1/2 for abs(u) >= 2, nu{abs(x - lambda) >= 2 eps_n} <= 16 C^2 eps_n.

Multiplicities. Apply this to an orthonormal eigenbasis e_i of D_F, with lifts xi_i. The inner estimate makes the Gram matrix of the xi_i tend to the identity, and concentration keeps it so after projecting each xi_i to the spectral window of radius 2 eps_n around its eigenvalue. So each window around an eigenvalue of multiplicity m contains at least m eigenvalues of D_n, for large n. The total is at most N, by the previous step, so equality holds. Mapping eigenvectors to the e_i gives W_n with norm(W_n D_n W_n* - D_F) <= 2 eps_n. ■

Corollary 15.8 (finite limits of connected commutative triples). Let T_n be metric spectral triples with A_n = C(X_n), X_n connected, converging in Lambda_spec to a triple with finite-dimensional Hilbert space. Then the limit is (C, C^N, D_F), and the convergence is characterized by Theorem 15.7. Every Hermitian D_F arises: on the circle of length l, the operator -i d/dx + diag(lambda_i) on L^2(S^1_l; C^N) has the block diag(lambda_i) on constants and gap at least 2 pi/l - norm(D_F). This construction concerns ungraded triples; a grading or real structure imposes its own compatibility conditions.

Proof. A metric triple is faithful: a self-adjoint a acting as 0 has [D, a] = 0, so a = t 1 by ker L = R 1, and then t = 0. So the limit algebra is finite-dimensional, and Theorem 15.2 gives C. ■

Corollary 9.10 fits the converse: at fixed mu every eigenvalue diverges, the spectra have no bounded sequences, and no limit exists. Theorem 9.8 lies outside the metric, because the diameter diverges.

Grading and real structure

The metric ignores gamma and J. They pass to the block exactly.

Proposition 15.9. In Theorem 15.3 let Sigma = -Sigma, and let D anticommute with a grading gamma and satisfy J D = eps' D J for an antiunitary J with J^2 = eps and J gamma = eps'' gamma J. Then P commutes with gamma and J, and gamma_F = W gamma W*, J_F = W J W* satisfy the same relations exactly with D_B = W D W*, and with D_F up to defects bounded by 2 Delta. The tunnel of Theorem 15.3 controls them: for DN_T(omega) <= 1 and DN_P(xi, eta) <= 1,

```
abs(<omega, gamma xi> - <W omega, gamma_F eta>) <= 1/g^2 + rho,     abs(<omega, J xi> - <W omega,
↪  J_F eta>) <= 1/g^2 + rho.
```

For Delta > 0, W omega need not be a pseudo-state of T_F; the normalized c W omega is one, and adds at most Delta to these bounds.

Proof. Spectral projections of D for a symmetric set commute with gamma and J, because gamma and J map the lambda-eigenspace to the eigenspaces of -lambda and eps' lambda. Decompose along P and Q as in Theorem 15.3, using additivity of J_F. ■

So gamma added to the families converges along the same tunnels, and the KO-signs of the parent reach the block unchanged when Delta = 0. For the interval, P Gamma P = sigma_3 (Theorem 9.6). Composition with the antilinear J gives antilinear functionals, which the definition of the separation does not cover; a statement of convergence for J needs an extension of the pseudo-state formalism, which we have not developed.

Unbroken symmetry and the metric class

Proposition 15.10 (referee). Let (A, H, D, J) be a finite-dimensional real triple with order zero, order one and J D = eps' D J, and let u be a unitary in A. If D commutes with u J u J^-1, then [D, u] = 0. If moreover the triple is metric, u is scalar, and the inner gauge transformation is trivial.

Proof. Put v = J u J^-1. From [D, u v] = 0, we get [D, u] v = - u [D, v], so a := u* [D, u] = - [D, v] v*. Now [D, v] = eps' J [D, u] J^-1 commutes with A by order one, and v* commutes with A by order zero. So a commutes with u, and [D, u^n] = n a u^n by induction. Then u^n D u^-n = D - n a. D is bounded, so a = 0. For a metric triple, the self-adjoint parts of u lie in ker L = R 1. ■

Counterexample 15.11 (referee; order one is needed). Let H = M_2(C) with the Hilbert-Schmidt product, A = M_2(C) acting by left multiplication, J x = x*, and D x = x - (1/2) tr(x) I. Applying [D, L_a] to I gives a - (1/2) tr(a) I, so [D, L_a] = 0 forces a scalar, and the triple is metric. Yet every conjugation x -> u x u* = L_u J L_u J^-1 commutes with D. This triple fails order one.

What this says about T0

Corollary 15.8 describes one notion of convergence: the state space of the whole algebra C(K_l) must converge in the Monge-Kantorovich metric. Under it, a collapsing connected parent yields a point, and the finite data survive as (C, C^N, D_F, gamma_F, J_F). Three limits on its reach:

1. **The truncation is a different object.** Along Theorem 9.9, the compression at each l is exactly Connes-Lott, with distance 1/m_f, while the parent has diameter l. The compression is not a quantum isometry, so this metric cannot see it. This is the towerless phenomenon itself: the two-sheet distance is unrelated to the size of the internal space.
2. **The class excludes a kind of target.** A metric spectral triple has ker L_T = R 1, so no nonscalar element of A commutes with D. The finite triple of the Standard Model lies outside the class: D_F commutes with the nonscalar subalgebra C_F = {(lambda, lambda, 0)}, which keeps the photon massless, and colour is unbroken (Chamseddine-Connes-Marcolli 2007, section 2.6). Its almost-commutative product lies outside as well, through the constant copy of such an element. A_F is moreover a real algebra, so applying the complex framework also needs a complexification. Among finite real first-order triples, the metric class has no nontrivial unbroken inner gauge transformation (Proposition 15.10); without order one this fails (Counterexample 15.11).
3. **The theorem concerns the scalar algebra.** Theorem 15.2 restricts limits of the whole scalar algebra C(K). Morita equivalence does not remove that restriction: C(K) and C(K, M_n) are Morita equivalent but not isomorphic, and their limits differ, and a bundle E can carry matrix-valued connections while C(K) still acts by scalars. Replacing the represented algebra by End(E) is a change of data, and a possible enlargement of T0. In that enlargement, Farsi-Latrémolière Theorem 4.6 makes an even metric finite (B, H_F, D_F) with noncommutative B the limit of B (x) C(K_l) with K_l collapsing, with nonzero multiplicity space ker D_K; there B is put in by hand.

Theorem 15.2 is a real constraint within its scope: this metric, on scalar function algebras. T0 also allows truncations, where it does not apply; lemma 16 treats these.

Part 16

Noncommutative finite real triples from commutative data: truncating the coarse correspondence

Proved by Claude Opus 5.5 without Astra or Fable, in response to the challenge that the T0 note overread lemma 15 and the Farsi-Latrémolière limits. Signs and the level-two algebra were checked with numpy on random data (section "Numerical check"). The term "coarse correspondence" is that of Connes, *Noncommutative Geometry* (1994), chapter V, for the bimodule of Hilbert-Schmidt operators, and the form D = Delta + J Delta J^-1 is in Krajewski (arXiv:hep-th/9701081) and Paschke-Sitarz (arXiv:q-alg/9612029); all checked against the sources. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): the examples and Corollary 16.4 are correct; amendments to Proposition 16.1, Theorem 16.2, Corollary 16.3 and the caveats are incorporated below; Remark 16.5 and Example 16.6 come from the referee's assessment of what the lemma achieves. The two-sheet section was refereed separately (Terra 6.1, 3 Nov 2026; report), with one error in Corollary 16.8 and one in Example 16.9, corrected below.

Outcome. Compressing functions on a connected commutative space to a fixed spectral window produces a noncommutative finite algebra with no limit taken. With the standard reality J a J^-1 = a*, the inherited structure fails order zero exactly when that algebra is noncommutative (Proposition 16.1). With the parent Hilbert space doubled to H_0 (x) conj(H_0), order zero and order one hold for every such algebra (Theorem 16.2, Corollary 16.3); these two conditions are formal consequences of the tensor square, available for any finite even representation (Remark 16.5). The KO-6 signs and the count dim H_F = 2 k^2 are reached by a product with a Clifford point, which is an input and fails finite orientability (Corollary 16.4), and more faithfully by a parent with two sheets, which gives the form of the Chamseddine-Connes classification (Theorem 16.7, Corollary 16.8, Example 16.9). This selects no particular algebra and gives no dynamics.

Setting

Let (A_0, H_0, D_0, gamma_0) be an even spectral triple, A_0 commuting with gamma_0, with D_0 of compact resolvent. Let Sigma = -Sigma be a finite subset of R and P_0 = 1_Sigma(D_0), of rank k, so that P_0 commutes with gamma_0. Put V = P_0 H_0, graded by gamma_V, with d = D_0 restricted to V, and

```
Q(a) = P_0 a P_0 on V,     O_F = Q(A_0),     A_F = C*(O_F) in B(V).
```

O_F is an operator system, in general not an algebra, and A_F consists of even operators. Compression is standard: for functions it is the Toeplitz compression, and its use in truncating spectral triples is due to D'Andrea-Lizzi-Martinetti and Connes-van Suijlekom. The latter note that the C*-algebra generated by PAP is in many examples a noninformative full matrix algebra. Since Q(1) = 1, A_F is the bicommutant of O_F, so its structure is set by the operators commuting with all compressed functions, such as the grading in example (b) and j in Example 16.9. That is where the information lies, and the fact that A_F is noncommutative is modest.

Examples of A_F.

(a) *Chiral interval, lowest level.* For condition (c) with M = 0 and Sigma = {+/- pi/(2l)}, O_F = C + C = A_F (lemma 05).

(b) *Chiral interval, next level.* For Sigma = {+/- pi/(2l), +/- 3 pi/(2l)}, V has basis (c_1, 0), (c_2, 0), (0, s_1), (0, s_2), with c_r = (2/l)^{1/2} cos((r - 1/2) pi x/l) and s_r = (2/l)^{1/2} sin((r - 1/2) pi x/l). With C_n = (1/l) int f cos(n pi x/l),

Q(f) = (Q_L, Q_R), Q_L = [[C_0 + C_1, C_1 + C_2], [C_1 + C_2, C_0 + C_3]], Q_R = [[C_0 - C_1, C_1 - C_2], [C_1 - C_2, C_0 - C_3]].

The values C_0, C_1, C_2, C_3 are independent and determine Q(f), so O_F has dimension 4. Each of Q_L and Q_R alone ranges over all symmetric 2 x 2 matrices, which generate M_2(C), so A_F projects onto each summand. For f_1 = 1/2 + cos(pi x/l) + cos(2 pi x/l) + cos(3 pi x/l) and f_2 = 1/2 - cos(pi x/l) + cos(2 pi x/l) - cos(3 pi x/l),

```
Q(f_1) = ([[1, 1], [1, 1]], 0),     Q(f_2) = (0, [[1, -1], [-1, 1]]).
```

M_2(C) is simple, so multiplying Q(f_1) and Q(f_2) by elements of A_F gives M_2(C) + 0 and 0 + M_2(C). Hence A_F = M_2(C) + M_2(C), of dimension 8. The finite algebra of the interval itself is noncommutative at the second level.

(c) *Twistor space.* For the level-one Toeplitz compression on CP^3, Q(x_a) = Gamma_a/5 (Proposition 10.1), and these generate A_F = M_4(C) on V = C^4. Here V is the kernel of the Dolbeault-Dirac operator twisted by O(1), so d = 0 on V.

The standard correspondence

Proposition 16.1 (order zero fails with inherited reality). Let J_0 be a real structure on H_0 (antiunitary, J_0^2 = +/-1, with the grading sign), with J_0 a J_0^-1 = a* for all a in A_0 and J_0 D_0 = eps' D_0 J_0, and let J_F = J_0 restricted to V. Then J_F Q(a)* J_F^-1 = Q(a), and the data (A_F, V, d, J_F) satisfy order zero if and only if A_F is commutative. The same holds for a product with a fixed finite factor that itself satisfies order zero, with the compatible product real structure.

Proof. J_0 preserves V because Sigma = -Sigma. Then J_0 P_0 a* P_0 J_0^-1 = P_0 a P_0. Order zero would make Q(a) commute with Q(b)° = Q(b) for all a, b, so O_F and hence A_F would be commutative; the converse is immediate, since J_F A_F J_F^-1 = A_F. For the product, take the unit in the finite factor: [1 (x) Q(a), 1 (x) Q(b)] = 1 (x) [Q(a), Q(b)]. ■

Together with Amendment 2 of lemma 07 (first order fails for C + C with inherited J_2), this is the correct form of the obstruction. It concerns the standard correspondence L^2(A_0) with the pointwise real structure.

The coarse correspondence

Let conj(H_0) be the conjugate Hilbert space, with canonical antiunitary xi -> xi^c, and with gamma^c, D^c the conjugate operators, gamma^c xi^c = (gamma_0 xi)^c and D^c xi^c = (D_0 xi)^c. On H = H_0 (x) conj(H_0), which is the coarse correspondence of A_0, equivalently the Hilbert-Schmidt operators on H_0, define for homogeneous xi, eta

```
pi(a) = a (x) 1,
DD = D_0 (x) 1 + gamma_0 (x) D^c,
Gamma = gamma_0 (x) gamma^c,
J (xi (x) eta^c) = (-1)^{|xi| |eta|} eta (x) xi^c.
```

Theorem 16.2. (A_0, H, DD, J, Gamma) is a real even spectral triple with signs J^2 = 1, J DD = DD J, J Gamma = Gamma J (KO-dimension 0). For even a, b with bounded commutators,

```
a° := J pi(a)\* J^-1 = 1 (x) (a\*)^c,     [pi(a), b°] = 0,     [[DD, pi(a)], b°] = 0,
```

and norm([DD, pi(a)]) = norm([D_0, a]). So if the original triple is metric, the new triple is metric with the same Connes distance on A_0. If the Weyl exponent p = lim log N_0(R)/log R of D_0 exists, that of DD is 2 p, since N_0(R/sqrt(2))^2 <= N_DD(R) <= N_0(R)^2.

Proof. J is antilinear, and J^2 = 1 because the sign is applied twice with the degrees exchanged. Gamma has eigenvalue (-1)^{|xi| + |eta|}, symmetric under the exchange. For the Dirac operator,

```
J DD (xi (x) eta^c) = (-1)^{(|xi| + 1) |eta|} eta (x) (D_0 xi)^c + (-1)^{|xi|} (-1)^{|xi| (|eta| +
↪  1)} D_0 eta (x) xi^c
                    = (-1)^{|xi| |eta|} [D_0 eta (x) xi^c + (-1)^{|eta|} eta (x) (D_0 xi)^c] = DD J
                     ↪  (xi (x) eta^c).
```

For even a, a* preserves degree, so J pi(a)* J^-1 (eta (x) xi^c) = eta (x) (a* xi)^c, which gives the formula for a° and order zero. Since a is even, [DD, pi(a)] = [D_0, a] (x) 1, which commutes with 1 (x) (b*)^c. Finally DD^2 = D_0^2 (x) 1 + 1 (x) (D^c)^2, because gamma_0 anticommutes with D_0. An eigenbasis of D_0 decomposes DD into finite-dimensional invariant blocks, so DD is essentially self-adjoint on the algebraic tensor product, with compact resolvent, and the spectrum is +/- (lambda_i^2 + lambda_j^2)^{1/2}. ■

Geometrically, when H_0 = L^2(S_X) on an even-dimensional spin manifold X, a charge conjugation commuting with D_X identifies conj(H_0) with H_0, H is L^2(S_{X x X}) and DD is the ordinary product Dirac operator. (The grading may acquire an overall sign under this identification.) So the data are a product-manifold Dirac operator on X x X with a restricted observable algebra C(X) (x) 1, and a J covering the flip of the two factors, which differs from the pointwise spin real structure. They count as commutative Kaluza-Klein data only with that restriction and that J: they are neither the ordinary spin triple of X x X with its full function algebra nor a finite-rank twist over X. The restriction is essential: on all of C(X x X), the compressions generate A_F (x) conj(A_F), and order zero generally fails again.

Corollary 16.3 (truncation). Let P = P_0 (x) conj(P_0). It commutes with DD, Gamma and J. It is a joint cutoff in the two factors, and need not be a spectral projection of DD: for absolute eigenvalues 1, 5, 7, sqrt(73) of D_0 with P_0 selecting the first three, the selected pair (5, 7) and the excluded pair (1, sqrt(73)) both give DD^2 = 74. The compression of pi(A_0) is O_F (x) 1; passing to the generated algebra gives the finite real even triple

```
T_F = (A_F, V (x) conj(V), D_F, J_F, Gamma_F),     D_F = d (x) 1 + gamma_V (x) d^c = Delta + J_F
↪  Delta J_F^-1,     Delta = d (x) 1,
```

with A_F acting on the left. It has KO-dimension 0, and satisfies order zero and order one for every A_F, noncommutative or not, with dim H_F = k^2. The family is exact at each truncation; no limit is taken.

Proof. The proof of Theorem 16.2 applies verbatim on V with arbitrary even a, b in B(V), since it uses only the grading and the tensor structure. For J_F Delta J_F^-1, apply J to (d (x) 1) J (xi (x) eta^c): the signs combine to (-1)^{|xi|}, giving gamma_V (x) d^c. ■

So H_F has the bimodule form of Connes' finite geometries, with A_F on the left and its opposite on the right, and D_F has the first-order form of Krajewski and Paschke-Sitarz.

Example (a) gives (C + C, C^2 (x) C^2, m (sigma_1 (x) 1 + sigma_3 (x) sigma_1)). There [[D_F, e], e°] = 0 for e = diag(1, 0), which repairs the first-order failure of Counterexample 08.4 at the price of doubling. Example (b) gives (M_2(C) + M_2(C), C^16, D_F).

Corollary 16.4 (KO-6 by a Clifford point). The product of T_F with (C, C^2, 0, sigma_1 K, sigma_3) is

```
(A_F, V (x) conj(V) (x) C^2, D_F (x) 1, J_F (x) sigma_1 K, Gamma_F (x) sigma_3),
```

with signs (+, +, -), order zero, order one and dim H = 2 k^2. It fails the usual zero-dimensional orientability axiom: every zero-cycle built from the left and opposite actions acts as T (x) 1_2, while the grading is Gamma_F (x) sigma_3.

Proof. (sigma_1 K)^2 = 1 and sigma_1 K sigma_3 = -sigma_3 sigma_1 K, while the D term is untouched. a° is a° of T_F tensored with 1. ■

Remark 16.5 (the order conditions are formal). For every finite even representation (A, V, d, gamma), the map V -> V (x) conj(V), d -> d (x) 1 + gamma (x) d^c gives a real bimodule with order zero and order one, using no prior finite real structure. So these two conditions follow from choosing the tensor square, and say nothing specific about A_F. The construction does not recover an arbitrary prescribed finite (H, D, J). What is specific to T0 is that A_F is generated by compressed functions, that the tensor square is itself the truncation of a commutative parent in the restricted sense above, and families such as the following.

Example 16.6 (a tower-decoupling family, from the referee). Let D_0 be the Dolbeault-Dirac operator on CP^3 twisted by O(1), with V = ker D_0 = C^4, and let DD_l = DD/l on the doubled parent. Its kernel is V (x) conj(V), and its gap grows like 1/l. Here P is a spectral projection. Compression followed by algebra generation gives the constant finite data A_F = M_4(C), dim H_F = 16, D_F = 0, and the Clifford point gives dim H = 32 with the KO-6 signs. Order one is automatic, since D_F = 0. The whole triple converges to (C, C^16, 0) by Theorem 15.3, consistently with lemma 15: the two statements concern the whole algebra and the truncation.

Two sheets: the form of the classification

Chamseddine and Connes (*Why the Standard Model*, arXiv:0706.3688, sections 2-3) classify irreducible pairs (A, J) with order zero and a separating vector: the centre of the complexified algebra is C, with A_C = M_k(C) on itself and J x = x*, or C + C, with A_C = M_k(C) + M_k(C) on itself and J(x, y) = (y*, x*). With centre C, J gamma = -gamma J is impossible. The pair of Corollary 16.4 is therefore reducible: the projection 1 (x) (1 + sigma_1)/2 commutes with A_F and J. The irreducible form comes from a parent with two sheets. At the finite level the bimodule (W (x) conj(V)) (+) (V (x) conj(W)) with its adjoint-flip J is already stated in Remark 2.9 of that paper; what we add is its realization from parents and compressed functions.

Theorem 16.7 (two-sheet coarse parent). Let (A_X, H_X, D_X, gamma_X) and (A_Y, H_Y, D_Y, gamma_Y) be even spectral triples. On H = (H_X (x) conj(H_Y)) (+) (H_Y (x) conj(H_X)), for even a in A_X and b in A_Y, put

```
pi(a, b) = (a (x) 1) (+) (b (x) 1),
DD = (D_X (x) 1 + gamma_X (x) D_Y^c) (+) (D_Y (x) 1 + gamma_Y (x) D_X^c),
Gamma = (gamma_X (x) gamma_Y^c) (+) (-gamma_Y (x) gamma_X^c),
J (xi (x) eta^c) = (-1)^{|xi| |eta|} eta (x) xi^c,
```

where J maps each summand to the other. Then J^2 = 1, J DD = DD J and J Gamma = -Gamma J, the KO-6 signs. Moreover

```
(a, b)° = (1 (x) (b\*)^c) (+) (1 (x) (a\*)^c),
```

so the first summand is an (A_X, A_Y)-bimodule and the second an (A_Y, A_X)-bimodule, and order zero and order one hold.

Proof. The computation of Theorem 16.2 applies with D_Y in place of the second factor: on xi (x) eta^c in the first summand, J DD and DD J both give (-1)^{|xi| |eta|} [D_Y eta (x) xi^c + (-1)^{|eta|} eta (x) (D_X xi)^c]. The sign in Gamma on the second summand gives J Gamma = -Gamma J. For a°, apply J, then pi(a, b)*, then J, as in Theorem 16.2: an element of the first summand goes to the second, where b* acts on the left, and returns with b* on the right factor. Since a and b are even, [DD, pi(a, b)] = ([D_X, a] (x) 1) (+) ([D_Y, b] (x) 1), which commutes with the opposite actions. ■

Corollary 16.8 (truncation; the form of the classification). For symmetric windows with V_X, V_Y of dimensions k_X, k_Y, and even subalgebras A_{F,X} in B(V_X), A_{F,Y} in B(V_Y), the compression

```
T_F = (A_{F,X} + A_{F,Y}, (V_X (x) conj(V_Y)) (+) (V_Y (x) conj(V_X)), D_F, J_F, Gamma_F)
```

has the KO-6 signs, order zero and order one, and dim H_F = 2 k_X k_Y. If gamma_{V_X} lies in A_{F,X} and gamma_{V_Y} in A_{F,Y}, the real orientability relation holds: with c = (gamma_{V_X}, gamma_{V_Y}) and z = (1, -1), Gamma_F = pi(z c) c°, a zero-cycle with coefficients in the algebra and its opposite. This is weaker than Gamma_F in pi(A_F). If V_X and V_Y are ungraded with k_X = k_Y = k and the generated algebras are full, then (A_F, H_F, J_F) is exactly the irreducible pair with centre C + C: identify both summands with M_k(C), so that J_F(x, y) = (y*, x*); an invertible matrix in each summand is a separating vector, and a projection commuting with both full matrix actions is scalar on each summand, with J equating the scalars. In all cases D_F commutes with the two sheet projections, so the sheets are not connected. It need not commute with the whole centre: for V_X = C^2, gamma_X = sigma_3, d_X = sigma_1, A_{F,X} = C + C acting diagonally and V_Y = C, the central element (diag(1, 0), 0) does not commute with D_F.

Proof. Theorem 16.7 applies on the truncation with arbitrary even operators. For orientability, pi(c) c° = (gamma_{V_X} (x) gamma_{V_Y}^c) (+) (gamma_{V_Y} (x) gamma_{V_X}^c), and z flips the sign of the second summand. The last statements are direct. ■

Example 16.9 (twistor level one, and M_2(H) + M_4(C)). Let D_0 be the Dolbeault-Dirac operator on CP^3 twisted by O(1), with kernel H^0(CP^3, O(1)) = C^4, and let j be the antiunitary with j^2 = -1 commuting with the Gamma_a of Proposition 10.1, which makes C^4 a quaternionic space H^2. Use this

space twice, as W (with the functions of S^4 pulled back along the twistor projection) and as V (with the functions of CP^3).

1. A real function f on S^4 pulls back to a function invariant under w -> j w. So j Q(f) j^-1 = Q(f), and Q(f) is a Hermitian element of End_H(W). Since Q(x_a) = Gamma_a/5, the real algebra generated is End_H(W) = M_2(H), of real dimension 16.
2. For real functions on CP^3, Q(w* B w / w* w) = (B + tr(B) 1)/5 for Hermitian B, so Q is onto the Hermitian matrices, and the real algebra generated is End_C(V) = M_4(C), of real dimension 32.

Hence Corollary 16.8, with the parent gradings, which are trivial on the kernels, gives the real algebra A = M_2(H) + M_4(C) on H_F = Hom(V, W) (+) Hom(W, V) = E (-) E* of dimension 32, with J(x, y) = (y*, x*). These are the algebra, bimodule and real structure of Chamseddine and Connes. The generation is of real algebras from real functions: complex generation on the S^4 side would give M_4(C).

The nontrivial grading of W in that construction is not obtained from this parent. An even parent has [P gamma P, P a P] = 0 for every compressed function, while Gamma_5 = 5 Q(x_5) anticommutes with Q(x_1). So the grading has to be added after truncation, as abstract finite data, fixing an axis. The full algebra is then a graded prealgebra rather than the algebra of an even triple, with even part H + H + M_4(C), and Gamma_F(xi, eta) = (Gamma_5 xi, -eta Gamma_5), which is Proposition 3.2 of that paper; the real orientability relation holds on the even part. Similarly d = mu n^a Gamma_a with n orthogonal to the axis is the compression of a bounded real multiplication operator, but adding that operator to the parent Dirac operator keeps neither the kernel as a spectral projection nor the parent grading as an odd perturbation, so d too is abstract data. It commutes with the two sheet projections and with the centre R + C of the full prealgebra, and not with the centre of the even algebra.

This bears on the step left open in canonical section 4, where M_2(H) in M_4(C) is a nested pair on one four-space, with "the independent E, F module roles" to be derived. The example closes that abstract finite-module problem after adding a second sheet: the nested actions on one four-space become independent summands acting on two Hom spaces. It does not turn four global sections into rank-four fibres over S^4, derive the two sheets from the fibration, or produce the Dirac operator connecting the two pieces that the selection of A_F = C + H + M_3(C) requires. The fibre-rank problem remains open.

The inputs are the second sheet, the restriction of one algebra to base functions, real algebra generation, the four-dimensional window, the grading axis and d. The content beyond the formal bimodule is the concrete compression calculation in items 1 and 2, already in the corpus, and its explicit match with the classified data.

Caveats

These results should be read as narrowly as the note they correct.

1. **Metric.** T_F is usually not a metric spectral triple. For A_F = B(V_+) + B(V_-) with V_+ of dimension at least 2 and d odd, the commutant of d in A_F contains the pairs (x, B^-1 x B) with x commuting with B B*, when B is the invertible block of d. In example (b), (diag(1, 0), diag(1, 0)) has L = 0, so d leaves a residual U(1) x U(1) of U(2) x U(2) unbroken. The Standard Model finite triple has the same feature, through M_3(C).
2. **Spectrum.** The eigenvalues of D_F are +/- (lambda_i^2 + lambda_j^2)^{1/2}. In example (a) they are +/- sqrt(2) m with Connes distance 1/m, so the doubled spectrum differs from the masses in d, and the mass relations of Theorem 9.6 and lemma 11 would have to be redone.
3. **Parent.** The doubled parent is a choice. Its algebra sees one factor of X x X, its J covers the flip, which is outside the spin real structure of X x X, and generating A_F from O_F is an additional operation. We have no dynamical reason for it. For an ordinary compact spin X of dimension n, its Weyl exponent is 2 n with metric dimension n, which bears on the counting of 9, 10 and 11 and has not been analyzed.
4. **KO-6.** The Clifford point is an input, as F_0 is in Theorem D, and it fails finite orientability. In Theorem 16.7 the sign comes from the block structure of two sheets, which is itself an input. The

documents checked here establish neither a construction of two sheets from one connected parent nor a no-go theorem. The grading-reversing structure of lemma 08 is a candidate we have not tested.

5. **Count.** For k = 4, dim H = 32, the value in the classification of Chamseddine-Connes, and k = 4 occurs in examples (b) and (c). For k = 4 in examples (b) and (c) alone this is a count and nothing more: the classification's algebra is M_2(H) + M_4(C), acting differently, and those examples do not produce it. Example 16.9 does, as abstract data with the listed inputs.

Consequence for T0 and T0b

T0 asks for finite real triples from commutative Kaluza-Klein data under specified notions of convergence. For one specified truncation procedure (compression, then generation, on the doubled parent with restricted algebra and flip J), this lemma gives a noncommutative finite algebra with an explicit bimodule, and a family with the tower decoupled (Example 16.6). It does not establish the target under every meaning of ordinary Kaluza-Klein data. For T0b, it gives examples with the requested signs and first-order relations; with two sheets it reaches the form of the classification, and in Example 16.9 the real algebra M_2(H) + M_4(C), bimodule and real structure of Chamseddine and Connes, with the grading added as abstract data. It leaves open an operator connecting the two pieces and hence the selection of A_F, a parent that yields the grading and nonzero mass data, and the beyond-first-order and twisted frameworks. See the background note.

Numerical check

Python (numpy), run on 3 Nov 2026 in a temporary directory, not kept. On random odd d and even a, b for dim V_+ = dim V_- = 2 with J = M K: J^2 = 1, J DD = DD J, J Gamma = Gamma J, DD Gamma = -Gamma DD, order zero and order one all hold to machine precision, with norm([a, b]) = 7.67; the KO-6 signs of Corollary 16.4 hold. For Theorem 16.7 and Example 16.9, with k = 4, W graded by Gamma_5 and V ungraded: J^2 = 1, J Gamma = -Gamma J, J D = D J, D Gamma = -Gamma D, D = Delta + J Delta J^-1, order zero and order one for random a, b, and Gamma_F = pi(z c) c° all hold to machine precision; j^2 = -1 commutes with the Gamma_a, and x_a(j w) = x_a(w); the real algebras generated by Gamma_a, by i Gamma_a and by all Hermitian matrices have real dimensions 16, 32 and 32; and sum_a x_a Gamma_a = 2 (P_w + P_{jw}) - 1. For example (b), by quadrature with 23 test functions: dim O_F = 4 and the generated algebra has dimension 8; the two displayed values Q(f_1) and Q(f_2) were reproduced by quadrature.

Part 17

Explicit masses in the reduced triple, the chiral limit and the set of reductions

Routine part of STRATEGY T10. Proved by Claude Opus 5.5 without Astra or Fable. The identities of Theorems 17.1 and 17.6 were checked with numpy on random finite models (section "Numerical check"). Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): of 10 items, 1 correct, 4 needed amendments, and 5 were wrong as first written (Corollary 17.4, Remark 17.7, Proposition 17.8, Proposition 17.9 and the consequence for T10, each by overreach). All corrections are incorporated below. Proposition 17.10 is the referee's.

Outcome. The fermion action of a product M x F, restricted to its physical subspace, is exactly the action of the odd triple M x T on an unconstrained spinor, with the Yukawa block becoming the constant matrix d in D_M + gamma_5 d (Theorem 17.1). A paired Dirac sector gives one Dirac action for either sign s, up to a change of Grassmann variables (Corollary 17.2); other sectors, such as a neutral one, have Majorana-type actions. On a subalgebra commuting with D_F there are no scalar fluctuations, so the mass is explicit (Proposition 17.3). At M = 0 the symmetry of the quark sector is U(N_f) x U(N_f) (Proposition 17.5). The admissible involutions of the quark sector are the Sigma in U(N_f) with Sigma* M Hermitian, which are exactly the stationary points of the mass term Re tr(M Sigma*); at M = 0 they form a homogeneous space of the chiral group (Theorem 17.6). All of this is kinematic. A pion kinetic term does not come from the fixed spectral action (Proposition 17.10). The condensate requires a measure on gauge fields, infinite volume and an order of limits (Proposition 17.8).

"D = 9" below is the formal KO-label 4 + 5 of Theorem D. The metric dimension of M x T is 4. The label exists only where an involution of sign s = -1 exists: charged paired sectors have one, and the strict light spectrum with one or three unpaired neutrinos has none, since its H_0 has odd dimension (lemma 06, Result 7).

Setting

Let M be a compact Riemannian spin 4-manifold with (D_M, J_M, gamma_5), where J_M^2 = -1, J_M D_M = D_M J_M, J_M gamma_5 = gamma_5 J_M and gamma_5 D_M = -D_M gamma_5. Let F = (A_F, H_F, D_F, J_F, gamma_F) be a finite real triple with the KO-6 signs. On L^2(S) (x) H_F put

```
D = D_M (x) 1 + gamma_5 (x) D_F,     J = J_M (x) J_F,     Gamma = gamma_5 (x) gamma_F,
```

with J^2 = -1, J D = D J and J Gamma = -Gamma J. Let H^+ be the Gamma = +1 subspace. The fermion action is (1/2) <J xi, D xi> on Grassmann variables xi in H^+ (Chamseddine-Connes-Marcolli 2007).

Let U be an admissible involution in the sense of Theorem D: U^2 = 1, U unitary, U gamma_F = -gamma_F U, [U, a] = [U, a°] = [U, D_F] = 0 and U J_F = s J_F U. On H_0 = {gamma_F = +1} put d = U D_F and j = U J_F, restricted to H_0, so that j^2 = s and j d = s d j.

The reduced action

Theorem 17.1. Define R : L^2(S) (x) H_0 -> H^+ by R(phi (x) x) = phi_+ (x) x + phi_- (x) U x, with phi_+- the chiral parts of phi. Then R is unitary, and

```
<J R psi, D R psi'> = <jj psi, DD psi'>,     DD = D_M (x) 1 + gamma_5 (x) d,
```

where jj = J_M (x) j for s = +1 and jj = gamma_5 J_M (x) j for s = -1, with jj^2 = -s and jj DD = s DD jj. Hence the action is (1/2) <jj psi, DD psi> on an unconstrained Grassmann spinor psi. Moreover DD^2 = D_M^2 (x) 1 + 1 (x) d^2, and if f(D^2/Lambda^2) is trace class, Tr f(D^2/Lambda^2) = 2 Tr f(DD^2/Lambda^2).

Proof. R is unitary because U exchanges the two gamma_F-eigenspaces. Let UU = 1 (x) U, which commutes with D and is an involution, so that <J xi, D xi'> = <UU J xi, UU D xi'>.

Put r x = (x + U x)/sqrt(2). Then D_F r = r d, since D_F r x = (D_F x + U D_F x)/sqrt(2) with U D_F x = d x and D_F x = U d x. Also (1 + UU) R/sqrt(2) = 1 (x) r. Since UU commutes with D, this gives UU D R = R DD.

For the real structure, J_F x = U j x and U J_F U = s J_F. As J_M preserves chirality,

```
UU J R(phi (x) x) = J_M phi_+ (x) j x + s J_M phi_- (x) U j x = R((J_M phi_+ + s J_M phi_-) (x) j
↪  x) = R jj (phi (x) x).
```

Therefore <J R psi, D R psi'> = <UU J R psi, UU D R psi'> = <R jj psi, R DD psi'> = <jj psi, DD psi'>. The square follows from gamma_5 D_M = -D_M gamma_5. Finally, D is the direct sum of its restrictions to the eigenspaces of UU, on which it is equivalent to D_M + gamma_5 d and D_M - gamma_5 d, and conjugation by gamma_5 sends the first to minus the second, so their squares are unitarily equivalent. ■

So the doubling is removed by the chirality projection, and the physical space is exactly the Hilbert space of M x T.

Corollary 17.2 (a paired sector gives one Dirac fermion, for both signs). Let H_0 = span(e_L, conj(e_R)) with m > 0 and psi = (psi_1, psi_2). For (d_+, j_+) = (m I, sigma_1 K) and (d_-, j_-) = (diag(m, -m), [[0, 1], [-1, 0]] K) of lemma 06, the action is

```
S_+ = <J_M psi_2, (D_M + gamma_5 m) psi_1>     and     S_- = <gamma_5 J_M psi_2, (D_M + gamma_5 m)
↪  psi_1>
```

respectively, with S_-(psi_1, psi_2) = S_+(psi_1, gamma_5 psi_2). After the charge-conjugation matrix is absorbed, the left argument is an independent Euclidean variable psi-bar, in the Grassmann-bilinear sense of Chamseddine-Connes-Marcolli, section 4. With R_5 = exp(i pi gamma_5/4), R_5 (D_M + gamma_5

m) R_5 = D_M + i m. So the action is that of one Dirac fermion of explicit mass m, whichever sign s is used, up to the displayed invertible change of Grassmann variables.

Proof. The total form is antisymmetric, so on Grassmann variables the two cross terms are equal, which cancels the factor 1/2. The rotation uses R_5 D_M R_5 = D_M. ■

This agrees with Theorem D, item 5: at the level of the classical fermion action, s labels two coordinate descriptions of one bilinear form, and no anomaly or theta shift follows from changing s alone. The rotation R_5 is a formal chiral change of both independent variables; in a quantized theory its Jacobian must be included.

An admissible reduction need not have this paired form. For gamma_F = sigma_3, J_F = sigma_1 K, D_F = m sigma_1 and U = sigma_1 we have s = +1, H_0 is one-dimensional, and the action is the Majorana-type (1/2) <J_M psi, (D_M + gamma_5 m) psi>. The neutral sector of the strict light spectrum has this form with m = 0.

Proposition 17.3 (no Higgs on a commuting subalgebra). Let B in A_F be a unital subalgebra with [D_F, B] = 0, and let U commute with B and B°. The inner fluctuation of D by C^infinity(M) (x) B contains only Clifford one-forms, D_A commutes with UU, and D_A reduces to

```
DD_A = D_M(A) + gamma_5 (x) d,
```

with d constant on M. If [D_F, B] != 0, the same reduction holds, and the fluctuation permits a finite scalar field d_A(x) in place of d.

Proof. For a = f (x) alpha and b = g (x) beta, we have a [D, b] = f [D_M, g] (x) alpha beta + f g gamma_5 (x) alpha [D_F, beta], whose second term vanishes. The term J A J^-1 is a Clifford one-form whose finite coefficients lie in B°. Since U commutes with B and with B°, Theorem 17.1 applies pointwise with D_M(A) in place of D_M. In the other case, U commutes with [D_F, beta], so the fluctuation preserves the reduction. ■

Corollary 17.4 (Yukawa couplings to explicit masses). In the representation of lemma 06, Result 5, with B = {(lambda, Q_lambda, m)} = C + M_3(C), u_L and u_R carry the same left character lambda and the same opposite colour action, d_L and d_R carry the left character conj(lambda) and the same opposite colour action, and e_L and e_R pair as well. So the kept mass blocks commute with B, in the strict projection of lemma 06 and also when only a vectorlike top pair is removed. Hence DD_A = D_M(A) + gamma_5 d, with d the direct sum of the kept mass matrices, and no Higgs field.

Two qualifications apply. First, U(B) contains U(1) x U(3), so before unimodularity or an inherited restriction from the Standard Model gauge group, two independent abelian gauge fields act on the surviving charged fermions; the reading "photon and gluons" needs that restriction. Second, a canonically normalized radial field couples to a fermion of fixed mass m_f with strength m_f/v, which tends to 0 if v tends to infinity at fixed gauge coupling. Under the CCM relation (lemma 11), keeping all Dirac masses fixed while M_W tends to infinity is impossible: some Dirac mass must become unbounded. It need not be the top; top dominance is a further hypothesis. The relation holds in the asymptotic matching regime, and the regime near the cutoff is T5b.

The chiral limit

Consider N_f quark flavours, with q_L, q_R in C^{N_f} (x) C^3, and q-bar_L, q-bar_R = J q_L, J q_R. Let D_F act on (q_L, q_R) by [[0, M], [M*, 0]] (x) 1_3, and by the complex conjugate block on the antiparticles. Let M_3(C) act on the antiparticles on the left and on the particles through its opposite action.

Proposition 17.5 (the symmetry restored at M = 0). Let G(M) be the group of unitaries on this sector commuting with both actions of M_3(C), gamma_F, J_F and D_F. Then

```
G(M) = {(V_L, V_R) in U(N_f) x U(N_f) : V_L M = M V_R},
```

acting by V_L, V_R on q_L, q_R and by the complex conjugates on q-bar_L, q-bar_R. So G(0) = U(N_f) x U(N_f), G(m I) = U(N_f)_V for m != 0, and G(M) = U(1)^{N_f}_V for M positive diagonal with distinct entries. With m_u = m_d = 0 and m_s != 0 in one block, G contains U(2) x U(2). If V is also required to commute with the electric charge, through lambda in B, then each V_L, V_R is diagonal between u and d, and G(0) = U(1)^4 for two flavours.

Proof. The colour identity separates particles from antiparticles, since its left action is zero on particles and the identity on antiparticles. The commutant of the colour matrix algebra is scalar on colour, so V = V' (x) 1_3 on each. The grading separates L and R, and J_F fixes the antiparticle part. Commutation with D_F gives V_L M = M V_R, whose adjoint with unitarity gives the other block equation. ■

The nonabelian flavour transformations are not colour gauge transformations, and at M = 0 the axial part is an ungauged symmetry of the reduced triple. If the central U(1) of U(3) is retained in the gauge group, it contains a common vector phase, which an imposed unimodularity condition can remove. The Lagrangian symmetry U(1)_A is anomalous in the quantum theory, which the classical data do not show.

Theorem 17.6 (the set of reductions). Let B contain only the colour algebra. The admissible involutions of sign s on this sector are exactly the operators, tensored with 1_3,

```
U_{Sigma,s} = [[0, Sigma], [Sigma\*, 0]] on (q_L, q_R),     s [[0, conj(Sigma)], [Sigma^T, 0]] on
↪  (q-bar_L, q-bar_R),
```

with Sigma in U(N_f) and Sigma* M Hermitian. This condition holds if and only if Sigma is a stationary point of V_M(Sigma) = -Re tr(M Sigma*) on U(N_f). The reduced mass operator on H_0 = (q_L, q-bar_R) is

```
d_{Sigma,s} = (Sigma M\* (x) 1_3) (+) (s conj(Sigma\* M) (x) 1_3).
```

The group G(0) acts by Sigma -> V_L Sigma V_R*. Hence:

1. At M = 0 the set is all of U(N_f), for each s, and the action is transitive with stabilizer of Sigma = 1 the diagonal U(N_f): the reductions form the homogeneous space (U(N_f) x U(N_f))/U(N_f)_V.
2. At M = m I with m > 0 it is {Sigma : Sigma^2 = 1, Sigma* = Sigma}, the disjoint union of the Grassmannians of C^{N_f}.
3. For M positive diagonal with distinct entries, it is the 2^{N_f} matrices diag(+-1, …, +-1).
4. For M invertible, the unique Sigma with Sigma* M positive is the polar part of M. It is the unique global minimum of V_M, and it is the unique Sigma for which the first block of d_{Sigma,s} is positive.

With the electric charge in B, the admissible set is the intersection of this set with the commutant of the charge. For two differently charged flavours this keeps diagonal phases at M = 0, and only signs +-1 for positive nonzero diagonal masses.

Proof. An admissible U commutes with both colour actions, so it acts on each of q, q-bar as U' (x) 1_3; it reverses the grading, so it is off-diagonal between L and R; U^2 = 1 with U unitary gives the form [[0, Sigma], [Sigma*, 0]] with Sigma unitary. Then U J_F = s J_F U fixes the antiparticle block. Commutation with D_F is Sigma M* = M Sigma* and Sigma* M = M* Sigma. If H = Sigma* M is Hermitian, then Sigma M* = Sigma H Sigma* is Hermitian, so the two are equivalent. Along Sigma exp(i t X), X Hermitian, the derivative of Re tr(M Sigma*) at t = 0 is Im tr(H X), which vanishes for all Hermitian X exactly when H is Hermitian. The formula for d is direct, using q-bar_R = J q_R. Item 1 is immediate. For item 2 the condition is Sigma* = Sigma. For item 3, H^2 = M* M forces H to be diagonal with entries +- m_i. Item 4 is von Neumann's trace inequality, Re tr(M Sigma*) <= tr |M|, with equality exactly at the polar part, and the first block is Sigma H Sigma*. ■

Remark 17.7 (scope). Item 1 is an equivariant statement: the reductions form a homogeneous space of the chiral group, with the same stabilizer as in the symmetry breaking of QCD. The mass condition has an elementary origin, since it says that H = Sigma* M is Hermitian, and for invertible M the positive choice is the polar decomposition. Its equality with the stationarity equation of -Re tr(M Sigma*) is exact, and it captures all stationary points, including saddles and maxima. Vacuum alignment (Dashen) asks which stationary point minimizes a physical energy, and the data here contain no such energy; the stationarity calculation is our derivation, and Dashen is cited for the vacuum-selection idea alone.

So the theorem shows the shape of the set of reductions. Identifying a reduction with a condensate direction would need an explicit map to a quantum order parameter, which we do not have. In QCD, the anomalous singlet axial direction also means that the full classical U(N_f) cannot be identified without qualification with the pion manifold.

For invertible M, every stationary point other than the polar part has at least one negative eigenvalue of Sigma* M; for singular M there may be a continuum of minima, as for M = diag(1, 0) and Sigma = diag(1, exp(i theta)). A negative entry in the reduced mass block comes from choosing another reduction of one fixed even triple, and is not a physical mass-sign change at fixed fermion measure. A physical sign change is different: for one Dirac flavour in a background with n_0 zero modes, the determinants at m and -m differ by the factor (-1)^{n_0}, the factor associated with a shift of theta by pi.

Spacetime-dependent Sigma

Proposition 17.10 (referee; no kinetic term from the fixed action). Let U(x) be a pointwise admissible involution on M, with c^mu = [D_M, x^mu] and B_mu = (U d_mu U) restricted to H_0. With R defined pointwise,

```
R^-1 UU D R = D_M + c^mu P_- B_mu + gamma_5 d(x),
```

where P_- = (1 - gamma_5)/2. This operator is generally not self-adjoint. So:

1. If Sigma(x) only changes coordinates for the fixed operator D, the spectral action of D does not depend on Sigma, and conjugating D by a smooth unitary preserves its spectrum. Neither produces a two-derivative term.
2. A kinetic term needs an additional choice of operator. For D_new = D_M + gamma_5 d_Sigma(x) on flat M, with M = m I and Sigma* = Sigma, Sigma^2 = 1, the heat expansion of Tr f(D_new^2/Lambda^2) contains

   ```
   (N_c f(0) m^2/(4 pi^2)) int tr(d_mu Sigma\* d^mu Sigma),
   ```

   tracing over both reduced blocks. The sign is positive when f(0) > 0. At M = 0 this operator has no dependence on Sigma.

Proof. The first display is direct differentiation in Theorem 17.1. Since B_mu is anti-Hermitian and P_- c^mu = c^mu P_+, the derivative term has adjoint c^mu P_+ B_mu. For the coefficient, D_new^2 = -(nabla^2 + E) with E containing -i gamma^mu gamma_5 d_mu d, and tr over spinors of the square of that term is 4 tr(d_mu d d^mu d); then the a_4 coefficient (1/16 pi^2) tr(E^2/2) gives (1/(8 pi^2)) tr_{H_0}(d_mu d d^mu d), and the two blocks of d give 2 N_c m^2 tr(d_mu Sigma d^mu Sigma). ■

So the idea of a pion as a spacetime-dependent reduction, in the form we first proposed, does not produce a kinetic term: the kinetic term needs a nonzero m and a new operator, or a quantum effective action. On the even triple, D_M + gamma_5 S(x) with S built from Phi = m Sigma(x) is self-adjoint for every unitary Sigma. Its kinetic term is the Higgs-type term of an inner fluctuation, available when A_F contains flavour algebras acting on L and R, and then flavour gauge fields accompany it. The coefficient is proportional to N_c m^2, as in constituent-quark estimates of f_pi^2 (quoted from memory); here it is a classical spectral-action coefficient, and m is not derived.

What the spectral data say about the condensate

Proposition 17.8. Let D_M(A) have eigenvalues lambda_n, with n_0 zero modes, on M of volume V, and for m > 0 put

```
c_V(m; A) = (1/V) sum_n m/(lambda_n^2 + m^2) = (1/V) d/dm log |det (D_M(A) + gamma_5 m)|,
```

which is formal on a compact four-manifold and needs ultraviolet regularization, with subtraction of divergent terms including m Lambda^2, m^3 log Lambda and possibly curvature terms.

1. With a subtraction compatible with the pairing +-lambda whose finite remainder is O(m), for example odd and analytic in m, at fixed A on compact M, c_V(m; A) = n_0/(V m) + O(m) as m -> 0. A

subtraction merely regular at m = 0 may leave a constant. So at finite volume and fixed background, no nonzero limit exists beyond the zero-mode term.

2. Let rho_m(lambda) = lim_{V -> infinity} <(1/V) sum_n delta(lambda - lambda_n)>*m for a measure* <.>m on gauge fields. Assume that the family is equicontinuous at 0 with rho_m(0) -> rho_0(0), that the tails and the ultraviolet subtraction are controlled uniformly in m, and that the infinite-volume limit may be taken inside the integral. Then lim*{m -> 0} lim*{V -> infinity} = pi rho_0(0). This has the form of the Banks-Casher relation as stated by Giusti and Lüscher (the original, Nucl. Phys. B 169, eqs. (49), (50), has neither the factor pi nor an order of limits with the volume); the sign and flavour normalization of a condensate depend on convention, and the hypotheses here are ours.

Equicontinuity alone is insufficient: rho_m(lambda) = (1 + (1/2) sin(1/m)) exp(-lambda^2) gives no limit.

Proof. Item 1: for a cutoff preserving the pairs +-lambda, the determinant has absolute value m^{n_0} times the product of (lambda^2 + m^2) over lambda > 0, and the nonzero eigenvalues are bounded away from 0. Item 2: m/(lambda^2 + m^2) is pi times the Poisson kernel at scale m. ■

So the spectral triple supplies the operator D_M + gamma_5 m, the density rho_A of each fixed operator, and the continuity of m -> 0 at fixed background, since DD^2 = D_M^2 + m^2. The gauge-averaged density, the regulated determinant or Pfaffian measure, the sequence of volumes and the order of limits need further input. A spectral action may contribute to a proposed gauge weight, but a single compact spectral triple does not specify the functional integral or the thermodynamic limit.

A proof of existence of QCD with fermions would supply a measure. It would not by itself give rho_0(0) > 0. The points to check when it appears are those in STRATEGY T10: whether m_q = 0 is included, and whether infinite volume is reached. With the anti-self-adjoint operator slashed-D = -i D_M and a compatible regulator, det(slashed-D + m) = m^{n_0} times the product over lambda > 0 of (lambda^2 + m^2), which is positive for m > 0; this positivity is part of the method of Vafa and Witten.

The chiral limit on the interval

Proposition 17.9. In the scalar-algebra realization F_0 x I_l of Theorem 08.7, DD = D_M (x) 1 + gamma_5 (x) (D_0 + m Gamma_0), so the interval mass is an explicit mass term with the internal grading. The limit m -> 0 has two distinct routes:

1. *Localization (lemma 09).* With bulk mass -mu and mu l > 1, m = lambda_0 = mu sech(kappa_0 l), of order 2 mu exp(-mu l), which is positive at finite parameters. At fixed mu > 0, l -> infinity gives m -> 0 as an exponentially small residue. The relevant control is the dimensionless suppression: an absolute limit needs a scaling hypothesis, since mu = 2/l gives m -> 0 at fixed mu l, and mu l -> infinity does not ensure m -> 0 if mu grows fast enough. The localization and overlap mechanism is that of domain-wall fermions (Kaplan; Shamir), though this is neither lattice operator.
2. *Boundary Yukawa (lemma 12).* At X = 0 the zero modes are exact (Theorem 12.2), with boundary types (a) and (b). For flavour-degenerate bulk masses, D_0 commutes with U(N_Q) x U(N_U) on the whole tower, and at any l the boundary coupling X preserves exactly the stabilizer {(V_Q, V_U) : V_Q X = X V_U}. So X acts as a spurion located on the wall x = l, and the symmetry is exact at finite l when X = 0. In the towerless limit X = l Y_hat, M_F is proportional to X, and for N_Q = N_U the stabilizer is G(M_F) of Proposition 17.5. At finite l, M_F is not proportional to X: for one flavour with zero bulk masses, the light eigenvalue is arctan(abs(X))/l.

The boundary-coupled operator of route 2 is a different construction from the tensor-product truncation of Theorem 08.7, and an identification of the two at finite l would need a proof.

Proof. Route 1 is Theorem 9.4 and Lemma 9.5, and the scaling examples are direct. Route 2 follows from the invariance of the domain of D_X (lemma 12), Theorem 12.7 and Example 12.5. ■

Caveats

1. **Classical data.** Everything here is at the level of the classical fermion operator. Anomalies, the running of masses and confinement do not appear.

2. **QED.** With the photon present, the axial group reduces as stated in Proposition 17.5, and the set of reductions reduces as stated in Theorem 17.6.
3. **Regime.** Corollary 17.4 inherits the regime of the CCM relation.
4. **Label.** A reduction is a rewriting of one even triple. It gives no parent nine-dimensional geometry, in agreement with Theorem D, and the KO-5 label exists only where an s = -1 involution exists.

Consequence for T10

The reduced action is (1/2) <jj psi, (D_M(A) + gamma_5 d) psi> on an unconstrained spinor, whenever the chosen admissible involution exists. A paired charged sector gives one Dirac action of explicit mass for either sign, up to a change of variables; a neutral unpaired sector gives a Majorana-type action and has only the KO-7 reduction, so the formal label 9 is available only for sectors with the required s = -1 involution. The symmetry restored at m_u = m_d = 0 is U(2) x U(2) on the colour-only algebra, and the reductions there form the homogeneous space U(2), with the mass condition picking out the stationary points of the mass term. The spectral data state the operator, the symmetry, this homogeneous space and the density of each fixed operator. The condensate and the pions as dynamical objects need the QCD measure and a quantum effective action.

Numerical check

Python (numpy, scipy), run on 3 Nov 2026 in a temporary directory, not kept. On a random finite model with KO-4 signs for M (dimension 8) and a quark sector with N_f = 3 and random complex M, for three sign patterns of Sigma and both s: U^2 = 1, U gamma_F = -gamma_F U, [U, D_F] = 0 and U J_F = s J_F U hold to machine precision; R is unitary; the form <J R psi, D R psi'> equals <jj psi, DD psi'> to about 10^-14 and is antisymmetric; j^2 = s and j d = s d j; the formula for d_{Sigma,s} holds, after a first version of it was found wrong by this check; finite differences confirm that V_M is stationary at these Sigma; a generic unitary Sigma gives norm([U, D_F]) of about 10. R_5 (D_M + gamma_5 m) R_5 = D_M + i m holds to machine precision. The spin trace in Proposition 17.10 was evaluated with explicit Dirac matrices (tr of the square equals 4 |a|^2).

Part 18

T5b: vector masses in the spectral action at fixed cutoff

Prompt: prompts/18 (STRATEGY T5b). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5 with Python (numpy, scipy), by computations independent of the kernel derived here: a direct heat trace with constant a reproduces equation (2) for five choices of (m_1, m_2), to about 0.03%; a direct heat trace on Fourier modes with a_2(x_1) = a cos(q x_1), including the F_{12} and [T, V] vertices, reproduces the kernel (37) at three values of (m, q), to 0.1-0.3%, the size of the discretization error. The closed forms (16)-(19), the bound (28), the limits (27), the neutral matrix (30) recomputed from Standard Model charges, and the identities behind (31) also check. The largest value of the Taylor mass is M_W^2 = 0.275 Lambda^2, at m_t^2 = 2.02 Lambda^2. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): all ten numbered results are correct, with small amendments incorporated below. The referee added Propositions 11 to 14 and Remark 15, which widen the scope in both directions; the ratio in Proposition 14 was added by Claude.

Scope. Within the three-generation, zero-Majorana, top-only family at fixed cutoff, these results refute an infinitely heavy W, Z, t limit for the stated quadratic mass definitions. They do not rule out other finite data (Proposition 14), a different action or constraint, a joint limit with Lambda -> infinity, or a quantum effective theory. They do not show unbroken SU(2) x U(1) on the light space (Remark 15).

For the heat cutoff, on $\mathbb{R}^4$, we obtain the exact quadratic vector form without expanding in m/Λ. In the three-generation Standard Model finite representation with only the top Dirac mass nonzero, both the mass defined by the q^0/q^2 Taylor ratio and the unique positive timelike pole mass satisfy $M_W \sim \Lambda^2/m_t$ as $m_t/\Lambda \to \infty$ at fixed Λ. The broken vectors consequently do not decouple by becoming heavy, although the heavy fermion

contribution disappears from the gauge coefficients. These statements concern the classical bosonic quadratic action about a prescribed constant background; moreover, the positive Laplace class considered below does not itself select a finite nonzero stationary background when radial variations of that background are allowed.

1. Theorem (exact quadratic kernel).

Let

$$D_A = -i\gamma^\mu(\partial_\mu + iA_\mu) + \gamma_5 V, \qquad V = V_0 = V_0^\dagger,$$

on $\mathbb{R}^4$, with Hermitian Euclidean gamma matrices, $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$, $\gamma_5^2 = 1$, and four spin components. Let the finite trace be the literal trace on H_F, including every stated multiplicity, without a subsequent projection; this agrees with the bosonic trace of Chamseddine-Connes-Marcolli, which counts particles, antiparticles and four spin components. Take A smooth and compactly supported.

We use the vacuum-subtracted action, or equivalently its quadratic variation, since the constant-background trace on $\mathbb{R}^4$ has infinite volume. Write

$$A_\mu(q) = a_\mu(q)T, \qquad T = T^\dagger, \qquad q^\mu a_\mu(q) = 0, \qquad \int_q = \int \frac{d^4q}{(2\pi)^4}.$$

Diagonalize V_0:

$$V_0 e_i = m_i e_i, \qquad m_i \in \mathbb{R}.$$

The signed eigenvalues are needed; their squares alone do not determine the answer.

For $S_t = \operatorname{Tr} e^{-tD_A^2}$, define

$$b(u) = u(1-u), \quad d_{ij} = m_i - m_j,$$

$$I_t(a,b;q^2) = \int_0^1 e^{-t[(1-u)a+ub+u(1-u)q^2]}\, du.$$

Then

$$S_t^{(2)} = \int_q |a_T(q)|^2\, C_{t,T}(q^2),$$

where

$$\boxed{C_{t,T}(q^2) = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2\left[\left(\frac{2}{t} + d_{ij}^2 + q^2\right) I_t(m_i^2, m_j^2; q^2) - \frac{e^{-tm_i^2} + e^{-tm_j^2}}{t}\right].} \tag{1}$$

For several generators, replace $|T_{ij}|^2$ by $\operatorname{Re}((T_a)_{ij}(T_b)_{ji})$.

In particular, writing $C(q^2) = C_0 + q^2 C_1 + O(q^4)$, we obtain

$$\boxed{C_{0,t} = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2\left[\left(\frac{2}{t} + d_{ij}^2\right) I_t(m_i^2, m_j^2; 0) - \frac{e^{-tm_i^2} + e^{-tm_j^2}}{t}\right],} \tag{2}$$

and

$$\boxed{C_{1,t} = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2 \int_0^1 e^{-t[(1-u)m_i^2 + um_j^2]}\left[1 - 2b(u) - td_{ij}^2 b(u)\right] du.} \tag{3}$$

These formulas also apply, after integration over $t = s/\Lambda^2$, to

$$f(z) = \int_{(0,\infty)} e^{-sz}\, d\nu(s), \quad 0 < f(0) = \nu((0,\infty)) < \infty, \quad \int (s^{-2} + s^2)\, d\nu(s) < \infty, \tag{4}$$

where ν is positive. These hypotheses suffice for the Euclidean action density and the Taylor coefficients used below. They do not, by themselves, guarantee analytic continuation to every negative q^2.

Proof. With $\nabla_\mu = \partial_\mu + iA_\mu$,

$$D_A^2 = -\nabla^2 + V_0^2 - \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu} + \gamma^\mu\gamma_5[A_\mu, V_0].$$

Let $P_0 = -\partial^2 + V_0^2$ and write $D_A^2 = P_0 + P_1 + P_2$, according to degree in A. Duhamel expansion and cyclicity give

$$S_t^{(2)} = -t\,\mathrm{Tr}(e^{-tP_0}P_2) + \frac{t^2}{2}\int_0^1 \mathrm{Tr}\,\left(e^{-(1-u)tP_0}P_1 e^{-utP_0}P_1\right) du. \tag{5}$$

The linear vertex from j, p to $i, p+q$ is

$$T_{ij}\left[(2p+q)\cdot a + \gamma^{\mu\nu}q_\mu a_\nu + (m_j - m_i)\gamma^\mu\gamma_5 a_\mu\right].$$

Completing the momentum square produces

$$(4\pi t)^{-2}e^{-t[(1-u)m_i^2 + um_j^2 + b(u)q^2]}.$$

For transverse a, the averaged spin trace of the vertex product is

$$4\left(\frac{2}{t} + d_{ij}^2 + q^2\right)|a|^2.$$

The P_2 term contributes

$$-\frac{1}{4\pi^2 t}\sum_{ij}|T_{ij}|^2 e^{-tm_i^2}|a|^2.$$

Symmetrizing in i, j proves (1). Differentiation proves (2)–(3).

The divided-difference structure is explicit:

$$I_t(a,b;0) = \frac{e^{-ta} - e^{-tb}}{t(b-a)}, \qquad I_t(a,a;0) = e^{-ta}, \tag{6}$$

and

$$\int_0^1 u(1-u)e^{-t[(1-u)a+ub]}\,du = \frac{1}{t^2}\partial_a\partial_b I_t(a,b;0). \tag{7}$$

Thus (2) uses a first divided difference, while (3) also uses its mixed derivative. The full momentum dependence follows from the convergent series

$$I_t(a,b;q^2) = \sum_{n=0}^{\infty}\frac{(-q^2/t)^n}{n!}\partial_a^n\partial_b^n I_t(a,b;0). \tag{8}$$

Repeated-node values are obtained continuously.

For completeness, the mass coefficient for (4) can be written entirely in dimensionless variables. Put

$$r_i = m_i/\Lambda, \quad x_i = r_i^2, \quad z(u) = (1-u)x_i + ux_j, \quad \Phi_1(z) = \int s^{-1}e^{-sz}\,d\nu(s).$$

The bracket replacing that in (2) is

$$\Lambda^2\left[2\int_0^1 \Phi_1(z(u))\,du + (r_i - r_j)^2\int_0^1 f(z(u))\,du - \Phi_1(x_i) - \Phi_1(x_j)\right]. \tag{9}$$

The corresponding bracket in (3) is

$$\int_0^1 \left[(1-2b)f(z) + (r_i - r_j)^2 b f'(z)\right] du. \tag{10}$$

Each unweighted integral is a divided difference of an antiderivative. The moment assumptions justify the integrations and differentiations. □

On a finite flat torus, (5) remains exact, but the Gaussian momentum integral must be replaced by the spin-structure-dependent lattice sum. Explicitly, for a transverse polarization a, its two-vertex contribution per unit volume is

$$\frac{2t^2}{\mathrm{Vol}}\sum_{ij,p}|T_{ij}|^2\int_0^1 e^{-t[(1-u)(p^2+m_i^2)+u((p+q)^2+m_j^2)]}\left(|(2p+q)\cdot a|^2+(d_{ij}^2+q^2)|a|^2\right)du,$$

and the quadratic-vertex contribution is

$$-\frac{4t}{\mathrm{Vol}}\sum_{ij,p}|T_{ij}|^2e^{-t(p^2+m_i^2)}|a|^2.$$

Consequently, the continuum formulas below are statements on $\mathbb{R}^4$, or in the infinite-volume limit. A finite torus can have additional holonomy dependence, including for constant commuting connections.

2. Lemma (gauge invariance and the Goldstone direction).

Let scalar fluctuations include the gauge orbit of V_0. Then the nonzero vector coefficient C_0 in (2) is compatible with gauge invariance.

For one generator, parameterize the orbit by

$$V(x)=e^{i\pi(x)T}V_0e^{-i\pi(x)T}.$$

With our conventions,

$$a_\mu\longmapsto a_\mu-\partial_\mu\alpha,\qquad \pi\longmapsto\pi+\alpha.$$

The invariant combination is

$$B_\mu=a_\mu+\partial_\mu\pi. \tag{11}$$

Proof. Gauge covariance gives

$$D_{A^U,V^U}=UD_{A,V}U^{-1}.$$

Applying $U=e^{-i\pi T}$ transforms V to V_0 and a to B. Hence

$$S[a,e^{i\pi T}V_0e^{-i\pi T}]=S[B,V_0]. \tag{12}$$

This establishes the Goldstone completion directly, without a derivative expansion.

For comparison, the longitudinal vector kernel is obtained from (1) by replacing the q^2 in its prefactor by $(1-2u)^2q^2$ inside the integral. If $[T,V_0]=0$, its integral vanishes by

$$\int_0^1[2+tq^2(1-2u)^2]e^{-tq^2u(1-u)}\,du=2.$$

For a broken generator the longitudinal kernel need not vanish; it belongs to the invariant combination (11).

If V_0 is not stationary, a Cartesian scalar Hessian alone need not annihilate the linear gauge-orbit tangent. The quadratic term in $e^{i\pi T}V_0e^{-i\pi T}$, contracted with the scalar tadpole, must also be retained. Equation (12) includes that term. Transverse vectors do not mix quadratically with scalar fluctuations on this translation-invariant background. □

3. Corollary (the asymptotic normalization).

Let all finite eigenvalues be small relative to Λ, let f satisfy (4), and let Q below be positive definite on the chosen generator space (otherwise pass to the quotient by its null space). Define

$$Q_{ab}=\mathrm{Re}\,\mathrm{Tr}(T_aT_b),\qquad K_{ab}=\mathrm{Re}\,\mathrm{Tr}([T_a,V_0]^\dagger[T_b,V_0]).$$

At leading order,

$$C_0 = \frac{f(0)}{8\pi^2}K, \qquad C_1 = \frac{f(0)}{12\pi^2}Q,$$

and therefore

$$\boxed{\mathcal{M}^2_{\text{Taylor}} = \frac{3}{2}Q^{-1/2}KQ^{-1/2}.} \tag{13}$$

This reproduces Proposition 13.5.

For the full three-generation Standard Model representation,

$$Q_{11} = 12, \qquad K_{11} = \sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2),$$

so

$$\boxed{8M_W^2 = \sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2).} \tag{14}$$

Proof. In (2), the orbital expression

$$\frac{2}{t}I_t(a,b;0) - \frac{e^{-ta}+e^{-tb}}{t}$$

starts at order $t(a-b)^2$. The leading term is consequently d_{ij}^2. In (3), the leading integral is $\int_0^1(1-2u(1-u))du = 2/3$, proving (13).

For $T_1 = \sigma_1/2$ on each left doublet, one generation of particles contributes 2 to Q_{11}. Charge conjugates and three generations give 12. A doublet with Dirac masses m_u, m_d contributes $\frac{1}{2}(m_u^2+m_d^2)$ to the particle commutator trace, before color multiplicity. This proves (14).

Equivalently, the gauge coefficient is

$$\frac{f(0)g_2^2}{2\pi^2} = \frac{1}{4}.$$

For comparison with CCM's $\operatorname{Tr} F(D/\Lambda)$, take $F(v) = f(v^2)$; then $F(0) = f(0)$. The normalization and mass constraint are precisely equations (4.10) and (4.16), with the latter stated in Lemma 4.6 of the checked preprint. CCM describes the relation as "valid at unification scale." Chamseddine–Connes–Marcolli, pp. 37–39. □

The leading bound $M^2 \le 6\|V_0\|^2$ in Lemma 14 follows from (13) and $K \le 4\|V_0\|^2 Q$. Formula (13), rather than an exact finite-cutoff identity, is its applicable approximation.

4. Proposition (the exact charged-vector Taylor mass).

Let the finite data be the full three-generation Standard Model representation, including charge conjugates. Let the Majorana matrix and every Dirac mass except one up-type color triplet be zero. Write

$$m = \mu\Lambda, \qquad x = \mu^2, \qquad r = e^{-x}.$$

Let $f(z) = e^{-z}$. Define

$$H(y) = \int_0^1 e^{-yu(1-u)}\,du,$$

$$g(y) = (2+y)H(y) - 2,$$

$$h_x(y) = (2+x+y)\int_0^1 e^{-xu-yu(1-u)}\,du - 1 - e^{-x}. \tag{15}$$

Then the full charged-vector kernel is

$$\boxed{C_W(q^2) = \frac{\Lambda^2}{4\pi^2}\left[\frac{9}{2}g(y) + \frac{3}{2}h_x(y)\right], \qquad y = q^2/\Lambda^2.} \tag{16}$$

Define

$$A(x) = h_x(0) = \frac{2[1-(1+x)e^{-x}]}{x},$$

$$w(x) = 3\int_0^1 u^2 e^{-xu}\,du = \frac{3[2-(x^2+2x+2)e^{-x}]}{x^3}. \tag{17}$$

Continuous values are understood at $x = 0$. The particle-trace kinetic coefficient is

$$K_W(x) = \frac{9}{2} + \frac{3}{2}w(x) > 0.$$

The Taylor mass, defined by $M^2_{W,\mathrm{Taylor}} = C_W(0)/C'_W(0)$, is

$$\boxed{\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{9A(x)}{4K_W(x)} = \frac{3[1-(1+x)e^{-x}]}{x[3+w(x)]}.} \tag{18}$$

Its limiting behavior is

$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{3}{8}x - \frac{23}{128}x^2 + O(x^3), \qquad \frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} \sim \frac{1}{x} \quad (x\to\infty). \tag{19}$$

Consequently,

$$\frac{m}{M_{W,\mathrm{Taylor}}} \longrightarrow \sqrt{\frac{8}{3}} \quad (\mu\to 0), \qquad \frac{m}{M_{W,\mathrm{Taylor}}} \sim \mu^2 \quad (\mu\to\infty). \tag{20}$$

Proof. On the particle space, $Q_{11} = 6$. The selected quark doublet contributes $3/2$, while the remaining doublets contribute $9/2$. Diagonalizing the massive left-right block produces eigenvalues $+m, -m$. The charged generator connects these to the massless down states; every such pair has $d^2_{ij} = m^2$. Formula (1) therefore gives h_x for the selected doublet and g for the remaining doublets. Charge conjugation supplies the factor two in (16).

Integration by parts gives

$$h_x(y) = 2\int_0^1 u\,[x + 2y(1-u)]e^{-xu-yu(1-u)}\,du, \tag{21}$$

and

$$\boxed{h'_x(y) = 2\int_0^1 u^2[1-y(1-u)]e^{-xu-yu(1-u)}\,du.} \tag{22}$$

The boundary term proving (22) is $\partial_u[u(1-u)e^{-xu-yu(1-u)}]$. Thus

$$h'_x(0) = \frac{2}{3}w(x), \qquad g'(0) = \frac{2}{3}.$$

This proves (18). Expanding the elementary expressions at zero and taking their large-x limits proves (19)–(20). □

The same Taylor result extends to every f in (4). Put

$$\ell = f(0), \qquad w_f(x) = \int w(sx)\,d\nu(s), \qquad A_f(x) = \int \frac{A(sx)}{s}\,d\nu(s).$$

Then

$$\boxed{\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{9A_f(x)}{4\left(\frac{9}{2}\ell + \frac{3}{2}w_f(x)\right)}.} \tag{23}$$

All denominators are positive, and dominated convergence gives

$$M^2_{W,\mathrm{Taylor}} \sim \frac{3}{8}m^2 \quad (x \to 0), \qquad \boxed{\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} \sim \frac{\int s^{-2}\, d\nu(s)}{\ell x} \quad (x \to \infty).} \tag{24}$$

Indeed, $A(z)/z \to 1$, $w(z) \to 1$ at zero, while $zA(z) \to 2$, $w(z) \to 0$ at infinity.

If the other Dirac masses are merely small and nonzero, the answer depends on those additional parameters and is given by (1). In particular, the vanishing limit in (19) assumes they are zero: masses of surviving light doublets can leave a residual vector mass.

5. Theorem (the charged-vector pole).

Under the heat-cutoff hypotheses of Proposition 4, define a timelike pole by analytic continuation of the quadratic inverse propagator:

$$C_W(-M^2_{W,\mathrm{pole}}) = 0. \tag{25}$$

For every $x > 0$, this equation has exactly one solution with $M^2_{W,\mathrm{pole}} > 0$. It is simple and has positive residue in the convention that the Euclidean propagator near the pole is a positive multiple of $(q^2 + M^2_{W,\mathrm{pole}})^{-1}$.

Let $z(x) = M^2_{W,\mathrm{pole}}/\Lambda^2$. Its exact specification is the unique positive root

$$\boxed{\frac{9}{2}g(-z) + \frac{3}{2}h_x(-z) = 0.} \tag{26}$$

Furthermore,

$$z(x) \sim \frac{3}{8}x \quad (x \to 0), \qquad z(x) \sim \frac{1}{x} \quad (x \to \infty), \tag{27}$$

and both mass definitions obey

$$\boxed{0 < M^2_{W,\mathrm{Taylor}},\ M^2_{W,\mathrm{pole}} \le \frac{\Lambda^2}{e}.} \tag{28}$$

Proof. Let

$$F_x(y) = \frac{9}{2}g(y) + \frac{3}{2}h_x(y).$$

Equation (22) implies $h'_x(y) > 0$ for $y \le 0$, and also $g'(y) > 0$, since $g = h_0$. Hence F_x is strictly increasing on that half-axis. We have

$$F_x(0) = \frac{3}{2}A(x) > 0.$$

As $y \to -\infty$, the prefactors in (15) become negative and both integrals grow exponentially, so $F_x(y) \to -\infty$. This proves existence, uniqueness, simplicity, and the residue sign.

For $y \le 0$, equation (22) gives $g'(y) \ge 2/3$, and therefore

$$F'_x(y) \ge 3.$$

The mean-value theorem yields

$$z(x) \le \frac{1}{2}A(x).$$

The Taylor denominator gives the same bound for $M^2_{W,\mathrm{Taylor}}/\Lambda^2$. Finally,

$$A(x) = 2\int_0^1 xu\, e^{-xu}\, du \le \frac{2}{e},$$

proving (28).

As $x \to 0$, the bound implies $z(x) \to 0$, and $F'_x(y) \to 4$ near the root. As $x \to \infty$, $z(x) \to 0$, $h'_x(y) \to 0$ near the root, and $F'_x(y) \to 3$. Together with $F_x(0) \sim 3x/2$ and $F_x(0) \sim 3/x$, respectively, these prove (27). □

The pole mass governs the linearized dispersion relation of this analytically continued quadratic action. The Taylor mass governs its two-derivative truncation; they need not coincide near the cutoff. We assert no closed-form inversion of (26), no classification of complex poles, and no Lorentzian unitarity theorem.

The pole proof also extends to positive measures supported in a compact subset of $(0,\infty)$, using

$$F_{f,x}(y) = \int \frac{1}{s}\left[\frac{9}{2}g(sy) + \frac{3}{2}h_{sx}(sy)\right] d\nu(s).$$

It gives the analogues of (24) for the pole mass. For the larger class (4), only the Euclidean and Taylor conclusions have been established here; its moment assumptions alone do not justify that continuation.

6. Proposition (unbroken coefficients, neutral vectors, and weak angles).

Retain the heat-cutoff Standard Model family of Proposition 4. Normalize color generators by $\operatorname{tr}_{\mathbf{3}} t_a t_b = \frac{1}{2}\delta_{ab}$, and use $Q = T_3 + Y$. Define the full-trace coefficients by

$$S^{(2)}_{\text{unbroken}} = \int \left(\kappa_{\text{em}} F_{\text{em}}^2 + \kappa_s \sum_a G_a^2\right).$$

Then

$$\boxed{\kappa_{\text{em}} = \frac{1}{24\pi^2}\frac{16}{3}(5+r), \qquad \kappa_s = \frac{1}{24\pi^2}2(5+r), \qquad \frac{\kappa_{\text{em}}}{\kappa_s} = \frac{8}{3}.} \tag{29}$$

Both coefficients approach 5/6 of their all-massless values.

Let K denote the particle-trace neutral q^2 coefficient in coordinates (A_3, A_Y). Then

$$\boxed{\begin{aligned} K_{33} &= \frac{21}{4} + \left(\frac{3}{4} - \frac{3x}{8}\right) r, \\ K_{YY} &= \frac{103}{12} + \left(\frac{17}{12} - \frac{3x}{8}\right) r, \\ K_{3Y} &= -\frac{1}{4} + \left(\frac{1}{4} + \frac{3x}{8}\right) r. \end{aligned}} \tag{30}$$

The matrix K, the charged coefficient K_W, and the unbroken coefficients are positive for every $x \geq 0$.

Define the neutral-current angle from the current of the massive direction after diagonalizing the two-derivative neutral kinetic form:

$$J_Z \;\propto\; T_3 - \sin^2\theta_{\text{NC}}\, Q.$$

Then

$$\boxed{\cos^2\theta_{\text{NC}} = \frac{K_{YY} + K_{3Y}}{K_{33} + 2K_{3Y} + K_{YY}} = \frac{5}{8}.} \tag{31}$$

A different continuation, namely

$$\cos^2\theta_{\text{match}} := \frac{K_{YY}}{K_W + K_{YY}},$$

satisfies

$$\boxed{\cos^2\theta_{\text{match}}(0) = \frac{5}{8}, \qquad \cos^2\theta_{\text{match}}(\infty) = \frac{103}{157}.} \tag{32}$$

Thus these definitions cease to describe one common electroweak angle.

Proof. Photon and color generators commute with V_0. Their coefficients therefore follow from (3) with $d_{ij} = 0$ on each invariant mass block. The charge traces give (29).

A Dirac species of mass m, with neutral charges q_L^a, q_R^a, contributes to K

$$e^{-x}\left[q_L^a q_L^b + q_R^a q_R^b - \frac{x}{2}(q_L^a - q_R^a)(q_L^b - q_R^b)\right]. \tag{33}$$

Applying (33) to the three top colors, and retaining all other states, proves (30).

Write

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad B_\infty = \begin{pmatrix} 21/4 & -1/4 \\ -1/4 & 103/12 \end{pmatrix}, \quad U = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 17/12 \end{pmatrix}.$$

Then

$$K = B_\infty + rU - \frac{3xr}{8} vv^T.$$

We have $B_\infty \geq 5I$, $U \geq 0$, and $xr \leq 1/e$. Hence

$$K \geq \left(5 - \frac{3}{4e}\right) I > 0.$$

Positivity of K_W follows from (17).

The neutral mass direction is v, while the photon direction is $(1, 1)$. The kinetic-orthogonal massive field is proportional to $K^{-1}v$, giving the first expression in (31). Directly,

$$K_{33} + K_{3Y} = 5 + r, \qquad K_{YY} + K_{3Y} = \frac{5}{3}(5 + r),$$

which proves (31). Taking limits proves (32). □

The neutral mass also tends to zero. More precisely,

$$\boxed{\frac{M^2_{Z,\mathrm{Taylor}}}{\Lambda^2} = \frac{9}{4} xr\, v^T K^{-1} v \sim \frac{2}{3} xe^{-x} \quad (x \to \infty).} \tag{34}$$

For $x > 0$ there is likewise one positive massive neutral pole, besides the photon pole. To prove this, put $B = B_\infty + rU$. The exact neutral kernel, apart from the common factor $\Lambda^2/(4\pi^2)$, is

$$B\, g(y) + \frac{3}{2} xr\, H(y) vv^T. \tag{35}$$

After congruence by $B^{-1/2}$, its massive eigenvalue is

$$g(y) + \beta H(y), \qquad \beta = \frac{3}{2} xr\, v^T B^{-1} v > 0.$$

Its timelike root obeys

$$z - 2 + \frac{2}{H(-z)} = \beta. \tag{36}$$

The left side increases from zero to infinity, because

$$\frac{d}{dz}\left(z - 2 + \frac{2}{H(-z)}\right) = 1 - \frac{2\,\mathbb{E}_z[b]}{H(-z)} \geq \frac{1}{2},$$

where $\mathbb{E}_z$ uses density proportional to $e^{zb(u)}$. The pole is therefore unique and has positive residue. Since $v^T B_\infty^{-1} v = 8/27$,

$$\frac{M^2_{Z,\mathrm{pole}}}{\Lambda^2} \sim \frac{2}{3} xe^{-x}.$$

In particular, M_W^2/M_Z^2 does not continue to equal $5/8$.

For every f in (4), equations (29)–(33) extend by replacing 1 by $\ell = f(0)$, r by $f(x)$, and xr by $-xf'(x)$. Positivity follows by averaging the positive heat-cutoff matrices. The ratio $8/3$, the neutral-current value $5/8$, and the matching limit $103/157$ persist. At the endpoint both neutral masses vanish, so the distinguished massive neutral direction—and hence its angle—requires continuation from finite x.

7. Counterexample (positive cutoff does not ensure positive broken kinetic coefficients).

Let $H_F = \mathbb{C}^2$,

$$V_0 = m\sigma_3, \qquad T = \sigma_1, \qquad x = m^2/\Lambda^2,$$

with the same spacetime and trace conventions. For the heat cutoff,

$$C_T(q^2) = \frac{\mathrm{Tr}(T^2)\Lambda^2}{8\pi^2} e^{-x} \left[(2 + 4x + y)H(y) - 2\right]. \tag{37}$$

Its q^2 Taylor coefficient is proportional to

$$e^{-x}(1 - x).$$

It vanishes at $x = 1$ and becomes negative for $x > 1$. Thus the two-derivative truncation has a wrong-sign vector kinetic term in this example.

Nevertheless, the full quadratic form has a unique positive real timelike pole with positive residue, satisfying

$$\boxed{4x < \frac{M^2_{\mathrm{pole}}}{\Lambda^2} < 4x + 2.} \tag{38}$$

Consequently $M_{\mathrm{pole}} \sim 2m$ diverges at fixed cutoff.

Proof. The generator connects the eigenvalues $m, -m$, so $d^2_{ij} = 4m^2$ while both squared eigenvalues equal m^2. Substitution in (1) proves (37). Its Taylor ratio is

$$M^2_{\mathrm{Taylor}} = \frac{6m^2}{1 - x},$$

where this is a positive mass only for $x < 1$.

The pole equation is (36) with $\beta = 4x$. The monotonicity proof there gives uniqueness and the residue sign. Evaluating (37) at $y = -4x$ gives $2H(-4x) - 2 > 0$, whereas at $y = -4x - 2$ it gives -2; this proves (38).

For general f satisfying (4), the Taylor kinetic coefficient in this example is proportional to

$$f(x) + xf'(x) = \frac{d}{dx}[xf(x)]. \tag{39}$$

It must be negative somewhere: $xf(x) > 0$ for $x > 0$, but $xf(x) \to 0$ at infinity under (4). A nonnegative derivative everywhere would contradict these facts. □

This counterexample concerns arbitrary finite data; the top-only Standard Model family is outside it. It also distinguishes a wrong-sign **Taylor truncation** from a negative-residue pole of the **full** form factor. We have not proved absence of complex-pole or other nonlocal problems in either model.

8. Counterexample (heavy fermions do not dynamically select the centralizer).

Let the data be those of Proposition 4, and let

$$\mathfrak{g}_{V_0} = \{X \in \mathfrak{g}_{\mathrm{SM}} : [X, V_0] = 0\}.$$

For nonzero top mass this is the represented unbroken gauge algebra $\mathfrak{su}(3) \oplus \mathfrak{u}(1)_{\mathrm{em}}$. The limit $x \to \infty$ at fixed Λ does **not** produce a theory containing only these vector directions by quadratic mass decoupling.

Proof. For fixed y, equation (21) implies

$$h_x(y) \longrightarrow 0.$$

Consequently

$$\boxed{C_W(q^2) \longrightarrow \frac{\Lambda^2}{4\pi^2}\frac{9}{2} g(q^2/\Lambda^2).} \tag{40}$$

For positive y,

$$g(y) = 4y \int_0^1 u(1 - u)e^{-yu(1-u)}\, du > 0.$$

Thus a charged generator outside $\mathfrak{g}_{V_0}$ retains a finite nonzero quadratic kinetic form and a nonvanishing propagator at every fixed positive momentum. Its mass approaches zero. Equation (35) similarly leaves the positive matrix $B_\infty g(y)$, including the additional neutral direction. These facts contradict decoupling of the broken vectors through diverging masses or vanishing propagators. □

A narrower decoupling statement is true. If we **restrict the connection from the outset** to the centralizer, the finite mass blocks are invariant and the heat trace factorizes. On the heavy block its contribution carries e^{-sm^2/Λ^2}, which tends to zero. The restricted action therefore converges to the light-block spectral action. This proves decoupling of a finite fermion block under that restriction; it does not prove dynamical elimination of the other gauge fields.

The amendment to Lemma 03 Part C is therefore:

- Its transverse kinetic weights, including $w_L(x,0) = w(x)$, and its two weak-angle formulas are correct under their stated conventions.
- They must be accompanied by the mass coefficient (2), including the quadratic-vertex subtraction.
- A centralizer-only endpoint does not follow from those weights. In the present exact quadratic family, the broken-vector masses vanish while their remaining kinetic coefficients are positive.

9. No-go theorem (finite radial vacuum for the positive Laplace class).

Let f satisfy (4), and suppose the allowed scalar configurations include

$$V = cV_*, \qquad c > 0, \qquad V_* \neq 0,$$

with $A = 0$. Let the action be the literal spectral action, without an added scalar potential or a constraint fixing c. Then no finite $c > 0$ is stationary along this radial direction.

Proof. On $\mathbb{R}^4$, the constant-background action density is

$$\mathcal{U}(c) = \frac{\Lambda^4}{4\pi^2} \int s^{-2} \operatorname{Tr}_{H_F} e^{-sc^2V_*^2/\Lambda^2}\, d\nu(s).$$

Differentiating gives

$$\boxed{\mathcal{U}'(c) = -\frac{c\Lambda^2}{2\pi^2} \int s^{-1} \operatorname{Tr}_{H_F} \left(V_*^2 e^{-sc^2V_*^2/\Lambda^2}\right) d\nu(s) < 0.} \tag{41}$$

The inequality is strict because $V_* \neq 0$ and ν is nonzero and positive. The same argument applies on a finite flat torus before performing its positive momentum sum. □

Thus the masses above are conditional quadratic-background quantities. In the top-only family, the ordinary Higgs radial variation supplies the direction used in this theorem. A finite symmetry-breaking minimum of the polynomial asymptotic scalar potential cannot automatically be regarded as a minimum of this full positive-Laplace spectral action.

10. Corollary (comparison of heavy limits).

Let the background family and mass definitions be those above.

At fixed Λ, with $\mu \to \infty$:

quantity, heat cutoff	behavior
m_t	$\mu\Lambda \to \infty$
$M_{W,\mathrm{Taylor}}$, $M_{W,\mathrm{pole}}$	$\sim \Lambda/\mu \to 0$
$M_{Z,\mathrm{Taylor}}$, $M_{Z,\mathrm{pole}}$	$\sim \sqrt{2/3}\,\Lambda\mu e^{-\mu^2/2} \to 0$
m_t/M_W	$\sim \mu^2 \to \infty$

(42)

At fixed $0 < \mu < \infty$, with $\Lambda \to \infty$, all four vector masses in (42) instead grow linearly with Λ, as does m_t. A controlled asymptotic family can also satisfy

$$\Lambda \to \infty, \qquad \mu \to 0, \qquad \mu\Lambda \to \infty;$$

then $M_W, m_t \to \infty$ while

$$m_t/M_W \to \sqrt{8/3}.$$

Proof. The first assertions follow from (19), (27), and (34)–(36). At fixed positive μ, their dimensionless mass functions are fixed and positive, so multiplication by Λ proves the joint-limit assertions. The final assertion follows from the small-μ limits. □

Accordingly, the obstruction has three distinct scopes. The CCM identity cannot be extrapolated to $\mu \to \infty$ at fixed cutoff because it belongs to the asymptotic truncation. The exact top-only quadratic action obstructs a fixed-cutoff heavy-W limit for the heat cutoff under both mass definitions, and for class (4) under the Taylor definition. A family of actual nonzero stationary vacua of the unmodified positive-Laplace action encounters the separate obstruction (41). Counterexample 7 excludes extending the first of these exact mass conclusions to arbitrary finite representations.

11. Proposition (the radial derivative for general f; referee). On flat $\mathbb{R}^4$, for a sufficiently integrable real cutoff f, with λ_j^2 the eigenvalues of V_*^2,

$$\mathcal{U}'(c) = -\frac{c\Lambda^2}{2\pi^2} \sum_j \lambda_j^2 \int_{c^2\lambda_j^2/\Lambda^2}^{\infty} f(z)\, dz. \tag{43}$$

For nonnegative f, a finite $c > 0$ is stationary exactly when every tail associated with a nonzero λ_j vanishes. So if f has nonzero support arbitrarily far out, there is no finite stationary point on this ray; if f has compact support ending at R, there is a stationary plateau once every nonzero $c^2\lambda_j^2/\Lambda^2 \geq R$. Sign-changing cutoffs can have cancellations. A nonconstant cutoff flat near zero lies outside (4), since a nonzero positive measure gives $f'(0) < 0$; (43) covers the nonnegative ones. A radial path with an additional fixed noncommuting mass matrix need not have the form cV_*. □

12. Proposition (the top-only Taylor mass for general f; referee). For nonnegative smooth compactly supported f with $f(0) > 0$, the formulas of Proposition 4 hold with

$$A_f(x) = \frac{2}{x} \int_0^x z f(z)\, dz, \qquad w_f(x) = \frac{3}{x^3} \int_0^x z^2 f(z)\, dz,$$

by approximating f with its derivatives by real combinations of e^{-nz}. Hence

$$\frac{M_{W,\mathrm{Taylor}}^2}{\Lambda^2} \sim \frac{\int_0^\infty z f(z)\, dz}{f(0)\, x}.$$

This includes cutoffs flat near zero. For a nonanalytic cutoff there need be no analytic continuation defining the pole. □

13. Proposition (several heavy doublets; referee). For the heat cutoff, let P be the total particle-trace weight of complete massless doublets, and let heavy doublet i, of weight $d_i > 0$, have one Dirac member of mass m_i and one massless member. Then

$$C_W(y) = \frac{\Lambda^2}{4\pi^2}\Big(P g(y) + \sum_i d_i h_{x_i}(y)\Big), \qquad \frac{M_{W,\mathrm{Taylor}}^2}{\Lambda^2} = \frac{3}{2}\, \frac{\sum_i d_i A(x_i)}{P + \sum_i d_i w(x_i)}.$$

For finitely many doublets, if $P > 0$ and all $x_i \to \infty$, the Taylor mass tends to zero. If $P = 0$ and all $x_i = x$, then $M_{W,\mathrm{Taylor}}^2 \sim m^4/(2\Lambda^2)$, the form factor has a unique positive pole with $M_{W,\mathrm{pole}}^2 \sim m^2/2$, and the whole unnormalized kernel tends to zero at every fixed Euclidean momentum. The divergent Taylor and pole scales differ, and neither alone establishes a healthy limiting gauge theory. □

14. Proposition (the fully massive family; referee; the complete proof by Astra 6 follows the sketch). Give both members of every Standard Model doublet the same Dirac mass m, with no Majorana masses, and let z_W, z_Z be the charged and massive neutral poles in units of Λ^2. The charged form factor is proportional to $e^{-x}[g(y) + 2xH(y)]$, and

$$2x < z_W < 2x + 2, \qquad \tfrac{16}{5}x < z_Z < \tfrac{16}{5}x + 2.$$

So both pole masses diverge with m at fixed cutoff, and

$$\frac{M^2_{W,\mathrm{pole}}}{M^2_{Z,\mathrm{pole}}} \longrightarrow \frac{5}{8} \qquad (x \to \infty),$$

as also for $x \to 0$, where $z_W \sim 3x$ and $z_Z \sim \frac{24}{5}x$. In between the ratio differs, reaching about 0.66 near $x = 3$ (Python, solving the pole equations). The kernel tends to zero at every fixed Euclidean momentum, and the charged Taylor kinetic coefficient is proportional to $e^{-x}(1 - x/2)$, negative for $x > 2$.

Proof. The species with equal mass have eigenvalues $\pm m$. The charged generator connects them with $d^2 = 0$ and $d^2 = 4m^2$ in equal weights, which gives the charged form. For the neutral generators the diagonal charge has $d = 0$ and the off-diagonal charge $(q_L - q_R)/2 = T_3 v/2$ has $d^2 = 4m^2$; on the particle trace the kernel is $e^{-x}[\mathrm{diag}(6, 10)g + 12xHvv^T]$, so $\beta = 12x\, v^T \mathrm{diag}(6, 10)^{-1} v = \frac{16}{5}x$, and for the charged direction $\beta = 2x$. Equation (36) gives the bounds, since $H(-z) > 1$. □

Complete proof (Astra 6, 3 November 2026; written here from its validation report, item 11, so that the lemma is self-contained). Take the heat cutoff, three generations with right-handed neutrinos, no Majorana masses, and one common positive Dirac mass m for every species. Let $x = m^2/\Lambda^2$, $y = q^2/\Lambda^2$, $H(y) = \int_0^1 e^{-yu(1-u)}\, du$ and $g(y) = (2+y)H(y) - 2$. Every mass eigenvalue is $+m$ or $-m$, so the exact quadratic kernel of Theorem 1 for a generator entry is proportional to $e^{-x}[g(y) + ((m_i - m_j)^2/\Lambda^2)H(y)]|T_{ij}|^2$. A charged generator connects equal and opposite mass signs with equal weights, the average of $(m_i - m_j)^2$ is $2m^2$, and the charged kernel is proportional to $e^{-x}[g(y) + 2xH(y)]$. In the neutral particle trace, with $v = (1, -1)^T$, the charge and mass-commutator traces give $Q = \mathrm{diag}(6, 10)$ and $C_{\mathrm{neutral}}(y) \propto e^{-x}[Qg(y) + 12xH(y)vv^T]$, whose determinant factors as $\det Q\, g(y)[g(y) + 12x\, v^T Q^{-1} v\, H(y)]$; since $v^T Q^{-1} v = 1/6 + 1/10 = 4/15$, the massive neutral factor has coefficient $16x/5$. Charge conjugation multiplies the common normalization and moves no root. Both massive pole equations therefore have the form $(2 + \beta - z)H(-z) = 2$ with $\beta_W = 2x$ and $\beta_Z = 16x/5$. Let $h(z) = H(-z)$ and $F(z) = z - 2 + 2/h(z)$. For $z \geq 0$, $h(z) \geq 1$ and $0 < h'(z) = \int_0^1 u(1-u)e^{zu(1-u)}\, du \leq h(z)/4$, so $F'(z) = 1 - 2h'(z)/h(z)^2 \geq 1 - 1/(2h(z)) \geq 1/2$; with $F(0) = 0$ and $F(z) \to \infty$, each $\beta > 0$ has a unique positive solution, at which $h(z) > 1$, so $z = \beta + 2 - 2/h(z)$ gives $\beta < z < \beta + 2$: the displayed bounds. For small z, $h(z) = 1 + z/6 + O(z^2)$ and $F(z) = \frac{2}{3}z + O(z^2)$, so $z_W \sim 3x$, $z_Z \sim 24x/5$ and the ratio tends to $5/8$; for $x \to \infty$ the bounds give $z_W = 2x + O(1)$, $z_Z = 16x/5 + O(1)$, again $5/8$. At fixed Euclidean momentum every kernel entry is e^{-x} times a function growing at most linearly in x, hence tends to zero. Finally $H'(0) = -1/6$ and $g'(0) = 2/3$, so the charged Taylor kinetic coefficient is proportional to $e^{-x}(\frac{2}{3} - \frac{x}{3}) = \frac{2}{3}e^{-x}(1 - x/2)$, negative for $x > 2$. A quadrature and root calculation (Python 3, SciPy, not kept) gave $z_W = 7.4996$, $z_Z = 11.3746$, ratio 0.6593 at $x = 3$. □

This is a counterexample to any general no-go for diverging pole locations at fixed cutoff. It has no light fermions, and the physical quality of that limit is open.

15. Remark (the light space does not carry unbroken $SU(2) \times U(1)$; referee). In the top-only limit the broken masses vanish, but this does not restore the electroweak group on the whole light space. The light space retains b_L after removing t_L. If P_{light} projects off the top mass block, then on b_L we have $P_{\mathrm{light}} T_1 P_{\mathrm{light}} = P_{\mathrm{light}} T_2 P_{\mathrm{light}} = 0$ and $P_{\mathrm{light}} T_3 P_{\mathrm{light}} = -\frac{1}{2}$, which do not obey the $SU(2)$ relations. Correspondingly $K_W = 9/2$, $K_{33} = 21/4$, $K_{3Y} = -1/4$ in the limit, while a linearly realized unbroken action would have equal weak coefficients and no mixing. The remaining complete massless doublets carry the ordinary representation, and their contribution is gauge invariant. If both members of every removed doublet are heavy, the retained light space can be invariant under the full group; a full limiting theory still needs control of the remaining fields, normalizations, interactions and vacuum. "The heavy fermion leaves the loop" is shorthand for the suppression in this classical trace.

The light-space statement should be read together with the separate role of the top (lemma 21): among single charged species only an up-type quark can be removed while keeping cos^2 theta_NC = 5/8 and the ratio 8/3, and removing it leaves the five other quarks and the leptons, on which SU(3) x U(1)_em acts with b_L paired with b_R.

16. Remark (the CCM relation as the small-mass form of a weighted sum rule; Claude). For

the heat cutoff and doublets with one massive member, Proposition 13 reads

$$8\kappa M_{W,\mathrm{Taylor}}^2 = \sum_f N_f m_f^2\, \rho(m_f^2/\Lambda^2), \qquad \rho(x) = \frac{A(x)}{x} = \frac{2[1-(1+x)e^{-x}]}{x^2},$$

with $N_f = 2d_f$ and $\kappa = (P + \sum_i d_i w(x_i))/6$. As $x \to 0$, $\rho, \kappa \to 1$, which is the CCM relation (14). The relation is therefore the leading term of an expansion, and its range is $m \ll \Lambda$. For the top alone, $\kappa = (3+w)/4$; m/M_W departs from $\sqrt{8/3}$ by 5% at $m \approx 0.45\Lambda$, and M_W peaks at about 0.52Λ at $m \approx 1.42\Lambda$ (Python). The maximum is unique, at $x = 2.024$ with $M_W/\Lambda = 0.5242$ (referee, by a sign argument and Python). A massive diagonal contribution to the trace is exponentially suppressed above the cutoff, while the mixed loop with a massless partner has the algebraic $\rho \sim 2/x^2$ at large x; a vacuum with no other mass leaves the W mass with no source. Whether the classical action is meaningful for $m > \Lambda$ is a separate question, on which these formulas do not decide. In the top-only heat-cutoff family, the ratio differs from $\sqrt{8/3}$ for every $x > 0$, since $3 + w - 4\rho$ vanishes at zero and is increasing, so the limit with M_W and m_t diverging together and $m_t/M_W \to \sqrt{8/3}$ needs $\Lambda \to \infty$ faster than m_t (Corollary 10). This necessity is for that family: with one complete massless lepton doublet and one member of each other doublet of common mass m, the weighted sum equals the unweighted CCM sum at some $x_* > 0$ (referee).

As algebraic checks, we used **Python 3 with SymPy** to verify the integration-by-parts identities, the displayed small-x Taylor series, and $v^T B_\infty^{-1} v = 8/27$. The arguments above provide the proofs; these computations introduce no fitted constants.

Consequence for T5b. At fixed cutoff, the specified top-only Standard Model spectral action does not realize an infinitely heavy W, Z, t family: the top grows while both vector masses tend to zero, for the heat cutoff and the stated quadratic mass definitions. A joint limit with $\Lambda \to \infty$ permits all three background mass scales to diverge, but neither centralizer-only decoupling nor a stationary Higgs vacuum follows from that scaling. These conclusions concern the classical bosonic action and contain no quantum corrections or claims about the physical Standard Model at those scales. Other finite data behave differently: with every doublet heavy and no light fermions, the pole masses diverge at fixed cutoff with M_W^2/M_Z^2 -> 5/8 (Proposition 14). With a massless complete doublet retained, the Taylor mass tends to zero (Proposition 13). So in the families treated here, a heavy W and Z at fixed cutoff is found only without a massless complete doublet. Light nonzero masses are not treated: they leave a residual vector mass that depends on them.

Part 19

Inner fluctuations of the two-point data without order one

Step 1 of the next steps in the background note, for STRATEGY T0b. Proved by Claude Opus 5.5 without Astra or Fable; every identity checked with numpy on random data (section "Numerical check"). Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): Theorem 19.1 and the comparison with arXiv:1304.7583 are correct; Corollary 19.2 item 3 and the Wilson phase after the table were wrong as first written, and these and the amendments are incorporated below.

Outcome. The truncated two-point data satisfy order zero and fail order one, for each natural real structure. So the inner fluctuations of Chamseddine-Connes-van Suijlekom (arXiv:1304.7583), with their quadratic term, are the applicable ones. The result factorizes. With the inherited pointwise real structure, the fluctuated operator is m |Phi|^2 sigma_1: a real gauge-neutral radial field, with trivial gauge group. With the reflection real structure, which has the KO-6 signs, the link is m Phi^2: all of C is reached, and the gauge group U(1) acts with charge 2. Along a gauge orbit the quadratic term keeps the modulus of the link fixed. This parallels the fixed singular value of the covariant projection of lemma 14, which the fixed compression of lemma 13 lacks. So the first-order failure of Counterexample 08.4 points to the framework of arXiv:1304.7583 for these truncated data.

Setting

Let A = C + C act diagonally on H = C^2, with D = m sigma_1, m > 0, and gamma = sigma_3. For a real structure J = C K put x-hat = J x J^-1. We use two choices.

- **J_0 = K** (or sigma_3 K, the other sign of lemma 08), inherited from pointwise reality on the interval. Then x-hat = conj(x) on A, and J_0 commutes with gamma.
- **J_r = sigma_1 K**, inherited from the reflection real structure sigma_1 R_l K of Counterexample 08.4. Then diag(x_1, x_2)-hat = diag(conj(x_2), conj(x_1)), J_r^2 = 1, J_r D = D J_r and J_r gamma = -gamma J_r: the KO-6 signs.

Both satisfy order zero, since A is commutative and x-hat lies in A. Both fail order one: for e = diag(1, 0), [[D, e], e-hat] = m sigma_1 for J_0 and -m sigma_1 for J_r.

For a universal one-form A = sum_j a_j delta(b_j), the fluctuated operator of arXiv:1304.7583 is

```
D' = D + A_(1) + A~_(1) + A_(2),
A_(1) = sum a_j [D, b_j],   A~_(1) = sum a_j-hat [D, b_j-hat] = eps J A_(1) J^-1,   A_(2) = sum
↪  a_j-hat [A_(1), b_j-hat],
```

with J D J^-1 = eps D. Gauge transformations are U = u J u J^-1 for unitary u in A, and U D(A) U* = D(gamma_u(A)), where gamma_u(A) = u delta(u*) + u A u*.

Put

```
phi_1 = sum a_{j,1} (b_{j,2} - b_{j,1}),     phi_2 = sum a_{j,2} (b_{j,1} - b_{j,2}),
```

so that A_(1) = m [[0, phi_1], [phi_2, 0]], which is self-adjoint exactly when phi_2 = conj(phi_1). The pair (phi_1, phi_2) takes every value in C^2, as a = diag(x, y) and b = diag(0, 1) show.

The fluctuated operator

Theorem 19.1. For every universal one-form,

```
J_0:   D' = m [[0, |1 + phi_1|^2], [|1 + phi_2|^2, 0]],
J_r:   D' = m [[0, (1 + phi_1)(1 + conj(phi_2))], [(1 + phi_2)(1 + conj(phi_1)), 0]].
```

For a self-adjoint form, with Phi = 1 + phi_1,

```
J_0:   D' = m |Phi|^2 sigma_1,          J_r:   D' = m [[0, Phi^2], [conj(Phi)^2, 0]].
```

Proof. For diagonal b, [X, b] for off-diagonal X = [[0, x], [y, 0]] is [[0, x (b_2 - b_1)], [y (b_1 - b_2), 0]].

For J_0: a-hat and b-hat are the conjugates. Then A~*(1) = m [[0, conj(phi_1)], [conj(phi_2), 0]], and the first row of* $A(2)$ is m phi_1 sum conj(a_{j,1}) (conj(b_{j,2}) - conj(b_{j,1})) = m phi_1 conj(phi_1); the second row is m phi_2 conj(phi_2). So the (1,2) entry of D' is m (1 + phi_1 + conj(phi_1) + |phi_1|^2) = m |1 + phi_1|^2, and likewise (2,1).

For J_r: a-hat = diag(conj(a_2), conj(a_1)) and b-hat = diag(conj(b_2), conj(b_1)). Then A~*(1) = sigma_1 conj(*$A(1)$) sigma_1 = m [[0, conj(phi_2)], [conj(phi_1), 0]], and the first row of A_(2) is m phi_1 sum conj(a_{j,2}) (conj(b_{j,1}) - conj(b_{j,2})) = m phi_1 conj(phi_2), and the second is m phi_2 conj(phi_1). This gives the stated factorization. ■

Corollary 19.2 (gauge action and image). Let u = diag(e^{i alpha}, e^{i beta}) and psi = alpha - beta. Under gamma_u, Phi -> e^{i psi} Phi.

1. For J_0: U = 1. The image of the self-adjoint forms is {t sigma_1 : t >= 0}, so the link is real, and t is a gauge-neutral field. The value t = 0 is reached at Phi = 0.
2. For J_r: U = diag(e^{i psi}, e^{-i psi}), and the link m Phi^2 goes to e^{2 i psi} m Phi^2. So U(1) acts with charge 2 in the variable of the link, and the image is all of C, with Phi -> Phi^2 two-to-one away from 0. The image statement is pointwise: a globally prescribed smooth link need not have a global smooth square root.

3. The gauge orbit of the zero form has Phi = e^{i psi}. It gives D' = D for J_0, whose operator orbit is the single point D, and link m e^{2 i psi} for J_r. Without A_(2), the link is m (2 cos psi - 1) for J_0, of modulus m (1 - psi^2 + O(psi^4)), although U = 1; and m (2 e^{i psi} - 1) for J_r, of modulus m (5 - 4 cos psi)^{1/2} = m (1 + psi^2 + O(psi^4)), off the orbit of D.

Proof. gamma_u(0) has phi_1 = u_1 conj(u_2) - 1 = e^{i psi} - 1. The rest is direct. ■

Item 3 shows, in the smallest example, why the quadratic term is needed: it restores gauge covariance.

Comparison with the parent

Proposition 19.3. Consider the chiral interval at M = 0, with m = pi/(2l), and the links of lemmas 13 and 14.

1. *With no real structure*, the finite fluctuation is m (1 + phi_1), all of C, while the fixed compression of a gauge field gives only the line m + i B (Proposition 13.2).
2. *With pointwise reality*, the parent has no scalar fluctuation at all: the algebra C(I) is commutative with J f J^-1 = conj(f), it satisfies order one, and A + J A J^-1 = 0 for a scalar one-form. The truncated data do have a fluctuation, the neutral radial field t. So the finite fluctuation has a gauge-neutral radial parameter, while the specified pointwise-real parent has no surviving self-adjoint scalar inner fluctuation. This does not establish a dynamical scalar field or its action, and it says nothing about parent fields in other models. Compression is not multiplicative, so there is no canonical identification of finite potentials with parent potentials.
3. *With the reflection real structure*, the parent satisfies order one, since J_r f J_r^-1 = conj(f) composed with the reflection commutes with [D, g]. One finds J_r D_{A_5} J_r^-1 = D_{A_5 o R}, with R the reflection, so that the symmetrized field A_5 + A_5 o R survives, and the fixed compression gives m + i B^s with B^s = 2 B, because the weight sin(pi x/l) is even about l/2. The modulus along a pure-gauge field behaves as follows.

object	modulus of the link along a gauge orbit
covariant projection (lemma 14, Theorem 3)	constant: lambda W_l^* has fixed singular value
finite data with A_(2) (Corollary 19.2)	constant: m
fixed compression (Proposition 13.2), with B the integral of the symmetrized coefficient for J_r	(m^2 + B^2)^{1/2}, growing at second order along paths where B has a linear term
finite data with J_r, linear terms only	m (5 - 4 cos psi)^{1/2}, growing at second order

Proof. Item 1 is lemma 13. For item 2, a represented self-adjoint parent one-form is E h(x) with h = i a and a real, and K (i E a) K^-1 = -i E a, so the real fluctuation cancels. An individual commutator need not change sign, since K [D, x] K^-1 = [D, x] = E: self-adjointness is the step that matters. For item 3, K changes the sign of i A_5, the reflection changes the sign of the derivative and composes A_5 with R, and sigma_1 E sigma_1 = -E for E = [[0, 1], [-1, 0]]. The table collects Theorem 3 of lemma 14, Corollary 19.2 and Proposition 13.2. ■

The table states a parallel and no identification. The variables differ: for constant unsymmetrized A_5 = a, the transported link is m e^{i a l} = m + i (pi a/2) + O(a^2), with Wilson phase a l, against m + i (2 a/pi) in the compression. For the reflection-real fluctuation from the same a, these become m e^{2 i a l} and m + i (4 a/pi). The phase of a link depends on the frame, while its modulus does not. In the parent, the covariance defect is repaired by moving the projection, and in the finite data by the quadratic term.

Reading

Spectral compression can break order one, and it can also break order zero: on the circle with J = K, compression to the zero mode preserves both, while compression to the modes -1, 0, 1 sends e^{ix} to a truncated shift S, with [S, S*] != 0, so that order zero fails. Where a represented algebra, a preserved real structure and order zero hold, as here, the framework of arXiv:1304.7583 applies to the truncated data as

they stand. So the first-order failure of Counterexample 08.4 has an escape, and this calculation is a first use of it. Two points were open. Whether the parallel in the table extends to an exact statement for a fluctuated parent and its truncation: this holds on gauge orbits, by Corollary 22.6 of lemma 22, and remains open for the nonunitary part. And whether the neutral radial parameter t is related to the singlet of the grand-symmetry model (Devastato-Lizzi-Martinetti, where sigma comes from a Majorana-operator fluctuation in a larger algebra). This is an analogy only: in arXiv:1304.7583 the quadratic term does not generate new scalar fields, and here too it depends on the same two parameters. An identification would need a map of algebras, representations, fields and gauge actions.

Numerical check

Python (numpy), run on 3 Nov 2026 in a temporary directory, not kept. On random complex a_j, b_j, with m = 1.3: the three real structures have the stated signs; the formulas of Theorem 19.1 hold to about 10^-15, without assuming self-adjointness; D(gamma_u(0)) = U D U* holds to 10^-16 for both real structures, and fails by 4.5 for the sample when A_(2) is dropped; U D(A) U* = D(gamma_u(A)) holds for a general form; the values of [[D, e], e-hat] are as stated.

Part 20

Sheet-connecting operators and boundary gluing

By Astra 6 (gpt-6-astra, effort xhigh), 3 Nov 2026; prompt in prompts/20_connecting_operator.md. Reproduced as delivered, with links made relative and one link to a temporary copy of a source written as plain text. The sections "Numerical check" and "Readings" at the end are by Claude Opus 5.5. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): Results 1 to 4 and 6 to 11 correct, with one amendment to Result 7 given in the note after its proof; the readings are corrected below.

A rank-one operator can connect the two finite summands with the required real signs and grading, and the Chamseddine–Connes maximal algebra follows from its intertwining algebra. A boundary coupling supplies such an operator in an explicit interval model. However, agreement of scalar functions at a wall generally does not survive compression as the finite order-one restriction: an exact one-flavour counterexample exhibits this failure. The interval construction realizes the classified rank-one block when its matrix algebra, grading and coupled channels are supplied as inputs; it does not derive those inputs from Example 16.9. The three requested obstruction checks hold in the scopes specified below.

1. Theorem — Finite connecting blocks and their admissible algebras

Hypotheses and conventions. Let

$$E = \mathrm{Hom}_{\mathbb{C}}(V, W), \qquad E^* = \mathrm{Hom}_{\mathbb{C}}(W, V),$$

with Hilbert–Schmidt inner products, conjugate-linear in the first argument. Here E^* denotes the second Hom space; operator adjoints also carry a superscript $*$.

Let $c : E \to E^*$ be the antiunitary occurring in the real structure:

$$J(x, y) = (c^{-1}y, cx), \qquad \Gamma = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}, \qquad h = -cgc^{-1}.$$

In Example 16.9, after choosing the grading of W,

$$cx = x^*, \qquad gx = \gamma_W x, \qquad hy = -y\gamma_W.$$

For the more general graded flip of Corollary 16.8, c includes its stated grading sign.

Let $\mathcal{A}^{\mathrm{ev}}$ act diagonally through π_1, π_2. Write $\rho_i(b)$ for its opposite action on the two summands. Assume that the sheet-diagonal D_F already satisfies self-adjointness, oddness, $JD_F = D_F J$, and order one on $\mathcal{A}^{\mathrm{ev}}$, as in Corollary 16.8.

For a complex-linear map $T : E \to E^*$, put

$$S_T = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}, \qquad D = D_F + S_T, \qquad \tau(T) = cT^*c.$$

Then D has the required signs and satisfies order one on a unital involutive subalgebra $B \subseteq \mathcal{A}^{\mathrm{ev}}$ precisely when

$$\boxed{T = \tau(T), \qquad hT + Tg = 0,} \tag{1}$$

and

$$\boxed{(T\pi_1(a) - \pi_2(a)T)\rho_1(b) = \rho_2(b)(T\pi_1(a) - \pi_2(a)T) \quad (a, b \in B).} \tag{2}$$

Self-adjointness is automatic from the definition of S_T. In bases related by c, the first equation in (1) says that the matrix of T is complex symmetric.

In the ordinary adjoint-flip convention, for $a = (a_W, a_V)$,

$$\pi_1(a)x = a_W x, \quad \pi_2(a)y = a_V y, \quad \rho_1(a)x = xa_V, \quad \rho_2(a)y = ya_W.$$

Thus (2) is an entirely explicit equation on Hom spaces.

For an operator $R : E \to E^*$, define exactly the commutation algebra used in section 4 of Chamseddine–Connes:

$$\mathcal{A}(R) = \left\{ a \in \mathcal{A}^{\mathrm{ev}} : \begin{array}{l} \pi_2(a)R = R\pi_1(a), \\ \pi_2(a^*)R = R\pi_1(a^*) \end{array} \right\}. \tag{3}$$

For fixed T satisfying (1), its maximal admissible algebras are the inclusion-maximal members of

$$\boxed{\{\mathcal{A}(R) : \ hR + Rg = 0, \quad T = R + \tau(R)\}\,.} \tag{4}$$

Consequently, maximal dimension for fixed T is obtained by maximizing the dimension of the algebras in (4). The distinction between T and R is necessary.

Proof. Block multiplication gives (1). The lower off-diagonal block of the order-one equation is exactly (2); adjoints give the other block.

To prove (4), suppose (2) holds. Average over the compact unitary group of the finite-dimensional real C^*-algebra B:

$$R_0 = \int_{U(B)} \pi_2(u)T\pi_1(u)^*\, du.$$

Then R_0 intertwines the left actions. Equation (2) implies that

$$T - \pi_2(u)T\pi_1(u)^*$$

intertwines the opposite actions. Therefore $Q_0 = T - R_0$ intertwines the opposite actions.

The involution τ exchanges these two intertwining conditions. Since $\tau(T) = T$,

$$R = \frac{1}{2}(R_0 + \tau(Q_0))$$

intertwines the left actions and satisfies $T = R + \tau(R)$. Averaging preserves oddness because B is even; τ also preserves oddness. Hence $B \subseteq \mathcal{A}(R)$ for an R occurring in (4).

Conversely, on $\mathcal{A}(R)$, the term R intertwines the left actions and $\tau(R)$ intertwines the opposite actions. Their sum satisfies (2). This proves the characterization and the maximality statement.

For an explicit reason why $\mathcal{A}(T)$ alone is insufficient, take ungraded $W = V = \mathbb{C}^2$, $g = 1$, $h = -1$, and

$$B = \{(\mathrm{diag}(\alpha, \beta), \mathrm{diag}(\alpha, \beta))\}.$$

Let $e_{ij} \in E$, $f_{ij} = ce_{ij}$, and define

$$Te_{11} = f_{12}, \qquad Te_{12} = f_{11},$$

with all other images zero. This T is symmetric and odd. The first arrow preserves the right B-character; the second preserves the left character. Each therefore satisfies order one. But the first arrow fails to intertwine the left actions when $\alpha \neq \beta$. ■

2. Theorem — The classified algebra is realized by a rank-one block

Hypotheses. Let

$$W = W_+ \oplus W_-, \qquad \dim_{\mathbb{H}} W_+ = \dim_{\mathbb{H}} W_- = 1, \qquad V = \mathbb{C}^4,$$

and use the adjoint-flip convention of Result 1. Thus

$$\mathcal{A}^{\mathrm{ev}} = \mathbb{H}_+ \oplus \mathbb{H}_- \oplus M_4(\mathbb{C}).$$

Choose unit vectors $w \in W_+$, $v \in V$, and $\kappa \neq 0$. Put

$$e = wv^* \in E, \qquad f = vw^* = ce,$$

and

$$\boxed{T_\kappa x = \kappa\,(w^* x v)\, vw^* = \kappa\,|f\rangle\langle e|x.} \tag{5}$$

Then T_κ has complex rank one, satisfies (1), and has intertwining algebra

$$\boxed{\mathcal{A}(T_\kappa) = \{(\lambda \oplus q, \lambda \oplus m) : \lambda \in \mathbb{C},\ q \in \mathbb{H},\ m \in M_3(\mathbb{C})\}.} \tag{6}$$

Here the first λ acts on the quaternionic line W_+ as

$$\lambda \longmapsto \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}$$

in a basis starting with w, and the second acts on $\mathbb{C}v \subset V$.

The algebra (6) has real dimension 24. Up to an automorphism of $\mathcal{A}^{\mathrm{ev}}$, it is the unique maximal-dimensional involutive subalgebra admitting a nonzero sheet-connecting Dirac block with order one. The off-diagonal self-adjoint operator S_{T_κ} has complex rank two.

Proof. Since $ge = e$ and $hf = -f$, T_κ is odd. Its matrix has the single symmetric entry κ in c-paired bases, so $\tau(T_\kappa) = T_\kappa$.

The two intertwining equations in (3) require that $\mathbb{C}w$ and $\mathbb{C}v$ be reducing subspaces for the respective actions, with the same scalar action on these lines. A quaternion preserving $\mathbb{C}w$ belongs to the displayed copy of $\mathbb{C}$. On V, the orthogonal complement $v^\perp$ carries an arbitrary $M_3(\mathbb{C})$-action. The other quaternionic summand is unrestricted. This proves (6), and $2 + 4 + 18 = 24$.

We reproduce the dimension argument in Chamseddine–Connes, section 4, whose local source defines (3) at line 875. A nonzero connecting block implies, by Result 1, a nonzero intertwiner R for B. Compress R on either side by projections in the commutants of the full even algebra. Some nonzero compression remains, and its support and range can be placed in irreducible representation spaces

$$K : \mathbb{C}^2 \longrightarrow \mathbb{C}^4.$$

These compressions enlarge the intertwining algebra.

If K has rank two, its image is a reducing two-plane in $\mathbb{C}^4$. The matrix action on that plane is determined by the quaternion action. The remaining two-plane contributes at most $M_2(\mathbb{C})$. Including the unused quaternionic summand gives

$$\dim_{\mathbb{R}} B \leq 4 + 4 + 8 = 16.$$

If K has rank one, its support and range are complex lines. The intertwining algebra is (6), of dimension 24. Unitary changes of these lines, and exchange of the quaternionic summands, give the stated uniqueness.

Finally, T_κ and T_κ^* have orthogonal source and target summands, giving rank $S_{T_\kappa} = 2$. ■

The connecting condition here is relative to the sheet projections of the original prealgebra: $[S_{T_\kappa}, e_1] \neq 0$. On the selected algebra (6), this particular Majorana block commutes with the algebra. Additional sheet-diagonal Dirac terms are needed for the usual nontrivial finite one-forms and Yukawa data.

3. Lemma — Boundary preservation is an intertwining condition

Hypotheses. Use the domain of Lemma 12:

$$Q_R(0) = 0, \quad U_L(0) = 0, \quad Q_R(l) = XU_R(l), \quad U_L(l) = -X^*Q_L(l). \tag{7}$$

Let multiplication act through smooth matrix functions F_Q, F_U, with the same matrix on both chiral components of each field.

Multiplication preserves this domain precisely when

$$\boxed{F_Q(l)X = XF_U(l), \qquad F_U(l)X^* = X^*F_Q(l).} \tag{8}$$

These equations define a unital involutive algebra. For scalar functions and $X \neq 0$, they reduce to

$$f_Q(l) = f_U(l). \tag{9}$$

For $X = 0$, no equality between the boundary values is required.

Proof. At $x = l$, the traces $Q_L(l)$ and $U_R(l)$ can be chosen independently. Applying multiplication to the first coupled condition in (7) gives

$$F_Q(l)XU_R(l) = XF_U(l)U_R(l)$$

for every $U_R(l)$. This is the first equation in (8). The other condition gives the second equation. Conversely, these equations preserve both boundary conditions; the conditions at zero are automatically preserved.

Taking the adjoint of the second equation gives

$$F_Q(l)^*X = XF_U(l)^*.$$

Thus (8) has precisely the two-equation form (3). Closure under products and adjoints follows directly. ■

For matrix-valued functions, "agreement at the wall" therefore means agreement through X on the coupled boundary channels. It need not mean equality of the entire boundary matrices.

4. Counterexample — Scalar gluing does not imply finite order one

Hypotheses. Let $N_Q = N_U = 1$, both bulk masses vanish, and $X = t > 0$. Set

$$\theta = \arctan t \in (0, \pi/2), \qquad m = \theta/l.$$

Take the full domain-preserving scalar algebra

$$\mathcal{A}_{\text{glue}} = \{(f_Q, f_U) \in C^\infty([0,l])^2 : f_Q(l) = f_U(l)\}. \tag{10}$$

The bulk operator is

$$D(Q_L, Q_R; U_L, U_R) = (Q_R', -Q_L'; U_R', -U_L')$$

on (7). Use physical chirality

$$\Gamma(Q_L, Q_R; U_L, U_R) = (Q_L, -Q_R; U_L, -U_R)$$

and the antiunitary

$$\mathcal{J}(Q_L, Q_R; U_L, U_R) = (\overline{U_R}, -\overline{U_L}; -\overline{Q_R}, \overline{Q_L}). \tag{11}$$

This parent satisfies the signs $(+,+,-)$, order zero and order one. Its lowest symmetric spectral window gives

$$A_{\mathrm{fin}} = \mathbb{C} \oplus \mathbb{C}, \qquad D_{\mathrm{fin}} = m\sigma_1, \qquad J_{\mathrm{fin}} = \sigma_1 K, \qquad \Gamma_{\mathrm{fin}} = \sigma_3, \tag{12}$$

and these finite data fail order one.

Proof. The boundary form proves self-adjointness, as in Lemma 12.1. Equation (11) preserves (7), squares to one, commutes with D, and reverses Γ. The opposite algebra acts by the two scalar functions exchanged. Hence it commutes both with the multiplication algebra and with its differential commutators.

The lowest window is spanned by the orthonormal chiral vectors

$$\begin{aligned} u_+(x) &= l^{-1/2}(\cos mx, 0; -\sin mx, 0), \\ u_-(x) &= l^{-1/2}(0, \sin mx; 0, \cos mx). \end{aligned} \tag{13}$$

Direct differentiation gives $Du_+ = mu_-$, $Du_- = mu_+$. The boundary conditions hold because $\tan(ml) = t$. Solving the same elementary ODE gives the next absolute eigenvalue $(\pi - \theta)/l$, so any symmetric cutoff between m and this value selects exactly (13).

Compression of (10) is

$$Q(f_Q, f_U) = \begin{pmatrix} \dfrac{1}{l}\displaystyle\int_0^l (f_Q \cos^2 mx + f_U \sin^2 mx)\, dx & 0 \\ 0 & \dfrac{1}{l}\displaystyle\int_0^l (f_Q \sin^2 mx + f_U \cos^2 mx)\, dx \end{pmatrix}. \tag{14}$$

Let f_Q be a nonnegative bump supported sufficiently near zero, and let $f_U = 0$. The wall equality holds, while the difference of the diagonal entries is

$$\frac{1}{l}\int_0^l f_Q(x)\cos(2mx)\, dx > 0.$$

Together with the identity, this gives the full diagonal algebra $\mathbb{C}^2$.

On (13), (11) gives $J_{\mathrm{fin}} = \sigma_1 K$. For $p = \mathrm{diag}(1, 0)$,

$$p^\circ = 1 - p, \qquad [[D_{\mathrm{fin}}, p], p^\circ] = -D_{\mathrm{fin}} \neq 0.$$

Thus finite order one fails.

There is also a precise distinction between the original field sheets and the two finite chiral summands. The projection onto the Q-field compresses to

$$\begin{pmatrix} \frac{1}{2} + \frac{\sin 2\theta}{4\theta} & 0 \\ 0 & \frac{1}{2} - \frac{\sin 2\theta}{4\theta} \end{pmatrix}, \tag{15}$$

which is not a projection. The coupled spectral window does not preserve the original field decomposition. ■

This proves failure of statement 2(c) for the full scalar gluing algebra, even when the parent has all the required real signs and order conditions. It also specifies the qualification needed in 2(b): the finite summands require an identification, such as continuation from the uncoupled zero modes.

5. Theorem — An exact interval parent supplies the rank-one connector

Hypotheses. Use the Hom representation, c, g, h of Result 1. Set

$$k = cgc^{-1}, \qquad h = -k.$$

Choose a unit vector $e \in E$ with $ge = e$, let $f = ce$, and put

$$S = |f\rangle\langle e|, \qquad X = tS^* : E^* \to E, \qquad t > 0. \tag{16}$$

Take two massless interval fields, Q valued in E and U valued in E^*, with boundary conditions (7).

As candidate multipliers, take **constant** elements of $\mathcal{A}^{\mathrm{ev}}$, acting by π_1 on Q and π_2 on U. Their domain-preserving algebra is

$$B = \mathcal{A}(S). \tag{17}$$

Define

$$\hat{\Gamma}(Q_L, Q_R; U_L, U_R) = (gQ_L, -gQ_R; kU_L, -kU_R), \tag{18}$$

$$\widehat{J}(Q_L, Q_R; U_L, U_R) = (c^{-1}U_R, -c^{-1}U_L; -cQ_R, cQ_L). \tag{19}$$

Let $n = \dim_{\mathbb{C}} E$, $\theta = \arctan t$, $m = \theta/l$, and choose

$$m < \Lambda_0 < (\pi - \theta)/l.$$

Then the spectral projection

$$P = 1_{[-\Lambda_0, \Lambda_0]}(D_X)$$

has rank $2n$. Under an explicit identification $P\mathcal{H} \simeq E \oplus E^*$, its finite data are

$$\boxed{\left(B,\ E \oplus E^*,\ \begin{pmatrix} 0 & mS^* \\ mS & 0 \end{pmatrix},\ J,\ \begin{pmatrix} g & 0 \\ 0 & -k \end{pmatrix}\right).} \tag{20}$$

They satisfy self-adjointness, oddness, order zero, order one, and the signs $(+, +, -)$. The connecting block is $T = mS$, of complex rank one.

Proof. Result 3, with $X = tS^*$, gives (17). The rank-one condition makes $\mathbb{C}e$ and $\mathbb{C}f$ reducing subspaces for B, with the same action on the two lines.

The coupled channels have exactly the eigenvectors (13), with 1 replaced by e in the Q-components and by f in the U-components. Every orthogonal Q-channel has boundary type (a), and every orthogonal U-channel has type (b). They contribute $2n - 2$ constant chiral zero modes; their next absolute eigenvalue is π/l. The coupled channels contribute the two eigenvalues $\pm m$. This proves the window and dimension assertions.

The identification in (20) sends e and f to these two coupled chiral vectors. It sends $e^{\perp} \subset E$ to constant Q_L-profiles and $f^{\perp} \subset E^*$ to constant U_R-profiles. Since B acts identically on the coupled lines, this identification intertwines its representations exactly. Direct application of (18) and (19) gives the displayed finite grading and real structure.

On the parent, (19) squares to one and commutes with the real differential expression. The relations $kf = f$ and $ge = e$ ensure grading preservation of the boundary conditions. The identity

$$cX = X^*c^{-1}$$

ensures their preservation by $\widehat{J}$. These establish the real signs on the operator domain.

The opposite representation is the original right Hom action, so order zero holds. Every admissible constant multiplier commutes with D_X, including preservation of its domain. Order one therefore holds on the parent and on (20). ■

The wall coupling can also be converted explicitly into a bounded bulk term. On the coupled field channels, let

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad E_{\text{spin}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad U(x) = \exp(mxF).$$

Apply the same field rotation to both spin components. It maps the uncoupled boundary domain to the coupled domain and gives

$$U^* D_X U = D_0 + mF \otimes E_{\text{spin}}. \tag{21}$$

Compression of the added bounded term to the uncoupled zero modes is exactly $m(S+S^*)$. Thus the parent supplies the connector through an explicit change of domain and an explicit unitary transport.

Two specializations specify the extent of this construction.

- **A commutative example.** Let $E = E^* = \mathbb{C}$, $g = 1$, $c = K$, and start with constant pairs $\mathbb{C}^2$. Boundary preservation selects their common scalar:

  $$B = \{(\lambda, \lambda)\} \simeq \mathbb{C}.$$

  The finite triple is

  $$(\mathbb{C}, \mathbb{C}^2, m\sigma_1, \sigma_1 K, \sigma_3).$$

  This is the simplest successful instance of 2(a)–(c), with the candidate observable algebra restricted to constant functions on each sheet.

- **The classified block.** Take $E = \mathrm{Hom}(\mathbb{C}^4, \mathbb{H}^2)$, the chosen quaternionic grading, and $e = wv^*$ from Result 2. Equations (17) and (20) give

  $$B = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \qquad \dim_{\mathbb{C}} H_{\text{fin}} = 32,$$

  and the rank-one connecting block (5) with $\kappa = m$. This parent contains the Hom fibres and constant matrix algebra as inputs. It is not a construction from scalar functions on the twistor parent.

An exact tower-decoupling family is obtained by fixing $m > 0$ and setting

$$t(l) = \tan(ml), \qquad 0 < ml < \pi/2.$$

The finite mass remains exactly m, while the next absolute eigenvalue is $\pi/l - m \to \infty$.

The usual zero-dimensional orientability axiom fails in these examples: the linked vectors e, f have identical left and opposite characters, whereas their grading eigenvalues are opposite. Every represented zero-cycle acts identically on these two lines. The claims above concern real even spectral triples with the stated order conditions and signs.

6. Proposition — Point evaluation bounds rank by the boundary-channel dimension

Hypotheses. Let K_i be finite-dimensional windows of smooth sections, let

$$\mathrm{ev}_i : K_i \to F_i$$

be evaluation at the proposed wall, and suppose a finite coupling is defined by

$$T = \mathrm{ev}_2^* \; X \; \mathrm{ev}_1 \,. \tag{22}$$

Then

$$\operatorname{rank} T \le \min\{\operatorname{rank} X, \dim F_1, \dim F_2\}. \tag{23}$$

A single scalar evaluation channel gives rank at most one. A point with several fibre components can give larger rank.

Proof. Equation (23) follows by factoring (22) through the range of X.

For a counterexample to an unrestricted rank-one claim, take Lemma 12 with

$$N_Q = N_U = N, \qquad M_Q = M_U = 0, \qquad X = tI_N.$$

It is the orthogonal sum of N copies of Result 4. Its lowest chiral block is

$$T = \frac{\arctan t}{l} I_N,$$

of rank N, although the coupling occurs at the single point $x = l$. ■

For an exact boundary-domain perturbation, formula (22) is an additional hypothesis. The operators D_X and D_0 have different domains, so their difference is not automatically a point-evaluation perturbation. The exact interval mass $\arctan(t)/l$ also distinguishes the finite operator from its small-t approximation t/l. An extended wall carries a space of boundary sections and supplies no rank-one conclusion by itself.

7. Proposition — Point gluing does not select the algebra in the unchanged twistor window

Hypotheses. Let $Q_i(f) = P_i M_f P_i$, where P_i project onto finite-dimensional spaces of smooth sections on compact manifolds. Let Z_i be closed subsets with empty interior. Impose any scalar gluing condition which allows independent functions vanishing near Z_1 and Z_2, including equality at two marked points.

Then the image of this glued algebra under the **unchanged separate compressions** $Q_1 \oplus Q_2$ equals the image of the unrestricted pair algebra.

Proof. We first show

$$Q_i(C_c^\infty(M_i \setminus Z_i)) = Q_i(C^\infty(M_i)). \tag{24}$$

If the left image were a proper subspace of the right image, finite-dimensional linear algebra would give a linear functional separating them. Applied to $Q_i(f)$, that functional has the form

$$f \longmapsto \int_{M_i} f(x)\, r(x)\, dx$$

for a smooth density r, because matrix entries of $Q_i(f)$ are integrals against products of smooth sections.

Vanishing for every test function supported outside Z_i forces $r = 0$ there. Since that complement is dense and r is continuous, $r = 0$ everywhere, a contradiction. The argument also applies over the real numbers.

By (24), arbitrary compressed values on either sheet can be obtained using functions vanishing near its wall. Such pairs satisfy the gluing condition independently, proving the assertion. ■

In Example 16.9, both analytic section spaces come from $\mathbb{CP}^3$; one observable algebra is restricted to pullbacks from S^4. The corresponding matrix-entry densities on S^4 are smooth after integration along the twistor fibres, so the same proof applies.

Neither closed space comes with the boundary required by Lemma 12. One may choose a base point $p \in S^4$, a point $q \in \mathbb{CP}^3$, or a twistor fibre, and impose scalar agreement there. In the original level-one windows, the generated real algebra remains

$$M_2(\mathbb{H}) \oplus M_4(\mathbb{C}).$$

Thus this operation derives no further algebra reduction.

A marked base point does provide a quaternionic line through

$$\Pi_p = \frac{1}{2}(1 + p_a \Gamma_a).$$

Choosing a complex line inside that quaternionic line, a complex line in the other four-space, and the rank-one map between the corresponding Hom vectors gives the data of Result 2. The grading $2\Pi_p - 1$ and the rule selecting this connector are then additional finite data. The stabilizer calculation derives $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ from those choices.

To obtain an analytic parent for Example 16.9, further work must specify an interface or interval collar, its boundary trace spaces, self-adjoint coupling conditions, a compatible real structure and grading, and the spectral window of the coupled operator. Result 7 concerns the unchanged windows. A new coupled window, boundary degrees of freedom, or a different observable representation lies outside its hypotheses.

Referee amendment. For a general subset upstairs in the twistor space, add the hypothesis that its image in S^4 has empty interior, so that independent base test functions exist.

8. Proposition — Scope of the Hořava–Witten comparison

Hypotheses. Compare chiralities using one common ten-dimensional chirality convention, and retain the unbroken boundary gauge group $E_8 \times E_8$.

The same-chirality interval analogue is type **(a) at both ends**, or type **(b) at both ends** after reversing the chirality convention. To model two separately localized multiplets, take two fields of that same type, with opposite signs of their localization masses. Lemma 12 at $X = 0$ contains one type-(a) and one type-(b) field, so its displayed pair has opposite chiralities.

Proof and scope. Hořava–Witten impose $\Gamma_{11}\chi = \chi$ on the boundary gaugino; the boundary vector multiplets are intrinsic boundary fields. Their chirality projector is stated in section 2, immediately before the boundary Yang–Mills Lagrangian. Labels based on outward normals change sign between the endpoints; this does not change the comparison in a fixed ten-dimensional convention.

Type (a) in Lemma 09 retains the L-zero mode at both ends, while type (b) retains R. Different mass signs localize two such fields near opposite ends. Type (c) instead produces opposite-chirality wall profiles.

On a space containing only the two same-chirality modes, an operator odd for that chirality grading has zero matrix elements between them. Moreover,

$$\mathrm{Hom}_{E_8\times E_8}((248,1),(1,248)) = 0,$$

because the first factor acts trivially on the target and has no invariant vector in its adjoint representation. Thus these boundary multiplets do not supply a gauge-invariant bilinear connector of the interval mass-block type. ■

Bulk propagation can define effective boundary-to-boundary interactions. Identifying one with the finite T requires a specified field, representation and grading, potentially after compactification or symmetry breaking. The Hořava–Witten comparison alone supplies no such identification.

9. No-go theorem — Corollary 11.2 under its matching-scale hypothesis

Verdict and hypotheses. The corrected Corollary 11.2 is correct. Assume a fixed finite set of Dirac mass parameters, N_g generations, and the matching-scale relation

$$8M_W^2 = \sum_{g=1}^{N_g}(m_{\nu,g}^2 + m_{e,g}^2 + 3m_{u,g}^2 + 3m_{d,g}^2). \tag{25}$$

For $N_g \neq 3$, the coefficient 8 is an explicitly assumed normalization; it does not follow automatically by replicating the three-generation gauge trace.

Then $M_W \to \infty$ is incompatible with keeping every Dirac mass bounded. More precisely,

$$\frac{m_{\max}}{2\sqrt{2}} \le M_W \le \sqrt{N_g}\, m_{\max}. \tag{26}$$

Proof. The weighted sum in (25) lies between $m_{\max}^2$ and $8N_g m_{\max}^2$, proving (26). Thus a fixed nonzero gauge coupling and $v \to \infty$, with all $y_f v$ fixed, contradict (25).

The conclusion concerns $m_{\max}$. One fixed species need only be unbounded along a subsequence. For example, with $M_W = t$, put

$$\alpha(t) = \frac{\pi}{4}(1 + \sin t), \quad m_e = 2\sqrt{2}\, t \cos\alpha(t), \quad m_\nu = 2\sqrt{2}\, t \sin\alpha(t).$$

Their squared sum is $8t^2$, while each mass vanishes on an unbounded sequence. ■

Scope and escapes. Equation (25) belongs to the stated normalization of the truncated asymptotic spectral action at matching. It concerns Dirac Yukawa masses, including the neutrino Dirac parameters, rather than physical seesaw eigenmasses. Changing the action or normalization, controlling the full finite-cutoff action, varying the particle content, or allowing Dirac masses to grow changes the problem. Extrapolation to a heavy-mass limit at fixed cutoff requires a separate calculation; the theorem directs attention to that calculation.

10. Theorem — Proposition 16.1 holds for inherited pointwise reality

Verdict and hypotheses. The corrected Proposition 16.1 is correct. Let

$$J_0 a J_0^{-1} = a^*, \qquad [P, J_0] = 0, \qquad Q(a) = PaP|_{P\mathcal{H}},$$

and let $J_F = J_0|_{P\mathcal{H}}$. Let A_F be the involutive algebra generated by the compressed functions. Then order zero holds for this inherited structure exactly when A_F is commutative.

Proof.

$$J_F Q(b)^* J_F^{-1} = P J_0 b^* J_0^{-1} P = Q(b).$$

Order zero therefore requires

$$[Q(a), Q(b)] = 0$$

for every pair of generators. Their generated algebra is then commutative.

Conversely, the opposite operation reverses products and fixes each compressed generator. On a commutative generated algebra its image lies in the same algebra, proving order zero. ■

For a product with a fixed finite factor, the unit of that factor gives the same necessity. Sufficiency also requires that the finite factor satisfy order zero and that the product real structure be compatible. The commutativity conclusion concerns the compressed function algebra; the separate finite factor may be non-commutative.

Scope and escapes. The theorem concerns compression followed by algebra generation with this inherited pointwise reality and a J_0-invariant window. The coarse correspondence supplies a separate opposite action and escapes the hypothesis, as demonstrated positively by Theorem 16.2. Other representations or real structures require their own order-zero calculation.

11. Proposition — The stated inherited grading cannot equal Γ_5

Verdict and hypotheses. The grading obstruction in Example 16.9 is correct under the following simultaneous requirements:

1. Every parent observable under consideration is even: $[\gamma, a] = 0$.
2. The finite grading is inherited as $g = P\gamma P|_{P\mathcal{H}}$.
3. This compression is an actual grading: $g^2 = 1$.
4. The retained observables include $Q(x_a) = \Gamma_a/5$.

Under these hypotheses, $g = \Gamma_5$ is impossible.

Proof. Even without initially assuming $[P, \gamma] = 0$, the grading identity implies

$$0 = P - (P\gamma P)^2 = P\gamma(1-P)\gamma P = \big((1-P)\gamma P\big)^*\big((1-P)\gamma P\big).$$

Hence $(1-P)\gamma P = 0$, and self-adjointness gives $[P,\gamma] = 0$. Therefore

$$[g, Q(a)] = P[\gamma, a]P = 0$$

for every retained even observable.

But

$$[\Gamma_5, Q(x_1)] = \frac{1}{5}[\Gamma_5, \Gamma_1] = \frac{2}{5}\Gamma_5\Gamma_1 \neq 0.$$

Thus the proposed grading and all the stated coordinate compressions cannot simultaneously be inherited in this manner. ■

Scope and escapes. The conclusion does not exclude Γ_5 as additional finite data or as a grading of a prealgebra. Possible changes include compressing a graded prealgebra with odd observables, restricting to the algebra commuting with the proposed grading, or constructing an additional grading. Changing the window alone does not remove the contradiction while retaining all four hypotheses. In the specified Dolbeault kernel, the inherited grading is already scalar.

The three checks above agree with the corrected repository statements. The recorded author is Claude Opus 5.5, and the existing independent Terra checks appear in the Lemma 11 report, the Proposition 16.1 report, and the Example 16.9 report. The present response adds a sequential Codex check; no subagents or additional model consultations were used.

Consequence for T0b and T11

For T0b, an exact interval parent supplies a rank-one connector with KO-6 signs, and a parent with chosen Hom fibres and constant matrix multipliers realizes the classified connecting block and its algebra. Deriving that algebra, grading and connector from the commutative twistor parent remains open: scalar wall agreement in the original window does not perform the selection. For T11, same-chirality boundary multiplets require the type-(a) or type-(b) comparison, and the unbroken Hořava–Witten boundary representations supply no identified finite mass connector.

Numerical check

By Claude Opus 5.5, in Python (numpy), 3 Nov 2026, in a temporary directory, not kept. On W = V = C^4 with random elements:

- T_kappa x = kappa K x K with K = v w*, the rank-one map (5): J D = D J, J Gamma = -Gamma J, D Gamma = -Gamma D; complex rank 1; real dimension of A(T_kappa) inside the 40-dimensional H + H + M_4(C) equal to 24, by linear algebra.
- K of rank two on W_+: dimension 16, as in the proof of Result 2, and x -> K x K has rank 4.
- On C + H + M_3(C): order zero, and order one with defect about 10^-15, and [D, a] = 0 with defect about 10^-15. For a random element off this algebra the order-one defect is about 40.
- The example at the end of Result 1: order-one defect 0, and a nonzero defect of left intertwining.
- Result 4: the values in (15), with a quadrature. By hand: the eigenvectors (13), their boundary conditions, the action of (11), and the value of the commutator.
- Equation (21): by hand, including the compression to m (S + S*).
- Pi_p is a projection of rank 2 and 2 Pi_p - 1 = p_a Gamma_a; the bound (26) on a random sample.

Against the source: (3) is equation (comrel) of arXiv:0706.3688, read in the LaTeX source. The proof there also takes an intertwiner distinct from the Dirac block, in line with the distinction between T and R in Result 1. The other statements are checked by the referee.

Readings

By Claude Opus 5.5, as proposals, corrected after the referee report.

1. The connector is the Majorana block. In the Standard Model dictionary (Chamseddine-Connes-Marcolli 2007, section 2, quoted from memory), W_+ holds (nu_R, e_R) and V = C^4 holds the lepton line

and the three colours, so that with w along nu_R and v along the lepton line, the vector e = w v* is nu_R, and the off-diagonal block of D_F acts on nu_R alone. After this identification, and an overall grading convention if needed, T = m S in Result 5 has the form of the one-generation Majorana block (equation (2.34) there, as the referee checked), with m = arctan(t)/l. In this window m < pi/(2l), so the first level of the tower bounds it, and for t of order one m is of the order of that level. As a pattern, this agrees with the placement of the Majorana mass near the cutoff in that model. Three identical copies give three equal masses, and a general three-generation Majorana matrix needs additional flavour data.

2. The connector as a marked pair of points and a coupling. Since T_kappa x = kappa K x K with K = v w*, the data of the connector are a complex line w in W = C^4, a complex line v in V = C^4, that is, a point of CP^3 on each sheet, and the coupling kappa. In Example 16.9, the point w lies over a point p of S^4 in the twistor fibration, and Pi_p = (1 + p_a Gamma_a)/2 projects onto its quaternionic line. Since Q(x_a) = Gamma_a/5, the grading is

```
2 Pi_p - 1 = p_a Gamma_a = 5 Q(p . x),
```

the compression of the height function at p. So one marked point fixes the quaternionic line and the complex line w, and a second one fixes v. Sp(2) acts transitively on the unit vectors of H^2, and U(4) on those of C^4, so at fixed nonzero |kappa| all line choices are conjugate, in agreement with the uniqueness in Result 2. The modulus |kappa| remains, rephasing w and v changes K x K by a phase, and other finite data can introduce relative moduli. The inputs are then: mark a point on each sheet, with a coupling and a prescription for evaluation and normalization. This has the form (22) with one channel per sheet.

The height function 5 Q(p . x) fails to commute with the other coordinate compressions, so it can serve as additional finite grading data only after restricting to its even algebra. This evades the inherited-grading hypothesis of Result 11, and it does not derive the grading from the even parent. Result 7 shows that scalar agreement at the points does not do this in the unchanged windows, so a coupled window with the data listed after Result 7 is needed. Whether a parent with an interface at those points yields this connector is open.

3. The wall coupling as a Wilson line. Equation (21) says that the wall coupling X = tan(m l) is unitarily equivalent to the uncoupled boundary conditions with a constant matrix connection m F, of relative endpoint rotation angle theta = m l, and the mass is that angle over the length, k = theta/l. On an interval, open-path transport depends on endpoint frames, and the physical quantity is the transport together with the boundary conditions. The comparison with gauge-Higgs unification (Y. Hosotani, Phys. Lett. B 126 (1983) 309; Hosotani, Noda and Takenaga, arXiv:hep-ph/0403106) is an analogy at this stage: the calculation supplies a fixed matrix connection, with no Higgs fluctuation or effective potential. It bears on the question of mass as internal momentum in the steer for the next paper, and compares with the link of lemma 19 and the covariant projection of lemma 14.

4. Next calculation. The connector commutes with B, so it gives no finite one-forms and no Higgs field. The next test is one interval parent carrying both the Dirac Yukawa block of lemma 12, odd for g and diagonal in the sheets, and this connector, for instance at opposite walls: whether order one survives on B, and what the joint form of (21) is.

Part 21

The separate role of the top: removal identities and counting

Prompted by the observation that the top has a separate role, which matters more than the loss of an electroweak partner for b_L. Proposition 21.1 and the counts below are exact calculations by Claude Opus 5.5, in Python with exact fractions. The readings in the last section are proposals, and nothing there is derived. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): one error, in Proposition 21.2, and several amendments, incorporated below. Proposition 21.4 and Corollary 21.5 (added 4 Nov 2026) were refereed by Claude Opus 5.5 (effort max, 5 Nov 2026; report): the counts correct, two amendments (the group SO(6(n_u + n_d)); the SU(3)^3 trace) and one error in the window clause, which held in reading (a) only; all applied.

Outcome. The spectral action detects the top: in the exact quadratic action of lemma 18, among single charged species only an up-type quark can be removed while keeping both the neutral-current value cos^2 theta_NC = 5/8 and the ratio 8/3 of the photon and gluon coefficients. Removing it leaves 84 of the 96 states of the finite Hilbert space. Those 84 are the seed of an SO(30), inside SO(32), whose adjoint contains the charged scalar counts of three generations, and 84 is also the number of on-shell components of the three-form of eleven-dimensional supergravity.

Removal identities

For a Dirac species with charges q_L, q_R in the coordinates (T_3, Y), equation (33) of lemma 18 gives a contribution to the neutral kinetic matrix K with weight e^{-x}. The off-diagonal charge is (q_L - q_R)/2 = (T_3/2)(1, -1), which is orthogonal to the photon direction (1, 1). So the combinations K_33 + K_3Y and K_YY + K_3Y depend on x only through the factor e^{-x} of each massive species, and cos^2 theta_NC = (K_YY + K_3Y)/(K_33 + 2 K_3Y + K_YY) is determined by the sums of T_3 Q and Y Q.

Proposition 21.1. Per species, with colour multiplicity, on the particle trace for the first two columns and with charge conjugates and both chiralities for the last two:

species	sum T_3 Q	sum Y Q	Q^2 coefficient	colour coefficient
up-type quark	1	5/3	16/3	2
down-type quark	1/2	1/6	4/3	2
charged lepton	1/2	3/2	4	0
neutrino	0	0	0	0
three generations	6	10	32	12

Hence the ratios of the whole, 5/3 and 8/3, are those of an up-type quark alone. As one species is removed, that is, as its mass tends to infinity at fixed cutoff in the setting of lemma 18:

removed	cos^2 theta_NC	photon/gluon coefficient
up-type quark	5/8	8/3
down-type quark	59/92	46/15
charged lepton	17/28	7/3

So among single charged species only an up-type quark preserves both values. For an up-type quark they hold at every x, as in Proposition 6 of lemma 18. Within one generation, the down-type quark and charged lepton together contribute exactly what the up-type quark contributes: (1, 5/3; 16/3, 2).

Proof. Arithmetic with the Standard Model charges, Y = Q - T_3. The last statement is direct addition. ■

The scope is that of lemma 18: classical bosonic trace, with these definitions of the coefficients and of the angle, three generations, and the other species with the mass assignments of that calculation. Other definitions of the weak angle do not stay at 5/8 (Proposition 6 there). At the endpoint where the neutral mass vanishes, the massive neutral direction is no longer distinguished, so its angle is read as the limit from finite mass, as lemma 18 states.

The state count in the action

Proposition 21.2. With three generations and nu_R, dim H_F = 96 (Chamseddine-Connes-Marcolli). With only an up-type Dirac mass m and no Majorana mass, the constant-background density of lemma 18 has finite trace Tr exp(-V_0^2/Lambda^2) = 84 + 12 e^{-x} for the heat cutoff. So the coefficient of the vacuum density moves from 96 to 84 along the radial direction, with no finite nonzero stationary point on this free radial ray (No-go theorem 9 and Proposition 11 of lemma 18). This concerns the literal flat-space action with no added potential or constraint. Compact support can give a plateau; sign-changing cutoffs,

additional terms, constraints and paths with an additional fixed noncommuting mass lie outside it. At m = 0 the derivative is zero.

Proof. The top block has 3 colours, 2 chiralities and charge conjugates, so 12 states, each with V_0^2 = m^2. ■

This also gives the top-only family a place in the radial problem, as the endpoint of a descending direction. A radial direction giving other species masses descends further, so this is the endpoint for that direction only.

SO(30)

Let the five quarks other than the top, in three colours, with their conjugates, form the vector 30 = (5, 3) + (5-bar, 3-bar) of SU(5) x SU(3) inside SO(30). The adjoint, of dimension 435, is

```
Lambda^2(30) = (24, 8) + (24, 1) + (1, 8) + (1, 1) + (10, 6) + (15, 3-bar) + conjugates of the last
↪  two.
```

With Q = 2/3 for u, c and Q = -1/3 for d, s, b, that is, Q = diag(3/5, 3/5, -2/5, -2/5, -2/5) + (1/15) I_5, which includes the U(1) commuting with SU(5), the charges are:

representation	charge: number
(15, 3-bar)	-2/3: 6; +1/3: 6; +4/3: 3
(10, 6)	-2/3: 3; +1/3: 6; +4/3: 1
(24, 1) + (1, 1)	+1: 6; -1: 6; 0: 13

Three generations of scalar partners in a supersymmetric Standard Model with nu_R need, for antisquarks in the 3-bar, 6 of charge -2/3 and 6 of charge +1/3, and for sleptons 6 of charge +1, 6 of charge -1 and 12 neutral real states. So (15, 3-bar) and (24, 1) + (1, 1) contain exactly the charged counts. What is left over in them is three triplets of charge +4/3 and one neutral state. The counting of 15 diquarks and 25 mesons, with the charge +4/3 states, and the remark that the top is unable to form them, go back to an early preprint (hep-ph/0512065), which finds neither SO(30) nor a role for SO(32). What SO(30) adds is one group whose adjoint contains both families. It also sits in SO(32), with 496 -> (435, 1) + (1, 1) + (30, 2) under SO(30) x SO(2), which hints at a role for SO(32); we have not explored this. These are representation and charge counts: identifying the entries with physical scalar partners needs a field construction, spin and chirality assignments, interactions and dynamics.

Proposition 21.4 (the counting forces the top out and leaves the charge 4/3 half-filled). Let n_u up-type and n_d down-type light quarks in three colours, with their conjugates, span the vector of SO(6(n_u + n_d)), whose 6(n_u + n_d) real dimensions are those of (n_u + n_d, 3) + conjugate. In the colour antitriplet of the adjoint, the symmetric diquarks S^2(n_u + n_d) (x) 3-bar have n_d(n_d + 1)/2 states of charge -2/3, n_u n_d of charge +1/3 and n_u(n_u + 1)/2 of charge +4/3. Three generations of antisquarks need 6 and 6 in the first two charges, and the only solution in nonnegative integers is (n_u, n_d) = (2, 3). The charge +4/3 count is then 3. With the top included, (n_u, n_d) = (3, 3) gives (6, 9, 6), and the top would need 36 = 30 + 6 real dimensions rather than the 32 of SO(32), whose extra plane is colour-neutral.

Proof. n_d(n_d + 1)/2 = 6 gives n_d = 3, then n_u n_d = 6 gives n_u = 2, and 2 x 3/2 = 3. For n_u = n_d = 3 the three counts are 6, 9, 6. A coloured quark adds 6 real dimensions to the vector; 32 - 30 = 2. ■

Corollary 21.5 (the charge-4/3 sector). Under the bijection of Proposition 21.4 between the (6, 6) part and the antisquarks of three generations, the three antitriplets of charge +4/3 are half of what a chiral multiplet pairing would need: a Dirac quark of charge 4/3 in each generation needs two complex scalars, so three generations need 6. If the three scalars are read as superpartners, their three fermions are left-handed Weyl fields in (3-bar, 4/3), SU(2) singlets, and are anomalous on their own (with T(3) = 1/2 and A(3) = 1, the SU(3)^3, SU(3)^2 U(1), U(1)^3 and gravitational U(1) traces are -3, 2, 64/3 and 12. The meson sector,

six complex scalars of charge +-1 and neutral ones, cancels none of them for any chirality of its fermions: they are colour singlets, their cubic charge sum is an integer, and their linear charge sum has absolute value at most 6. With any other chirality assignment of the three multiplets, the SU(3)^3 trace is odd, so the exotic fermions stay anomalous). The readings compatible with the counting are: (a) the three scalars have no fermion partner, as the Higgs doublet has none in the Standard Model, so supersymmetry is a property of the (6, 6) part alone; (b) the top is included, which completes the exotic to 6 and breaks the +1/3 count to 9. In the window of the interval parent (lemma 38, Proposition 11.4, the shadow B_ij -> (m_i - m_j) B_ij, for distinct masses), the same-flavour diquarks drop out and the antitriplet counts become (3, 6, 1) in reading (a). One state of charge +4/3 remains, the combination of u and c antisymmetric in flavour, with coefficient proportional to m_u - m_c, in the (Lambda^2 5, 3bar) of S^2 V. In reading (b), with six flavours, the counts become (3, 9, 3), and three states of charge +4/3 remain, uc, ut and ct. In both readings the window has 3 states of charge -2/3, so the match with the antisquarks in Proposition 21.4 holds in the parent and fails in the window. This records a conflict between the count of three generations and the anomaly of the exotic sector; the premise of the whole counting is that five light quarks generate every squark and slepton as a composite, scalars that the finite geometry of Connes et al. does not contain.

Provenance. Stated by the user and Claude Fable 5.1 in conversation on 4 November 2026; the arithmetic checked by hand. The remark that the top cannot form the diquarks is from hep-ph/0512065; the chirality reading of the charge-4/3 states as an additional quark per generation is in the abstract of the EPJC paper (Rivero 2024), where the half-multiplet problem is stated.

Direction (referee, 5 Nov 2026). The selection of (2, 3) belongs to the parent: with the flavour-antisymmetric counts n_d(n_d - 1)/2, n_u n_d, n_u(n_u - 1)/2, which are those of the window and of the sign s = -1 (USp(30)), the condition (6, 6) has no integer solution (n_d = 4 forces n_u = 3/2), so five light quarks with the top excluded are a property of Lambda^2 V with walls reducing to M_15(C) (lemma 38, Theorem 11.3), and the question is whether that wall condition can be derived. The anomaly is a parity: in reading (a) the three exotic antitriplets give an odd SU(3)^3 trace, which no partners for the rest of the SO(30) adjoint cancel (octets contribute 0, sextets multiples of 7); in reading (b) the six charge-4/3 states pair vectorlike but the three extra charge-1/3 antitriplets td, ts, tb leave an odd trace again. In both readings an odd number of antitriplets lies beyond the antisquarks, 3 = 15 - 12 and 9 = 21 - 12, so a supersymmetric reading of the diquark sector needs partnerless scalars or an even excess. With four generations the symmetric counting has no solution, since n_d(n_d + 1)/2 = 8 has no integer root.

The two families of 84

Proposition 21.3. Count four states per Dirac fermion and colour, and take Dirac neutrinos. Then

```
96 = 84 + 12:   five quarks and all leptons (60 + 12 + 12), and the top (12);
96 = 84 + 12:   all electrically charged fermions (72 + 12), and the neutrinos (12).
```

The two families of 84 overlap in 72 states, and each is the other with the top and the neutrinos exchanged. Without nu_R, the first family has 78 states and the second stays at 84. In eleven dimensions, the three-form C_3 has C(9, 3) = 84 on-shell components, and its dual C_6 has C(9, 6) = 84. The 84 in H_F is a complex finite-Hilbert-space dimension, and the three-form count is a real bosonic polarization count, so an identification needs a map of representations and degrees of freedom.

Readings, as proposals

These are proposals to test, with no derivation behind them.

1. **Seiberg duality.** For SU(3) with five flavours, N_f = 5 lies in the conformal window 3 N_c/2 < N_f < 3 N_c, and the magnetic dual is SU(N_f - N_c) = SU(2) with five flavours and a 5 x 5 gauge-singlet meson chiral superfield coupled by W = M q q-tilde (N. Seiberg, Nucl. Phys. B 435 (1995) 129, arXiv:hep-th/9411149, checked by the referee). The removal of the top is what gives N_f = 5, and SU(2) is the magnetic group there. To become a statement this needs an N = 1 supersymmetric theory with the required massless matter and dynamics, for example an emergent one, with an operator and

anomaly map. Five-flavour ordinary QCD alone does not supply the duality, and the magnetic SU(2) needs a separate identification with any electroweak group.

2. **Branes.** The first family of 84 could relate to C_3 or to an M2 or M5 brane, and the second family to the dual object. The exchange of the top and the neutrinos would then be the content of the duality on the Standard Model side. The counts match, and equality of the number is insufficient: a representation and dynamics must relate the finite fermionic states to the bosonic form or brane degrees of freedom. The pairing of item 1 is one candidate. The first family also needs Dirac neutrinos for its 84, and the second does not.
3. **Relation to the odd labels.** The removal of the top leaves the five quarks and the leptons, on which SU(3) x U(1)_em acts with b_L paired with b_R. So the loss of the electroweak partner of b_L has no role at the label 9, where SU(2) is absent. A statement here needs the required involution on this finite sector, including its action on D, J, the algebra and its opposite; the vectorlike charges alone do not establish it, and the neutral states still need treatment.

Numerical check

Python with exact fractions, run on 3 Nov 2026 in a temporary directory, not kept: all tables above, and the dimension $435 = 224 + 1 + 105 + 105$ with $105 = 60 + 45$ and $224 = 192 + 24 + 8$.

Part 22

The finite Dirac operator as a boundary condition on an interval

For STRATEGY T0, T0b, T10 and T13, following lemma 12 and lemma 20. Proved by Claude Opus 5.5, with the identities checked in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): two errors and one gap as first written, and amendments; all are incorporated below.

Outcome. Let (H_F, D_F, J_F, Gamma_F) be finite data of signs (+, +, -), and let l ||D_F|| < pi/2. Then D_F, with J_F and Gamma_F, is exactly the lowest spectral window of the pure derivative on an interval of length l, with no bulk mass and no Yukawa term, on the domain fixed by the boundary coupling tan(l D_F). So all the eigenvalues of D_F, Yukawa and Majorana blocks alike, are quantized internal momenta k with tan(k l) equal to a singular value of the coupling. The input is the finite triple: the content is that a consistent parent exists, with these properties:

- Preservation of the domain forces the observables at the wall into C_F, the commutant of D_F in A_F, while A_F acts freely in the bulk.
- The parent satisfies order one for any finite data with order zero.
- The compression is the exact average Q_l(a) = (2l)^-1 int_{-l}^{l} e^{ixD_F} a e^{-ixD_F} dx, so it tends to the finite representation as l -> 0, with defects of order l^2.
- The scalar inner fluctuations of the parent are pure gauge under its allowed endpoint gauge group. On the moving window, they act by the gauge orbit of D_F through the transport along the interval. This reproduces the quadratic term of Chamseddine-Connes-van Suijlekom on the orbit, and it leaves the singular values fixed, so the radial mode of the Higgs field is absent from this parent.

With the family X(l) = tan(l M) and l -> 0, the couplings of the light fermions vanish as m l at fixed mass, the tower decouples, and the broken gauge directions go to pi/(2l). The parent algebra is matrix-valued, so T0 for a commutative parent remains open.

Setting

Let H_F = H_+ (+) H_- be finite-dimensional, Gamma_F = diag(1, -1), let c : H_+ -> H_- be antiunitary, and put J_F(a, b) = (c^-1 b, c a), so that J_F^2 = 1 and J_F Gamma_F = -Gamma_F J_F. Let D_F = [[0, M*], [M, 0]] with M : H_+ -> H_-. Then J_F D_F = D_F J_F is equivalent to M = c M* c. By the Takagi factorization, in bases related by c, we may write M = V |M| with a unitary V satisfying V = c V* c.

Let l > 0 with l ||D_F|| < pi/2, and define X : H_- -> H_+ by

```
tan(l D_F) = [[0, X], [X\*, 0]],   that is,   X = tan(l |M|) V\*.
```

Since J_F commutes with real functions of D_F, c X = X* c^-1. Conversely, every X with c X = X* c^-1 arises from exactly one such D_F, namely D_F = l^-1 arctan of the block operator. In the limit, X = l M* + O(l^3).

The parent. On H_l = L^2([0, l]; C^2) (x) (H_+ (+) H_-), write psi = (Q_L, Q_R; U_L, U_R), with Q valued in H_+ and U in H_-. Let (D psi)_L = psi_R' and (D psi)_R = -psi_L' on both fields, the operator of lemma 12 with no bulk mass, on the domain

```
Q_R(0) = 0,   U_L(0) = 0,   Q_R(l) = X U_R(l),   U_L(l) = -X\* Q_L(l).
```

Call it D_X. Put Gamma^ psi = (Q_L, -Q_R; U_L, -U_R) and J^ psi = (c^-1 U_R, -c^-1 U_L; -c Q_R, c Q_L). Let (E psi)_L = psi_R and (E psi)_R = -psi_L, so that D_X = E d/dx on its domain.

In the Standard Model form of the classification, with E = Hom(V, W) and W = W_+ (+) W_-, one takes H_+ = E_+ (+) E_-* and H_- = E_- (+) E_+*, with c the adjoint map. At X = 0, the Q-fields have boundary type (a) and the U-fields type (b), in the terms of lemma 09; for X != 0 the modes continue from those zero modes, which generally lift. A single matrix X encodes the Yukawa blocks and the Majorana block together. Its entries need not keep the original block pattern, since the tangent brings in higher powers of D_F: for M = [[a, b], [b, 0]] real symmetric, X_22 = (l^3/3) a b^2 + O(l^5).

The window

Theorem 22.1. D_X is self-adjoint with compact resolvent, and J^^2 = 1, J^ D_X = D_X J^, J^ Gamma^ = -Gamma^ J^, Gamma^ D_X = -D_X Gamma^. Let R(x) = exp(x Gamma_F D_F), and define W : H_F -> H_l so that for phi = (a, b), the L-components (Q_L, U_L) of W phi at x are l^{-1/2} R(x)(a, 0), and the R-components (Q_R, U_R) are l^{-1/2} R(x)(0, b). Then W is an isometry and

```
D_X W = W D_F,    J^ W = W J_F,    Gamma^ W = W Gamma_F.
```

With sigma_k the singular values of M and theta_k = l sigma_k, the signed spectrum of D_X is (n pi +- theta_k)/l, n in Z, with multiplicities added when values coincide; a zero singular value gives a zero mode of multiplicity two. So the spectrum of |D_X| consists of the sigma_k and of (n pi +- theta_k)/l for n >= 1, and ran W is the spectral subspace of |D_X| <= ||D_F||, and the next level is at least (pi - l ||D_F||)/l.

Proof. Self-adjointness is Lemma 12.1. Since Gamma_F D_F is anti-Hermitian, R(x) is unitary, and with C(x) = cos(x |M|), C'(x) = cos(x |M*|), S(x) = V sin(x |M|),

```
R(x)(a, 0) = (C(x) a, -S(x) a),    R(x)(0, b) = (S(x)\* b, C'(x) b).
```

Isometry follows from the unitarity of R(x) and the orthogonality of the L- and R-components. The conditions at 0 hold since S(0) = 0. At l, using V* cos(l |M*|) = cos(l |M|) V*, we get X C'(l) = S(l)* and X* C(l) = S(l), which are the conditions at the wall. For the derivative, the L-part of D W phi is d/dx R(x)(0, b) = R(x) Gamma_F D_F (0, b) = R(x)(M* b, 0), and the R-part is -d/dx R(x)(a, 0) = R(x)(0, M a); together these are W D_F phi. The relations for Gamma^ are direct. For J^, use c |M| c^-1 = |M*| and V = c V* c, which give c^-1 C' = C c^-1 and c S(x)* = S(x) c^-1. The signs of the parent follow in the same way, using c X = X* c^-1 for the domain. For the spectrum, constant unitaries on Q and on U commute with E d/dx and bring X to diag(tan theta_k), which reduces to Example 12.5 in each channel. ■

Corollary 22.2 (conjugated form). D_X = R (D_0 + E Gamma_F D_F) R*, with R acting pointwise on the pair (Q, U) and D_0 the uncoupled operator, X = 0. So after this unitary change of domain, D_F is a constant matrix connection, odd for the grading, with relative endpoint rotation R(l). This is equation (21) of lemma 20, for general D_F.

Proof. D(R phi) = R(D phi + E R* R' phi), and R* R' = Gamma_F D_F. For phi in the uncoupled domain, the computation at the wall in the proof above shows that R phi satisfies the coupled conditions. ■

So in this family a fermion mass is a momentum: D_X has no mass term, and the light eigenvalue k = theta/l is fixed by tan(k l) = tan(theta), a singular value of X. In the finite description, the condition relating k to l is lost and the value appears as free data in D_F.

The parent as a real spectral triple

Let (A_F, H_F, D_F, J_F, Gamma_F) be finite data as above, with an even representation pi = pi_+ (+) pi_- of a unital finite-dimensional real or complex *-algebra A_F, faithful, unital and satisfying order zero, with pi°(b) = J_F pi(b)* J_F^-1. Order one is not assumed. Let

```
C_F = {a in A_F : [pi(a), D_F] = 0},    A_l = {f in C^infinity([0, l], A_F) : f(l) in C_F},
```

with f acting pointwise by pi_+(f(x)) on Q and pi_-(f(x)) on U.

Theorem 22.3.

1. A function f in C^infinity([0, l], A_F) preserves Dom D_X if and only if f(l) lies in C_F.
2. (A_l, H_l, D_X, J^, Gamma^) is a real even spectral triple, with [D_X, pi(f)] = E pi(f') and J^ pi(g)* J^^-1 = pi°(g) pointwise.
3. It satisfies order zero and order one, with signs (+, +, -).

So A_F acts freely in the bulk, and the boundary condition restricts the observables at the wall to the commutant of D_F. In the Standard Model data with nonzero Yukawa blocks, this is the algebra {(lambda, lambda, m)} of the photon and gluons; with vanishing Yukawa blocks the commutant is larger. Order one holds in this parent whatever the finite data, for its pointwise algebra: the finite condition appears only on the truncation, and arbitrary compressions need not preserve either order condition.

Proof. Item 1: at l, the traces Q_L(l) and U_R(l) are free, and applying f to the conditions gives pi_+(f(l)) X = X pi_-(f(l)) and pi_-(f(l)) X* = X* pi_+(f(l)), as in Result 3 of lemma 20. That is [pi(f(l)), tan(l D_F)] = 0, equivalent to [pi(f(l)), D_F] = 0 because l ||D_F|| < pi/2 makes D_F a function of tan(l D_F). At 0 the conditions are preserved since pi is even. Item 2: the commutator is the Leibniz rule. Conjugation by J^ acts on (Q, U) as J_F does, and pi(g) acts identically on the two spin components, which gives the second formula. This preserves the domain since J_F commutes with D_F. Item 3: pi(f(x)) and pi°(g(x)) commute at each x by finite order zero, and E commutes with both; so [E pi(f'), pi°(g)] = E [pi(f'), pi°(g)] = 0. ■

This is consistent with Counterexample 4 of lemma 20, whose finite data A_F = C + C, D_F = m sigma_1 fail order one while the parent satisfies it.

The compression

Let Q_l(T) = W* T W for a bounded operator T on H_l. For T(x) a bounded function with values in the even operators on H_F, acting pointwise and identically on both spin components, we write T for the multiplication operator.

Theorem 22.4.

```
Q_l(T) = l^-1 int_0^l cos(x ad_{D_F})(T(x)) dx,      Q_l(E T) = -l^-1 int_0^l sin(x ad_{D_F})(T(x))
↪ dx.
```

For constant a in A_F,

```
Q_l(a) = sinc(l ad_{D_F})(pi(a)) = (2l)^-1 int_{-l}^{l} e^{ix D_F} pi(a) e^{-ix D_F} dx,
```

so in an eigenbasis of D_F with eigenvalues mu_i, the matrix elements are pi(a)_{ij} sin(l (mu_i - mu_j))/(l (mu_i - mu_j)). Moreover:

1. Q_l(A_l) = Q_l(C^infinity([0, l], A_F)), so the image of the parent algebra contains Q_l(a) for every a in A_F.
2. ||Q_l(a) - pi(a)|| <= (l^2/6) |||[D_F, [D_F, pi(a)]]|| <= (2/3) l^2 ||D_F||^2 ||a||.
3. Q_l([D_X, f]) = [D_F, Q_l(f)] for f in A_l.

4. With a' = Q_l(a) and b' = J_F Q_l(b)* J_F^-1, |||[a', b']|| <= (8/3) l^2 ||D_F||^2 ||a|| ||b||, and |||[[D_F, a'], b'] - [[D_F, pi(a)], pi°(b)]|| <= (16/3) l^2 ||D_F||^3 ||a|| ||b||.

So as l -> 0 with D_F fixed, D_F, J_F and Gamma_F stay exact, and the compressed observables converge to pi(A_F) with rate l^2. The finite order conditions hold in the limit exactly when they hold for the finite data.

Proof. From the definition of W, the expectation of T splits into the L- and R-parts, so Q_l(T) is l^-1 int of the Gamma_F-even part of R(x)* T(x) R(x). Put K = Gamma_F D_F. For even Z, ad_K Z = Gamma_F [D_F, Z], which is odd, and ad_K (Gamma_F [D_F, Z]) = -[D_F, [D_F, Z]]. So the even part of e^{-xK} T e^{xK} is cos(x ad_{D_F})(T). Likewise, E exchanges the L- and R-parts with a sign, so Q_l(E T) is l^-1 int of Gamma_F times the odd part of R* T R, and that odd part is -Gamma_F sin(x ad_{D_F})(T). The formula for constant a follows from cos(x ad_D)(Z) = (e^{ixD} Z e^{-ixD} + e^{-ixD} Z e^{ixD})/2.

Item 1: the map f -> Lambda(Q_l(f)), for a linear functional Lambda, is integration against a smooth density. If it vanishes on A_l, which contains all f of compact support in [0, l), the density vanishes on [0, l), hence everywhere by continuity. Item 2: by Taylor's formula with integral remainder, ||cos(x ad_D)(Z) - Z|| <= (x^2/2) ||ad_D^2 Z||, since cos(x ad_D) has norm at most 1 by the average formula; averaging over x gives l^2/6. Item 3: P commutes with D_X. It also follows from the two formulas by integration by parts, where the boundary term at l vanishes because f(l) is in C_F. Item 4: J_F commutes with D_F, so b' is the same average of pi°(b). Expand both commutators with the bound of item 2 and |||[D_F, Z]|| <= 2 ||D_F|| ||Z||. ■

The convergence is a statement about the maps Q_l, on the operator systems, in the setting of Connes and van Suijlekom (arXiv:2004.14115, sections 1 and 3). The generated algebra is not continuous at l = 0. An analytic example, from the referee: let M = [[0, 0, 1], [0, 0, 1], [1, 1, 0]], D = [[0, M], [M, 0]], with A_F = C^3 acting diagonally and equally on the two blocks. With P_1, P_2 the first two channel projections, [Q_l(P_1), Q_l(P_2)]_{12} = -l^2/3 + O(l^4), so the compressions of commuting projections generate a noncommutative algebra for small nonzero l. On the one-generation Standard Model data, a numerical closure with tolerance 10^-7 gives complex dimension 15 for the algebra generated by pi(B), with B = C + H + M_3(C), and 30 for Q_l(B) at l = 0.3; these are numerical values with no certified rank.

Scalar fluctuations

Theorem 22.5. Let h be smooth with values in the anti-Hermitian elements of A_F. The self-adjoint one-forms of the parent are exactly the E pi(h). Let u' = -h u, u(0) = 1, and let w = u(l), a unitary of A_F. Then

```
D_X + E pi(h) = pi(u) D_{X_w} pi(u)\*,      X_w = pi_+(w)\* X pi_-(w),
```

and the window of D_{X_w} carries pi(w)* D_F pi(w). With the real structure, for A = E pi(h) we have J^ A J^^-1 = E J_F pi(h) J_F^-1, and D_X + A + J^ A J^^-1 has the window U_w* D_F U_w, with U_w = pi(w) J_F pi(w) J_F^-1, and this commutes with J_F. Moreover, with u~(x) = u(x) w*, which lies in A_l since u~(l) = 1, and U~ = pi(u~) J_F pi(u~) J_F^-1,

```
D_X + A + J^ A J^^-1 = U~ D_X U~\*.
```

So every scalar connection in this model is gauge equivalent to the original parent under the allowed endpoint gauge group (referee).

Proof. A one-form is sum f_1 [D_X, f_2] = E pi(sum f_1 f_2'), and E pi(h) is self-adjoint exactly when h is anti-Hermitian, since E is anti-Hermitian and commutes with pi(h) and pi is faithful. Every smooth such h arises: with b(x) = int_l^x h, we have b(l) = 0, so b is in A_l, and [D_X, b] = E pi(h). Next, pi(u)* D pi(u) = D + E pi(u* u'), and pi(u)* E pi(h) pi(u) = E pi(u* h u), so the sum is D with no connection term. At 0 the conditions are unchanged. At l, the substitution psi = pi(u) phi gives the conditions with X_w. Then tan(l pi(w)* D_F pi(w)) has the block X_w. For general w, this operator need not commute with J_F, so we use the spectrum and the isometry of Theorem 22.1 without its real-structure part. For the real fluctuation, J^ commutes with E, u(x) is replaced by pi(u) J_F pi(u) J_F^-1, using finite order zero,

and U_w commutes with J_F, so the real window follows. The last formula follows by differentiating, with the endpoint condition. ■

Corollary 22.6 (the quadratic term as transport). By Chamseddine-Connes-van Suijlekom (arXiv:1304.7583, section III), for unitary v in A_F, with U = v J v J^-1, one has U D U* = D + A_(1) + A~*(1)* + *A*(2), for the universal form v delta(v*). Taking v = w*, on the gauge orbit the fluctuation of the parent, which satisfies order one and has no quadratic term, gives on the moving window exactly the fluctuated operator of that framework (equations (10)-(11) of section III.2 there), quadratic term included, for finite data that fail order one. This answers the first open point of lemma 19 on these gauge orbits and with the moving window. For the two-point data with J = sigma_1 K and w = diag(e^{i alpha}, e^{i beta}), it gives the upper link m e^{-2 i (alpha - beta)}, which is m e^{2 i psi} of Corollary 19.2 with psi defined from v = w*. Theorem 22.5 and the covariant projection of lemma 14 share a transport principle. The fixed compression, as in lemma 13, is a different object: by Theorem 22.4, it gives Q_l(E pi(h)) = -[D_F, l^-1 int_0^l x pi(h) dx] + O(l^3 ||D_F||^3 ||h||) for h fixed, so that for constant h it is -(l/2)[D_F, h], while the first derivative of the transported operator is -l [D_F, h]. The difference is the contribution of the moving spectral projection.

No-go theorem 22.7 (the radial mode). *Hypotheses:* the parent above, on one interval, with fixed X, no bulk masses, and scalar inner fluctuations from the even algebra A_l, with the moving spectral window. *Conclusion:* the fluctuations leave all singular values of the finite Dirac operator fixed. In the usual Standard Model representation with nonzero Yukawa data, the normalized shifted Higgs quaternion runs through the unit quaternions, a three-dimensional gauge orbit, with no radial mode. Degenerate data can have a smaller orbit. A fixed compression can show changes in singular values at second order, so the window has to be named.

Proof. Theorem 22.5: D_F and pi(w)* D_F pi(w) are unitarily equivalent. The same holds with U_w. ■

Known escapes.

These lie outside the hypotheses, and each needs its own construction.

- The coupling X as a field on the wall, with its own kinetic term and potential.
- Connections odd for the grading, as in Corollary 22.2 and lemma 23: the fluctuation K -> K + delta K changes the singular values.
- Bulk masses that do not commute with the gauge action.
- Two or more internal dimensions, where a connection has curvature, although curvature alone does not prove a radial mode.
- An additional composite or other degree of freedom; a quantum effective action on the same configuration space does not by itself add a radial coordinate.
- A fluctuation of the finite data alone, as with the radial parameter of lemma 19, which is not the moving window.

Direction. On the interval, the even connection is pure gauge up to its boundary effect, so the finite orbit is all it can reach. The physical Higgs modulus must then be sought at the wall, in odd directions, or in more dimensions. At the 10 -> 9 endpoint this agrees with T10, where the Higgs is frozen.

The gauge sector and the 10 -> 9 family

Proposition 22.8. Consider the product of M^4 with this parent, with algebra of functions f(y, x) valued in A_F and f(y, l) in C_F. Then the gauge one-forms sum f_1 d_mu f_2 take values in C_F at x = l. Assume, in addition, a flat product, a positive five-dimensional Yang-Mills kinetic form, the trivial background, unrestricted endpoint variation in the allowed directions, and no additional boundary kinetic or mass terms. Then the directions in the Lie algebra of C_F have masses n pi/l, n >= 0, and the orthogonal directions have masses (n - 1/2) pi/l, n >= 1. The physical gauge algebra comes from the skew-adjoint elements through the real gauge representation, with its kernel and any unimodularity condition.

Proof. The first statement is direct evaluation at x = l. The rest is the interval spectrum in the gauge A_5 = 0, with Neumann conditions at 0, and with Neumann or Dirichlet conditions at l. ■

In the Standard Model data with nonzero Yukawas, M_W = M_Z = pi/(2l). The Lie algebra of C_F has dimension 10 before unimodularity; with the unimodularity condition of Chamseddine-Connes-Marcolli (arXiv:hep-th/0610241, section 2.5), the massless modes are the photon and the gluons. These are the boundary conditions of Higgsless models (Csaki, Grojean, Murayama, Pilo and Terning, Phys. Rev. D 69 (2004) 055006), and the fermion formula tan(m L) = M_D L is equation (5.6) of Csaki, Grojean, Hubisz, Shirman and Terning (Phys. Rev. D 70 (2004) 015012); paper/LITERATURE.md, section C. Boundary spectral geometry for the Standard Model has been studied by Chamseddine and Connes (arXiv:1008.3980, section 5). What is added here, as far as our searches went, is the combination: the real structure, the finite triple as an exact window, the wall algebra C_F, the compression formula and the fluctuations. A wider claim of novelty would need a fuller comparison.

Corollary 22.9 (the 10 -> 9 family). Fix a block M_0 for the light fermions and a block Theta/l on an orthogonal channel, compatible with J_F and the grading, with 0 < Theta < pi/2, for the top and if wanted for the Majorana block. Let D_F(l) have M(l) = M_0 (+) Theta/l, and let l -> 0. Then:

1. X(l) has the light block tan(l |M_0|) V_0* = l M_0* + O(l^3) and a heavy block with singular values tan(Theta): the light couplings vanish as m l at fixed mass m.
2. The heavy channel has mass Theta/l, the broken gauge directions have mass pi/(2l), and the other levels diverge.
3. The massless gauge sector is C_F. Let P_L be the projection reducing D_F(l) to the fixed light block D_L. For constant a, P_L Q_l(a) P_L = sinc(l ad_{D_L})(P_L pi(a) P_L), with ||P_L Q_l(a) P_L - P_L pi(a) P_L|| <= (2/3) l^2 ||D_L||^2 ||pi(a)||, also when pi(a) mixes light and heavy channels. The limiting light compression need not be a representation, since products can pass through the heavy subspace. The full estimate of Theorem 22.4 does not apply, because l ||D_F(l)|| tends to Theta. For example, with D_F(l) = sigma_1 (x) diag(m, Theta/l) and pi(a) = 1 (x) sigma_1, one has Q_l(a) = sinc(Theta - l m) pi(a), which tends to sinc(Theta) pi(a).
4. With l = pi/(2 M_W), we get X_q ~ (pi/2) m_q/M_W, which has the form of y_q = sqrt(2) m_q/v -> 0 at fixed mass, as the coupling vanishes while the mass stays. Identifying X_q with the four-dimensional Yukawa coupling is a hypothesis.
5. The ratio is m_t/M_W = 2 Theta/pi < 1, and M_W/M_Z = 1.

So the 10 -> 9 proposal is realized at this level as a limit l -> 0, with W, Z and top at the tower scale and the light fermions keeping explicit masses. Item 5 does not reach m_t/M_W = sqrt(8/3), which belongs to the interior. This is a statement in the stated family: lowest level, no bulk mass, strict wall algebra. Bulk masses for the top, and a changed gauge wall condition, are directions beyond it.

A numerical comparison for the interior (Python). A separate model, with a finite wall scalar and Neumann conditions at 0, lies outside Proposition 22.8 and No-go theorem 22.7. In it, the mass of a broken direction satisfies t tan t = s, with t = M l and s = g^2 v^2 l^2/4 (for the Z, s/cos^2 theta_0), which has the form of M_k tan(M_k pi R) = -V in Csaki et al. (arXiv:hep-ph/0305237, section on mixed boundary conditions, read in the LaTeX source), with the sign convention of V there. If g'^2/g^2 = 3/5 is taken at all scales, M_W^2/M_Z^2 goes from 5/8 as s -> 0 to 1 as s -> infinity. It equals the measured 0.777 at M_W l = 1.03, with 1/l = 78 GeV and a next W level at 280 GeV; whether collider searches allow this would need the couplings, branching fractions and an applicable limit, which we have not analyzed. The top needs M_W l < 0.96 for m_t/M_W = sqrt(8/3), or M_W l < 0.73 for the measured ratio, in the lowest level with no bulk mass. So in the flat tree-level toy without bulk masses or running, the top and gauge requirements are incompatible. It does show the interior interpolating between the spectral-action value 5/8 and the endpoint value 1. The directions are bulk masses, warping and running.

What is input and what is derived

The finite data (A_F, H_F, D_F, J_F, Gamma_F) are all inputs, so the existence of the window is close to a reparametrization: X = tan(l D_F). The derived statements are the wall algebra C_F for that input, the order conditions of the parent, the compression formula with its rate, the form of the fluctuations, and the gauge spectrum under the stated assumption. The window identifies D_F, J_F and Gamma_F exactly, while the represented algebra is obtained through compression and is not an exact representation of A_F at

$l > 0$. In particular, this does not select A_F. T0 asks for a commutative parent: a natural next step is a two-stage parent, compressing functions on S^4 and CP^3 at level one to get A_F-valued functions on the interval, and then taking this window, where the difficulty is the same interface problem as after Result 7 of lemma 20.

Numerical check

By Claude Opus 5.5, in Python (numpy and scipy), 3 Nov 2026, in a temporary directory, not kept.

- Random complex symmetric M on C^5, l ||M|| = 1.3: D u_+(a) = u_-(M a) and D u_-(b) = u_+(M* b) by finite differences, to about 10^-9; boundary conditions and J^ to about 10^-15; norm of W to 10^-13.
- Q_l(a) = sinc(l ad)(a), by quadrature, to 2 x 10^-8. The formula for Q_l(E T), with a nonconstant T, to 10^-14.
- One-generation Standard Model data with Yukawas and the rank-one Majorana block, with signs, order zero and order one exact on B = C + H + M_3(C), and commutant of D in B of real dimension 20. At l = 0.1 and 0.03, ||Q_l(a) - a + (l^2/6)[D, [D, a]]|| is 10^-5 and 4 x 10^-8, and the order-one defect of Q_l is 1.5 x 10^-4 and 1.2 x 10^-5, which scales as l^2.
- The dimensions of the generated algebras quoted above, 15 and 30, from a numerical closure with tolerance 10^-7, not retained and not certified; at smaller l the numerical rank depends on the tolerance.
- The roots of t tan t = s, and the values quoted in the comparison for the interior.
- On the two-point data with J = sigma_1 K: D + A_(1) + A~*(1)* + *A*(2) = U D U* to 4 x 10^-16, and the link of U_w* D U_w equal to m e^{-2 i psi}.
- The bounds of Theorem 22.4, items 2 and 4, on 30 random samples of the one-generation data: largest ratios to the bound 0.33, 10^-4 and 10^-3.

Part 23

Two gradings on an interval: masses as principal angles, the Higgs modulus, and exact W and Z masses

For STRATEGY T0b, T5b, T10 and T13, following lemma 22. Proved by Claude Opus 5.5, with the identities checked in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): four errors and one gap as first written, and amendments, all incorporated below. The simpler form of the mass formulas, and the proof of item 3 of Theorem 23.6, are by the referee.

Outcome. A self-adjoint boundary condition for the derivative on an interval is a grading at each wall. The whole finite Dirac operator is then the relative position of two gradings: its singular values are the principal angles between them, divided by l, and

```
D_F = (1/21) Gamma_0 log(Gamma_1 Gamma_0).
```

With a bulk prealgebra, this gives four results.

1. Aligned walls compress a prealgebra stable under the grading to its even part, with the grading inherited, which is the step that Example 16.9 had to add by hand.
2. Odd scalar fluctuations move the second grading, so they change the singular values and supply the radial mode that No-go theorem 22.7 excludes for even fluctuations. For M_2(H) the field space is S^4, and the Higgs modulus is half the geodesic distance between the points chosen by the two walls.
3. With a wall connector on one channel and a five-dimensional Yang-Mills action with common normalization, the masses are exact: sin^2(M_W l) = sin^2(2 theta)/2 and sin^2(M_Z l) = (4/5) sin^2(2 theta), for universal fermions of mass theta/l. At small theta this gives M_W^2 = 2 m^2 and M_Z^2 = (16/5) m^2, the constants in the lower bounds of the fully massive family of lemma 18, with M_W^2/M_Z^2 tending to 5/8 at both ends.
4. The heat-cutoff spectral action on the interval is a Jacobi theta function of theta, decreasing on [0, pi/2]. This family lies outside the hypotheses of the radial no-go of lemma 18: the radial direction is

compact, with a stationary point at m = pi/(2l), the Connes-Lott value, where M_W = M_Z = 0.

Everything finite is still an input, and the parent algebra is matrix-valued. For a commutative parent, lemma 24 finds the window exactly, with spectral walls, while direct compression of scalar functions gives only R + R on the S^4 factor. The formulas for gauge masses depend on the assumed action. The relation to measured values fails in the flat toy, as stated at the end.

Two gradings

Let H be a finite-dimensional Hilbert space, and let Gamma_0, Gamma_1 be self-adjoint involutions on H. On L^2([0, l]; C^2) (x) H, with psi = (psi_L, psi_R), let D = E d/dx, where (E psi)_L = psi_R and (E psi)_R = -psi_L, on the domain

```
Gamma_0 psi_L(0) = psi_L(0),   Gamma_0 psi_R(0) = -psi_R(0),   Gamma_1 psi_L(l) = psi_L(l),
↪  Gamma_1 psi_R(l) = -psi_R(l).
```

Call it D(Gamma_0, Gamma_1), and let Gamma^ act as +1 on psi_L and -1 on psi_R.

Theorem 23.1.

1. D(Gamma_0, Gamma_1) is self-adjoint with compact resolvent, and it anticommutes with Gamma^.
2. Suppose Gamma_0 and Gamma_1 have eigenspaces of equal dimensions, and that -1 is not an eigenvalue of Gamma_1 Gamma_0. Let theta_k in [0, pi/2) be the principal angles between the +1 eigenspaces. Then K = (1/2l) log(Gamma_1 Gamma_0), with the principal branch, is anti-Hermitian, Gamma_0 K Gamma_0 = -K, and D_F = Gamma_0 K is self-adjoint, odd for Gamma_0, with singular values theta_k/l on the +1 eigenspace. Moreover Gamma_1 = e^{lK} Gamma_0 e^{-lK}.
3. With H_+, H_- the eigenspaces of Gamma_0, and X the H_- -> H_+ block of tan(l D_F), as in lemma 22, D(Gamma_0, Gamma_1) = D_X, and the spectral part of Theorem 22.1 gives the spectrum and the window, with no real structure needed. Conversely, every D_F odd for Gamma_0 with l ||D_F|| < pi/2 arises from exactly one such Gamma_1.
4. Let an antiunitary J_1 satisfy J_1^2 = -1, J_1 Gamma_i J_1^-1 = -Gamma_i for i = 0, 1. Then (J^ psi)_L = J_1 psi_R, (J^ psi)_R = -J_1 psi_L defines a real structure of signs (+, +, -) for D(Gamma_0, Gamma_1). With J_F = Gamma_0 J_1 up to sign, J_F^2 = 1, J_F Gamma_0 = -Gamma_0 J_F and J_F D_F = D_F J_F, which are the signs of lemma 22.

So J_F itself does not anticommute with Gamma_1 in general: J_F Gamma_1 J_F^-1 = -Gamma_0 Gamma_1 Gamma_0, the reflection of Gamma_1 through Gamma_0.

Proof. Item 1: the boundary form is [<psi_R, phi_L> - <psi_L, phi_R>] evaluated between 0 and l, which vanishes on the domain because L-values and R-values lie in orthogonal eigenspaces at each wall, and these are half of the boundary data.

Item 2: U = Gamma_1 Gamma_0 is unitary and Gamma_0 U Gamma_0 = Gamma_0 Gamma_1 = U^-1, so K is anti-Hermitian and anticommutes with Gamma_0. The eigenvalues of U are e^{+-2 i theta_k}, by the standard two-projection decomposition, and K then has eigenvalues +-i theta_k/l. Since e^{lK} Gamma_0 e^{-lK} = e^{2lK} Gamma_0 = U Gamma_0 = Gamma_1, the last formula follows.

Item 3: in the notation of Theorem 22.1, R(l) = e^{lK} carries H_+ to the graph {(C a, -S a)} = graph(-X*), and H_- to graph(X). These are the +1 and -1 eigenspaces of Gamma_1, which is the condition at the wall. The converse uses the same formula, with uniqueness from injectivity of the exponential on the stated range.

Item 4: J^ commutes with E by direct computation, and J^^2 = -J_1^2 = 1. The domain is preserved since J_1 exchanges the eigenspaces of each Gamma_i. For the last statement, J_1 U J_1^-1 = U, so as J_1 is antilinear, J_1 K J_1^-1 = K in the sense of the principal branch, and J_F = Gamma_0 J_1 then satisfies J_F K J_F^-1 = -K and commutes with D_F = Gamma_0 K. ■

When theta_k = pi/2 the channel has type (c) of lemma 09, with no zero mode and lowest level pi/(2l), the Connes-Lott mass of Theorem A in the paper. There the principal logarithm and the isolated window cease to be unique.

The bulk prealgebra

Let P be a *-subalgebra of End(H), which need not commute with Gamma_0. Let

```
A(Gamma_0, Gamma_1) = {f in C^infinity([0, 1], P) : [f(0), Gamma_0] = 0, [f(1), Gamma_1] = 0},
```

acting pointwise and identically on both spin components.

Theorem 23.2.

1. A function f in C^infinity([0, 1], P) preserves the domain exactly when it lies in A(Gamma_0, Gamma_1). The commutators are [D, f] = E f', and Gamma^ commutes with every f, including those with values odd for Gamma_0.
2. Under the hypotheses of Theorem 23.1, item 2, with W the isometry of Theorem 22.1, Q(T) = 1^-1 int_0^1 E_0(R(x)* T(x) R(x)) dx, where E_0(Z) = (Z + Gamma_0 Z Gamma_0)/2 is the conditional expectation onto the even part.
3. *Aligned walls.* If Gamma_1 = Gamma_0, then D_F = 0, the window consists of constant L-values in H_+ and constant R-values in H_-, and Gamma^ restricts on it to Gamma_0. For an ambient constant multiplier a in P, Q(a) = E_0(a), and the image of the boundary algebra A(Gamma_0, Gamma_0) is E_0(P), by compactly supported profiles. If Gamma_0 P Gamma_0 = P, then E_0(P) is the even part of P, so the compressions of the prealgebra are exactly its even part, with the grading inherited from the even parent operator Gamma^. Without that hypothesis this can fail: for H = C^2, Gamma_0 = sigma_z and P generated by sigma_x + sigma_z, E_0(P) = span{1, sigma_z}, while the even part of P is C.
4. *Order conditions.* Let J_1 be as in Theorem 23.1 and let P° = {J_1 b* J_1^-1 : b in P} commute with P. Then the parent has order zero and order one for A(Gamma_0, Gamma_1).

Proof. Item 1: as in Theorem 22.3, using that the conditions at a wall say that L- and R-values lie in the eigenspaces of Gamma_i. Item 2: as in the proof of Theorem 22.4. Item 3 is the case R = 1. Item 4: [D, f] = E f', and pointwise commutation gives both conditions. ■

Item 3 evades the inherited-grading obstruction of Result 11 of lemma 20, because hypothesis 4 there fails: for odd a, here Q(a) = 0, while there the coordinates are retained as Q(x_a) = Gamma_a/5. So at the level of a matrix-valued parent this addresses the question left in step 3 of the framework note: the prealgebra is the bulk algebra, the even algebra is what the window sees, and the grading is a boundary condition. The commutative-parent and retained-coordinate questions remain open, and Theorem 4 of lemma 24 shows a real difference: scalar functions compressed directly give R + R on the S^4 factor, and H + H needs the matrix algebra generated at the intermediate stage.

Remark 23.3 (the twist). On H = E (+) E*, with J(x, y) = (y*, x*), the operator J_1 = Gamma_0 J is the one that anticommutes with both gradings. For an odd anti-Hermitian h in P, J_1 h J_1^-1 is y -> +y h on E*, where J alone gives y -> -y h. So the opposite representation b° = J_1 b* J_1^-1 incorporates on the W-factor the grading automorphism sigma(b) = gamma_W b gamma_W. In this parent the twist comes from the requirement that J_1 preserve both boundary conditions. Twisted spectral triples for the grand symmetry (Devastato, Lizzi and Martinetti, JHEP 01 (2014) 042, arXiv:1304.0415; Devastato and Martinetti, Math. Phys. Anal. Geom. 20 (2017) 2, arXiv:1411.1320, Eqs. (1.8)-(1.11)) use a different automorphism: their rho exchanges left and right spinorial indices and algebra summands, while conjugation by gamma_W is inner on M_2(H) and fixes its centre. So this is a structural analogy between involutive automorphisms, and an identification needs a comparison of the representations, opposites and first-order conditions.

Odd fluctuations and the radial mode

Theorem 23.4. Let h be smooth with values in the anti-Hermitian elements of P, and let u' = -h u, u(0) = 1, w = u(1). Then

```
D(Gamma_0, Gamma_1) + E h = u D(Gamma_0, w\* Gamma_1 w) u\*.
```

So scalar fluctuations move the second grading along its orbit under the unitary group of P. If h has an odd part, the principal angles between Gamma_0 and w* Gamma_1 w change in general.

Proof. u* D u = D + E u* u', and u* E h u = E u* h u, so the connection cancels. Since u(0) = 1, the wall at 0 is unchanged. At l, psi = u phi gives the stated grading for phi. ■

This lies outside No-go theorem 22.7, whose hypotheses include the even algebra: an even connection gives a w commuting with Gamma_0, and the angles are kept, while an odd connection changes them. For a real fluctuation, add the conjugate one-form: D + E h + J^(E h)J^^-1, with generator of the form (L_F, R_F) in Example 23.5. The invariant is a double coset. With H_i = Stab(Gamma_i) in the connected unitary group G of P, a gauge transformation removing h has the form u(x) c, and it preserves both endpoints exactly when c is in H_0 and w c in H_1, that is, when w is in H_1 H_0. So the scalar moduli of the parent are the double cosets H_1 G / H_0, the relative position of the two gradings. This is with the moving window, as in lemma 22.

Example 23.5 (M_2(H) and S^4). Let W = H^2 = C^4, with the Clifford generators Gamma_a of lemma 16, and for a unit vector p in R^5 put Gamma_p = p_a Gamma_a. Each grading of W with one-dimensional quaternionic +1 and -1 eigenspaces is such a Gamma_p, so these gradings form S^4 = Sp(2)/(Sp(1) x Sp(1)), the quaternionic projective line. For two points, Gamma_q Gamma_p = p . q + (p wedge q), with eigenvalues e^{+-i d}, where d is the geodesic distance; so all the principal angles equal theta = d/2.

Take H = Hom(V, W) (+) Hom(W, V), J(x, y) = (y*, x*), Gamma_0(x, y) = (gamma x, -y gamma) with gamma = Gamma_{p_0}, and K(x, y) = (theta/l)(F x, y F), with F an odd anti-Hermitian element of M_2(H), F^2 = -1. Then:

- D_F = Gamma_0 K has all singular values equal to theta/l: a universal Dirac mass, for all colours.
- The second grading is Gamma_1(x, y) = (Gamma_p x, -y Gamma_{p'}), where p is at distance 2 theta from p_0 and p' is its mirror image through p_0. J_1 = Gamma_0 J satisfies Theorem 23.1, item 4, while J alone does not.
- The coordinates (theta, q) on S^4, with F determined by the unit quaternion q, are the polar coordinates of a Higgs quaternion: theta is the radial coordinate, bounded by pi/2, and q gives three angular directions, undefined at the poles. At theta = pi/2 the finite logarithm and the lowest window are degenerate. Quotienting by the stabilizer leaves the radial interval. A canonically normalized field needs the scalar kinetic term, which we have not computed.

With bulk masses, by Theorem 12.6, the light masses are about (tan(theta)/l) Phi_Q Phi_U when small compared with the tower thresholds, with a universal matrix before flavour data; general finite-angle eigenvalues need the secular equation. This field space is the coset of the minimal composite Higgs model (Agashe, Contino and Pomarol, Nucl. Phys. B 719 (2005) 165, sections 2-3, with their h/f_pi corresponding to 2 theta). Their construction uses a warped interval and additional fermion and boundary data, so the shared coset does not identify the actions or potentials. The appearance here of the S^4 of the twistor window is an observation, and no identification between the two roles of S^4 is made.

The weak bosons

Take P = M_2(H) (+) M_4(C), acting on E = Hom(V, W) on the left and on the right. Assume the following.

- A five-dimensional Yang-Mills field valued in sp(2) (+) su(4), with the trace form Tr_W + Tr_V, so with one coupling (the normalization of the spectral action, g_3 = g_2 = sqrt(5/3) g_Y).
- At x = 0: Neumann for the stabilizer of p_0, Dirichlet for its complement. At x = l: Neumann for the stabilizer of p_l and of a connector, that is, the a with [a_W, Gamma_p] = 0, a_W w = lambda w and a_V v = lambda v, as in Results 2 and 5 of lemma 20, for a line w in the +1 eigenspace of Gamma_p and a line v in V; Dirichlet for the complement.
- The connector is prescribed gauge-breaking wall data here. If it is instead a term in the fermion boundary operator, as for a unit vector e in the +1 eigenspace of Gamma_1, f = J_1 e, K_c = |f><e| - |e><f| and Gamma_{1,c} = e^{eta K_c} Gamma_1 e^{-eta K_c}, then J_1 still anticommutes with Gamma_{1,c}, the parent order conditions are kept, and the affected fermion tower changes, so the universal fermion formula must be recalculated for that channel.

Theorem 23.6. With theta the angle above, the modes that are massless for all theta are exactly nine, the photon and the eight gluons, with three additional electroweak zero modes at theta = 0 and theta = pi/2. The lowest charged and neutral massive modes satisfy

```
sin^2(M_W l) = sin^2(2 theta)/2,
sin^2(M_Z l) = (4/5) sin^2(2 theta),
```

equivalently tan^2(M_W l) = sin^2(2 theta)/(1 + cos^2(2 theta)) and tan^2(M_Z l) = 4 sin^2(2 theta)/(1 + 4 cos^2(2 theta)).

The photon has 1/e^2 = (8/3)/g^2. Moreover:

1. As theta -> 0, with m = theta/l: M_W^2 -> 2 m^2 and M_Z^2 -> (16/5) m^2, so M_W^2/M_Z^2 -> 5/8.
2. As theta -> pi/2: M_W, M_Z -> 0, with M_W^2/M_Z^2 -> 5/8, while m -> pi/(2l).
3. At theta = pi/4: M_W l = pi/4, tan(M_Z l) = 2, and M_W^2/M_Z^2 = ((pi/4)/arctan 2)^2 = 0.5032…, the global minimum. So M_W^2/M_Z^2 stays in [0.5032, 0.625), with 5/8 as a limit at the ends, where the ratio itself is undefined.
4. m/M_W goes from 1/sqrt(2) at theta -> 0 to infinity at theta -> pi/2. It equals sqrt(8/3) at theta = 1.0589.

Proof. In the gauge A_5 = 0 the field is free, with cos(k x) on the Neumann directions at 0 and sin(k x) on the Dirichlet ones, and the condition at l holds in the frame rotated by Ad exp(theta F). The generators split according to the frequency of Ad F. Write L, R for su(2) on W_- and W_+, S = (L + R)/sqrt(2), A = (L - R)/sqrt(2), and O for the odd generators; Ad F has frequency 0 on S, and frequency 2 on the plane (A, O). So S = s cos kx, A = alpha cos kx, O = beta sin kx.

At l, in the rotated frame, A' = cos(2 theta) A + sin(2 theta) O and O' = -sin(2 theta) A + cos(2 theta) O, with O' = 0 at l, which gives beta = alpha tan(2 theta) cot(k l), and so A'(l) = alpha cos(k l)/cos(2 theta).

Charged directions. The connector forces R^+- to vanish at l, with L^+- Neumann. So S(l) = A'(l), and d/dx (S + A') = 0 at l. Substituting gives sin^2(kl) (1 + cos^2(2 theta)) = sin^2(2 theta) cos^2(kl).

Neutral directions. Add B = b cos kx for the generator T_15 = diag(1, 1, 1, -3)/(2 sqrt 6) of su(4). With Y = sqrt(3/5) R + sqrt(2/5) B and Z' = -sqrt(2/5) R + sqrt(3/5) B, the connector forces Z' to vanish at l, with L and Y Neumann. The Neumann conditions give b = -sqrt(3) s, and Z' = 0 gives s = alpha/(4 cos(2 theta)). Substituting in the condition for L gives sin^2(kl) (1/4 + cos^2(2 theta)) = sin^2(2 theta) cos^2(kl).

Massless modes and limits. At k = 0, a zero mode is a constant in the intersection of the two stabilizers. This consists of a_W = diag(q, q) on W_+ and W_-, identified by F, with q preserving the line of w, together with a_V = lambda (+) m, and with the trace condition this has dimension 1 + 8. In it, L = R and B/L = sqrt(2/3), which gives 1/e^2 = (1 + 1 + 2/3)/g^2. The limits follow by expanding. A second derivation, from the referee, uses the principal angles between the Neumann subspaces at the two walls: for each charged direction the squared sine of the angle is sin^2(2 theta)/2, and in the neutral sector the single nonzero value is (4/5) sin^2(2 theta); it also covers cos(2 theta) = 0, where the divisions above need a limit. For item 3, with a = sin(2 theta), the ratio is [arcsin(a/sqrt 2)/arcsin(2a/sqrt 5)]^2, and this decreases with a on (0, 1], since x/(sqrt(1 - x^2) arcsin x) increases. ■

The constants in item 1 are the values ||[D_F, a]||^2/||a||^2 in the Hilbert-Schmidt norm, for universal mass m, for a = L^+- and for the direction orthogonal to the photon and Z', which has A-component of squared length 4/5. These are the coefficients of the lower bounds for the poles of the fully massive family in Proposition 14 of lemma 18 (2 x < z_W, (16/5) x < z_Z); the small-x pole coefficients there are 3 and 24/5. The limit 5/8 is common to both, with different behaviour: the finite-action poles grow in the heavy limit of lemma 18, while the interval masses here tend to zero at the antipodal point. The spectral-action Taylor mass for universal m is M_W^2 = 3 m^2, which exceeds the value here by the factor 3/2 of Proposition F in the paper: there the mass matrix carries the relative heat coefficients of the action, and here we have assumed a five-dimensional Yang-Mills action, which gives the commutator norm alone. A direct matching claim needs the full parent spectral action, with its tower and boundary conditions.

Item 2 gives a geometric form of the behaviour that lemma 18 found: heavy fermions with light vectors. At antipodal points Gamma_p = -Gamma_{p_0}, so both walls have the same stabilizer in sp(2) again, and the fermion channels have type (c) with mass pi/(2l).

The spectral action on the interval

Proposition 23.7. For the heat cutoff and universal theta, with s = l Lambda, the trace over a channel is

```
Tr_channel exp(-D^2/Lambda^2) = 2 sum_{n in Z} exp(-(n pi + theta)^2/s^2).
```

This is proportional to the Jacobi function theta_3(theta, exp(-s^2)). It is strictly decreasing on [0, pi/2], with a maximum at theta = 0 and a minimum at theta = pi/2, where it is stationary, and it is symmetric under theta -> pi - theta. In the towerless regime s -> 0 at fixed x = theta^2/s^2, it tends to 2 e^{-x}, the form of the vacuum density in lemma 18 and Proposition 21.2.

Proof. The eigenvalues in a channel are +-(n pi + theta)/l, by Theorem 22.1, and the sum is over Z after combining the signs. By Poisson summation the sum equals (s/sqrt(pi)) sum_{k in Z} exp(-k^2 s^2) e^{2 i k theta}, and the Jacobi triple product writes this as a constant times the product over n >= 1 of (1 + 2 q^{2n-1} cos(2 theta) + q^{4n-2}), with q = exp(-s^2). Each factor is positive and strictly decreasing in theta on [0, pi/2]. ■

In the form of the Jacobi function, the trace is (2 s/sqrt(pi)) theta_3(theta, exp(-s^2)). This family lies outside the hypotheses of No-go theorem 9 of lemma 18, which concerns finite internal data and a free ray V = c V_*: here the operator family is different, the radial direction is a compact orbit of relative gradings, and the tower, which the finite description omits, supplies a second branch with a stationary point at theta = pi/2. In the present family the literal action has its minimum at antipodal gradings, m = pi/(2l), with M_W = M_Z = 0 by Theorem 23.6, where the lowest eigenspace is degenerate. This is a vacuum with massless vectors, so it is not the physical point. The sign of the term agrees with the fermion loop in the Hosotani mechanism, where gauge-boson loops enter with the opposite sign. So an interior minimum could arise from that competition, which the classical spectral action does not contain. We have not computed it.

The one-loop potential in the same family

From the exact spectra, the products over the towers are, relative to theta = 0 and in Euclidean momentum x = p l,

```
fermion channel:  1 + sin^2(theta)/sinh^2(x),
charged vectors:  1 + sin^2(2 theta)/(2 sinh^2(x)),
neutral vector:   1 + (4/5) sin^2(2 theta)/sinh^2(x).
```

So in the gauge-boson and fermion-loop form of the Hosotani mechanism, assuming the usual one-loop formula with three polarizations per vector, and with n_f = 2 x (number of physical Dirac species),

```
8 pi^2 l^4 V(theta) = int_0^infinity x^3 dx [ 3 log(1 + s^2/(2 sinh^2 x)) + (3/2) log(1 + (4/5)
↪  s^2/sinh^2 x) - n_f log(1 + sin^2(theta)/sinh^2 x) ],
```

with s = sin(2 theta). A Dirac top with three colours has n_f = 6, which needs a separately specified fermion sector, since the universal connection gives masses to all: one generation has n_f = 16 and three have n_f = 48. With I = int x^3/sinh^2 x dx = (3/2) zeta(3) and I' = int x^3/cosh^2 x dx = (9/8) zeta(3), near theta = 0 the bracket is (10.8 - n_f) I theta^2 + o(theta^2), and with epsilon = pi/2 - theta, near theta = pi/2 it is V(pi/2) + (10.8 I + n_f I') epsilon^2 + o(epsilon^2), where we have written V for the left side.

Proposition 23.8 (referee). In the massless flat family with this content, for every n_f > 0, theta = pi/2 is the unique global minimum on [0, pi/2], since the gauge logarithms are nonnegative and vanish there, and the negative fermion logarithm is minimized at sin^2(theta) = 1. For n_f >= 10.8, V is strictly decreasing on the interior. Indeed, with t = sin^2(theta) and J(z) = int_0^infinity x^3 dx/(sinh^2 x + z), the derivative with respect to t is, up to a positive factor,

```
(1 - 2t) [6 J(2t(1 - t)) + (24/5) J((16/5) t (1 - t))] - n_f J(t).
```

For t <= 1/2, both arguments in the bracket are at least t, and J decreases; for t > 1/2 both contributions have the required sign.

For 0 < n_f < 10.8, on grids of theta (Python), for n_f in {1, 2, 3, 4, 6, 8, 10}, V has one interior maximum, at about 0.33 for n_f = 8, and no interior minimum. Theorem 9 of lemma 24, by Astra 6, proves this for all n_f in that range: exactly one interior stationary point, a strict maximum, using a log-concavity property of the integral.

These are negative results in an exact scope: one interval, no bulk masses, fermions of frequency 1 under F, this gauge content, one loop, this condition at the connector wall. As a formal leading approximation, not uniform at x = 0, V has the form B sin^2(2 theta) - A sin^2(theta) with A, B > 0, whose interior stationary point cos(2 theta) = A/(4B) is a maximum. *Bulk masses.* With bulk masses, the fermion product becomes cos^2(theta) + sin^2(theta) G(x), with G = (w_Q coth w_Q - c_Q)(w_U coth w_U - c_U)/x^2 and w = (c^2 + x^2)^{1/2} in the secular form of lemma 12. For a = c_Q + c_U != 0, G = 1 - a/x + O(x^-2), so the potential has angle-dependent ultraviolet divergences, and a finite value depends on a regulator and subtraction: as the referee showed, an earlier scan of general pairs here did not establish a potential, and it is withdrawn. The exceptional case c_U = -c_Q = c has a common bulk mass on the two mixed components and G = 1 + w^2/(x^2 sinh^2 w), which decays exponentially. Keeping the flat gauge products as a separate tree-level assumption, a scan for c in {0, 1/2, 1, 3/2, 2, 3, 5}, n_f in {1, 3, 6, 8, 10, 16, 48}, on a grid of 721 values of theta, finds no interior minimum (Python). General bulk masses need a gauge-invariant completion, a regulator and wall counterterms.

So an interior minimum needs another change: a negative coefficient of sin^2(2 theta), as from fermions of frequency 2 under F, which the bimodule E = Hom(V, W) does not contain; general bulk masses or other gauge form factors, as with warping; a wall scalar with its own potential; or tree-level terms at the walls. With h/f_pi = 2 theta, the leading flat form has the angular form alpha cos(h/f_pi) - beta sin^2(h/f_pi) of Agashe, Contino and Pomarol (section 2.1, Eqs. (19)-(23)) with spinorial SO(5) fermions, with alpha = A/2 and beta = -B, up to a constant. Their form factors can give the sign and competition needed for an interior minimum, so the flat results here do not exclude that construction. This points away from a pure flat Wilson line, toward those changed form factors or a wall scalar with its own potential, for the physical interior.

Comparison with measured values in the flat toy

These are numerical comparisons (Python), with no running and no bulk masses, in lowest levels, using the Particle Data Group 2026 values quoted by the referee, m_t = 172.60 GeV as a direct-measurement mass, M_W = 80.3625 GeV and M_Z = 91.1879 GeV, so m_t/M_W = 2.1478 and M_W^2/M_Z^2 = 0.7767. The ratio m_t/M_W reaches sqrt(8/3) at theta = 1.0589 and the measured value at theta = 1.1626, where M_W l = 0.541, so 1/l is about 148.5 GeV. There, M_W^2/M_Z^2 = 0.581. The wall-scalar model of lemma 22 covers [0.625, 1], and this family covers [0.5032, 0.625); in the wall-scalar model, fitting the measured vector ratio gives M_W l = 1.028, where the lowest no-bulk-mass fermion channel has m/M_W < 1.528. So neither flat family fits both ratios on its own, and the flat one-loop potential above does not select an interior point. Whether a tower at 1/l of this size is allowed would need the couplings and an applicable experimental limit, which we have not analyzed. The directions are bulk masses for the light fermions, warping, running of the couplings, and families combining a Wilson line with a wall scalar. These negative statements hold in the flat tree-level toy only.

Open questions

1. The interior vacuum: the effective potential with a wall scalar, or with warping, or with general bulk masses, with a gauge-invariant completion, a regulator and wall counterterms. The flat cases above have none on the grids.
2. The relation between the two roles of S^4: the base of the twistor window, and the field space of the Higgs quaternion.
3. A commutative parent: functions on S^4 x I and CP^3 x I, with the gradings as local boundary conditions.

4. The factor 3/2 between the five-dimensional normalization and the spectral action: whether the spectral action on the parent itself, with its tower, reproduces the formula of Theorem 23.6.
5. The comparison with twisted spectral triples, Remark 23.3.
6. The contribution of a real connector to the fermion tower, the degenerate antipodal window, and the scalar kinetic normalization needed for a canonically normalized Higgs field.

Numerical check

By Claude Opus 5.5, in Python (numpy and scipy), 3 Nov 2026, in a temporary directory, not kept.

- Random complex symmetric M on C^4 with l ||M|| = 1.25: Gamma_1 is an involution, Gamma_1 = exp(2 l K) Gamma_0; D_F = (1/2l) Gamma_0 log(Gamma_1 Gamma_0) to 3 x 10^-15; the principal angles over l equal the singular values of M to five digits.
- Sp(2) x SU(4), of dimension 25, with the Neumann and Dirichlet subspaces built from the stated stabilizers: the two-point boundary problem was solved as a linear system on a grid in k l. For theta = 0.2, pi/4, 1.06 and 1.3, the two lowest nonzero roots are (0.279, 0.356), (0.785, 1.107), (0.647, 0.868), (0.373, 0.479), equal to the formulas of Theorem 23.6 to three digits, with other roots at pi/2 and above. The intersection of the two stabilizers has dimension 9 at each theta tested.
- Small theta: M_W^2/m^2 = 2.000 and M_Z^2/m^2 = 3.200.
- The universal connection on E (+) E*: J D = D J, Gamma_0 D = -D Gamma_0, all |eigenvalues| equal to theta/l, Gamma_1 = (left Gamma_p, right mirror), J_1^2 = -1, J_1 Gamma_1 = -Gamma_1 J_1, while J alone fails; J_1 h J_1^-1 is right multiplication by +h on E*, for odd h.
- The theta function: monotone on a grid for s = 0.3, 1, 2, with the towerless value at s = 0.3 equal to e^{-x} to five digits; Poisson summation and the triple product to 10^-16.
- The ratio M_W^2/M_Z^2 on a grid: range [0.5032, 0.6250], symmetric under theta -> pi/2 - theta to 10^-15, minimum at pi/4. At theta = 1.1625, M_W l = 0.541, and at theta = 1.0589, M_W l = 0.648.
- The one-loop integrals by adaptive quadrature (scipy), and by Gauss-Legendre quadrature on 400 nodes for the scan with c_U = -c_Q. The referee recomputed the derivative with scipy, found one sign change for each n_f in {1, 2, 3, 4, 6, 8}, with the stationary angle about 0.3382 at n_f = 8, and checked I = (3/2) zeta(3).

Part 24

Commutative cylinders, spectral walls, and the two-stage compression

By Astra 6 (gpt-6-astra, effort xhigh), 3 Nov 2026; prompt in prompts/24_commutative_parent.md. Reproduced as delivered, with links made relative. The section "Numerical check" at the end is by Claude Opus 5.5. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): no error in the displayed equations, and one gap in the general statement on compression maps. The record above the section "Referee amendments" stays as delivered, and those amendments apply to it.

A commutative cylinder admits an exact realization of the **operator window** of lemmas 22 and 23 when the two finite gradings are prescribed through nonlocal spectral boundary conditions. The higher K-modes then decouple as the radius of K tends to zero. Two qualifications remain essential: these boundary conditions restrict scalar functions at the walls, and compression of scalar functions through both stages differs from generating a matrix algebra between the stages. In particular, aligned walls give $\mathbb{R} \oplus \mathbb{R}$ on the quaternionic factor under direct scalar compression, whereas the intervening matrix-algebra construction gives $\mathbb{H} \oplus \mathbb{H}$. Local boundary gradings preserve all scalar functions, but their reduction to the unchanged four-section window has a precise restriction proved below. The radial no-go of lemma 22 survives rechecking in its stated scope. For the massless one-loop potential of lemma 23, an analytic argument strengthens the numerical statement to all $n_f > 0$.

1. Theorem — An exact spectral boundary construction.

Hypotheses and conventions. Let

$$K = \mathbb{CP}^3, \qquad \mathcal{H}_K = L^2(K, \Lambda^{0,*}T^*K \otimes \mathcal{O}(1)),$$

and let

$$A = \sqrt{2}(\bar\partial + \bar\partial^*), \qquad \gamma_K = (-1)^{\deg}.$$

Use the Fubini–Study metric at a fixed reference scale. Write

$$V = \ker A \simeq \mathbb{C}^4, \qquad \Pi = P_K, \qquad \Delta_K = \inf(\operatorname{spec}|A| \setminus \{0\}) > 0.$$

Here $\gamma_K|_V = 1$. The radius parameter $r > 0$ replaces A by A/r, after the usual unitary identification of the L^2-spaces.

These are the analytic data of Example 16.9. Its S^4 option uses functions pulled back to this same twistor space. An actual operator on S^4 with the required kernel and compression map would need to be specified separately.

Let

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

On $L^2(I_\ell; \mathbb{C}^2 \otimes \mathcal{H}_K)$, set

$$\mathcal{D}_r = E\partial_x + Z\,A/r, \qquad \hat\gamma = Z\gamma_K.$$

All unspecified tensor factors are identities.

Choose self-adjoint involutions Γ_0, Γ_1 on V, each with two-dimensional positive and negative eigenspaces, and assume

$$-1 \notin \operatorname{spec}(\Gamma_1\Gamma_0).$$

Extend Γ_i by zero on $V^\perp$, writing $C_i = \Pi\Gamma_i\Pi$. Define $\operatorname{sgn} A = 0$ on V, and impose

$$\chi_0\psi(0) = \psi(0), \qquad \chi_\ell\psi(\ell) = \psi(\ell),$$

where

$$\chi_0 = ZC_0 - S\operatorname{sgn} A, \qquad \chi_\ell = ZC_1 + S\operatorname{sgn} A. \tag{1}$$

Then:

1. The H^1 realization of $\mathcal{D}_r$ is self-adjoint with compact resolvent and anticommutes with $\hat\gamma$.
2. $\Pi \otimes 1$ reduces the operator, including its domain.
3. On this reducing subspace, the operator is exactly $D(\Gamma_0, \Gamma_1)$ of lemma 23.
4. Every eigenvalue on the complementary subspace satisfies

$$|\lambda| \geq \Delta_K/r. \tag{2}$$

Proof. Put $B = SA/r$. Since $ES = Z$,

$$\mathcal{D}_r = E(\partial_x + B).$$

The operators χ_i are self-adjoint involutions and anticommute with E. Their positive eigenspaces are therefore maximal isotropic for the boundary form

$$[\langle\psi, E\phi\rangle]_0^\ell.$$

On $V^\perp$, condition (1) selects the negative spectral subspace of B at 0 and its positive spectral subspace at ℓ. These are APS conditions with the inward-normal sign reversed between the two walls. On $\ker B = \mathbb{C}^2 \otimes V$, condition (1) supplies a Lagrangian subspace:

$$\Gamma_i\psi_L = \psi_L, \qquad \Gamma_i\psi_R = -\psi_R.$$

For q in the complementary domain, integration by parts gives

$$\|\mathcal{D}_r q\|^2 = \|q'\|^2 + \|Aq/r\|^2 + [\langle q, Bq\rangle]_0^\ell \geq \|q'\|^2 + \frac{\Delta_K^2}{r^2}\|q\|^2. \tag{3}$$

The boundary term is nonnegative by the spectral choices.

An eigenbasis of A decomposes the complementary problem into finite-dimensional interval problems with maximal isotropic endpoint conditions. Each is self-adjoint. Estimate (3) identifies the domain of their self-adjoint direct sum with the stated H^1 domain and gives compactness of its resolvent. The finite-dimensional tangential kernel contributes an ordinary self-adjoint interval problem. This proves self-adjointness, compactness, and (2).

Every boundary projector commutes with Π, proving reduction. Finally, $\{\gamma_K, A\} = 0$ implies $\{\hat{\gamma}, \mathcal{D}_r\} = 0$, and $\hat{\gamma}$ preserves both boundary subspaces. ■

Define

$$K_F = \frac{1}{2\ell}\log(\Gamma_1\Gamma_0), \qquad D_F = \Gamma_0 K_F, \qquad R(x) = e^{xK_F},$$

with the principal logarithm. Let $P_\pm = (1 \pm \Gamma_0)/2$. The finite-window embedding is

$$W_\ell v(x) = \ell^{-1/2}\begin{pmatrix} R(x)P_+ v \\ R(x)P_- v \end{pmatrix}. \tag{4}$$

Direct differentiation gives

$$\mathcal{D}_r W_\ell = W_\ell D_F, \qquad \hat{\gamma} W_\ell = W_\ell \Gamma_0.$$

Thus the prescribed finite operator and grading are exact, provided $\Delta_K/r > \|D_F\|$. The next eigenvalue in absolute value is at least

$$\min\left\{\frac{\Delta_K}{r}, \frac{\pi - \ell\|D_F\|}{\ell}\right\}. \tag{5}$$

---

2. Proposition — Local walls and spectral walls have different scalar domain algebras.

Hypotheses. Use the product differential expression of Result 1.

For the local option, let $G_i(y)$ be smooth self-adjoint bundle involutions and impose

$$G_i\psi_L = \psi_L, \qquad G_i\psi_R = -\psi_R \quad \text{at } x = i.$$

Assume

$$[G_i, \gamma_K] = 0, \qquad [G_i, \sigma_A(y,\xi)] = 0 \quad (\xi \in T_y^*K), \tag{6}$$

where σ_A is the tangential principal symbol.

Then these local conditions give a self-adjoint elliptic realization with compact resolvent, and **every** smooth scalar function on $K \times I_\ell$ preserves its domain.

For the spectral option (1), the maximal smooth scalar $*$-algebra preserving the domain is

$$\mathcal{A}_{\mathrm{APS}} = \{f \in C^\infty(K \times I_\ell) : f(\,\cdot\,,0), f(\,\cdot\,,\ell) \text{ are constant on } K\}. \tag{7}$$

The same conclusion holds for the subalgebra pulled back from $S^4 \times I_\ell$.

Proof. For the local option, the boundary involution is ZG_i. It anticommutes with the normal matrix E. Condition (6) makes it anticommute with the principal symbol of the adapted tangential operator SA/r. Hence its positive eigenspace is an elliptic Lagrangian boundary condition. The corresponding local ellipticity and adjoint-domain criteria are described in Bär–Ballmann, Corollary 4.18 and Example 4.20. Primary reference

Scalar multiplication commutes pointwise with G_i, so it preserves the endpoint conditions and the H^1 domain. Moreover,

$$[\mathcal{D}_r, f] = E\,\partial_x f + Z[A, f]/r$$

is bounded for each $r > 0$.

For spectral walls, preservation by a scalar $*$-algebra is equivalent to

$$[M_{f_i}, \chi_i] = 0.$$

Independence of the spin matrices Z, S gives

$$[M_{f_i}, C_i] = 0, \qquad [M_{f_i}, \operatorname{sgn} A] = 0.$$

Since $C_i^2 = \Pi$, the first equation implies

$$[M_{f_i}, \Pi] = 0. \tag{8}$$

Consequently multiplication by f_i preserves $H^0(K, \mathcal{O}(1))$. If s is such a section, then

$$0 = \bar\partial(f_i s) = (\bar\partial f_i)s.$$

The sections of $\mathcal{O}(1)$ have no common zero, so $\bar\partial f_i = 0$ everywhere. A holomorphic function on compact connected $\mathbb{CP}^3$ is constant. Constants plainly preserve (1), proving (7). ■

The hypotheses in (6) are a sufficient local ellipticity condition. An arbitrary bundle grading supplies a pointwise isotropic condition, but self-adjointness on an H^1 domain and compact resolvent still require an ellipticity check. Likewise, prescribing a grading only on V leaves the boundary problem incomplete until a condition on $V^\perp$ is supplied.

---

3. Proposition — Compression of a local grading and the unchanged level-one window.

Hypotheses. Let G be a self-adjoint bundle involution acting by multiplication, and let

$$g = \Pi G \Pi|_V, \qquad Q_K(f) = \Pi M_f \Pi|_V.$$

Keep the level-one space and scalar representation of Example 16.9 unchanged.

Then

$$1_V - g^2 = \big((1-\Pi)G\Pi\big)^*\big((1-\Pi)G\Pi\big). \tag{9}$$

In particular, g is a grading exactly when $[G, \Pi] = 0$. In that event,

$$[g, Q_K(f)] = 0 \quad \text{for every scalar } f. \tag{10}$$

For either the full $\mathbb{CP}^3$ scalar algebra or the S^4 pullback algebra, this forces

$$g = 1_V \quad \text{or} \quad g = -1_V. \tag{11}$$

Proof. Expand $\Pi G^2 \Pi = \Pi$ using $1 = \Pi + (1-\Pi)$; this gives (9). Vanishing of its positive right-hand side implies $(1-\Pi)G\Pi = 0$, and self-adjointness gives the other off-diagonal block.

Since scalar multiplication commutes with G, reduction then gives (10). The compressed $\mathbb{CP}^3$ functions generate $M_4(\mathbb{C})$. The compressed real S^4 functions generate $M_2(\mathbb{H})$, whose complex-linear commutant on $\mathbb{C}^4$ is also $\mathbb{C}1$. Thus a self-adjoint involution satisfying (10) is ± 1. ■

For **aligned local walls** with $[G, A] = 0$, the finite zero-mode window is explicitly

$$v \longmapsto \ell^{-1/2} \begin{pmatrix} P_+(g)v \\ P_-(g)v \end{pmatrix}.$$

The induced grading is g, and scalar compression is

$$P_+(g)Q_K(f)P_+(g) + P_-(g)Q_K(f)P_-(g) = Q_K(f).$$

By (11), the unchanged four-section window has trivial grading. Its generated algebra remains $M_2(\mathbb{H})$ or $M_4(\mathbb{C})$, according to the scalar algebra chosen. The globally constant choices $G = \pm 1$ give concrete self-adjoint examples with this window and a separated higher spectrum.

There is also an explicit local field illustrating failure of reduction. Add a trivial auxiliary $\mathbb{C}^4$ bundle and put

$$G(y) = \sum_{a=1}^{5} y_a \Gamma_a^{\mathrm{aux}},$$

where $y \in S^4$ is the twistor projection and the Γ_a satisfy $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$. This is a local involution satisfying (6), but now the tangential kernel has dimension 16, and

$$g = \frac{1}{5} \sum_{a=1}^{5} \Gamma_a^{\mathrm{section}} \otimes \Gamma_a^{\mathrm{aux}}.$$

Using $\mathrm{tr}(\Gamma_a \Gamma_b) = 4\delta_{ab}$,

$$\mathrm{tr}(g^2) = \frac{16}{5} \neq 16.$$

Thus g is not a grading. Equation (9) even gives

$$\|(1 - \Pi)G\Pi\| \geq \frac{2}{\sqrt{5}},$$

independently of r. This is a bound on boundary-trace mixing; it does not identify the actual low spectral window of the coupled local problem.

A local $G(y)$ specifies a **field of gradings**. It does not select a marked point p. In the tautological quaternionic line bundle,

$$\Pi_y = \frac{1 + y_a \Gamma_a}{2},$$

the natural field $2\Pi_y - 1$ acts as $+1$ on the fibre $\operatorname{ran} \Pi_y$. By contrast, the finite grading

$$\Gamma_p = p_a \Gamma_a = 5Q_K(p \cdot y)$$

uses a separately chosen point p. Its action on global section coefficients is nonlocal.

The restriction proved here concerns the unchanged level-one space, a local multiplication grading, and its compression as an actual involution. A different coupled window, additional fibres, or spectral boundary conditions remain available directions.

---

4. Theorem — The composed compression map and the aligned-wall algebra.

Hypotheses. Use the exact spectral construction of Result 1. For a scalar multiplier $f(y, x)$, let

$$T_f(x) = Q_K(f(\,\cdot\,, x)), \qquad \mathcal{E}_0(T) = \frac{1}{2}(T + \Gamma_0 T \Gamma_0).$$

Use real-valued functions and real algebra generation when discussing $M_2(\mathbb{H})$.

Then the compression directly from the scalar cylinder to the finite window is

$$\boxed{\Phi_\ell(f) = \frac{1}{\ell} \int_0^\ell \mathcal{E}_0\big(R(x)^* T_f(x) R(x)\big)\, dx.} \tag{12}$$

It is independent of r.

For aligned walls $\Gamma_1 = \Gamma_0 = \Gamma_p$, direct scalar compression on the S^4 factor generates

$$\mathbb{R} \oplus \mathbb{R}. \tag{13}$$

If instead one first generates $M_2(\mathbb{H})$ and then applies the interval compression, the result is

$$\mathbb{H} \oplus \mathbb{H}. \tag{14}$$

Proof. Insert (4) on both sides of the multiplication operator. Orthogonality of the two interval spin components gives exactly (12).

For real S^4 functions, Example 16.9 implies

$$Q_K(C^\infty(S^4, \mathbb{R})) = \mathrm{span}_{\mathbb{R}}\{1, \Gamma_1, \dots, \Gamma_5\}. \tag{15}$$

Indeed all compressed functions are quaternionic Hermitian matrices; that space has real dimension six, and the displayed six matrices belong to the image and are independent.

For aligned walls $R = 1$. Clifford multiplication gives

$$\mathcal{E}_p(\Gamma_a) = p_a \Gamma_p.$$

Therefore the image of direct scalar compression is

$$\mathrm{span}_{\mathbb{R}}\{1, \Gamma_p\}.$$

This space is already the algebra $\mathbb{R} \oplus \mathbb{R}$, proving (13).

On the other hand,

$$\mathcal{E}_p(M_2(\mathbb{H})) = \{a \in M_2(\mathbb{H}) : [a, \Gamma_p] = 0\} = \mathbb{H} \oplus \mathbb{H},$$

proving (14). The noncommuting products needed to generate the quaternionic diagonal blocks have been inserted between the two compressions in this second procedure. ■

On the single $\mathbb{CP}^3$ factor, real scalar compression is onto the Hermitian 4×4 matrices. A chosen $2+2$ aligned grading therefore gives all Hermitian block-diagonal matrices and generates

$$M_2(\mathbb{C}) \oplus M_2(\mathbb{C}).$$

For the **chosen two-sheet coarse correspondence** of Example 16.9, with

$$H_0 = \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W, V), \qquad \Gamma_0(\xi, \eta) = (\Gamma_p \xi, -\eta \Gamma_p),$$

the same calculation gives

$$\boxed{\mathrm{alg}_{\mathbb{R}}(\text{direct aligned scalar compression}) = \mathbb{R} \oplus \mathbb{R} \oplus M_4(\mathbb{C}).} \tag{16}$$

The V-side matrices act on the left of η and commute with its right grading, so that factor survives unchanged.

Endpoint restrictions do not change these finite-dimensional images. If a linear functional vanishes on compressions of functions supported in $0 < x < \ell$, its smooth matrix-entry density vanishes in the interior and hence everywhere. Finite-dimensional separation then gives equality with the unrestricted image, exactly as in Result 7 of lemma 20.

At the intermediate stage, however, spectral walls still give scalar endpoint values. Even after algebra generation, those values remain scalar. Enlarging them to the full stabilizers of Γ_0, Γ_1, as in lemma 23, is another change of the parent algebra. Equality of finite compression images does not identify the parent gauge groups or one-forms.

5. Theorem — Radius collapse, interval collapse, and controlled mixing.

Hypotheses. First use the exact construction of Result 1. Identify the varying Hilbert spaces by metric rescaling. For resolvent formulas, fix units in which the nonreal spectral parameter is i.

Then, at fixed ℓ,

$$\left\|(\mathcal{D}_r - i)^{-1} - \iota_\Pi (D(\Gamma_0, \Gamma_1) - i)^{-1} \iota_\Pi^*\right\| \le \frac{r}{\Delta_K}. \tag{17}$$

Thus radius collapse gives the entire matrix interval operator in generalized norm-resolvent sense.

If D_F is fixed, $\ell \to 0$, and

$$\Gamma_1(\ell) = e^{\ell \Gamma_0 D_F} \Gamma_0 e^{-\ell \Gamma_0 D_F},$$

then $r \to 0$ and $\ell \to 0$ decouple both towers and leave the finite operator D_F. No relation between the two rates is required in this exactly reducing construction.

Proof. The resolvent is block diagonal with respect to Π. Estimate (2) bounds its complementary block by r/Δ_K, proving (17). For the finite-window limit, (5) bounds the complementary resolvent by

$$\max\left\{\frac{r}{\Delta_K}, \frac{\ell}{\pi - \ell\|D_F\|}\right\},$$

which tends to zero. ■

A useful mixing statement requires additional hypotheses. Suppose a transformed problem has a common domain and the form

$$\widetilde{\mathcal{D}}_r = \mathcal{D}_r + V_r, \qquad V_r = V_r^*, \qquad \|V_r\| \le b_r.$$

Write

$$v_r = \|(1 - \Pi) V_r \Pi\|.$$

For an eigenvector $\psi = p + q$, with $p = \Pi\psi$, $q = (1 - \Pi)\psi$, and $|\lambda| \le L$, assume

$$g_r = \Delta_K / r - b_r - L > 0.$$

Then

$$\boxed{\|q\| \le \frac{v_r}{g_r}\|p\|, \qquad \|\text{Schur correction at } \lambda\| \le \frac{v_r^2}{g_r}.} \tag{18}$$

Proof. The complementary eigenvalue equation gives

$$q = -(\mathcal{D}_{r,Q} + V_{QQ} - \lambda)^{-1} V_{QP} p.$$

The inverse has norm at most g_r^{-1}. Substituting this formula into the p-equation gives the second bound. ■

For bounded b_r, v_r, both errors are $O(r/\Delta_K)$. If a common-domain construction instead produces terms of order $1/\ell$, sufficient conditions include

$$\frac{r v_r}{\Delta_K} \to 0, \qquad \frac{r v_r^2}{\Delta_K} \to 0,$$

together with convergence of the Π-diagonal perturbation.

An arbitrary local $G_i(y)$ generally fails $[G_i, \Pi] = 0$. Although the differential expression commutes with Π, its domain then does not. Such a boundary change is not automatically a bounded perturbation satisfying (18). The explicit local field in Result 3 has nonvanishing boundary mixing at every radius. No convergence to the prescribed two-gradings operator is established here for that nonreducing local problem.

At the level of observables, radius collapse leaves (12) unchanged. For an x-independent $T = Q_K(f)$,

$$\|\Phi_\ell(f) - \mathcal{E}_0(T)\| \le \ell \|D_F\| \, \|T\|. \tag{19}$$

If T is even for Γ_0, the stronger formula of Theorem 22.4 applies:

$$\Phi_\ell(f) = \operatorname{sinc}(\ell \operatorname{ad}_{D_F})(T),$$

$$\|\Phi_\ell(f) - T\| \le \frac{\ell^2}{6} \|[D_F, [D_F, T]]\|. \tag{20}$$

Equation (19) follows by integrating $\|R^* T R - T\| \le 2x \|D_F\| \|T\|$; equation (20) follows by taking the even part of the conjugation series.

Consequently there is convergence of the **compression maps on their scalar operator systems**. The limit is $\mathcal{E}_0 Q_K$. It does not turn that image into every element of $\mathcal{E}_0(P)$; Result 4 computes the difference explicitly. Algebra generation remains a separate operation.

---

6. Proposition — The second sheet, reality, even algebra, and connector remain prescribed data.

Hypotheses. Consider the scalar cylinder construction above and, separately, its extension using the two-sheet coarse correspondence of Example 16.9.

The resulting distinctions are:

Structure	What follows under the stated choices
Second sheet and the two Hom spaces	They are inputs in the coarse correspondence. A single four-section cylinder window does not supply them.
$J_1^2 = -1$, $J_1 \Gamma_i = -\Gamma_i J_1$	These are additional compatibility conditions on the prescribed gradings and antiunitary.
$\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$	This is the derived even part after choosing the quaternionic real form, the grading, and intermediate algebra generation. Direct aligned scalar compression gives (16).
Connector and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$	The connector's lines and coupling are chosen. Their intertwining algebra is then derived.

Proof. The one-factor construction starts with $V = \mathbb{C}^4$, and (4) has domain V. The two-sheet construction instead starts with

$$(W \otimes \overline{V}) \oplus (V \otimes \overline{W}).$$

Geometrically, its analytic realization uses product factors and a restricted scalar algebra, as Theorem 16.7 specifies. It is additional correspondence data over the scalar spaces.

For reality, suppose an antiunitary $\mathcal{J}_1$ on the tangential Hilbert space satisfies

$$\mathcal{J}_1^2 = -1, \quad \mathcal{J}_1 A = -A \mathcal{J}_1, \quad [\mathcal{J}_1, \gamma_K] = 0,$$

and its restriction J_1 to V anticommutes with both Γ_i. Then

$$\widehat{J} = E \mathcal{J}_1$$

preserves (1) and obeys

$$\widehat{J}^2 = 1, \qquad [\widehat{J}, \mathcal{D}_r] = 0, \qquad \widehat{J} \widehat{\gamma} = -\widehat{\gamma} \widehat{J}.$$

On the window,

$$J_F = \Gamma_0 J_1.$$

These identities follow directly from the stated commutation relations. They specify sufficient conditions; the natural $\mathcal{O}(1)$ cylinder has not supplied such an extension or its order-zero property here.

In particular, the quaternionic antiunitary j used to recognize $M_2(\mathbb{H})$ commutes with the Γ_a. It does not itself satisfy the required anticommutation with Γ_p.

The even-algebra assertion is the centralizer calculation in Result 4 after intermediate generation.

Finally, choose

$$w \in W_+, \quad v \in V, \quad \|w\| = \|v\| = 1, \quad \kappa \neq 0,$$

and prescribe

$$T_\kappa \xi = \kappa (w^* \xi v)\, v w^*.$$

Its intertwining conditions require the lines $\mathbb{C}w, \mathbb{C}v$ to be reducing, with the same scalar action. A quaternion preserving $\mathbb{C}w$ lies in its chosen copy of $\mathbb{C}$; $v^\perp$ carries arbitrary $M_3(\mathbb{C})$, and W_- carries arbitrary $\mathbb{H}$. Hence

$$\mathcal{A}(T_\kappa) = \{(\lambda \oplus q, \lambda \oplus m)\} \simeq \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),$$

as in Result 2 of lemma 20. ■

A marked $p \in S^4$ fixes a quaternionic line, but choosing w also requires a complex line within it. The second line v, the coupling, and an interface realizing the connector remain necessary. Scalar agreement at marked points has unchanged compression image by Result 7 of lemma 20, so it supplies none of this selection by itself.

---

7. No-go theorem — Recheck of the radial statement in lemma 22.

Hypotheses. Let the parent be exactly the one-interval model of lemma 22: fixed wall coupling X, zero bulk mass, the even algebra $\mathcal{A}_\ell$, scalar inner fluctuations Eh with $h^* = -h$, and the **moving spectral window**. For real fluctuations, include the conjugate one-form and assume the stated order-zero representation.

Then every fluctuated finite operator has the same singular values as D_F. This is an exact finite-amplitude spectral statement; it uses no action or loop expansion.

Proof. Solve

$$u' = -hu, \qquad u(0) = 1, \qquad w = u(\ell),$$

and set $v(x) = u(x)w^*$. Then

$$v(\ell) = 1, \qquad v(0) = w^*.$$

Since h is even, w preserves the boundary splitting at 0; the identity at ℓ preserves the fixed coupled wall. Thus v preserves the original domain, and

$$v D_X v^* = D_X + Eh.$$

For the real fluctuation, the analogous implementing unitary is

$$v J_F v J_F^{-1};$$

order zero separates the two factors in its derivative.

The full operators are unitarily equivalent, so their moving spectral windows are unitarily equivalent. In the finite identifications of lemma 22, the operators are $w^* D_F w$, or $U_w^* D_F U_w$ in the real case. Their singular values coincide. ■

The moving-window hypothesis matters. For example, with $D_F = m\sigma_1$ and $h = i\sigma_3$, fixed compression gives

$$Q_\ell(Eh) = -\frac{1 - \cos(2m\ell)}{2m\ell}\sigma_2.$$

Adding a multiple of this operator to the fixed $m\sigma_1$ changes its singular values at second order.

The verdict agrees with the author's amended theorem and Terra's independent endpoint-preserving proof. In the usual nondegenerate Standard Model representation, the normalized shifted Higgs quaternion lies on the unit-quaternion orbit; its norm is fixed.

The known escapes are a variable wall coupling, odd connection directions, suitable bulk masses, additional internal directions with nontrivial connection data, or additional elementary or composite fields. A quantum effective action on the same configuration space does not add a radial coordinate. The proposed $K \times I_\ell$ parent has tangential directions and tangential one-forms, so this theorem does not decide its full fluctuation space.

---

8. Lemma — A logarithmic concavity property of the one-loop integral.

Hypotheses. Let

$$\mathcal{J}(z) = \int_0^\infty \frac{x^3}{\sinh^2 x + z}\, dx, \qquad z > 0,$$

and define

$$e(z) = -\frac{z\mathcal{J}'(z)}{\mathcal{J}(z)}.$$

Then $0 < e(z) < 1$, and e is nondecreasing.

Proof. Positivity and

$$0 < \frac{z}{\sinh^2 x + z} < 1$$

show that $e(z)$, expressed as a weighted average of this fraction, lies strictly between zero and one.

Put $x = e^v$, $z = e^u$. Then

$$\mathcal{J}(e^u) = \int_{\mathbb{R}} \exp\left(4v - \log(\sinh^2(e^v) + e^u)\right)\, dv. \tag{21}$$

The function

$$b(v) = 2\log\sinh(e^v)$$

is convex, because, with $x = e^v$,

$$b''(v) = 2x(\coth x - x\ \operatorname{csch}^2 x) > 0.$$

The last inequality is equivalent to $\sinh x \cosh x > x$.

The function $(u, v) \mapsto \log(e^{b(v)} + e^u)$ is therefore convex. The integrand in (21) is jointly log-concave. Marginal integration preserves log-concavity by Prékopa's theorem, so $u \mapsto \log \mathcal{J}(e^u)$ is concave. We use precisely Theorem 6, p. 342, of Prékopa's original paper.

Finally,

$$\frac{d}{du}\log \mathcal{J}(e^u) = -e(e^u).$$

Concavity makes this derivative nonincreasing, proving the assertion. ■

---

9. Theorem — The massless one-loop potential has no interior local minimum.

Hypotheses. Let the action be exactly the one-loop determinant displayed in lemma 23: a flat interval, zero bulk masses, frequency-one fermions, the stated charged and neutral gauge towers and multiplicities, and the prescribed connector wall condition. Assume that no fermion connector modifies the displayed fermion tower, and add no wall potential or other angle-dependent term.

Let $n_f > 0$, set $t = \sin^2\theta$, and write

$$\mathcal{V}(t) = 8\pi^2\ell^4 V(\theta).$$

Then:

1. $t = 1$, equivalently $\theta = \pi/2$, is the unique global minimum.
2. If $n_f \geq 54/5$, the potential is strictly decreasing on the interior.
3. If $0 < n_f < 54/5$, there is exactly one interior stationary point, and it is a strict maximum.

Proof. Define

$$F(z) = \int_0^\infty x^3 \log\left(1 + \frac{z}{\sinh^2 x}\right) dx, \qquad F'(z) = \mathcal{J}(z).$$

The exact potential is

$$\mathcal{V}(t) = 3F(2t(1-t)) + \frac{3}{2}F\Big(\frac{16}{5}t(1-t)\Big) - n_f F(t). \tag{22}$$

Its gauge terms are nonnegative and vanish at $t = 1$. Since F is strictly increasing, the fermion term is uniquely minimized there. This proves item 1.

Differentiate:

$$\mathcal{V}'(t) = (1-2t)\left[6\mathcal{J}(2t(1-t)) + \frac{24}{5}\mathcal{J}\Big(\frac{16}{5}t(1-t)\Big)\right] - n_f\mathcal{J}(t). \tag{23}$$

For $t \geq 1/2$, this is strictly negative.

For $0 < t < 1/2$ and $c \geq 2$, put

$$R_c(t) = (1-2t)\frac{\mathcal{J}(ct(1-t))}{\mathcal{J}(t)}.$$

Let $a = ct(1-t)$. Then $a \geq t$, and Result 8 gives $e(a) \geq e(t)$. Consequently,

$$\begin{aligned}\frac{d}{dt}\log R_c(t) &= -\frac{2}{1-2t} - \frac{1-2t}{t(1-t)}e(a) + \frac{e(t)}{t} \\ &\leq -\frac{2}{1-2t} + \frac{e(t)}{1-t} \\ &< -\frac{2}{1-2t} + \frac{1}{1-t} < 0.\end{aligned}$$

Thus every R_c is strictly decreasing. Since $\mathcal{J}(0)$ is finite and positive,

$$R_c(0+) = 1, \qquad R_c(1/2-) = 0.$$

Therefore

$$H(t) = 6R_2(t) + \frac{24}{5}R_{16/5}(t)$$

decreases strictly from $54/5$ to zero. Equation (23) becomes

$$\mathcal{V}'(t) = \mathcal{J}(t)\,[H(t) - n_f] \quad (0 < t < 1/2).$$

For $n_f \geq 54/5$, its sign is negative. For $0 < n_f < 54/5$, it changes sign exactly once, from positive to negative. Since $t = \sin^2\theta$ is strictly increasing on the interior, the same stationary-point classification holds in θ. ■

This proves the massless statement for the entire parameter range, including the previously numerical range. No numerical computation is used in this proof.

The recorded author and Terra referee calculations agree on the sampled massless cases. The logarithmic-concavity proof above is an additional single-agent derivation; it has not received a fresh independent model review in this turn. The recorded agreement therefore should not be described as a multi-model review of this new proof.

The escapes remain changes to the stated determinant: frequency-two fermions, modified form factors through warping or other geometry, suitable bulk-mass sectors, wall potentials, and wall kinetic or interaction terms. The full effective action of the proposed $K \times I_\ell$ parent, including its K-tower, has not been computed here.

10. Proposition — Scope of the bulk-mass scans.

Hypotheses. Retain the flat gauge terms of (22) as an independent assumption. Replace the fermion factor by the expression in lemma 23:

$$1 + t(G(x) - 1), \qquad G(x) = \frac{(w_Q \coth w_Q - c_Q)(w_U \coth w_U - c_U)}{x^2}, \quad w_i = \sqrt{x^2 + c_i^2}.$$

If $c_U = -c_Q = c$, then the potential is finite and $t = 1$ is its unique global minimum for every $n_f > 0$. The reported absence of additional interior **local** minima at nonzero c remains a numerical statement here.

If $a = c_Q + c_U \neq 0$, the displayed unrenormalized integral has angle-dependent ultraviolet divergences.

Proof. For opposite masses,

$$G(x) = 1 + \frac{w^2}{x^2 \sinh^2 w}, \qquad w = \sqrt{x^2 + c^2}.$$

The added fraction is positive and decays exponentially at infinity. The negative fermion logarithm is strictly decreasing in t. The gauge logarithms are nonnegative and vanish at $t = 1$, proving the global-minimum assertion.

For general masses,

$$G(x) = 1 - \frac{a}{x} + \frac{a^2}{2x^2} + O(x^{-3}),$$

and hence

$$\log(1 + t(G - 1)) = -\frac{ta}{x} + \frac{t(1-t)a^2}{2x^2} + O(x^{-3}).$$

Multiplication by the integration measure $x^3 dx$ produces angle-dependent divergences. A finite potential therefore requires a specified completion, regulator, and counterterms. ■

The opposite-mass scan in lemma 23 samples finitely many masses, multiplicities, and angles. It cannot exclude unsampled local minima. General bulk masses and altered gauge form factors remain directions for constructing an interior vacuum; these conclusions leave that question open.

Consequence for T0 and T0b

For T0, a commutative scalar representation with prescribed spectral walls gives an exact interval operator window and controlled tower decoupling, while local walls preserve the full scalar algebra under the ellipticity hypotheses above. The derivation of a nontrivial local grading and the desired algebra from one coupled scalar window remains open, and intermediate algebra generation must be stated explicitly. For T0b, the second sheet, compatible reality, quaternionic grading choice, and connector remain inputs; their algebraic consequences can be derived once those choices are supplied. Neither the radial theorem nor the specified one-loop potential closes the commutative-parent problem.

Referee amendments

From Terra 6.1. These apply to the text above.

- **Result 2.** Write G_0 at x = 0 and G_l at x = l. Equation (6) supplies the principal-symbol condition of Bar-Ballmann, Corollary 4.18 and Theorem 4.11; their Example 4.20 assumes more. On the unaugmented Dolbeault bundle, the Clifford representation is irreducible, so a fibre endomorphism commuting with every sigma_A is scalar, and a smooth involution of this form is +1 or -1. Additional fibres and other local elliptic projectors lie outside this sufficient criterion.
- **Result 3.** The statement on marked points concerns the natural field displayed there. A local field can encode a chosen point: on the auxiliary bundle, G_p(y) = p_a Gamma_a^aux is a local constant field whose coefficients contain p.

- **Result 4.** Equation (16) has the two Hom spaces and the two scalar algebras of Example 16.9 as inputs. The endpoint argument proves equality of the finite-dimensional linear images, and does not identify the parent algebras, norms, gauge groups or one-forms.
- **Result 5, gap in the final assertion.** Convergence of compression maps needs a common domain and a collapse map. With aligned walls, D_F = 0 and f_l(y, x) = 6 (x/l)(1 - x/l), both endpoint values vanish while Phi_l(f_l) = 1. A correct form: for F(y, s) fixed on K x [0, 1] and f_l(y, x) = F(y, x/l), Phi_l(f_l) tends to E_0 Q_K(int_0^1 F(., s) ds). For x-independent ambient multipliers the limit is E_0 Q_K, by (19). The rate discussion after (18) also needs control of the denominator: a sufficient addition is r (b_r + L) <= (1 - epsilon) Delta_K. The case b_r = O(1/l) follows from the stronger estimate |D_{r,Q}| >= (Delta_K^2/r^2 + pi^2/(4 l^2))^{1/2}, from the one-ended Poincare inequality on the positive and negative B-spectral components.
- **Result 6.** At the finite level, if the gradings are Gamma_p, Gamma_q, a unit n orthogonal to p and q gives J_1 = Gamma_n j, with J_1^2 = -1 and the required anticommutation. Extending this choice to the parent remains a separate task.
- **Lemma 8 and Theorem 9.** An independent check of the key monotonicity, without log-concavity or Prekopa's theorem: with rho(s) = (arsinh sqrt(s))^3/(2 sqrt(s (1 + s))), J(z) = int_0^infinity rho(s)/(s + z) ds; with L(v) = log rho(e^v), L'(v) = 3 tanh(x)/(2x) - (1 + tanh^2 x)/2 for v = 2 log sinh x, which is strictly decreasing. For z = e^u, with the probability density in y proportional to e^y rho(e^{u+y})/(1 + e^y), e(e^u) is the mean of 1/(1 + e^y), and its derivative in u is Cov(1/(1 + e^y), L'(u + y)) > 0, as both are strictly decreasing. Theorem 9 is for theta in [0, pi/2]. At theta = pi/2 the determinant has its continuous value, while the principal-logarithm window is degenerate. Its hypotheses are the flat massless one-loop determinant, its gauge multiplicities and connector wall condition, frequency-one fermions, no fermion-connector modification, and no other angle-dependent term.
- **Result 10.** The divergence is stated for nonzero fermion multiplicity, with no cancellation by additional sectors. The opposite-mass scan in lemma 23 is a numerical result on its sampled parameters.

Numerical check

By Claude Opus 5.5, in Python (numpy, scipy), 3 Nov 2026, in a temporary directory, not kept: E_p(Gamma_a) = p_a Gamma_p to 10^-15, which gives equation (13); tr(g^2) = 16/5 for the auxiliary field of Result 3; on grids, e(z) lies in (0, 1) and is nondecreasing for z in [10^-3, 50]; H(t) is strictly decreasing on (0, 1/2), with H near 54/5 at small t and near 0 at t = 1/2; for n_f in {1, 3, 6, 8, 10}, H - n_f has one sign change, with the stationary angle 0.34 at n_f = 8, which agrees with the referee of lemma 23. Results 1, 2, 5 and 6 were read and not separately computed.

Part 25

The weak-boson sum rule and the effective dimension of the interval

For STRATEGY T5b, T13 and T1, following lemma 18, lemma 22 and lemma 23. Proved by Claude Opus 5.5, with checks in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): two errors and two gaps as first written, and amendments, all incorporated below.

Outcome. Unfolded, the interval parent is a circle of length 2l, and the parent operator is -i d/dx (x) 1 + 1 (x) D_F: the tower is D_F shifted by the momenta k_n = n pi/l. So the exact quadratic kernel of lemma 18 applies termwise over the tower. For small masses, at leading order, for the heat cutoff, the Taylor mass of a vector with constant profile in the transported frame is

```
M^2 = ((3 - d_eff)/2) ||[D_F, T]||^2 / ||T||^2,      d_eff(s) = d log S / d log s,      S(s) =
↪  sum_{n in Z} exp(-n^2 pi^2/s^2),   s = l Lambda,
```

in Hilbert-Schmidt norms, where d_eff is the spectral dimension at the scale of the cutoff of the unfolded circle, equivalently of the paired Neumann and Dirichlet interval: 0 for l Lambda « 1 and 1 for l Lambda » 1. For the Standard Model representation, at the same order,

```
M_W^2 = ((3 - d_eff)/24) sum_f N_f m_f^2,       m_t/M_W = (8/(3 - d_eff))^{1/2} for the top alone,
```

with M_W^2/M_Z^2 = 5/8 for all s. So the relation 8 M_W^2 = sum N_f m_f^2 of the asymptotic spectral action is the zero-dimensional value, with m_t/M_W = sqrt(8/3). A resolved interval gives 12 M_W^2 = sum N_f m_f^2, with m_t = 2 M_W, as five-dimensional covariance requires for this operator on the circle. The endpoint m_t = 2 M_W is known in flat gauge-Higgs models with suitable representations (references below); in the inspected records the referee did not find the coefficient (3 - d_eff)/24 or its crossover, and priority remains open pending a wider search. The factor 3/2 in Proposition F of the paper, the finite-action coefficient relative to the Yang-Mills commutator quotient, is the ratio of these two endpoints. The crossover is at l Lambda about 2.

Unfolding

Let D_X = E d/dx be the parent of lemma 22, and let D_0 + E Gamma_F D_F be its form after the transport of Corollary 22.2, where D_0 is the operator with X = 0.

Proposition 25.1. On the circle x in [-l, l] with periodic conditions, for psi in the domain of D_0, put

```
Psi_Q(x) = Q_L(|x|) + i sgn(x) Q_R(|x|),      Psi_U(x) = U_R(|x|) - i sgn(x) U_L(|x|).
```

After multiplying by 1/sqrt(2), this is a unitary from the parent Hilbert space onto L^2(circle) (x) (H_+ (+) H_-), which maps the H^1 domain of D_0 onto the periodic H^1 domain, and it carries D_0 + E Gamma_F D_F to

```
-i d/dx (x) 1 + 1 (x) D_F.
```

So the signed spectrum is k_n + mu_i, with k_n = n pi/l for n in Z and mu_i the eigenvalues of D_F. An even operator T = T_Q (+) T_U on H_F, constant in x in this transported frame and acting identically on both spin components, is carried to 1 (x) T.

Proof. Since Q_R(0) = Q_R(l) = 0 and U_L(0) = U_L(l) = 0, the functions Psi are continuous and periodic, and every smooth function on the circle arises from its even and odd parts. For x > 0, -i d/dx Psi_Q = Q_R' - i Q_L' = (D psi)_L + i (D psi)_R on Q, and -i d/dx Psi_U = -i U_R' - U_L' = (D psi)_R - i (D psi)_L on U, which are the transforms of D_0 psi; x < 0 is the same with the odd parts. For the connection term, with M* the block of D_F, the term E Gamma_F D_F sends Psi_Q to M* Psi_U and Psi_U to M Psi_Q. ■

In this form D_F is a constant connection on the circle, with holonomy exp(2 i l D_F) up to the convention for parallel transport, and the two walls are the fixed points of the reflection that carries D_F to -D_F. With U = exp(-i pi Gamma_0/4), the identity of lemma 23 reads Gamma_1 Gamma_0 = exp(2 l Gamma_0 D_F) = U exp(2 i l D_F) U*. This form of the interval as a circle with a reflection is the usual orbifold description. It also gives a one-line proof of the spectrum in Theorem 22.1.

The frame. By Corollary 22.2, D_X = R (D_0 + E Gamma_F D_F) R*, with R(x) = exp(x Gamma_F D_F). A generator T constant after transport is R(x) T R(x)* in the original frame of D_X, which generally varies with x and has an odd bulk component. Conversely, a constant generator in the original frame need not preserve the coupled wall, if it is broken. So the field used here requires an admissible enlarged gauge algebra, as in the prealgebra P of lemma 23, and it is generally not a field of the even algebra of lemma 22.

The kernel over the tower

Let A_mu(y) = a_mu(y) T on M^4 = R^4, with T even and constant along x, and consider D = -i gamma^mu (d_mu + i A_mu) + gamma_5 V with V = -i d/dx (x) 1 + 1 (x) D_F. Theorem 1 of lemma 18 has this form, with V_0 = V, whose eigenvalues are k_n + mu_i, and T_{n i, n' j} = delta_{n n'} T_{ij}.

Theorem 25.2. With d_ij = mu_i - mu_j, for the heat cutoff t = 1/Lambda^2, the kernel is

```
C_t(q^2) = (1/(8 pi^2)) sum_{n in Z} sum_{ij} |T_ij|^2 [ (2/t + d_ij^2 + q^2) I_t((k_n + mu_i)^2,
↪  (k_n + mu_j)^2; q^2) - (e^{-t (k_n + mu_i)^2} + e^{-t (k_n + mu_j)^2})/t ],
```

with I_t as in lemma 18. The sum converges, and it holds without expansion in D_F. Expanding in D_F, with C_t = C_0 + q^2 C_1 + …, and with t = 1, so with k_n = n pi/(l Lambda) dimensionless,

```
C_0 = (1/(8 pi^2)) ||[D_F, T]||^2 sum_n e^{-k_n^2} (1 - (2/3) k_n^2) + O(||D_F||^4),
C_1 = (1/(8 pi^2)) (2/3) ||T||^2 sum_n e^{-k_n^2} + O(||D_F||^2).
```

So, with <k^2> = sum k_n^2 e^{-k_n^2} / sum e^{-k_n^2}, at fixed s and for fixed nonzero T,

```
M^2 = C_0/C_1 = (3/2 - <k^2>) ||[D_F, T]||^2/||T||^2 + O(||D_F||^4 / Lambda^2).
```

Proof. V and T split into finite-dimensional blocks indexed by n, so equation (1) of lemma 18 applies to symmetric finite sets of blocks, and the limit is justified by Gaussian bounds, uniform for D_F in a bounded neighborhood of zero. The same bounds justify the Taylor expansion under the sum. Terms odd in k cancel between n and -n: exactly, the bracket at q = 0 is d_ij^2 int_0^1 E(u) [1 - u(1-u)(2k + mu_i + mu_j)^2] du, with E(u) = exp(-(1-u)(k + mu_i)^2 - u (k + mu_j)^2), whose next term is d_ij^2 e^{-k^2} ((2k^3 - 5k)/3)(mu_i + mu_j), and the bracket at order q^2 begins e^{-k^2} (2/3 - (2k/3)(mu_i + mu_j)). For the leading term of the expansion, put a_i = (k + mu_i)^2 = k^2 + p mu_i + mu_i^2 with p = 2k. To second order in mu,

```
I(a_i, a_j) = e^{-k^2} [1 - (p/2)(mu_i + mu_j) - (mu_i^2 + mu_j^2)/2 + (p^2/6)(mu_i^2 + mu_j^2 +
↪  mu_i mu_j)],
```

and e^{-a_i} = e^{-k^2} [1 - p mu_i - mu_i^2 + p^2 mu_i^2/2]. Therefore

```
2 I(a_i, a_j) - e^{-a_i} - e^{-a_j} = -(p^2/6) d_ij^2 e^{-k^2},
```

and adding d_ij^2 I gives d_ij^2 e^{-k^2} (1 - p^2/6) = d_ij^2 e^{-k^2} (1 - (2/3) k^2). Finally, sum_ij |T_ij|^2 d_ij^2 = ||[D_F, T]||^2. For C_1, at D_F = 0 the bracket in equation (3) of lemma 18 has d_ij = 0 and integrates to 2/3. ■

Differentiating directly, <k^2> = (1/2) d log S/d log s = (s/2) d log S/ds, with S(s) = sum_n exp(-n^2 pi^2/s^2). This S is the heat trace of the Dirac operator of the unshifted circle of length 2l at t = 1/Lambda^2, equivalently the paired Neumann and Dirichlet trace on the interval, with S_N = (S + 1)/2 and S_D = (S - 1)/2 separately. So d_eff = d log S/d log s = -2 d log Tr e^{-t D^2}/d log t is its spectral dimension at that scale. For nonzero D_F, the parent trace has e^{-t (k_n + mu_i)^2}, and S is the unperturbed trace used in the leading expansion. Hence, at leading order,

```
M^2 = ((3 - d_eff)/2) ||[D_F, T]||^2/||T||^2.
```

The two limits. We first take the coefficient at small D_F/Lambda at fixed s, and then the limit in s. Explicitly, d_eff ~ (4 pi^2/s^2) exp(-pi^2/s^2) as s -> 0, and d_eff = 1 - 4 s^2 exp(-s^2) + O(s^2 exp(-2 s^2)) as s -> infinity.

1. For s « 1, only n = 0 contributes. Then d_eff = 0, the factor is 3/2, and this is the towerless regime of lemma 18 and Proposition F of the paper.
2. For s » 1, the sum is a Gaussian integral, with <k^2> = 1/2. Then d_eff = 1, and the factor is 1. This is forced by covariance on the unfolded circle: V is the fifth component of a five-dimensional Dirac operator with constant A_5 = D_F, and the local five-dimensional heat coefficient contains F_{MN} F^{MN}, with F_{5 mu} = i [D_F, A_mu] up to convention and with the same coefficient as F_{mu nu}^2. So the mass is the Hilbert-Schmidt commutator norm alone, as at small angle for the Yang-Mills parent of Theorem 23.6. On a general space with boundary, the bulk coefficient alone does not exclude boundary contributions; here the operator with these transported fields is unitarily equivalent to the circle problem, which is the reason they are absent.

In table form, at small masses:

s = l Lambda	0.3	1	1.5	2	2.5	3	5
(3 - d_eff)/2	1.5000	1.4990	1.3935	1.1414	1.0240	1.0022	1.0000

Corollary 25.3 (the sum rule). For the three-generation Standard Model representation, with the literal particle and antiparticle trace, with Dirac masses small compared with Lambda and with 1/l, and with T a charged generator of SU(2)_L, at leading order,

```
M_W^2 = ((3 - d_eff)/24) sum_f N_f m_f^2,
```

which at d_eff = 0 is 8 M_W^2 = sum N_f m_f^2, and at d_eff = 1 is 12 M_W^2 = sum N_f m_f^2. For the top alone,

```
m_t/M_W = (8/(3 - d_eff))^{1/2},
```

from sqrt(8/3) = 1.633 at d_eff = 0 to 2 at d_eff = 1. The neutral direction scales with the same factor, so M_W^2/M_Z^2 = 5/8 at all s to this order, and the massless directions keep zero mass coefficient, while their kinetic coefficients can change. These are leading-order statements: already in the towerless top-only family, M_W^2 = (3/8) m_t^2 - (23/128) m_t^4/Lambda^2 + … (equation (19) of lemma 18).

Proof. The trace gives ||T||^2 proportional to Q_11 = 12 and |||[D_F, T]||^2 proportional to K_11 = sum N_f m_f^2, which gives the coefficient. At d_eff = 0 this is the CCM relation (hep-th/0610241, Eq. (4.16)), as recovered in Proposition 4 and Remark 16 of lemma 18 at small x. The factor is common to all generators in Theorem 25.2. ■

In terms of Yukawa couplings, with m_t = y_t v/sqrt(2) and M_W = g v/2, this reads y_t^2 = 4 g^2/(3 - d_eff), so y_t = 2 g/sqrt(3) at d_eff = 0, the top-dominance value of CCM (Eq. (5.27) there), and y_t = sqrt(2) g at d_eff = 1.

Remark (positive Laplace cutoffs). For f(z) = int e^{-tz} dnu(t) as in equation (4) of lemma 18, with t measured in units of 1/Lambda^2, the same computation gives at leading order

```
M^2 = ((3 - d_nu)/2) ||[D_F, T]||^2/||T||^2,     S_nu(s) = int S(s/sqrt(t)) dnu(t),     d_nu = d
↪ log S_nu/d log s = 2 <t k^2>_nu,
```

where the average has weights dnu(t) e^{-t k_n^2}. Under the moment hypotheses there, dominated convergence gives the same endpoints, and the crossover and its corrections depend on nu. A general smooth cutoff, such as one flat near zero, need not belong to this class.

Scope

- This is the Taylor quotient of a prescribed constant-profile vector, at leading order in D_F, about a prescribed background and not a demonstrated vacuum. Taylor and pole masses differ in general. Calling it the leading term of a lowest-mode mass also needs an admissible gauge field, a positive kinetic form and an isolated lowest branch; on the circle, translation invariance separates the zero Fourier sector, within which the generator matrix must still be diagonalized. For finite angles in the Yang-Mills parent, the exact masses are in Theorem 23.6, which expands as M_W^2 = 2 m^2 - (4/3) m^4 l^2 + … and M_Z^2 = (16/5) m^2 - (64/75) m^4 l^2 + …, in agreement with d_eff = 1 at leading order. That theorem is for its own Yang-Mills boundary problem, and it does not establish the full finite-cutoff gauge spectrum of the spectral action.
- T must be even and constant in the transported frame, so it preserves both walls there, and it needs the enlarged gauge algebra noted above. Vectors that are Dirichlet at a strict wall, as in Proposition 22.8, are a different boundary problem.
- All fields have flat profiles on one interval, with no bulk masses. In particular, for the top-only statement, the kinetic term counts all twelve doublets of three generations on the same interval, with the same S. With bulk masses to localize light fields, the spectra and overlaps change, and the kinetic coefficient must be recomputed from all eigenstates and overlaps.
- The spectral action is taken as a function of the fermion operator at a matching scale, with no running.
- For a boundary with reflection, the walls contribute heat-kernel boundary terms in general. Here the unfolding removes them for this operator and these generators.
- Other priority: the endpoint m_t = 2 M_W appears at tree level in Cacciapaglia, Csaki and Park, Fully radiative electroweak symmetry breaking, arXiv:hep-ph/0510366, section 5, for a suitable rank-four representation, and a limit approaching 2 M_W in Adachi and Maru, arXiv:1804.06012, Eq. (3.23).

These are representation-dependent mechanisms, and they do not give the spectral-action crossover. Universal propagation alone does not impose it in ordinary universal extra dimensions (Cordero-Cid et al., arXiv:1108.2926, section 2).

Reading

This connects three items of the project.

1. *The emergence of a dimension.* The heat dimension of the interval, which distinguishes the labels 10 and 11, enters the mass relation directly. As the cutoff resolves the interval, the coefficient moves from the finite-space value to the five-dimensional value.
2. *Proposition F and lemma 13.* The factor 3/2 is the finite-action coefficient relative to the Yang-Mills commutator quotient, and in this constant-profile calculation the two are the two ends of one function. Proposition F also contains a bounded-fermion against tower-scale obstruction and overlap conditions, which keep their separate hypotheses.
3. *The measured ratio.* With the Particle Data Group 2026 listings, m_t = 172.60 +- 0.27 GeV as a direct-measurement Monte Carlo mass, and M_W = 80.3625 +- 0.0077 GeV, the ratio is 2.148, which would formally correspond to d_eff = 1.27, outside the range. With a running mass of about 163 GeV (Yoon and Peskin, arXiv:1810.12352, section 3.3 and Eq. (23), for a pole mass of 173 GeV, with about 147 GeV at 2 TeV), it is 2.03, which corresponds to d_eff = 1.06. Dividing a running mass by an on-shell W mass is only a formal orientation and not a matching test at a common scale, and no running from the matching scale has been computed, so we draw no conclusion from this comparison. The known spectral-action value y_t = 2 g/sqrt(3) is the d_eff = 0 boundary condition. A tower near the weak scale, as in the separate unrefereed estimate of the steer note, item 9, would use the d_eff = 1 boundary condition y_t = sqrt(2) g, at a matching scale of that order, in a specified effective theory with thresholds and a renormalization prescription. In a one-loop orientation with universal bulk fields (the steer note, item 11, no referee), the running approaches an infrared quasi-fixed point near 211 to 232 GeV, nearly independently of the boundary value, so that family does not support the closeness of 2 M_W to the running top mass.

Numerical check

By Claude Opus 5.5, in Python (numpy), 3 Nov 2026, in a temporary directory, not kept. The kernel of lemma 18, with quadrature in u, was summed over the tower, with explicit 4 x 4 level blocks for the operator on the interval, independently of the unfolding.

- Random complex K_1 and Hermitian T_Q, T_U on C^3, with ||K|| = 10^-3 min(s, 1): the ratio of C_0/C_1 to the Hilbert-Schmidt quotient agrees with 3/2 - <k^2> to six digits, for s in {0.3, 1, 1.5, 2, 2.5, 3, 5, 10}.
- Three generations, top only, with T = sigma_x/2 on the U-fields: m_t/M_W agrees with (8/(3 - d_eff))^{1/2} to six digits on the same grid, from 1.63299 at s = 0.3 to 2.00000 at s = 5.
- The neutral direction, with weights for lepton and colour: M_Z^2/m^2 goes from 4.801 to 3.200 for universal mass, and M_W^2/M_Z^2 stays at 0.625 to four digits.
- Finite mass in the top-only family: M^2/m^2 at s = 5 is 0.2500, 0.2498, 0.2477, 0.2409, 0.2241, 0.1478 for m/Lambda = 0.02, 0.1, 0.3, 0.6, 1, 2, and at s = 0.5 it is 0.3749, 0.3732, 0.3591, 0.3145, 0.2277, 0.0555, the latter matching lemma 18.

Part 26

The interior vacuum and the matching scale in the flat parent

For STRATEGY T13, T5b and T12, following lemma 23, lemma 24, lemma 25 and the steer note, items 9 and 11. Proved by Claude Opus 5.5; the numerical statements are in Python and each is labelled. Refereed by Terra 6.1 (effort max, 3 Nov 2026; first report): three errors and two gaps as first written, and amendments. Proposition 26.3c and Result 26.8 were added after that report and answer its gap on the wall mass, and a

second report corrected one error in the summary. The textual corrections of both are incorporated below, and the calculations they call for are listed under "Outstanding".

Outcome.

1. Pure frequency-2 fermion channels give the flat family an interior global minimum, and we classify it for n_2 >= n_c (Theorem 26.2). With frequency-2 channels alone it sits exactly at theta = pi/4, where M_KK = 4 M_W.
2. In the displayed channels of the multiplets containing Q_L and U_R, a pure channel carries an exact massless coloured exotic, and a wall mass of any finite positive size leaves a coloured state of charge 2/3 below M_W (Proposition 26.3c). So a top at 2 M_W in such a channel has a lighter partner, whose couplings are not analyzed. In E there is no such partner: m_1 = 2 M_W at theta = 1.1377, with M_KK = 444 GeV, and the vacuum then needs a separate sector.
3. In the step-threshold model of the parent, without frequency-2 multiplets, the left-right difference of the couplings runs only logarithmically (Proposition 26.5). The couplings then meet at mu_* = 2.2 x 10^14 GeV, with M_KK = 3.1 x 10^13 GeV (Result 26.6, one loop, conditional on the spectrum).
4. At a low tower, with the hypercharge coupling fitted by a wall kinetic term, the finite interval shifts M_W^2/M_Z^2 below its four-dimensional value by 0.05 at the top-compatible point, and by less than 0.006 only above about 1.2 TeV (Result 26.8, tree level).
5. These give two regimes of one family, stated at the end with hypotheses and escapes.

Setting

We keep the flat interval of length l and the gauge content, couplings and connector wall of Theorem 23.6. A fermion channel is *pure of frequency q* when its two wall gradings differ by a rotation through q theta in the channel. Its spectrum is +-(n pi + q theta)/l, n in Z, so that its product over the tower, relative to theta = 0, is 1 + sin^2(q theta)/sinh^2(x), with x = p l.

The bimodule E = Hom(V, W) is pure with q = 1 (Proposition 23.7). Let n_q >= 0 be twice the number of physical Dirac species in such channels, each colour counted once. With t = sin^2(theta), u = t (1 - t), and F, J as in Results 8 and 9 of lemma 24, the one-loop potential of the form displayed in lemma 23 is

```
8 pi^2 l^4 V = 3 F(2u) + (3/2) F((16/5) u) - n_1 F(t) - n_2 F(4u),        (1)
```

```
dV/dt = (1 - 2t) G(u) - n_1 J(t),   G(u) = 6 J(2u) + (24/5) J((16/5) u) - 4 n_2 J(4u),        (2)
```

up to the factor 8 pi^2 l^4. Equation (1) assumes precisely these towers and multiplicities; a full representation can contain others. We use J(0) = (3/2) zeta(3) and J(1) = (9/8) zeta(3), so J(0)/J(1) = 4/3.

Lemma 26.1. Let R(u) = [6 J(2u) + (24/5) J((16/5) u)] / (4 J(4u)) on (0, 1/4]. Then R is strictly increasing, with R(0+) = 27/10 and R(1/4) = n_c, where

```
n_c = [6 J(1/2) + (24/5) J(4/5)] / (4 J(1)) = 2.8855033... (numerical quadrature).
```

Proof. For c < 4, d/du log[J(c u)/J(4u)] = [e(4u) - e(c u)]/u, with e(z) = -z J'(z)/J(z). By Result 8 of lemma 24 and the covariance argument in the referee amendment there, which is strictly positive, e is strictly increasing. So both ratios J(c u)/J(4u), for c = 2 and c = 16/5, strictly increase. ■

The classification

Theorem 26.2. Let n_1, n_2 >= 0 in the family (1), with theta in [0, pi/2].

1. *n_1 = 0.* The potential is a function of u.
   - If n_2 <= 27/10, it increases with u: theta = 0 and theta = pi/2 are the two global minima, degenerate at n_2 = 27/10, where the leading term is (99/25) theta^4 log(1/theta^2), and theta = pi/4 is the maximum.
   - If 27/10 < n_2 < n_c, there are exactly two global minima, at theta_* and pi/2 - theta_, *with theta_* in (0, pi/4), and theta = pi/4 is a local maximum.

- If n_2 >= n_c, theta = pi/4 is the unique global minimum, with a quartic leading displacement at n_2 = n_c.

2. *n_1 > 0 and n_2 >= n_c.* Let n_* = (4/3)(4 n_2 - 54/5).
   - If n_1 < n_*, there is exactly one stationary point in the open interval. It is the strict global minimum, and it lies in (pi/4, pi/2).
   - If n_1 >= n_, *V decreases strictly on (0, pi/2), with unique global minimum at theta = pi/2, degenerate at n_1 = n_.*

The endpoints are stationary in theta in every case, since dt/d(theta) vanishes there. The range n_1 > 0, n_2 < n_c is not classified here.

Proof. Item 1: dV/du = 4 J(4u) (R(u) - n_2), so Lemma 26.1 gives the three cases. Item 2: since n_2 >= n_c >= R(u), G(u) = -4 J(4u) (n_2 - R(u)) <= 0 on (0, 1/4], with equality only at u = 1/4 and n_2 = n_c. For t <= 1/2, (2) gives dV/dt <= -n_1 J(t) < 0. For t > 1/2 write dV/dt = J(t) [Psi(t) - n_1], with Psi(t) = (2t - 1) |G(u)|/J(t). Here 2t - 1 increases, 1/J(t) increases, and |G(u)| increases with t, since u decreases with t and both factors J(4u) and n_2 - R(u) decrease in u. So Psi increases strictly from Psi(1/2) = 0 to Psi(1) = (4 n_2 - 54/5) J(0)/J(1) = n_*. ■

No numerical computation enters the proof, apart from the value of n_c. A Python check on a grid of 11600 pairs, n_2 from 3 to 60 and n_1 from 0 to 199, with theta on 20001 points, finds the stated stationary points in each case.

For the remaining integer counts: n_2 = 0 is Theorem 9 of lemma 24. For n_1 = 0 and n_2 = 1, 2, Theorem 26.2 gives both endpoints as global minima. For n_1 > 0 and n_2 = 1, 2, a scan of n_1 up to 200 (Python) finds no interior local minimum, which we have not proved. So at integer counts every interior minimum found has theta >= pi/4. The window 27/10 < n_2 < n_c, with its minima at small theta, contains no integer.

Proposition 26.2b (referee; no global minimum at small angle). For n_1 > 0 and 0 < t < 1/2, V(1 - t) - V(t) = -n_1 [F(1 - t) - F(t)] < 0, so no point of (0, pi/4) is a global minimum. With item 1 of Theorem 26.2 for n_1 = 0, this covers all integer counts. It does not exclude metastable minima at small angle in the cases left to scans. ■

Corollary 26.3 (the spectrum at the minimum). Write g(y) = arcsin(sin(y)/sqrt(2)), so that M_W l = g(2 theta) by Theorem 23.6, for theta in (0, pi/2), where M_W > 0. On [pi/4, pi/2), m_1 l = theta and m_2 l = pi - 2 theta for pure channels.

1. At theta = pi/4, the minimum for n_1 = 0 and n_2 >= n_c: M_KK := pi/l = 4 M_W, m_1 = M_W, m_2 = 2 M_W, tan(M_Z l) = 2 and M_W^2/M_Z^2 = 0.5032…
2. As theta increases from pi/4 to pi/2, m_1/M_W increases from 1 to infinity and m_2/M_W decreases strictly from 2 to sqrt(2).
3. M_KK >= 4 M_W for all theta in (0, pi/2), and M_KK/M_W grows without bound as theta -> pi/2 and as theta -> 0.

Proof. Item 1 follows from Theorem 23.6. For item 2, g'(y) = cos(y)/(1 + cos^2 y)^(1/2) is positive and decreasing on (0, pi/2), so g is concave with g(0) = 0, and y/g(y) increases. With y = pi - 2 theta, this gives m_2/M_W = y/g(y). Item 3 follows from sin^2(M_W l) <= 1/2. ■

Evaluating the minimum (Python, on the root given by the proof, recomputed by the referee), with M_W = 80.36 GeV:

n_1	n_2	theta_*	M_KK (GeV)	m_1/M_W	m_2/M_W	M_W^2/M_Z^2
0	>= 3	0.7854	321	1	2	0.503
6	8	0.941	342	1.27	1.71	0.525
14	8	1.138	444	2.00	1.52	0.576
16	8	1.186	491	2.31	1.50	0.586
16	6	1.446	1437	8.23	1.42	0.621
48	12	1.502	2597	15.5	1.42	0.624

A frequency-1 top with m_1/M_W = 2 needs theta = 1.13774, on the line n_1 = 2.71387 n_2 - 7.71725, with M_KK = 444 GeV. A larger tower needs n_1 close to n_* from below. With even counts this reaches 29 TeV at (n_1, n_2) = (604, 116), and the frequency-1 fermions then sit near M_KK/2, as in item 2 of Theorem 23.6.

Remark (the quadratic coefficient). The coefficient of theta^2 in (1) is (54/5 - n_1 - 4 n_2) (3/2) zeta(3), which does not vanish for integer n_1, n_2. So in (1) the quadratic coefficient has no exact cancellation. This concerns that coefficient only: F(z) = J(0) z + (1/4) z^2 log z + O(z^2), so a theta^4 log(theta^2) term remains at a tuned small angle. For added bosons the statement needs a stated normalization: a real scalar determinant carries 1/2 and a real vector 3/2, and mixed boundary projections can give other prefactors.

Where frequency 2 lives

With multiplicities for the eigenvalues of F: the 4 has +-1, each twice; the 5 has 0 three times and +-2 once each; the 10 has 0 four times and +-2 three times each. So E = (4, 4bar) has no frequency-2 channel. For a top with Q_L in (2, 2) and U_R in the same multiplet, the charges Y = R_3 + (B - L)/2 require colour triplets with B - L = 4/3 for the 5, smallest in the 15, so (5, 15), of 75 components; or B - L = -2/3 for the 10, smallest in the 6, so (10, 6), of 60 components, with Q_L at R_3 = +1/2 and U_R at R_3 = +1. Also Lambda^2 E = (10, 6) (+) ((1 (+) 5), 10bar). The (5, 15) with light fermions localized is the pattern of Contino, Da Rold and Pomarol.

Proposition 26.3b (the frequency-2 plane does not separate the top). Take the (10, 6), with F the generator J_45 of so(5). The frequency-2 planes are (A^a, O^a), with A^a = (L^a - R^a)/sqrt(2) and O^a in (2, 2). The component O^+ is the left-handed top, with (T_3L, R_3) = (+1/2, +1/2), and its partner A^+ contains the right-handed top R^+ with weight 1/sqrt(2), together with the triplet component L^+. Let phi = 2 theta, A' = cos(phi) A + sin(phi) O, O' = -sin(phi) A + cos(phi) O, and S = (L + R)/sqrt(2). Consider the self-adjoint problem, by the referee: at x = 0, psi_R is Neumann on S, A and Dirichlet on O, and psi_L has the complementary conditions.

1. If at x = l, psi_R is Neumann on S and A' and Dirichlet on O', then (O^+, A^+) is a pure frequency-2 channel, and S_R is an exact massless coloured mode of charge 2/3.
2. If at x = l, psi_R is Dirichlet on O' and on S + A', and Neumann on S - A', the determinant is proportional to cos(a) (2 sin^2 a - sin^2 phi), with a = m l. So the angle-dependent tower obeys sin^2(m l) = sin^2(2 theta)/2, the equation of the charged vectors: m_t = M_W at every theta. There is also an angle-independent half-integer tower.
3. In the (5, 15) the same holds with the chiralities exchanged: the frequency-2 plane contains U_R and (t_L + X)/sqrt(2), with X the exotic of Y = 7/6.

Proof. These are the commutation relations of so(5) and the boundary problem, solved below as the limits of Proposition 26.3c. ■

These statements concern the displayed channel. A gauge-invariant completion needs conditions on complete wall multiplets, and the other modes of the multiplet also enter the potential.

Proposition 26.3c (a wall mass leaves a light state). In the problem above, normalize u = (S + A')/sqrt(2), v = (S - A')/sqrt(2), and let a wall Weyl field chi couple to g_u at x = l with parameter w: the self-adjoint operator is H(f, g, chi) = (-g', f', w g_u(l)), with domain f_u(l) = w chi, g_{O'}(l) = 0, f_v(l) = 0 and the stated conditions at x = 0 (referee). Let omega = w^2 l and a = m l. Eliminating chi for m != 0 gives f_u(l) = (omega/a) g_u(l). With s = sin(2 theta), the nonzero eigenvalues solve

```
2 a Delta sin(a) = omega cos(a) (Delta + sin^2 a),   Delta = sin^2 a - s^2,        (3)
```

where omega = 0 gives Delta sin(a) = 0, item 1 above, with the decoupled wall Weyl zero mode counted separately, and omega -> infinity gives cos(a) (2 sin^2 a - s^2) = 0, item 2. For every theta in (0, pi/2) and every omega in (0, infinity), there is an eigenvalue with 0 < m < M_W.

At theta = pi/4 equation (3) factorizes as cos(a) (omega cos(2a) - a sin(2a)) = 0: the state m = pi/(2l) = 2 M_W does not depend on omega, and the other equation has a root in (0, pi/4) for every omega > 0, moving from 0 to M_W l = pi/4.

Proof. On x in [0, l], with f the left and g the right components and the massless bulk operator, f' = m g and g' = -m f. The conditions at 0 give f_O = alpha cos(m x), g_O = -alpha sin(m x), f_A = beta sin(m x), g_A = beta cos(m x), f_S = gamma sin(m x), g_S = gamma cos(m x). These enter three linear conditions at x = l, g_{O'}(l) = 0, f_v(l) = 0 and f_u(l) = (omega/a) g_u(l), and (3) is the vanishing of their determinant, which we use before any division, so that it covers cos(2 theta) = 0 as well. For the root, let u_W = M_W l = arcsin(s/sqrt(2)) <= pi/4. On 0 < a < u_W,

```
omega = 2 a tan(a) (s^2 - sin^2 a)/(s^2 - 2 sin^2 a)
```

is continuous and strictly increasing from 0 to +infinity (referee), and it takes every value in (0, infinity). ■

So in the displayed charge-2/3 channel a state lies below M_W for every finite positive wall mass, and in the infinite-mass limit the lowest root of this isolated channel is M_W. This does not exclude identifying a higher eigenstate with the top: equation (3) has the eigenvalue m = 2 M_W when omega = 4 u_W sin(2 u_W)/(1 + 2 cos(2 u_W)). For example, at theta = 1 this is omega = 2.04, with another state at 0.85 M_W (Python, referee). That higher state has nonzero O_L and R_R components; its physical identification and the couplings of the lighter state need analysis. This compares with the top-mass limits for low representations in flat gauge-Higgs models (Cacciapaglia, Csaki and Park; Adachi and Maru; massless exotics and their wall masses in Hosotani, Oda, Ohnuma and Sakamura). A complete gauge-invariant wall multiplet, other boundary mixings, bulk masses and the other towers need separate analysis, and they can add still lighter states.

The vacuum exists in case 2 as well. Let n_d be twice the number of Dirac species with the product 1 + sin^2(2 theta)/(2 sinh^2 x), in place of n_2. Then G(u) acquires -2 n_d J(2u), and for n_d >= 27/5 we have G <= 0 on (0, 1/4], since J(16u/5) <= J(2u). So with n_1 = 0, theta = pi/4 is the unique global minimum, by the argument of item 1 of Theorem 26.2. One Dirac colour triplet has n_d = 6. A scan (Python) gives minima at 0.8585 and 1.0011 for (n_d, n_1) = (12, 2), (12, 6), and no interior stationary point for n_d = 6 and n_1 = 2, 6. For E with n_1 and such a sector, m_1/M_W = 2 lies on the line n_d = 5.292 + 0.6857 n_1; for example (n_1, n_d) = (4, 8) gives theta = 1.1423, with m_1/M_W = 2.026. Such a sector has its own state at M_W in that isolated channel.

The tower of the parent in the step-threshold model

Take the conditions of Theorem 23.6 at theta = 0, and let S = mu/M_KK with M_KK = pi/l. A massive vector, with its fifth component and ghosts, contributes -(7/2) tr Q^2 per level, the trace over real components.

generators	conditions (x = 0, x = l)	tower	SU(3)	SU(2)	U(1), GUT normalization
gluons	N, N	n	-21/2		
W_L	N, N	n		-7	
O = (2, 2), Y = +-1/2	D, D	n		-7/2	-21/10
R^+-, Y = +-1	N, D	n - 1/2			-21/5
X = 3 + 3bar, Y = +-2/3	N, D	n - 1/2	-7/2		-28/5
hypercharge combination	N, N	n			0
orthogonal neutral combination	N, D	n - 1/2			0

The totals -21/2 for SU(2) and -14 for SU(3) are -(7/2) C_2 of Sp(2) and SU(4). The Higgs is the zero mode of A_5 along O, counted in the Standard Model coefficients, and the nonzero modes of A_5 are the gauge partners counted in the vector towers. Three complete massless bulk generations of E add F = 8 to each

coupling on the integer tower, in the limit theta -> 0; at nonzero angle, frequency-one masses are shifted to (n pi +- theta)/l. With the fermions on the walls, F = 0.

So above M_KK, with b = (41/10, -19/6, -7), we take as a model

```
alpha_i^-1(mu) = alpha_i^-1(M_Z) - (b_i/2 pi) log(mu/M_Z) - (1/2 pi) [ (B_i + F) I_n(S) + C_i
↪  I_h(S) ],        (4)

B = (-21/10, -21/2, -21/2),   C = (-49/5, 0, -7/2),
```

where I_n(S) is the sum of log(S/n) over n <= S, and I_h(S) the sum of log(S/(n - 1/2)) over n - 1/2 <= S. For large S, I_n ~ S - (1/2) log(2 pi S) and I_h ~ S - (1/2) log 2, so I_h - I_n ~ (1/2) log S + (1/2) log pi.

The model (4) omits the frequency-2 multiplets used for the vacuum above. A complete five-dimensional Dirac (10, 6) has level contributions (296/15, 24, 40/3), before its threshold splitting and zero-mode subtractions, and these are not a universal F. Boundary kinetic counterterms are also generated by loops and need a renormalization prescription (Carena, Tait and Wagner). So (4) is the running of this spectrum, and of the parent only as far as this spectrum is the parent's.

Proposition 26.4. In (4), the differences of the alpha_i^-1 are independent of F. ■

Proposition 26.5 (the left-right difference in Sp(2) is logarithmic). Define alpha_R^-1 = (5/3) alpha_1^-1 - (2/3) alpha_3^-1, the coupling of SU(2)_R in 1/g_Y^2 = 1/g_R^2 + (2/3)/g_4^2. Then, from (4),

```
alpha_R^-1(mu) - alpha_2^-1(mu) = D_0 - (44/3)/(2 pi) log(mu/M_Z) + (7/pi) [I_h(S) - I_n(S)],
↪  (5)
```

with D_0 = alpha_R^-1(M_Z) - alpha_2^-1(M_Z), and it has no power of S. With the inputs (59.0, 29.6, 8.47), D_0 = 63.09.

Proof. The zero modes give (5/3)(41/10) + (2/3)(7) + 19/6 = 44/3. The integer tower gives (5/3)(-21/10 + F) - (2/3)(-21/2 + F) - (-21/2 + F) = 14, and the half-integer tower gives (5/3)(-49/5) - (2/3)(-7/2) = -14. ■

The cancellation concerns the difference within Sp(2); differences involving colour keep a power of S. This is the differential running of orbifold models (Nomura, Smith and Weiner). Without the tower the difference vanishes at 5 x 10^13 GeV. With a tower, for S » 1, it vanishes at

```
log(mu_LR/M_Z) = (3/23) [ 2 pi D_0 + 7 log(pi) - 7 log(M_KK/M_Z) ].        (6)
```

So at M_KK = 321 GeV, the difference is 61.69 at S = 1 in (5), where I_h = log 2, against 60.15 from the Standard Model extrapolation alone, and by (6) it vanishes near 2 x 10^24 GeV. It lies above the Planck mass, 1.2 x 10^19 GeV, for M_KK below about 2 x 10^8 GeV.

Result 26.6 (the meeting point in (4); Python, one loop). The two conditions alpha_1 = alpha_2 = alpha_3 in (4) have the solution

```
M_KK = 3.15 x 10^13 GeV,   mu_* = 2.21 x 10^14 GeV,   S = 7.04,
```

with alpha^-1 = 46.0 for F = 8 and 52.6 for F = 0. It is the only solution with S >= 1 (referee): eliminating log(mu/M_Z) between the left-right equation and the equation for alpha_2 = alpha_3 leaves a continuous function of S whose derivative between thresholds is (315 N_h - 161 N_n)/(44 S) > 0, with N_n, N_h the numbers of levels below S. Shifts of 0.1 in alpha_1^-1 or alpha_2^-1 and of 0.15 in alpha_3^-1, one at a time, move M_KK within 2.9 to 3.4 x 10^13 GeV, and all three together within 2.66 to 3.70 x 10^13 GeV. Larger effects are possible: one additional light colour-triplet Weyl fermion changes alpha_3^-1 by about -1.4 up to that scale, the Standard Model gauge part of the two-loop terms by about (-0.20, -0.27, +0.39) (referee), and finite wall terms and split thresholds are not included. So in this model the relation g_3^2 = g_2^2 = (5/3) g_Y^2 holds at one scale with no added thresholds, and that scale fixes the tower. An interval Pati-Salam model with a compactification threshold of order 10^13 GeV, with a different intermediate group and matter, is in Abe, Goto and Kawamura.

The top against the matching scale

In a flat parent with the Higgs as A_5, the tree-level relation at small angle, for a pure channel of frequency q, is y_q = (q/2) g, since m_q = q theta/l and M_W = sqrt(2) theta/l. The same covariant derivative fixes the local bulk vertices. Physical four-dimensional gauge and Yukawa couplings still receive different finite corrections and thresholds (Lim and Maru). So the absence of separate running holds for the local bulk coupling, in a stated matching scheme. This differs from item 11 of the steer note, where the Higgs is an independent bulk field, y_t runs with its own power law, and the matching sits at the crossings. Item 11 stays correct for that content, and the two calculations have different actions.

At finite angle a mass ratio does not give the Higgs coupling. With v_ang = 2 sqrt(2) theta/(g l), the coupling is sqrt(2) dm/dv_ang. At theta = pi/4, the pure frequency-2 branches have slopes corresponding to |y| = g and are degenerate, so the coupling is a matrix on that subspace, while the tower with m = M_W has dm/d(theta) = 0 there (referee).

Result 26.7 (illustration; Python, one loop, Standard Model below M_KK). The running mass m_t solving y(m_t) v/sqrt(2) = m_t, from the boundary value y(M_KK) = r g_2(M_KK):

M_KK (GeV)	r = 1/2	r = 1/sqrt(2)	r = 1	r = 2/sqrt(3)	r = sqrt(2)
321	63	87	119	136	164
10^4	74	101	136	153	179
10^9	99	129	163	176	195
3.1 x 10^13	114	143	172	183	196

The measured value in the same convention is 163 GeV, and from it y_t/g_2 equals sqrt(2) at 265 GeV, 2/sqrt(3) at 2.8 x 10^5 GeV and 1 at 1.3 x 10^9 GeV. These are boundary-value calculations, and they are predictions of no spectrum: at low M_KK a real parent has other thresholds below M_KK, so Standard Model running there is an illustration. Reading the entry 164 GeV as the pure channel at theta = pi/4 fails for three reasons: the Higgs coupling at that angle is not sqrt(2) g, as above; the channel has its massless exotic and a second fermion at M_KK/2; and Proposition 26.3c gives a lighter partner for any wall mass.

Result 26.8 (the neutral problem with a wall term; Python, tree level). Let the bulk couplings of Sp(2) and SU(4) be separate, and let a wall kinetic term r at x = l act on the hypercharge combination Y-hat = cos(phi) R + sin(phi) B, with cos^2(phi) = alpha_2^-1/(alpha_2^-1 + (2/3) alpha_3^-1). We fix these, as an estimate, by Standard Model running to M_KK, with alpha_Y^-1 = (1 + r/l)(alpha_2^-1 + (2/3) alpha_3^-1). The charged sector is unchanged. At x = 0, S, A, B are Neumann and O is Dirichlet. At x = l, with L_l = (S + A')/sqrt(2), R_l = (S - A')/sqrt(2), p = cos(phi), q = sin(phi): O' = 0, -q R_l + p B = 0, d_x L_l = 0 and d_x(p R_l + q B) = r m^2 (p R_l + q B). The lowest neutral root a = M_Z l solves, with t = r/l (referee),

```
(1 + t a cot a) sin^2 a = (sin^2(2 theta)/2) (1 + p^2 + t a cot a),
```

whose r = 0, one-coupling limit is Theorem 23.6, and which our 4 x 4 determinant reproduces.

theta	M_KK (GeV)	m_1/M_W	M_W^2/M_Z^2	four-dimensional value at M_KK	r/l
0.7854	321	1.00	0.629	0.762	1.63
1.1377	444	2.00	0.709	0.761	1.60
0.30	615	0.73	0.735	0.759	1.56
1.30	677	3.48	0.739	0.759	1.55
0.15	1199	0.71	0.750	0.756	1.48
0.08	2236	0.71	0.751	0.752	1.42
1.53	4378	26.5	0.749	0.749	1.35

So within this matching prescription the finite interval shifts the ratio by -0.13 at theta = pi/4 and by -0.05 at the theta where m_1/M_W = 2. In the rows of the table the shift falls below 0.006 from about

1.2 TeV, where m_1/M_W is either near 0.71 or above 8, as a numerical statement for this prescription. The four-dimensional number is a reference within this prescription. A physical precision comparison needs the coupling definitions, wavefunction overlaps, threshold matching and radiative corrections of the model, which can enter at the same order as a small shift. We have not done an electroweak fit, and the measured on-shell value, 0.7767, is not comparable with these columns without that.

Two regimes

We state the outcome in the exact scope of (1) and (4): one interval, flat profiles, this gauge content, one loop, no bulk masses. This is one independent review of a statement about one family, and broader consensus on its negative parts remains open. It leaves T13 open.

Low tower. With frequency-2 channels of sufficient count, the one-loop potential has a natural interior minimum, at theta >= pi/4 and M_KK from 321 GeV upward (Theorem 26.2). In E, m_1 = 2 M_W at theta = 1.1377, with M_KK = 444 GeV, and the frequency-2 sector that stabilizes it has its own light states in the displayed channels (Propositions 26.3b, 26.3c). A state at 2 M_W in a frequency-2 channel has a lighter partner, whose couplings are not analyzed. In (5), the left-right difference at the tower is about 60, so if only a hypercharge wall term and separate bulk couplings are admitted, that wall term is of that size in alpha^-1, comparable with twice the bulk value, as an estimate; general non-universal wall kinetic terms are another way to fit the couplings. The weak angle is then an input, and the finite interval shifts M_W^2/M_Z^2 by the amounts of Result 26.8. Whether a tower at this scale is allowed needs its couplings and an applicable experimental limit, which we have not analyzed.

High tower. The relation of one coupling holds in the model (4) at mu_* = 2.2 x 10^14 GeV, with M_KK = 3.15 x 10^13 GeV, and the weak angle runs down from it. With the boundary values of Result 26.7, r = 1 gives a running mass near 172 GeV and r = 1/2 gives 114 GeV. The small-angle relation M_W l = sqrt(2) theta then requires theta = 5.7 x 10^-12, and Proposition 26.2b shows that no such angle is a global minimum of (1) at integer counts. Metastable minima, in the cases left to scans, are not excluded by it.

Escapes. All of these are model-dependent: wall kinetic terms and separate bulk couplings, which give up the relation as a prediction; a wall scalar with its own potential; warping, whose spectra, counterterms and matching the flat sums do not determine; bulk masses; antiperiodic conditions; larger representations with brane mixing; boundary interactions.

Direction. Warping and larger representations are the standard directions for items 2 and 4 of the outcome. In this project the next computations are the same family in a warped interval, with the spectral action of lemma 25 at the level of its heat trace, and the items below.

Outstanding. The low-tower potential (1), the wall-mass spectrum (3), the wall kinetic term of Result 26.8 and the step-threshold running (4) use different simplified spectra, so their conclusions have not been combined into one model. A combined model needs: complete gauge-invariant wall multiplets, with their determinants in (1) and their thresholds in (4); the coupling matrix at finite angle; the couplings of the light partner in (3); and the matching of the neutral sector at one loop. These calculations are open.

Priority

The sources are listed in LITERATURE, section J. The referee's search covered orbifold differential running, Pati-Salam interval scales, flat large-representation tops, SO(5) vector and adjoint embeddings, exotics and Wilson-line minima, and found no prior occurrence of the classification with n_c or of the pair (M_KK, mu_*) from (4), with forward citation chains incomplete. So priority for those exact statements is open. Hosotani et al. find theta_H = pi/2, corresponding to 2 theta = pi/2, in a warped model with other matter.

Numerical check

By Claude Opus 5.5, in Python (numpy, scipy), 3 Nov 2026, in a temporary directory, not kept; the referee recomputed the items marked in the report.

- J(0) and J(1) by quadrature against the closed forms, to 15 digits. The function e(z) is nondecreasing and R(u) is increasing on grids, with R(0+) = 2.7001 at u = 10^-5 and R(1/4) = 2.88550.
- Theorem 26.2 on the grid described after its proof, with no failure. For n_1 = 0: n_2 = 2.75 has minima at 0.1527 and 1.4180 and a maximum at pi/4, and n_2 = 2.90 has the minimum at pi/4.
- Equation (3) against a direct 3 x 3 determinant of the boundary conditions, for theta = 0.3, pi/4, 1.0, 1.1377 and omega from 10^-3 to 10^4: the lowest roots agree to four digits, and the lowest root lies below M_W l in every case.
- Equations (4), (5): the difference computed directly from (4) agrees with (5) to 13 digits at S = 1000. For S above 10^8 the closed sums lose precision in floating point, so (6) is used there.
- Result 26.6 by nested root finding, over a scan of 1401 values of M_KK. The inputs are those of the steer note, with no thresholds at the top or W.
- Result 26.7: one-loop top Yukawa equation of the steer note with no tower, below M_KK. The asymptotics of I_n, I_h and their difference, against the closed sums at S = 10.3, 100.3, 1000.3, to about 10^-3 or better.
- Result 26.8: the 4 x 4 determinant, with root finding on 6401 points. Its limit r = 0, one coupling against Theorem 23.6 at theta = 0.2, pi/4, 1.06, 1.1377, 1.3.

Part 27

The axial anomaly on the reduced action, and the two-flavour discrete term

For STRATEGY T10 and T13, following lemma 17 and open question 6 of the steer note. Proved by Claude Opus 5.5, using the index theorem and Witten 1982 as references. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): two errors of interpretation, in Propositions 27.4 and 27.5, and amendments, all incorporated below.

Outcome.

1. On the reduced operator D_M + gamma_5 d, with masses commuting with the gauge field, the regulated axial trace has an exact form at every cutoff: each flavour block contributes its index, weighted by f(m^2/Lambda^2) (Theorem 27.1). The regulated mass term equals it identically (Proposition 27.3). As Lambda -> infinity this is the anomaly with the coefficient 2 N_c tr(T^3 Q^2) = 1, on the action of Theorem 17.1 (Corollary 27.2).
2. The operator of Proposition 17.10 at M = 0, in independent quark and antiquark variables, is the massless Dirac operator with a pure-gauge connection on one chirality. So for two flavours and a reduction Sigma : S^4 -> SU(2) in the nontrivial class, the spectral action is unchanged, and the Weyl Pfaffian transported along connections from 0 to that pure gauge has the relative sign (-1)^{N_c} (Proposition 27.5). The same sign appears in Witten's anomaly, in the discrete term of the pion sigma model, and in the mod-two index on S^5 of the research state. The sigma field represents a class in the reduced spin bordism of SU(2) in dimension four, and the anomaly a class in Omega_5^Spin(BSU(2)); these agree through the anomaly comparison and are distinct groups. The corpus S^5 is an internal space, and the S^5 here is the compactified S^4 x R of an auxiliary flow.

Setting

Let M be a closed Riemannian spin 4-manifold. Following Corollary 17.2, a paired sector gives one Dirac fermion: we take the operator

```
DD = D_M(A) + gamma_5 d,   d = diag(m_f) on E = C^{N_c} (x) C^{N_f},   m_f >= 0,
```

on L^2(S) (x) E, with action psi-bar DD psi, and with A the gauge connection of the commuting subalgebra: colour and electromagnetism, so that A is diagonal in the flavour index, with charge Q_f on flavour f. For topologically nontrivial backgrounds, E is replaced by the colour and electromagnetic bundles E_f, with charges normalized so that the flavour representations are globally defined. Let T be a Hermitian matrix on C^{N_f}, diagonal in the same basis, with eigenvalues tau_f. Then d and T commute with A, and Theorem 17.1 gives DD^2 = D_M(A)^2 + d^2.

Let f be a cutoff with f(DD^2/Lambda^2) trace class, for example in the positive Laplace class of lemma 18. Choose gamma_5 = +1 on S^+, and let ind_f = n_{f,+} - n_{f,-} be the index of D_f^+ : S^+ (x) E_f -> S^- (x) E_f.

The rotation psi -> exp(i alpha gamma_5 T) psi, psi-bar -> psi-bar exp(i alpha gamma_5 T) sends DD to D_M + gamma_5 d exp(2 i alpha gamma_5 T), since gamma_5 anticommutes with D_M. So the classical action is invariant at d = 0. Its Jacobian, regulated by the spectrum of DD, is exp(-2 i alpha Tr(gamma_5 T f(DD^2/Lambda^2))), up to the orientation convention.

The axial trace

Theorem 27.1. For every Lambda > 0,

```
Tr(gamma_5 T f(DD^2/Lambda^2)) = sum_f tau_f f(m_f^2/Lambda^2) ind_f.
```

Proof. On block f, DD^2 = D_f^2 + m_f^2, with D_f = D_M(A) on that block. The operator D_f anticommutes with gamma_5, so it maps the eigenspace of D_f^2 with eigenvalue mu > 0 isomorphically between the two chiralities. So these eigenspaces contribute zero to the trace with gamma_5, and only ker D_f remains, with weight f(m_f^2/Lambda^2) and signed dimension ind_f. ■

No expansion in m/Lambda enters. By the index theorem,

```
ind_f = (1/8 pi^2) int tr_f(F wedge F) - (N_c/24) p_1(M),
```

with F Hermitian, and with F = Q_f e F_em + g G on block f, tr_f(F wedge F) = N_c Q_f^2 e^2 F_em wedge F_em + g^2 tr_c(G wedge G), since G is traceless. Reversing the chirality convention reverses the index.

Corollary 27.2 (the anomaly on the reduced action). Let f be continuous at 0, and let Lambda -> infinity at fixed masses. If tr T = 0, with the flavours in the same colour bundle, then

```
2 Tr(gamma_5 T f(DD^2/Lambda^2)) -> f(0) 2 N_c tr(T Q^2) (e^2/8 pi^2) int F_em wedge F_em,
```

with no gluonic and no gravitational term. For two flavours with T = T^3 and Q = diag(2/3, -1/3), the coefficient is 2 N_c tr(T^3 Q^2) = 1, so that the right side is f(0) (e^2/16 pi^2) int F_em,mu nu F-tilde_em^mu nu, with F-tilde^mu nu = (1/2) epsilon^mu nu rho sigma F_rho sigma. A singlet T = 1 on N_f flavours gives in addition the gluonic term f(0) 2 N_f (g^2/8 pi^2) int tr_c(G wedge G) and the gravitational term -f(0) (N_c N_f/12) p_1(M). ■

The normalization f(0) = 1 is the usual convention for a regulator, and it is the value that makes the Jacobian the anomaly. For a smooth cutoff class in which the heat expansion can be used, with E and Omega defined by the Laplace-type square, the local form, with a smooth function chi on M inserted, follows at Lambda -> infinity from the a_4 coefficient, (1/16 pi^2) tr(gamma_5 T (E^2/2 + Omega^2/12)), which gives the integrands above. The other a_4 terms and the lower coefficients have vanishing gamma_5 spin trace in this problem, and the factor exp(-t d^2) commutes with the heat kernel and changes only higher orders in t. So the algebraic coefficient quoted in the steer note, item 6, now has the regulated trace and measure that it lacked.

Proposition 27.3 (the mass term equals it at every cutoff). Let m_f > 0 for all f. Then

```
Tr(f(DD^2/Lambda^2) d T DD^-1) = Tr(gamma_5 T f(DD^2/Lambda^2)) = sum_f tau_f f(m_f^2/Lambda^2)
↪ ind_f.
```

Proof. DD^-1 = (D_M + gamma_5 d)(D_M^2 + d^2)^-1. The term with D_M has zero trace, by conjugation with gamma_5, since all other factors commute with gamma_5. The remaining term is Tr(f gamma_5 T d^2 (D_M^2 + d^2)^-1). On the nonzero spectrum of D_f it vanishes by the pairing of Theorem 27.1, and on ker D_f the factor m_f^2/m_f^2 is 1. ■

Since {gamma_5 T, D_M + gamma_5 d} = 2 d T, the insertion is psi-bar d T psi in this mass convention, pseudoscalar in a conventional scalar-mass basis. So if the Jacobian and the composite mass insertion are given the same specified regularization, this is the integrated Ward identity in a fixed background at every Lambda, with a common factor f(m_f^2/Lambda^2) on both sides. The trace identity alone

constructs neither an independently regulated measure nor a local current, and the local identity has a current-divergence term as well. The surviving zero modes contribute the signed count n_{f,+} - n_{f,-}, whereas the scalar density of Proposition 17.8 sees the unsigned count n_0 = n_{f,+} + n_{f,-}.

Proposition 27.4 (a heavy flavour at fixed cutoff). With the weights of Theorem 27.1, for T^3 on a doublet (t, b) with m_b/Lambda -> 0, the electromagnetic coefficient is

```
2 N_c (1/2) (Q_t^2 f(x) - Q_b^2 f(0)) / f(0),   x = m_t^2/Lambda^2.
```

It equals 1 at x = 0. For the heat cutoff it is (4 exp(-x) - 1)/3, which tends to -N_c Q_b^2 = -1/3 as x -> infinity, with gluonic and gravitational regulated traces proportional to f(x) - f(0). ■

This is a trace in the fixed-cutoff family of lemma 18, where the top is heavy at fixed Lambda. An explicitly massive flavour breaks the axial symmetry, so this calculation alone does not establish a Wess-Zumino term in the low-energy theory. Such a term can be required once the retained symmetry, its background fields or spurions, and the anomaly-matching prescription are specified, as in D'Hoker and Farhi (Nucl. Phys. B 248 (1984) 59 and 77, named in the steer note, and not read here). The weights depend on the chosen regulator, and Proposition 27.3 carries them on both sides of the Ward identity. Making f part of a finite-scale spectral action, as in lemmas 18 and 25, does not by itself identify the regulator of the fermion measure with that action or make these weights observable.

The two-flavour discrete term

Take N_f = 2 and M = 0, with electromagnetism off, so that SU(2)_L x SU(2)_R is a symmetry of the reduced sector (Proposition 17.5). By Theorem 17.6, at M = 0 the admissible involutions are U_Sigma, with Sigma in U(N_f), exchanging the two eigenspaces of gamma_F by Sigma. Let Sigma(x) be a smooth map, and take the pointwise reduction of Proposition 17.10. Its operator at d = 0 is

```
D_M + c^mu P_- B_mu,   B_mu = Sigma d_mu Sigma^*,        (1)
```

up to the convention for which side carries Sigma, in independent quark and antiquark variables. On the full H_0 = (q_L, q-bar_R) the two blocks of (U_Sigma d U_Sigma) are Sigma d Sigma^* and Sigma^T d conj(Sigma). On the chirality selected by P_- this is the massless Dirac operator with the pure-gauge connection B of SU(2), and on the other chirality it has no flavour connection. The operator is generally not self-adjoint. The colour connection is unaffected.

Let Sigma : S^4 -> SU(2) generate pi_4(SU(2)) = Z_2, on the round S^4, with colour background zero, so that D_M has no kernel. Such a Sigma cannot be joined to 1 through maps to SU(2). So let Pf_-(A_t) be the square root of Witten 1982 for one Weyl doublet, transported along the path of connections A_t = t B, with the endpoint fibres compared by the gauge transformation.

Proposition 27.5. For the operator (1), with N_c doublets of the chirality selected by P_-, the transport along A_t gives

```
Pf_-(B)^{N_c} = (-1)^{N_c} Pf_-(0)^{N_c}.
```

The spectral action of the fixed operator D does not depend on Sigma. The fixed classical form is unchanged as a form, while its expression in the fixed reduced variables acquires the connection in (1).

Proof. The display counts N_c Weyl doublets, on which we use the main result of Witten 1982: (det i D-slash)^(1/2) of one doublet has the opposite sign after a gauge transformation in the nontrivial class of pi_4(SU(2)) (Eq. (6), p. 325 there, read on 3 Nov 2026). The last sentences are item 1 of Proposition 17.10. ■

This is a relative sign for the stated Pfaffian transport and endpoint identification. The pointwise R_Sigma of Theorem 17.1 changes fermion coordinates, and pulling the original measure through that change preserves the value of the same integral, because the Jacobian of the measure accompanies the transformed action. So the sign is the comparison between that pulled-back measure and the Pfaffian continued along connections, and for N_c odd they differ between the two classes of reductions. Matching this to a pion field needs the sigma model and its anomaly-matching assumptions. The coupling that results has the sign of the discrete term in the N_f = 2 sigma model: Lee, Ohmori and Tachikawa (arXiv:2009.00033, section 2.5 and Eq.

(2.29), read in the LaTeX source) define the coupling as (-1)^{N_c [sigma]}, with [sigma] the class in the reduced spin bordism group of SU(2) in dimension four, which is Z_2, on spin four-manifolds. The original discussion is Witten 1983 (*Global aspects of current algebra*), which we have not read. Proposition 17.10 constructs neither a pion action nor its measure: a kinetic term needs a new operator or a quantum effective action, and Proposition 27.5 supplies only the sign.

Remark 27.6 (the relation with the S^5 index). By Witten 1982, for the flow from A = 0 to A = B, the sign equals (-1) to the number of zero modes of the five-dimensional Dirac operator on S^4 x R. The two ends compactify, with the appropriate gauge trivializations, to the nontrivial SU(2) bundle on S^5, and footnote 2 there describes the invariant zero mode for a special connection. The general protected statement is the parity of the kernel. For isospin j the parity is that of 2 tr T^2 = 2 j (j + 1)(2j + 1)/3, which is odd exactly at j = 2k + 1/2 (LITERATURE, section G). So the discrete term, with N_c colours, has the parity of N_c times the mod-two index of the doublet on the nontrivial SU(2) bundle on S^5. After choices of identification, that bundle is the one in section 6.12 of the research state, and the operator D_{S^5,V} there has the same mod-two index, on the generator of Omega_5^Spin(BSU(2)) = Z_2.

The scope of this identification is the bundle and the index. In the research state the S^5 is an internal space, the base of the chain 7 -> 5. In Proposition 27.5 the S^5 is the compactification of spacetime and the auxiliary flow parameter, with the flavour group of the quarks as structure group. We have not proved a relation between those two spaces, or between their SU(2) groups, and open question 6 of the steer note stays open for that part.

Caveats

- The background is fixed in Theorem 27.1 and Proposition 27.3. A measure on gauge fields and the order of limits are separate, as in Proposition 17.8.
- T, d and A commute by hypothesis. So T^3 and the W are excluded: this is the commuting subalgebra of Proposition 17.3. Off-diagonal d is outside Theorem 27.1, where DD^2 has a term gamma_5 [d, D_M].
- Proposition 27.5 has colour background zero and M = 0. For a nonzero colour background the integral is complex in general, and the statement on the sign is on the class of Sigma through the flow, which we have not developed.
- The sign in Proposition 27.5 is a statement on the fermion measure at one configuration. A sum over Sigma needs a sigma model that is not constructed here.
- The regulated Jacobian by the spectrum of DD is one choice. It is gauge invariant under the commuting group, which is why the anomaly falls entirely on the axial current.

Consequence for T10

The chiral limit now has, on the reduced action: the anomaly with its coefficient, the Ward identity at every cutoff, the top against the cutoff, and the discrete term. Still to be supplied are a measure on gauge fields, the map to a quantum order parameter, and the kinetic term of a pion field, which need a quantum effective action.

Numerical check

By Claude Opus 5.5, in Python (fractions, numpy), 3 Nov 2026, in a temporary directory, not kept. These check the arithmetic and the pairing, and the proofs stand without them.

- 2 N_c tr(T^3 Q^2) = 1 and tr T^3 = 0, in exact fractions. The third-generation coefficient for the heat cutoff at x = 0, 0.1, 1, 5, 50 is 1.0000, 0.8731, 0.1572, -0.3243, -0.3333.
- On T^2 x T^2 with fluxes n_1, n_2 and Landau spectra, truncated to 2000 levels: Tr(gamma_5 exp(-t(D^2 + m^2))) = exp(-t m^2) n_1 n_2 to 15 digits, at (n_1, n_2, t, m) = (1, 1, 0.3, 0.7), (2, 3, 0.05, 1.2), (4, 1, 1, 0).
- tr T^2 half-integer at j = 2k + 1/2 and integer at j = 2k + 3/2 for k up to 20, in exact fractions.

Part 28

Two sheets as the two ends of one connected commutative parent

For STRATEGY T0 and T0b, following lemma 16 (Example 16.9), lemma 24 (Proposition 6) and lemma 09. Propositions 28.1 to 28.4 and 28.13 are by Claude Opus 5.5. Results 28.8 to 28.12 and 28.14 to 28.21 are by Astra 6, job 28 (report, with the full proofs; prompt), checked by Claude Opus 5.5 by hand and in Python. Proposition 28.13 and the numerics are refereed by Terra 6.1 (report), with its corrections applied. The window is a conjecture with numerical support; the numbers come from a radial solve in Python (numpy and scipy), described in the section Numerical check; the script is not kept.

Correction, 3 Nov 2026. The first version of Result 28.5 and of the numbers in Remark 28.7 (commits dd1b14c, 530fa06) used the wrong sign of the spin connection in the Python slice operator. That changed the singular values and the boundary fibre parts, so those numbers are withdrawn, including the next singular value 1.04, the value 1.35 on the representation 16, and the boundary fibre parts -0.011 to -0.853. The exponents at the origin and the multiplicities are unaffected. Astra 6 did not reproduce the old boundary values and its proofs do not use them. We found the error when that operator gave a core zero mode in the representation 20, against the exact count of Proposition 28.11. The singular values, fibre parts and densities below are recomputed, and the script tests the operator against exact statements: {A_0, A_1} = 0, A_0^2 equal to the Casimir formula on S^4, D^2 on (0, 0)-forms with values in O(1) at the Fubini-Study radius equal to j(j + 4), and A_2 equal to its curvature contraction (the last test proposed by Terra 6.1).

Update, 4 Nov 2026. Lemma 29 proves Conjecture 28.6 in case (b) for the metric class of its Setting 29.0, and in case (a) at the Fubini-Study radius, refereed by Terra 6.1. Item 3 below and the Status of Proposition 28.13 describe the position before that lemma.

Outcome.

1. *The mechanism is proved at the level of compression maps.* A smooth function on the disc bundle X has independent values on the zero section S^4 and on the boundary CP^3, and at the zero section it is a function on S^4 alone (Proposition 28.8). For prescribed windows over this X, real functions generate exactly M_2(H) + M_4(C) at every localization scale (Theorem 28.21). In Example 16.9 the second sheet and the restriction to S^4 were inputs, so this is a connected source for them.
2. *The two ends of an actual local operator are understood.* There is a self-adjoint operator with a local wall (Theorem 28.9). The core carries exactly four states in the representation 4 (Proposition 28.11). At the wall, large mass keeps a tangential mixing (Lemma 28.12). For the pull-back connection the boundary mode then has the fibre part -w(rho) in closed form, with w(rho) = 2(rho^2+2)^2/(9+2(rho^2+2)^2), which reaches the level-one value of Example 16.9 only as rho -> infinity (Proposition 28.13). This is a statement on the finite wall model, and its convergence from the full operator is numerical. A connection preserving the line summands in a collar, with the Fubini-Study metric at the wall, gives the level-one value in that model.
3. *The isolated window of the whole operator stays a conjecture* (Conjecture 28.6): what is missing is the normal reductions with their errors, the matching, and a coercive estimate uniform in the mass and in all representations of Sp(2) (Lemma 28.14).
4. *The algebra M_2(H) + M_4(C) is the algebra of the limit map.* Generated algebras need not converge when compression maps do (Counterexample 28.17). At the sample points resolved numerically, the core mode has a nonzero density breaking the quaternionic structure, empirically of order lambda^2; if it is nonzero in the continuum, Proposition 28.16 gives M_4(C) for the core block at those masses. This is numerical evidence and does not establish the exact algebra at any finite mass.

Setting

Let G = Sp(2), K = Sp(1)_a x Sp(1)_b and H = Sp(1)_a x U(1)_b, so that G/K = S^4, K/H = S^2 and G/H = CP^3. Let S be the bundle G x_K (1, 2), so that CP^3 = P(S). The normal bundle of the zero section is G x_K (1, 3), and away from it X = (0, 1] x CP^3, with radial coordinate x and S^2 fibres of radius rho(x), rho(x) ~ x at x = 0. Charges of U(1)_b are normalized so that the doublet of Sp(1)_b

has +-1. The isotropy representation is (2)_{+-1} (+) (1)_{+-2}. The spinors of X restrict on a slice to Sigma_6 = Sigma^+ (+) Sigma^-, with c(d_x) = Z the chirality of Sigma_6, where

```
Sigma^+ = (1)_2 (+) (1)_0 (+) (2)_{-1},   Sigma^- = (2)_1 (+) (1)_0 (+) (1)_{-2}.
```

The representation 4 of G restricts to (2)_0 (+) (1)_1 (+) (1)_{-1}, that is (2, 1) (+) (1, 2) of K. Let Gamma_a be the Clifford generators on 4, with Gamma_5 commuting with K and equal to +-1 on the two summands, and let x_a(g) be defined by lambda(g) Gamma_5 lambda(g)^-1 = x_a(g) Gamma_a.

Proposition 28.1 (the twist).

1. For k != 0, O(k) on CP^3 does not extend to a line bundle on X, since H^2(X) = H^2(S^4) = 0 while O(k) has degree k on the fibre S^2.
2. The pullback of S to X restricts on CP^3 to O(1) (+) O(-1), as the tautological splitting. So E = S^3(S), which is pulled back from S^4 and has trivial connection along the fibres, restricts to O(3) (+) O(1) (+) O(-1) (+) O(-3). The summand O(3) gives the Dolbeault operator on O(1) of Example 16.9, since Sigma (x) O(3) = Lambda^{0,*} (x) O(1).
3. On E the section Phi(v) = phi(|v|) (v-hat . T), with T the spin-3/2 generators and phi odd and smooth, is smooth and G-equivariant, with zero set exactly S^4. In the frame of the slice it is phi(x) tau, with tau diagonal in the summands, with eigenvalues +-3/2, +-1/2.

So we take the Callias-type operator T = D_X(E) + i Phi on spinors of X tensored with E, and the self-adjoint operator D-hat = [[0, T^*], [T, 0]], with the grading Gamma-hat = diag(1, -1). At x = l we take the local condition Gamma-hat Z = -1 on the doubled spinor, that is u_even(l) = 0 for T and v_odd(l) = 0 for T^*. It is a chirality condition, elliptic and local, and it preserves the domain under all smooth functions on X.

Proposition 28.2 (multiplicities). On a slice, Hom_H(4, Sigma^+ (x) E) and Hom_H(4, Sigma^- (x) E) each have dimension 5. For the summand O(3), the dimensions are 1 and 0. So on 4, the operator on the summand O(3) with its U(1) connection vanishes for every G-invariant metric on the slice, and its kernel contains H^0(CP^3, O(1)) = 4 for every fibre radius. This concerns the separate line connection. The pull-back connection of E does not preserve the summands, and its slice operator is invertible for small fibres (Proposition 28.10).

Proof. Tensoring the lists above with the charges of E gives the multiplicities. The operator is odd under Z and maps the multiplicity space of 4 into itself. ■

Proposition 28.3 (the zero section). The fibre operator c_N . nabla + i Phi on R^3 with values in E has zero modes (1, 3) (+) (1, 1) of K, by the Callias count for the positive weights 3/2 and 1/2, with one sign of Phi. On S^4 this gives the Dirac operator twisted by (1, 3) (+) (1, 1). Then Hom_K(4, S^- (x) (1, 3)) has dimension 1, through (1, 2), and Hom_K(4, S^+ (x) (1, 3)) = 0, so the twisted operator vanishes on that copy of 4, in agreement with its index 4. The untwisted part (1, 1) has eigenvalue 2 on the representation 4. Every map in Hom_K(4, S^- (x) (1, 3)) annihilates (2, 1) in 4. These are the twistor spinors of S^4, as sections of S.

Proof. The representation count is as in Proposition 28.2, and the Callias count is a reference, with the exact count in Proposition 28.11. For the last sentence, the first-order operator from (1, 2) to (2, 3) in the slice limit is the twistor operator, and its kernel is the 4. ■

On a representation V the twisted operator has D^2 = Cas_G(V) + 3/2 - Cas_K(tau), for the twist tau, with Cas_G(4) = 5/2, Cas_G(16) = 15/2, Cas_G(20) = 21/2, Cas_G(40) = 29/2, Cas_K(j_a, j_b) = 2 j_a(j_a+1) + 2 j_b(j_b+1). With the twist (1, 3) this gives 0, 5, 8, 12, and with the twist (1, 1) on the representation 4 it gives 4. The slice operator reproduces these (Result 28.5).

Proposition 28.4 (the compression). Let s_v(g) = F lambda(g)^-1 v be a section in the multiplicity space of 4 on a slice, with B = F^* F, which commutes with lambda(H). Write t-hat = 2 i t_3 of Sp(1)_b. Then, with normalization by the norm of s_v,

```
Q(x_a) = (tr(B Gamma_5)/tr B) Gamma_a/5,
```

and for the functions y_T(g) = tr(T Ad(g) t-hat) with T in the Lie algebra sp(2), Q(y_T) is a multiple of T, proportional to tr(B t-hat)/tr B.

Proof. End_H(4) is spanned by 1, Gamma_5 and t-hat. Since int lambda(g) Gamma_5 lambda(g)^-1 x_a = int x_a x_b Gamma_b = Gamma_a/5, the part with Gamma_5 gives the first formula. Equivariant maps from 5 to End(4) are multiples of Gamma_a, and tr(Gamma_a Ad(g) t-hat) = 0, so 1 and t-hat contribute nothing. The second formula is Schur's lemma on the representation 10. ■

Consequently, a mode with F supported on (1, 2) and K-symmetric has tr(B Gamma_5)/tr B = +-1 and tr(B t-hat) = 0. That is the zero section: Q(x_a) = +-Gamma_a/5, and the part of a function that depends on the fibre gives zero. On the boundary with the pure level-one mode, B is the projection on (1)_{-1}, so that tr(B t-hat)/tr B = +-1. Example 16.9 has these values.

The operator and its two ends

Proposition 28.8 (endpoint restrictions). Restriction C^infinity(X, R) -> C^infinity(S^4, R) + C^infinity(CP^3, R) is surjective, and in normal coordinates a(y, v) = a(y, 0) + O(|v|), uniformly on bounded subsets of C^1(X). (Report, Proposition 1.)

Theorem 28.9 (a self-adjoint realization). Take the connection metric g = dr^2 + h(r)^2 pi^* g_{S^4} + rho(r)^2 g_{S^2} with h smooth, even and positive at 0, rho odd and smooth with rho(r) = r + O(r^3), and a product near r = l. Let Phi_M = M b(|v|^2) v . J with b smooth and positive, T_M = D_X^E + i Phi_M, and D-hat_M the doubled operator. At r = l set Z = -i c(dr), chi = Gamma-hat Z, and impose chi psi = -psi. Then D-hat_M on this H^1 domain is self-adjoint, elliptic and odd, with compact resolvent, and every smooth scalar function preserves the domain. No boundary condition enters at the zero section: after the outer condition is fixed, removing S^4 creates no further self-adjoint extension. (Report, Theorem 2, using Bär-Ballmann, Corollary 4.18.)

Proposition 28.10 (three slice operators). For the standard complex structure on CP^3: (1) the kernel of the Dolbeault-Dirac operator on O(1) is C^4 for every Hermitian metric; (2) the Levi-Civita Dirac operator twisted by the separate line bundle O(3) with its invariant connection vanishes on the 4 for every invariant squashing, which gives a kernel of dimension at least four, with any further kernel in other representations at a locally discrete set of squashings; (3) the Levi-Civita Dirac operator twisted by the full pull-back connection on E is invertible for small fibres, since (D^E_rho)^2 >= 1/(2 rho^2) - C_0 - C_1 rho^2. (Report, Proposition 3.) In Python, on the representation 4, the smallest |eigenvalue| of that operator is 3.235, 1.842, 1.200, 0.722, 0.148 at rho = 0.3, 0.5, 0.707, 1, 2. So the window comes from the competition between the normal operator and the mass.

Proposition 28.11 (normal model). Let Q = sigma . d_y + y . J on L^2(R^3; C^2 (x) C^4), with J Hermitian of spin 3/2. Then ker Q = 0 and ker Q^* = 3 (+) 1 under the normal Sp(1), with a positive gap above the zero modes; changing the sign of the mass exchanges the two kernels. The proof uses Q^* Q = -Laplacian + (y . J)^2 - sigma . J >= 0, with equality excluded since sum_i J_i^2 = 15/4, and the Callias index for the positive eigenbundles of n . J, of degrees 3 and 1. The induced bundle over S^4 is S^2(S) (+) 1, the operator on Sigma (x) S^2(S) has exactly four zero modes, in one chirality, and on the parallel inclusion S -> S (x) S^2(S) the twisted equation is a multiple of the twistor equation. So the kernel consists of harmonic spinors of the twisted operator, which are twistor spinors as sections of S. Rescaling gives the localization scale r = O(M^{-1/2}) at the core and O(M^{-1}) at the wall. (Report, Proposition 4.) In Python, in an oscillator basis: Q^* Q starts at 1.02, QQ^* has four near-zero values, with total spin 0 and three values near spin 1, then 1.02.

Lemma 28.12 (tangential mixing). On a product collar, normalized stable profiles for mass eigenvalues M q_i overlap by 2 sqrt(q_i q_j)/(q_i + q_j), independently of M, which is sqrt(3)/2 for q = 1/2, 3/2. So large mass does not remove the coupling between the stable channels O(3) and O(1), which in the pull-back connection has order 1/rho. (Report, Lemma 5.)

Proposition 28.13 (the boundary operator on the representation 4). Take the sign of Phi for which the right singular vector of T is at the zero section. Its left singular vector v is at the wall, in the domain Z v = +v of Theorem 28.9 for T^*. A wall profile exp(q(x - l)), q > 0, for T^* = i Z d/dx + D_6 + i phi tau,

with phi > 0, has q = -phi Z tau at leading order. So the stable channels of v are (Z, tau) = (+, -3/2) and (+, -1/2) twice, and they map to the channels (-, -1/2) twice. In unitary bases of these, the compression of D_6(rho) with the overlaps of Lemma 28.12 is

```
A(rho) = ( 3/(2 rho)   1/rho   rho + 1/rho )
         (     0       sqrt 2     -sqrt 2  )
```

with the matrix elements computed in Python, and reproduced by Terra 6.1 in exact symbolic arithmetic from the bracket and Clifford data. Its kernel is spanned by c = (-(2/3)(rho^2 + 2), 1, 1). The channel tau = -3/2 and one channel tau = -1/2 are supported on (1)_{-1} of the 4; the other channel tau = -1/2 is supported on (1)_{+1}. All three lie in Gamma_5 = -1, and their target spaces are mutually orthogonal, so the cross terms vanish in the traces of Proposition 28.4. Hence for the mode c

```
tr(B Gamma_5)/tr B = -1,    tr(B t-hat)/tr B = -w(rho),    w(rho) = 2(rho^2 + 2)^2 / (9 + 2(rho^2 +
↪  2)^2),
```

for every rho, with horizontal scale 1; for horizontal scale h, replace rho^2 by rho^2/h^2. So w = 8/17 as rho -> 0, w = 2/3 at the Fubini-Study radius rho = 1 (the nearly Kähler radius is 1/sqrt 2), and w -> 1 as rho -> infinity, the level-one value. The nonzero singular values of A are the next wall values on the 4: 1.863 and 2.789 at rho = 1, with the core value 2 in between. With the opposite sign of Phi, the right singular vector is at the wall, in the channels Z = -1, tau > 0, with fibre part +w(rho), and its left singular vector at the zero section.

For the connection preserving the line summands, the first row is replaced by (0, 1/rho, rho + 1/rho), the kernel is (1, 0, 0), so w = 1 for every rho, and the block on tau = -1/2 has singular values with sum of squares rho^2 + 2/rho^2 + 6 and product sqrt 2 (rho^2 + 2)/rho, so it is invertible for every rho > 0, with D^2 = 3, 6 at rho = 1. This is the collar repair of the report: an invariant connection preserving the four line summands in a collar extends into X with a cutoff, with the pull-back connection near the zero section. At the Fubini-Study metric the stable boundary model is then a sum of line-twisted Dirac operators, with four zero modes on O(3) and none on O(1), since H^q(CP^3, O(-1)) = 0 for all q.

Status. The kernel and the traces are exact for this finite block, which is the leading formal wall operator; the next correction is O(1/M) in the fixed-block expansion and does not change the limiting kernel, since the kernel is separated from the other singular values. That the boundary mode of T_M converges to it as M -> infinity is a numerical statement here (Result 28.5), since a uniform global reduction is unproved: the difference falls as phi_0 grows, from 0.037 to 0.009 at wall radius 0.5 and from 0.008 to 0.002 at wall radius near 1, between phi_0 = 16 and 32. The value 2/3 tests the factor sqrt 3/2, since without it w(1) would be 3/5.

Lemma 28.14 (a sufficient spectral criterion). Let U_M : C^8 -> Dom D-hat_M be an isometry with ||D-hat_M U_M|| <= eps_M -> 0 and ||D-hat_M psi|| >= delta ||psi|| for psi orthogonal to the range, with delta > 0 independent of M. Then for large M, D-hat_M has exactly eight eigenvalues in (-delta, delta), and the spectral space is within O(eps_M/delta) of the range. *Proof* (the Astra report, Lemma 6, written out at the request of the second Terra reading of lemma 29, 5 November 2026). D-hat_M is self-adjoint with compact resolvent (Theorem 28.9). Let P be its spectral projection for (-delta, delta). If rank P > 8, the range of P meets the orthogonal complement of the range of U_M nontrivially, against the complement estimate; if rank P < 8, the range of U_M meets ker P nontrivially, against ||D-hat_M U_M|| < delta. So rank P = 8. The spectral theorem gives ||(1 - P) U_M|| <= eps_M/delta, and the equality of the two finite ranks gives a bound of the same order for the difference of the two projections. ■ The missing steps for Conjecture 28.6 are the normal reduction with its errors, and the coercive estimate on every representation of Sp(2).

Conjecture 28.6 (the window). Take Theorem 28.9 with the sign of Proposition 28.13 and with (a) the pull-back connection and wall radius rho, or (b) the collar connection of Proposition 28.13 and the Fubini-Study metric at the wall. For large M the hypotheses of Lemma 28.14 hold, with the eight states being one copy of the 4 at the zero section in one Gamma-hat degree and one at the boundary in the other. For a family of metrics and mass profiles with a uniform positive decay rate, the off-diagonal block is exponentially small in l. The compression of a smooth function tends to (Q_B(a|*{S^4}), Q_Y(a/*{CP^3})), with Q_B

and Q_Y(x_a) = -Gamma_a/5, and Q_Y(y_T) a multiple of T, equal to the level-one value of Example 16.9 times w(rho) in case (a), and equal to it in case (b). This concerns compression maps, in line with Counterexample 28.17.

Result 28.5 (Python, the radial system). We solve T on the multiplicity spaces, with Hermitian Clifford generators, so that c(dx) = i Z in the skew convention of Theorem 28.9. For horizontal scale h, after absorbing the half-density h^2 rho, T = i Z d/dx + A_0/h + A_1/rho + (rho/h^2) A_2 + i phi tau, with -s(x) A_E/rho added for a collar connection. We take h = 1, so that T = i Z d/dx + D_6(rho(x)) + i phi(x) tau, with D_6 built from the structure constants of sp(2), rho(x) = rho_inf tanh(x/rho_inf), phi(x) = phi_0 tanh(x/w_0), w_0 = 0.3, in units with S^4 of radius 1, by the quadratic form of T^* T with P1 elements on a graded grid. This rho is not exactly a product near the wall. Rayleigh quotients of the exact form bound the values from above, and with three-point quadrature they approximate those bounds. Small entries have a mesh floor that depends on the parameters: at (phi_0, l) = (6, 4) the lowest value is 0.00548, 0.00401, 0.00355 for 2500, 5000, 10000 elements, so such entries do not establish a zero continuum value. At x = 0 the exponents are the values for R^3: +-1, +-2 (three times), +-3 on the 4.

phi_0	l = 2	l = 3	l = 4	next singular value, l = 4
3	0.406	0.159	0.066	1.293
4	0.288	0.088	0.028	1.452
5	0.200	0.045	0.011	1.665
6	0.132	0.021	floor	1.818
8	0.049	floor	floor	1.852

(rho_inf = 1, representation 4; the next value at larger phi_0 is 1.86 to 1.87.) The lowest value falls with the exponent of the constant-coefficient system, the smallest |Re kappa| of i Z (D_6(rho) - i phi tau): measured between l = 3 and 4, 0.877, 1.155, 1.464 at phi_0 = 3, 4, 5, against 0.856, 1.151, 1.476, as numerical evidence for these profiles, without a general tunnelling theorem. At large phi, kappa = phi/2 - 2/rho + O(1/phi) (phi = 64: 30.07 at rho = 1, 31.07 at rho = 2). The formula 2 mu exp(-mu l) of lemma 09 belongs to a separable interval and does not carry over.

On the two modes, at rho_inf = 1, l = 3, with the sign of Proposition 28.13, the boundary mode being the left singular vector; the core fibre parts at phi_0 = 16, 40 are below what the computation resolves:

phi_0	zero section: position, Q(x_a)/(Gamma_a/5), tr(B t-hat)/tr B	boundary: position, Q(x_a)/(Gamma_a/5), tr(B t-hat)/tr B, Z, tau
10	0.39, -1.0000, -3e-7	2.93, -1.0000, -0.645, 0.944, -1.117
16	0.28, -1.0000, 1e-11	2.96, -1.0000, -0.658, 0.976, -1.146
40	0.16, -1.0000, 1e-11	2.99, -1.0000, -0.663, 0.996, -1.162

Against Proposition 28.13, at l = 3:

rho_inf	wall radius	-w(rho)	phi_0 = 16	phi_0 = 32	next value, phi_0 = 16	second singular value of A
0.5	0.500	-0.529	-0.493	-0.521	1.992	1.992
0.7	0.700	-0.579	-0.562	-0.575	1.966	1.966
1.0	0.995	-0.665	-0.658	-0.663	1.864	1.865
1.5	1.446	-0.788	-0.785	-0.787	1.660	1.661
2.0	1.810	-0.861	-0.858	-0.860	1.557	1.556
3.0	2.285	-0.921	-0.918	-0.919	1.487	1.486

Collar connection. With rho(l) = 1 and s(x) = (1 + tanh((x - l + 1)/0.15))/2 multiplying the vertical connection term to be removed, at phi_0 = 8, 16, 32: the boundary mode has tr(B t-hat)/tr B = -1.0000 and Q(x_a)/(Gamma_a/5) = -1.0000, the next value is 1.733, that is sqrt 3, and the core mode is unchanged. The switch is nonzero, though tiny, at x = 0, whereas the smooth extension needs a cutoff vanishing near the zero section, so these rounded values support the limiting model and do not establish exact finite-mass equality.

Other representations. The multiplicity is zero on 1, 5, 10, 14, 35. At phi_0 = 16, rho_inf = 1, l = 3, the lowest singular values are 2.188 (wall) and 2.235 (core) on the 16, 1.920 (wall) and 2.828 (core) on the 20, and 3.381 (wall) and 3.463 (core) on the 40, the core values being sqrt 5, sqrt 8 and sqrt 12 up to the mass. The multiplicity zero follows from the central character. The finite wall operator A_V(rho) has no zero at 1500 sampled radii in [0.1, 20] on the 16, 20, 40, 64, a sampled statement; its smallest singular value tends to the core value as rho -> 0 and falls slowly at large rho on the 16 and the 40 (0.33 and 0.67 at rho = 20). Higher representations were not computed, and no estimate in the representation is proved.

Core block at finite mass. The density of Proposition 28.16 is nonzero at these points, stable in the mesh to five digits:

phi_0, l	lambda	tr(B t-hat)/tr B, integrated	max of abs tr(B t-hat) over max of tr B	sup of the pointwise ratio	integrated value over lambda^2
4, 3	0.0882	-5.71e-3	6.48e-3	0.849	-0.73
5, 3	0.0453	-1.84e-3	1.97e-3	0.856	-0.89
6, 3	0.0212	-4.55e-4	4.73e-4	0.858	-1.01

The pointwise ratio reaches its maximum at the outer wall, where the core density is small, so the breaking comes from the tail of the core mode at the wall. Over phi_0 = 3 to 6, l = 2 to 4, the integrated value over lambda^2 lies between -0.4 and -1.0, an empirical observation with no O(lambda^2) asymptotic proved. These values support the hypothesis of Proposition 28.16 at the sample points, conditional on the density being nonzero in the continuum.

The compression and the algebra

Theorem 28.15 (compression, conditional on localization). Assume an eight-dimensional spectral window splitting under Gamma-hat as W_M (+) V_M with dimensions 4, 4. Then scalar compression is exactly block diagonal. If the isometries differ by at most eta_M from model embeddings with angular compressions Q_B, Q_Y and normal measures with first moments at most C M^{-1/2} at the core and C M^{-1} at the wall, then

```
|| Q_M(a) - (Q_B(a|_B) (+) Q_Y(a|_Y)) || <= C ||a||_{C^1} M^{-1/2} + 4 ||a||_infinity eta_M.
```

(Report, Theorem 7.) For a spectral wall as in lemma 24, the domain has constant boundary values, and if a|Y = *c then any boundary-localized family has* //Q{Y,M}(a) - c|| <= C M^{-1} ||d_r a||_infinity. So localization just inside a spectral wall does not recover arbitrary boundary functions for a fixed smooth a, and the limit needs a local wall or an explicitly different comparison algebra.

Proposition 28.16 (the exact symmetry test). For an equivariant copy of 4, B(r) = F(r)^* F(r) = diag(b_a(r) I_2, b_+(r), b_-(r)). All real scalar functions compress to operators commuting with the quaternionic antiunitary j if and only if b_+(r) = b_-(r) for almost every r. If in addition the 5 component is nonzero, the block generates M_2(H). If both the 5 and the 10 components are nonzero, it generates M_4(C). A vanishing integral of tr(B(r) t-hat) alone does not decide this. If a radial h has Q_M(h) = alpha I_W + beta I_V with alpha != beta, then the block projections lie in the generated algebra. (Report, Proposition 8.) At the limit, the real linear image is Herm_2(H) + Herm_4(C), of real dimension 6 + 16 = 22, and the generated real algebra has dimension 16 + 32 = 48.

Counterexample 28.17 (generation is discontinuous under convergence of compression maps). Let R_0 = Q_B E_fib, with E_fib the fibre average on CP^3, R_1 = Q_Y, and R_eps = (1 - eps) R_0 + eps R_1. These are unital completely positive, ||R_eps - R_0|| <= 2 eps, and real functions generate M_4(C) under R_eps for every eps > 0, and M_2(H) at eps = 0. So the correct limit statement concerns specified compression maps, and it gives no convergence of the generated algebras. (Report, Counterexample 9.)

Theorem 28.21 (prescribed windows on the same X). Let P_y = (1 + y_a Gamma_a)/2 and p_z <= P_{pi(z)} the tautological rank-one projector. With normalized radial profiles chi_0, chi_l supported in disjoint eps-neighbourhoods of the two ends, U_0 v = sqrt 2 chi_0 P_y v and U_l v = 2 chi_l p_z v are isometries into L^2(X; C^4) (+) L^2(X; C^4). Real functions compressed by them generate M_2(H) + M_4(C) at every eps, and the compression differs from Q_B (+) Q_Y by at most C eps ||a||_{C^1}. (Report, Theorem 13.) This proves the connected-space mechanism without Conjecture 28.6, with the window as input. Allowing nonlocal smoothing perturbations, (1 - P) D (1 - P) + P d P makes such a window reducing, which enlarges the operator class and keeps the window as input.

The finite data

Proposition 28.18 (charge conjugation and the wall). With Hermitian generators J_i, C = C_7 (x) j_E has C^2 = -1 and commutes with T_M, D-hat_M and Gamma-hat as a differential expression, and CZ = -ZC. So it exchanges the two local wall conditions and does not preserve the chosen domain. If a pointwise parent antiunitary preserves a window and commutes with real multiplication, every real compressed multiplier commutes with its restriction, which cannot hold on a generated full M_4(C). (Report, Proposition 10.) The complex boundary block therefore requires this loss of the inherited symmetry, or a different real correspondence.

Remark 28.7 (why one block is quaternionic and the other complex). Proposition 28.18 proves the signs argument of the first version, with one correction of wording: j_E commutes with the anti-Hermitian Lie algebra action and reverses the Hermitian generators, so that C commutes with i Phi. Numerically (Python, phi_0 = 16, l = 3, rho_inf = 1), exchanging the wall condition exchanges (Z, tau, tr(B t-hat)/tr B) = (0.976, -1.146, -0.658) and (-0.976, 1.146, 0.658) on the boundary mode, with the same singular values, and leaves the zero-section mode unchanged. The formal parent symmetry does not by itself prove that the finite-mass core space is preserved, since C changes the wall domain, and the breaking computed in Result 28.5 shows the size of the effect at the sample points.

Proposition 28.19 (tunnelling and the grading). If the window consists of one copy of the 4 in each Gamma-hat degree, then d_M = [[0, conj(lambda_M) I_4], [lambda_M I_4, 0]] after equivariant unitary identifications, by Schur's lemma, and lambda_M = (1/4) Tr(U_{Y,M}^* T_M U_{B,M}). The inherited grading is I_W (+) (-I_V), the auxiliary grading of the doubling, which labels the two ends under localization. The radial Clifford involution Z does not extend across the zero section, so it does not serve as an interval chirality. This grading leaves all four core states in one degree, so W = H (+) H, for the even algebra H + H + M_4(C), still needs a chosen quaternionic grading, such as a marked Gamma_p. (Report, Proposition 11.)

Proposition 28.20 (the doubled bimodule). With H = W (+) V, Theorem 16.2 gives K = H (x) conj(H), of dimension 64, with DD = d (x) 1 + Gamma-hat (x) d^c. Its odd part is exactly K^- = Hom(V, W) (+) Hom(W, V), with the desired bimodule actions, and order zero and order one hold exactly on the full K, for its actual even algebra. If lambda != 0, then DD maps K^- into K^+, P_- DD P_- = 0 and DD^2 = 2 lambda^2. So K^- is not a reducing subspace. On K^- the inherited tensor grading is constant; the sheet-sign grading of Theorem 16.7 gives J(x, y) = (y^*, x^*), J^2 = 1, J Gamma_6 = -Gamma_6 J, with zero Dirac operator. The tunnelling map W -> V does not provide the map Hom(V, W) -> Hom(W, V) of lemma 20. (Report, Proposition 12.)

Balance against Proposition 6 of lemma 24

(Report, Proposition 14, with the numerical statements updated.)

Structure	What follows
Connected scalar parent	Supplied by the chosen disc bundle X.
Independent endpoint functions	Derived, by restriction and extension.
Base-function restriction at the collapsed end	Derived geometrically; quantitative for localized states.
Four core states	Proved for the linear normal model and the resulting twisted operator on S^4.
Four boundary states	Proved for the separated collar model; for the pull-back connection, the finite block of Proposition 28.13, with the convergence numerical.
Isolated 4 + 4 window	Conjecture 28.6.
Limiting quaternionic core compression	Derived once convergence to the core model is established.
Exact finite-mass quaternionic block	Requires b_+ = b_-; numerically not satisfied at the sample points resolved.
Exact level-one boundary compression	Requires rho -> infinity or the separated collar connection.
End grading	Inherited from the chosen self-adjoint doubling.
Quaternionic grading axis	Input.
The two Hom spaces	The odd part of the coarse correspondence; the selection is additional.
KO-6 reality and order conditions	Exact after the specified correspondence and grading choices.
Nonzero connector of lemma 20	Input.
Mass value and dependence on l	Radial matching; equivariance gives only lambda I_4; the exponent is computed numerically in Result 28.5.

Choices that stay inputs: X, E = S^3(S), the hedgehog, the wall condition, the collar connection in case (b), and the doubling of Theorem 16.2. A double with two identical collapsed zero sections removes the outer boundary condition, but the same local core construction gives two quaternionic endpoint maps, so obtaining a complex block needs different structure at one end. T0 stays open, and this is a candidate for its first half.

Direction. The next analytic step is the coercive estimate of Lemma 28.14, in case (b) first, where both end models have exact kernels. Since Phi commutes with the Clifford symbol, D + iM Phi is a concentrating pair in the sense of Maridakis (arXiv:1510.07004, Definition 2.1 and Lemma 2.2), with [D, Phi] of order zero; the symbol condition goes back to Prokhorenkov and Richardson (arXiv:math/0307251). Two things are missing there. The spectral separation theorem of arXiv:2110.11654 (section 2, Assumption 3) needs a leading endomorphism independent of the normal direction, and this fails here, since (n . J)^2 depends on n for spin 3/2; so a separate core analysis or an anisotropic extension is needed. And the wall adds a boundary layer: on a half-line ||(d_t + M) f||^2 = ||f'||^2 + M^2 ||f||^2 - M |f(0)|^2, and exp(-Mt) is a zero mode away from the zero set of the mass. (Read by Terra 6.1 in arXiv HTML; the LaTeX sources were not available to it.)

Numerical check

By Claude Opus 5.5, in Python (numpy, scipy), 3 Nov 2026, in a script that ran the tests below as assertions and printed the tables of Result 28.5; by the rule of the project the script is not kept (removed on 4 Nov 2026; Terra 6.1 had reproduced its matrix of Proposition 28.13 in exact arithmetic). The slice operator on each multiplicity space is Hermitian, odd under Z, and closes on Hom_H. The structure constants reproduce [t_1, t_2] = t_3 in Sp(1)_a and Sp(1)_b. The decomposition D_6(rho) = A_0 + A_1/rho + A_2 rho holds. The form uses three-point Gauss quadrature on each element. The exact tests are {A_0, A_1} =

0, A_0^2 = 0, 4 on the 4, 5, 9 on the 16, 0, 8 on the 20, 12, 16 on the 40 (the value 0 on the 20 has the twist (1, 5) and gives no core zero mode), A_2 = -(i/8) sum C_ab^alpha c_a c_b c_alpha, and at rho = 1 with the line connection the values D^2 = 0, 5, 5, 12 on the summand O(3) of the 4, 16, 20, 40, against j(j + 4) for (0, 0)-forms in O(1); the whole summand also has the value 10 on the 20. The closed form of Proposition 28.13 is tested to ten digits, and the core mode density in the mesh.

Part 29

The eight-state window on the disc bundle

Written by Claude Opus 5.5 on 4 November 2026, for STRATEGY T0, without an Astra job. It continues lemma 28 and answers the prompt 29_window_coercive_estimate.md. Refereed by Terra 6.1 (report): case (b) stands with amendments, and the two errors it found in case (a), the definition of A_eff and the sign of the fibre part, are corrected below. The referee ran no computation. A fresh reading of the corrected case (a) in the referee report on lemma 30 found one more sign and one missing step in Proposition 29.10; these are corrected. A second, fresh reading by Terra 6.1 (report, 5 November 2026, with the first reports withheld) found two errors, both repaired below: the range and quantifiers of delta (delta = 0 was a counterexample to the wording) and the sign of the Clifford commutator in the IMS step (the identity itself was right); it confirmed the chain of case (b) with amendments, the corrected case (a) with two added steps, and listed what a stand-alone paper needs (section "Second reading" below).

Outcome

1. **Conjecture 28.6 holds in case (b)** for the class of metrics and mass profiles of Setting 29.0: for each fixed 0 < delta < sqrt 3 there is M_delta such that, for M >= M_delta, D-hat_M has exactly eight eigenvalues in (-delta, delta) (Theorem 29.7).
2. **Conjecture 28.6 holds in case (a) at the Fubini-Study radius** rho = 1, with fibre part -2/3 (Theorem 29.11). At another wall radius, case (a) follows from one kernel condition K(rho), which reduces at each fixed radius to finitely many representations by ellipticity; an explicit bound identifying those representations remains to be written.
3. **The off-diagonal block is exponentially small**, |lambda_M| <= C exp(-(1 - eta)(M/2) int f dr), by an exact tunnelling identity and Agmon estimates (Proposition 29.8).
4. **The compression converges** to (Q_B(a|*{S^4}), Q_Y(a/*{CP^3})) with error O(M^{-1/2}) (Theorem 29.9). The statement concerns compression maps, in line with Counterexample 28.17.

The proof is short because Sp(2)-equivariance gives exact sums of squares at the core and at the wall. Each estimate holds on all of L^2 of a fixed manifold, with the boundary operators elliptic on CP^3 or S^4, so uniformity over the representations of Sp(2) needs no separate argument. The spectral separation hypothesis of Maridakis, which fails for this mass (historical/reviews/2026-11-03_terra_check_28.md, item 6), is never used. General forms of two identities are in the literature: the square with its zero-order cross term in the bulk (Maridakis, arXiv:1510.07004, Lemma 2.2), and a horizontal and vertical factorization with a curvature contraction (Kaad and van Suijlekom, arXiv:1710.03182, Proposition 18, under other hypotheses).

Scope. These are spectral statements for a chosen metric family, with flat fibres near the zero section. They give the finite window from the operator, for the sign of Proposition 28.13. They leave open: other metrics in the class of Theorem 28.9, including product-collar modifications of Bryant-Salamon-type metrics, and the complete Bryant-Salamon geometry (Theorem 28.9 assumes a product metric near a finite wall); the connector of Proposition 28.20; the quaternionic grading of Proposition 28.19; any dynamics selecting M, l or rho. Finite-mass generated algebras stay outside the statement (Counterexample 28.17).

Setting

Setting 29.0. Let X be the disc bundle of lemma 28, G = Sp(2), K = Sp(1)_a x Sp(1)_b, H = Sp(1)_a x U(1)_b, E = S^3(S), Phi_M = eps M f(r) n . J, and T_M = D_X^E + i Phi_M, with D-hat_M the

doubled operator and the wall condition of Theorem 28.9: Z u = -u for T_M, and Z v = +v for T_M^*. Assume:

- (H1) g = dr^2 + pi^* g_{S^4} + rho(r)^2 g_{S^2}, with S^4 of radius 1, rho(r) = r on [0, 2e], and rho constant, equal to rho_l, on the collar [l - 2a, l]. Case (b) and Theorem 29.11 take rho_l = 1, the Fubini-Study metric.
- (H2) f(r) = r on [0, 2e], f = f_0 > 0 on the collar, and f > 0 on (0, l]. This is M b(|v|^2) v . J with b positive.
- (H3) The connection on E is the pull-back connection on r <= 2e. In case (b) it preserves the four line summands on the collar, with any invariant interpolation in between. In case (a) it is the pull-back connection throughout. We use radial parallel gauge: nabla_{d/dr} = d/dr. An invariant dr-component a(r) q could otherwise be added to a line-preserving connection; it is removed by radial parallel transport, which changes the displayed formulas by a phase on each line.
- (H4) eps = -gamma_b, where gamma_b is the chirality of Sigma_4 whose K-type is (1, 2). This is the sign of Proposition 28.13: its right singular vector is at the zero section. The opposite sign exchanges T_M and T_M^* throughout. In the convention of the radial computation of lemma 28, gamma_b = +1.

On the collar put q = eps n . J, with eigenvalues +-3/2, +-1/2 on the line summands, and m = M f_0. In general q = -gamma_b n . J. In the notation of Proposition 28.13 and of the radial computation of lemma 28, q = -tau, and T_M = i Z d/dx + A + i m q on the collar, with x increasing towards the wall. The line q = +3/2 is the summand O(3), and q > 0 are the stable channels of Proposition 28.13.

Let C_R = ||A_2||, the norm on Sigma_6 (x) E of the curvature contraction A_2 = -(i/8) sum C_ab^alpha c_a c_b c_alpha; it does not depend on V, and C_R = 1 in Python (a finite C_R is all the proofs use). Let mu_1 > 0 be the smallest positive eigenvalue of Q^* Q for Q = sigma . d_y + y . J of Proposition 28.11. It is positive since the resolvent is compact, because (y . J)^2 >= |y|^2 / 4, and ker Q = 0: with M = 1, Q^* Q = (-Delta + |y|^2/4 - 3/2) + ((y . J)^2 - |y|^2/4) + (3/2 - sigma . J), three nonnegative terms; equality in the first forces a Gaussian times a constant vector, and equality in the second would then force J_i^2 v = v/4 for all i, against sum_i J_i^2 = 15/4 (referee). Python gives 1.02.

The zero section

Proposition 29.1 (exact splitting). Identify sections over the tube r <= 2e with K-equivariant maps s : G -> F, F = Sigma_4 (x) L^2(R^3; C^2 (x) C^4), with c(e_a) = gamma_a (x) 1 and c(f_alpha) = gamma_5 (x) sigma_alpha. Let D_H = i sum gamma_a e_a, with e_a acting as left-invariant derivatives on G, and Q^ = gamma_5 sigma . d_y + eps M y . J. Then on this tube T_M = T_0 + R, with T_0 = D_H + i Q^ and R Hermitian of order zero with |R| <= C_R |y|, and

```
T_0^* T_0 = D_H^2 + Q^^* Q^,    T_0 T_0^* = D_H^2 + Q^ Q^^*,
```

with D_H^2 commuting with Q^^* Q^ and Q^ Q^^*.

Proof. The fibres are flat and totally geodesic by (H1), so the Levi-Civita connection of X differs from the connection associated with the principal K-bundle G -> S^4 only by the O'Neill tensor. Its component is the curvature of N evaluated at v. This gives the zero-order term R, which equals r A_2 in the radial form D_6 = A_0 + A_1/rho + A_2 rho, of norm at most C_R |y|. For the pull-back connection, the remaining operator is D_H + i gamma_5 sigma . d_y + i Phi_M. The operator Q^ acts on the values of s, and it is K-equivariant, since sigma . d_y and y . J are invariant under the simultaneous rotation of y, of C^2 and of C^4. On the ambient space of fibre-valued functions on G, Q^ commutes with each e_a, because it does not depend on the point of G; the individual e_a need not preserve the K-equivariant subspace, but both Q^ and the summed operator D_H do. Since gamma_a anticommutes with gamma_5 and commutes with y . J, D_H Q^ = Q^^* D_H, where Q^^* = -gamma_5 sigma . d_y + eps M y . J. Therefore

```
T_0^* T_0 = (D_H - i Q^^*)(D_H + i Q^) = D_H^2 + Q^^* Q^ + i (D_H Q^ - Q^^* D_H) = D_H^2 + Q^^* Q^,
```

and likewise for T_0 T_0^*. Finally D_H Q^^* Q^ = Q^ Q^^* D_H, so D_H^2 commutes with both. ■

In radial form. T_0 = i Z d/dx + A_0 + A_1/rho + i phi q, and the cross terms vanish by {A_0, Z} = 0, [A_0, tau] = 0 and {A_0, A_1} = 0. these three identities were asserted on the representations 4, 16, 20, 40 (Python, script not kept).

Proposition 29.2 (the model spectrum). On the total space of N, with f = r and T_0 as above, let Pi be the projection onto ker T_0. Then:

1. ker T_0 is one copy of the 4 of G; ker T_0^* = 0.
2. ||T_0 u||^2 >= min(4, mu_1 M) ||u||^2 for u orthogonal to ker T_0, and ||T_0^* v||^2 >= min(4, mu_1 M) ||v||^2 for all v in the domain.

Proof. Write Q^_gamma for Q^ on chirality gamma. By Proposition 28.11, Q^_gamma has kernel exactly when gamma = -eps; this is the rank-four K-representation W_0 = (1, 3) + (1, 1), and on the orthogonal complement Q^^* Q^ >= mu_1 M, after the rescaling y -> M^{-1/2} y. The same bound holds on the other chirality, and for Q^ Q^^* with the chiralities exchanged, since Q^* Q and Q Q^* have the same nonzero spectrum. The fibrewise projection P_W onto W_0, on both chiralities of Sigma_4, commutes with D_H, by Proposition 29.1. (The projection onto the fibre kernel of Q^ also selects gamma = -eps, so it does not commute with D_H, which is odd; it commutes with D_H^2.) On chirality -eps, Q^ vanishes on the range of P_W and is bounded below by sqrt(mu_1 M) on its complement; on chirality +eps, it is bounded below by sqrt(mu_1 M) throughout. On Sigma^gamma (x) W_0, D_H is the Dirac operator of S^4 twisted by the homogeneous bundle of W_0, so by Parthasarathy's formula D_H^2 = Cas_G(V) + 3/2 - Cas_K(tau), with Cas_K = 2 j (j + 1) on Sp(1)_b, that is 0 and 4 on W_0. Only V odd under -1 occur, with Cas_G(V) >= 5/2 and Cas_G(V) >= 15/2 for V not the 4. So D_H^2 >= 4 except for D_H^2 = 0 at V = 4, twist j = 1. This kernel needs Hom_K(4, Sigma^gamma (x) (1, 3)) != 0, which holds exactly for Sigma^gamma = (1, 2), where (1, 2) (x) (1, 3) contains (1, 2) once. By (H4), gamma_b = -eps, so T_0 sees this chirality and T_0^* the other. Decomposing u by chirality and by P_W, and using the sum of squares, gives both bounds. The connection metric on the total space N is complete, and the doubled model is a symmetric Dirac-type operator plus a smooth Hermitian zero-order potential, hence essentially self-adjoint on C_c^infinity(N); the identities and estimates extend from this core to its closure, and no boundary condition enters at infinity. The S^4 spectrum has a direct check in the literature: Dolan's homogeneous-instanton calculation (hep-th/0304037, section 2(ii), equations (40)-(46)) gives four zero modes for isospin 1 and the next squared value 5, with the untwisted squared spectrum starting at 4 (referee). The equivariant Callias calculation that identifies ker Q^* as 3 + 1, and not only its dimension, is stated here with the conventions of Proposition 28.11 and is one of the items a stand-alone paper must display. ■

Lemma 29.3 (core estimate). Let u be in H^1 with support in r < 2e. Then

```
|| |y| u || <= (2/M) ||T_0 u|| + sqrt(10/M) ||u||,
```

and the same holds with T_0^*. So ||T_M u|| >= (1 - 2 C_R/M) ||T_0 u|| - C_R sqrt(10/M) ||u||, and likewise for T_M^* with T_0^*. In particular, for such u with Pi u = 0, and for all such v,

```
||T_M u|| >= (2 - o(1)) ||u||,    ||T_M^* v|| >= (2 - o(1)) ||v||,    M -> infinity.
```

Proof. Q^^* Q^ = -Laplacian_y + M^2 (y . J)^2 - eps M gamma_5 sigma . J. The eigenvalues of sigma . J are 3/2 and -5/2, and (y . J)^2 >= |y|^2/4. Hence M^2 || |y| u ||^2 / 4 <= ||Q^ u||^2 + (5/2) M ||u||^2 <= ||T_0 u||^2 + (5/2) M ||u||^2, by Proposition 29.1. The rest is the triangle inequality with |R| <= C_R |y|, and Proposition 29.2. ■

The bulk

Lemma 29.4 (bulk estimate). Let w be in H^1 with support in a compact subset of 0 < r < l, and let f_1 be the minimum of f on that support. Then

```
||T_M w||^2, ||T_M^* w||^2 >= (M^2 f_1^2 / 4 - C_Phi M) ||w||^2,
```

with C_Phi the supremum of the operator norm of the Clifford contraction sum_i c(e_i) nabla_{e_i} (f n . J). With 0 < r_1 < r_2 < l and a radial cutoff psi equal to 1 on [r_1, r_2], with compact support in 0 < r < l, we have for u in Dom T_M,

||u||^2_{L^2(r_1 <= r <= r_2)} <= C M^{-2} (||T_M u||^2 + ||u||^2).

Proof. ||(D + i Phi) w||^2 = ||D w||^2 + ||Phi w||^2 + <w, i [D, Phi] w>, with [D, Phi] of order zero and pointwise bounded independently of V, and Phi^2 >= M^2 f^2 / 4. For the second statement, choose 0 <= psi <= 1 equal to one on the compact set under consideration; the first estimate applied to psi u with T_M (psi u) = psi T_M u + [D, psi] u gives ||psi u||^2 <= 2 (||T_M u||^2 + ||d psi||_infinity^2 ||u||^2) / (M^2 f_psi^2/4 - C_Phi M), with f_psi the minimum of f on the support of psi. Applied to compact sets containing the supports of the d chi_i and of chi_b, this gives the O(M^{-2}) bounds used in the transition regions below. ■

The interpolation of the connection in (H3) takes place in this region, where it enters only through C_Phi.

The wall

Lemma 29.5 (line-twisted operators at the Fubini-Study metric). Let D_k be the Dirac operator on CP^3 twisted by O(k) with its invariant connection, k odd, at rho = 1. Then D_k^2 = C_2(L) + 3/2 - (3/8) k^2 on the SU(4)-type L, with C_2 normalized by C_2(4) = 15/8. Consequently:

1. On O(3): ker D_3 is one copy of the 4 of Sp(2), in Z = +1; and D_3^2 >= 5 on the V-isotypic part for V not the 4.
2. On O(1): ker D_1 = 0; D_1^2 >= 3 on the 4 and D_1^2 >= 6 on the V-isotypic part for V not the 4.

So the smallest nonzero |eigenvalue| on O(3) + O(1), over all V, is kappa_w = sqrt 3, attained on the 4; the lowest wall value on the 16 is sqrt 5.

Proof. The form of D_k^2 is Parthasarathy's formula for SU(4)/U(3), since the twist is a character of U(3). The Sp(2)-invariant connection on a line summand is unique, because the isotropy representation has no trivial summand, so it is the Chern connection. The normalization: j(j + m + 3) is the eigenvalue of 2 box on (0,0)-forms in O(m) on the type [m + j, 0, j], with C_2([m + j, 0, j]) - C_2([m, 0, 0]) = j (j + m + 3), which fixes the coefficient of C_2; scalar curvature 12 gives the constant 3/2; and the term in k^2 is fixed by the kernel of D_3 on the 4. Two checks: D_0^2 = 4 on the type 6, the Kirchberg limit R/3 at R = 12 (Ann. Global Anal. Geom. 4 (1986) 291-325); and the values in the Python table below. The kernel statements are H^q(CP^3, O(1)) = C^4 at q = 0 only, and H^q(CP^3, O(-1)) = 0, with K^{1/2} = O(-2). The types of odd 4-ality with C_2 < 55/8 are the 4, its conjugate, the 20 and its conjugate (enumerated in Python). The 4 occurs on O(3) once, by the multiplicity on the slice, and this copy is the kernel; since each 20 restricts to Sp(2) as 4 + 16, neither 20 occurs on O(3). So V != 4 has C_2 >= 55/8 on O(3), giving D_3^2 >= 5. On O(1), the 4 and its conjugate restrict only to the 4 of Sp(2), so V != 4 has C_2 >= 39/8, giving D_1^2 >= 6; and on the 4 Frobenius reciprocity gives the relevant SU(4) types 4 and 20, each in both chiralities, with Casimirs 15/8 and 39/8, hence the two squared values 3 and 6 on the slice block. The chirality of ker D_3 is that of (0,0)-forms, and the slice block fixes it as Z = +1 for q = +3/2 (Python). The line-twisted spectrum on odd complex projective spaces is in print: Ben Halima, Comment. Math. Univ. Carolin. 49 (2008), Theorem 1, with the tautological line O(-1) and scalar curvature 3; taking its exponent m = -k and multiplying its squared eigenvalues by 4 for scalar curvature 12 reproduces the zero and the gap 5 for k = 3 and the gap 3 for k = 1 (referee); the Sp(2) branching and the Z-sign are recorded here in the present conventions. ■

Lemma 29.6 (wall estimate in case (b)). Let w be in H^1 with support in the collar, vanishing near its inner end, and p(x) = sqrt(3 m) exp((3/2) m (x - l)), so that the kernel of T_M^* on the half-cylinder is p (x) ker D_3. Then:

1. If Z w = -w at r = l, then ||T_M w||^2 >= min(m^2/4, 3) ||w||^2.
2. If Z w = +w at r = l, and w is orthogonal to p (x) ker D_3, then ||T_M^* w||^2 >= min(m^2/4, 3) ||w||^2.

Proof. On the collar, A commutes with q and with d/dx, and anticommutes with Z. Writing w = w_+ + w_- by Z, integration by parts gives, for T_M,

||T_M w||^2 = ||(d + m q) w_+||^2 + ||A w_+||^2 + ||(d - m q) w_-||^2 + ||A w_-||^2,

where the sum of the real parts of the mixed terms is the boundary term obtained by integrating d/dx <w_+, A w_->, which vanishes because w_+(l) = 0 (the individual mixed terms need not be purely imaginary). For T_M^* the same holds with q replaced by -q and with w_-(l) = 0. There are three one-dimensional cases on (-infinity, l]. (i) If g(l) = 0, then ||(d +- m q) g||^2 = ||g'||^2 + m^2 q^2 ||g||^2. (ii) With g(l) free, ||(d + m |q|) g||^2 = ||g'||^2 + m^2 q^2 ||g||^2 + m |q| |g(l)|^2. (iii) With g(l) free, ||(d - m |q|) g||^2 is the form of -d^2 + m^2 q^2 with a Robin condition at l, whose spectrum consists of 0, on the profile sqrt(2 m |q|) exp(m |q| (x - l)), and of [m^2 q^2, infinity). Cases (i) and (ii) give the lower bound m^2/4. Case (iii) arises exactly in the channels (Z, q) = (-, q > 0) for T_M and (+, q > 0) for T_M^*. On those, expand in eigenvectors of A^2: the term ||A w||^2 is at least 3 ||w||^2, except on ker D_3 in Z = +1, which by Lemma 29.5 only arises for T_M^* at q = 3/2, where the Robin gap (9/4) m^2 applies. ■

The window

Theorem 29.7 (the window, case (b)). Under Setting 29.0 with rho_l = 1 and connection (b), for each fixed 0 < delta < sqrt 3 there is M_delta such that, for M >= M_delta, D-hat_M has exactly eight eigenvalues in (-delta, delta). The spectral subspace splits under Gamma-hat as W_M (+) V_M, each a copy of the 4 of Sp(2), with W_M within O(M^{-1/2}) of the cut-off core modes in the domain of T_M, and V_M within O(M^{-1/2}) of the cut-off wall modes p (x) ker D_3 in the domain of T_M^*.

Proof. We verify Lemma 28.14. Take radial functions chi_c, chi_b, chi_w with chi_c^2 + chi_b^2 + chi_w^2 = 1, chi_c = 1 on r <= e with support in r < 2e, chi_w = 1 on [l - a, l] with support in the collar. Let U_M span (chi_c u_i, 0) and (0, chi_w v_k), where u_i is a basis of ker T_0 and v_k of p (x) ker D_3, with Gram-Schmidt. Radial functions preserve the wall condition.

Trial states. T_M (chi_c u_i) = [D, chi_c] u_i + R chi_c u_i, of norm O(M^{-1/2}), since u_i has Gaussian decay at the scale M^{-1/2}; and T_M^* (chi_w v_k) = [D, chi_w] v_k, which is O(exp(-(3/2) m a)). So eps_M = O(M^{-1/2}).

Coercivity. Since [D, chi] = c(d chi) is skew-adjoint with c(d chi)^* c(d chi) = |d chi|^2 (its square is -|d chi|^2: for instance [D, x] = i Z on the collar), while sum_i chi_i d chi_i = 0,

```
||T_M u||^2 = sum_i ||T_M chi_i u||^2 - sum_i || |d chi_i| u ||^2,
```

and likewise for T_M^*. The functions d chi_i vanish outside the bulk region e <= r <= l - a. So by Lemma 29.4,

```
(1 + C M^{-2}) ||T_M u||^2 >= ||T_M chi_c u||^2 + ||T_M chi_w u||^2 - C M^{-2} ||u||^2,    ||chi_b
↪  u||^2 <= C M^{-2} (||T_M u||^2 + ||u||^2).
```

If u is orthogonal to every chi_c u_i, then Pi (chi_c u) = 0 exactly, since <u_i, chi_c u> = <chi_c u_i, u>, so Lemmas 29.3 and 29.6 (1) give ||T_M u||^2 >= (3 - o(1)) ||u||^2. If v is orthogonal to every chi_w v_k, then chi_w v is orthogonal to p (x) ker D_3, so Lemmas 29.3 and 29.6 (2) give the same for T_M^*. Since ||D-hat (u, v)||^2 = ||T_M u||^2 + ||T_M^* v||^2, Lemma 28.14 gives the count and the distance to the range. The window is G-invariant and close to the range of U_M, so as representations W_M and V_M are copies of the 4. ■

Remark. The gap is set by the wall at sqrt 3, with the core next at 2. On the 16, 20 and 40 the lowest values are sqrt 5, sqrt 5, sqrt 12.

Proposition 29.8 (tunnelling). Let T_M u = lambda v and T_M^* v = lambda u be a singular pair of the window, with unit vectors. By Proposition 28.19, the four singular values coincide. For a radial chi equal to 1 near r = 0 and 0 near r = l,

```
lambda (<u, chi u> - <v, chi v>) = <v, c(d chi) u>.
```

For every eta in (0, 1) there are M_eta and C_eta with

```
|lambda_M| <= C_eta exp(-(1 - eta)(M/2) int_e^{l - a} f dr),    M >= M_eta.
```

Proof. The function chi u lies in Dom T_M, since chi vanishes near the wall. So <T_M^* v, chi u> = <v, T_M chi u>, and T_M chi u = chi lambda v + c(d chi) u. This gives the identity.

For the decay of u, let xi = psi exp(g) be radial, with psi = 0 on r <= e/2 and psi = 1 on r >= e, with g = 0 on r <= e, g' = (1 - eta)(M/2) f on [e, r_*], and g constant on r >= r_*. Since xi^2 u lies in Dom T_M,

```
||T_M (xi u)||^2 = lambda^2 ||xi u||^2 + || |d xi| u ||^2.
```

Choose the partition so that the derivative of the Agmon weight is supported where the bulk cut-off is one. The IMS lower bound, with no core piece and with Lemma 29.6 (1), or Proposition 29.10 in case (a), on the collar, reads int (M^2 f^2/4 - C M) chi_b^2 |xi u|^2 + kappa_w^2 ||chi_w xi u||^2 - sum_i || |d chi_i| xi u ||^2 <= lambda^2 ||xi u||^2 + || |d xi| u ||^2, and the transition terms are controlled by the bulk estimate of Lemma 29.4. On the bulk the excess over (g')^2 is at least (eta/4)(M^2 f^2 / 4) for M large, with f >= f_1 > 0, so ||exp(g) u||*{L^2(r >= e)} <= C_eta. For v we use the same argument from the wall inward, with g-tilde = 0 on [l - a, l], with -g-tilde' = (1 - eta)(M/2) f, and with Lemma 29.3 replacing Lemma 29.6 on the core, since T_M^* has no low states there. Take chi with |d chi| <= 2/s, supported on [r_, r_ + s] in the bulk. By Theorem 29.7, <u, chi u> -> 1 and <v, chi v> -> 0, so the identity gives |lambda| <= (4/s) ||u||{[r_, r_ + s]} ||v||{[r_, r_ + s]}. Run the weights with eta' = eta/2, and take s so small that (1 - eta')(I - int{r_}^{r_ + s} f) >= (1 - eta) I, with I = int_e^{l - a} f.* This gives the stated exponent. The absorption uses lambda -> 0, supplied by the trial states. The factor 1/2 holds on every channel, since (n . J)^2 >= 1/4; for the adjoint, the cut-off vanishes near the wall and the core estimate replaces the wall estimate. For a family with l -> infinity the collar widths, the cut-off derivatives, the connection bounds and the positive lower bound of f on the bulk have to be uniform, and the excluded end lengths are absorbed in the spare eta (referee). ■

For a family with l -> infinity, M >= M_eta, with uniform bounds on the end geometry, the cutoffs, the connection and the positive bulk mass, the constants are uniform, so lambda is exponentially small in l at the rate (1 - eta)(M/2) inf f. This is the uniform decay rate required by Conjecture 28.6, as amended by the referee.

Theorem 29.9 (compression). Under Theorem 29.7 or Theorem 29.11, for a in C^1(X),

```
|| Q_M(a) - (Q_B(a|_{S^4}) (+) Q_Y(a|_{CP^3})) || <= C ||a||_{C^1} M^{-1/2},
```

where Q_B(a) = int a(x) 2 P_x d mu(x), with P_x = (1 - x . Gamma)/2, and Q_Y(a) = int a(z) B(z) d mu(z). In case (b), B(z) = 4 p_z, the level-one map of Example 16.9. In case (a) at radius rho, B(z) = 2((1 + w) p_z + (1 - w) p_{jz}), with p_{jz} the quaternionic partner of p_z and w = w(rho); so Q_Y(x_a) = -Gamma_a/5, Q_Y(y_T) is w(rho) times the level-one value, and the signed fibre part is -w(rho). Restriction to the two ends is surjective (Proposition 28.8), so real functions on X generate M_2(H) + M_4(C) under the limit map, by lemma 10 and Example 16.9, for every w(rho) > 0.

Proof. Theorem 28.15, with eta_M = O(M^{-1/2}) from Theorems 29.7 and 29.11, after choosing equivariant unitary identifications of the window spaces with the model copies of the 4. The normal first moments are O(M^{-1/2}) at the core, by the Gaussian decay of the model, and O(M^{-1}) at the wall, by the profiles. At the core, F in Hom_K(4, (1, 2) (x) (1, 3)) kills (2, 1), so B = 2 P_x in the normalization Q_B(1) = I. The sign in P_x is the value tr(B Gamma_5)/tr B = -1 of Proposition 28.13. At the wall, the pointwise inner product of different line summands vanishes, so the density is a sum over channels with weights from Proposition 28.13. ■

Case (a)

On the collar in case (a), A = A_d + A_o, where A_d = D_line preserves the lines and A_o = A_E/rho has order zero, with [A_o, q] != 0. In Python, the block of A_E between q = 3/2 and q = 1/2 has norm sqrt 3 and ||A_E|| = 2, on the full fibre, hence for all V. Let P be the projection onto {sum_{q > 0} p_q g_q : Z g_q = -g_q}, with p_q = sqrt(2 m q) exp(m q (x - l)), and Y the same with Z = +1, with Pi_Y the projection onto the second space. Let F = Pi_Y (A_d + A_o) P, from the range of P to Y, and let A_eff be the operator on the sum of both ranges with blocks F and F^*. Identified with sections over CP^3 of the lines q > 0, it is, by Lemma 28.12,

```
A_eff = D_3(rho) (+) D_1(rho) + (sqrt 3 / 2) B(rho),
```

with B(rho) the block of A_o between the stable channels, of norm sqrt 3 / rho. This is self-adjoint and elliptic on the fixed manifold CP^3, so its nonzero spectrum has a gap kappa_a(rho) > 0, uniformly in V. On the 4, F^* is the matrix A(rho) of Proposition 28.13, from Y to the range of P.

Condition K(rho). The kernel of A_eff(rho) is exactly one copy of the 4, in Z = +1, so F is injective and ker F^* is that copy.

Proposition 29.10 (wall estimate in case (a)). Assume K(rho). For w as in Lemma 29.6, with the kernel p (x) ker D_3 replaced by the kernel of the half-cylinder operator T_M^*, the two statements of Lemma 29.6 hold with the constant 3 replaced by kappa_a(rho)^2 - O(1/m). The half-cylinder kernel is exactly one copy of the 4 for m large, with fibre part tending to -w(rho) at rate O(1/m).

Proof. T_d = i Z d/dx + A_d + i m q preserves the block structure: T_d P = Pi_Y T_d P, since T_d (p_q g) = p_q A_d g. Also T_d maps the complement of P to the complement of Y, because for y in Y the function T_d^* y = A_d y lies in P. On the complement of P, the sum of squares of Lemma 29.6, with the channels with Robin kernel removed, gives the bound m/2 both for T_d : P^perp -> Y^perp and for its adjoint, so this block is bijective; for m > 2 ||A_o|| its bounded perturbation B_22 = (1 - Pi_Y)(T_d + A_o)(1 - P) is bijective with ||B_22^{-1}|| <= (m/2 - ||A_o||)^{-1}, which is the surjectivity the Schur complement F^* - B_21^* (B_22^*)^{-1} B_12^* needs, with B_12 = Pi_Y A_o (1 - P) and B_21 = (1 - Pi_Y) A_o P (referee). So with w = w_1 + w_2 and w_1 = P w,

```
||T_M w|| >= ||(1 - Pi_Y) T_M w|| >= (m/2) ||w_2|| - ||A_o|| ||w||,
||T_M w|| >= ||Pi_Y T_M w|| >= ||F w_1|| - ||A_o|| ||w_2||.
```

Eliminating ||w_2||, one gets (1 + 2 ||A_o|| / m) ||T_M w|| >= ||F w_1|| - (2 ||A_o||^2 / m) ||w||. Under K(rho), F is bounded below by kappa_a(rho), and F^* is bounded below on the complement of its kernel. The same elimination holds for T_M^*, with P and Y and F and F^* exchanged. On the half-cylinder T_M is thus injective with closed range, so T_M^* is surjective; its kernel is the kernel of the Schur complement of the block T_d + (1 - Pi_Y) A_o (1 - P), whose inverse has norm at most (m/2 - ||A_o||)^-1. So this Schur complement differs from F^* by at most ||A_o||^2 / (m/2 - ||A_o||), and the graph of the Schur solution gives ||P_{ker T_M^*} - P_{ker F^*}|| = O(1/m). This projection estimate turns orthogonality to the exact half-cylinder kernel into orthogonality to ker F^* up to O(1/m), which gives the adjoint bound. Off the 4, this is invertible. On the 4 the block is a 2 x 3 matrix of rank 2, so its kernel has dimension exactly 1, within O(1/m) of (-(2/3)(rho^2 + 2), 1, 1). ■

Theorem 29.11 (the window in case (a)). Under Setting 29.0 with the pull-back connection, K(1) holds, and for each fixed 0 < delta < min(2, kappa_a(1)) there is M_delta such that for M >= M_delta the conclusion of Theorem 29.7 holds at the Fubini-Study radius, with the wall modes replaced by the kernel of Proposition 29.10, and with fibre part tending to -w(1) = -2/3. Here kappa_a(1) >= sqrt 5 - 3/2 off the 4, and it equals 1.863 on the 4, the squared values there being (45 +- 3 sqrt 33)/8. At another wall radius the same conclusion holds under K(rho).

Proof. By Lemma 29.5, on V != 4 we have |D_3 (+) D_1| >= sqrt 5, while ||(sqrt 3/2) B(1)|| = 3/2. So A_eff is invertible there, with lower bound sqrt 5 - 3/2 > 0. On the 4, F^* is A(1), with kernel of dimension one in Z = +1, the vector (-2, 1, 1), and nonzero singular values 1.863 and 2.789, by Proposition 28.13. So K(1) holds. The proof of Theorem 29.7 then applies with Proposition 29.10 in place of Lemma 29.6. The trial wall states are cut-offs of the exact half-cylinder kernel: on the 4-isotypic component the adjoint kernel equation is v' = (m Z q + i Z A_4) v, with A_4 a fixed finite Hermitian matrix anticommuting with Z, so the coefficient is Hermitian, every L^2 solution on the negative half-cylinder lies for large m in its positive spectral subspace, whose eigenvalues are at least m/2 - ||A_4||, and its mass beyond distance a is O(exp(-(m/2 - ||A_4||) a)); the cut-off defect is exponentially small. The finite matrix A(rho) of Proposition 28.13 is used here as given; its derivation from the geometric data, with bases and signs, is not in the live text and is an obligation for a stand-alone paper (referee). ■

Other radii. For K(rho), only finitely many V can carry a kernel, those with |D_3(rho) (+) D_1(rho)| <= 3/(2 rho), by ellipticity. In Python, on the 16, 20, 40 and for rho in [0.2, 4], the compressed block has smallest singular value at least 1.30. A check of K(rho) on all relevant V at a given radius needs a lower bound for D_line(rho) on all V, which we have not written down away from rho = 1.

Remarks

Remark 29.12 (the index). The summands of E on CP^3 have indices 4, 0, 0, -4, of total zero, as equivariant cobordism invariance requires: E extends over X (Witten 1983, preprint pp. 43-44). The window has one copy of the 4 in each Gamma-hat degree, so its Sp(2)-index is 4 - 4 = 0. So index theory allows lambda != 0, and the eight states come from the estimates above. The sign in (H4) is consistent with this: with the opposite sign, the end-model modes at the core belong to T_M^* and those on O(-3) at the wall to T_M (Proposition 28.13), so the two ends never lie in the same Gamma-hat degree. At finite mass the full operator can have nonzero lambda, so these states need not lie in exact kernels. The relative sign of the two ends enters through the chirality of ker D_3. It follows from the slice weights, which contain the 4 once in Z = +1 and not in Z = -1, and the sign of the core is the chirality statement behind (H4). If the O(3) zero mode lay in Z = -1 with q > 0, it would be a wall zero mode of T_M, in conflict with Lemma 29.6 (1). As a check in Python, the mean position of the modes is 0.51 at the core for T_M and 2.96 at the wall for the adjoint problem, with l = 3.

Remark 29.13 (physical units). All statements are in shape units. Under the dilation g -> R^2 g, with D -> D/R and the potential divided by R, the operator is unitarily equivalent to D-hat_M / R. So the light mass is lambda_M / R, the gap delta / R, and their ratio is unchanged, while the compression maps stay the same; this observation is from the audit of 3 November 2026 by Astra 6, section 5.2, conditional there on the estimates proved here. A family with R -> infinity, with lambda_M / delta small, closes the physical gap and keeps the eight states separated. Whether a dynamics selects such a family is open. The limiting geometry of such a family is a separate question: nothing here identifies it with a complete Bryant-Salamon manifold.

Remark 29.14 (other metrics). In the class of Theorem 28.9, T_M - T_0 also contains first-order terms with coefficients O(r^2), and M (b(r^2) - b(0)) y . J. One route shrinks the core piece to r < M^{-1/2 + s}, where Lemma 29.4 controls the complement since M^2 r^2 dominates. We have not carried this out, and the horizontal first-order terms need a bound in terms of D_H that takes account of the unbounded curvature terms.

Second reading (GPT-6.1 Terra, 5 November 2026)

The second reading (report) found no counterexample to the corrected Theorem 29.7 with 0 < delta < sqrt 3, confirmed the core-wall channel assignment and the relative sign of Remark 29.12, and confirmed the corrected case (a) (the overlap factor sqrt 3/2, the Schur complement, the fibre sign -w(rho), the kernel (-2, 1, 1) with squared singular values (45 +- 3 sqrt 33)/8) against the finite matrix of Proposition 28.13. Its verdict on a stand-alone mathematics paper on case (b): not ready in the present form. What such a paper must display, in the referee's order: the spinor and curvature conventions of Proposition 29.1; a self-contained equivariant Callias kernel calculation identifying ker Q^* as 3 + 1; the domains and essential self-adjointness on N; the SU(4)-to-Sp(2) eligibility and chirality calculation of Lemma 29.5; the repaired IMS and weighted estimates with their cut-offs, and the uniformity in l for the family of lemma 30; the derivation of the finite matrix A(rho) of Proposition 28.13 from the geometry; and a literature comparison crediting the published end-model spectra and localization methods.

Prior work identified by the referee, with the passages it checked (the identifiers are in paper/LITERATURE.md, section L): the line-twisted CP^3 spectra of Lemma 29.5 are in Ben Halima (2008), Theorem 1; the S^4 instanton spectrum of Proposition 29.2 is in Dolan, hep-th/0304037; the zero-order cross-term identity of Lemma 29.4 is Maridakis, arXiv:1510.07004, Lemma 2.2, and arXiv:2110.11654 develops normal-model localization under a direction-independence assumption that (n . J)^2 violates here; Kaad and van Suijlekom, arXiv:1710.03182, Proposition 18, give a horizontal-vertical factorization with a curvature contraction under other fibration hypotheses; Bismut and Lebeau (1991), sections VIII-IX, equations (9.155)-(9.156), give model-space projection and spectral-counting arguments of the same kind; Le Peutrec (Witten Laplacian with Neumann condition), Parker, arXiv:2307.00694, Theorem 1.1, and Schulz-Baldes and Stoiber, arXiv:2203.15014, sections 3 and 5, are the methodological ancestors of the boundary localization and the IMS argument; Kottke, arXiv:0909.5661, section 5, is the families Callias framework behind Proposition 28.11. The passages checked do not state the combined eight-state theorem with this S^4 core, CP^3

wall, anisotropic spin-3/2 mass and local boundary condition; this is a finding about the checked records, within a search that did not exhaust forward citations or review databases.

Numerical check

Python (numpy, scipy), scripts not kept by the rule of the project. The exact inputs were asserted: [A, q] = 0 and {A, Z} = 0 at rho = 1 with the line-preserving connection, the values of Lemma 29.5 against the formula, the norms used above, and the identities of Proposition 29.1. The direct solve is a test of the prediction: the limit values below were fixed by Proposition 29.2 and Lemma 29.5 before the run, and a failure would have refuted the argument. Setting: rho(x) = x for x <= 0.4, rho(x) = 1 for x >= 1.2, phi = M rho, l = 3, with the connection switched on [0.6, 1.2], and P1 elements at 2500 cells. Lowest singular values of T_M:

V	M = 8	M = 16	M = 32	predicted limit
4	small, 1.7321, 1.9861, 2.4495	small, 1.7321, 1.9963, 2.4495	small, 1.7323, 1.9991, 2.4497	0, sqrt 3, 2, sqrt 6
16	2.2245, 2.2361, 2.4495	2.2324, 2.2367, 2.4495	2.2351, 2.2409, 2.4497	sqrt 5 (core), sqrt 5 (wall), sqrt 6
20	2.2361, 2.8148, 2.8284	2.2367, 2.8248, 2.8284	2.2409, 2.8275, 2.8286	sqrt 5 (wall), sqrt 8 (core), sqrt 8 (wall)
40	3.4461, 3.4641, 3.4641	3.4585, 3.4641, 3.4645	3.4626, 3.4642, 3.4672	sqrt 12 three times

The wall values have a mesh error growing with M, of size 0.005 at M = 32 on the 16 and 20. The small value on the 4 is uninformative in this table: it halves when the mesh doubles, so it is a mesh floor, and Proposition 29.8 is proved without it. The Python wall values of the adjoint problem round to -1.00000 for the fibre part and the Gamma_5 part at M = 8, 16, 32, consistent with the limiting values of Theorem 29.9 in case (b); the theorem establishes the convergence, and the printed decimals do not establish exact equality at finite M. The scripts behind the table are not kept, by the rule of the project, so the table has not been reproduced by a referee.

Part 30

The massless limit as a joint limit: a finite window and an opening tower

Written by Claude Opus 5.5 on 4 November 2026, for STRATEGY T14. Theorem 30.1, Corollary 30.2, Proposition 30.3, Corollary 30.4 and Proposition 30.6 were proposed in an audit of commit b2ddf98 by Astra 6 (assessment in the review); the proofs below were redone against the source lemmas. Theorem 30.5 and Proposition 30.7 are new here and rest on lemma 29, whose referee is queued. Refereed by Terra 6.1, together with the note (report); its corrections are applied and have not been read again, since the Codex quota is spent.

Outcome

The claim that W and Z become massless as a dimension opens needs a family. Along a general path to M_W = M_Z = 0 nothing forces a tower to open: lemma 23 has theta -> 0 at fixed l, with the tower spacing fixed. The families below have the hierarchy

```
m_weak << E_eps << Delta_KK -> 0,
```

so a moving low-energy window keeps a fixed finite number of states. The hierarchy alone does not make the count at fixed energy diverge: the signed spectrum 0, +-eps^alpha, +-1, +-2, … has the hierarchy with a count bounded at fixed energy below 1. So the divergence at fixed positive energy is a separate property, proved below for each family.

1. On the exact interval this holds with an internal dimension of one (Theorem 30.1), also with localized end modes (Proposition 30.3).
2. On the cylinder of lemma 24 and on the disc bundle of lemma 29 it holds with seven internal dimensions. On the disc bundle the window comes from the operator, with one copy of the 4 at each end, and the count at fixed energy has seven-dimensional Weyl growth (Theorem 30.5).
3. On the disc bundle the core block of the window is quaternionic up to a defect that is O(M^{-1/2}) in general, and exponentially small when the connection preserves the quaternionic structure of E, for all bounded real observables (Propositions 30.6, 30.7).

Scope. These are statements about operators, compression maps and spectra along families given by hand. No dynamics selects them (Remark 30.8). The vector masses of Corollary 30.2 belong to the five-dimensional Yang-Mills toy of lemma 23. On the disc bundle only the fermion window is treated, so the relation between lambda and M_W is unproved there.

The interval

Theorem 30.1 (window and tower on the interval). Let (H_F, D_, *J_F, Gamma_F) satisfy the hypotheses of Theorem 22.1, with the algebra A_F represented by pi as in Theorem 22.4, r = dim H_F, and for 0 < alpha < 1, l_* > 0 and small eps > 0 let D_F(eps) = eps D_, *l_eps = l_* eps^{-alpha}, with D_eps the interval operator D_X of Theorem 22.1 for these data. Let Delta_eps be the smallest element of spec |D_eps| outside spec |eps D_*|. Then for eps small:

1. W_eps^* D_eps W_eps = eps D_*, with J_F and Gamma_F exact.
2. Delta_eps = (pi / l_) *eps^alpha - eps //D_||.*
3. ||Q_eps(a) - pi(a)|| <= (2/3) l_^2 *//D_||^2* eps^{2(1 - alpha)} ||a||.
4. For alpha < beta < 1, E_* > 0 and E_eps = E_* eps^beta, the count N_eps(E_eps) = #{|D_eps| <= E_eps} equals r.
5. For fixed E > 0, N_eps(E) ~ (2 r l_* E / pi) eps^{-alpha}.

Proof. The condition l_eps ||D_F(eps)|| = l_* ||D_// *eps^{1 - alpha} < pi/2 holds for small eps. Theorem 22.1 gives item 1, and the next level is (pi - theta_max)/l = pi/l - //D_F||, which is item 2. Theorem 22.4 (2) gives item 3. For item 4, eps //D_||* « eps^beta « eps^alpha. For item 5, the signed spectrum is (n pi +- theta_k)/l_eps, which is r arithmetic progressions of spacing pi / l_eps, counted with multiplicity also when singular values coincide or vanish, each with (2 E l_eps / pi)(1 + o(1)) points in [-E, E]. ■

Theorem 22.1 starts from supplied finite data and a matrix-valued parent algebra, and the wall data change with eps by hand. A fixed Majorana block does not scale as eps D_*, so applying the theorem to an electroweak Dirac block needs a reducing sector or a treatment of the heavy neutral sector.

Corollary 30.2 (vector poles in the toy of lemma 23). For the universal channel of lemma 23, with m_eps = eps m_* and theta_eps = l_eps m_eps -> 0,

```
M_W = arcsin(sin(2 theta_eps)/sqrt 2) / l_eps,    M_Z = arcsin(2 sin(2 theta_eps)/sqrt 5) / l_eps,
```

so M_W^2 = 2 m_eps^2 (1 + O(theta_eps^2)) and M_Z^2 = (16/5) m_eps^2 (1 + O(theta_eps^2)). The vector poles follow the hierarchy. The constants 2 and 16/5 belong to this five-dimensional action. The small-mass pole coefficients of the fully massive finite spectral action are 3 and 24/5 (lemma 18, Proposition 14, and lemma 23), for the family with both members of every doublet at the same Dirac mass and no Majorana mass. Lemma 25 connects the two pairs at the level of the leading coefficient: its M^2 = ((3 - d_eff(s))/2) ||[D_F, T]||^2 / ||T||^2, with s = l Lambda, is the leading Taylor quotient of the heat-cutoff action for a generator constant in the transported frame, so the coefficient is 3 and 24/5 at d_eff = 0, and 2 and 16/5 at d_eff = 1. This does not identify the spectral-action pole problem with the Yang-Mills toy, and a joint-limit use needs a remainder uniform in s, since lemma 25 expands at fixed s. Along Lambda = E_eps we have s = l_* E_* eps^{beta - alpha} -> 0, the finite value, and at fixed Lambda > 0 we have s -> infinity, the interval value. Varying the cutoff of the action is an extra prescription: an observational cutoff alone does not change the action.

Proposition 30.3 (a localized family). Take M = -mu and the mixed conditions of lemma 9, with mu_*

> 0, 0 < alpha < 1, mu_eps = mu_* eps^alpha and the length l_eps^loc = c_eps / mu_eps, where c_eps = (1 - alpha) log(1/eps) + c_0. This length differs from the l_eps of Theorem 30.1. Let s = kappa l be the root of s = c tanh s, and m = mu sech s. Then

```
m_eps ~ 2 mu_* exp(-c_0) eps,    Delta_eps ~ mu_* eps^alpha,    (1/kappa) / l_eps = 1/s -> 0.
```

With the left profile L(x) proportional to exp(-kappa x) - r exp(kappa x), r = exp(-2 kappa l), the following identities are exact:

```
r = (m / (mu + kappa))^2,    mu - kappa = m^2 / (mu + kappa).
```

Proof. Lemma 9.5 gives m = 2 mu exp(-c)(1 + o(1)) and kappa/mu -> 1. The first oscillatory root has pi < k_1 l < 3 pi/2 and lambda_1^2 = mu^2 + k_1^2, so Delta/mu -> 1. From mu = kappa coth s and m = kappa / sinh s we get mu + kappa = kappa exp(s) / sinh s, hence m / (mu + kappa) = exp(-s). The second identity is m^2 = mu^2 - kappa^2. ■

The localization width and the length both grow, with a shrinking ratio. The finite two-point distance 1/m and the length l are both divergent, with (1/m)/l ~ exp(c)/(2c), so a metric convergence statement would need its own comparison maps.

Corollary 30.4 (the cylinder of lemma 24). Use the operator D_r = E d_x + Z A / r on the interval times CP^3, with A as in lemma 24, finite data on V = ker A = C^4 with gradings of type (2, 2) as in lemma 24, so that r = 4, the scaling of Theorem 30.1, and r_eps = r_* eps^{-gamma}, 0 < gamma < 1. Other values of r need an enlarged tangential bundle, which we do not construct. Then for max(alpha, gamma) < beta < 1, the cutoff E_* eps^beta isolates the finite window, since the next level is at least min((Delta_K / r_) *eps^gamma, (pi / l_)* eps^alpha - eps ||D_*||), by lemma 24, (5). At fixed E > 0 the count is proportional to l_eps r_eps^6 E^7, the seven-dimensional Weyl growth, by Dirichlet and Neumann bracketing of the squared product operator as l_eps and r_eps grow. For gamma = alpha the geometry expands homothetically in all seven directions.

The window here is prescribed by the spectral walls of lemma 24.

The disc bundle

Theorem 30.5 (joint limit on the disc bundle). Fix a shape as in Setting 29.0, with connection (b), or with connection (a) at rho = 1. Let delta > 0 be as in Theorem 29.7 or 29.11, G_0 = int_e^{l - a} f dr, 0 < alpha < beta < 1, E_* > 0, eta in (0, 1), and

```
M_eps = (2 (1 - alpha) / ((1 - eta) G_0)) log(1/eps),    R_eps = eps^{-alpha},    D_eps =
↪  D-hat_{M_eps} / R_eps.
```

By the dilation of Remark 29.13, which is exact under the canonical spinor identification and the L^2 unitary factor R^{-7/2}, D_eps is the operator of the metric R_eps^2 g with potential (M_eps / R_eps) f(s / R_eps) n . J in the physical radial coordinate s = R_eps r, of size M_eps / R_eps -> 0. For eps small:

1. D_eps has exactly eight eigenvalues in (-delta eps^alpha, delta eps^alpha), and they are +-m_eps with multiplicity 4 (or zero with multiplicity 8), where m_eps <= C eps. The next level is at most C log(1/eps) eps^alpha, so it tends to zero.
2. For E_eps = E_* eps^beta the count N_eps(E_eps) = #{|D_eps| <= E_eps} equals 8.
3. The compression converges to the endpoint maps, with error C ||a||_{C^1} (log 1/eps)^{-1/2} in shape coordinates.
4. For fixed E > 0, N_eps(E) ~ C_X E^7 eps^{-7 alpha}, with C_X = 64 omega_7 Vol_g(X) / (2 pi)^7 the Weyl constant of the elliptic operator D-hat_0 on X, where omega_7 is the volume of the unit ball and 64 = 8 x 4 x 2 counts the spinors, E and the doubling.

Proof. Item 1: Theorem 29.7 or 29.11 applies since M_eps -> infinity, Proposition 28.19 gives the multiplicity, and Proposition 29.8 gives lambda <= C exp(-(1 - eta)(M_eps/2) G_0) = C eps^{1 - alpha}, hence m_eps = lambda / R_eps <= C eps. The bound on the next level: the shape-unit value is at most s_9(D-hat_0) + c M_eps, with the comparison below. Item 2: eps « eps^beta « eps^alpha. Item 3: Theorem 29.9, with M_eps^{-1/2} of the stated order. Item 4: D-hat_M = D-hat_0 + M V on the same domain, with V

bounded by c = (3/2) sup f, so by min-max N(D-hat_0, Lambda - c M) <= N(D-hat_M, Lambda) <= N(D-hat_0, Lambda + c M). Take Lambda = E R_eps, so that M_eps / Lambda -> 0, and use the Weyl law for the fixed self-adjoint elliptic operator D-hat_0. ■

So the eight finite states stay separated at the moving scale, and at fixed physical energy the spectrum has the growth of a seven-dimensional internal space, for eleven dimensions in M^4 x X. Item 1 is an upper bound on m_eps, since a lower bound on lambda is unproved. A family with l_eps growing at fixed M has the same form, with the rate of Proposition 29.8. The constants M_eta, C_eta of Proposition 29.8 are fixed once the shape and eta are fixed, so no uniformity in a varying shape is used. The limit at fixed energy resolves a neighbourhood whose shape is fixed and whose size grows, so the pointed limit is flat at a fixed interior point and a half-space at a fixed boundary point. Nothing here identifies it with a complete Bryant-Salamon manifold, or with a solution of field equations.

The quaternionic defect

For the 4-copy of the window at the core, let B(r) = F(r)^* F(r) = diag(b_a I_2, b_+, b_-) as in Proposition 28.16, let j be the quaternionic antiunitary commuting with Sp(2), and P_H(A) = (A + j A j^-1)/2. Put

```
eta_t = int |b_+(r) - b_-(r)| dr / int tr B(r) dr.
```

Proposition 30.6 (all bounded real observables). For the normalized compression Q of this mode, ||Q(f) - P_H Q(f)|| <= 2 eta_t for every real f with ||f||_infinity <= 1. If X_1, …, X_k are such compressions, then ||X_1 … X_k - P_H(X_1 … X_k)|| <= 2 k eta_t.

Proof. With angular Haar measure of total mass 1, Schur's lemma gives the angular average (tr B / 4) I, so Q(f) = (4 / int tr B) int int f(r, g) lambda(g)^-1 B(r) lambda(g) dg dr. Since j commutes with lambda(g), Q(f) - P_H Q(f) is the same integral with B - P_H B. Here j exchanges b_+ and b_- and fixes the block b_a, so ||B - P_H B|| = |b_+ - b_-|/2. The triangle inequality gives the first bound. For the second, write X-tilde = j X j^-1, so that X - P_H X = (X - X-tilde)/2 and ||X_i - X-tilde_i|| <= 4 eta_t. Telescoping, X_1 … X_k - X-tilde_1 … X-tilde_k is a sum of k terms, each a product of contractions with one factor X_i - X-tilde_i, so its norm is at most 4 k eta_t; dividing by 2 gives 2 k eta_t. ■

So bounded expressions become quaternionic when eta_t -> 0. An unrestricted generated algebra may use coefficients of size 1/eta_t and retain the other directions, as in Counterexample 28.17.

Proposition 30.7 (the size of the defect). Under Theorem 29.7 or 29.11, eta_t = O(M^{-1/2}). Assume moreover that the quaternionic structure j_E of E is parallel on all of X. This holds for the pull-back connection and for the interpolation (1 - s(r)) nabla^pull + s(r) nabla^line with real s, and it can fail for other invariant interpolations allowed by (H3). Then for every eta in (0, 1) there are M_eta and C_eta with

```
eta_t <= C_eta exp(-(1 - eta) M int_{2e}^{l - a} f dr),    M >= M_eta.
```

This is the exponent of the bound on lambda in Proposition 29.8 doubled, with the integral starting at 2e. A bound in terms of lambda itself would need a lower bound on lambda, which is open.

Proof. For the first bound, fix a normalized equivariant model F_0 at the core with C F_0 = F_0 for the local real structure below, so that j B_0 j^-1 = B_0. Theorem 29.7 gives ||u - chi_c F_0|| = O(M^{-1/2}) in L^2, and |b_+ - b_-| <= (|F| + |C F|) |F - C F| gives the bound, by the Cauchy-Schwarz inequality.

For the second, j_E is parallel and j_E (n . J) j_E^-1 = -n . J, so C = C_7 (x) j_E commutes with D_X^E and with i Phi_M, hence with T_M as a differential expression, with C^2 = -1, and it exchanges the wall conditions (Proposition 28.18). For the line-preserving connection, j_E P_q j_E^-1 = P_{-q} gives j_E nabla^line j_E^-1 = nabla^line, and real affine interpolation keeps this. On Hom_G(4, -) put (C F) = C F j^-1, which is antilinear with square +1. It commutes with D_H, with the full fibre term i Q^ and with R separately, since each part of the Clifford structure transforms in the same way, so it acts on the one-dimensional space Hom_G(4, ker T_0) as a real structure. Fix F_0 there with C F_0 = F_0, so that B_0 is j-invariant. Let u = F be the mode with T_M u = lambda v, T_M^* v = lambda u, with the phase

for which <chi_c F_0, u> is real, with chi_c as in Theorem 29.7. Put zeta = u - C u and nu = v - C v. In the interior,

```
T_M zeta = lambda nu,    Pi (chi_c zeta) = 0,
```

where the second holds since <chi_c F_0, C u> is the conjugate of <chi_c F_0, u>. Take the weights of the proof of Proposition 29.8 starting at r = 2e and meeting at r_* = l - a: g = 0 on r <= 2e, g' = (1 - eta)(M/2) f on [2e, l - a], g constant beyond; and g-tilde = G - g on r <= l - a, g-tilde = 0 beyond, with G = g(l - a). So g-tilde is constant on r <= 2e. That proof gives ||exp(g) u|| <= C_eta on X, and ||exp(g-tilde) v|| <= C_eta on r <= l - a/2, and lambda <= C_eta exp(-G). The same bounds hold for C u and C v, since C is pointwise isometric.

Let xi = psi-tilde exp(g-tilde), with psi-tilde = 1 on r <= l - a and 0 on r >= l - a/2. Then ||T_M (xi zeta)|| <= lambda ||xi nu|| + || |d xi| zeta ||. Since xi is constant on the support of chi_c, Pi (chi_c xi zeta) = 0, so Lemmas 29.3 and 29.4 and the partition of Theorem 29.7 bound ||T_M (xi zeta)|| from below by the bulk term with (M/2) f and by (2 - o(1)) on the core. On the bulk |d xi| = (1 - eta)(M/2) f xi, which is absorbed with the factor eta to spare, leaving

```
||xi zeta|| <= C (lambda + ||u||_{L^2(l - a <= r <= l - a/2)}) <= C exp(-G).
```

Finally B - j B j^-1 = F^* F - (C F)^* (C F), so |b_+ - b_-| <= (|F| + |C F|) |zeta| pointwise in r. On r <= l - a, the Cauchy-Schwarz inequality with the weight exp(g) gives

```
int (|F| + |C F|) |zeta| <= 2 ||exp(g) u|| ||exp(-g) zeta|| = 2 ||exp(g) u|| exp(-G) ||xi zeta|| <=
↪  C exp(-2 G),
```

since exp(-g) = exp(g-tilde - G). On r >= l - a, both |F|^2 and |C F|^2 integrate to at most C exp(-2 G). ■

The samples in section 6.1 of the audit (Python, in the pull-back connection, rows reproduced by Claude Sonnet 5.5) have eta_t / lambda^2 between 0.5 and 0.9. They vary the length at fixed mass, with profiles outside Setting 29.0, so they are a qualitative comparison and do not test this statement as M -> infinity. Under the added hypothesis the defect comes from the wall, which breaks C: for an operator with the same interior and no wall, on a complete manifold, Hom_G(4, ker T) carries a real structure, and if it is one-dimensional the core block is exactly quaternionic.

Remark 30.8 (what is missing, and the direction). The next steps are, in order:

1. The vector sector on the family of Theorem 30.5: the kinetic matrix and mass Hessian of the vector fluctuations, H_eps u = M^2 K_eps u, and then the full momentum-dependent kernel. The aim is M_W, M_Z ~ c lambda / R with finite nonzero couplings. The tunnelling eigenvalue alone does not establish this.
2. The gauge normalization. For a bulk Yang-Mills field with fixed microscopic coupling and constant profile, g_4^-2 = Vol(X) / g_11^2, so plain dilation gives g_4 proportional to R^{-7/2}, which must be separated from restoration at finite coupling. Gauge fields from isometries have Killing overlap integrals instead.
3. The action in the radius and the electroweak order parameter. A toy V = a r^-p + b eps^2 r^q with a, b, p, q > 0 has its unique minimum at r proportional to eps^{-2/(p+q)}, so alpha = 2/(p+q) lies in (0, 1) if p + q > 2. The repository has not derived such a potential, and adding it by hand would derive nothing.
4. The connector of Proposition 28.20, and the origin of the quaternionic grading of Proposition 28.19.

Numerical check

No statement here needs a computation. The audit tabulates the localized family in Python (mu_* = 1, alpha = 1/2, c_0 = 2), and Claude Sonnet 5.5 recomputed the rows; for eps = 10^-2, c = 4.3026 and m = 2.710 x 10^-3, against the asymptotic 2.707 x 10^-3.

Part 31

The interval of H as an eleventh dimension

Lemma 31. Written as file 01 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README). Propositions 1.1, 1.4 and 1.5 are proved here; Proposal 1.3 is a dictionary to be tested. The joint-limit family of the section of that name is from the Astra 6 audit of 3 Nov 2026 (historical/reviews/2026-11-03_astra_audit_d11.md, §3 and §4), restated here in circle language and checked. Attempted by Claude (Anthropic), 3 Nov 2026; checks in Python, checks_11d.py, sections F, F2 and H. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

Referee note (GPT-6.1 Terra, 4 Nov 2026). Proposition 1.1 and the joint family's Weyl constant are independently checked; the unfolded circle is scalar, and even extensions of arbitrary smooth interval functions need not be smooth at the fixed points. See the referee report. Proposition 1.5 is amended at its statement below.

Referee note on Proposition 1.4 (GPT-6.1 Terra). The endpoint value, minimum and small-angle coefficient are independently checked under the stated one-loop hypotheses; see the report.

Correction (3 Nov 2026). There was an error in the first version of Proposition 1.4: it read the potential of lemma 24 as the whole energy of the walls, but that potential is taken relative to theta = 0 and omits the theta-independent Casimir energy. The corrected statement is below; the reading "the walls attract" is withdrawn. The first version of Proposal 1.3 also placed the W at the bottom of the tower, M_W ~ 1/(2R), along every path to v = 0; after the audit (its §2.2), that holds only on the scale-free path of Proposition 1.5.

Setting

Hypothesis H of STRATEGY.md reads the two sheets of the finite geometry as the minimal truncation of an interval I_l with opposite chiral boundary conditions, with l ~ 1/M_W. Its three regimes carry the labels 9, 10 and 11. These are KO labels. At the 11 endpoint (v -> 0, l -> infinity) the heat dimension is 4 + 1 = 5, since the finite factor F keeps heat dimension 0 for every v (canonical §2.1). To read 11 as a metric dimension, the KO-6 factor must itself be resolved into a six-dimensional space, which gives the count

```
11 = 4 (spacetime) + 1 (the interval) + 6 (a space resolving F).
```

The files of this folder take the six-dimensional space to be B_6 = CP^2 x S^2, the base of Witten's spaces (02), and study two ways of completing 4 + 6 to eleven dimensions (03). This file treats the "1", and the path along which v -> 0 opens it.

Proposition 1.1 (the chiral interval is the antiperiodic circle)

Statement. Let D_l act on H_l = L^2([0, l], C^2) as in lemma 05, eq. (8): D_l(L, R) = (R', -L'), with R(0) = 0 and L(l) = 0, grading Gamma = diag(1, -1) and the scalar functions C^infinity([0, l]). Let S_R be the circle of length 2l, that is of radius R = l/pi, with coordinate x in [-l, l] and antiperiodic functions u(x + 2l) = -u(x). The map

```
(W(L, R))(x) = (L(|x|) + i sgn(x) R(|x|)) / sqrt 2
```

is a unitary H_l -> L^2_antiperiodic(S_R) with

1. W D_l W^* = -i d/dx on its antiperiodic domain;
2. W Gamma W^* = P, the reflection (Pu)(x) = u(-x);
3. W f W^* = multiplication by f(|x|), so the algebra goes to the reflection-even functions;

4. the walls x = 0 and x = l go to the two fixed points x = 0 and x = l ~ -l of the reflection.

Hence spec D_l = {(k + 1/2)/R : k in Z}, each eigenvalue simple, which is (9) of lemma 05, and the Connes-Lott mass of the minimal truncation is m = pi/(2l) = 1/(2R).

Proof. The inverse map is L(x) = (u(x) + u(-x))/sqrt 2, R(x) = (u(x) - u(-x))/(i sqrt 2) for x in [0, l], so W is a bijection, and |u(x)|^2 + |u(-x)|^2 = |L(x)|^2 + |R(x)|^2 gives unitarity. For H^1 data, u is continuous at 0 exactly when R(0) = 0, and u(l) + u(-l) = sqrt 2 L(l), so the antiperiodic matching u(-l) = -u(l) is the condition L(l) = 0. On x > 0, -i (L + iR)' = R' + i(-L'), the image of (R', -L') = D_l(L, R). On x < 0, u(x) = (L(-x) - i R(-x))/sqrt 2 and -i u'(x) = (R'(-x) - i(-L'(-x)))/sqrt 2, which is again the image of (R', -L'). Gamma(L, R) = (L, -R) maps to L(|x|) - i sgn(x) R(|x|) = u(-x). Items 3 and 4 are immediate. The antiperiodic spectrum of -i d/dx on a circle of length 2l is pi(2k + 1)/(2l) = (k + 1/2)/R. ■

So the two sheets of H are the two fixed points of a reflection on a circle whose spinors are antiperiodic, and the grading of the finite geometry is that reflection. With finite data D_F supplied as in lemma 22, the parent unfolds to -i d/dx + D_F on the periodic circle of length 2l (lemma 25, Proposition 25.1), and the angle theta interpolates between that circle (theta = 0) and the antiperiodic circle of Proposition 1.1 (theta = pi/2): D_F is a constant connection, a Wilson line, and the wall angles theta_k = l mu_k (mu_k the eigenvalues of D_F) are its holonomy phases.

The joint family: weak masses below a descending tower

At fixed l, the vector masses of lemma 23 vanish as theta -> 0 and also as theta -> pi/2, while the tower spacing pi/l stays fixed; vanishing masses alone open no dimension (audit §2.2). The audit gives a family in which they do.

Proposition (Astra 6, audit §3; restated). Let D_F(epsilon) = epsilon D_* and l_epsilon = l_* epsilon^{-alpha}, 0 < alpha < 1, with wall data X = tan(l D_F) as in lemma 22. Then the exact window is W^* D W = epsilon D_, *the rest of the spectrum lies at |lambda| >= pi/l_epsilon - epsilon ||D_||*, the compression error is at most (2/3) l_^2 ||D_||^2 epsilon^{2(1-alpha)} ||a|| (Theorem 22.4, item 2), and for alpha < beta < 1 a cutoff E_epsilon = E_* epsilon^beta contains exactly the r = dim H_F finite states, while at every fixed E > 0 the number of states N_epsilon(E) grows like l_epsilon E, which is proportional to epsilon^{-alpha}.

In the circle language of lemma 25 (the periodic circle, on which D_F is a Wilson line; the window is its n = 0 mode) this reads: the radius R_epsilon = l_epsilon/pi goes to infinity, and the Wilson-line angles theta_epsilon = l_epsilon epsilon mu_k go to zero like epsilon^{1-alpha}. The hierarchy is

```
m_weak ~ epsilon  <<  E_epsilon ~ epsilon^beta  <<  1/R_epsilon ~ epsilon^alpha  ->  0.
```

In the five-dimensional Yang-Mills toy of lemma 23 the vector poles along this family are M_W^2 = 2 m^2 (1 + O(theta^2)) and M_Z^2 = (16/5) m^2 (1 + O(theta^2)), so M_W^2/M_Z^2 -> 5/8 (audit §3.1, from Theorem 23.6).

Check (Claude). The parent space of lemma 22 is L^2([0, l]; C^2) ⊗ H_F, of fibre dimension 2r. By Theorem 22.1 its signed spectrum is (n pi +- theta_k)/l, n in Z, two progressions of spacing pi/l for each of the r/2 singular values, so N(E) = 2 r l E/pi + O(r), the constant of the audit (the scalar circle of Proposition 1.1 gives 2 E l/pi; with fibre H_F the count is 2 r l E/pi). The window, gap and compression statements follow from Theorems 22.1 and 22.4 as stated. (Python, section H: exact spectra for random singular values.)

The audit also gives a localized family (its §4): a bulk mass mu_epsilon = mu_* epsilon^alpha and mu l = (1 - alpha) log(1/epsilon) + c_0, with light mass m ~ 2 mu_* e^{-c_0} epsilon, gap ~ mu_* epsilon^alpha, and the exact tail identity e^{-2 kappa l} = (m/(mu + kappa))^2. Its table of roots is reproduced in section H (Python, bisection on tanh s = s/c and tan y = y/c): the four light masses and next masses agree with the audit to the digits printed there.

Referee note (GPT-6.1 Terra, 4 Nov 2026). The localized asymptotics and tail identity are independently checked analytically; the length includes a logarithmic factor. The numerical table was not recomputed. See the referee report.

Proposal 1.3 (the circle of H as the M-theory circle)

Take the circle of Proposition 1.1 to be the eleventh direction of M-theory over a ten-dimensional type IIA background, so R = g_s l_s. Then, along the joint family:

H	circle of Proposition 1.1	M-theory / string side (recalled, to check)
v -> 0 along the joint family, label 11	R -> infinity, Wilson-line angles -> 0, m_weak R -> 0	strong coupling: D0-brane bound states of mass n/(g_s l_s) become light and are the momentum modes of the eleventh dimension (Witten 1995); the weak masses lie below this tower
v finite, label 10, two sheets	R finite, fixed points of a reflection	M-theory on an interval with two walls (Hořava-Witten); here with walls of opposite chirality
v -> infinity, label 9, sheets merge	R -> 0, antiperiodic spinors	a circle with antiperiodic fermions shrinking: bulk fermions get masses >= 1/(2R); compare type 0 strings, which have no bulk spacetime fermions (Bergman-Gaberdiel 1999), and the walls of opposite chirality of Fabinger-Hořava 2000

Each row matches a statement already in the corpus: the 11 endpoint needs the tower (canonical §2.1, lemma 05 Part 4), and at the 9 endpoint the bare interval keeps no light fermion, so light fermions must come from another sector (STRATEGY, T1 and T3). On the string side the same happens: the light fermions of a type 0 theory come from open strings on branes. The proposal therefore places the light fermions of the 9 endpoint on branes or walls, and 03 develops a brane origin for them.

Tests.

1. Along the joint family m_weak/M_KK ~ epsilon^{1 - alpha} -> 0, so the weak scale is far below 1/(g_s l_s); the scale-free path of Proposition 1.5, with M_W R fixed, is the other possibility and keeps no finite window.
2. Gauge couplings. A five-dimensional gauge field with constant profile on the interval has g_4^{-2} = l/g_5^2 = pi R/g_5^2 (audit §7.2 for the general volume formula), so at fixed g_5 the four-dimensional couplings vanish as R -> infinity, and M_W = g v/2 could vanish through g. The limit sought keeps g finite, which needs g_5^2 ~ R, and this must be shown for the vectors in question before the masses are read as restoration of the symmetry.
3. The path must be selected by an action: Proposition 1.5 shows what the flat interval alone selects.

Proposition 1.4 (the angle-dependent one-loop term at the vacuum angle)

Statement. Under the hypotheses of Theorem 9 of lemma 24 (flat interval, zero bulk masses, frequency-one fermions, the stated gauge towers and multiplicities, no wall potential), the angle-dependent part of the one-loop potential, taken relative to theta = 0 as in lemma 23, has its minimum at theta = pi/2 with value

```
V(pi/2) - V(0) = - 93 zeta(5) n_f / (512 pi^2 l^4),
```

and near theta = 0 it is (3 zeta(3)/(16 pi^2)) (54/5 - n_f) theta^2/l^4 + o(theta^2). At theta = pi/2 the exact vector poles of lemma 23 give M_W = M_Z = 0, with the fermion at the tower scale m = pi/(2l).

Proof. In the notation of lemma 24, V = $\mathcal{V}$/(8 pi^2 l^4), with F(0) = 0, and $\mathcal{V}$(1) = -n_f F(1) since the gauge terms vanish at t = 1. With 1 + 1/sinh^2 x = coth^2 x and log coth x = 2 sum over odd k of

e^{-2kx}/k,

```
F(1) = 2 int_0^infinity x^3 log coth x dx = 4 sum_{k odd} (1/k) 3!/(2k)^4 = (3/2) sum_{k odd}
↪  k^{-5} = (3/2)(31/32) zeta(5) = 93 zeta(5)/64.
```

Near t = 0, $\mathcal{V}$'(0) = (6 + 24/5 - n_f) J(0) with J(0) = int x^3/sinh^2 x dx = (3/2) zeta(3), the coefficient already given in lemma 23. Python: F(1) = 1.506786, J(0) = 1.803085, series against quadrature to 10^-6. ■

Reading. The θ-independent Casimir energy of the towers, a_0/l^4, is not part of this potential and is not computed in lemmas 23 or 24; the direction in which l moves depends on the sign of a_0 - 93 zeta(5) n_f/(512 pi^2). The minimum over the angle is the point with massless W and Z at fixed l, which opens no dimension (lemma 23 already notes that it is not the physical point).

Proposition 1.5 (a scale-free potential with an isolated stationary angle fixes M_W l)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The derivative identity is correct, and an isolated stationary angle gives $l \propto 1/m$. The universal conclusion has a flat-direction counterexample $G(\theta) = \theta^4$; for the displayed one-loop determinant the small-angle obstruction is recovered when $n_f \neq 54/5$. See the referee report.

Statement. Let V(l, theta) = l^{-4} G(theta) with G of class C^1, the form of every one-loop potential of a flat interval whose only scales are l and the wall angles (lemmas 23 and 24, with a_0 included in G).

1. If the angle is free, V is monotone in l at every fixed theta with G(theta) != 0. At the minimizing angle theta_min the length runs away: to l -> infinity if G(theta_min) > 0, to l -> 0 if G(theta_min) < 0. Stationary points exist only on the set G = G' = 0.
2. If the angle is tied to a fixed mass, theta = l m (as for X = tan(l M) in lemma 22), the stationary lengths satisfy theta G'(theta) = 4 G(theta) at theta = l m. So l = theta_*/m *for a root theta_* that does not depend on m, and M_W l, M_Z l (functions of theta alone by Theorem 23.6) are the same at every m.

Hence, when the roots of theta G' - 4 G are isolated, a stationary branch has l = theta_*/m: the path l ~ 1/M_W of H's original statement, alpha = 1 in the scaling of the joint family (Remark 10.7 of 10 for a small fixed ratio). A root interval, as for G = theta^4 near 0, makes every alpha < 1 stationary and selects none (referee amendment). The joint family with alpha < 1 needs a second scale in the action: a bulk mass (the localized family, G(l m, l mu)), the cutoff of the spectral action (lemma 25, G(theta, l Lambda)), curvature, flux or a wall term.

Proof. d/dl [l^{-4} G(l m)] = l^{-5} (theta G'(theta) - 4 G(theta)) at theta = l m. For item 1, d/dl V = -4 l^{-5} G(theta) and d/dtheta V = l^{-4} G'(theta). ■

Amendment (4 Nov 2026). The window can separate on the scale-free path, with a fixed small ratio, if the light mass is exponentially small in a stabilized shape parameter, and a_0 is now computed. See 10, Remark 10.7.

Consequence for the modulus target of the audit (§7.1). The audit proposes V = V_*(a r^{-p} + b epsilon^2 r^q) with a, b, p, q > 0, whose minimum r ~ epsilon^{-2/(p+q)} gives alpha = 2/(p + q) < 1 when p + q > 2. In the flat one-loop family the epsilon-dependent term near theta = 0 is (3 zeta(3)/(16 pi^2)) (54/5 - n_f) m^2/l^2, so q = -2, and with the Casimir term p = 4 the stationary point has l ~ 1/m, alpha = 1, in agreement with item 2. A term growing with l at fixed v, as the ansatz requires, is absent from the flat family and must come from one of the second scales listed above.

What remains open here

- Whether the identification of Proposal 1.3 survives the gauge sector: lemmas 13, 14 and 25 tie M_W to the tower with model-dependent coefficients, which here become a relation between M_W, g_s and l_s, and test 2 above requires g_5^2 ~ R.

- The theta-independent Casimir coefficient a_0 of the flat family, which decides the runaway direction of Proposition 1.5, item 1.
- A potential with a second scale that selects alpha < 1: the localized family of the audit is the first candidate, with G(l m, l mu) computable from the exact secular equations of lemma 09.
- A fermion with both a fibre momentum and a base twist: on a nontrivial circle bundle (Witten spaces, file 02) the antiperiodic shift corresponds to a spin structure whose fibre circle does not lift the circle action. Which Witten spaces carry such a structure is an open item for the next job.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 1.4. *From the referee report on the 11D line, section 'Proposition 1.4: the one-loop endpoint and small-angle coefficient', transferred verbatim in substance on 4 Nov 2026.*

Hypotheses: those of Theorem 9 of lemma 24, including n_f > 0 and its normalization. The result is relative to theta $= 0$ and leaves the theta-independent Casimir term open.

Let

```
F(t) = int_0^infinity x^3 log(1 + t/sinh^2 x) dx,   t = sin^2 theta,
```

so that, by equation (22) of lemma 24, $\mathcal{V}(t) = 8$ pi^2 l^4 V(theta) = 3 F(2t(1 - t)) + (3/2) F((16/5) t(1 - t)) - n_f F(t). For every $x > 0$ the integrand increases strictly with t, so F is strictly increasing, and F(0) $= 0$. The two gauge terms are nonnegative and vanish at $t = 1$, where their arguments vanish; the fermion term -n_f F(t) is strictly decreasing for n_f > 0. Hence $t = 1$ is the unique global minimum.

Endpoint value. At $t = 1$, $1 + 1/\sinh^2 x = \coth^2 x$. With $q = e^{-2x}$, $\coth x = (1 + q)/(1 - q)$, so

```
log coth x = log(1 + q) - log(1 - q) = 2 sum_{k >= 1, k odd} e^{-2kx}/k,     int_0^infinity x^3
↪  e^{-2kx} dx = 3!/(2k)^4.
```

All terms are positive, so the series may be integrated term by term, and

```
F(1) = 2 int_0^infinity x^3 log coth x dx = 4 sum_{k odd} (1/k) 3!/(2k)^4 = (3/2) sum_{k odd}
↪  k^{-5} = (3/2)(1 - 2^{-5}) zeta(5) = 93 zeta(5)/64.
```

Since $\mathcal{V}(0) = 0$ and $\mathcal{V}(1) =$ -n_f F(1), division by 8 pi^2 l^4 gives the stated endpoint value V(pi/2) - V(0) $=$ -93 zeta(5) n_f/(512 pi^2 l^4).

Small-angle coefficient. Differentiation under the integral at $t = 0$ is allowed because $x^3/\sinh^2 x$ is integrable on (0, infinity). With $1/\sinh^2 x = 4 e^{-2x}/(1 - e^{-2x})^2 = 4 \sum_{n \geq 1} n e^{-2nx}$,

```
F'(0) = int_0^infinity x^3/sinh^2 x dx = 4 sum_{n >= 1} n 3!/(2n)^4 = (3/2) sum_{n >= 1} n^{-3} =
↪  (3/2) zeta(3),
```

which is J(0) in the proof of Proposition 1.4. The two gauge arguments 2t(1 - t) and (16/5) t(1 - t) have derivatives 2 and 16/5 at $t = 0$, with prefactors 3 and 3/2, so

```
V'(0) = (3 * 2 + (3/2)(16/5) - n_f) F'(0) = (54/5 - n_f)(3/2) zeta(3).
```

With $t = \sin^2 \text{theta} = \text{theta}^2 + o(\text{theta}^2)$, V(theta) - V(0) $= \mathcal{V}'(0)$ theta^2/(8 pi^2 l^4) + o(theta^2) $=$ (3 zeta(3)/(16 pi^2))(54/5 - n_f) theta^2/l^4 + o(theta^2), the claimed coefficient. The exact vector formulas of lemma 23 vanish at theta $=$ pi/2, where the lowest fermion mass is pi/(2l).

Counterexample (referee) to the universal conclusion of Proposition 1.5. *From the referee report on the 11D line, section 'Proposition 1.5: scale-free potentials and the joint path', transferred verbatim in substance on 4 Nov 2026.*

Verdict of the referee: an error in the universal conclusion, and correct with an amendment for isolated stationary angles. The derivative identities in the proof hold. The universal conclusion, that every stationary family has constant angle and that a second scale is necessary for the existence of a joint family, fails by the counterexample below.

At fixed m > 0 put theta = l m. Then

```
d/dl V(l, l m) = l^{-5} (theta G'(theta) - 4 G(theta)).
```

If a continuous stationary branch chooses an isolated positive root theta_, *then theta_* is constant along the branch and l = theta_*/m. A root interval gives a flat direction instead.

Counterexample. Let G(theta) = theta^4 on a neighbourhood 0 <= theta < delta, with a C^1 extension to the angle domain. There

```
V(l, l m) = l^{-4} (l m)^4 = m^4
```

is independent of l. For any 0 < alpha < 1, the family m = epsilon, l = epsilon^{-alpha} has theta = l m = epsilon^{1 - alpha} < delta for all sufficiently small epsilon, so it is stationary, and l m -> 0. The vector mass formula of Theorem 23.6 gives M_W l ~ sqrt 2 l m -> 0. The potential leaves this path undetermined and does not dynamically select it. A smooth extension can also be chosen with G(theta) >= theta^4 on the rest of the fundamental interval; then V(l, l m) >= m^4, with equality on l m < delta, and the example is a flat family of minima.

The specified flat one-loop determinant. By Proposition 1.4, with a_0 the theta-independent Casimir coefficient,

```
G(theta) = a_0 + c theta^2 + o(theta^2),    theta G'(theta) - 4 G(theta) = -4 a_0 - 2 c theta^2 +
↪  o(theta^2),
```

where c = (3 zeta(3)/(16 pi^2))(54/5 - n_f) is proportional to 54/5 - n_f (the expansion of G' follows from that of $\mathcal{V}$', since $\mathcal{V}$ is C^1 at t = 0). If c != 0, there is no sequence of positive stationary angles tending to zero, whether or not a_0 = 0: for a_0 != 0 the right side tends to -4 a_0, and for a_0 = 0 it equals -2 c theta^2 (1 + o(1)). In particular this covers every integer n_f under the displayed determinant. The tuned case c = 0 requires its higher-order term to be checked. The comparison with the audit's ansatz with q > 0, in the paragraph on the modulus target, is correct: the displayed quadratic contribution is l^{-4} c (l m)^2 = c m^2/l^2.

Free angle. For a free angle, d/dl V = -4 l^{-5} G(theta); a nonzero minimized G produces the stated runaway, and a zero minimum leaves the length flat. At an endpoint of a constrained angle interval, the stationarity condition on G' must be understood one-sided.

Scope and direction. The corrected fixed-angle result concerns isolated stationary roots of a scale-free C^1 potential. A flat modulus, a tuned higher-order potential, or additional dimensionful terms leave other families possible. A mechanism that selects a joint path with finite gauge couplings remains to be constructed.

Independent proof of the asymptotics of the localized family (audit, its §4). *From the referee report on the 11D line, section 'Audit section 4 and file 01: localized joint family', transferred verbatim in substance on 4 Nov 2026.*

The asymptotic light mass, the gap, the localization ratio and the exact tail identity follow analytically from the interval equations. The localized length includes a logarithmic factor, and so differs from the pure-power length of the boundary-angle family above. The numerical table of section H was not recomputed by the referee.

Let c = mu l, s = kappa l, and s = c tanh s, the light-mode condition kappa = mu tanh(kappa l). For c > 1 there is a positive solution, and as c -> infinity it has s -> infinity and s/c = tanh s -> 1. Moreover

```
0 < c - s = c (1 - tanh s) = 2c e^{-2s}/(1 + e^{-2s}) <= 2c e^{-2s} -> 0.
```

Therefore

```
m = mu sech s = 2 mu e^{-s}/(1 + e^{-2s}) ~ 2 mu e^{-s} ~ 2 mu e^{-c}.
```

With mu = mu_* epsilon^alpha and c = (1 - alpha) log(1/epsilon) + c_0, e^{-c} = e^{-c_0} epsilon^{1 - alpha}, which gives m ~ 2 mu_* e^{-c_0} epsilon and l = c/(mu_* epsilon^alpha). The localization width is 1/kappa, so its ratio to l is 1/s ~ 1/c.

The first oscillatory root y_1 = k_1 l of tan y = y/c satisfies pi < y_1 < 3 pi/2, since y/c > 0 is small for large c. The gap above the light pair is therefore

```
Delta = sqrt(mu^2 + k_1^2) = mu sqrt(1 + (k_1 l/c)^2) ~ mu = mu_* epsilon^alpha.
```

Finally, s = c tanh s reads mu = kappa coth s, and with it m = mu sech s = kappa/sinh s. Then

```
m/(mu + kappa) = 1/(sinh s (coth s + 1)) = 1/(cosh s + sinh s) = e^{-s},    m^2 = kappa^2/sinh^2 s
  = kappa^2 (coth^2 s - 1) = (mu - kappa)(mu + kappa),
```

and the first identity squared is e^{-2 kappa l} = (m/(mu + kappa))^2. This proves both displayed tail identities.

Part 32

Witten spaces as the hypercharge quotient of the five-dimensional representation

Lemma 32. Written as file 02 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README). Theorem 2.1 and Propositions 2.2 to 2.4 and 2.6 are proved here, Theorem 2.1 and Proposition 2.6 conditionally as the referee notes say; Proposition 2.5 is a negative result in a stated scope, refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5. Attempted by Claude (Anthropic), 3 Nov 2026, with a remark from the steer of the same day on the mixing of u and d. Checks: checks_11d.py, sections A to D, Python with exact fractions. Referee pending.

Referee note (GPT-6.1 Terra, 4 Nov 2026). The exclusion in Proposition 2.5 has an independent analytic proof. The required-bidegrees lemma and the general assertion that u^c, d^c must have the same flux image require a common ten-dimensional chirality; the exclusion itself also holds without that extra assumption. See the referee report; other reviewed items are noted separately.

Referee note on Theorem 2.1 and Propositions 2.2, 2.4 (GPT-6.1 Terra). The quotient and tensor-centre calculations are independently checked. The cited M^{111} identification is conditional on the recalled fibration; the original quotient metric needs rescaling to be Sasaki-Einstein, and the action on holomorphic functions is dual to the action on U. See the report.

Referee note on Proposition 2.3 (GPT-6.1 Terra). The Gysin, Smith normal form and spin calculations are independently checked; the cited Kreck-Stolz paper was unavailable. See the report.

Setting

Witten's spaces M^{pqr} = G/H, with G = SU(3) x SU(2) x U(1) and H = SU(2) x U(1) x U(1), are the seven-dimensional spaces of least dimension with the isometry group of the Standard Model, so 4 + 7 = 11 (Witten 1981; paper/LITERATURE.md §F). Castellani, D'Auria and Fré gave them Einstein metrics; Killing spinors exist only for p = q, where the AdS4 vacuum has N = 2 and the space is M^{111} (Fabbri, Fré, Gualtieri, Termonia, hep-th/9903036). Fabbri et al. (hep-th/9907219, §4) write the cone over M^{111} as the Kähler quotient of C^5 by a circle acting with weights 2 on a triplet U and -3 on a doublet V, with D-term 2|U|^2 = 3|V|^2, and M^{111} as a circle bundle over P^2 x P^1. In later notation it is M^{3,2}.

These weights are six times the hypercharges of the 5-bar of SU(5). The rest of the file develops that observation.

Theorem 2.1 (M^{111} is the hypercharge quotient of the 5-bar)

Statement. Let 5bar = C^3 + C^2 carry the Standard Model action on the left-handed fields d^c in (3bar, 1)_{1/3} and L in (1, 2)_{-1/2}; write U for d^c and V for L, so 6Y = 2 on U and -3 on V. Let mu_Y =

(2|U|^2 - 3|V|^2)/6 be the moment map of hypercharge, and

Y_{3,2} = (mu_Y^{-1}(0) ∩ S^9) / U(1)_Y.

1. Y_{3,2} is a smooth, simply connected seven-manifold, the circle bundle of O(3, 2) over CP^2 x CP^1 (up to orientation), and its cone is the Kähler quotient C^5 // U(1)_Y of Fabbri et al.
2. The cone is Calabi-Yau (Gorenstein, sum of weights 0) because tr Y = 0 on the 5-bar, that is, because U(1)_Y lies in SU(5).
3. The group acting effectively on Y_{3,2} through the quotient is (U(3) x U(2))/U(1)_Y. Its subgroup S(U(3) x U(2))/U(1)_Y is PSU(3) x SO(3), and the overall phase of U(5) supplies one further circle, the Reeb (R-symmetry) direction.
4. The functions on Y_{3,2} that are polynomial in U, V and their conjugates have U(1)_Y-charge zero; the holomorphic ones are spanned by the monomials of degree (3k, 2k) in (U, V), whose span as tensors is Sym^{3k}(3bar) x Sym^{2k}(2) and as functions its dual Sym^{3k}(3) x Sym^{2k}(2): for k = 1 the functions form the (10, 3) of Fabbri et al.

Proof. On mu_Y^{-1}(0) ∩ S^9 we have |U|^2 = 3/5 and |V|^2 = 2/5, a product of round spheres S^5 x S^3. The circle acts by (lambda^2 U, lambda^{-3} V); a fixed point would need lambda^2 = lambda^3 = 1, so lambda = 1 and the action is free. The quotient of S^5 x S^3 by the full torus T^2 of phases of U and V is CP^2 x CP^1, and Y_{3,2} -> CP^2 x CP^1 is a principal bundle for T^2/U(1)_Y. A character lambda_U^m lambda_V^n of T^2 is trivial on U(1)_Y exactly when 2m - 3n = 0, so the characters of T^2/U(1)_Y are generated by (m, n) = (3, 2), whose line bundle is O(3) ⊠ O(2) up to the sign convention; this gives item 1 and item 4. Simple connectivity follows from the exact homotopy sequence of U(1) -> S^5 x S^3 -> Y_{3,2}. Item 2: the sum of weights is 3 * 2 + 2 * (-3) = 0. Item 3: an element (g, h) of U(3) x U(2) acts trivially on the quotient exactly when (g, h) = (lambda^2, lambda^{-3}), because mu_Y^{-1}(0) spans C^5. The intersection of SU(3) x SU(2) with U(1)_Y consists of lambda^6 = 1, generated by (e^{2 pi i/3} I_3, -I_2), the centre Z_3 x Z_2. ■

So the isometry group of the Witten space with Killing spinors, SU(3) x SU(2) x U(1) (Fabbri et al.), is the centralizer of hypercharge in U(5), divided by hypercharge. The SU(3) x SU(2) part is the Standard Model one; the remaining circle is the R-symmetry, and hypercharge itself has been quotiented. Witten assigned the U(1) of G to hypercharge; in this realization hypercharge is a different U(1) (03, Proposal 3.1).

Proposition 2.2 (uniqueness of the count 7)

Statement. (a) Among the quotients Y_{a,b} = (S^5 x S^3)/U(1), with weights b on C^3 and -a on C^2 and gcd(a, b) = 1, a, b > 0, the cone is Calabi-Yau only for (a, b) = (3, 2). (b) For a complex representation of dimension n and a free circle action, (S^{2n-1} ∩ mu^{-1}(0))/U(1) has dimension 2n - 3; this is 7 only for n = 5. Among representations of SU(5), with hypercharge as the circle, only the 5 and the 5-bar give a seven-dimensional quotient.

Proof. (a) The Calabi-Yau condition is 3b - 2a = 0. (b) Dimension count: 2n - 1 for the sphere, minus one for the moment map, minus one for the circle. The irreducible representations of SU(5) of dimension 5 are the 5 and the 5-bar. ■

Witten's count reads: 11 = 4 + (2 * 5 - 3). The seven internal dimensions come from the smallest representation of SU(5), with hypercharge as the circle. For the 10 the quotient has dimension 17, and for the 5 + 10 it has 27.

Proposition 2.3 (topology of Y_{a,b})

Statement. Let Y_{a,b} -> B = CP^2 x CP^1 be the circle bundle with Euler class a h_1 + b h_2, a != 0. Then H^2(Y) = Z + Z/gcd(a, b), b_3(Y) = 0, H^4(Y) is a group of order a^2, cyclic when gcd(a, b) = 1, and Y is spin exactly when a is odd and b is even. For (a, b) = (3, 2): H^2 = Z, H^4 = Z/9, spin. The pullback H^2(B) = Z^2 -> H^2(Y) = Z is (x, y) -> 2x - 3y up to sign: the two divisor classes h_1 = [{l(U) = 0}] and h_2 = [{m(V) = 0}] go to 2 and -3, the values of 6Y on d^c and L.

Proof. The Gysin sequence H^{k-2}(B) -> H^k(B) -> H^k(Y) -> H^{k-1}(B) -> H^{k+1}(B), with the first map the cup product with c = a h_1 + b h_2 and H^*(B) = Z[h_1, h_2]/(h_1^3, h_2^2). In degree 2 the cokernel of 1 -> (a, b) is Z^2/(a, b). In degree 4 the map H^2(B) -> H^4(B) has matrix [[a, 0], [b, a]] in the bases (h_1, h_2), (h_1^2, h_1 h_2), with determinant a^2 and Smith invariants (gcd(a, b), a^2/gcd(a, b)); since H^3(B) = 0 this gives H^4(Y), and the same matrix is injective, so H^3(Y) = 0. For spin: w_2(B) = h_1 mod 2, and the kernel of H^2(B; Z/2) -> H^2(Y; Z/2) is spanned by c mod 2; so w_2(Y) = 0 exactly when (1, 0) = (a, b) mod 2. For (3, 2), Z^2/(3, 2) -> Z by (x, y) -> 2x - 3y is an isomorphism. ■ (Python: Smith invariants and the spin condition, section C.)

The product member CP^2 x S^2 x S^1 of Witten's family (a = b = 0) is not spin.

Proposition 2.4 (the Z_6 of the Standard Model)

Statement. (a) Every monomial in U and V, and every state of the exterior algebra Λ^*(5bar), satisfies 2t + 3d + 6Y = 0 mod 6, with t the triality (1 for 3, -1 for 3bar) and d = 2j mod 2. (b) The U(1)_Y-neutral monomials have t = 0 and d = 0. (c) Every tensor field on Y_{3,2} carries a representation of (U(3) x U(2))/U(1)_Y, on which SU(3) x SU(2) acts through PSU(3) x SO(3). So no tensor Kaluza-Klein mode on Y_{3,2}, on a background invariant under the displayed action, is a colour triplet or a weak doublet; spinors are open.

Proof. (a) holds for U (t = -1, d = 0, 6Y = 2) and V (t = 0, d = 1, 6Y = -3) and is additive. (b) U^alpha V^beta is neutral when 2 alpha = 3 beta, so 3 | alpha and 2 | beta. (c) The effective group of Theorem 2.1 acts on the frame bundle and on every tensor bundle, and its intersection with SU(3) x SU(2) is the centre Z_6 (Theorem 2.1, item 3). ■

(a) is the statement that the Standard Model group is (SU(3) x SU(2) x U(1))/Z_6 = S(U(3) x U(2)). In the quotient realization the Z_6 is the kernel of SU(3) x SU(2) -> Isom(Y_{3,2}). States with fractional hypercharge are those charged under the quotient circle, and they cannot be functions or tensors on Y_{3,2}. For spinors the isometry group may lift to a double cover; whether fermionic Kaluza-Klein modes on Y_{3,2} can carry half-integer weak isospin or triality is open (next job, item 2 of the README).

Chiral zero modes on the base with abelian fluxes

Witten's argument makes Kaluza-Klein fermions on an odd-dimensional space vectorlike, for gauge fields acting by isometries. The even base B_6 = CP^2 x CP^1 escapes it: abelian fluxes on B_6 give chiral four-dimensional fermions through the index of the twisted Dirac operator. This section asks which fluxes give one generation.

Setting. Ten-dimensional Weyl fermions on M^4 x CP^2 x CP^1, each with definite charges under some U(1) factors with fluxes on B_6. Since CP^2 is not spin, a fermion is a spin^c spinor, Λ^{0,*} ⊗ K^{1/2} ⊗ N with N the flux line bundle of its charges, and the four-dimensional chiral content is the virtual representation H^*(O(alpha, beta)) with (alpha, beta) = (-3/2, -1) + pairing of the charges with the fluxes. The cohomology is H^0(O(alpha)) = Sym^alpha(3bar) for alpha >= 0, H^2(O(alpha)) = Sym^{-alpha-3}(3) for alpha <= -3, and zero otherwise on CP^2; on CP^1, H^0(O(beta)) = Sym^beta(2) for beta >= 0 and H^1(O(beta)) = Sym^{-beta-2}(2) for beta <= -2. Each Standard Model Weyl field must come from one ten-dimensional field, from the whole cohomology of one line bundle.

Lemma (required bidegrees). Writing P_s = (alpha + 3/2, beta + 1) for the pairing of the charges of field s (up to the common sign of charge and chirality):

field	P_s	note
Q (3, 2)	(-5/2, 2)	H^2(O(-4)) x H^0(O(1))
u^c, d^c (3bar, 1)	(5/2, 1)	H^0(O(1)) x H^0(O(0))
L (1, 2)	(+-3/2, 2)	
e^c, nu^c (1, 1)	(+-3/2, 1)	

The second entry is 2j + 1 and the first is +-5/2 on colour triplets and +-3/2 on singlets. (Python, section D, by enumeration of the line bundles with these cohomologies and the Riemann-Roch check.)

Proposition 2.5 (no generation from fluxes along hypercharge and B - L) *Scope note (4 Nov 2026, after the referee).* The statement on equal images and the linearity of 2j + 1 assume a common ten-dimensional chirality for the fields. The conclusion holds without it: in the span of 6Y and 3(B - L) the charges of L are -3 times those of Q, so their second pairings are P_2(L) = -3 P_2(Q), against |P_2| = 2 on both doublets.

Referee note on the third-charge search (GPT-6.1 Terra, 4 Nov 2026). The bounded search below has an independent analytic proof. The same argument extends to any real third charge and independent source chiralities; this stronger extension was checked by Claude Sonnet 5.5 (second check: the vectors y, b, p, w have rank 4 for every sign pattern), so it stands as a no-go in the same scope with two model identifiers. See the referee report.

Statement. In the setting above, with the canonical spin^c root and fluxes only in U(1) directions spanned by Y and B - L (any flux quanta), no choice reproduces one generation, with or without nu^c. More generally, the pairing must give u^c and d^c the same image, and 2j + 1 = (2, 1, 1, 2, 1, 1) on (Q, u^c, d^c, L, e^c, nu^c) must be a linear function of the fluxed charges.

Proof. The pairing M is linear in the charges, and M c_s = P_s with c_s the charges of the left-handed field s. With only Y and B - L fluxed, c_u - c_d = (-6, 0) in the coordinates (6Y, 3(B - L)), so M vanishes on the hypercharge direction and P_s = 3(B - L)_s m for one vector m. The second entries 2 on Q and 1 on u^c would then need 3(B - L) to take values in the ratio 2 : 1 on (Q, u^c), against 1 : -1. In general, 2j + 1 is not in the span of 6Y = (1, -4, 2, -3, 6, 0) and 3(B - L) = (1, -1, -1, -3, 3, 3): from Q and u^c the coefficients are -1 and 3, and then d^c gives -5 against 1. ■ An exhaustive search adding a third integer charge z in [-3, 3]^6 to Y and B - L finds no solution either (Python, section D).

Scope and escapes. Abelian fluxes on CP^2 x CP^1 with the Fubini-Study complex structure; one ten-dimensional Weyl field per Standard Model field; canonical spin^c root; Dolbeault index as the chiral content. Known escapes: nonabelian bundles (an SU(2)_R doublet bundle), several ten-dimensional fields per four-dimensional field, a changed common twist, orbifolds and walls, and non-Kaluza-Klein (brane) states. The result points to the next proposition.

Proposition 2.6 (one generation from a universal charge, the grading and B - L) **Referee note (GPT-6.1 Terra, 4 Nov 2026).** The index calculation works with the stated cohomology convention. For the colour action of the earlier quotient, reverse the first-component coefficients to obtain the labelled colour representations; global flux quantization remains to be specified. See the referee report.

Statement. Let the fluxed charges be F (one on every field: a change of the common spin^c twist), chi (+1 on Q and L, -1 on u^c, d^c, e^c, nu^c) and B - L. Then one generation with nu^c is the exact index content for the pairing

```
P^{(1)} = c_0 chi + c_1 (B - L),   P^{(2)} = (3F + chi)/2,
```

with (c_0, c_1) = (-9/4, -3/4) or (-3/2, -3). The bidegrees are Q: (-4, 1); u^c, d^c: (1, 0); L: (-3, 1) or (0, 1); e^c, nu^c: (0, 0) or (-3, 0). Without chi, or without B - L, there is no solution.

Proof. Substitution, and the two failures by the span test (Python, section D2). ■

Reading.

1. u^c and d^c see the same line bundle O(1, 0), and so do e^c and nu^c. The flux treats them as SU(2)_R doublets with a common internal profile; hypercharge Y = T_3R + (B - L)/2 enters through T_3R, which carries no flux. This is the left-right pattern of Pati-Salam, produced here by the index.
2. chi is the finite grading gamma_F on the particle basis. The flux therefore couples to the grading, an axial abelian connection on B_6; in the language of the finite geometry this is a fluctuation proportional to gamma_F, the structure of the two gradings of lemma 23.

3. **Mass and charge bases.** Because u^c and d^c share one zero-mode profile, overlap integrals on B_6 cannot separate them; their masses come from data that break SU(2)_R (Higgs or wall couplings), and the basis that diagonalizes the masses is set by those data, independently of the basis in which the charges are diagonal. The steer of 3 Nov 2026 remarks that u and d should still mix, since mass eigenvalues are not charge eigenvalues, and that at the 9 label, with the left-handed interactions off, charge eigenvalues can coincide. Proposition 2.6 puts the coincidence in the flux charges: equal flux charges for u^c and d^c, unequal hypercharges through T_3R. At the 9 label the electroweak part of the algebra is removed (lemma 21, STRATEGY T2), and what distinguishes u^c from d^c in the light theory is then the vectorlike charge alone. Making this precise needs the SU(2)_R-breaking wall data and the 9D reduction of this index; listed as item 4 of the README.

What this file establishes, and the direction

Referee note (GPT-6.1 Terra, 4 Nov 2026). The M^{111} identification is conditional on the recalled fibration and metric normalization. The flux examples establish a particular construction and scoped obstructions; a general necessity for the grading flux remains open. See the wording audit.

The Witten space with Killing spinors is the hypercharge quotient of the 5-bar (Theorem 2.1), the count 7 is fixed by SU(5) (Proposition 2.2), and the Z_6 of the Standard Model is the kernel of the isometry action (Proposition 2.4). The price is that hypercharge is no longer an isometry, and no tensor Kaluza-Klein mode on an invariant background carries triality, weak isospin 1/2 or fractional hypercharge. Kaluza-Klein chirality on the base needs fluxes that see the grading and leave T_3R free (Propositions 2.5 and 2.6). Both results point to states outside the Kaluza-Klein spectrum of Y_{3,2} for the quarks and leptons: 03 takes them to be wrapped branes.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 2.5 (the exclusion). *From the referee report on the 11D line, section 'Proposition 2.5: hypercharge and B - L fluxes', transferred verbatim in substance on 4 Nov 2026.*

Hypotheses: one abelian line bundle per field, the canonical formal shift, fluxes in the span of Y and B - L. The exclusion holds even if the ten-dimensional Weyl fields have different chiralities.

Let M_2 be the component of the flux pairing M along CP^1, and use the charges c = (6Y, 3(B - L)). A weak doublet occurs for a line bundle O(beta) on CP^1 only when beta = 1 or beta = -3, which give H^0(O(1)) = 2 and H^1(O(-3)) = 2 respectively. The formal K^{1/2} shift on CP^1 is -1, so M_2(c) = beta + 1 and |M_2(c)| = 2 for each doublet. This remains true if a right-handed mode is conjugated to the left-handed convention, since conjugation reverses the sign of the pairing.

For the left-handed doublets,

```
c_Q = (1, 1),   c_L = (-3, -3) = -3 c_Q.
```

Linearity gives M_2(c_L) = -3 M_2(c_Q), so |M_2(c_L)| = 3 |M_2(c_Q)|, whereas both absolute values must be 2. This contradiction excludes a generation with or without nu^c, for arbitrary flux quanta in the two specified directions. It uses neither the colour component nor a choice of signs for the individual doublets.

Amendment to the required-bidegrees lemma. On CP^2 x CP^1, K^{1/2} is a formal notation: K = O(-3, -2) has no line-bundle square root. A genuine spin^c structure and its charged twists must give integral final bidegrees. With a common ten-dimensional chirality, Serre duality relates a right-handed conjugate to the sign-reversed pairing, and the convention P^{(2)} = 2j + 1 of the table is consistent. If the ten-dimensional chiralities can be chosen independently, the table is incomplete. For example, O(-4, -3) has H^3 = H^2(O(-4)) ⊗ H^1(O(-3)) = (3, 2) and pairing (-5/2, -2); a suitable ten-dimensional chirality makes this a left-handed Q, and (-5/2, -2) is neither the table entry (-5/2, 2) nor its overall negative.

Likewise, with independent chiralities and independent flux directions, u^c can arise from H^0(O(1, 0)) = 3bar, while d^c arises by conjugating a right-handed H^2(O(-4, 0)) = 3 with opposite abelian charges. Their pairings in the left-handed charge convention are (5/2, 1) and (5/2, -1). The equality of their images, and the unsigned linear formula P^{(2)} = 2j + 1, are therefore conditional on the common-chirality convention; the direct doublet obstruction above uses neither.

Scope and direction. This concerns the Dolbeault cohomology of a single abelian line bundle for each field on CP^2 x CP^1, with the canonical formal shift and fluxes in the span of Y and B - L. Nonabelian bundles, additional charges or a common twist, multiple source fields, walls, orbifolds and brane states remain possible directions. This is an independent check of the scoped obstruction and makes no claim about those alternatives.

Independent proof of the third-charge exclusion in Proposition 2.5. *From the referee report on the 11D line, section 'Proposition 2.5: adding a third charge', transferred verbatim in substance on 4 Nov 2026.*

Status as the referee stated it: the argument proves the original bounded search, a third integer charge in [-3, 3]^6 under the common-chirality convention. The same argument covers an arbitrary real third charge and independently chosen ten-dimensional chiralities; the referee stated that extension as a conjecture awaiting an independent check by another model, and the referee note at Proposition 2.5 records that check by Claude Sonnet 5.5.

Write, on (Q, u^c, d^c, L, e^c, nu^c),

```
y = 6Y = (1, -4, 2, -3, 6, 0),    b = 3(B - L) = (1, -1, -1, -3, 3, 3).
```

(Here b is a charge vector, unrelated to the Euler-class coefficient b of Proposition 2.3.) For the stated colour convention the necessary CP^2 pairings are

```
p = (-5/2, 5/2, 5/2, p_L, p_e, p_nu),   |p_L| = 3/2,
```

and the CP^1 pairings have the form

```
w = (2 s_Q, s_u, s_d, 2 s_L, s_e, s_nu),   s_i in {+1, -1}.
```

The signs allow independently chosen source chiralities. Reversing the colour convention reverses p and leaves the argument unchanged.

Suppose that p and w both lie in the span of y, b and z, for some third charge z. The Q and L entries of y and of b have ratio -3, so every vector of span(y, b) has L entry equal to -3 times its Q entry, while |w_L/w_Q| = 1; hence w lies outside span(y, b). Then span(y, b, w) has dimension three and lies in span(y, b, z), so the two spans coincide, and

```
p = a y + b_0 b + t w
```

for real a, b_0, t. Apply the combination 2Q + u^c + d^c, the linear form v -> 2 v_Q + v_u + v_d. It vanishes on y (2 - 4 + 2 = 0), on b (2 - 1 - 1 = 0) and on p (-5 + 5/2 + 5/2 = 0), and takes the value 4 s_Q + s_u + s_d on w. So

```
0 = t (4 s_Q + s_u + s_d).
```

The factor in parentheses is nonzero, since |4 s_Q| = 4 > 2 >= |s_u + s_d|; hence t = 0. Equality of the u^c and d^c entries of p = a y + b_0 b, -4a - b_0 = 2a - b_0, gives a = 0, and their common value -b_0 = 5/2 gives b_0 = -5/2. The resulting L entry is -3 b_0 = 15/2, contradicting |p_L| = 3/2. The same four fields suffice without nu^c.

Scope and direction. This concerns the one-line-bundle-per-field Dolbeault construction, the specified Y and B - L charges and the canonical formal shift. A fourth charge direction, a changed common twist, nonabelian bundles, multiple source fields, walls, orbifolds and brane states remain directions to investigate.

Independent proof of Proposition 2.3. *From the referee report on the 11D line, section 'Proposition 2.3: topology of the circle bundle', transferred verbatim in substance on 4 Nov 2026.*

The calculation is independent of the specific M^{111} identification. The cited Kreck-Stolz paper was unavailable to the referee, so no direct comparison with that paper is claimed.

Let B = CP^2 x CP^1, H^*(B; Z) = Z[h_1, h_2]/(h_1^3, h_2^2), and let the Euler class be e = a h_1 + b h_2, with a != 0. The Gysin sequence gives

```
H^2(Y) = Z^2/<(a, b)> ~= Z ⊕ Z/g,   g = gcd(a, b).
```

The map H^2(B) -> H^4(B), cup product with e, sends h_1 to a h_1^2 + b h_1 h_2 and h_2 to a h_1 h_2 (since h_2^2 = 0); in the bases (h_1, h_2) and (h_1^2, h_1 h_2) its matrix is

```
[[a, 0], [b, a]].
```

Its determinant is a^2 != 0, so its kernel is zero, and since H^3(B) = 0 the Gysin sequence gives H^3(Y) = 0. The greatest common divisor of its entries is g, so its Smith invariants are g and a^2/g, and H^4(Y), its cokernel, is Z/g ⊕ Z/(a^2/g): a group of order a^2, cyclic when g = 1.

Spin. The vertical tangent line is trivial, spanned by the generator of the circle action, so w_2(Y) = pi^* w_2(B) = pi^*(h_1 mod 2), using w_2(CP^2) = h_1 mod 2 and w_2(CP^1) = 0. The mod-two Gysin sequence H^0(B; Z/2) -> H^2(B; Z/2) -> H^2(Y; Z/2), whose first map is the cup product with e mod 2, shows that the kernel of the pullback on H^2 is the span of e mod 2. Thus Y is spin exactly when h_1 ≡ e mod 2, that is, when a is odd and b is even. For (a, b) = (3, 2), the primitive homomorphism (x, y) -> 2x - 3y is onto Z and has kernel generated by (3, 2), so it identifies H^2(Y) with Z; and H^4(Y) ≅ Z/9.

Independent proofs for Theorem 2.1 and Propositions 2.2 and 2.4: freeness, the neutral-monomial congruence, the dual action and the Sasaki-Einstein normalization. *From the referee report on the 11D line, section 'Theorem 2.1 and Propositions 2.2 and 2.4: quotient and tensor action', transferred verbatim in substance on 4 Nov 2026.*

Hypotheses: the identification with M^{111} is conditional on the recalled description of M^{111} as the circle bundle of class 3h_1 + 2h_2; the cited paper could not be accessed by the referee.

Freeness and the bundle. On the moment level in the unit sphere, |U|^2 = 3/5 and |V|^2 = 2/5. Both factors are nonzero, so the stabilizer of a point under (lambda^2 U, lambda^{-3} V) has lambda^2 = lambda^3 = 1, hence lambda = 1, and the action is free. The quotient has dimension 9 - 1 - 1 = 7 and is simply connected by the homotopy sequence of U(1) -> S^5 x S^3 -> Y_{3,2}. The residual character of the two Hopf phases has primitive weight (3, 2); its circle bundle over CP^2 x CP^1 therefore has Euler class ±(3h_1 + 2h_2). Principal circle bundles over this base are classified by their Euler classes, which gives the conditional identification with M^{111}. The zero sum 3(2) + 2(-3) = 0 gives the Gorenstein holomorphic cone with trivial canonical bundle; a Ricci-flat metric needs a separate argument.

The congruence of Proposition 2.4. The congruence 2t + 3d + 6Y ≡ 0 (mod 6) holds on U, where (t, d, 6Y) = (-1, 0, 2), and on V, where (t, d, 6Y) = (0, 1, -3), and extends additively to tensor products and to conjugates, which reverse all three quantities. For a monomial in U, V and their conjugates, let Delta_U be the number of factors U minus the number of factors Ubar, and Delta_V likewise for V. Neutrality under U(1)_Y reads

```
2 Delta_U - 3 Delta_V = 0,
```

so 3 | Delta_U and 2 | Delta_V. The triality of the monomial is -Delta_U ≡ 0 (mod 3) and d ≡ Delta_V ≡ 0 (mod 2): zero triality and integer weak isospin. This extends item (b) of Proposition 2.4 from the holomorphic monomials U^alpha V^beta to all monomials in U, V and their conjugates.

Dual action. If U is the vector representation 3bar and projective space parametrizes its lines, the conventional pullback action on holomorphic coordinate functions is the dual 3: functions of degree 3k transform as Sym^{3k}(3). Treating U as formal field components gives Sym^{3k}(3bar) instead. The convention must be fixed consistently when identifying the 10 versus the 10bar and the later cohomology table.

Sasaki-Einstein normalization. With the standard unit-sphere quotient metric, the two transverse Fubini-Study factors have relative scales 3/5 and 2/5. CP^2 is the base of the Hopf circle on the sphere of radius sqrt(3/5), with Ricci constant 6/(3/5) = 10, and CP^1 is the base of the Hopf circle on the sphere of

radius sqrt(2/5), a round sphere of radius sqrt(2/5)/2 with Ricci constant 4/(2/5) = 10; so each has Ricci constant 10. A seven-dimensional Sasaki-Einstein metric needs transverse Ricci constant 8. The Sasaki metric obtained by the standard transverse homothety of factor 5/4 has the same underlying circle bundle and the required Einstein normalization. Thus the existence of the Sasaki-Einstein metric on that manifold is compatible with the quotient, and the initial round-sphere quotient metric is identified with it only after the rescaling.

Independent proof of Proposition 2.6, with the colour convention. *From the referee report on the 11D line, section 'Proposition 2.6: flux assignments for one generation', transferred verbatim in substance on 4 Nov 2026.*

Verdict of the referee: correct with amendments, conditional on global flux quantization. The bidegrees and the two linear solutions are correct with the cohomology convention of the file. With the colour action on the quotient fixed as in Theorem 2.1, that convention is conjugated and the first-component coefficients must be reversed.

The pairings. The charges of the left-handed fields are F = 1 on all; chi = +1 on Q, L and -1 on u^c, d^c, e^c, nu^c; and B - L = 1/3, -1/3, -1/3, -1, 1, 1 on Q, u^c, d^c, L, e^c, nu^c. For (c_0, c_1) = (-9/4, -3/4), direct substitution in P^{(1)} = c_0 chi + c_1 (B - L) and P^{(2)} = (3F + chi)/2 gives

```
P_Q = (-5/2, 2),   P_{u^c, d^c} = (5/2, 1),   P_L = (-3/2, 2),   P_{e^c, nu^c} = (3/2, 1);
```

for instance P_Q^{(1)} = -9/4 - 1/4 and P_L^{(1)} = -9/4 + 3/4. For (c_0, c_1) = (-3/2, -3), the quark pairings are unchanged (-3/2 - 1 = -5/2 and 3/2 + 1 = 5/2), and the first components for L and for e^c, nu^c are +3/2 and -3/2 respectively. Subtracting the formal shift (3/2, 1) yields exactly the listed integral bidegrees. On CP^2, the nonzero cohomology of a line bundle O(k) is H^0(O(k)) for k >= 0, or H^2(O(k)) ≅ H^0(O(-k - 3))^* for k <= -3; CP^1 has the analogous H^0/H^1 formula. So O(-4, 1) gives (3, 2), O(1, 0) gives 3bar, O(-3, 1) and O(0, 1) give (1, 2), and O(0, 0) and O(-3, 0) give a singlet, in the convention of the Setting. All listed choices have even total Dolbeault degree (0 or 2), so the common-chirality convention gives the indicated chiral index.

The colour convention. If the base is P(3bar), as in the quotient of Theorem 2.1, then H^0(O(1)) = 3 and H^2(O(-4)) = 3bar, and the listed quark bundles carry the conjugates of the labelled colour representations. Replacing (c_0, c_1) by (9/4, 3/4) or (3/2, 3), with the second component unchanged, reverses the first pairings and exchanges O(1) with O(-4) for the quarks; this gives the labelled Q and u^c, d^c with the colour action of the quotient. Alternatively, the original table is valid for P(3), with the corresponding conjugate colour convention.

Without chi. The required weak-doublet pairings have absolute value 2. Write P^{(2)} = alpha F + beta (B - L), so P_Q^{(2)} = alpha + beta/3 and P_L^{(2)} = alpha - beta. If the two signs for Q and L agree, then beta = 0 and |alpha| = 2, and u^c gets |P^{(2)}| = 2, the wrong absolute value for a weak singlet. If they disagree, say alpha + beta/3 = 2 and alpha - beta = -2, then beta = 3, alpha = 1, and u^c gets P^{(2)} = 1 - 1 = 0.

Without B - L. Q and L have identical charges F = 1, chi = +1, so their pairings coincide, while they require colour pairings of absolute values 5/2 and 3/2 respectively.

These obstructions use only absolute values, so they also allow independent source chiralities.

Global data. The final line-bundle degrees are integral, but K^{1/2} itself is formal and the displayed coefficients include fractional flux components. An actual background requires a specified global gauge group, charge lattice, spin^c determinant and compatible flux quantization. The index calculation alone establishes neither those data nor the physical interpretation of the grading as a gauge connection.

Part 33

The two families of 84 and the M2 and M5 sides

Lemma 33. Written as file 04 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN). Attempted by Claude (Anthropic), 4 Nov 2026. Propositions 4.1 to 4.3 and Theorem 4.4 and Propositions 4.5, 4.6 are proved here by hand (branching rules and counting); the negative results 4.3 and 4.6 were refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5, with the amendments below; readings are labelled as proposals.

Referee note (GPT-6.1 Terra, 4 Nov 2026). Proposition 4.1 and the colour obstructions in Propositions 4.2(a) and 4.3 are independently checked. The tensor-factor conclusion of 4.2(b) needs a factor-preserving map, and the "exactly" clause in 4.3 also needs nonzero triplets: the trivial nine-dimensional representation gives 84 singlets. See the referee report; the other checked results are noted below.

Referee note on Proposition 4.6 (GPT-6.1 Terra). The stated sector and block obstructions hold in the scopes in the referee report. The Chevalley involution is complex-linear; an antiunitary J needs an additional conjugation. Its identification with physical M2/M5 duality remains a proposal.

The question

Lemma 21, Proposition 21.3: with three generations and Dirac neutrinos, dim H_F = 96 = 84 + 12 in two ways,

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F1 = all states except the top          (the top: 3 colours x 4 = 12 states),
F2 = all states except the neutrinos    (the neutrinos: 3 generations x 4 = 12 states),
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where 4 counts chirality and particle/antiparticle. The two families overlap in the 72 states of the charged fermions other than the top. Lemma 21 proposed relating one family to C_3 (or the M2-brane) and the other to C_6 (or the M5-brane), with the exchange of the top and the neutrinos as the content of the duality on the Standard Model side, and recorded that equal numbers are insufficient. The files 02 and 03 showed that colour-equivariant identifications through the little group or through Kaluza-Klein modes on the Witten space fail (Proposition 3.8, Corollary 3.9). This file asks the question in the Hořava-Witten wall group E8 and in the finite geometry.

1. What singles out the two blocks

Proposition 4.1. Consider blocks of 12 states made of whole Dirac species (all colours, both chiralities, particles and antiparticles; 12 states for a quark species, 4 for a lepton species). The blocks whose removal keeps both spectral-action values of lemma 21, cos^2 theta_NC = 5/8 and kappa_em/kappa_s = 8/3, are exactly: one up-type quark (u, c or t), and the three neutrinos together. So F1 (with the top, the heavy up-type quark) and F2 are the two kinds of solutions: a charged one and a neutral one.

Proof. A 12-state block of whole species is one quark species or three lepton species. By Proposition 21.1, with the totals (sum T_3 Q, sum Y Q; Q^2, colour) = (6, 10; 32, 12) and cos^2 theta_NC = (sum Y Q)/(sum T_3 Q + sum Y Q), removing a quark keeps both values only for an up-type quark, (1, 5/3; 16/3, 2). For three lepton species with k charged leptons, each contributing (1/2, 3/2; 4, 0), the angle becomes (10 - 3k/2)/(16 - 2k), which is 5/8 only for k = 0; neutrinos contribute zero to all four sums. ■

Reading at the 9 label. At the 9 label the light gauge algebra is colour plus electromagnetism (canonical §3, lemma 21). H_F then splits into three parts: the 72 charged light states (F1 ∩ F2), the 12 neutrino states, which carry no charge of that algebra and are free, and the 12 top states, which are charged and infinitely heavy. F1 = charged light + free light; F2 = charged light + charged heavy. The exchange of

lemma 21 exchanges the block that decouples through zero charge with the block that decouples through infinite mass.

2. Colour obstructions

Proposition 4.2. (a) As representations of colour SU(3), F1 = 10 (3 + 3bar) + 24 (1) and F2 = 12 (3 + 3bar) + 12 (1); they are not isomorphic. Hence no unitary of H_F commuting with colour maps F1 onto F2, and no automorphism of the finite spectral triple of the Standard Model does. (b) Any tensor-factor-preserving bijection of the top block onto the neutrino block that is linear in each index exchanges the colour index of the top (r, g, b) with the generation index of the neutrinos (e, mu, tau).

Proof. (a) Counting: F1 has the five quarks other than the top (5 x 2 triplet pairs, counting chirality) and 24 leptonic states; F2 has the six quarks and 12 charged-lepton states. Colour acts through A_F. (b) The top block is C^3_colour ⊗ e_3 ⊗ C^4 (third generation, 4 for chirality and conjugation) and the neutrino block is e_lepton ⊗ C^3_generation ⊗ C^4 in the Pati-Salam form C^4_colour ⊗ C^3_generation of the up-type sector; a map of the first onto the second that respects the tensor factors sends C^3_colour to C^3_generation. ■

So an exchange needs a structure in which colour and generation can be exchanged; a structure in which colour acts and generations do not admits none.

Proposition 4.3 (complex case of Proposition 3.8). For every homomorphism SU(3) -> SU(9), the colour content of Λ^3 C^9 is not that of F1 or F2. Λ^3 C^9 contains only 1, 3 and 3bar exactly when C^9 is 9 (1), 3 + 6 (1) or 3bar + 6 (1); for 3 + 6 (1), Λ^3 C^9 = 15 (3) + 6 (3bar) + 21 (1), and for 9 (1) it is 84 (1).

Proof. C^9 is a sum of irreducibles of dimensions 1, 3, 6, 8 with total 9. With a sextet: 6 + 3 (1) has the 6 in Λ^1(6) ⊗ Λ^2(3 (1)); 6 + 3 has Λ^3(6) = 10 + 10bar. With an octet: 8 + 1 has Λ^3(8) = 1 + 8 + 10 + 10bar + 27. With two triplets T, T' (each 3 or 3bar) and three singlets: Λ^1(T) ⊗ Λ^1(T') ⊗ Λ^1(1) contains T ⊗ T', which is 6 + 3bar, 6bar + 3 or 8 + 1. With three triplets: T ⊗ T' ⊗ T'' contains a 10, a 10bar, a 15 or a 15bar. With nine singlets there is no triplet. With one triplet and six singlets, Λ^3(3 + 6 (1)) = Λ^3(3) + 6 Λ^2(3) + 15 (3) + 20 (1) = 15 (3) + 6 (3bar) + 21 (1). Neither family has unequal numbers of 3 and 3bar. ■

Together with Proposition 3.8 (real nine-dimensional representations) this excludes, in that scope, the colour-equivariant route through Λ^3 of a linear nine-dimensional space. The positive construction below places the two blocks, and leaves the two families unidentified.

3. Two commuting gradings of E8

E8 is the gauge group of each wall in Hořava-Witten theory, and it also models the C-field topologically (Witten 1996, "On flux quantization in M-theory"; Diaconescu, Moore and Witten 2000; recalled). Over C, e8 has an inner automorphism of order three with fixed algebra sl(9):

```
e8 = sl(9) ⊕ Λ^3 C^9 ⊕ Λ^6 C^9          (80 + 84 + 84),
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with [Λ^3, Λ^3] ⊂ Λ^6 (wedge product), [Λ^3, Λ^6] ⊂ sl(9) (contraction) and [Λ^6, Λ^6] ⊂ Λ^3. In the U-duality algebra e8(8) of eleven-dimensional supergravity on T^8, the decomposition under gl(8) by level (West's level, recalled) places the C_3 shifts at level 1 (Λ^3 of 8, 56 states), the C_6 shifts at level 2 (Λ^6 of 8, 28), the dual graviton at level 3 (8) and gl(8) at level 0; then Λ^3 C^9 = 56 (level 1) + 28 (level -2) and the grading above is the level mod 3. So M-grade b = 1 is one unit of M2 charge (or minus an M5), b = 2 is one unit of M5 charge (or minus an M2), and b = 0 holds the metric and the dual graviton (levels 0 and ±3). The bracket Λ^3 × Λ^3 -> Λ^6 is the algebraic form of the term C_3 ∧ G_4 in G_7 = dC_6 + (1/2) C_3 ∧ G_4, equivalent by duality to the Chern-Simons coupling; Λ^3 × Λ^6 -> sl(9) adds levels 1 + 2 = 3, the dual graviton. Under SO(9) ⊂ SL(9), 248 = 36 + 44 + 84 + 84, the SO(9) content of so(9), the graviton, C_3 and C_6. We call the three pieces the gravity, C_3 and C_6 sectors.

A second element of order three, the centre of SU(3) in E8 ⊃ (SU(3) × E6)/Z_3, gives the family grading e8 = (su(3) ⊕ e6) ⊕ (3, 27) ⊕ (3bar, 27bar), with family grade a = 0, 1, 2. With this SU(3) as family

group, (3, 27) holds three families of E6, each containing a generation 16 ⊂ 27, and (3bar, 27bar) holds their conjugates.

Lemma 4.A (the two gradings commute). Take SU(9) ⊃ SU(3) × SU(6) × U(1) with C^9 = (3, 1)_{-2} + (1, 6)_{1}, and let s_f be the centre of this SU(3). Then s_f commutes with the SU(9) grading, its centralizer is (SU(3) × E6)/Z_3, the U(1) is the Cartan of the SU(2) in E6 ⊃ SU(6) × SU(2) with charge q = 6 T_3, and the joint pieces g_{a,b} are

	b = 0 (gravity)	b = 1 (C_3)	b = 2 (C_6)
a = 0	su(3) + su(6) + u(1): 44	(1,1)_{-6} + (1,20)_{3}: 21	(1,1)_{6} + (1,20)_{-3}: 21
a = 1	(3, 6bar)_{-3}: 18	(3, 15)_{0}: 45	(3, 6bar)_{3}: 18
a = 2	(3bar, 6)_{3}: 18	(3bar, 6)_{-3}: 18	(3bar, 15bar)_{0}: 45

so that b ≡ a + q/3 = a + 2 T_3 (mod 3).

Proof. Λ^3 of (3,1)_{-2} + (1,6)_{1} is (1,1)_{-6} + (3bar,6)_{-3} + (3,15)_{0} + (1,20)_{3}; the adjoint is (8,1) + (1,35) + (1,1) + (3,6bar)_{-3} + (3bar,6)_{3}; Λ^6 is the conjugate of Λ^3. The column a = 0 sums to 86 = 8 + 78 and the row a = 1 to 81 = 3 x 27, which matches E8 ⊃ SU(3) × E6 with E6 ⊃ SU(6) × SU(2): 27 = (15, 1) + (6bar, 2) and 78 = (35, 1) + (1, 3) + (20, 2), the SU(2) triplet being (1,1)_{-6}, the u(1) and (1,1)_{6}. ■

4. Placement of the generation

Let SU(5) ⊂ SU(6) ⊂ E6 be the Georgi-Glashow group, with 15 = 10 + 5 and 6bar = 5bar + 1. The commutant of SU(5) in E6 is SU(2) × U(1), and its SU(2) is the SU(2) of the lemma; we call it SU(2)_I. With SO(10) × U(1)_psi and SU(5) × U(1)_chi, 27 = 16_1 + 10_{-2} + 1_4, 16 = 10_{-1} + 5bar_3 + 1_{-5} and 10 = 5_2 + 5bar_{-2}. The Cartan T_3 of SU(2)_I vanishes on the singlets 10 and 5, so it is proportional to psi + chi: T_3 = t on 5bar_16 and on N = 1_{E6}, and T_3 = -t on 5bar_10 and on nu^c = 1_16, with t = ±1/2. The SU(2)_I doublets are (d^c, D^c), (L, H_d) and (N, nu^c).

Theorem 4.4 (the generation in the three sectors). For the generation 16 ⊂ 27 of family grade a = 1:

Referee note (GPT-6.1 Terra, 4 Nov 2026). The branching and counts are independently checked. The fields t_L, t^c are in the left-handed basis; t^c is an antitop field, so the top particle's two chiralities occupy different sectors. See the referee report.

1. the 10 = (Q, u^c, e^c) and the 5 = (D, H_u) lie in g_{1,1}, the C_3 sector;
2. (d^c, L) and N lie in g_{1,1+2t}, and (D^c, H_d) and nu^c lie in g_{1,1-2t}: one pair in the C_6 sector and the other in the gravity sector, according to the sign t; the two choices differ by the Weyl reflection of SU(2)_I;
3. the antiparticles lie in the conjugate pieces g_{2,-b}.

So every top state of a particle (t_L in Q, t^c in u^c), and the up-type Higgs, lie in the C_3 sector, and the anti-top in the C_6 sector. The left-handed particle fields of the neutrinos (a = 1) lie outside the C_3 sector, nu_L and nu^c in different sectors (C_6 and gravity); their conjugates do not, since conj(nu_L) lies in C_3 for t = +1/2. With conjugates, the top block has (C_3, C_6, gravity) = (6, 6, 0) and the neutrino block (3, 3, 6), for both signs of t; a conjugation-closed block always has equal C_3 and C_6 counts. With three families and antiparticles, the 96 states of H_F split as 45 (C_3) + 45 (C_6) + 6 (gravity: nu^c and its conjugate) for t = +1/2, and 33 + 33 + 30 for t = -1/2.

Proof. b ≡ a + 2 T_3, with T_3 = 0 on the 10 and the 5, and T_3 = ±t on the doublet members; for antiparticles a = 2 and T_3 changes sign. The counts follow per family: for t = +1/2, C_3 holds the 10 and the 5 of antiparticles (15), C_6 their conjugates, gravity nu^c and its conjugate; for t = -1/2, C_3 holds the 10 and the conjugate of nu^c (11), C_6 their conjugates, gravity the 5bar and the 5 (10). ■

5. Yukawa couplings as brackets

Proposition 4.5. Suppose the cubic couplings of the matter come from the structure constants of e8, as in heterotic and Hořava-Witten compactifications, and both gradings are symmetries, so that a coupling needs grades summing to (0, 0) mod 3. Take H_u in the 5 ⊂ g_{1,1} and H_d in 5bar_10 ⊂ g_{1,1-2t}. Then, for either sign t:

Referee note (GPT-6.1 Terra, 4 Nov 2026). The grade selection and bracket targets are independently checked. In the literal exact $SU(3)_f$ triplet realization, the family tensor is antisymmetric and does not by itself give an isolated top mass; the physical Yukawa matrix needs additional data. See the referee report.

1. the up-type Yukawa couplings Q u^c H_u have all three fields in the C_3 sector;
2. the down-type, charged-lepton and Dirac-neutrino couplings Q d^c H_d, L e^c H_d and L nu^c H_u have one field in each of the three sectors;
3. in Dirac form, the top mass is ad() : g_{1,1} -> g_{2,2}, the wedge Λ^3 -> Λ^6 with the Higgs trivector, while the Dirac-neutrino mass is a bracket that passes through the gravity sector: Λ^6 -> sl(9) (levels 1 + 2 = 3) for t = +1/2, sl(9) -> Λ^3 for t = -1/2.

Proof. Grades from Theorem 4.4: (1, 1) three times for the up type; for the others the b-grades are {1, 1 + 2t, 1 - 2t} = {1, 2, 0}. The Dirac masses pair the left-handed field with the conjugate of the right-handed one, through the Killing form between g_{a,b} and g_{-a,-b}. ■

The family grading alone allows only epsilon_{ijk} in exact family SU(3); in compactifications the family label is a cohomology label and the couplings are intersection numbers. The M-grading survives when its element commutes with the structure group of the bundle: for the standard embedding, s_M = s_f exp(2 pi i (2/3) T_3), a product of the centre of the holonomy SU(3) and an element of SU(2)_I ⊂ E6, which does.

Reading (proposal). The up-type Yukawa couplings are the only Standard Model Yukawa couplings inside one sector, and that sector's bracket is the C_3 ∧ C_3 -> C_6 term, metric-independent; the other couplings use one field of each sector, including the gravity sector. The top, the only fermion with a Yukawa coupling of order one and the one singled out by the spectral action in lemma 21, takes its mass from the C_3 bracket.

6. The Chevalley involution and the real structure J

Referee note (GPT-6.1 Terra, 4 Nov 2026). This heading describes a proposed physical identification. The Chevalley involution is complex-linear, and an antiunitary J needs additional conjugation; constructing the finite triple remains open. See the wording audit.

Proposition 4.6.

1. No automorphism of e8 that preserves the family grading maps the C_3 sector of the matter, g_{1,1} (dimension 45), onto its C_6 sector, g_{1,2} (dimension 18).
2. The Chevalley involution, the element -1 of the Weyl group of E8, maps g_{a,b} to g_{-a,-b}. It normalizes SU(9), acting as complex conjugation (Λ^3 <-> Λ^6), and on the placed H_F it exchanges every state with its conjugate: it is the real structure J of the finite geometry. The family grade a is the four-dimensional chirality in the left-handed basis (a = 1 for the left-handed fields, a = 2 for their conjugates), and J inverts it, as KO-dimension six requires (J gamma = -gamma J).
3. J preserves F1 and F2, which are unions of species with their conjugates.
4. The Weyl reflection of SU(2)_I fixes s_f and replaces b by 2a - b: on the matter (a = 1) it fixes the C_3 sector and exchanges the C_6 and gravity sectors, together with (d^c, L) <-> (D^c, H_d) and nu^c <-> N.

Proof. (1) Dimensions. (2) -1 acts on a Cartan subalgebra by -1, so it inverts every element of the maximal torus containing s_f and s_M, and maps the joint eigenspace for (omega^a, omega^b) to that for (omega^{-a}, omega^{-b}); it maps the root system A_8 of SU(9) to itself. (3) Immediate. (4) The reflection inverts T_3 and commutes with SU(3) × SU(6). ■

Negative result (scope: e8 with these two gradings, automorphisms preserving the pair up to

inversion). The Chevalley involution, read as the M2/M5 exchange (a proposal), maps each block to its conjugate and leaves the top and neutrino blocks in place. An exchange as in lemma 21 must therefore come from an automorphism that neither preserves nor inverts the family grading; for aligned families, Proposition 5.1 also excludes every element of W(E8) mapping F1 onto F2. The colour-family exchanges inside E8 ⊃ SU(3)^4, where the four SU(3) factors are permuted by the normalizer, are the remaining candidates (PLAN, step 4). What the gradings do give is a placement of the left-handed fields (a = 1): the top on the M2 side, the neutrinos on the M5 and gravity sides; with conjugates the counts are those after Theorem 4.4.

7. Agreement with files 02 and 03, and the finite geometry

- **nu^c as the gravitational state.** nu^c is the only Standard Model state with Y = 0, so on the Witten space it is the only one that can be a Kaluza-Klein mode (Proposition 2.4 with Proposal 3.1), while every hypercharged state is a brane state. In e8 with t = +1/2 it is the only Standard Model state in the gravity sector, and every hypercharged state lies in the C_3 or C_6 sector. The two placements agree for t = +1/2; for t = -1/2 the gravity sector holds (d^c, L) instead. We take the agreement as selecting t = +1/2 (proposal).
- **The finite geometry.** On H_F, the family grading is the chirality, the Chevalley involution is J, and the Yukawa blocks of D_F are brackets with the Higgs: the top block maps the C_3 sector to the C_6 sector, the Majorana block acts inside the gravity sector (nu^c with nu^c, t = +1/2). A finite triple built on the graded e8, with D_F = ad(Higgs), is the construction to attempt next (PLAN, step 5).
- **The two families.** F1 removes a block of the C_3 sector (with its conjugate in C_6): the states joined by the cubic C_3 coupling. F2 removes the block that meets the gravity sector. In this sense, for the left-handed fields, the top belongs to the M2 side and the neutrinos to the M5 and gravity sides, which is the correspondence lemma 21 asked for, at the level of the removed blocks; with conjugates the blocks have counts (6, 6, 0) and (3, 3, 6). The two families of 84 are not themselves the 84 of C_3 and the 84 of C_6 (Propositions 3.8, 4.3, 4.6).

8. Remark: a discrete symmetry from the M-grading

For t = +1/2, s_M acts on the 27 by omega^b: omega on the 10 and on the 5 (H_u, D), omega^2 on (d^c, L) and N, and 1 on (D^c, H_d) and nu^c. It allows Q u^c H_u, Q d^c H_d, L e^c H_d, L nu^c H_u, N H_u H_d, the Majorana mass of nu^c and the Weinberg operator L L H_u H_u, and forbids the mu term H_u H_d, the dimension-five operators Q Q Q L and u^c u^c d^c e^c, and the trilinear couplings L L e^c, L Q d^c, u^c d^c d^c; the bilinear L H_u is allowed. This is the pattern of E6-inspired models in which the singlet N supplies the mu term. It is a symmetry of the low-energy theory only if the breaking of E6 commutes with s_M; for t = -1/2 the charges change. Stated as a remark; the results above do not use it.

Scope of the remark (second check, 4 Nov 2026). The selection rule is that of the single element s_M, and it is a symmetry only if the breaking of E6 commutes with s_M. The family grading would forbid (nu^c)^2, L L H_u H_u and L H_u, which s_M allows, since their family grades are 2, 1 and 2.

9. What remains open

1. Colour-family exchange in E8 ⊃ SU(3)^4 acting on the placed states: does an element map the top block onto the neutrino block and F1 onto F2? (PLAN, step 4.)
2. A finite real spectral triple on the graded e8 with D_F = ad(Higgs): order zero and order one, and comparison with C + H + M_3(C) (PLAN, step 5).
3. Hořava-Witten dynamics: whether an M5-brane moving into a wall (small-instanton transition) can remove one of the 12-state blocks of Proposition 4.1 (PLAN, step 6).
4. The E11 level decomposition beyond E8, where C_3, C_6 and the dual graviton sit at levels 1, 2, 3: test the assignment 10 -> level 1, (d^c, L) -> level 2, nu^c -> level 0 or 3 (PLAN, step 7).

References for this file

Recalled without rereading in this session: E. Witten, "On flux quantization in M-theory and the effective action", hep-th/9609122; D.-E. Diaconescu, G. Moore and E. Witten, "E8 gauge theory, and a derivation

of K-theory from M-theory", hep-th/0005090; P. West, "E11 and M theory", hep-th/0104081, for the level decomposition and the commutator of the level-one generators; E. Cremmer and B. Julia (1979) for E8(8) in three dimensions; E. B. Vinberg and A. G. Elashvili (1978) for the Z_3-graded e8 and trivectors of C^9; the E6 subgroup chains are standard (R. Slansky, Phys. Rep. 79 (1981) 1).

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 4.2(a), with the scope of 4.2(b). *From the referee report on the 11D line, section 'Propositions 4.2 and 4.3: complex colour representations', transferred verbatim in substance on 4 Nov 2026.*

The colour multiplicities are

`F1 = 10 (3 + 3bar) + 24 (1),    F2 = 12 (3 + 3bar) + 12 (1):`

each quark species of 12 states gives two triplets and two antitriplets (particle and antiparticle, two chiralities); F1 holds five quark species and the 24 lepton states, and F2 holds the six quark species and the 12 charged-lepton states. An intertwiner preserves the isotypic components, so no colour-commuting unitary exchanges F1 and F2. Even a colour-group automorphism preserves the dimension of the invariant subspace, which is 24 for F1 and 12 for F2.

For 4.2(b), the stated conclusion holds for a tensor-factor-preserving bijection identifying the top's colour factor with the neutrinos' family factor. Linearity in each index alone does not imply this: every linear map on the tensor product is multilinear in its arguments, and a unitary may entangle the three-dimensional index with the four-dimensional chirality/conjugation index. Such a unitary need not be a flip of tensor factors.

Counterexample (referee) to the first version of Proposition 4.3. *Same section of the referee report, transferred verbatim in substance on 4 Nov 2026.*

The first version stated that the exterior cube contains only 1, 3 and 3bar exactly for 3 + 6 (1) or its conjugate. The trivial homomorphism gives C^9 = 9 (1) and Λ^3 C^9 = 84 (1), which also contains only those representations. The statement above lists 9 (1); equivalently, the corrected statement adds the requirement that the exterior cube contain a nonzero triplet.

Independent proof of Proposition 4.3 (corrected). *Same section of the referee report, transferred verbatim in substance on 4 Nov 2026.*

The irreducible SU(3) representations of dimension at most nine are 1, 3, 3bar, 6, 6bar and 8: the irreducible of highest weight (p, q) has dimension (p + 1)(q + 1)(p + q + 2)/2, and the next values are 10 and 15. So C^9 is a sum of irreducibles of dimensions 1, 3, 6, 8 with total 9, and the cases are the following.

1. 8 + 1. Λ^3(8 + 1) = Λ^3 8 + Λ^2 8, and Λ^2 8 contains a 10: identify 8 with sl_3(C); the vector E_13 ∧ E_12 is killed by ad E_12 and ad E_23 (the only nonzero bracket, [E_23, E_12] = -E_13, produces E_13 ∧ E_13 = 0) and has weight 2 alpha_1 + alpha_2 = (3, 0), so it generates a 10.
2. 6 + 3 (1). Λ^1(6) ⊗ Λ^2(3 (1)) = 3 (6) is a summand, so the exterior cube has a 6.
3. 6 + 3 or 6 + 3bar. Λ^3 6 is a summand and contains both 10 and 10bar: in 6 = Sym^2 C^3, with coordinates x, y, z, the highest-weight vectors

   `x^2 ^ xy ^ xz,    x^2 ^ xy ^ y^2`

   are killed by the raising operators x d/dy and y d/dz and have weights (3, 0) and (0, 3) respectively. Their dimensions already sum to C(6, 3) = 20, so Λ^3 6 = 10 + 10bar. The conjugate cases are analogous.
4. Two triplets T, T' (each 3 or 3bar) and three singlets. The cube contains T ⊗ T' ⊗ 1, hence T ⊗ T', which is 6 + 3bar, 6bar + 3 or 8 + 1: a 6, 6bar or 8.

5. Three triplets. The cube contains T ⊗ T' ⊗ T'', which contains a 10, 10bar, 15 or 15bar (3 ⊗ 3 ⊗ 3 contains 10, and 3 ⊗ 3 ⊗ 3bar contains 6 ⊗ 3bar, which contains 15; the others are conjugate).

6. One triplet and six singlets:

   ```
   Λ^3(3 + 6 (1)) = Λ^3 3 + 6 Λ^2 3 + 15 (3) + 20 (1) = 15 (3) + 6 (3bar) + 21 (1),
   ```

   using Λ^3 3 = 1, Λ^2 3 = 3bar, C(6, 2) = 15 and C(6, 3) = 20. Its conjugate has the multiplicities of 3 and 3bar reversed.

7. The trivial representation 9 (1), with Λ^3 C^9 = 84 (1).

None gives either target: both targets contain equally many 3 and 3bar, as well as nonzero triplets, while cases 1 to 5 contain a 6, 6bar, 8, 10, 10bar, 15 or 15bar, case 6 and its conjugate have unequal numbers of 3 and 3bar, and case 7 has no triplet. Only cases 6 and 7 contain nothing beyond 1, 3 and 3bar, which is the "exactly" clause.

Scope and direction. This concerns the full exterior cube of a complex nine-dimensional linear SU(3) representation and the stated colour action on F1 and F2. Other dimensions, non-equivariant identifications, nonlinear or brane constructions, and maps that do not preserve the colour action remain possible directions.

Independent proof of Lemma 4.A and of the grades in Theorem 4.4. *From the referee report on the 11D line, section 'Lemma 4.A and Theorem 4.4: the two E8 gradings', transferred verbatim in substance on 4 Nov 2026.*

Let 9 = (3, 1)_{-2} ⊕ (1, 6)_1. The exterior-power expansion Λ^3(A ⊕ B) = Λ^3 A ⊕ (Λ^2 A ⊗ B) ⊕ (A ⊗ Λ^2 B) ⊕ Λ^3 B, with Λ^3 3 = 1, Λ^2 3 = 3bar, Λ^2 6 = 15 and Λ^3 6 = 20, gives

```
Λ^3 9 = (1, 1)_{-6} ⊕ (3bar, 6)_{-3} ⊕ (3, 15)_0 ⊕ (1, 20)_3.
```

The adjoint, 9 ⊗ 9bar minus the trace, contains (8, 1)_0 ⊕ (1, 35)_0 ⊕ (1, 1)_0 ⊕ (3, 6bar)_{-3} ⊕ (3bar, 6)_3, and Λ^6 9 is the conjugate of Λ^3 9. The family centre gives grade a = 1 on 3, a = 2 on 3bar and a = 0 on singlets. Comparing the three columns (b = 0 on sl(9), b = 1 on Λ^3, b = 2 on Λ^6) yields

```
b == a + q/3 = a + 2 T_3   (mod 3),
```

including the singlet terms of charges ±6: for instance (1, 1)_{-6} has a + q/3 = -2 ≡ 1, and (3bar, 6)_{-3} has a + q/3 = 2 - 1 = 1. The grade-zero dimensions are 8 + 78 = 86, and the grade-one pieces form 3 ⊗ 27 (18 + 45 + 18 = 81), in agreement with the stated SU(6) × SU(2) ⊂ E6 branching.

Grades of the generation. Under SU(5), 15 = 10 + 5 has T_3 = 0. In the quoted E6 charge normalization, T_3 = (psi + chi)/8 gives +1/2 on 5bar_16 and N, and -1/2 on 5bar_10 and nu^c (for example (1 + 3)/8 on 5bar_3 ⊂ 16_1 and (1 - 5)/8 on 1_{-5} ⊂ 16_1); reversing the SU(2)_I Cartan exchanges the two signs. With t = +1/2, the left-handed 16 has grades 10_{b=1} ⊕ 5bar_{b=2} ⊕ 1_{b=0}. Adding its conjugate, with a = 2 and T_3 reversed, gives 15, 15 and 2 states per family in grades 1, 2 and 0, hence 45 + 45 + 6. With t = -1/2 the corresponding counts are 11, 11 and 10, hence 33 + 33 + 30.

Left-handed basis. Both t_L ∈ Q and t^c ∈ u^c lie in b = 1, and t^c is the left-handed conjugate of t_R. The right-handed top state lies in b = 2, and the antitop likewise has chiral components in both sectors. The phrase "every top state of a particle" in Theorem 4.4 therefore holds in the left-handed basis. The labels C_3 and C_6 express this algebraic grading; a physical M2/M5 identification requires the corresponding field construction.

Independent proof of Proposition 4.6, with the block argument. *From the referee report on the 11D line, section 'Proposition 4.6: the graded E8 exchange', transferred verbatim in substance on 4 Nov 2026.*

Dimensions. The branching table gives dim g_{1,1} = 45 and dim g_{1,2} = 18, so no linear automorphism maps the first whole space onto the second.

Chevalley involution. The Chevalley involution theta sends each root space g_alpha to g_{-alpha}; consequently it sends g_{a,b} to g_{-a,-b}. It exchanges the two exterior-power sectors and preserves F1 and F2, which include each root together with its negative.

Real structure. Let kappa be coefficient conjugation in a real Chevalley basis. Since theta maps the real span of that basis to itself, theta and kappa commute, and J = theta kappa is antilinear with J^2 = 1. With a compatible positive Hermitian form it is antiunitary. On the placed 16 + 16bar states, the grading gamma with signs +1, -1 on family grades 1, 2 obeys J gamma = -gamma J, since J maps family grade a to -a. This establishes these two KO-6 signs for the constructed operators; the spectral-triple algebra, the order conditions and JD = DJ remain to be established. The complex-linear theta alone cannot be the antiunitary J.

The SU(2)_I reflection. It sends T_3 to -T_3, so b = a + 2 T_3 becomes b' = a - 2 T_3 = 2a - b (mod 3). It fixes b = 1 at a = 1 and exchanges b = 0 and b = 2, as stated.

Blocks. The dimension argument excludes an exchange of the whole joint-graded sectors. The exclusion of the top and neutrino blocks has an independent argument under the specified three-family embedding. In g_1 = 3_f ⊗ 27 their grade-one parts are

```
T ∩ g_1 = <e_t> ⊗ U_t,  dim U_t = 6;      N ∩ g_1 = 3_f ⊗ U_nu,  dim U_nu = 2,
```

with e_t the top's family line, U_t ⊂ 27 spanned by the t_L and t^c states, and U_nu ⊂ 27 spanned by nu_L and nu^c. Both subspaces have dimension 6, so dimensions alone do not separate them. An automorphism phi preserving the family grading preserves g_0 = sl_3 ⊕ e_6 and its two nonisomorphic simple ideals, so it acts on g_0 as a pair (alpha, beta) of automorphisms of sl_3 and e_6. Its restriction to g_1 satisfies phi([x, v]) = [phi(x), phi(v)], so it is an isomorphism of g_1 onto g_1 with the g_0-action twisted by (alpha, beta). That twisted module is isomorphic to 3_f ⊗ 27 only when the twists preserve 3_f and 27, so there are A in GL(3) and B in GL(27) implementing alpha on 3_f and beta on 27; then (A ⊗ B)^{-1} phi is an endomorphism of the irreducible g_0-module 3_f ⊗ 27, a scalar by Schur's lemma. So phi restricted to g_1 is a multiple of A ⊗ B. It sends a subspace supported on one family line to another subspace supported on one family line, and cannot give the neutrino subspace, whose family support is all of 3_f. The same argument applies when the family grading is inverted, with the conjugate module g_2.

Scope and direction. This concerns complex-linear automorphisms of the specified graded e8, the aligned 3_f ⊗ 16 embedding, and the top and neutrino blocks just described. Automorphisms that neither preserve nor invert the family grading, other embeddings or state spaces, and dualities not represented by such an automorphism remain open directions. A physical M2/M5 identification also requires its action on the fields and on the finite triple.

Independent proof of Proposition 4.5, with the family tensor. *From the referee report on the 11D line, section 'Proposition 4.5: Yukawa grades and family dependence', transferred verbatim in substance on 4 Nov 2026.*

The grade-selection rules and bracket targets hold as algebraic statements; by themselves they establish neither a physical top mass nor its observed family hierarchy.

Grades. The up-type triple Q u^c H_u has grades (1, 1) three times, so both Z_3 sums vanish. For the down-type, charged-lepton and Dirac-neutrino triples the second grades are {1, 1 + 2t, 1 - 2t} = {1, 2, 0}, with sum 3 ≡ 0, and the family grades are again 1, 1, 1, with sum zero. The bracket with H_u ∈ g_{1,1} adds (1, 1), so it sends grade b = 1 to b = 2, as claimed. For the neutrino it sends b = 2 to b = 0 when t = +1/2 (L lies at b = 1 + 2t = 2), and b = 0 to b = 1 when t = -1/2 (L lies at b = 0). These are necessary grade conditions; a particular bracket or four-dimensional coupling may still vanish.

The family tensor. In the literal 3_f ⊗ 27 realization with exact SU(3)_f invariance, the family part of a cubic involving three grade-one triplets is proportional to epsilon_{ijk}, the invariant of 3 ⊗ 3 ⊗ 3. A Higgs expectation value h_k therefore gives a family matrix proportional to

```
Y_{ij} = epsilon_{ijk} h_k.
```

In the aligned family basis its diagonal entries, including the nominal top-top entry, vanish. For nonzero h this matrix has rank two and two equal nonzero singular values, since, up to the family-index convention,

```
(Y^† Y)_{jl} = sum_{i,k,m} conj(epsilon_{ijk} h_k) epsilon_{ilm} h_m = ||h||^2 delta_{jl} - h_j
↪  conj(h_l),
```

that is Y^† Y = ||h||^2 I - h h^†, with eigenvalue ||h||^2 twice and 0 on h. Thus this exact one-Higgs family tensor alone gives no single isolated top mass.

The Z_3 family grading alone allows more tensors than epsilon_{ijk}; epsilon_{ijk} follows from the full SU(3)_f invariance. A compactification in which generation labels are cohomology labels and couplings are overlap integrals may evade the literal triplet-tensor restriction, but its mode functions and intersections must be supplied before the physical Yukawa matrix is identified.

Independent proof of the selection rule of section 8. *From the referee report on the 11D line, section 'File 04, section 8: the discrete grade', transferred verbatim in substance on 4 Nov 2026.*

The remark holds as a selection rule for the single element s_M, assuming it survives the breaking. For t = +1/2 the grades are b(10) = b(H_u) = 1 (Q, u^c, e^c and H_u), b(5bar_16) = b(N) = 2 (d^c, L and N), and b(H_d) = b(nu^c) = 0. Adding grades modulo three gives zero for

```
Q u^c H_u (1 + 1 + 1),  Q d^c H_d (1 + 2 + 0),  L e^c H_d (2 + 1 + 0),  L nu^c H_u (2 + 0 + 1),
N H_u H_d (2 + 1 + 0),  (nu^c)^2 (0),  L L H_u H_u (2 + 2 + 1 + 1),  L H_u (2 + 1),
```

and grade 2 for

```
H_u H_d (1 + 0),  Q Q Q L (1 + 1 + 1 + 2),  u^c u^c d^c e^c (1 + 1 + 2 + 1),
L L e^c (2 + 2 + 1),  L Q d^c (2 + 1 + 2),  u^c d^c d^c (1 + 2 + 2).
```

Additional unbroken symmetries may impose further restrictions. In particular, the family Z_3 has to be considered separately if it is also retained.

Part 34

The exchange of the top and the neutrinos as a colour-family transposition in E8

Lemma 34. Written as file 05 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, step 4). Attempted by Claude (Anthropic), 4 Nov 2026. Proposition 5.1 (a negative result in its scope) and Proposition 5.2 are proved by hand, refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md); an exploratory search in Python with exact fractions found the same results (no isometry for three families, signed coordinate permutations for four); the reruns of 4 Nov 2026 (Python, exact integers) are not kept, by the rule that checking scripts are not saved. Readings are labelled as proposals. Proposition 5.6, stated in Reading 3b (c) by Claude Sonnet 5.5, was proved and refereed by Claude Opus 5.5 (effort max, 5 Nov 2026; historical/reviews/2026-11-05_opus_check_05_21_34.md).

Referee note (GPT-6.1 Terra, 4 Nov 2026). Proposition 5.1 has an independent analytic check of its neighbour and pair counts, and of its no-isometry conclusion in the aligned-root scope. See the referee report. Proposition 5.4 is addressed at its statement below.

Setting

Referee note (GPT-6.1 Terra, 4 Nov 2026). The M2/M5 identification is a proposal, and the colour-family conclusion of Proposition 4.2 needs a factor-preserving map. See the wording audit.

Proposition 4.6 showed that the M2/M5 exchange of the graded e8 acts as J and cannot exchange the top with the neutrinos, and Proposition 4.2 that any such exchange must trade the colour index of the top for the generation index of the neutrinos. This file tests the colour-family exchanges inside E8.

Use E8 ⊃ SO(16) ⊃ SO(10) × SO(6), with the roots of E8 in the coordinates of R^8: ±e_i ± e_j, and (1/2)(±1, …, ±1) with an even number of minus signs. The 128 half-integer roots form the chiral spinor of SO(16), which under SO(10) × SU(4) is (16, 4) + (16bar, 4bar): four families of the Standard Model with ν^c, with their conjugates. Write a root as (χ | w | φ), with χ the signs of the three colour coordinates, w those of the two weak coordinates and φ those of the three family coordinates; a particle has an even number of minus signs among χ, w and an even number among φ. The four family patterns are 𝟙 = (+++) (family 0) and the three patterns with two minus signs (families 1, 2, 3); the colour patterns with one minus sign are the quark colours, the pattern −𝟙 is the lepton. Then, for a colour pattern c with one minus sign and a family f,

state	root (times 2)
u_L, d_L	(c | −+ | f), (c | +− | f)
u^c, d^c	(−c | ++ | f), (−c | −− | f)
ν_L, e_L	(−𝟙 | −+ | f), (−𝟙 | +− | f)
ν^c, e^c	(𝟙 | ++ | f), (𝟙 | −− | f)

and antiparticles are the negatives. Three families give the 96 roots of H_F, closed under negation; the four family slots are permuted by the Weyl group of SU(4), so the choice of three is immaterial. The same 96 roots are (16, 3) + (16bar, 3bar) in E8 ⊃ E6 × SU(3) ⊃ SO(10) × U(1) × SU(3), the embedding of file 04.

Proposition 5.1 (the two families of 84 are not isometric)

Statement. Let H_F be the 96 roots of three families, the top block T the 12 roots of the up-type quark of one family f_t with its conjugate, and the neutrino block N the 12 roots of the neutrinos of the three families with their conjugates; F1 = H_F − T and F2 = H_F − N. Then F1 and F2 have the same numbers of pairs at each inner product (42 at −2, 816 at −1, 1812 at 0, 816 at +1), but no isometry of R^8 maps F1 onto F2. In particular no element of the Weyl group of E8 does, and no isometry of H_F maps T onto N.

Proof. For a root v of a set S let d_S(v) be the number of roots of S at inner product +1 with v; it is preserved by isometries. Two half-integer roots have inner product +1 exactly when their signs agree in six of the eight places.

1. d_{H_F} = 22 for every root. H_F is a single orbit of the group generated by the Weyl group of SO(10) (transitive on the 16, a minuscule representation), the permutations of the three families and the negation. For ν^c of family 0, (𝟙 | ++ | 𝟙), the roots of H_F with exactly two minus signs are: two in the first five places with family 𝟙 (10), two among the family places (2, the patterns of families 1 and 2 in H_F), and antiparticles with one minus sign in each block (5 x 2). The total is 22.
2. Removing N. For u_L = (c | −+ | f): the inner product with ν_L(g) is (1 + f · g)/4 and with ν^c(g) is (1 + f · g)/4, where f · g = 3 for g = f and −1 otherwise; so u_L has exactly 2 roots of N at +1 (its own family's ν_L and ν^c), and the same holds for u^c and the conjugates. For d_L = (c | +− | f): the inner product with ν_L(g) is (−3 + f · g)/4, which is −1 for g ≠ f, so the two roots −ν_L(g), g ≠ f, are at +1, and ν^c(f) is at +1: 3 in all; the same count holds for d^c, e_L, e^c and the conjugates. So in F2, d = 20 on the 36 up-type quark roots and d = 19 on the 48 others.
3. Removing T. For a neutrino of the top's family, ν_L(f_t) = (−𝟙 | −+ | f_t): its inner products with t_L(c) = (c | −+ | f_t) and with t^c = (−c | ++ | f_t) are (−1 + 2 + 3)/4 = 1 and (1 + 0 + 3)/4 = 1, six roots at +1; for g ≠ f_t, ν_L(g) has inner product (−1 + 2 − 1)/4 = 0 with t_L and (1 + 0 − 1)/4 = 0 with t^c. The same holds for ν^c. So in F1, d = 16 on the 4 roots of the neutrinos of the top's family and d = 22 on the 8 roots of the other neutrinos (the up-type quarks of the other families have 2 roots of T at +1 and the down-type quarks and charged leptons 3, by inner products of the same kind as in item 2, so d = 20 and d = 19).

F1 has roots with d = 16 and d = 22, and F2 has none. The sums of d agree, 4 · 16 + 48 · 19 + 24 · 20 + 8 · 22 = 48 · 19 + 36 · 20 = 1632 = 2 x 816, which is the equality of the pair counts at +1. Both sets are closed under negation, so the pairs at −1 are as many as those at +1 and the pairs at −2 are the 42 antipodal pairs; the 1812 pairs at 0 are the remainder of the 3486 pairs. ■

Scope. The roots of H_F in E8, through the embedding above, which is unique up to W(E8) for the aligned (16, 3) + (16bar, 3bar); isometries of R^8 acting on the roots. Escapes: maps that are not linear on the roots (bijections of states with no geometric meaning), other realizations of H_F in a larger group, and a fourth family (Proposition 5.2). The tau neutrino carries the obstruction: in F1 the neutrino of the top's own family is the only state whose neighbourhood changes by six.

Proposition 5.2 (four families: the transposition of colour and family)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The Weyl map and block exchange are independently checked. Family-zero lepton species blocks are fixed setwise; ν^c and e^c individually go to their negatives. See the referee report.

Statement. Let P be the permutation of coordinates that exchanges the three colour coordinates with the three family coordinates and the two weak coordinates with each other, and M = −P. Then M is an involution in the Weyl group of E8 that preserves the 128 roots of the four families and SU(2)_L, and acts on the up-type and down-type states as the transposition of the four Pati-Salam colours (r, g, b, lepton) with the four families (1, 2, 3, 0), with the lepton paired with family 0 and the colour c paired with the family −c. In particular:

1. the up-type quark of family 0 (t_L and t^c, three colours, with conjugates) goes onto the neutrinos of families 1, 2, 3 (ν_L and the conjugate of ν^c), and back;
2. the down-type quark of family 0 goes onto the charged leptons of families 1, 2, 3;
3. nu_L and e_L of family 0 are fixed, nu^c and e^c of family 0 go to their negatives, u^c and d^c go to conjugate roots, and the quarks of families 1, 2, 3 go to quarks of families 1, 2, 3 with colour and family transposed.

So the four-family sets 128 − T(0) and 128 − N(1, 2, 3), each of 116 roots, are exchanged.

Proof. P is an even permutation of coordinates and −1 is in the Weyl group of E8, so M is in it; M^2 = P^2 = 1. Both preserve the set of half-integer vectors with an even number of minus signs. P maps (χ | w_1 w_2 | φ) to (φ | w_2 w_1 | χ), so M maps it to (−φ | −w_2 −w_1 | −χ). The weak part (−+) goes to (−+) and (++) to (−−), so T_3 of SU(2)_L, (w_2 − w_1)/2, is preserved and T_3 of SU(2)_R is reversed. Then t_L(c) = (c | −+ | 𝟙) goes to (−𝟙 | −+ | −c) = ν_L(−c); t^c = (−c | ++ | 𝟙) goes to (−𝟙 | −− | c) = −ν^c(−c); b_L(c) = (c | +− | 𝟙) goes to (−𝟙 | +− | −c) = e_L(−c); ν_L(0) = (−𝟙 | −+ | 𝟙) and e_L(0) are fixed; u_L(c, f) = (c | −+ | f) goes to u_L(−f, −c), with −f a quark colour and −c a family pattern. ■

The search found further signed permutations with the same effect on the two blocks; P itself maps T(0) onto N(1, 2, 3) up to conjugation, and M = J ∘ P, with J = −1 the real structure of Proposition 4.6.

Proposition 5.3 (the flip in a four-family finite geometry)

Referee note (GPT-6.1 Terra, 4 Nov 2026). Choose the factor isomorphism ι unitary, or unitary up to a common scalar. An arbitrary isomorphism need not make F unitary. With this amendment the flip and block calculation are independently checked; see the report.

Statement. Let H_f = (C^2_L ⊕ C^2_R) ⊗ C^4_c ⊗ C^4_f, H = H_f ⊕ conj(H_f), with J the antilinear exchange of the two summands and γ equal to +1 on C^2_L and −1 on C^2_R in H_f and to its negative on conj(H_f), the signs of KO-dimension six. Fix a unitary isomorphism ι: C^4_c -> C^4_f sending the lepton line ℓ to the family line e_0 and the colour space C^3 onto the span of e_1, e_2, e_3, and let F be the flip x ⊗ y -> ι^{-1}(y) ⊗ ι(x) on C^4_c ⊗ C^4_f, extended by the identity on C^2_L ⊕ C^2_R and to conj(H_f) by J F J^{-1}. Then:

1. F is unitary, F^2 = 1, and F commutes with γ and J.
2. For the up-type Dirac blocks T_top = Y ⊗ P_q ⊗ P_0 and T_ν = Y ⊗ P_ℓ ⊗ (1 − P_0), where Y maps the up state of C^2_L to the up state of C^2_R (and Y^* back), P_ℓ and P_q = 1 − P_ℓ are the lepton and quark projectors of C^4_c and P_0 the projector on e_0, one has F T_top F^{-1} = T_ν, and likewise for the down-type blocks of family 0 and the charged leptons of families 1, 2, 3.

3. So the heavy limits D + t T_top and D + t T_ν, for D commuting with F, are conjugate by F: their light spaces, of 116 states each, correspond. The light algebras A ∩ {T}' correspond when the algebra is invariant under F; the Pati-Salam and Standard Model algebras act on C^4_c alone and are not.

Proof. F acts on the factor C^4_c ⊗ C^4_f only, where it conjugates a ⊗ b to ι^{-1} b ι ⊗ ι a ι^{-1}, so P_q ⊗ P_0 goes to P_ℓ ⊗ (1 − P_0) by the choice of ι. γ and Y act on the other factor, and J F J^{-1} on conj(H_f) is the conjugate flip. ■

F equals M on the left-handed states and M followed by the particle-antiparticle exchange on the right-handed ones (second check): F exchanges T_top and T_nu inside H_f, while M sends right-handed states to antiparticle roots and so flips gamma on 64 states; F lies outside W(E8). It also makes reading 2 below precise: F exchanges the projector P_ℓ, which the Majorana direction selects in colour space, with P_0, which the top Yukawa coupling selects in family space; as operators, it maps the top Yukawa block to the Dirac blocks of the neutrinos of families 1, 2, 3, and it does not map the Majorana block to a Dirac block, since the Majorana block joins H_f to conj(H_f).

Proposition 5.4 (no three-slot assignment admits the exchange)

Referee note (GPT-6.1 Terra). The no-isometry conclusion is correct by an independent analytic argument, but the neighbour counts in the statement below are false for misaligned slots. The corrected formulas and a counterexample are in the referee report.

Statement. Place the top in slot 0, and let the up-type quarks, the down-type quarks, the charged leptons and the neutrinos each occupy three of the four family slots, independently (the up-type quarks in slots containing 0). This includes the aligned families of Proposition 5.1 and the configurations of the 9 label, where SU(2)_L is off and does not tie the slots of up-type and down-type quarks. For none of the 192 configurations S is there an isometry of R^8 mapping S − T onto S − N. S − N has d in {13, 17, 19, 20}, and {19, 20} only when the down-quark and charged-lepton slot sets coincide (48 of the 192); each neutrino root of S − T has d = 16 + 6 (1_A(f) − 1_0(f)), with A the slot set of the up-type quarks, so the values are 16 and 22 in the patterns (16, 22, 22) and (16, 16, 22).

Proof. Exhaustive computation of the neighbour counts d (Python, exact integers, rerun twice on 4 Nov 2026 and not kept; also the analytic proof in the referee report); the counts of S − T and S − N differ for every configuration, so no isometry exists. ■ This is a computer-assisted statement; Proposition 5.1 is its hand-proved aligned case.

Correction (4 Nov 2026). The conclusion holds on rerun; the sentence of the first version of the Statement, "In every case S - N has d = 19 and d = 20 only, while S - T has d = 16 on the neutrino roots of one slot and d = 22 on those of the two others", was wrong for misaligned configurations and has been removed. See 10, section on corrections.

Scope. It assumes that each family occupies one slot, so that every state is a single root. Four slots then cannot be reduced to three while keeping the exchange as an isometry of the roots.

Proposal 5.5 (four slots, three zero modes)

Referee note (GPT-6.1 Terra, 4 Nov 2026). General bundle zero modes are outside the root-slot hypotheses. The index gives net chirality, and split bundles do not automatically reduce to three distinct root slots; the Weyl action on a specified bundle and its zero modes remains open. See the referee report.

In heterotic compactifications on a Calabi-Yau threefold with E8 ⊃ SO(10) × SU(4), 248 = (45, 1) + (1, 15) + (16, 4) + (16bar, 4bar) + (10, 6), the SU(4) is the structure group of a rank-four bundle V, and the families are the zero modes, the 16s coming from H^1(X, V); their net number is the index |c_3(V)|/2, which is 3 in the heterotic Standard Models built on SU(4) bundles (Braun, He, Ovrut and Pantev, hep-th/0501070; recalled without rereading). In that reading the four slots of this file are the fibre of V, and each family is a zero mode, a section of V that in general involves all four slots. Propositions 5.1 and 5.4 do not apply to it, since they assume one slot per family.

The transposition M of Proposition 5.2 then exchanges the Pati-Salam SU(4), a gauge group, with the structure group SU(4) of V: colour and holonomy trade places, and SO(10) goes to the commutant SO(10)' of the other SU(4). The top-neutrino exchange becomes a statement about the fibre, which an index map must carry to the zero modes. The test is an explicit bundle. For a split bundle V = L_0 + L_1 + L_2 + L_3 (line-bundle models), slot i carries n_i = −χ(L_i) families with n_0 + … + n_3 = 3; such models fall under Proposition 5.1 only for n = (1, 1, 1, 0) up to order; for every n_i >= 0 with sum 3 the degree multisets still differ (second check, 40 cases), so the obstruction persists for aligned split bundles. A non-split V, whose zero modes are spread over the slots, is the case to compute.

Proposition 5.6 (the so(9) of the Chevalley involution is conjugate to the Spin(9) of Reading 3)

Let h be a Cartan subalgebra of e8 contained in the regular sl(9) of lemma 33, section 3, and let theta be the Chevalley involution of e8 for h: in a Chevalley basis, theta acts as -1 on h and maps e_alpha to -e_{-alpha}. Let sigma be the involution of the root table of this lemma, +1 on h and on the integer roots and -1 on the half-integer roots; its fixed algebra is so(16)_sigma and its -1 eigenspace is the 128 of the four families. Then:

1. theta preserves sl(9) and acts on it by an outer involution, whose fixed algebra so(9)_theta is conjugate in SL(9) to the so(9) of lemma 33.
2. The fixed algebra k_theta of theta on e8 is isomorphic to so(16). As so(9)_theta-modules, k_theta = 36 + 84 and p_theta = 44 + 84, where p_theta is the -1 eigenspace. The 44 equals p_theta ∩ sl(9) and lies in the gravity sector. The two 84 are the graphs {x + theta x} and {x - theta x}, x in Λ^3 C^9, and each meets the C_3 and C_6 sectors equally.
3. so(9)_theta acts on the 16-dimensional vector of k_theta by the spinor.
4. so(9)_theta is conjugate in E8 to the Spin(9) of Reading 3, the image of Spin(9) under the spinor 16 inside so(16)_sigma, in the class with 128 -> 44 + 84. Under both, 248 = 36 + 44 + 84 + 84.
5. In E8 both connected subgroups are isomorphic to SO(9). The other spinor class, with an irreducible 128 and 248 = 36 + 84 + 128, is isomorphic to Spin(9).

Proof. (1) theta maps the root system A_8 to itself, since it acts as -1 on h, so it preserves sl(9). On sl(9) it acts as -1 on a Cartan subalgebra, and -1 lies outside W(A_8), so the involution is outer. Since 9 is odd, every outer involution of sl(9) has the form X -> -B^{-1} X^T B with B symmetric; its fixed algebra so(B) has dimension 36 and is conjugate in SL(9) to the so(9) of lemma 33. The conjugating element lies in SL(9), which commutes with the centre of SU(9), so it preserves the three sectors.

(2) The fixed algebra of theta is spanned by the e_alpha - e_{-alpha}, so it has dimension 120, the number of positive roots; it is the complexified maximal compact subalgebra of the split form e8(8), which is so(16) (Cartan's classification; Helgason, chapter X). On sl(9), theta has fixed part so(9)_theta (36) and -1 part 44, which is h plus the span of the e_alpha + e_{-alpha} for alpha a positive root of A_8. theta maps Λ^3 to Λ^6. For Y in so(9)_theta, theta [Y, x] = [Y, theta x], so the maps x -> x ± theta x from Λ^3 are so(9)_theta-equivariant and injective, and their images are the fixed and -1 parts of Λ^3 + Λ^6, each isomorphic to Λ^3 C^9 = 84.

(3) As k_theta-modules, k_theta = Λ^2 W with W the vector 16, so Λ^2(W restricted to so(9)_theta) = 36 + 84. The 16-dimensional representations of so(9) are 16, 9 + 7 · 1 and 16 · 1, with exterior squares 36 + 84, 36 + 7 · 9 + 21 · 1 and 120 · 1. Only the spinor fits.

(4) Complex e8 has two conjugacy classes of involutions, with fixed algebras so(16) (dimension 120) and e7 + sl(2) (dimension 136) (Cartan; Kac's classification of automorphisms of finite order). So there is g in E8 with g theta g^{-1} = sigma; g maps k_theta to so(16)_sigma and p_theta to the 128, and g(so(9)_theta) is a spin(9) in so(16)_sigma, acting on the 16 by the spinor by (3) and on the 128 as 44 + 84 by (2). Let rho_1 and rho_2 be two embeddings of spin(9) in so(16)_sigma through the spinor, both with 128 -> 44 + 84. The spinor is the only 16-dimensional irreducible representation and its invariant symmetric form is unique up to scale, so rho_2 = Ad(A) rho_1, up to an inner automorphism of so(9), for some A in O(16). If det A = -1, then Ad(A) exchanges the two half-spinors, and rho_2 would act on the 128 through the other half-spinor of rho_1, which is the irreducible 128. So A lies

in SO(16), and the image in E8 of either lift of A to Spin(16) conjugates rho_1 to rho_2. Hence the class with 128 -> 44 + 84 is a single class under the subgroup Spin(16)/Z_2 of E8, and g(so(9)_theta) lies in it.

(5) The modules 36, 44 and 84 are tensor representations, on which -1 of Spin(9) acts trivially, so in the class with $248 = 36 + 44 + 84 + 84$ the connected subgroup of E8 is SO(9), of trivial centre. The 128 has half-integer weights, on which -1 acts as -1, so in the other class the subgroup is Spin(9). ■

The statement is the one recorded by Claude Sonnet 5.5 on 4 November 2026 in Reading 3b (c); the proof is by Claude Opus 5.5 as referee (5 November 2026, historical/reviews/2026-11-05_opus_check_05_21_34.md), who also computed, from weight multisets with exact fractions, the restrictions $80 = 36 + 44$ and $\Lambda\hat{}3 = 84$ for the so(9) of sl(9), and $120 = 36 + 84$ with the two half-spinors $44 + 84$ and 128 for the spinor Spin(9) in SO(16); both classes have Dynkin embedding index 2, and the vector class so(9) $\subset$ so(16) through $9 + 7 \cdot 1$ has index 1 and $248 = 36 + 21 \cdot 1 + 7 \cdot 9 + 8 \cdot 16$. The proof needs no claim that the decomposition $36 + 44 + 84 + 84$ by itself is a single conjugacy class.

Readings (proposals)

1. **Three families are the obstruction.** The exchange of lemma 21 is a symmetry of the E8 geometry exactly in the four-family completion, where the colour-family square is 4 x 4 and the top belongs to the family paired with the lepton colour. With three families in single slots the square is 4 x 3 and the exchange is not an isometry (Propositions 5.1 and 5.4, in their scope of one root per state). Lemma 21's $96 = 84 + 12$ in two ways is the three-family shadow of $128 = 116 + 12$ in two ways, exchanged by M.
2. **The two heavy directions of the algebra chain.** Canonical §2.2 and §3 obtain the chain Pati-Salam -> Standard Model -> colour plus electromagnetism from two heavy operators: the Majorana direction, which selects the lepton among the four Pati-Salam colours and ν_R among the SU(2)_R states, and the top Yukawa coupling, which selects the heavy family and the up-type state. M exchanges the lepton colour with family 0 and keeps the up-type direction (F, its finite-geometry counterpart, differs from it on right-handed states): it exchanges the direction selected by the first heavy operator with the direction selected by the second. If the 11 label is the grand algebra M_2(H) + M_4(C) (lemma 28) and the 9 label colour plus electromagnetism, M exchanges the two steps of 11 -> 10 -> 9. This needs a finite geometry in which the generation space is C^4 with an algebra exchangeable with M_4(C), and a Dirac operator compatible with the exchange.
3. **Four families and eleven dimensions.** The 128 of the four families is the chiral spinor of SO(16) $\subset$ E8; under Spin(9) $\subset$ SO(16), embedded through its spinor, it decomposes as $44 + 84$ in the class with 128 -> 44 + 84 (the other chirality of the spinor embedding gives an irreducible 128), the graviton and the three-form of eleven-dimensional supergravity (Pengpan and Ramond, hep-th/9808190; Kostant's cubic Dirac operator on F4/Spin(9); recalled). The two families of 116 are not Spin(9) subrepresentations; whether the exchange M has a Spin(9) meaning is open.

3b. **The arithmetic $44 = 32 + 12$ and the two placements of the top (observation of the user, 4 Nov 2026).** Observation with an open question; nothing beyond (a) to (d) is claimed.

(a) With four families, $128 = 44 + 84$ under the Spin(9) of Reading 3, and $128 - 84 = 44 = 32 + 12$: removing F1 (three families minus the top, 84 states) leaves the fourth family plus the top block, and removing F2 (three families minus the neutrinos) leaves it plus the neutrino block. In this count the top goes with the fourth family into the graviton number 44, and dually the neutrinos do: the four-family completion of the $96 = 84 + 12$ of lemma 21 in two ways. M of Proposition 5.2 exchanges the two blocks of 12 that fill the 44, T(0) and N(1, 2, 3); it sends each family to the states of one colour (family 0 to the leptons), so it does not carry one set of 44 onto the other.

(b) The fit is numerical in the present scope. Proposition 3.8 excludes a colour-equivariant identification of Λ^3 R^9 with either family of 84 for any linear action of colour on the 9. The colour SU(3) lies in no such Spin(9) (Claude Opus 5.5, 4 Nov 2026; checked by Claude Sonnet 5.5): the vector 16 of SO(16) is 3 + 3bar + 10 (1) under colour, while the spinor 16 of Spin(9) restricted to an SU(3) inside Spin(9) is 16 (1), 2 (3 + 3bar) + 4 (1) or 8 + 8, by the three real nine-dimensional representations of Proposition

3.8. The stabilizer in so(16) of the split 44 + 84 is spin(9) itself (the intermediate subalgebras would be submodules of 36 + 84, and so(16) is irreducible on the 128), so the split is not colour-covariant; and no set of 84 roots spans the 84, since a root partition has the rank-8 Cartan subalgebra in its stabilizer. Also, colour has 248 = 8 + 78 (1) + 27 (3 + 3bar), while the 44 of any so(9) of e8 with 248 = 36 + 44 + 84 + 84 contains a 6 or a 27, so colour lies in no such so(9), including that of file 04. Scope: the regular colour SU(3) of the root table, in this class. Escapes: other colour embeddings; identifications that are not equivariant. A strengthening (Sonnet, exact integers, not kept): no isometry of R^8 maps the set (fourth family + top) onto (fourth family + neutrinos), since up to family permutations their degree multisets differ, {10^16, 12^12, 13^16} against {10^24, 13^16, 16^4}.

(c) The record holds two placements of the top, from different subgroups of E8, and they differ: under the Z_3 grading by SU(9) of file 04 (Lemma 4.A, Theorem 4.4) the left-handed top lies in the C_3 sector and ν^c in the gravity sector (t = +1/2, proposed in section 7 there); under the Spin(9) count above the top goes with the fourth family into the 44. The two placements use two frames for conjugate subgroups. The Spin(9) of Reading 3 is conjugate in E8 to the SO(9) of SU(9)/Z_3 (Proposition 5.6), and inside E8 both are isomorphic to SO(9), with Spin(16)/Z_2 in place of SO(16). In the four-family frame the 128 is spanned by the half-integer root vectors, and the count 44 = 32 + 12 is numerical. For the theta-fixed so(9), every root vector e_beta of H_F splits as (e_beta + theta e_beta)/2 + (e_beta - theta e_beta)/2, with the first component in so(16)_theta and the second in the 128 of theta. So there is no contradiction, and no identification between the sectors is claimed. The two groups themselves are related: the Chevalley involution theta restricted to sl(9) is an outer involution with fixed algebra the so(9) of file 04, which is conjugate in E8 to the Spin(9) of this reading, with k = so(9) + 84 and p = 44 + 84. The 44 = p ∩ sl(9) lies inside the gravity sector, and the two 84 are the graphs {x ± theta x : x in Λ^3 C^9}, each meeting the C_3 and C_6 sectors (Proposition 5.6; stated by Claude Sonnet 5.5 on 4 Nov 2026, proved by Claude Opus 5.5 on 5 Nov 2026). How the 96 states of the three-family frame sit against the conjugated Spin(9) has not been worked out.

(d) Recalled analogy: Gell-Mann's colour-flavour SU(8) of N = 8 supergravity put the spin-1/2 fermions in Λ^3(8) = 56 with 8 = 3 + 5 (Gell-Mann, "From renormalizability to calculability?", 1983, and the SU(8) of Cremmer and Julia; recalled without rereading); here it is Λ^3 of 9 = 84 with the C-field, and Proposition 4.3 plays the role of the known failure of that content.

4. **NCG.** In the finite geometry with three generations the exchange is not an automorphism (Proposition 4.2). A Pati-Salam finite geometry with C^4_colour ⊗ C^4_family, where the flip of the two factors exchanges the σ field with the top Yukawa direction, is the candidate in which it is (PLAN, step 5).

What remains open

- A finite spectral triple, or a Hořava-Witten configuration, with four families in which M is a symmetry and the fourth family is decoupled in a controlled way, so that the three-family statement of lemma 21 follows as a limit.
- Whether the fourth family can be made heavy by the same mechanism as the top (lemma 21 counts only one heavy up-type quark).
- Where the 96 states sit against the theta-fixed so(9) (lead from the referee of Proposition 5.6): the component of a root vector e_beta in the 128 of theta is (e_beta + e_{-beta})/2, in the 44 when beta lies in the gravity sector and in the 84 otherwise, so with the counts of lemma 33, Theorem 4.4, the projection of H_F to the 128 of theta has 3 dimensions in the 44 (the nu^c) and 45 in the 84, the top among them, for t = +1/2 (15 and 33 for t = -1/2): for this so(9) the top meets only the three-form 84, which agrees with lemma 33, section 7, and differs from the numerical count of Reading 3b (a). sigma and theta commute, with joint fixed algebra so(8) + so(8) and three further joint eigenspaces of dimension 64, the decomposition 248 = (28, 1) + (1, 28) + (8v, 8v) + (8s, 8s) + (8c, 8c), in which the 96 states of p_sigma split 48 + 48 between k_theta and p_theta; this Spin(8) x Spin(8) frame is where the question can be computed, after transport by an explicit g with g theta g^{-1} = sigma.
- The global form: inside E8 the class with 128 -> 44 + 84 is SO(9) and the class with the irreducible 128 is Spin(9) (Proposition 5.6, item 5), so the graviton and three-form class acts through SO(9) and a gravitino class needs the spin cover; where this lemma writes SO(16) ⊂ E8 the subgroup is

Spin(16)/Z_2, and the Z_2 decides which representations of the 96 states can occur when the two placements of the top are related to brane charges.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent check of Proposition 5.1 (the neighbour table). *From the referee report on the 11D line, section 'Proposition 5.1: aligned three-family root configurations', transferred verbatim in substance on 4 Nov 2026.*

Write a half-integer root as (s | f)/2 with five colour and weak signs s and a family sign pattern f. Distinct family patterns have inner product -1 on the family part and identical ones have 3. For two particle roots in the same slot, inner product +1 means that the five-sign patterns have Hamming distance two. For distinct slots, a particle root is a +1 neighbour exactly when the five-sign patterns are equal, and an antiparticle root is a +1 neighbour exactly when the five-sign patterns have distance four; no antiparticle in the same slot is a +1 neighbour. Let U, D, E, N denote the up-quark, down-quark, charged-lepton and neutrino blocks of one slot, each with its antiparticles. The number of +1 neighbours of one root of the row block among all roots of a column block, for the same slot and for a different slot, is

source	U	D	E	N
U	4 / 2	3 / 3	1 / 1	2 / 0
D	3 / 3	4 / 2	2 / 0	1 / 1
E	3 / 3	6 / 0	0 / 2	1 / 1
N	6 / 0	3 / 3	1 / 1	0 / 2

The entries follow from the root types and colour choices of the Setting and the Hamming criterion; for instance a neutrino has six same-slot up-quark neighbours and none in another slot, and a charged lepton has six same-slot down-quark neighbours and none in another slot. In three aligned slots each row sums to 10 + 2 x 6 = 22, the degree in H_F. Removing all three neutrino blocks gives degree 20 on the 36 up-quark roots and 19 on the 48 down-quark and charged-lepton roots. Removing the top block instead gives degree 22 - 6 = 16 to the neutrinos of its slot, 22 to the other eight neutrino roots, 20 to the 24 remaining up-quark roots and 19 to the 48 others. So F1 has degrees 16 and 22 that F2 lacks; both sets are closed under negation with centroid zero, so an affine Euclidean isometry between them would fix the origin and preserve degrees, and none exists. The degree sums 4(16) + 48(19) + 24(20) + 8(22) = 48(19) + 36(20) = 1632 give 816 unordered pairs at inner product +1 in each set, 816 at -1 by negation, 42 antipodal pairs at -2, and the remaining C(84, 2) - 42 - 2(816) = 1812 pairs at inner product zero. Scope: the indicated half-root embedding, three aligned slots, complete species blocks and Euclidean isometries.

Independent check of Proposition 5.2. *From the section 'Proposition 5.2: four-family colour-family transposition'.*

P exchanges three pairs of colour and family coordinates and the two weak coordinates; negating all eight coordinates is an even sign change, and signed permutations with an even number of sign changes preserve the D_8 roots and lie in W(D_8), a subgroup of W(E8). So M = -P is in W(E8), M^2 = 1, and M preserves the half-integer roots with an even number of minus signs. For a quark colour pattern c and the all-plus family pattern 1,

```
M(c | -+ | 1) = (-1 | -+ | -c) = nu_L(-c),     M(-c | ++ | 1) = (-1 | -- | c) = -nu^c(-c),
```

and the three patterns -c are the other three family patterns; with negatives this is the whole neutrino block, and M^2 = 1 gives the reverse map. The weak root e_4 - e_5 is fixed, so the SU(2)_L subalgebra is preserved; the same calculation exchanges the down quarks of family zero with the charged leptons. For a family-zero lepton, M(nu_L) = nu_L but M(nu^c) = -nu^c, and likewise for the charged lepton: the Dirac species blocks of family zero are invariant setwise, and their roots are fixed only up to sign.

Counterexample (referee) to Proposition 5.3 without unitarity. *From the section 'Proposition 5.3: finite-geometry flip'.*

The factor identification iota must be unitary, or unitary up to one nonzero scalar, for the flip F to be unitary. With orthonormal bases, iota(l) = 2 e_0 and the three colour basis vectors sent isometrically to e_1, e_2, e_3 give F(l (x) e_1) = 2 e_1 (x) e_0, which changes the norm while F stays an involution. With unitary iota, F is unitary and involutive, F(P_q (x) P_0)F^{-1} = P_l (x) (1 - P_0), F commutes with gamma (which acts on the chirality factor) and with J by its conjugate extension to the second summand; the two Dirac blocks have rank 12 (two chiralities, three colour or family states, two conjugate summands) and kernels of dimension 128 - 12 = 116. If a fixed self-adjoint D commutes with F, the families D + t T_top and D + t T_nu are conjugate for every t, with a bounded spectral sector of dimension 116 in the heavy limit; the commutant algebras are conjugate when the represented algebra is itself flip-invariant, which the displayed Standard Model and Pati-Salam colour algebras acting on the colour factor alone are not.

Independent proof of Proposition 5.4 for all 192 assignments, without enumeration. *From the section 'Proposition 5.4: misaligned three-slot configurations'.*

Let A, B, C, D be the three-slot sets of the up quarks, down quarks, charged leptons and neutrinos, with 0 in A the slot of the top. The neighbour table above applies to every slot assignment. For a root in slot f, after removing all neutrinos,

source	d_{S - N}
U, f in A	20
D, f in B	17 + 2 1_C(f)
E, f in C	13 + 6 1_B(f)

(for a down quark, 3\|A\| + [4 + 2(\|B\| - 1)] + 2 1_C(f) = 17 + 2 1_C(f)). So S - N has degrees in {13, 17, 19, 20}, and 16 and 22 are absent. After removing the top, every neutrino root in slot f in D has

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d_{S - T} = 16 + 6 (1_A(f) - 1_{{0}}(f))  in {16, 22}:
```

the three down-quark slots contribute 9, the three charged-lepton slots 3, the two other neutrino slots 4, and the up quarks six exactly when f is in A, less six for the removed top when f = 0. Both sets are closed under negation, so their centroids are zero, and an isometry between them would preserve these degrees, which is impossible. This covers all 3 x 4^3 = 192 choices. The universal counts stated in the first version of the proposition are false when slots are misaligned: for A = C = {0, 1, 2} and B = D = {0, 1, 3}, the down-quark roots in slot 3 have degree 17 and the charged-lepton roots in slot 2 have degree 13 in S - N, and the neutrino degrees in slots 0, 1, 3 of S - T are 16, 22, 16; so S - N has degrees other than 19 and 20, and more than one neutrino slot can have degree 16. Scope: the specific half-root realization, whole colour multiplets, one root slot per species and Euclidean isometries; general zero modes of a nonsplit SU(4) bundle and maps without an isometric root action are outside it (Proposal 5.5).

Independent proof of the index statement in Proposal 5.5, with the net-chirality caution. *From the referee report on the 11D line, section 'Proposal 5.5: four-slot bundles and three families', transferred verbatim in substance on 4 Nov 2026.*

Verdict of the referee: correct with amendment as a possible escape. General bundle zero modes are sections, and the isometry hypotheses of Propositions 5.1 and 5.4, which concern individual fibre roots, do not automatically apply to them. The index counts net chirality, and the proposed Weyl map on zero modes remains to be constructed.

Index. On a Calabi-Yau threefold X, for a rank-four SU(4) bundle V, c_1(V) = 0 gives ch(V) = 4 - c_2(V) + c_3(V)/2, and c_1(X) = 0 gives Td(X) = 1 + c_2(X)/12. Riemann-Roch gives

```
chi(V) = int_X ch(V) Td(X) = (1/2) int_X c_3(V).
```

When H^0(V) = H^3(V) = 0, this is -h^1(V) + h^2(V), with h^2(V) = h^1(V^*) by Serre duality, since K_X is trivial. Thus index three gives three net chiral families. Exactly three 16 zero modes additionally requires the opposite-chirality cohomology H^1(V^*) to vanish.

Split bundles. For a split bundle V = L_0 + L_1 + L_2 + L_3, the integers n_i = -chi(L_i) are signed indices; their sum being three does not force one zero mode in each of three distinct slots. Even in that special distribution, applying Proposition 5.1 to normalized global sections requires an identification of the physical zero-mode metric with the root configuration.

The Weyl map. The fibre Weyl transposition M can exchange the abstract subgroups, but a symmetry of a particular compactification must also act compatibly on the bundle, the connection, the zero-mode spaces, their inner products and any Wilson-line data. Establishing such an action is an open construction; the escape from the root no-go alone does not establish it.

Part 35

The sign of lemma 06 as an orientifold type, and E8 walls from the configuration 15 + 1

Lemma 35. Written as file 08 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, task C, next step of 07). Attempted by Claude (Anthropic), 4 Nov 2026. Proposition 8.1 is proved (it adds a commutant statement to lemma 06); Proposition 8.2 is proved from Proposition 7.1; Remark 8.3 rests on recalled string-theory rules and on the dictionary of 06; readings are proposals. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

1. Setting

In type I' the two ends of the interval are orientifold planes. A stack of n Chan-Paton labels (k D8-branes with their images, n = 2k) on an orientifold plane of type O^- carries SO(n), and on one of type O^+ carries USp(n) (recalled; the physical O8 planes of the SO(32) web are O8^-). Lemma 06 factors a KO-6 finite triple into an odd one through a grading-reversing unitary involution U, commuting with the algebra and with D, with U J = s J U; s = −1 gives KO-5 (label 9) and s = +1 gives KO-7 (label 11), and the antiunitary j induced on H_+ satisfies j^2 = s and commutes with d^2 = D^2|{H+} (lemma 06, proof of Result 2 and the check of the electrodynamics data). STRATEGY T2c: with a realistic seesaw sector only s = +1 occurs.

2. Proposition 8.1 (the factorization sign fixes orthogonal or symplectic symmetry)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The antiunitary commutants are independently checked. They need not preserve d or the remaining data, and the Chan-Paton identification is a proposal. See the referee report.

Statement. Let E be an eigenspace of d^2 in H_+, of dimension n. The unitaries of E that commute with j form O(n) if s = +1, and USp(n) if s = −1, in which case n is even.

Proof. j preserves E because it commutes with d^2. An antiunitary with j^2 = +1 is a real structure: E = E_R ⊗ C with E_R its fixed points, and a unitary commutes with j exactly when it preserves E_R, which gives O(n). An antiunitary with j^2 = −1 is a quaternionic structure: E is a right H-module of quaternionic dimension n/2, and the unitaries commuting with j form USp(n). ■

The even dimension for s = −1 is lemma 06's own statement (Kramers degeneracy of the singular values). The proposition adds the symmetry group.

Reading (proposal). The groups are those of Chan-Paton labels on an orientifold plane: s = +1 with O^- (SO), s = −1 with O^+ (USp). So the two odd factorizations of the modern triple, which carry the labels 11 and 9, have the form of the two kinds of orientifold planes. The seesaw sector selects s = +1 (T2c), which is the O^- type of the SO(32) web used in 06 and 07: the physical neutrino sector and the type I' walls of the dictionary agree in type. Two refinements from the second check: the symmetries of the pair (j, d) are smaller, O(n_+) x O(n_-) for s = +1 and U(n/2) for s = -1, since for s = -1 the antiunitary j swaps the +-mu eigenspaces of d; and there is a tension of labels, since here label 9 goes with O^+ while section 6b of 07 puts label 9 on a far O8^- wall (the two are the KO sign and the position of the top's brane, see 10).

3. Proposition 8.2 (wall channels come with doubled multiplicity)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The stated count is for one signed eigenvalue. The lowest $|D|$ multiplicity is $2k$ in the interior and at zero, and $4k$ at $\pi/2$; interpreting the signed count as Chan-Paton labels requires further data. See the referee report.

Statement. In the parent of lemma 22, k channels at a common angle θ contribute, at the eigenvalue θ/l of the window, an eigenspace of dimension k if 0 < θ < pi/2 and of dimension 2k at the ends θ = 0 and θ = pi/2.

Proof. By Proposition 7.1 each channel has the spectrum {(n pi ± θ)/l}; the value θ/l occurs once (n = 0, sign +) for 0 < θ < pi/2, and twice at the ends: at θ = 0 as the zero mode of multiplicity two, at θ = pi/2 as (0 + pi/2)/l = (pi − pi/2)/l. ■

Reading (proposal). k D8-branes in the bulk carry k Chan-Paton labels and k D8-branes on an orientifold plane carry 2k, with their images; the parent reproduces this count, with the ends of the angle interval as the walls. With Proposition 8.1, a stack of channels at a wall has a symmetry of type SO(2k) or USp(2k), as a stack of D8-branes on an O8^- or O8^+ plane. The rank-reduced nine-dimensional theory with one O8^- and one O8^+ (duality 3 of file 07) would then be a parent whose two walls carry opposite signs s: one sheet of type 11 and the other of type 9.

4. Remark 8.3 (E8 walls from 15 + 1; conditional)

Correction (4 Nov 2026). The far-wall group is reached at the string-scale radius R = R_*, with the tower fixed, and only its charged states are D0-branes. See 10.

Referee note (GPT-6.1 Terra, 4 Nov 2026). The brane and slope arithmetic is independently checked conditional on the recalled rule. Wall enhancement requires tuning $z = 0$; for $7 + 7$ the two bulk positions must sum to L. The E_2 identification with the physical electroweak sector needs further data. The cited primary sources were unavailable to the referee; the slope rule and the E_{n+1} table were confirmed against Polchinski-Witten and Bergman-Gaberdiel-Lifschytz in the second check. See the referee report.

Conditions: the dictionary of 06 (fifteen D8-branes for the five light quarks in three colours, the sixteenth for the top) and the recalled rule that an O8^- plane with n <= 7 D8-branes on it, at infinite coupling, carries E_{n+1} (E_4 = SU(5), E_5 = SO(10), E_6, E_7, E_8), while n >= 8 admits no such point (Seiberg, hep-th/9608111; Bergman, Gaberdiel and Lifschytz, hep-th/9711098; recalled).

1. Dilaton bookkeeping: the dilaton is e^{phi} = z^{−5/6} with z piecewise linear along the interval (conformal coordinate; Polchinski and Witten, hep-th/9510169) and slope proportional to 8 − n_left, where n_left counts the D8-branes to the left (O8^- carrying −8 units), so the coupling grows toward the wall with fewer branes; this reproduces the constant dilaton of 8 + 8, the SU(2) at the empty wall of the SO(32) point (the self-dual radius of the heterotic string) and 7 + 7 for E8 x E8 (extrapolated from the single-wall rule; neither Polchinski-Witten nor Bergman-Gaberdiel-Lifschytz states it). In the configuration 15 + 1 the wall with fifteen branes has no E point (15 > 7), and the coupling can diverge only at the far wall. With the top's brane on the far wall (the label-9 configuration of Proposition 7.2,

n = 1) the far wall enhances to E_2 = SU(2) x U(1), the electroweak group; with the top's brane in the bulk it is empty and enhances to E_1 = SU(2). The light-quark wall keeps SO(30).
2. If branes move in colour triples (a light flavour is three D8-branes), a wall holds 3j branes, or 3j + 1 with the top's brane. The reachable wall groups are E_1, E_2 (the top's brane alone), E_4 = SU(5) (one flavour), E_5 = SO(10) (one flavour and the top's brane), E_7 (two flavours) and E_8 (two flavours and the top's brane); E_6 (five branes) is not reachable.
3. Hence an E8 wall needs the top's brane together with two light flavours, and two E8 walls are impossible with one top brane: E8 x E8 requires breaking a colour triple.

In this reading the top is what completes E8 at a wall (7 = 2 x 3 + 1), and with colour triples the proposed E8 x E7 arrangement has seven branes at one wall, six at the other and a bulk triple at x_b, while two E8 walls require relaxing the colour-triple rule (referee replacement): with that arrangement, the slopes are proportional to +1 on (0, x_b) and −2 on (x_b, L), so z vanishes at both walls when x_b = 2L/3. The E_n series also contains E_3 = SU(3) x SU(2) (two branes at a wall), which colour triples do not reach.

Reading (proposal). At strong coupling the wall that holds the top's brane in the label-9 configuration carries SU(2) x U(1), and the wall of the five light quarks in three colours carries SO(30). The electroweak group then appears at the top's wall, where the M-theory circle opens locally, and the quarks' flavour group at the other wall.

5. What remains open

- The bulk symmetry: k D8-branes in the bulk carry U(k), while Proposition 8.1 gives O or USp on every eigenspace of d^2; the relation between the window's degenerate levels in the bulk and U(k) needs the parent's own structures in addition to j.
- The dilaton: a spectral-triple counterpart of the type I' dilaton profile (a coupling varying along the interval, with slope set by the brane charge to one side) and the infinite-coupling points of Remark 8.3. Proposition 9.1 of 09 writes the profile as a trace of a function of |D_F|, only implicitly after 10, Proposition 10.1, and the gradient is not the second scale in the scope of Propositions 10.5 and 10.6 there.
- The physical status of E8 x E7 with colour triples, and whether the top's brane at an E8 wall has a counterpart in the finite geometry.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 8.1, with its caution. *From the referee report on the 11D line, section 'Proposition 8.1: antiunitary commutants', transferred verbatim in substance on 4 Nov 2026.*

On each finite-dimensional eigenspace E of d^2, the commutant of j in the unitary group of E is O(n) for j^2 = 1, and USp(n) for j^2 = -1, with even complex dimension n in the latter case.

For j^2 = 1, the fixed points of j form a real subspace E_R with E = E_R ⊕ i E_R, and the inner product is real on E_R, since <jv, jw> = conj<v, w>. A real orthonormal basis of E_R is an orthonormal basis of E and identifies j with complex conjugation; the unitaries commuting with complex conjugation are the real orthogonal matrices, O(n). For j^2 = -1, a unit vector v and jv are orthogonal: antiunitarity gives <jv, j^2 v> = conj<v, jv> = <jv, v>, and j^2 v = -v gives <jv, j^2 v> = -<jv, v>, so <jv, v> = 0. The span of v and jv is j-invariant, and so is its orthogonal complement; induction gives a quaternionic orthonormal basis v_1, jv_1, ..., v_m, jv_m, with n = 2m. The unitaries commuting with j are then the compact symplectic group USp(n).

Caution. This commutant is defined using j and the d^2-eigenspace alone. It need not commute with d, with the represented algebra, or with other physical structures. For example, on C^2 let j be complex conjugation and d = diag(1, -1): the commutant on the d^2 = 1 space is O(2), and a generic rotation does not commute with d. Moreover O(n) contains disconnected transformations, while SO(n) is its identity component for n

>= 2. The orientifold comparison therefore requires an explicit Chan-Paton representation and compatibility with the rest of the data.

Independent proof of Proposition 8.2, with the multiplicities of |D|. *From the referee report on the 11D line, section 'Proposition 8.2: endpoint channel multiplicities', transferred verbatim in substance on 4 Nov 2026.*

For k independent channels at the same angle, the signed eigenvalue theta/l has multiplicity k in the interior and 2k at either endpoint. For 0 < theta < pi/2, only n = 0 in the positive progression gives +theta/l: (n pi + theta)/l = theta/l forces n = 0, and (n pi - theta)/l = theta/l forces theta = n pi/2, outside the interior. At zero both progressions have n = 0. At pi/2 the positive n = 0 and negative n = 1 branches coincide. This proves the claimed signed count.

The multiplicity of the corresponding lowest eigenvalue of |D| is 2k in the interior (the values ±theta/l, k each), 2k at zero, and 4k at pi/2 (the values ±pi/(2l), 2k each). At zero the two interior signed eigenvalues simply coalesce; at pi/2 the next level also meets the window. Thus the signed count alone does not establish a doubling of independent physical Chan-Paton labels or an orientifold projection.

Independent proof of the arithmetic of Remark 8.3. *From the referee report on the 11D line, section 'Remark 8.3: wall bookkeeping and enhancements', transferred verbatim in substance on 4 Nov 2026.*

The slope and brane-count arithmetic below is conditional on the recalled type I' rules; the referee could not access the primary sources, and the second check later confirmed the slope rule and the E_{n+1} table against Polchinski-Witten and Bergman-Gaberdiel-Lifschytz (note at the remark). An enhancement requires tuning the coupling to a zero of z, and identifying the resulting E_2 factor with the physical electroweak sector requires additional data.

Assume z' = c (8 - N), with c > 0, N(x) the number of D8-branes to the left of x, and z > 0 in the open interval.

- *8 + 8.* N = 8 in the interior, so the interior slope is zero.
- *16 + 0.* z' = -8c, and the empty far wall reaches z = 0 only when z(0) = 8cL.
- *7 + 7.* With the other two branes at x_1 < x_2 in (0, L), the slopes are c, 0 and -c on (0, x_1), (x_1, x_2) and (x_2, L). Both walls have z = 0 precisely when

  ```
  z(0) = 0,    x_1 + x_2 = L,
  ```

  since z(L) = z(0) + c x_1 - c (L - x_2). Then z is positive in the interior. Thus "7 + 7 for E8 x E8" includes a coupling and a position condition.
- *Isolated zeros.* With n branes at a wall where z = 0, the slope next to the wall is c (8 - n) at x = 0 and c (n - 8) at x = L, so z > 0 inside requires n <= 7: an isolated zero with a positive interior requires at most seven branes at that wall under this slope rule.
- *15 + 1.* With the single brane at L, the slope is c (8 - 15) = -7c, so the far-wall zero is at z(0) = 7cL. Assuming the recalled E_{n+1} rule gives E_2, with Lie algebra su(2) ⊕ u(1), at that tuned wall. The rule alone does not determine the Standard Model charges, couplings, global group or embedding.

Colour triples. If every light flavour moves as a triple and there is a single top brane, a wall holds 3j or 3j + 1 branes, so the allowed wall counts up to seven are 0, 1, 3, 4, 6, 7. Only a wall containing the top can have seven (7 = 2 x 3 + 1); two such walls require relaxing the triple constraint or the single-top assumption. For seven branes at 0, six at L, and the remaining triple at x_b, N = 7 on (0, x_b) and N = 10 on (x_b, L), so the slopes are +c and -2c. With z(0) = z(L) = 0, c x_b - 2c (L - x_b) = 0 gives x_b = 2L/3, and

```
z(x) = c x            for 0 <= x <= 2L/3,
z(x) = 2c (L - x)     for 2L/3 <= x <= L.
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The resulting E8 x E7 interpretation is conditional on the recalled enhancement rule and the proposed brane dictionary.

Part 36

The type I' dilaton as a spectral function, the SO(32) of the EPJC paper, and two more dualities

Lemma 36. Written as file 09 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, task C). Attempted by Claude (Anthropic), 4 Nov 2026, after the steer of the same day (the coincident-brane route is overused; the type I' dilaton and the colour-flavour idea are worth pursuing; compare with the SO(32) of the EPJC paper). Proposition 9.1 is proved from the recalled type I' slope rule; Proposition 9.2 is proved from representation theory and the recalled orientifold projections; Proposition 9.3 is proved (linear algebra and Connes's sign table); the comparison of section 1 is read from the arXiv text of the paper through a summarizing fetch; the rest are proposals. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

Correction (same day): the first version stated the slope rule for e^{−phi}. The piecewise-linear function is z, with e^{phi} = z^{−5/6}; Proposition 9.1 holds as stated with z in place of e^{−phi}, and infinite coupling is z = 0.

1. The SO(32) of A. Rivero, Eur. Phys. J. C 84 (2024) 1058 (arXiv:2407.05397)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The branching algebra is independently checked. The archived original abstract also states an additional chiral charge-+4/3 quark per generation. The full paper was unavailable for direct comparison, so its detailed attribution remains pending. See the referee report.

That paper uses the adjoint of SO(32) to classify the scalars of a supersymmetric Standard Model with three generations, as composites: pairs of quarks as colour triplets and quark-antiquark pairs as singlets and octets, with three extra scalars of charge +4/3. Its pattern is SO(2n) ⊃ SO(n_1) x SU(n_2) x SU(3) x U(1) with 2n = n_1 + 6 n_2, which for SO(32) and n_2 = 5 gives SO(2) x SU(5) x SU(3) x U(1): the 30 = 6 n_2 are the five light quarks u, c, d, s, b ("turtles") in three colours with conjugates, and the top is the non-participating "elephant". Its explicit route also passes through SO(32) ⊃ SU(16) x U(1), with 496 = 1 + 120 + 120bar + 255 (quarks from the 120s, leptons from the 255), and it mentions open strings, orientifolds and an SU(9) for M2 and M5.

Reading of the LaTeX source (Claude Sonnet 5.5, 4 Nov 2026; the earlier comparison used a summarizing fetch). The paper uses colour singlets for mesons and colour antitriplets for diquarks, and keeps only (1, 24, 1), (1, 15, 3bar) and (1, 15bar, 3) of its decomposition; colour octets and sextets lie outside it. The top is the favoured non-binding quark ("we have not a concrete argument"), and the postulates force the non-binding quark to be up-type. The chain is SU(16) x U(1) to SU(15) to SU(5) x SU(3), with SO(30) and SO(32) reached by "adding singlets". The identification of the top with the SO(2) factor, the O8^- plane and the Chan-Paton reading belong to these files, and "orientifold" in the paper means planar orientifold QCD. The footnote on the 84 + 12 split and SU(9) for M2 and M5 anticipates Proposition 21.3 and 04. The abstract specifies a chiral +4/3 quark per generation; the body has three scalars of charge +4/3 and leaves their fermion partner open. The paper gives (15, 3bar) as the antisymmetric square of one fundamental without stating chirality, so the question of Proposition 9.3 stays open; its remark on odd-parity diquarks is consistent with that proposition. The paper's selection of an up-type quark by its postulates and lemma 21's by the spectral-action values are independent arguments with the same outcome.

Coincidence. The chain SO(32) ⊃ SO(2) x SO(30) ⊃ SO(2) x SU(15) x U(1) ⊃ SO(2) x SU(5) x SU(3) x U(1), the 30 as five light quarks in three colours, and the top as the factor SO(2), are the same as lemma 21 and the dictionary of 06. Lemma 21 records the branching 496 -> (435, 1) + (1, 1) + (30, 2) and leaves SO(32) unexplored; the EPJC paper explores it through composites, and the files 06 to 09 through type I'

branes. What the 11D files add: the type I' realization of the SO(2) as the top's D8-brane and of the 30 as the Chan-Paton labels of fifteen D8-branes on an O8^- wall (06); the top angle as the brane's position and the one-loop vacuum on the far wall (Propositions 7.1, 7.2); the orientifold type O^- matched by the seesaw sign (Proposition 8.1); the far wall's enhancement to SU(2) x U(1) at infinite coupling (Remark 8.3); the dilaton (Proposition 9.1); the diquark counts as strings between the stack and its image, which need the O^- type (Proposition 9.2); and the real-structure condition under which these counts arise, as connection components along the interval, outside the finite inner fluctuations (Proposition 9.3). In type I' language the paper's other chain, SU(16) x U(1), is the configuration with all sixteen D8-branes coincident in the bulk, and the chain with SO(2) x SO(30) is the configuration with fifteen on a wall and one apart; the second is the one the corpus selects. The SU(9) of M2 and M5 in the paper is the grading of e8 used in 04.

2. Proposition 9.1 (the type I' coupling function is a trace of a function of the finite Dirac operator)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The integration is independently checked conditional on the recalled slope rule. The trace is over sixteen selected channels, and the full physical D_F Hilbert space enters only through a specified projection onto them; direct comparison with the cited source was unavailable to the referee. (Second check, against Polchinski-Witten.) The formula z(L) = z(0) - c (8L - sum x_k) is exact in the positions; the dictionary x_k = (2L/pi) l m_k holds only for constant z, and in general theta_k = (pi/2) W(x_k)/W(L) with W = int z^{-1/3} (10, Proposition 10.1). In the profile 15 + 1 the endpoint values 7cL and 8cL stand, and theta_t = pi/4 gives 7.358 cL in place of 7.5 cL. See the referee report.

Setting. Type I' on an interval [0, L], in the conformal coordinate x, with sixteen D8-branes at x_1, ..., x_16 and O8^- planes at the ends. Recalled rule (Polchinski and Witten, hep-th/9510169): the dilaton is e^{phi} = z^{−5/6} for a function z that is continuous and piecewise linear in x, with slope c (8 − N(x)), where N(x) is the number of D8-branes to the left of x and c > 0; the coupling diverges where z vanishes. Dictionary (06, 07): x_k = (2L/pi) θ_k with θ_k = l m_k the angles of sixteen channels of the finite Dirac operator (fifteen light-quark channels with colour, and the top).

Statement.

1. z(x) = z(0) + c Σ_k min(x/2, x_k − x/2) for x in [0, L], so z is z(0) plus c Tr g_x(l |D_F|) over the sixteen channels, with g_x(θ) = min(x/2, (2L/pi) θ − x/2): a trace of a function of the finite Dirac operator, as the spectral action is.
2. At the far wall, z(L) = z(0) − c (8L − Σ_k x_k).
3. Infinite coupling at the far wall (z(L) = 0) holds exactly when z(0) = (2cL/pi)(4 pi − l Σ_k m_k): the coupling at the light-quark wall is fixed by the sum of the sixteen channel masses, the trace of |D_F| on these channels.

Proof. *∫0^x N(y) dy = Σ_k (x − x_k)+*, so z(x) = z(0) + c (8x − Σ_k (x − x_k)+) = *z(0) + c Σ_k (x/2 − (x − x_k)+)*, and x/2 − (x − x_k)_+ = min(x/2, x_k − x/2). Item 2 is x = L with Σ_k (L − x_k) = 16L − Σ x_k. Item 3 sets z(L) = 0 and substitutes x_k = (2L/pi) l m_k. ■

Correction (4 Nov 2026). The dictionary x_k = (2L/pi) theta_k is exact only for constant z; the map from angles to positions is nonlinear. See 10, Proposition 10.1.

Reading (proposal). With the light quarks near θ = 0, Σ_k m_k ≈ m_t and the condition is z(0) ≈ (2cL/pi)(4 pi − θ_t): 7cL at label 9 (θ_t = pi/2) and 8cL at label 11 (θ_t -> 0). The coupling therefore varies along the interval on a scale set by the spectrum of D_F, which is the kind of second scale that Proposition 1.5 requires to select a joint family; whether the type I' coupling can play that role in the corpus's flat family is the next question. Since the slope decreases with x, z is concave, so z > 0 on [0, L) once z(0) > 0 and z(L) = 0; positivity of z(0) gives l Σ_k m_k < 4 pi, a bound on the trace of |D_F| in units of the tower scale, met when only the top is heavy (θ_t <= pi/2).

3. Proposition 9.2 (diquarks as strings between the stack and its image)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The representation and charge counts are independently checked, conditional on the recalled projections. The shifts describe the specified gauge-field channels; Λ^2 of

a line excludes that component only, and leaves the other open-string states and the strings to other branes. See the referee report.

Setting. V = C^5 ⊗ C^3, five light flavours in three colours, with Q = 2/3 on u, c and Q = −1/3 on d, s, b; the 30 of lemma 21 is V ⊕ Vbar. Fifteen D8-branes on an O8 plane carry the Chan-Paton space 30 (branes and images), with group SO(30) on an O8^- and USp(30) on an O8^+, and U(15) when the stack is moved off the plane (recalled).

Statement.

1. Under U(15), the adjoint of SO(30) is Λ²V ⊕ Λ²Vbar ⊕ (V ⊗ Vbar) and the adjoint of USp(30) is S²V ⊕ S²Vbar ⊕ (V ⊗ Vbar). The summands Λ²V or S²V and their conjugates are the strings between the stack and its image; V ⊗ Vbar are the strings with both ends on the stack.
2. Λ²V = (S²5, 3bar) ⊕ (Λ²5, 6) and S²V = (S²5, 6) ⊕ (Λ²5, 3bar).
3. In the colour antitriplet, Λ²V has the charge counts (−2/3: 6, +1/3: 6, +4/3: 3) of lemma 21, and S²V has (−2/3: 3, +1/3: 6, +4/3: 1). So the antisquark counts of three generations recorded in lemma 21 (6 of charge −2/3 and 6 of charge +1/3 in the 3bar) occur among the stack-image strings of an O8^- plane, while those of an O8^+ plane have the counts (3, 6, 1).
4. With the angles θ_k = l m_k of 06, a string from channel i to the image of channel j has the spectrum (n pi ± (θ_i + θ_j))/l, and a string from channel i to channel j has (n pi ± (θ_i − θ_j))/l, n in Z.
5. A single brane has no string to its own image on an O8^- plane (Λ² of a line vanishes) and one on an O8^+ plane.

Proof. 1: 30 = V ⊕ Vbar as a U(15) module, Λ²(V ⊕ Vbar) = Λ²V ⊕ Λ²Vbar ⊕ (V ⊗ Vbar) and likewise for S²; the orientifold projection keeps the antisymmetric part for SO and the symmetric part for USp, and the mixed summand V ⊗ Vbar is the U(15) adjoint. 2: Λ²(A ⊗ B) = (S²A ⊗ Λ²B) ⊕ (Λ²A ⊗ S²B) and S²(A ⊗ B) = (S²A ⊗ S²B) ⊕ (Λ²A ⊗ Λ²B), with Λ²3 = 3bar. 3: split 5 = 2 ⊕ 3 by charge; S²5 = S²2 ⊕ (2 ⊗ 3) ⊕ S²3 has dimensions 3, 6, 6 with charges 4/3, 1/3, −2/3, and Λ²5 = Λ²2 ⊕ (2 ⊗ 3) ⊕ Λ²3 has dimensions 1, 6, 3 with the same charges. 4: Proposition 6.1 with charges (+1, +1) and (+1, −1) under U(1)_i x U(1)_j, whose holonomy is e^{2 pi i (a_i ± a_j)}, and θ = pi a. 5: dimensions. ■

Reading (proposal). (11, Theorem 11.3 and Proposition 11.4, qualify the selection below: the window sees the counts (3, 6, 1) also at s = +1, through the commutator with the masses, and parent scalars with the counts (6, 6, 3) need M_15(C) at both walls.) The diquark counts of lemma 21 and of the EPJC paper are the stack-image strings of an O8^- plane, the type that Proposition 8.1 associates with the factorization sign s = +1 selected by the seesaw sector (T2c); the O8^+ alternative gives the wrong number of up-type antisquarks. So the seesaw sign and the scalar counts select the same orientifold type. Two features follow from the projection. Diquark masses go with the sum θ_i + θ_j and meson masses with the difference θ_i − θ_j, so when the stack leaves the wall the diquarks become massive and the mesons stay massless: the diquarks are tied to the orientifold. And the top's single brane has no string to its own image, while its strings to the stack, the (30, 2), have masses near m_t (Proposition 6.1); in the EPJC paper the top's exclusion is an input, and here the O8^- projection with one brane for the top removes the top-top channel only (item 5); the top-to-stack strings (30, 2) remain, with masses near m_t, so the rest of the exclusion is a mass statement, as in the paper.

4. Proposition 9.3 (the real structure decides between the two diquark counts)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The stated sign constraint excludes a nonzero antisymmetric block in the three KO labels. The signs permit all of S^2 V (second check: B ranges over exactly S^2 V for eps eps' = +1, with the diagonal block free); the algebra, grading and order conditions of a triple then select a subspace. Distinct-multiplet blocks remain related to their transposes. See the referee report.

Setting. E = V ⊕ Vbar with V as in Proposition 9.2, an antiunitary J on E that maps V onto Vbar and Vbar onto V (as particles and antiparticles are exchanged), J² = ε, and a Hermitian X on E with JX = ε' XJ, the condition that inner fluctuations obey in a KO-dimension with signs (ε, ε'). Let B be the block of X from Vbar to V, an element of V ⊗ V.

Statement.

1. B is symmetric (B in S^2V) if εε' = +1 and antisymmetric (B in Λ^2V) if εε' = −1.
2. In the KO-dimensions 5, 6, 7 of the finite part for the labels 9, 10, 11, (ε, ε') = (−1, −1), (+1, +1), (+1, +1), so εε' = +1: the diquark block of every J-compatible fluctuation lies in S^2V, whose colour antitriplet ($\Lambda^2$5, 3bar) has the counts (−2/3: 3, +1/3: 6, +4/3: 1).
3. For a connection on a circle whose real structure is complex conjugation (KO-dimension 1, (ε, ε') = (+1, −1)), the block lies in Λ^2V, whose colour antitriplet ($S^2$5, 3bar) has the counts (−2/3: 6, +1/3: 6, +4/3: 3) of lemma 21. The SO(32) Wilson line of Proposition 6.1 is of this type.

Proof. Choose a basis of V and let K: V -> Vbar be coordinatewise conjugation; J(v, w) = (ε K^{-1} w, K v) has $J^2 = \varepsilon$. Write X = [[A, B], [B*, D]]. The component of JX = ε' XJ that maps V to V gives ε K^{-1} B* = ε' B K, so B = εε' K^{-1} B* K^{-1} = εε' B^T in coordinates. Under the U(15)-equivariant identification of Hom(Vbar, V) with V ⊗ V, symmetric matrices form S^2V and antisymmetric ones Λ^2V. Items 2 and 3 use Connes's sign table, ε = (+, +, −, −, −, −, +, +) and ε' = (+, −, +, +, +, −, +, +) for KO-dimensions 0 to 7, and the counts of Proposition 9.2. On the circle, D = −i d/dθ and J = conjugation give JD = −DJ, and a Hermitian connection A with JA = −AJ is i times a real antisymmetric matrix, an element of so(n), where the generator of R(2 pi a) lies. ■

Item 2 is a negative statement with the following scope: the blocks of J-compatible Hermitian operators between particles and antiparticles, for one multiplet V, in the three labels. It is the finite-geometry form of the symmetry of Majorana mass matrices, and of the Fermi-statistics constraint on a Lorentz-scalar bilinear ψ_i ψ_j of one Weyl multiplet without derivatives (recalled), under which the (15, 3bar) vanishes. Known escapes: item 3 (connection components along a circle or an interval), blocks between two distinct multiplets, which J does not constrain, and derivative couplings.

Reading (proposal). The counts of lemma 21 and of the EPJC paper, (6, 6, 3), belong to the antisymmetric block. In the corpus's setting this block arises as the component of an SO(30) connection along the circle or interval: the Wilson-line moduli of the type I' stack, whose stack-image part is Proposition 9.2. The finite triple alone, in any of the three labels, gives the symmetric block and the counts (3, 6, 1). So the composite scalars with the counts of lemma 21 point to the interval parent of lemma 22 and its gauge field, and the selection of O8^- in Proposition 9.2 concerns these connection components. This narrows the spin gap of Propositions 6.1 and 9.2: the Λ^2V states are scalars as circle components of a nine-dimensional gauge field, and they come with that field's vector and gaugino partners.

5. Two more dualities between 9 and 11 (recalled; proposals)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The E_1 alternatives need correction: the canonical-orthogonal root lattices distinguish $\mathbb{P}^1 \times \mathbb{P}^1$ from $dP_1 = \mathbb{F}_1$, and the usual local engineering assigns the latter to $\widetilde{E}_1$. The dP_2 and brane duality claims remain conditional on the unavailable primary sources. The Weyl action is a gauge relabelling of a holonomy, subject to the full background data. See the report on the del Pezzo comparison and Weyl action.

1. **E-type walls and del Pezzo surfaces.** The E_{n+1} symmetry of an O8^- plane with n D8-branes at infinite coupling is the symmetry of the five-dimensional theory on a D4-brane probe, which M-theory realizes on a Calabi-Yau threefold with a shrinking del Pezzo surface dP_{n+1} (Seiberg, hep-th/9608111; Morrison and Seiberg, hep-th/9609070; Intriligator, Morrison and Seiberg, hep-th/9702198). So the far wall with the top's brane (n = 1, E_2 = SU(2) x U(1)) has an eleven-dimensional description by a local dP_2, and the empty far wall (E_1 = SU(2)) by dP_1 or P^1 x P^1; the light-quark wall (n = 15) has none. The Witten base CP^2 x CP^1 of 02 is built from the same pieces (CP^2 = dP_0, and P^1 factors).
2. **Colour-flavour exchange as a T-duality.** In the heterotic string on S^1, the Weyl group of the gauge group acts on the Wilson lines as part of the T-duality group O(17, 1; Z). The colour-family transposition M of 05 is an element of the Weyl group of E8, so it is a T-duality of the nine-dimensional E8 x E8 string, and through Hořava-Witten a symmetry of the E8 holonomy on a wall in eleven dimensions. As a Weyl element it acts as a global gauge transformation, so by itself it relabels gauge-equivalent vacua; it exchanges colour and family physically only relative to a structure that tells them apart, such as the placement of H_F in 05. The four-family requirement of Propositions 5.1 and 5.4

remains.

6. Open

Referee note (GPT-6.1 Terra, 4 Nov 2026). The finite KO signs restrict the symmetric block without producing the full representation. Light interval scalar modes, their boundary conditions and the physical string dictionary remain to be constructed; the detailed EPJC comparison is pending.

- Whether the type I' coupling, as a spectral function of D_F (Proposition 9.1), supplies the second scale of Proposition 1.5 in the flat family of lemmas 23 and 24.
- The local dP_2 of the top's wall and its relation to the Witten base CP^2 x CP^1.
- Spin: in type I' the strings of Proposition 9.2 belong to nine-dimensional vector multiplets, with one real scalar each (the relative position), the same gap as in Proposition 6.1. By Proposition 9.3 a finite triple in the labels 9 to 11 gives S^2V, so the Λ^2V scalars must come from the connection along the interval; the next computation is the inner fluctuations of the parent of lemma 22 with an SO(30) algebra, and whether its interval component carries Λ^2V with the counts of lemma 21.
- Whether the (15, 3bar) of the EPJC paper could be read as bilinears of two distinct multiplets (the paper itself gives one fundamental with itself) (for example left-handed quarks and conjugates of right-handed ones arranged into one 30), which Proposition 9.3 does not constrain.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 9.1, with the scope of its trace. *From the referee report on the 11D line, section 'Proposition 9.1: integrated coupling profile', transferred verbatim in substance on 4 Nov 2026.*

Hypothesis: the recalled slope rule. The Polchinski-Witten source could not be accessed by the referee; the comparison with it is the second-check note at the proposition.

Let N(x) = sum_k 1_{x_k < x}. Integrating z' = c (8 - N) almost everywhere, with z continuous, gives

```
z(x) = z(0) + c (8x - sum_{k=1}^{16} (x - x_k)_+) = z(0) + c sum_{k=1}^{16} min(x/2, x_k - x/2),
```

since int_0^x N(y) dy = sum_k (x - x_k)+, *and x/2 - (x - x_k)+* = min(x/2, x_k - x/2) (both sides equal x/2 for x <= x_k and x_k - x/2 for x > x_k). At x = L, with every x_k <= L, this becomes

```
z(L) = z(0) - c (8L - sum_k x_k).
```

With x_k = 2 l L m_k/pi, the dictionary of the Setting (exact only for constant z, by the correction after the proof), the zero condition z(L) = 0 is z(0) = (2cL/pi)(4 pi - l sum_k m_k), as stated. The slopes c (8 - N(x)) are nonincreasing, so z is concave; if z(0) > 0 and z(L) = 0, then z(x) >= (1 - x/L) z(0) > 0 on [0, L).

Scope of the trace. The trace is over the sixteen selected channel values. The equality with a spectral trace assumes a chosen sixteen-dimensional channel space with eigenvalues m_k, or an explicitly normalized projection selecting those values. The physical D_F has additional chirality, conjugation, colour and generation multiplicities, which cannot be dropped from its full trace without specifying that projection. The assignment of the sixteen brane positions to those channels is also part of the proposal. This profile alone does not derive a stabilizing term for the interval modulus. The slope, coordinate and e^phi-to-z conventions are those of the recalled rule.

Independent proof of Proposition 9.2, items 2 and 3, with the scope of items 4 and 5. *From the referee report on the 11D line, section 'Proposition 9.2: diquark representations and string channels', transferred verbatim in substance on 4 Nov 2026.*

Hypothesis: the recalled orientifold projections. For V = C^5 ⊗ C^3, the swap of V ⊗ V is the product of the swaps of the two tensor factors, so its eigenspaces combine the symmetric and alternating projectors on the factors:

```
Λ²V = (S²C^5 ⊗ 3bar) ⊕ (Λ²C^5 ⊗ 6),
S²V = (S²C^5 ⊗ 6) ⊕ (Λ²C^5 ⊗ 3bar),
```

with $\Lambda^2 3 = 3\text{bar}$ and $S^2 3 = 6$. Their dimensions are $15 * 3 + 10 * 6 = 45 + 60 = 105$ and $15 * 6 + 10 * 3 = 90 + 30 = 120$, which are C(15, 2) and C(16, 2). Splitting the flavour space into two charge-2/3 and three charge-(-1/3) directions, $S^2C^5 = S^2 2 \oplus (2 \otimes 3) \oplus S^2 3$ and $\Lambda^2 C^5 = \Lambda^2 2 \oplus (2 \otimes 3) \oplus \Lambda^2 3$ give the antitriplet counts (6, 6, 3) in S^2C^5 and (3, 6, 1) in Λ^2C^5, ordered by charges (-2/3, 1/3, 4/3). The full adjoints have dimensions $2(105) + 225 = 435 = \dim \mathrm{so}(30)$ and $2(120) + 225 = 465 = \dim \mathrm{usp}(30)$.

Scope of items 4 and 5. For the assumed Wilson-line action, the antisymmetric or symmetric block has weights a_i + a_j and the mixed block has weights a_i - a_j; hence their field-theory Kaluza-Klein momenta have the displayed shifts (n pi ± (θ_i ± θ_j))/l, with θ = pi a. This calculation leaves out the rest of the open-string spectrum, which also includes oscillator states and their orientifold projections. The sum-weight component with i = j is absent in Λ^2 and present in S^2; neutral diagonal modes remain. The claim of item 5, that a single brane has no string to its own image, holds in this sector. Absence of the same-line component leaves the strings between the top brane and other branes. Moving all stack branes by the same angle makes the mixed-block differences vanish; different individual positions can give nonzero differences. Identifying the resulting Chan-Paton modes with the proposed composites is an additional physical construction. The O8^-/O8^+ projection and global tadpole conditions were not independently verified, since the primary sources were unavailable to the referee.

Independent proof of Proposition 9.3, item 1, with the scope of item 2. *From the referee report on the 11D line, section 'Proposition 9.3: real-structure constraint on diquark blocks', transferred verbatim in substance on 4 Nov 2026.*

Choose an antiunitary identification K: V -> Vbar and write

```
J(v, w) = (eps K^{-1} w, K v),    X = [[A, B], [B*, D]].
```

Then JX(v, w) = (ε K^{-1}(B* v + D w), K(A v + B w)) and XJ(v, w) = (ε A K^{-1} w + B K v, ε B* K^{-1} w + D K v). The V -> V component of JX = ε' XJ, the coefficient of v in the first entry, gives

```
eps K^{-1} B* = eps' B K,   hence   B = epseps' K^{-1} B* K^{-1} = epseps' B^T,
```

the last equality in coordinates in which K is coordinatewise conjugation. Thus εε' = +1 forces $B \in S^2V$, and εε' = -1 forces $B \in \Lambda^2V$. For the displayed KO-5, KO-6 and KO-7 conventions the products are all +1. The zero block belongs to both subspaces, so "symmetric if εε' = +1" states the symmetry forced by the sign. For $V = C^5 \otimes C^3$ the decomposition of S^2V and the counts are those of the proof of Proposition 9.2: Λ^2C^5 has multiplicities 3, 6, 1 at charges -2/3, 1/3, 4/3, and S^2C^5, hence the colour-3bar part of Λ^2V, has 6, 6, 3.

Scope of item 2. A J-compatible finite fluctuation can occupy only a subspace of S^2V; the algebra, grading and order conditions determine which blocks actually occur. For example, with D = 0 every inner one-form vanishes, despite the symmetry condition. The KO signs alone therefore exclude a nonzero antisymmetric block and leave the full symmetric scalar spectrum to be established. For a circle operator D = -i d/dx with ordinary conjugation J, JD = -DJ. A Hermitian connection satisfying the same sign is purely imaginary and antisymmetric in a real basis; the complexified so(30) representation contains Λ^2V and its conjugate. Whether these components survive the particular interval boundary conditions as light scalars requires a separate construction. For distinct multiplets, an off-diagonal block B_12 may be chosen freely, while the full constraint still relates it to B_21 = εε' B_12^T.

Scope and direction. The exclusion concerns a Hermitian J-compatible operator between one multiplet and its conjugate, with the stated signs. Distinct multiplets, other real structures or signs, derivative couplings, and circle or interval connection components remain possible directions; each requires its own field and boundary data.

Independent proof of the branching in section 1. *From the referee report on the 11D line, section 'File 09, section 1: the SO(32) comparison', transferred verbatim in substance on 4 Nov 2026.*

The alternative chain follows from 32_C = 16_{+1} ⊕ 16bar_{-1} under U(16) ⊂ SO(32): so(32)_C = $\Lambda^2 32 = \Lambda^2 16 \oplus \Lambda^2 16\text{bar} \oplus (16 \otimes 16\text{bar})$, so

```
496 = 120_{+2} ⊕ 120bar_{-2} ⊕ 255_0 ⊕ 1_0.
```

Under SU(15), $16 = 15 \oplus 1$ and $120 = \Lambda^2(15 \oplus 1) = 105 \oplus 15$. With $15 = 5 \otimes 3$, the plethysm of Proposition 9.2 gives

```
105 = Λ²(5 ⊗ 3) = (15, 3bar) ⊕ (10, 6),
```

with $S^2 5 = 15$ and $\Lambda^2 5 = 10$, and $15 \otimes 15\text{bar} = (5 \otimes 5\text{bar}, 3 \otimes 3\text{bar}) = (24, 8) \oplus (24, 1) \oplus (1, 8) \oplus (1, 1)$. The antitriplet charge multiplicities are those checked in Proposition 9.2. These group products are Lie-algebra shorthand; global embeddings involve finite central quotients.

Scope. The SO(2) plane is a colour singlet. Assigning it to a top brane is a label in the proposed dictionary and does not by itself construct a colour-triplet top field. Chan-Paton bilinearity and matching representation counts likewise do not by themselves establish dynamical quark composites. The referee could not access the paper itself; the comparison with its text is the reading of the LaTeX source in section 1.

Counterexample (referee) to the unqualified E_1 surface identification in section 5, item 1. *From the referee report on the 11D line, section 'File 09, section 5: del Pezzo wall correspondence', transferred verbatim in substance on 4 Nov 2026.*

The two degree-eight surfaces have different root lattices orthogonal to the canonical class. On dP_1, let H^2 = 1, E^2 = -1, H · E = 0 and K = -3H + E. If D = aH + bE is orthogonal to K, then D · K = -3a - b = 0, so b = -3a and D^2 = a^2 - b^2 = -8a^2. There is no integral class with D^2 = -2. On P^1 x P^1, with ruling classes F_1, F_2 (F_1^2 = F_2^2 = 0, F_1 · F_2 = 1) and K = -2F_1 - 2F_2, the difference F_1 - F_2 is orthogonal to K and has square -2. These surfaces therefore cannot be interchanged in the root-lattice comparison. In the usual local engineering description (recalled), P^1 x P^1 gives E_1, whereas dP_1 = F_1 gives the distinct $\tilde{E}_1$ theory.

For dP_2, with exceptional classes E_1, E_2 and K = -3H + E_1 + E_2, the class E_1 - E_2 is orthogonal to K and has square -2; the orthogonal lattice has rank two, consistent with an A_1 factor and an additional abelian direction. The dP_2 comparison is compatible with the expected A_1 root. This computation alone does not construct the corresponding five-dimensional field theory or the type I' duality, and the brane-to-geometry map remains conditional on the cited dualities.

Sources. The distinction between the two degree-eight engineering models is also stated in a secondary geometric-engineering discussion. The Seiberg and Morrison-Seiberg primary sources were unavailable to the referee for direct comparison. The occurrence of CP^2 and CP^1 in the separate six-dimensional base does not by itself identify that compactification with a local dP_2.

Independent proof of the Weyl action in section 5, item 2. *From the referee report on the 11D line, section 'File 09, section 5: Weyl action on a Wilson line', transferred verbatim in substance on 4 Nov 2026.*

A Weyl transformation of E8 acts by conjugation on an E8 holonomy and gives a lattice automorphism in the heterotic circle frame. Whether it acts as a symmetry of a fixed compactification depends on all of its background and boundary data.

Let T ⊂ E8 be a maximal torus. An element w ∈ W(E8) = N(T)/T has a representative g ∈ N(T), and the constant gauge transformation by g sends a holonomy W ∈ T to g W g^{-1} ∈ T. Thus the holonomy alone determines the same gauge orbit. The corresponding isometry w ⊕ 1 of E8 ⊕ E8 ⊕ II_{1,1} is an element of the automorphism group of the nine-dimensional Narain lattice. Proposition 5.2 places the transposition M in that Weyl group.

If a bundle, a defect, an algebra embedding or a distinguished set of zero modes is held fixed, the same g need not preserve it. Transforming the entire configuration produces a gauge-equivalent description when the transformation is allowed; an exchange relative to fixed colour and family structures needs an explicit action on those structures. The root no-go applies to its stated three-slot configuration, while bundle zero modes remain an open escape.

Part 37

The type I' dilaton against the second scale of Proposition 1.5

Lemma 37. Written as file 10 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line (README, PLAN). Attempted by Claude Opus 5.5 on 4 November 2026, after the merge of branch 11D into master, to answer the last item left open in 09: whether the dilaton can be the second scale that Proposition 1.5 requires. Sources read from the arXiv LaTeX: J. Polchinski and E. Witten, hep-th/9510169, section 3; O. Bergman, M. Gaberdiel and G. Lifschytz, hep-th/9711098, section 3. Numbers in Python (numpy, scipy, mpmath), scripts not kept by the rule of the project. The Codex quota is spent, so no Terra referee is available. An independent reading by Claude Sonnet 5.5, with the error and amendments below, is in the review. The amendments were rechecked by Claude Opus 5.5 before being applied. Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report), who reread both sources: Propositions 10.2 and 10.4 correct with amendments; Proposition 10.1 correct with a qualification of "exactly when"; Proposition 10.3 conditional on the string identification; Proposition 10.5 an upper bound whose consequence for the potential is a hypothesis; Proposition 10.6, item 3, had an error, with the counterexample G_0 = theta^6, G_1 = theta^2 given below; all applied, together with the corrections to the Outcome, the far-wall normalization and the corrections section.

Correction (4 Nov 2026). The first version counted 21 gauge generators with integer towers. Lemmas 23, 24 and 26 use sp(2) (+) su(4), with 25 generators: 16 with integer towers at theta = 0 and 9 with half-integer towers. The bracket for one generation is 41.3125, so the near-cancellation 64 - 63 stated in the first version came from the wrong count, and it is withdrawn. So is the small value theta_* = 0.202 that followed from it.

Outcome

1. **In the scope stated below, the dilaton gradient does not supply the second scale.** Along any path that opens the circle, the relative variation of the profile is bounded by 6/R^2 (Proposition 10.5, an upper bound; it is zero in the chamber 8 + 8). Under the hypothesis that the effective potential has a regular joint expansion in theta and alpha'/l^2 around the flat family, the gradient enters as a power of alpha'/l^2, and it can matter only if the leading Casimir coefficient a_0 vanishes and, when a_1 also vanishes, only if the coefficient b of theta^2 vanishes too (Proposition 10.6, with the referee's counterexample at b = 0).
2. **a_0 is now computed**, which was the open item of 01: 8 pi^2 a_0 = (3 zeta(5)/16)(4 n_D' - 3 (n_i - (15/16) n_h) - n_s - 5 n_g), with n_i and n_h the generators with integer and half-integer towers at theta = 0. For the corpus content it is positive, so at v = 0 the interval runs to large length, in the direction H wants.
3. **The dictionary of 06 to 09 needs three corrections** (Propositions 10.1 to 10.4). The map from angles to positions is nonlinear, and linear as a function of every position exactly when the profile is constant. If the tower of Proposition 6.1 is identified with the type I' winding states, which needs the proposed string embedding and holds in the source at constant dilaton, then the length l of H and the type I' length are T-dual, and l -> infinity shrinks the type I' interval at fixed heterotic coupling. And the configuration 15 + 1 has a Romans mass in the bulk, which bounds length times coupling, so the classical profile has no regime resolved at the formal local scale g^{1/3} l_s.
4. **A scale-free action can still separate the window**, with a fixed small ratio, if the light mass is exponentially small in a stabilized shape parameter (Remark 10.7), which amends the last sentence of Proposition 1.5. This is a separated window at each radius; it is not the parametric hierarchy of Theorem 30.5 of lemma 30, where the ratio tends to zero.

The direction: the second scale must enter through a bulk energy depending on the electroweak order

parameter, through a cancellation a_0 = 0, or through a fixed shape with an exponentially small ratio, as in lemma 29. These targets are stated in Proposition 10.6 and Remark 10.7. None is derived.

The dictionary against the source

Polchinski and Witten give, in conformal gauge, g = Omega^2(x) eta with Omega = C z^{-1/6}, e^phi = z^{-5/6}, and z = 3 C (B mu_8 - nu_0 x)/sqrt 2, with nu_0 = (N(x) - 8) mu_8 piecewise constant. This confirms the slope rule recalled in 08 and 09. We write z for this function up to its constant, normalized so that its slope is exactly 8 - N(x), which fixes the normalization in the formula for R below, with x in [0, L].

Proposition 10.1 (the map from angles to positions). Put W(x) = int_0^x z^{-1/3} dy. By Polchinski-Witten, equations (mass) and (wilmap), the Wilson-line eigenvalue of a brane at x_k is lambda_k = W(x_k) / (2 W(L)), that is

```
theta_k = pi lambda_k = (pi/2) W(x_k) / W(L),
```

and with L = 2 pi, in their units alpha' = 2, R^2 = (3/4) int z^{1/3} dx / (int z^{-1/3} dx)^2. So the linear dictionary x_k = (2L/pi) theta_k of Proposition 9.1 holds as a function of every position exactly when z is constant (8 + 8), and asymptotically for large R; a particular configuration can satisfy it at its brane positions with z nonconstant, as fifteen branes at 0 and one at L do, with angles 0 and pi/2 and slope -7 in the bulk (referee). The normalization: the physical profile in e^phi = z_phys^{-5/6} is z_phys = (3 C mu_8/sqrt 2) h with h of slope 8 - N(x), and Polchinski and Witten set the heterotic alpha'_h = 2 and keep the type I' alpha' separate.

Proof. The string stretched from the wall to x_k has mass (1/(2 pi alpha')) int_0^{x_k} Omega^2 dx, since its area element is Omega^2 dx^0 dx^9, and Omega^2 is proportional to z^{-1/3}. A string winding from 0 to L and back has the mass 1/R. The ratio gives lambda_k; the formula follows from the brane-to-brane mass differences of their equations (14) and (15), since a string ending on the wall is not an available state in every configuration (referee). The formula for R is their equation (16), reduced as in their equation for the dilaton. ■

Consequence for Proposition 9.1: its items 1 and 2 hold as statements about z in terms of the positions x_k. Reading them as a trace of a function of |D_F| needs the map above, which depends on the whole configuration, so the coupling function is a trace only implicitly. Item 3 changes accordingly.

Proposition 10.2 (15 + 1 with infinite coupling at the far wall). Take fifteen branes at x = 0 and one at x_t, with z(L) = 0, and put rho = (z(x_t)/z(0))^{2/3} in [0, 1]. Then, with alpha' = 2,

```
theta_t = (pi/2) 8 (1 - rho) / (8 - rho),        R_*^2 = 14 (8 - rho^2) / (8 - rho)^2.
```

So R_* runs from sqrt 2, for the top's brane on the stack, to sqrt 7 / 2, for the top's brane on the far wall, which are the values (1/2)|n - 8|^{1/2} of Polchinski and Witten at n = 16 and 15. In heterotic string units, R_*/sqrt(alpha'h) lies in [sqrt(7/8), 1]. At each fixed Wilson angle, a nonnegative far-wall value of the normalized profile gives R_h >= R(theta): with A = h(0)^{2/3}, T = h(x_t)^{2/3}, E = h(L)^{2/3} and a = theta/pi, direct integration gives R_h^2 = 2 - a^2 + 28 E/(8A - T - 7E), and R_*^2 = 2 - (theta/pi)^2 is the value at E = 0 (referee). The critical point uses the classical profile at divergent coupling.

Proof. Integrate z^{1/3} and z^{-1/3} on the two segments, with slopes -7 and -8, and use z(x_t) = 8 (L - x_t). ■ Python agrees at x_t/L = 0.5, with theta_t = 0.5859 and R^2 = 1.9652, against 0.7854 for the linear dictionary (the referee's SciPy run gives 0.585918469 and 1.965216392).

So infinite coupling sits at the *smallest* radius of the chamber, at the string scale, and large R is large B, the weakly coupled region. Polchinski and Witten say this directly: "When B is large, the coupling e^{phi(x^9)} is small everywhere and the Type I' theory is weakly coupled; in this range the heterotic radius R is large".

Proposition 10.3 (T-dual lengths, conditional on the string identification). Suppose the tower of Proposition 6.1, n/R, is identified with the type I' winding tower. The source identifies a particular heterotic BPS momentum state with a type I' winding state ("a heterotic Kaluza-Klein state which is dual to a Type I' winding state", Polchinski and Witten); identifying the interval tower with those string states needs the proposed string embedding, and Wilson-line charges shift the momentum-to-winding dictionary (Bergman,

Gaberdiel and Lifschytz), so the identification is a hypothesis here (referee). With constant dilaton, the proper length L_p of the type I' interval and the radius satisfy L_p R_I = pi alpha' on the type I side, with alpha' that of type I'. So with l = pi R_I, measured in the type I' metric, l L_p = pi^2 alpha'. For the heterotic radius in the heterotic metric, l L_p = pi^2 alpha' sqrt(lambda_h), by Bergman, Gaberdiel and Lifschytz, with lambda_h the heterotic coupling.

The flat relations L_p R_I = pi alpha' and l L_p = pi^2 alpha' sqrt(lambda_h) hold at constant dilaton with these frame identifications; the source restricts its simple equations (1.2)-(1.3) to eight D8-branes at each wall, and in the chamber 15 + 1 the metric and coupling vary, so the applicable relation carries the Weyl factor, 1/R_h = (D/(pi alpha'_{I'})) int_0^L Omega^2 dx with L_p = int_0^L Omega dx. Equality of two lengths at a self-dual value does not identify the walls of the two descriptions (referee). Under the hypothesis, the walls of H, at distance l, and the O8 planes, at distance L_p, coincide only at the self-dual length. In particular, the reading "the interval opens" in section 6b of 07 has to be qualified: under the identification and at fixed heterotic coupling, l -> infinity shrinks the type I' interval and decompactifies the SO(32) circle; the large-R_h, fixed-lambda_h decompactification is a conditional reading, which the exact flat formulas do not prove throughout the chamber 15 + 1.

The endpoint of that limit is the ten-dimensional SO(32) theory (heterotic or type I), since at fixed lambda_h the type I' coupling falls as 1/(sqrt(lambda_h) R_h). *Recalled, and supported in the two sources only by remarks of Bergman, Gaberdiel and Lifschytz at the constant-dilaton point:* the eleven-dimensional limit of the rank-17 moduli space is M-theory on an interval times a circle, with E8 x E8 at the walls, which by Remark 8.3 is out of reach with colour-respecting moves. If so, label 11 of section 6b is not reached by theta -> 0, l -> infinity in the chamber 15 + 1.

Proposition 10.4 (Romans mass, length and coupling). On a segment with N branes to the left, the proper distance is linear in the inverse coupling:

```
L_p = (C_L l_s / |8 - N|) |e^{-phi_2} - e^{-phi_1}|,     C_L = 8 pi / 5,     l_s^2 = alpha'_{I'},
```

with C_L from the standard normalization of the Romans mass, |F_0| = |8 - N|/(2 pi l_s), whose sign depends on the orientation; from the source profile and its charge normalization, |d e^{-phi}/ds| = 5 |8 - N|/(8 pi l_s). For N = 8, F_0 = 0, the coupling is constant on the segment, and the division and the bound below do not apply. So in the configuration 15 + 1, g(0) L_p <= C_L l_s / 7 for the total length. Near an infinite-coupling wall the coupling is g = C_L l_s / (|8 - N| s) at proper distance s, and the region g >= g_0 has length (3/4) (C_L / |8 - N|) g_0^{-4/3} in the formal local scale l_11 = g^{1/3} l_s, which is 0.54 g_0^{-4/3} for N = 15. Therefore no region of the classical 15 + 1 background has both strong coupling and a length resolved at that scale. A standard eleven-dimensional lift has not been established for this massive background, and the strong-coupling end of the classical profile is uncontrolled; the source presents its broader M-theory conclusion as an expectation (referee).

Proof. Omega dx = C z^{-1/6} dz / (a C (8 - N)) integrates to a multiple of z^{5/6} = e^{-phi}. ■ This agrees with Bergman, Gaberdiel and Lifschytz, who conclude from the D0-brane bound states that they "expect neither to be applicable to the N != 0 backgrounds", meaning an M-theory interpretation or a matrix theory, and that "only the combination of eight D8-branes and an orientifold plane is known to arise in M-theory". Here their N is 8 - N. It holds wherever N != 8. The scope is the classical profile, which is not controlled where g is of order one, so this is a statement about the extrapolated background.

The far-wall electroweak group. Remark 8.3 places SU(2) x U(1) at the far wall at infinite coupling. By Bergman, Gaberdiel and Lifschytz, the states that become massless there are D0-branes at the wall, heterotic winding states. Let n be the number of branes at the strong wall, here n = 1 with the top's brane there, placed at their x^9 = 0 by reflecting the interval. Combining their equations (windingmass) and (D0mass), with N' = 8 - n, A = (B + 2 pi N')^{2/3} and b = B^{2/3},

```
M_D0 R = (N'/2) b / (A - b) = R^2 - R_*^2    (alpha' = 2),
```

which vanishes at B = 0, where R = R_*, and grows like R^2. This is the full D0 mass after conversion to the heterotic frame; the enhancement states with heterotic winding w_h = +-1 contain half a D0-brane (their equations (3.15)-(3.16) and section 4.2), so if those states are the W^+-, M_{W,h} R_h = (1/2)(R_h^2 - 7/4), with the same monotonic growth (referee). Only W^+ and W^- can be such states: a neutral Z is

a Cartan combination, and the rank of a nine-dimensional theory with sixteen supercharges is fixed, so no massive neutral vector arises this way. So in the chamber 15 + 1 with the top's brane on the far wall, if the W^+- are these D0 states, they are massless at the string-scale radius, with the tower fixed, and they become heavier as R grows: M_W l rises with l. This is the fixed-tower path of the audit (its section 2.2), and away from it the W moves the wrong way for hypothesis H. So on one path, in this scope, the dictionary can place the electroweak group at the far wall, or the opening of H on the circle of Proposition 6.1, but not both. Escapes: other chambers; the W as a Wilson-line string between D8-branes, as in lemma 23; the top's brane in the bulk, with the far wall giving E_1.

The test

Proposition 10.5 (the gradient is a string-scale effect). With L = 2 pi and alpha' = 2, let B = min h > 0 for the normalized profile h. For large R, R^2 = 3 B/(8 pi) + O(1), and the relative variation (max h - min h)/B across the interval is at most 16 pi / B ~ 6 / R^2, an upper bound, attained nowhere in the chamber 8 + 8, where the variation is zero. So along a joint family, R_eps ~ eps^{-alpha}, this variation falls as eps^{2 alpha}, the map of Proposition 10.1 linearizes, and the differences of the dilaton along the interval tend to zero; a finite limiting value of the coupling needs control of the constant C, since z_phys is proportional to C h, and a path can have R_h -> infinity with the coupling diverging uniformly (referee).

Proof. h is B plus a function of size at most 8 L = 16 pi. Insert in the formula for R. ■

If the effective potential has a regular, differentiable expansion in the relative variation, and with the frame identification relating l to R_h, the corrections from the gradient are powers of alpha'/l^2. Small variation alone gives no such expansion: a correction proportional to the square root of the variation has another power. So the regular expansion is a hypothesis of Proposition 10.6, which this estimate guides and does not establish (referee). In the supersymmetric type I' background there is no potential at all: D8 positions, B and C are flat directions. A potential needs broken supersymmetry, as in the flat family of lemmas 23 and 24.

The coefficient a_0. Write 8 pi^2 l^4 V = A_0 + 3 F(2u) + (3/2) F((16/5) u) - n_f F(t), the potential of lemma 24 with its theta-independent part A_0 restored, where n_f counts twice the Dirac species at frequency one (lemma 26). The relative terms vanish at theta = 0, and each tower at theta = 0 is half of a circle tower, periodic or antiperiodic:

```
A_0 = (3 zeta(5) / 16) (4 n_D' - 3 (n_i - (15/16) n_h) - n_s - 5 n_g),
```

with n_D' = 2 N_D counting twice all Dirac species, of any frequency (for frequency zero the tower is the theta = 0 tower), n_i and n_h the five-dimensional gauge generators with integer and half-integer towers at theta = 0, n_s the real bulk scalars with integer towers, and n_g = 1 if the five-dimensional graviton is included. This assumes that the bulk vacuum energy and the wall tensions are tuned away, since they carry the terms of order Lambda^5 l and Lambda^4.

Proof. A real degree of freedom on a periodic circle of length 2 l has V = -3 zeta(5)/(4 pi^2 (2 l)^4), and the interval tower n >= 0 gives half of this, -3 zeta(5)/(128 pi^2 l^4). In the normalization of lemma 24 this is -(3/16) zeta(5) per real degree of freedom. A half-integer tower has the factor -15/16 relative to an integer one, so a generator with mixed conditions contributes +3 (15/16) (3/16) zeta(5). For a fermion channel the sign is opposite, with four degrees of freedom per unit of n_f. The check: the shift to theta = pi/2 multiplies a tower by -15/16, so the difference is (31/16)(3/4) zeta(5) = 93 zeta(5)/64 = F(1), which is the value computed in Proposition 1.4. ■

In lemmas 23, 24 and 26 the gauge field is valued in sp(2) (+) su(4), with 25 generators. At theta = 0, sixteen have integer towers: 8 gluons, 3 W_L, 1 hypercharge combination with Neumann conditions at both walls, and the 4 odd generators O with Dirichlet conditions at both, whose interval components are the Higgs quaternion. Nine have mixed conditions and half-integer towers: 2 R^+-, 6 X and 1 orthogonal neutral direction. So n_i - (15/16) n_h = 121/16. For three generations, with 24 Dirac species, the bracket is 192 - 22.6875 = 169.3125 before gravity and bulk scalars, and A_0 = 32.92. So a_0 > 0: at v = 0 the length runs to infinity, in the direction of H, and the first half of the modulus target of the audit, a r^{-p} with a > 0 and p = 4, is present. For one generation the bracket is 41.3125.

Proposition 10.6 (when the gradient can be the second scale). Let V = l^-4 [G_0(theta) + (alpha'/l^2) G_1(theta) + …], theta = l m, with G_0(0) = a_0 and G_0 = a_0 + b theta^2 + … near 0.

1. If a_0 != 0, and the joint expansion in (theta, s), s = alpha'/l^2, with its derivatives is controlled near (0, 0), then at small m the leading balance is the scale-free one of Proposition 1.5: alpha = 1 or a runaway, and there is no stationary joint branch with theta, s -> 0. The term in alpha'/l^2 changes neither.
2. If a_0 = 0, G_1(0) = a_1 > 0 and b < 0, the truncated potential b m^2/l^2 + a_1 alpha'/l^6 has a minimum at l^4 = 3 a_1 alpha' / (|b| m^2), so alpha = 1/2: 1/l is the geometric mean of m and the string scale, up to constants, and m « 1/l « 1/l_s; with regular higher terms this is the leading asymptotic of the minimum, and in general not an exact equality.
3. If supersymmetry is exact at theta = 0, then V = 0 at theta = 0 at all orders, a_1 = 0, and V = m^2 l^-2 H(theta, alpha'/l^2). If b != 0 and the higher terms are regular, l is fixed by H independently of m, or runs away, and no joint family with theta, s -> 0 results, unless the stationarity condition has a root interval (a flat direction), as in the counterexample to the universal form of Proposition 1.5. If b = 0 the higher terms decide, and a joint family can result: for G_0 = theta^6 and G_1 = theta^2, V = m^6 l^2 + alpha' m^2 / l^4 is even in theta, vanishes at theta = 0 for every l, and has a strict minimum at l^6 = 2 alpha'/m^4, with l ~ m^{-2/3} and l m -> 0 (counterexample from the referee to the first version of this item, which stated the conclusion without the condition b != 0; it constructs no field theory producing this potential).

So in the scope of a potential with a regular joint expansion in theta and alpha'/l^2 around the flat family, with corpus content, where a_0 > 0, the gradient is not the second scale; the case a_0 = a_1 = b = 0 is outside the statement. Item 2 would need a Bose-Fermi balance at theta = 0 that is not a supersymmetry at theta = 0, with about 169 further real bosonic degrees of freedom for three generations.

Proof. Items 1 and 2: differentiate; in item 2, V = b m^2/l^2 + a_1 alpha'/l^6 has V'' = 8 |b| m^2/l^4 > 0 at the stationary point. Item 3: V is even in theta and vanishes at theta = 0 for every l, so V = theta^2 l^-4 H = m^2 l^-2 H. Stationarity is -2 H + theta H_theta - 2 s H_s = 0 with s = alpha'/l^2. With H(0, 0) = b != 0 and regular higher terms, the equation at small theta and s has no solution with both tending to zero, so its roots are angles independent of m, or there are none. With b = 0 the equation can balance powers of theta against powers of s, as the counterexample shows. ■

The flat family with the angle tied. For the top alone, n_f = 6: b is proportional to 54/5 - 6 > 0, and G > 0 on [0, pi/2] (Python: minimum 23.88, with A_0 = 32.92). So V has no stationary point with theta < pi/2, and l runs until theta leaves the fundamental domain of Proposition 7.1, where the top joins the tower; the periodically continued potential has its first stationary points far outside the window, near theta = 8.15 and 8.40, with further roots near 11.17, 11.71, 14.27, … (grid-and-root scan in Python, which proves no global count; periodicity of G with an interval where G' > 0 gives further roots for all large theta; referee). For a universal mass at three generations, n_f = 48, G changes sign and there is one stable stationary point, theta_* = 1.009, with M_W/m = 0.685. For a universal mass at one generation, n_f = 16, theta_* = 1.031, with M_W/m = 0.653 (Python). Top-only content therefore has no vacuum in the window in this scope, and adding a graviton does not change the classification. The stationary angles are of order one, so these vacua do not separate the window.

A localized channel. For a channel with bulk mass M = -mu and c = mu l, the spectrum of lemma 9 gives the one-loop term

```
8 pi^2 l^4 V_loc = -2 int_0^infinity x^3 log(1 + ((w + c)/(w - c)) exp(-2 w)) dx,    w = sqrt(x^2 +
↪ c^2),
```

from the secular function cosh(w) - (c/w) sinh(w). At c = 0 it is -(45/64) zeta(5) = -0.729, and for large c it is -4 c^3 exp(-2 c) up to corrections. Corrections at large c are of relative order 1/c. In between it has its unique minimum at c = 1 exactly, since differentiating the logarithm of the integrand gives d_c log Q = 2 (1 - c)/w (referee), of depth -1.3246 (Python, mpmath). This concerns the interaction term alone: the bulk term, -(4/15) c^5 in dimensional regularization, and the wall term are dropped as renormalizations of the bulk energy and wall tension, and with the bulk term included the minimum disappears. So the interaction

term of one fermion channel prefers c = 1, where the light pair merges with the tower (lemma 9.7). A large c_*, which is what a separated window needs, must come from other fields, such as the bosonic tower of the scalar that gives mu.

Other second scales

Remark 10.7 (amendment of Proposition 1.5, last sentence). On the scale-free path, m l is a fixed number. Proposition 1.5 says that on it the finite window never separates from the tower. That holds for theta_* of order one, which is what the flat family gives above. If m is exponentially small in a stabilized shape parameter, the ratio m/Delta is a small fixed number: a separated window at each radius, which is not the parametric hierarchy of Theorem 30.5, where the ratio tends to zero; and delta/R is a lower bound for the next level, not an established value of the gap. On the disc bundle of lemma 29, fix M and let R grow: m = lambda_M / R and Delta = delta / R both tend to zero, with lambda_M / delta exponentially small, and the count at fixed energy has seven-dimensional growth. So a potential R^-4 G(M), with M stabilized at a large M_* and G(M_*) > 0, *gives the hierarchy of lemma 30 with alpha = 1 and the fixed ratio lambda_{M_}*/delta, with no second scale; in lemma 30 the ratio tends to zero. The cost is that R then has no vacuum at finite value. The physical point at finite v needs one more term, and the stabilization at large M_* is a target, since the interaction term of one fermion channel prefers c = 1.

Remark 10.8 (a bulk energy). The ansatz V = a r^-p + b eps^2 r^q, with a > 0, has a minimum exactly when p + q > 0 and q b > 0, and then alpha = 2/(p + q), so alpha < 1 needs p + q > 2: for p = 4, q > -2 with q != 0. On the flat family the angle is a holonomy (Corollary 22.2), which no local bulk operator sees: at large l its effect is the Casimir term, q = -2, the borderline case alpha = 1. So a term falling more slowly than m^2/l^2 must come from elsewhere, and the natural place is a local bulk energy in the electroweak order parameter: for q = 1, V = a/l^4 + b v^2 M^3 l gives l^5 proportional to 1/v^2, so alpha = 2/5, if the bulk energy vanishes at v = 0. That requires the order parameter to be a bulk field with a bulk vacuum energy, and it departs from D_F as a pure boundary condition. Whether the spectral action provides this is open. In the heat cutoff, the dependence of the potential on D_F is exponentially small at large l Lambda, by Poisson summation over the tower, of relative size 4 exp(-(l Lambda)^2), so that action has no such term at this order. The factor (3 - d_eff)/2 of lemma 25 multiplies the term quadratic in the gauge field, a different term.

Corrections to the earlier files

- 09, Proposition 9.1 and its reading: subject to Proposition 10.1.
- 07, sections 4 and 6b: Proposition 7.2 stands as a statement on the circle; the reading "at distance proportional to a" in section 4 is corrected by Proposition 10.1, and the reading "the interval opens" at label 11 is qualified by Proposition 10.3, conditional on the string identification.
- 08, Remark 8.3 and its reading: the far-wall group is reached at R = R_*, with the tower at the string scale, by Propositions 10.2 and 10.4 and the section on the far wall. There is also a tension of labels: Proposition 8.1 pairs label 9 with O^+, and section 6b of 07 puts label 9 on a far O8^- wall. These are two different meanings of the label, the KO sign and the position of the top's brane.
- 06, section 2: "9, 10, 11" mixes KO labels with metric dimensions. In the string web, nine dimensions is finite R, and R -> 0 is T-dual to R -> infinity (with the Wilson-line transformation and the chosen dual frame: Polchinski and Witten qualify the simple inversion once a Wilson line is present), whereas in H the finite interior is labelled 10 and l -> 0 is labelled 9.
- 05, Proposition 5.4: the conclusion is correct on rerun (Python, exact integers, all 192 configurations). The sentence of its first version, "In every case S - N has d = 19 and d = 20 only, while S - T has d = 16 on the neutrino roots of one slot and d = 22 on those of the two others", since removed from the Statement of lemma 34, was wrong: S - N has only these values in 48 of the 192 configurations, and the other 144 show {13, 17, 19, 20}; the statement on S - T fails in 96 configurations, with the pattern 16, 16, 22. The values of d differ in every configuration, and that is what the proof needs.

Scope and escapes

Proposition 10.6 assumes a potential with a regular expansion in alpha'/l^2, around the flat family with its towers. It concerns the dilaton *gradient*. The following lie outside it: a dilaton potential from string tree level, in non-supersymmetric strings or with tadpoles, for which a term in positive powers of 1/g could grow with l at fixed v through the law of Proposition 10.4; a different frame relation between l and the four-dimensional Planck mass; wall-localized dynamics; and the other chambers of the moduli space, where 8 + 8 removes the Romans mass and E8 x E8 is available. The string-theory statements are read from the two sources for type I' and recalled for the moduli-space limits; only the first have been checked against the text. By AGENTS.md, a negative result needs an independent check by another model. This one has had a reading by Claude Sonnet 5.5, which found the generator count above, and a referee reading by GPT-6.1 Terra (5 November 2026), which found the error in item 3 of Proposition 10.6 and the hypotheses stated above; after the applied corrections, Propositions 10.5 and 10.6 have two independent readings in their stated scope, and the string-theory propositions 10.1 to 10.4 have one referee who reread the sources. The referee's A_0 for three generations is 32.9184, with the positivity and the runaway direction depending on the displayed field content and on the tuning of the bulk energy and the wall tensions.

Part 38

Inner fluctuations of the interval parent with an SO(30) algebra

Lemma 38. Written as file 11 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line (README, PLAN). Attempted by Claude Opus 5.5 on 4 November 2026, to answer the question left in 09: whether the gauge field along the interval of lemma 22 carries the antisymmetric block with the counts of lemma 21, given that Proposition 9.3 excludes it from the finite triple alone. Numbers in Python (numpy), scripts not kept by the rule of the project. The Codex quota is spent, so no Terra referee is available. An independent reading by Claude Sonnet 5.5, whose amendments are applied below, is in the review. Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report): the block calculations, the commutant, the one-forms and the compression formula correct; three corrections applied below, all on one point, that the representation pi is nonunital on the doubled space, so that the gauge action of Theorem 11.3 and the realization of the shadow of Proposition 11.4 as an inner fluctuation at nonzero mass need a unital completion, which is now stated as a hypothesis.

Outcome

1. **Yes, under a condition on the walls.** With a real algebra M_30(R) on the light block, the fluctuation along the interval is an so(30) connection, and its block between particles and antiparticles lies in Λ^2 V, with the counts (6, 6, 3) in the colour antitriplet (Proposition 11.2). Massless scalars in exactly Λ^2 V + conjugate appear when both walls reduce the algebra to M_15(C), the commutant of quark number, under the Yang-Mills action with its natural boundary conditions and a gauge action supplied by a unital completion of pi (Theorem 11.3): 210 real scalar and 225 real vector components at tree level.
2. **In lemma 22's own parent the massless scalars are absent.** Its algebra is free at x = 0, so the interval component has no zero mode, for any algebra (Theorem 11.3, item 2).
3. **A generator has a massless vector or a massless scalar, never both, on the interval**, in free five-dimensional Yang-Mills at trivial background. So the whole of Λ^2(30) of lemma 21 cannot be massless scalars with colour unbroken. The antisquark counts appear as scalars, while the slepton part (24, 1) + (1, 1) appears as vectors (Theorem 11.3, item 4), in the two-U(15)-wall model at trivial background; these are tree-level field counts, not a quantum massless spectrum.

4. **The finite window sees a symmetric shadow.** The connection acts on the window, by compression, through its commutator with the masses, B_ij -> (m_i - m_j) B_ij, which maps the (S^2 5, 3bar) of Λ^2 V onto the (Λ^2 5, 3bar) of S^2 V, with counts (3, 6, 1) (Proposition 11.4). This reconciles Propositions 9.2 and 9.3. The kernel is the same-flavour diquarks. The compression is an operator identity; its realization as an inner fluctuation of the algebra at nonzero mass needs a unital completion, since for invertible M every element of the corner algebra of lemma 22 vanishes at x = l (referee).
5. **SO(30) has a precise meaning in the finite geometry:** it is the commutant of the real structure j = J U of lemma 06 on the massless light block, for the sign s = +1, and USp(30) for s = -1 (Proposition 11.1).

Setting

Let V = C^5 ⊗ C^3, with Q = 2/3 on u, c and Q = -1/3 on d, s, b. In the finite Hilbert space of the five light quarks, with colour, the chirality gamma = +1 subspace is

```
H_+ = {q_L} ⊕ {conj(q_R)} = V ⊕ Vbar,
```

of dimension 30: the left-handed Weyl fields of fifteen Dirac fermions. H_- = {q_R} ⊕ {conj(q_L)}, and J_F exchanges H_+ and H_-, with c = J_F restricted to H_+, as in lemma 22. Let U be a unitary, grading-reversing and J-related as in lemma 06, with U q_L = q_R and U conj(q_R) = s conj(q_L), so U J_F = s J_F U. Then j = U^-1 c is an antiunitary on H_+, with j^2 = s, j q_L = s conj(q_R) and j conj(q_R) = q_L. Here U only defines j: the representation pi below does not commute with U, so this is not a doubling in the sense of lemma 06, in which U commutes with the algebra.

The interval parent is that of lemma 22: D_X = E d/dx on L^2([0, l]; C^2) ⊗ (H_+ ⊕ H_-), with Q in H_+, U-components in H_-, and the wall conditions Q_R(0) = U_L(0) = 0, Q_R(l) = X U_R(l), U_L(l) = -X^* Q_L(l).

The block

Proposition 11.1 (the commutant of j). Let s = +1. In the basis V ⊕ Vbar with j(v, w) = (K^-1 w, K v), the skew-Hermitian a with j a j^-1 = a form so(30), and

```
a = [[alpha, B], [conj(B), conj(alpha)]],    alpha^* = -alpha,   B^T = -B,
```

so, as real vector spaces, so(30) = u(15) ⊕ (Λ^2 V)_R, where u(15) commutes with the complex structure I = (i, -i), the quark number, and the second summand has complex dimension 105 and real dimension 210, with 435 = 225 + 210; the decomposition is not a direct sum of Lie algebras (referee). The Hermitian generator H = i a satisfies j H j^-1 = -H. For s = -1 the same computation gives usp(30) = u(15) ⊕ (S^2 V)_R, of real dimension 225 + 240 = 465.

Proof. Linear algebra, with Proposition 8.1 for the groups. For s = +1, j a j^-1 = a gives the block form, and a^* = -a gives alpha^* = -alpha and B^* = -conj(B), that is B^T = -B. Then j (i a) j^-1 = -i a. ■

So the SO(30) of lemma 21 is the symmetry of the massless block for the sign s = +1, the sign that Proposition 8.1 and STRATEGY T2c associate with label 11 and with the seesaw; and the counts of lemma 21 select the same sign:

	colour antitriplet	its charges (-2/3, +1/3, +4/3)	colour sextet
s = +1, Λ^2 V	(S^2 5, 3bar)	(6, 6, 3)	(Λ^2 5, 6)
s = -1, S^2 V	(Λ^2 5, 3bar)	(3, 6, 1)	(S^2 5, 6)

Fluctuations along the interval

Take on the light block D_F = 0, so X = 0, and let C_0, C_l be subalgebras of A = M_30(R), the real elements of End H_+. Let

```
A_1 = {f in C^infinity([0, 1], A) : f(0) in C_0, f(1) in C_1},    pi(f) = f on Q, 0 on the H_-
↪ components,
```

a nonunital representation on the doubled space: pi(1) is the projection onto H_+ (referee). A unital completion is pi~(f, r) = f (+) r 1 with a real scalar summand; it is used where a gauge action is needed below.

Proposition 11.2. Assume 1 lies in C_0 and C_1.

1. pi(f) preserves the domain, for every choice of C_0, C_1. Order zero and order one hold on this block.
2. The self-adjoint one-forms sum pi(f_i) [D_X, pi(g_i)] are exactly E a_x on the Q components, with a_x in C^infinity([0, 1], so(30)), with no condition at the walls. The fluctuated operator is

   ```
   D_A = E (d/dx + a_x ⊕ U a_x U^-1).
   ```

3. J-hat A J-hat^-1 = E (U a_x U^-1 on H_-), with sign +, as KO-dimension six requires. The block of a_x between Vbar and V is antisymmetric. The two statements agree since E is antisymmetric: the block of the Hermitian operator E a on C^2 ⊗ E is symmetric, as Proposition 9.3, item 2 requires, with B antisymmetric and the factor E carrying the other sign. So the interval component evades Proposition 9.3 through the escape in its item 3.

For D_F != 0 on this block, order one fails in the finite window: [[D_F, pi(a)], J pi(b) J^-1] has the blocks -U b U^-1 M a from H_+ to H_-, and -a M^* U b U^-1 from H_- to H_+, which vanish for all a, b only if M = 0. The corner representation still satisfies order one by the direct pointwise calculation; for invertible M its endpoint matrix corner is zero. Theorem 22.3 assumes a unital represented algebra, so a unital completion must be specified separately: for pi~(f, r) = f (+) r 1 and invertible M the endpoint condition is f(1) = r(1) 1, and the finite order-one failure remains, since with the original corner action a = b = 1 gives the off-diagonal blocks -M^* and -M (referee). So this algebra describes the massless block, and the masses need a treatment without order one, as in lemma 19.

Proof. (1) The conditions Q_R = 0 and U_L = 0 at the walls are preserved by any operator acting on the H_+ index. [D_X, pi(g)] = E g' on Q, and J-hat pi(b) J-hat^-1 acts on the H_- components only, so both commutators vanish. (2) Given T, choose tau with support inside (0, 1) and the same integral as T, which is a finite sum sum f_i g_i' of functions supported inside the interval (take a compactly supported scalar g with g' = 1 on the support of tau). Take f = 1 and g = int_0^x (T - tau), which vanishes at both walls, so g lies in A_1 and g' = T - tau. Self-adjointness of E T, with E skew, requires T skew. (3) J-hat E = E J-hat (direct from the formula for J-hat in Theorem 22.1), and c a c^-1 = U j a j^-1 U^-1 = U a U^-1. ■

Theorem 11.3 (zero modes of the interval component). Let k_0, k_1 be the skew parts of C_0, C_1, and assume a five-dimensional Yang-Mills action for the fluctuation on a flat product with M^4, as in Proposition 22.8, with the usual gauge action, which the corner representation pi does not supply by itself (pi(u) J-hat pi(u) J-hat^{-1} = 0) and a unital completion such as pi~ does, and with the Yang-Mills boundary variational problem at trivial background, unrestricted allowed endpoint variations and no wall terms (referee).

1. The gauge group is {g : I -> O(30), g(0) in K_0, g(1) in K_1}, with det g(0) = det g(1); if K_0 or K_1 contains reflections it does not act separately on SO(30). The holonomy of a_x transforms as P -> g(1) P g(0)^-1, so the moduli are the double coset K_1 SO(30) / K_0, and the massless four-dimensional scalars from a_x, before the quotient by the residual four-dimensional gauge transformations, form

   ```
   the orthogonal complement of k_0 + k_1 in so(30),
   ```

   while the vector zero modes form k_0 ∩ k_1. The action supplies the quadratic form and the natural boundary conditions; order one alone supplies no scalar kinetic term, and the complementary Dirichlet and Neumann conditions for A_x need a compatible gauge fixing.
2. In lemma 22's algebra, C_0 = A, so k_0 = so(30), and there are no such scalars, for any algebra of this kind.

3. If C_0 = C_l = M_15(C), the massless scalars are exactly Λ^2 V ⊕ Λ^2 Vbar, 210 real components with colour-antitriplet counts (6, 6, 3), and the massless vectors are u(15), 225 real components.

4. A generator in k_0 ∩ k_l has a massless vector and no massless scalar; a generator in the orthogonal complement of k_0 + k_l has a massless scalar and no massless vector; all others have neither. If the orthogonal projections onto k_0 and k_l commute (aligned walls), the towers are n pi/l in the first two cases, and (n - 1/2) pi/l for a generator in exactly one of k_0, k_l. For nonaligned walls the towers move, and the zero-mode statements are to be read as the subspaces above: k_0 spanned by T_1 and k_l by cos(phi) T_1 + sin(phi) T_2 give k l = n pi +- phi, which is the mechanism of lemma 23; a nontrivial holonomy can change the zero-mode spaces.

5. With colour unbroken at both walls, the (1, 8) of Λ^2(30) is a vector, so Λ^2(30) cannot consist entirely of massless scalars. In item 3, the (24, 8) + (24, 1) + (1, 8) + (1, 1) of lemma 21 are vectors, and the scalars are (15, 3bar) + (10, 6) + conjugates.

Proof. (1) A gauge parameter xi(x) with xi(0) in k_0, xi(l) in k_l shifts int a_x by xi(l) - xi(0), so the linear moduli are so(30)/(k_0 + k_l), and a constant a_x in the orthogonal complement is a zero mode of the quadratic form. (2), (3): substitution, with Proposition 11.1. (4) A_mu has Neumann or Dirichlet conditions at each wall according to membership in k_wall, and a_x the opposite, so the mode functions are cos(n pi x/l), sin(n pi x/l) or their half-integer versions. (5) Item 4, and the branching of lemma 21. ■

Scope of item 4. Free five-dimensional Yang-Mills at trivial background, with the gauge action and boundary problem stated in the hypotheses; the counts are tree-level field counts and not a quantum massless spectrum. Escapes: a circle without walls, nontrivial holonomy, brane-localized terms, further internal dimensions.

Note that the choice in item 3 is an input: the domain does not force the wall algebra to M_15(C), since by Proposition 11.2, item 1, every f preserves it at D_F = 0. It says that the walls conserve quark number and the bulk does not. On a circle, without walls, every generator has both a massless vector and a massless scalar, and the whole of Λ^2(30) appears, which is the nine-dimensional adjoint scalar used in 09. A wall condition, or the reflection-even algebra of Proposition 1.1, removes half. So the two readings differ: for the stack off the O8 plane, with holonomy I, diquarks are massive and mesons have massless scalars; on the interval with parity Ad(I), diquarks have massless scalars and mesons have none.

The shadow in the finite window

Proposition 11.4. Let s = +1, M = U Lambda with Lambda real for j and symmetric, and let a_tot = a ⊕ U a U^-1 be a constant bounded perturbation, with the window condition of Theorem 22.1. Then:

1. [D_F, a_tot] has the off-diagonal blocks U S and S U^-1, with S = [Lambda, a] real and symmetric, and [D_F, [D_F, a_tot]] = [Lambda, S] ⊕ U [Lambda, S] U^-1.

2. By Theorem 22.4, the compression of E a_tot to the window is -l^-1 int_0^l sin(x ad_{D_F})(a_tot) dx. For Lambda = diag(m_i) on flavours, on both V and Vbar, the block B_ij goes to

   ```
   - ((1 - cos(l (m_i - m_j))) / (l (m_i - m_j))) B_ij,    which is -(l/2)(m_i - m_j) B_ij to
   ↪ leading order, with the continuous value zero at equal masses,
   ```

   as a change of Lambda. The result is symmetric, as Proposition 9.3 requires of a finite J-compatible fluctuation.

3. For distinct masses, B -> (m_i - m_j) B_ij maps (S^2 5, 3bar) in Λ^2 V onto (Λ^2 5, 3bar) in S^2 V as vector spaces, with kernel the five same-flavour diquarks (five colour-antitriplet multiplets), and maps (Λ^2 5, 6) into (S^2 5, 6); distinct fixed masses break flavour SU(5), so the map is SU(5)-equivariant between the named representations only with the mass matrix read as a spurion (referee). In charges, (6, 6, 3) - (3, 0, 2) = (3, 6, 1). If the wall condition is imposed on the gauge unitaries g ⊕ U g U^-1, so that k_l is the commutant of Lambda in so(30), then with k_0 = u(15) the massless scalars of Theorem 11.3 are the same off-diagonal blocks. If it is imposed on pi(f) itself, as in the algebra of lemma 22, then k_l = 0 for invertible M.

4. For a universal mass, Lambda = m, the compression vanishes: S = 0, and M = m U is invariant under SO(30). Fifteen Dirac fermions of equal mass are thirty Majorana fermions of equal mass.

Proof. (1) Direct multiplication, using a_tot on H_- equal to U a U^-1. S is real and symmetric since a is real and antisymmetric and Lambda real and symmetric. (2) Item 1 gives sin(x ad)(a_tot) = (sin(x delta)/delta) ad(a_tot) on B_ij, with delta = m_i - m_j; integrate. (3) In the colour antitriplet, B is symmetric in flavour, and (m_i - m_j) B_ij is antisymmetric in flavour, hence total symmetric. Colour sextets are analogous. (4) S = 0. ■ Checked in Python on a small model, with V = C^3: reality, symmetry, the blocks of item 1 and their squares. Claude Sonnet 5.5 also checked the compression by direct quadrature of W^*(E a_tot)W, with five flavours in three colours, against Theorem 22.4 and the closed block (Python, 10^-15).

Reading. The commutator [D_F, a] is the form of the Higgs field in Connes' finite geometry, and E a_x is a Kaluza-Klein component along the interval. Theorem 22.4 (3), Q_l([D_X, f]) = [D_F, Q_l(f)], says that the second compresses to the first. Indeed E a_tot = [D_X, f_tot] with f(x) = (x - l) a and f(l) = 0. For invertible M, however, every element of the corner algebra of lemma 22 vanishes at l, hence so does every coefficient sum f_i g_i' of one of its one-forms, so a nonzero constant a_tot is not an inner one-form of that algebra, and f_tot is not one of its elements; the compression identity stands as an operator calculation, and its realization as an inner fluctuation at nonzero mass needs a unital completion and a construction of the real one-form there (referee). With that construction, in this parent, to first order and for the fixed compression, a finite Higgs one-form is the shadow of a gauge field along the interval, and it inherits the symmetry type of the finite triple by the commutator with masses. The moving window of Corollary 22.6 gives -l [D_F, h] and a quadratic term (lemma 19). This shadow is tangent to the O(30) orbit of Lambda, so it gives no radial mode, in agreement with No-go theorem 22.7; the odd generators of lemma 23 are absent here since M_30(R) maps H_+ to H_+. A generator commuting with the masses has no shadow. This explains the difference between the counts (6, 6, 3) of the parent and (3, 6, 1) of the finite triple, and it replaces the alternative O8^- against O8^+ of Proposition 9.2, for these blocks, by a parent and its window.

What remains open

- Whether a boundary condition, for example a wall term that conserves quark number, can derive M_15(C) at the walls, where at present it is chosen.
- The unital completion of pi and the gauge action it supplies, used as hypotheses in Theorem 11.3, and the construction of the nonzero-mass one-form whose compression is the shadow of Proposition 11.4 (referee).
- The counts of Theorem 11.3 are tree-level field counts in the two-U(15)-wall model at trivial background; other wall data, holonomy or further directions need separate calculations.
- The treatment of nonzero masses, where order one fails on the block: the quadratic terms of lemma 19 for this algebra.
- The sleptons: on an interval the (24, 1) + (1, 1) are vectors. Scalars in these representations need components along further internal directions, such as the base of 02.
- The colour sextets (10, 6) come with the antitriplets, and nothing removes them in this construction.
- The relation to the top: the (30, 2) of SO(32), with the second factor as the top's SO(2).

Part 39

Hypercharge from the three-form, matter from branes, and the two families of 84

Lemma 39. Written as file 03 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README). Propositions 3.2 and 3.5 and Corollary 3.9 are proved here, and Proposition 3.8 is a negative result in a stated scope, refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5; the numbered proposals are conjectures to test. Attempted by Claude (Anthropic), 3 Nov 2026. Checks: checks_11d.py, sections B, E and G, Python with exact fractions. Referee pending.

Referee note (GPT-6.1 Terra, 4 Nov 2026). Proposition 3.8 and Corollary 3.9 have been independently checked by hand in their stated scopes; the tensor-mode argument needs the displayed colour action to preserve the background. See the referee report.

Referee note on Proposition 3.5 (GPT-6.1 Terra). The classical field equations and curvature condition are independently checked; flux quantization, supersymmetry and stability remain open. See the report.

Referee note on Proposition 3.2 (GPT-6.1 Terra). The exterior-algebra representation and charge counts are independently checked. Their realization by wrapped branes remains a proposal; see the report.

Setting

02 realizes the Witten space with Killing spinors as Y_{3,2} = (5bar)//U(1)_Y. Its isometries give SU(3) x SU(2) x U(1)_R, and no tensor Kaluza-Klein mode on it, on an invariant background, carries triality, weak isospin 1/2 or fractional hypercharge (Proposition 2.4). The M-theory vacuum AdS4 x Y_{3,2} is the near-horizon geometry of N M2-branes at the apex of the cone C(Y_{3,2}), with N units of flux of *G_4 through Y_{3,2}. Lemma 21 proposed relating its two families of 84 states to the three-form C_3, to the M2 or M5 brane, and to their duality. This file connects the two.

Proposal 3.1 (hypercharge is the Betti vector of Y_{3,2})

In AdS4 x M^{111} the massless vectors are those of the isometries, SU(3) x SU(2) x U(1)_R, and one more from C_3 on the harmonic two-form (b_2 = 1), the Betti multiplet; the Kaluza-Klein states are neutral under it, and only non-perturbative states carry its charge (Fabbri et al., hep-th/9903036 and hep-th/9907219). By Proposition 2.3 the Betti class pairs with the divisors {l(U) = 0} and {m(V) = 0} as 2 and -3, the values of 6Y on d^c and L.

Proposal. Identify the Betti U(1) of Y_{3,2} with hypercharge. The gauge group of the vacuum is then

```
SU(3) x SU(2) (isometries) x U(1)_Y (from C_3) x U(1)_R (isometry),
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thirteen vectors, the Standard Model group and one R-symmetry. With massless W and Z this is the eleven-dimensional vacuum of the v -> 0 limit; the identification is consistent with three facts already established:

1. Tensor Kaluza-Klein modes on an invariant background have Y = 0, as they must by Proposition 2.4, which leaves them only PSU(3) x SO(3) representations; spinors are open.
2. The states with Y != 0 are non-perturbative: M5-branes wrapped on five-cycles and M2-branes wrapped on the two-cycle.
3. The M5 baryons of Fabbri et al. are det U on one five-cycle and det V on the other, with baryonic charges 2N and -3N and representations Sym^N(3bar) and Sym^N(2). For N = 1 they are U in (3bar, 1)_{1/3} and V in (1, 2)_{-1/2}: d^c and L, the 5-bar.

In this reading hypercharge is the homology class of a wrapped fivebrane, 6Y = [Sigma_5] in H_5(Y_{3,2}; Z) = Z, and Witten's 1981 chirality argument, which assumes gauge fields acting by coordinate transformations, no longer covers U(1)_Y. It still covers SU(3) x SU(2) for Kaluza-Klein fermions, so chiral doublets must also be brane states.

Proposition 3.2 (one generation is the odd part of the exterior algebra of the 5-bar)

Statement. As SU(3) x SU(2) x U(1)_Y representations,

```
Λ^1(5bar) + Λ^3(5bar) + Λ^5(5bar) = (d^c + L) + (Q + u^c + e^c) + nu^c = 16,
Λ^0 + Λ^2 + Λ^4 = the conjugates,
```

and the charge of SO(10)/SU(5) is X = 5 - 2k on Λ^k. The 32 states are the finite Hilbert space of one generation with nu^c (dim H_F = 96 for three).

Proof. Enumeration with U, V of charges 2, -3 (Python, section B); the identification of the 16 of SO(10) with the odd forms on C^5 is standard. ■

The new element is the reading of the generators: U and V are the coordinates whose zero loci are the baryonic five-cycles of Y_{3,2}, so a state of Λ^k carries the total class of k such cycles, which is its 6Y. For example Q = U^1 U^2 V^1 has class 2 + 2 - 3 = 1.

Conjecture 3.3 (a generation as a Fock space of wrapped fivebranes). At N = 1, the states of one generation and its conjugate are the states of an exterior algebra generated by five fermionic zero modes, one for each of the cycles {U^i = 0} and {V^A = 0}, with hypercharge the total homology class and X the number of fivebranes. Needed: the fermionic statistics of the wrapped states (from fermionic zero modes of the M5 worldvolume theory on Sigma_5), the regime N = 1, where the supergravity description fails and the M2-brane theory is the abelian quotient of 02, and three generations.

Proposal 3.4 (the Higgs is the position of the M2-brane, and the massless limit is the apex)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The stated coordinate stabilizer is independently checked under the proposed charge assignment. The FI quotient has residual $\mathbb{Z}_3$ or $\mathbb{Z}_2$ stabilizers on the respective exceptional loci, so it is a partial orbifold resolution. See the referee report.

Resolve the cone with a Fayet-Iliopoulos parameter: 2|U|^2 - 3|V|^2 = zeta. For zeta < 0 the locus U = 0 is the projective line of the doublet, P(V) = CP^1; for zeta > 0 the locus V = 0 is CP^2.

Statement (proved part). On the brane coordinates (U, V), on which the Betti U(1)_Y of Proposal 3.1 acts with charges (2, -3), the value V = (v, 0), U = 0 on the doublet line is fixed by SU(3) and by T_3 + Y, and moved by T_3 - Y. (In the quotient geometry, where U(1)_Y is divided out, the same point is fixed by T_3, which is T_3 + Y up to the quotient.) Q = T_3 + Y gives U the charge +1/3 of d^c and V the charges (0, -1) of (nu, e). (Python, section B2.)

Proposal. The scalar that breaks SU(2) x U(1)_Y to U(1)_em is the position of the M2-brane on the doublet line, with the quantum numbers of V, that is of H^* (or of a sneutrino). As zeta -> 0 and the brane moves to the apex, the full group is restored and the near-horizon geometry is AdS4 x Y_{3,2}. So the eleven-dimensional theory reached at v -> 0 is the Freund-Rubin vacuum on the Witten space with Killing spinors, with SU(3) x SU(2) from its isometries and U(1)_Y from C_3. The relation between M_W and zeta, or the brane position, is the next computation (README, item 3).

Which path this is. In AdS4 x Y_{3,2} the Kaluza-Klein scale is 1/L with L ~ N^{1/6} l_p, independent of zeta. At fixed N, the limit zeta -> 0 therefore restores the group at a fixed tower: it is the path of the Astra audit (§2.2) on which masslessness opens no dimension, and the eleven dimensions are present above 1/L at every v. A joint family in the sense of 01 (weak masses vanishing faster than the tower) needs L/l_p -> infinity, so N -> infinity, with the isometry couplings, g^{-2} proportional to the Killing-vector norms times the volume of Y in eleven-dimensional units, going to zero unless compensated (audit §7.2). Large N conflicts with item 3 of Proposal 3.1, which needs N = 1 for the baryons Sym^N(3bar) and Sym^N(2) to be d^c and L. This tension, between the regime in which the matter content is right and the regime in which the dimension opens, is the main obstruction of the M2 side in its present form, stated for the Freund-Rubin vacuum with N units of flux; escapes are N = 1 matter on a separate stack, a varying flux along the family, or a different vacuum.

Agreement with the audit (its Appendix A). The audit states independently that the circle bundle of M^{111} has c_1 = 3h + 2u, that the total space is spin, that its geometric U(1) is an R-symmetry, and that hypercharge belongs to the charged module and the gauge lift, since a fixed circle-charge sector acts by a scalar phase. This agrees with Theorem 2.1, Proposition 2.3 for (3, 2), and the placement of hypercharge in Proposal 3.1. The audit also notes that Toeplitz compression on the first holomorphic sector H^0(O(3, 2)), the (10, 3) of Theorem 2.1, item 4, generates the full M_30(C), a noninformative algebra in the sense of STRATEGY T0.

Proposition 3.5 (a magnetic completion: AdS5 x CP^2 x S^2)

Statement. On AdS5 x CP^2 x S^2 with product metric and four-form

```
G_4 = f_1 vol(CP^2) + f_2 J ^ vol(S^2),
```

with J the Kähler form of CP^2 and orthonormal-frame components f_1, f_2, the equations of eleven-dimensional supergravity hold with Ricci components

```
R_AdS = -(f_1^2 + 2 f_2^2)/6,   R_CP2 = f_1^2/3 + f_2^2/6,   R_S2 = 2 f_2^2/3 - f_1^2/6,
```

so a solution exists with round factors exactly when $|f_1| < 2|f_2|$.

Proof. dG_4 = 0, and *G_4 = vol(AdS5) ∧ (f_1 vol(S^2) + f_2 J), which is closed. G_4 ∧ G_4 is an eight-form on the six-dimensional factor, so it vanishes and* dG_4 + G_4 ∧ G_4/2 = 0. The Einstein equation R_MN = (G_MPQR G_N^PQR - g_MN G^2/12)/12 gives the displayed components and zero off the diagonal (Python, section G, by summing all frame components); each factor is Einstein, and the radii follow from the three values. ■

Supersymmetry, stability and flux quantization are not checked. The product with flux on KE_4 x S^2 appears in Pope and van Nieuwenhuizen (1989), and supersymmetric AdS5 solutions fibre the S^2 over the Kähler-Einstein base (Gauntlett, Martelli, Sparks and Waldram, hep-th/0402153); both recalled without rereading here.

Proposal 3.6 (two completions of one base)

The base B_6 = CP^2 x S^2 has isometry SU(3) x SU(2), and its dimension is the KO-dimension of the finite factor. It has two eleven-dimensional completions, both of the form 4 + 1 + 6:

	electric completion	magnetic completion
geometry	AdS4 x Y_{3,2}, Y a circle bundle over B_6	AdS5 x B_6
flux	*G_4 through Y: N M2-branes	G_4 through CP^2 and CP^1 x S^2: M5 charges (N_1, N_2)
the "1"	the fibre circle (Proposition 1.1 of 01)	the radial direction of AdS5
vectors from C_3	b_2(Y) = 1: U(1)_Y	b_2(B_6) = 2: U(1)_{h_1} x U(1)_{h_2}
further vectors	SU(3) x SU(2) x U(1)_R	SU(3) x SU(2)
v -> 0	the M2-brane at the apex	the infrared end of AdS5 moves to infinity

Under reduction along the fibre of class (3, 2), the combination of U(1)_{h_1} x U(1)_{h_2} along (3, 2) is replaced by the fibre isometry, and the quotient Z^2/(3, 2) = Z, (x, y) -> 2x - 3y, is hypercharge. The magnetic side therefore carries two abelian factors, a candidate for the fluxed pair needed by Proposition 2.6 together with the universal twist. The magnetic completion realizes the counting of H directly: four-dimensional spacetime on the M5 worldvolume, the interval as the conformal radial coordinate z in (z_UV, z_IR) with M_KK ~ 1/z_IR, as in Higgsless models on an interval, and v -> 0 as z_IR -> infinity.

Proposal 3.7 (the M2/M5 exchange acts on the Betti vector)

In AdS4, M2-branes on the two-cycle and M5-branes on five-cycles are electric and magnetic for the Betti vector (C_3 = A ∧ omega_2 and C_6 = Ã ∧ omega_5), with Dirac pairing given by the intersection of H_2 and H_5. Which of the two is electric depends on the boundary condition of the bulk vector, and the S operation of Witten's SL(2, Z) action on three-dimensional theories with a U(1) symmetry exchanges them (hep-th/0307041, recalled). With Proposal 3.1 this exchange acts on hypercharge as electric-magnetic duality.

Test against lemma 21. S maps charges to monopole charges; it does not permute the 96 states of H_F. The exchange of the top and the neutrinos in lemma 21 is a permutation, so it does not come from this S. In the brane reading, the 84 states of the second family (Q_em != 0) are the states whose fivebrane class 6Y is not cancelled by T_3, after the breaking of Proposal 3.4; the first family (all states except the top) has no reading here. As the origin of the exchange, this points to a property of the top (lemma 21, Proposition 21.1) that the present construction does not see, and away from the M2/M5 duality.

Proposition 3.8 (no colour-equivariant identification of the 84 with the three-form)

Statement. For every homomorphism SU(3) -> SO(9), Λ^3 R^9 ⊗ C is not isomorphic, as an SU(3) representation, to the colour content of either family of 84 in lemma 21: 10 (3 + 3bar) + 24 (1) for the five quarks and all leptons, 12 (3 + 3bar) + 12 (1) for the charged fermions. The same holds for Λ^6 R^9, the polarizations of C_6.

Proof. A real nine-dimensional representation has a self-conjugate complexification of dimension 9, which is 9 (1), 3 + 3bar + 3 (1) or 8 + 1 (the sextets and two triplet pairs already exceed 9). In the first case Λ^3 has only singlets. In the second, Λ^2(3) ⊗ 3bar = 3bar ⊗ 3bar contains the 6bar. In the third, Λ^3(8) = 1 + 8 + 10 + 10bar + 27. Both families contain only 1, 3 and 3bar. Λ^6 R^9 is isomorphic to Λ^3 R^9 under SO(9). (Python, section E, by weights.) ■

Scope and escapes. A linear action of colour on the nine transverse directions of the eleven-dimensional little group. Escapes: colour acting on the internal space, outside the little group, a nonlinear or brane realization, or a map that is not colour-equivariant, in which case the equality of the two 84s is an equality of numbers only.

Corollary 3.9 (the coloured states of the 84 are not Kaluza-Klein modes of C_3 on Y_{3,2}) Under the quotient realization, on a background invariant under the displayed action, every tensor Kaluza-Klein mode of C_3 (or of the metric) on Y_{3,2} carries a representation of PSU(3) x SO(3) (Proposition 2.4), so none is a colour triplet; spinor modes are open. The 60 coloured states of the first family, and the 72 of the second, are therefore not modes of C_3 on Y_{3,2}. Together with Proposition 3.8 this excludes, in their stated scopes, the two direct routes, the little group and the tensor Kaluza-Klein tower, for a colour-equivariant identification. The route left open is the brane one: wrapped M5 and M2 states, which carry triality through their hypercharge class (Proposition 2.4 (a)).

Summary of the direction

Referee note (GPT-6.1 Terra, 4 Nov 2026). The physical identifications in this summary are proposals. The colour exclusions have the stated linear-representation and tensor-mode scopes; other internal spaces, nonlinear actions and brane realizations remain directions. See the wording audit.

The M2 and M5 details of lemma 21 enter in a specific place. Hypercharge is the C_3 vector of the Witten space with Killing spinors, the 5-bar consists of the N = 1 fivebrane baryons, the Higgs is the M2 position, and the M2/M5 exchange is the electric-magnetic duality of hypercharge. The count 84 of C_3 does not survive as a colour-equivariant identification. The next computations are in the README.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 3.8. *From the referee report on the 11D line, section 'Proposition 3.8 and Corollary 3.9: colour and the three-form', transferred verbatim in substance on 4 Nov 2026.*

Hypothesis: a linear colour action on the nine transverse directions.

An irreducible complex SU(3) representation with highest weight (p, q) has dimension (p + 1)(q + 1)(p + q + 2)/2. The only irreducibles of dimension at most nine are 1, 3, 3bar, 6, 6bar and 8. The complexification

of a real representation is self-conjugate, so a sextet needs its conjugate ($12 > 9$) and two triplet pairs have dimension 12. Dimension nine therefore leaves exactly

```
9 (1),    3 + 3bar + 3 (1),    8 + 1.
```

The first exterior cube is trivial. In the second,

```
Λ^2 3 ⊗ 3bar = 3bar ⊗ 3bar = 6bar + 3
```

is a summand of the exterior cube. In the third, $\Lambda^3(8 + 1) = \Lambda^3 8 + \Lambda^2 8$, and $\Lambda^2 8$ contains a 10: identify 8 with sl_3(C); the vector E_13 ∧ E_12 is killed by the two positive simple root operators ad E_12 and ad E_23 and has highest weight (3, 0), so it generates a 10. Neither target colour representation contains a 6bar or a 10, and the trivial case contains no triplets.

The decomposition used in the proof above also holds:

```
Λ^3 8 = 1 + 8 + 10 + 10bar + 27.
```

E_13 ∧ E_12 ∧ E_23 is a highest-weight vector of weight 2 alpha_1 + 2 alpha_2 = (2, 2): the only nonzero brackets with the raising operators, [E_12, E_23] = E_13 and [E_23, E_12] = -E_13, produce a repeated factor E_13. E_13 ∧ E_12 ∧ H, with H a nonzero Cartan element satisfying alpha_2(H) = 0, has highest weight (3, 0). Conjugation supplies (0, 3), and the invariant Cartan three-form (X, Y, Z) -> tr(X [Y, Z]) supplies 1. The (1, 1)-weight space of $\Lambda^3 8$ has dimension five (E_13 ∧ H_1 ∧ H_2, E_12 ∧ E_23 ∧ H_i for a basis H_1, H_2 of the Cartan subalgebra, and E_13 ∧ E_alpha ∧ E_{-alpha} for alpha = alpha_1, alpha_2), while each of the two spaces reached from it by the simple raising operators, of weights (3, 0) and (0, 3), has dimension two (E_13 ∧ E_12 ∧ H_i and E_13 ∧ E_23 ∧ H_i). Their joint kernel therefore supplies a highest-weight 8. The dimensions sum to $1 + 8 + 10 + 10 + 27 = 56 = C(8, 3)$, exhausting the exterior cube. Orientation and the Hodge star give $\Lambda^6 R^9 \cong \Lambda^3 R^9$ equivariantly.

Independent proof of Corollary 3.9. *Same section of the referee report, transferred verbatim in substance on 4 Nov 2026.*

The colour centre acts trivially on Y_{3,2}: for omega = e^{2 pi i/3}, (omega I_3, I_2) is a transformation of the quotient circle, obtained from lambda = e^{4 pi i/3}, since (lambda^2, lambda^{-3}) = (e^{8 pi i/3}, 1) = (omega, 1). Its induced action on every tensor bundle is the identity. Thus tensor fields contain only colour representations with trivial centre, which excludes 3 and 3bar. The same argument applies to tensor modes of C_3 and of the metric. It is enough that the displayed colour action preserve the background metric and flux; an identification of the full isometry group is unnecessary.

Scope and direction. The first result concerns exterior powers of a linear real nine-dimensional colour representation; the second concerns tensor modes on the stated quotient with the stated action. Colour acting on other internal spaces, nonlinear or brane realizations, and maps without colour equivariance remain open directions.

Independent proof of Proposition 3.5. *From the referee report on the 11D line, section 'Proposition 3.5: magnetic product solution', transferred verbatim in substance on 4 Nov 2026.*

Scope: a classical local supergravity solution with scaled Einstein factors. Flux quantization, global consistency, supersymmetry and stability remain separate questions.

Choose an orthonormal frame e^1, …, e^4 on CP^2 and e^5, e^6 on S^2, with J = e^{12} + e^{34}, and write

```
G = f_1 e^{1234} + f_2 (e^{1256} + e^{3456}).
```

Then G^2 = G_MNPQ G^MNPQ = 4! (f_1^2 + 2 f_2^2) = 24 (f_1^2 + 2 f_2^2). The contraction G_aPQR G_a^PQR, with no sum over a, is 3! (f_1^2 + f_2^2) = 6 (f_1^2 + f_2^2) in a CP^2 direction (each such index lies in e^{1234} and in one of the two f_2 terms), 3! (2 f_2^2) = 12 f_2^2 in an S^2 direction, and zero in an AdS direction. The mixed contractions with a != b vanish, since no two components of G share three indices. Substitution in the Einstein equation R_MN = (G_MPQR G_N^PQR - g_MN G^2/12)/12 gives

```
R_AdS = -(f_1^2 + 2 f_2^2)/6,    R_CP2 = f_1^2/3 + f_2^2/6,    R_S2 = 2 f_2^2/3 - f_1^2/6;
```

for instance R_CP2 = (6 (f_1^2 + f_2^2) - 2 (f_1^2 + 2 f_2^2))/12. The forms J, vol(CP^2), vol(S^2) and the AdS volume form are parallel, so dG = d*G = 0. Also G ∧ G = 0, since it is an eight-form entirely on the six-dimensional compact factor.

Positive curvature of the S^2 factor, 2 f_2^2/3 > f_1^2/6, is equivalent to |f_1| < 2|f_2|. This also makes the other compact curvature positive and the AdS curvature negative. The radii are then determined by the three Einstein constants. Here "round factors" means the standard scaled Fubini-Study metric on CP^2 and the round metric on S^2.

Independent proof of Proposition 3.2. *From the referee report on the 11D line, section 'Proposition 3.2: exterior algebra of the five-bar', transferred verbatim in substance on 4 Nov 2026.*

The representation and charge count is the standard spinor construction. Identifying these algebraic states with wrapped-brane states remains conditional on Proposal 3.1 and Conjecture 3.3.

Write 5bar = U ⊕ V, with U = (3bar, 1)_{1/3} and V = (1, 2)_{-1/2}. The terms of Λ^3 5bar are

```
Λ^3 U = e^c,    Λ^2 U ⊗ V = Q,    U ⊗ Λ^2 V = u^c,
```

with respective 6Y charges 6, 4 - 3 = 1 and 2 - 6 = -4, using Λ^3 3bar = 1, Λ^2 3bar = 3 and Λ^2 2 = 1. Λ^1 5bar = d^c + L, and Λ^5 5bar = nu^c, since the determinant charge is 3(2) + 2(-3) = 0. The odd degrees therefore have dimension 5 + 10 + 1 = 16; the even degrees are their conjugates, since Λ^k 5bar is dual to Λ^{5-k} 5bar when the determinant is trivial, and have dimension 16. Three generations give 96 complex finite-Hilbert-space states. The conventional SO(10)/SU(5) charge X = 5 - 2k gives 3, -1 and -5 on the three odd degrees.

The divisor classes from Proposition 2.3 can be added formally with the same weights 2 and -3. A physical identification of those exterior generators with fermionic wrapped-fivebrane states requires the proposed brane construction and its statistics.

Independent proof of the proved part of Proposal 3.4, and the stabilizers of the FI quotient. *From the referee report on the 11D line, section 'Proposal 3.4: Higgs stabilizer and the FI quotient', transferred verbatim in substance on 4 Nov 2026.*

The coordinate charge calculation holds assuming the proposed action of the Betti U(1) on the brane coordinates; identifying that action with the electroweak one is part of the proposal. A nonzero FI parameter gives a partial orbifold resolution, with finite stabilizers along the projective loci below.

Stabilizer. On V = (v, 0), U = 0, take T_3 = diag(1/2, -1/2) and Y(V) = -1/2. The first component has (T_3 + Y) V = (1/2 - 1/2) V = 0 and (T_3 - Y) V = V, and SU(3) fixes U = 0. The charges Q = T_3 + Y of the two components of V are 0 and -1, and Q(U) = 0 + 1/3 = 1/3.

FI quotient. For the action lambda · (U, V) = (lambda^2 U, lambda^{-3} V) on the level 2|U|^2 - 3|V|^2 = zeta:

- At zeta < 0 every point has V != 0. Points with U != 0 as well have trivial stabilizer (lambda^2 = lambda^3 = 1), and the locus U = 0, V != 0 has stabilizer lambda^3 = 1. This Z_3 acts faithfully on the three transverse U coordinates, by the primitive cube root lambda^2. The coarse exceptional locus is {U = 0}/U(1) = CP^1, with transverse singularity C^3/Z_3.
- At zeta > 0, the locus V = 0, U != 0 has stabilizer lambda^2 = 1, a Z_2 acting by lambda^{-3} = -1, minus the identity, on the two transverse V coordinates. Its coarse exceptional locus is CP^2, with transverse singularity C^2/Z_2.

Thus the FI quotient needs further geometric data to be a smooth resolution.

Part 40

Dualities between nine and eleven dimensions, adapted to the spectral triple

Lemma 40. Written as file 07 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, task C). Attempted by Claude (Anthropic), 4 Nov 2026. Propositions 7.1 to 7.3 are proved here by hand; the string-theory statements of section 1 are recalled without rereading; the readings are proposals. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

1. The dualities (recalled)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The duality descriptions are conditional on the recalled sources. The M2 wrapping number and the auxiliary torus flux are separately specified; a map identifying them remains open. See the referee report.

1. **M-theory on T^2 and type IIB on S^1** (Schwarz, hep-th/9508143; Aspinwall, hep-th/9508154). The modulus tau of the torus is the IIB coupling and SL(2, Z) is IIB S-duality; the IIB radius scales as a negative power of the area A of the torus. M2-branes wrapped on T^2 are IIB momentum states, and momenta on T^2 are (p, q)-strings wound on S^1. Eleven dimensions at A -> infinity, ten (IIB) at A -> 0, nine in between.
2. **Sixteen supercharges, rank 17** (Polchinski and Witten, hep-th/9510169; Hořava and Witten, hep-th/9510209). The two heterotic strings on S^1 are T-dual through Wilson lines; their type I' form is an interval with two orientifold planes O8 and sixteen D8-branes whose positions are the Wilson-line eigenvalues; at strong coupling, M-theory on an interval times a circle, with E8 walls.
3. **Sixteen supercharges, lower rank.** The CHL string (rank 9) and M-theory on the Möbius strip; a rank-1 component related to the Klein bottle (Aharony, Komargodski and Patir, hep-th/0702195); further components with discrete theta angles (Montero and Parra de Freitas, arXiv:2209.03361).
4. **Light-cone M-theory.** On a light-like circle with momentum N, M-theory is the U(N) quantum mechanics of nine Hermitian matrices (Banks, Fischler, Shenker and Susskind, hep-th/9610043; Susskind). On T^2 with a light-like component of C_3, the torus becomes noncommutative and SL(2, Z) acts by Morita equivalence (Connes, Douglas and Schwarz, hep-th/9711162).

2. What each one offers the finite geometry

duality	9	11	the finite triple
1	the nine-dimensional theory	large torus	the lowest Landau level of a torus with flux p: algebra M_p(C), J from charge conjugation with doubling (Proposition 7.3); towers = Landau levels; p = wrapped M2-branes = IIB momentum

duality	9	11	the finite triple
2	type I' interval, or heterotic on S^1	Hořava-Witten on S^1	two sheets = two orientifold walls; D_F = Wilson lines = D8 positions (Propositions 6.1, 7.1, 7.2); A_F from Chan-Paton algebras; J from the orientifold
3	rank-reduced nine-dimensional theories	M-theory on non-orientable surfaces	J as an orientation reversal; the reflection with antiperiodicity of Proposition 1.1 is the Möbius-type ingredient
4	nine transverse matrices, Spin(9)	large N	A_F as the commutant of block-diagonal matrices: stacks of 1, 2, 3 give C + M_2(C) + M_3(C); H_F in off-diagonal blocks; D_F = Γ^i ad(X^i), masses = separations

Duality 2 is the one with an exact meeting point with the corpus, through the angle of the top channel.

3. Proposition 7.1 (the angle of a channel lives on an interval)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The spectral quotient and endpoint multiplicities are independently checked. Theorem 22.1 uses the strict interior; the antiperiodic endpoint is a limit, with a degenerate window. Equality of one channel spectrum does not by itself identify full physical backgrounds. See the referee report.

Statement. In the parent of lemma 22, a channel with singular value σ of the Yukawa block and angle θ = l σ has the signed spectrum {(n pi + θ)/l, (n pi − θ)/l : n in Z} (Theorem 22.1). As a set, this depends only on the class of θ in R modulo the group generated by θ -> −θ and θ -> θ + pi, whose fundamental domain is [0, pi/2]. At the endpoints:

1. θ = 0: the spectrum is n pi/l with a zero mode of multiplicity two (periodic);
2. θ = pi/2: the spectrum is (n + 1/2) pi/l, each value twice (antiperiodic), two copies of the spectrum of the chiral interval of Proposition 1.1; it is the limit X -> infinity of the wall data X = tan(θ), where the lowest eigenspace is degenerate, and in lemma 23 the point of antipodal gradings, where M_W = M_Z = 0 at fixed l with the fermion at pi/(2l).

Proof. The set is invariant under θ -> −θ (it exchanges the two progressions) and θ -> θ + pi (it shifts n). The orbit of θ meets [0, pi/2] in one point. The endpoint values are substitutions; for θ -> pi/2 the wall conditions Q_R(l) = X U_R(l), U_L(l) = −X^* Q_L(l) tend to U_R(l) = Q_L(l) = 0. The statements at θ = pi/2 are those of lemma 23 (Theorem 23.6 and the reading after Proposition 23.7). ■

So the parameter space of one channel of the finite Dirac operator, for a given parent, is an interval with two distinguished ends, and lemma 22's restriction l ||D_F|| < pi/2 is the choice of the fundamental domain.

4. Proposition 7.2 (the far end is the displaced D8-brane on the other wall)

Referee note (GPT-6.1 Terra, 4 Nov 2026). At $a = 1/2$ the full $SO(32)$ centralizer is $S(O(30) \times O(2))$, whose identity component is $SO(30) \times SO(2)$. The stated one-loop minimum uses the hypotheses of lemma 24 and $n_f > 0$; the D8 dictionary remains a proposal. (Second check.) In Spin(32)/Z_2 the component of

determinant -1 sends the lift of W_{1/2} to its negative, so the heterotic centralizer is connected; a -> a + 1 holds for SO(32) matrices, and the lift has period 2. See the referee report.

Statement. Put a = θ/pi in [0, 1/2]. By Proposition 6.1 the channel spectrum is that of the states (30)_{±1} of SO(32) on a circle of radius R = l/pi with the Wilson line W_a = 1_30 ⊕ R(2 pi a). The interval [0, 1/2] is the moduli interval of one Wilson-line eigenvalue modulo a -> −a and a -> a + 1; the unbroken group is SO(32) at a = 0 and SO(30) x SO(2) for 0 < a <= 1/2; at a = 1/2 the holonomy is −1 on the plane of SO(2), W_{1/2} = diag(1_30, −1_2), and the (30)_{±1} tower is (n + 1/2)/R, the antiperiodic tower of Proposition 1.1. The one-loop minimum of lemma 24, Theorem 9, θ = pi/2, is the point a = 1/2.

Proof. Proposition 6.1 and Proposition 7.1 with θ = pi a; the commutant of W_a in SO(32) for 0 < a < 1/2 is SO(30) x SO(2), and at a = 1/2 its identity component is still SO(30) x SO(2), since only two eigenvalues are −1. Lemma 24, Theorem 9, item 1, places the minimum at t = sin^2 θ = 1. ■

Reading (proposal). In type I' the Wilson line W_a is fifteen D8-branes, with images, on one orientifold plane and the sixteenth at distance proportional to a from it (proportional only at constant dilaton; see 10, Proposition 10.1); a = 1/2 puts it on the other orientifold plane (recalled: hep-th/9711098, hep-th/0108135). The dictionary of file 06 then reads: the top angle is the position of the top's D8-brane on the type I' interval, the two ends of the fundamental domain are the two orientifold walls, and the one-loop vacuum of the flat family of lemmas 23 and 24 carries the top brane to the far wall, where the tower of the stack-brane strings is antiperiodic. In supersymmetric type I' the D8 positions are flat directions; the potential of lemma 24 belongs to the non-supersymmetric flat family, the setting in which a force on the brane is expected.

Under the strong-coupling duality of item 2 of section 1, the D8 position is a Wilson-line eigenvalue of the heterotic string on S^1, and in eleven dimensions an E8 holonomy around the circle on a Hořava-Witten wall. So the same datum, the top angle, is a brane position at 9 and a holonomy at 11.

5. Proposition 7.3 (a torus with flux, doubled, has the colour form)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The signs, kernel dimensions and generated matrix corner check out, with a spin-line twist for nontrivial spin structures. The displayed action is degenerate and nonunital on the doubled Hilbert space, so the claimed unital finite triple needs an extra scalar summand. See the referee report.

Statement. Let T^2 be a flat torus of area A with a spin structure, L a line bundle of degree p >= 1 with a connection of constant curvature, and Lbar = L^{-1} with the conjugate connection. On H = L^2(S ⊗ L) ⊕ L^2(S ⊗ Lbar) put D = D_L ⊕ D_Lbar, gamma = gamma_S ⊕ gamma_S, and J(ψ, χ) = (Cχ, −Cψ), where C is the charge conjugation of the spin structure (antiunitary, C^2 = −1, C D = D C, C gamma_S = −gamma_S C, mapping S ⊗ Lbar to S ⊗ L). Let C^infinity(T^2) act by multiplication on the first summand only. Then:

1. J^2 = 1, J D = D J and J gamma = −gamma J, the signs of KO-dimension six, and the algebra satisfies order zero;
2. ker D = H^0(L) ⊕ H^1(Lbar) ≅ C^p ⊕ conj(C^p), with gamma = +1 on the first summand and −1 on the second; the other eigenvalues are the Landau levels, of order (p/A)^{1/2}, which go to infinity as A -> 0 at fixed p;
3. the compressions of the functions to the kernel generate M_p(C) on C^p ⊕ 0, so the window is the finite triple (M_p(C), C^p ⊕ conj(C^p), 0, J, gamma), with KO-6 signs and order zero: M_p(C) acts on the first summand and, through J, by its opposite on the second.

Proof. (1) J^2(ψ, χ) = (C(−Cψ), −C(Cχ)) = (ψ, χ) since C^2 = −1; JD = DJ and J gamma = −gamma J follow from the properties of C; order zero holds because the algebra acts on the first summand and J b J^{-1} on the second. (2) On a torus the spinor bundle is Λ^{0,*} up to a flat factor, so ker D_L = H^0(L) ⊕ H^1(L) = C^p ⊕ 0 by Riemann-Roch in genus one, and ker D_Lbar = 0 ⊕ H^1(Lbar) ≅ conj(C^p) by Serre duality; the Landau levels of a constant field of total flux p on area A scale as (p/A)^{1/2}. (3) The compression of a character e_{m,n} to H^0(L) is a nonzero multiple of a clock-and-shift Weyl operator U^m V^n with UV = e^{2 pi i/p} VU (Berezin-Toeplitz quantization of the torus; recalled; in the theta-function

basis the compression is a Gaussian factor times the Weyl word, checked for p = 2 to 5 in the second check), and these operators span M_p(C). ■

The construction is the torus analogue of lemma 16 (a doubled parent with restricted algebra and a J covering the flip; Example 16.9 for the twistor case), and its window at p = 3 has the form of the colour sector of the Standard Model triple: M_3(C) on the quarks and its opposite on the antiquarks. Berezin-Toeplitz quantization of the torus is known; the KO-6 doubling is the corpus device.

Reading for duality 1 (proposal). The area A -> 0 at fixed flux keeps the window and sends the tower to infinity, the towerless limit; A -> infinity brings the Landau levels down. In M-theory on T^2, flux and wrapping number of M2-branes on T^2 are IIB momentum states, so the dimension p of the window counts IIB momentum on the nine-dimensional circle, and the SL(2, Z) of the torus acts on the family by Morita equivalence of the noncommutative tori of Connes, Douglas and Schwarz when a C_3 background is on. Whether p = 3 for colour has a reason in this picture is open.

6. A remark on duality 4 (known)

For U(N) matrices X^i (i = 1, …, 9) that are block-diagonal, X^i = ⊕*a* *x_a^i* *1*{n_a} with distinct points x_a in R^9, the commutant of all X^i is ⊕*a* *M*{n_a}(C), and the fermion operator Γ^i ad(X^i) on the adjoint, with the sixteen-component spinor of Spin(9), vanishes on the diagonal blocks and has eigenvalues ±|x_a − x_b|, each with multiplicity 8 n_a n_b per ordered block (a, b), that is 16 n_a n_b per unordered pair. With stacks of sizes 1, 2, 3 the commutant is C + M_2(C) + M_3(C), the complex form of A_F, the off-diagonal blocks are bifundamentals as in the Krajewski diagrams of the finite geometry, and the masses are the distances between stacks; coincident stacks restore U(N), the analogue of massless W and Z, and large N is the eleven-dimensional limit. A quaternionic factor H needs an orientifold projection on the two-stack, and chirality needs more than the nine Hermitian matrices.

6b. The three labels in type I' variables (proposal)

Correction (4 Nov 2026). The length l of H and the type I' length are T-dual, and the limit at label 11 below ends at ten-dimensional SO(32). See 10, Proposition 10.3.

Hypothesis H has two parameters for the top channel: the interval length l and the angle θ = l m_t in [0, pi/2] (Proposition 7.1). In the type I' reading of Proposition 7.2 they are the length of the interval and the position of the top's D8-brane as a fraction of it. Then:

- **label 9:** θ = pi/2 at fixed l, the brane on the far orientifold wall: the top at the tower scale pi/(2l) (lemma 22, Corollary 22.9: W, Z and top at the tower scale in the 10 -> 9 family), the antiperiodic tower, and the one-loop minimum of the flat family (lemma 24);
- **label 10:** 0 < θ < pi/2, the brane between the walls, a window below the tower;
- **label 11:** θ -> 0 together with l -> infinity, the joint family of 01 (θ_epsilon ~ epsilon^{1 − alpha}): the brane returns to the stack, SO(32) is restored on that channel, and the interval opens.

In this reading the 9 and 11 endpoints are the two ends of the brane's path, the far wall and the stack, and the second scale required by Proposition 1.5 must fix how l grows as the brane returns. The flat one-loop potential pushes the brane to the far wall (label 9); reaching label 11 needs a term that pulls it back to the stack while the interval grows.

7. What this adds, and what remains open

Referee note (GPT-6.1 Terra, 4 Nov 2026). The brane and E_8-holonomy readings below are proposals. Proposition 7.3's displayed algebra action is nonunital; the corrected finite datum is in the referee report. Proposition 1.5 also permits flat directions.

Added: the angle of the top channel lives on an interval whose ends are the two orientifold walls of type I' (7.1, 7.2); the corpus's one-loop vacuum puts the top brane on the far wall, the SO(32) Wilson line diag(1_30, −1_2); the same datum is an E8 holonomy in the eleven-dimensional frame; a torus with flux gives the colour form of the finite triple with KO-6 signs (7.3).

Open: - A nine-dimensional spectral triple with sixteen D8 positions as the eigenvalues of D_F on the type I' interval, and its strong-coupling limit with E8 walls; the configuration 15 + 1 lies at the SO(32) end, while E8 needs seven D8-branes at an orientifold plane with divergent coupling. - The quaternionic factor H from a torus with even flux and a real structure of quaternionic type, and the full A_F from several tori or from a rank-three bundle with flux. - The lower-rank nine-dimensional components (duality 3), whose J-like orientation reversal may supply the real structure of the finite triple.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 7.1. *From the referee report on the 11D line, section 'Proposition 7.1: channel-angle fundamental domain', transferred verbatim in substance on 4 Nov 2026.*

The stated fundamental domain classifies the signed channel spectrum, including its endpoint multiplicities. The spectrum modulo pi/l consists of the two residues $\pm\theta/l$. Equality of two such multisets is equivalent to $\theta' \equiv \pm\theta \pmod{\pi}$: if $\theta' = \pm\theta + k\pi$ the progressions are relabelled, and conversely the residue sets $\{\pm\theta'\}$ and $\{\pm\theta\}$ modulo pi must agree. This gives the unique representative in [0, pi/2]. In the interior the two progressions are disjoint, since $n\pi + \theta = m\pi - \theta$ would need $2\theta \in \pi Z$. At zero they coincide on $\pi Z/l$; at pi/2 they coincide on $(Z + 1/2)\pi/l$. Thus every endpoint eigenvalue has multiplicity two, including the zero at the first endpoint.

The channel has two copies of the scalar antiperiodic circle spectrum at the second endpoint. Theorem 22.1 assumes the strict window $l\,||D_F|| < \pi/2$; $\theta = \pi/2$ is the corresponding boundary-condition limit, and its lowest level is degenerate. The massless electroweak vectors at that endpoint come from the specified two-grading model of lemma 23 and do not follow from the spectrum of an isolated channel. The endpoints require separate domains or limits, and the quotient of one spectrum need not identify complete physical backgrounds.

Independent proof of Proposition 7.2, with the full centralizer at a = 1/2. *From the referee report on the 11D line, section 'Proposition 7.2: Wilson-line endpoint and one-loop minimum', transferred verbatim in substance on 4 Nov 2026.*

For $0 < a < 1/2$, the nonreal conjugate eigenvalues $e^{\pm 2\pi i a}$ of the rotation force a commuting orthogonal transformation to preserve the two-plane of $R(2\pi a)$ and its complement, and to act on the plane by an element of O(2) commuting with a rotation by an angle other than 0 and pi, that is, by a rotation. Its centralizer in SO(32) is therefore SO(30) x SO(2). At $a = 1/2$ the holonomy is $\mathrm{diag}(1_{30}, -1_2)$; a commuting orthogonal transformation preserves both eigenspaces and is otherwise arbitrary on them, so the full centralizer is

$$S(O(30) \times O(2)) = \{(A, B) \in O(30) \times O(2) : \det A \det B = 1\},$$

with identity component SO(30) x SO(2) and a second component of pairs with $\det A = \det B = -1$. At $a = 0$ the centralizer is SO(32). Conjugation by a reflection in each subspace, an element of SO(32), sends a to -a, and $a \to a + 1$ leaves this SO(32) holonomy unchanged.

Minimum. For the determinant of lemma 24, write $t = \sin^2\theta$. Its two gauge terms are nonnegative and vanish at $t = 1$; the fermion term $-n_f F(t)$ is strictly decreasing for $n_f > 0$. Thus the unique global minimum is $t = 1$, that is $a = 1/2$. This conclusion assumes the flat massless determinant, its stated field content and wall conditions, and no additional angle-dependent term. It does not derive a force on a supersymmetric type I' brane, and the D8 interpretation is conditional on the proposed dictionary.

Independent proof of Proposition 7.3, items 1 and 2, and of the generation step in item 3. *From the referee report on the 11D line, section 'Proposition 7.3: doubled torus and the finite algebra', transferred verbatim in substance on 4 Nov 2026.*

Zero modes. Let S_0 be the flat spin line of the chosen spin structure, with $S_0^2 \cong O$, so that $S \otimes L = \Lambda^{\{0,*\}}(L \otimes S_0)$. The two zero-mode spaces are

H^0(L ⊗ S_0) and H^1(L^{-1} ⊗ S_0).

The first has dimension p: a negative-degree line bundle has no holomorphic sections, so by Serre duality (K trivial on the torus) H^1(L ⊗ S_0) = 0, and Riemann-Roch on a genus-one curve gives χ(L ⊗ S_0) = p. For the same reason H^0(L^{-1} ⊗ S_0) = 0, and Serre duality identifies H^1(L^{-1} ⊗ S_0) with the dual of H^0(L ⊗ S_0^{-1}) = H^0(L ⊗ S_0), hence with its complex conjugate after choosing the inner product. With the stated orientation these have opposite gradings. The untwisted formula of item 2, H^0(L) ⊕ H^1(Lbar), is literal for the trivial spin structure; the dimensions hold for all four.

J and the levels. The assumed charge conjugation C intertwines the two coupled Dirac operators and reverses the grading. Consequently the displayed J satisfies J^2 = 1, JD = DJ and J gamma = -gamma J. In a constant field B = 2 pi p/A, the nonzero Dirac levels have magnitudes sqrt(2|B| n), n >= 1, in the standard Dirac normalization.

Compression. For a reciprocal lattice vector k, split the position operator as x = R + , with guiding-centre part R and oscillator part , which commute with each other and satisfy [_1, _2] = -[R_1, R_2], of modulus 1/|B|. Then e^{ik · x} = e^{ik · R} e^{ik · }, and the oscillator-vacuum matrix element of e^{ik · } is e^{-|k|^2/(4|B|)}. So projecting multiplication by e^{ik · x} onto the lowest Landau level gives

P e^{ik·x} P = e^{-|k|^2/(4|B|)} e^{ik·R} P,

the nonzero Gaussian factor times a guiding-centre translation. For primitive reciprocal vectors k_1, k_2, with |k_1 x k_2| = (2 pi)^2/A and 1/|B| = A/(2 pi p), the two translations U = e^{ik_1 · R} P and V = e^{ik_2 · R} P are unitary on the lowest Landau level and form a clock-and-shift pair, UV = e^{±2 pi i/p} VU. Their p^2 products U^m V^n, 0 <= m, n < p, span M_p(C): the commutation relation forces tr(U^m V^n) = 0 unless m ≡ n ≡ 0 (mod p), so the p^2 products are orthogonal for the trace inner product. Flat spin and connection holonomies change only phases, and leave the nonzero Gaussian factor. Compression generates this algebra, and the compression map itself is not multiplicative on the function algebra.

Counterexample (referee) to the unital finite triple of Proposition 7.3, item 3, and the unital repair. *Same section of the referee report, transferred verbatim in substance on 4 Nov 2026.*

On V ⊕ Vbar, with V = H^0(L ⊗ S_0) ≅ C^p, the generated representation is (a) = a ⊕ 0. Thus (1) = 1_V ⊕ 0, and both the parent and the finite representations are degenerate on the second summand. They do not have the unital representation required in the finite-data setting of lemma 22. In particular, for p = 3, no complex-linear unital M_3(C) representation on C^3 ⊕ conj(C^3) can satisfy order zero with a unital opposite action: the two commuting unital actions would make this six-dimensional space a module for M_3(C) ⊗ M_3(C)^op ≅ M_9(C), whose nonzero modules have dimensions divisible by nine.

One repair is to unitize the represented corner:

A = M_p(C) ⊕ C, (a,) = a ⊕ I_{Vbar}.

In conjugate bases J (b,)^* J^{-1} = I_V ⊕ b^T, which commutes with (a,). This yields a unital finite KO-6 datum with D = 0 and order zero. It has an additional scalar summand; deriving the full Standard Model algebra and its representation remains open.

Second-check derivation (Claude Sonnet 5.5, 4 November 2026)

Spin lift of W_{1/2} and connectedness of the heterotic centralizer (Proposition 7.2). *From the second check on the 11D line (historical/reviews/2026-11-04_sonnet_second_check_11d.md, batch 4, item B2), adopted in the note at Proposition 7.2 and written out on 4 Nov 2026.*

Let G_1, ..., G_32 generate the Clifford algebra of R^32, with G_i G_j + G_j G_i = 2 _ij, and let pi: Spin(32) -> SO(32) be the covering map, under which g~ acts on vectors by v -> g~ v g~^{-1}. The rotation by angle 2 pi a in the plane of the coordinates 31 and 32 lifts to

W~_a = exp(pi a G_31 G_32) = cos(pi a) + sin(pi a) G_31 G_32,

using (G_31 G_32)^2 = -1. At a = 1/2 the lift of W_{1/2} = diag(1_30, -1_2) is W~ = G_31 G_32, up to the sign of the covering. The product of the reflections in e_1 and in e_31 lies in the component of

S(O(30) x O(2)) with det A = det B = -1, and lifts to g~ = G_1 G_31. Since G_1 anticommutes with G_31 and with G_32, it commutes with their product, and

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g~ W~ g~^{-1} = G_1 G_31 (G_31 G_32) G_31 G_1 = G_1 G_32 G_31 G_1 = G_32 G_31 = -W~.
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Every other element of that component is the product of this one with an element of SO(30) x SO(2), whose lifts are products of exp(φ G_i G_j/2) with i, j <= 30 or {i, j} = {31, 32}, all commuting with G_31 G_32. So every lift of the determinant -1 component conjugates W~ to -W~. The computation involves only G_1, G_31 and G_32 and holds in every even dimension.

Let = G_1 G_2 … G_32 be the volume element. It is central in Spin(32), ^2 = 1, != ±1, and pi( ) = -1. The heterotic group is the quotient of Spin(32) by {1, } (or by {1, - } for the other chirality convention; the argument is the same). The class of g~ centralizes the class of W~ exactly when g~ W~ g~^{-1} ∈ {W~, W~}. The case W~ would give pi(g~) W pi(g~)^{-1} = -W, which is impossible, since W and -W have the eigenvalue -1 with multiplicities 2 and 30. So pi(g~) lies in S(O(30) x O(2)). On the determinant -1 component g~ W~ g~^{-1} = -W~, and -1 differs from 1 and from . Hence the centralizer of the class of W~ in Spin(32)/Z_2 is the image of the preimage of SO(30) x SO(2) in Spin(32). That preimage is connected, since it contains the path exp(φ G_1 G_2/2), φ ∈ [0, 2 pi], from 1 to -1. So the heterotic centralizer is connected. Finally W~_{a+1} = -W~_a: the SO(32) holonomy has period 1 in a, and its lift, in Spin(32) and in Spin(32)/Z_2, has period 2.

Part 41

The axioms and the lift along the chain Pati-Salam, Standard Model, colour and electromagnetism

For STRATEGY T17, following lemma 06 (the representation of Result 5), lemma 17 (Corollary 17.4), lemma 20 (Theorem 20.2), lemma 21 and results/orientability_under_heavy_centralizer.md. Written by Claude Fable 5.1 on 4 November 2026. The three matrices, the orientation cycle and the traces were computed by hand and confirmed with numpy on the 32-state representation (one generation, Dirac neutrinos; script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): Lemma 41.1 correct; Theorem 41.2 and Proposition 41.4 correct with amendments; Remark 41.3 and Proposition 41.5 had errors, corrected below as the referee stated them (the generator of B, the state counts, the word “vectorlike”, the e_R obstruction); the referee recomputed every matrix and trace.

Outcome. Along the chain of heavy operators, A_PS = H_R + H_L + M_4(C) -> A_SM = C + H + M_3(C) -> B = C + M_3(C), the finite data keep the KO-6 signs, order zero and the first-order condition of Theorem 20.2, and they lose the other two axioms of a finite real spectral triple: the trace-form Poincaré duality fails at every step, with intersection forms of rank 2, 2 and 0 (the last because every range rho(p_i) rho(p_j)° in the light block has graded dimension zero, the bimodule balance behind the KO-7 doubling of lemma 06), and orientability holds at the Pati-Salam step, for a sheet-diagonal D of the form of Corollary 16.8, and fails at the two later ones, through the pair nu_R, J nu_R and the pair e_R, J e_R at the Standard Model step and through the central mass blocks at the third. The unimodularity trace of the abelian generator changes along the chain, (4i, 12i) on the full space, (2i, 12i) with the Majorana pair removed, (-2i, 6i) with the up-type sector removed and (-4i, 6i) on the light block, so the inherited lift and a lift reimposed later impose different hypercharge constraints, and only the inherited one gives the photon its charges.

Setting

One generation with Dirac neutrinos, in the convention of lemma 06, Result 5: F = C + C^3 (lepton line and colour), H_p = (C^2 (x) F)_L + (C^2 (x) F)_R, H = H_p + H_pbar, dim H = 32. For a = (lambda, q, m) in A_SM, the left action is q (x) I_4 on the left-handed particles, Q_lambda (x) I_4 = diag(lambda, conj lambda) (x) I_4 on the right-handed ones, and I_2 (x) B_{lambda, m} = I_2 (x) diag(lambda, m) on the antiparticles; J(p, pbar) = (K pbar, K p); gamma = +1 on p_L and pbar_R, -1 on p_R and pbar_L.

The opposite action is a° = J a* J^{-1}. The Pati-Salam parent acts by q_L (x) I_4 on p_L, q_R (x) I_4 on p_R and I_2 (x) m_4 on the antiparticles, and A_SM is its centralizer under the heavy Majorana block (Theorem 20.2, rank one, kappa != 0). The light block of the third step is the 18-dimensional space of d_L, d_R (three colours), e_L, e_R, nu_L and their conjugates, on which B = {(lambda, Q_lambda, m)} acts (lemma 06, Proposition 5), with the kept mass blocks commuting with B (lemma 17, Corollary 17.4).

Minimal projections, one per summand: p_R' = (1, 0, 0), p_L' = (0, 1, 0), p_4 = (0, 0, e_11) for A_PS; p_1 = (1, 0, 0), p_2 = (0, 1_H, 0), p_3 = (0, 0, e_11) for A_SM; for B the minimal projection of its summand C is (1, Q_1, 0) = (1, 1_H, 0) = p_1 + p_2, which is the identity on the nine light particles and on the light antileptons, and p_3 is that of M_3(C). The trace form is Q_ij = Tr(gamma rho(p_i) rho(p_j)°).

Poincaré duality

Lemma 41.1 (the trace form is skew). For a finite real triple with J^2 = +-1, J gamma = -gamma J, an algebra even for gamma and order zero, Q_ij = -Q_ji. *Proof.* For an antiunitary J, Tr(J A J^{-1}) = conj(Tr A). Then Q_ij = conj Tr(J gamma p_i p_j° J^{-1}) = -conj Tr(gamma p_i° p_j) = -conj Tr(gamma p_j p_i°) = -conj(Q_ji) = -Q_ji, using order zero to commute p_i° past p_j and the reality of Q. ■ (This is the observation of Ćaćić for KO-dimension 6; the corpus recorded it as rs_N7.)

Theorem 41.2 (the three forms). In the bases above,

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Q_PS = [[0, 0, -2], [0, 0, 2], [2, -2, 0]],   Q_SM = [[0, -2, -2], [2, 0, 2], [2, -2, 0]],   Q_B =
↪  [[0, 0], [0, 0]].
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Each has even rank: Q_PS and Q_SM have rank 2 with kernels spanned by p_R' + p_L' and by p_1 + p_2 - p_3, and Q_B vanishes. So the trace-form Poincaré duality fails at each step of the chain: at the first two because a skew form on an odd-rank K_0 is singular, at the third because every range rho(p_i) rho(p_j)° in the light block has graded dimension zero.

Proof. Each entry counts, with the sign of gamma, the states on which the left projection p_i and the right projection p_j° both act as the identity. For Q_SM: p_1 acts as the identity on all of p_R (eight states) and on the lepton line of pbar_L and pbar_R; p_2 on p_L; p_3 on one colour of pbar_L and pbar_R; and p_1°, p_2°, p_3° are the J-transports, so p_1° is the identity on pbar_R and on the lepton line of p_L, p_R, p_2° on pbar_L, p_3° on one colour of p_L, p_R. Then Q_12 counts the lepton line of pbar_L, two states of grading -1; Q_13 one colour of p_R, two states of grading -1; Q_23 one colour of p_L, two states of grading +1; the diagonal entries vanish by Lemma 41.1 (directly: Q_11 counts the lepton line of p_R and of pbar_R with opposite signs). Q_PS follows in the same way with p_R' the identity on p_R, p_L' on p_L and p_4 on one line of F on the antiparticles: Q_{R L} counts p_R against pbar_L, which is empty, and Q_{R 4}, Q_{L 4} count one line of p_R and of p_L. For Q_B, on the light block the projection (1, 1_H, 0) of the summand C is the identity on the nine light particles and on the three light antileptons, p_3 on one colour of dbar_L, dbar_R, and p_3° on one colour of d_L, d_R; Q_13 counts d_L and d_R of one colour, of opposite gradings, so it vanishes, and Q_11 vanishes by Lemma 41.1 (on the particles its range is nu_L, e_L, e_R with graded count +1, on the antiparticles the conjugates with -1). With the projection (1, 0, 0), which lies outside B, the form would be [[0, -1], [1, 0]]. The kernels are read off. ■

Remark 41.3 (the form is the bimodule balance). Q_ij is the graded dimension of the range of rho(p_i) rho(p_j)°, one colour of an isotypic component when p_j = e_11. It vanishes identically exactly when every such range has equal numbers of states of the two gradings, which is the bimodule balance condition behind the doubling of lemma 06 and is weaker than physical vectorlikeness: the light block of the third step keeps the unpaired nu_L, whose particle content restricted to SU(3) x U(1)_em has [V_L] - [V_R] = (1)_0, nonzero, while Q_B = 0, because the trace form sees the full real space, where the neutral class cancels against its conjugate and the characters lambda and conj(lambda) of the real summand C share one projection (referee; lemma 06 draws the same distinction in its no-go under [U, Q] = 0). On the full 32-state space with nu_R, where B acts and the Standard Model index restricts to zero, the form of B also vanishes, and there the two statements agree. At the first two steps the form has rank 2 because the quark doublets and the leptons carry the chiral asymmetry; the kernel vector p_1 + p_2 - p_3 of Q_SM is the

direction along which the graded counts of the three summands cancel.

Orientability

Proposition 41.4. (i) The Pati-Salam data are orientable: gamma = rho(p_L' - p_R') rho(1_4)° - rho(1_4) rho(p_L' - p_R')°. (ii) The Standard Model data are not, with or without a right-handed neutrino: nu_R and J nu_R carry the same left character lambda and the same right character lambda and have opposite gradings, so no sum of left-right products separates them (Stephan 2006, hep-th/0610097, for the KO-6 models with neutrinos); and e_R carries (conj lambda, lambda) with gamma = -1 while J e_R carries (lambda, conj lambda) with gamma = +1, on which a real cycle takes complex-conjugate values and cannot be -1 on one and +1 on the other, an obstruction that persists when the nu_R pair is removed (referee, checked numerically: removing the nu_R and e_R pairs restores orientability). (iii) The light block of the third step is not: the kept mass blocks are nonzero and commute with B (lemma 17, Corollary 17.4), and a nonzero D commuting with both actions commutes with every orientation cycle while anticommuting with gamma; directly, d_L and d_R of one colour carry the same characters (conj lambda on the left, e_11 on the right) and opposite gradings.

Proof. (i) On the particles rho(1_4)° is the identity and rho(1_4) vanishes, so the right side is p_L' - p_R', which is +1 on p_L and -1 on p_R; on the antiparticles rho(p_L' - p_R') vanishes and rho(1_4) is the identity, so the right side is -(p_L' - p_R')°, which is -1 on pbar_L and +1 on pbar_R. This is gamma. (ii) and (iii): a cycle c = sum rho(a_i) rho(b_i)° takes the same value on two states with the same pair of characters, while gamma takes opposite values. For the first sentence of (iii), if [D, rho(a)] = 0 = [D, rho(b)°] for all a, b then [D, c] = 0 for every cycle, and c = gamma would give D gamma = gamma D, which with D gamma = -gamma D forces D = 0; this is the corpus statement rs_N9 (historical/corpus/connes_kk_research_state.md, section 5.4), restated here with its proof. ■

The explicit model of results/orientability_under_heavy_centralizer.md shows the same loss under a heavy centralizer in a four-summand toy; here it happens in the physical chain, at the first heavy step when nu_R is present and at the second step in every case.

Unimodularity

Proposition 41.5. Let X_lambda be the generator of the abelian direction of the summand C, acting in each algebra as its own element: in A_SM the element (i, 0, 0), which acts as diag(i, -i) (x) I_4 on p_R and as i on the lepton line of the antiparticles; in B the element (i, Q_i, 0), which in addition acts as diag(i, -i) (x) I_4 on p_L, that is by i on nu_L and u_L and by -i on e_L and d_L. Let X_c be the common colour phase, i on the three colours of the antiparticles. The traces of the left action are

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full H, 32 states, A_SM:                          Tr X_lambda = 4i,   Tr X_c = 12i;
Majorana pair removed, 30 states, A_SM:            2i,                12i;
up-type sector removed, nu_R kept, 20 states, B:  -2i,                 6i;
light block of the third step, 18 states, B:      -4i,                 6i.
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So the unimodularity condition Tr A = 0 of Chamseddine-Connes-Marcolli on the one-form (the left action; the adjoint action u J u J^{-1} has determinant one for every unitary, so a condition on it would be empty), imposed on the parent, gives theta_lambda + 3 theta_c = 0 between the two abelian phases, the infinitesimal form of lambda det m = 1, and reproduces lambda^2 (det m)^4 at the 30-state step recorded by the corpus; the same condition reimposed on the light block gives -4 theta_lambda + 6 theta_c = 0, a different embedding of the surviving abelian generator. Under the adjoint action, normalized so that the electron has charge -1, the inherited condition gives the charges (2/3, -1/3, -1, 0) to (u, d, e, nu) and the reimposed one gives (1/6, -5/6, -1, 0): only the inherited lift gives the photon (referee). This is the condition Proposition 22.8 of lemma 22 uses, where the ten generators of u(C_F) reduce to the nine of the photon and the gluons.

Proof. Count the states with their phases. Full space: Q_i is traceless on p_R and X_lambda vanishes on p_L, so only the antilepton line contributes, two weak components in two sectors, 4i; X_c has 12i on the antiquarks. Removing nu_R (phase +i) and J nu_R (antilepton line, +i) gives 2i. For B, (i, Q_i, 0) is

traceless on p_L and on p_R by itself; removing the up-type sector removes u_L and u_R, three colours each with phase +i, and their conjugates, which carry no abelian phase, so the particle part gives -3i on p_L (nu_L at +i, d_L at -3i, e_L at -i) and -3i on p_R, and the antilepton line keeps 4i: -2i on 20 states; the antiquarks halve to 6i. Removing also nu_R and J nu_R takes i twice: -4i on 18 states. ■

The first version of this proposition used the element (i, 0, 0) on the spaces where only B acts and counted 26 states where there are 20; the referee corrected both (the element (i, 0, 0) gives the traces 4i, 2i, i, -i on the four spaces, the numbers of the first version).

Scope

One generation with Dirac neutrinos in the stated representation; the trace form on minimal projections as the test of Poincaré duality (a real Kasparov duality is a different question, as the corpus noted); the strict projections of lemma 06 for the third step; at the Pati-Salam step a sheet-diagonal D of the form of Corollary 16.8 (a Pati-Salam-symmetric Yukawa with no Majorana block), since with the Majorana block order one fails there, which is how Theorem 20.2 selects A_SM. With three generations the matrices multiply by three and nothing else changes. The direction the result points to: the finite triple of the main paper carries, along the chain, exactly the data that lemma 22 derives from the interval (signs, order zero, first order), and none that it does not; the two failed axioms are the ones a Kaluza-Klein parent would have to supply through its topology (orientability, Poincaré duality for the parent manifold), and the vanishing of the form at the third step is the bimodule balance of lemma 06, which coexists with the unpaired nu_L.

Part 42

The Higgs representation in the window: the tangent of the orbit of the marked lines

For STRATEGY T14, item (ii) of the corpus reading (CORPUS_SALVAGE item 13), following lemma 28 (the group data G = Sp(2), K = Sp(1)_a x Sp(1)_b, H = Sp(1)_a x U(1)_b, and Proposition 28.19), lemma 29 (the eight-state window) and lemma 20 (Theorem 20.2 and item 2 of its readings: the connector as a marked pair of points and a coupling). Written by Claude Fable 5.1 on 4 November 2026, by hand; the representation theory is standard and is checked against the isotropy representation stated in lemma 28. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): Proposition 42.2 correct (stabilizer dimensions and ad-eigenvalues recomputed), Proposition 42.1 correct with its escape added, Proposition 42.3 had an error in the count of the connector moduli, corrected below as the referee stated it.

Outcome. The window operator of lemma 29 carries no Higgs representation by itself: Sp(2)-equivariance makes its off-diagonal part a scalar (Proposition 28.19). The Higgs representation of the first-stage corpus note (section 6: T_p HP^1 = Hom_H(L, L^perp), the (2, 2) of Spin(4), one doublet of hypercharge 1/2 after marking a U(1)) is the tangent space of the orbit of the marked line that the connector of lemma 20 puts on the quaternionic sheet. That orbit is CP^3, and its tangent at the marked line is the isotropy representation (2)_{+-1} + (1)_{+-2} of lemma 28: a doublet of the weak Sp(1)_a with U(1)_b charge +-1, which is the Higgs doublet, and a singlet of charge +-2 along the twistor fibre, which is an SU(2)-singlet scalar direction of hypercharge 1. The moduli of the connector under Sp(2) are the Hermitian and the symplectic overlaps of the two marked lines, the modulus of the coupling, and the phase of kappa alpha^2 when the Hermitian overlap alpha is nonzero.

Setting

The window of lemma 29 is W + V, two copies of the 4 of G = Sp(2): the core copy W, on which the real functions of S^4 compress to M_2(H), and the wall copy V, on which the real functions of CP^3 compress to M_4(C) (Example 16.9, Theorem 28.21). The connector of Theorem 20.2 is the rank-one block T_kappa x = kappa K x K with K = v w*, that is, a complex line Cw in W, a complex line Cv in V and the coupling kappa; the centralizer of T_kappa is C + H + M_3(C) (lemma 20, Result 2). In the notation of lemma 28,

K = Sp(1)_a x Sp(1)_b is the stabilizer of a quaternionic line L in W = H^2, with Sp(1)_b acting on L and Sp(1)_a on L^perp, and H = Sp(1)_a x U(1)_b is the stabilizer of a complex line l in L, so that G/K = S^4, G/H = CP^3 and K/H = S^2 is the twistor fibre. Charges of U(1)_b are normalized so that the doublet of Sp(1)_b has charges +-1. In the flag algebra of the corpus note (section 5), marking l reduces H_R = Sp(1)_b to C and leaves H_L = Sp(1)_a, the weak isospin, on L^perp.

Statements

Proposition 42.1 (no Higgs representation in the equivariant window). If the window consists of one copy of the 4 in each degree of the end grading, then after equivariant identifications d_M = [[0, conj(lambda_M) I_4], [lambda_M I_4, 0]], and a phase makes lambda_M >= 0 (Proposition 28.19, by Schur's lemma). So the off-diagonal part of the window operator is a multiple of the identity, invariant under G, and it transforms under the isotropy group of any marked point trivially. The question of item (ii) of CORPUS_SALVAGE, whether that part transforms as T_p S^4, has the answer no for the equivariant window, for this reason. The escape (referee): as a representation of G, Hom(W, V) = 4 (x) 4 = 1 + 5 + 10 contains T_p S^4 = (2, 2) twice after restriction to the isotropy group of p (once in the 5, once in the 10), so a G-breaking off-diagonal block at p, such as the abstract d = mu n^a Gamma_a of Example 16.9 with n orthogonal to the axis, carries the representation of item (ii); the answer no is a statement about the G-invariant block, and the result points to the G-breaking blocks, which are the variation of the marked data described below.

Proposition 42.2 (the tangent of the marked line). Let w be a unit vector of W = H^2 = C^4, l = Cw its complex line, L = l + jl its quaternionic line, and p the point of S^4 = HP^1 under L. The G-orbit of l is CP^3 = G/H and the G-orbit of L is S^4 = G/K. The tangent spaces at l and L decompose under the stabilizers as

```
T_L S^4 = (2, 2) of Sp(1)_a x Sp(1)_b = Hom_H(L, L^perp),
T_l CP^3 = (2)_{+-1} + (1)_{+-2} under H = Sp(1)_a x U(1)_b,
```

where (2)_{+-1} is the restriction of T_L S^4 (the horizontal part, the doublet of the weak Sp(1)_a with U(1)_b charge +-1, one complex doublet of hypercharge 1/2 in the normalization Y = charge/2) and (1)_{+-2} is the tangent of the twistor fibre K/H = S^2 at l (one complex singlet of hypercharge 1, electric charge +1).

Proof. sp(2) = sp(1)_a + sp(1)_b + m with m = (2, 2) of real dimension 4, the isotropy representation of S^4; this is the decomposition of so(5) under so(4). Under H, sp(1)_b = u(1)_b + (1)_{+-2}, since the adjoint of Sp(1) restricted to its maximal torus has charges 0, +-2, and (2, 2) restricted to Sp(1)_a x U(1)_b is (2)_{+1} + (2)_{-1}, since the 2 of Sp(1)_b has charges +-1. The isotropy representation of G/H is sp(2)/h = (1)_{+-2} + (2)_{+-1}, as stated in lemma 28, with (1)_{+-2} tangent to the fibre K/H and (2)_{+-1} the lift of the tangent of G/K. The identification T_L HP^1 = Hom_H(L, L^perp) with the action t -> q_a t q_b^{-1} is the corpus note, section 6. ■

Proposition 42.3 (the moduli of the connector). Let w in W and v in V be the two marked unit vectors, with G acting on both copies of the 4 by the same representation, and let h and omega be the G-invariant Hermitian and symplectic forms on C^4 (omega(x, y) = h(jx, y)). The G-orbits of the pair of lines (Cw, Cv) are classified by the two real invariants

```
a = |h(w, v)|,   b = |omega(w, v)|,   a^2 + b^2 <= 1;
```

the orbit of the pair is determined by the position of Cv relative to the quaternionic line L of w: a^2 + b^2 is the squared norm of the component of v in L, a the overlap with l and b the overlap with jl. The G-orbits of the connector T_kappa itself are classified by a, b, |kappa| and, when alpha = h(w, v) != 0, the phase of kappa alpha^2: with the identity 1 in Hom(V, W) fixed by G, tr T_kappa(1) = kappa (w^* v)^2 = kappa alpha^2 is a G-invariant, unchanged by the rephasings that fix T_kappa (referee). So the connector has four real moduli when alpha != 0 and three when alpha = 0; a symmetry larger than the diagonal Sp(2), for instance an independent phase on V, removes the fourth.

Proof. G acts transitively on the unit sphere of H^2 with stabilizer Sp(1)_a (the stabilizer of w and jw), so the orbit of the pair is the Sp(1)_a-orbit of v. Write v = alpha w + beta jw + u with u in L^perp = H;

Sp(1)_a acts on u by the quaternionic scalars, transitively on each sphere of L^perp, and fixes alpha = h(w, v), beta = h(jw, v) = conj(omega(w, v)) up to the convention. Rephasing w by e^{i phi} and v by e^{i theta} sends (alpha, beta) to (e^{i(theta - phi)} alpha, e^{i(theta + phi)} beta), which removes both phases. So the invariants of the pair of lines are |alpha|, |beta|, and |u|^2 = 1 - |alpha|^2 - |beta|^2 is determined; here beta = omega(w, v) exactly. For the connector, the rephasings w -> e^{i phi} w, v -> e^{i theta} v, kappa -> e^{-2i(theta - phi)} kappa leave T_kappa = kappa K x K fixed and multiply alpha^2 by e^{2i(theta - phi)}, so kappa alpha^2 is invariant under them, and it is invariant under G because 1 is; T_kappa and T_{-kappa} on the same pair of lines with alpha != 0 have the same a, b, |kappa| and opposite invariants, so they lie in different orbits. ■

Reading

The Higgs doublet of the first-stage note is recovered in the window as the tangent of the orbit of the marked core line, projected to S^4: it is the representation in which the point p of S^4 under the connector moves. The connector of lemma 20 is an input of the window (Theorem 28.21 prescribes the windows; the connector is not derived), so the proposition says where the doublet sits once the connector is a field: in the variation of the marked lines, which is a variation of the finite data, in the moduli a, b, |kappa| and arg(kappa alpha^2), and in G-breaking off-diagonal blocks, and never in the equivariant window operator d_M. Beside the doublet, the twistor fibre contributes a charged singlet direction of hypercharge 1; whether this direction is a dynamical scalar, a gauge direction of the U(1)_b, or frozen by the collar connection of lemma 29 (which preserves the line summands, hence the fibre structure) is open, and it is the same question as the grading axis of Proposition 28.19, which marks the quaternionic line L. The dictionary note keeps one row from this: the Higgs doublet is the tangent of the base S^4 at the point marked by the connector, with the twistor fibre as a charged singlet beside it.

Scope

The window of lemma 29 with its Sp(2)-equivariance; the connector of Theorem 20.2 as a marked pair of lines and a coupling; the normalization Y = charge/2 of the corpus note. Nothing here makes the marked lines dynamical, computes a potential, or fixes the coupling; Proposition 42.3 counts moduli and does not identify them with a vacuum expectation value; Proposition 42.1 concerns the G-invariant block only. The direction it points to: a parent in which the connector's lines vary along S^4, that is, a Higgs field as a section over the base, for which the collar connection of lemma 29 and the engineered connections of T15 are the available tools.

Part 43

Which connections on a vertical kernel are Berry connections

For STRATEGY T15 (from CORPUS_SALVAGE items 1 and 2), following lemma 10 (Proposition 10.1 and the second-moment identity), lemma 29 (the collar connection of case (b)) and lemma 38. Written by Claude Fable 5.1 on 4 November 2026, by hand. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): Proposition 43.1 and Theorem 43.2 correct with amendments (the mean-curvature term, the transposed adjoint, the exclusion of vertical components), the two computations of Corollary 43.3 correct, and its last clause, which read lemma 29's collar connection as an instance, an error, withdrawn below. The construction was used in the corpus (historical/corpus/canonical_research.md, section 9, on N over S^5 with the flux (1, 1, 2)); this lemma supplies the general statement, and leaves the question of lemma 29's collar connection open.

Outcome. Let a fibrewise Dirac family on a bundle of compact fibres F_b have kernels K_b of constant rank, forming a bundle K over the base B, with the Berry connection induced from a unitary connection on the coefficient bundle E. Changing the coefficient connection by a horizontal one-form alpha changes the Berry connection by the fibrewise compression P alpha P. Every connection on K is a Berry connection of some coefficient connection exactly when the fibrewise compression Gamma(F_b, u(E)) -> u(K_b) is surjective at every point of B, and that holds exactly when the pairwise products s_i s_j^* of a frame of kernel sections

are linearly independent as sections of End(E) over the fibre. For scalar changes of the connection the same criterion reads: the pointwise inner products <s_i, s_j> are linearly independent functions on the fibre. On CP^3 with the kernel H^0(O(1)) this holds (sixteen independent functions z-bar_i z_j/|z|^2), so every connection on the rank-four kernel bundle is a Berry connection of an abelian change; for functions pulled back from the base S^4 the compressed image has real dimension six out of sixteen (Example 16.9 with Proposition 10.1), so scalar changes along the base reach six of the sixteen directions. The collar connection of lemma 29 is outside this statement: it acts at the wall, on the CP^3 copy, and it changes the fibre operator and moves the kernel, the effect excluded below (referee).

Setting

Let pi: P -> B be a fibre bundle with compact fibres F_b and a connection (a horizontal distribution), E -> P a Hermitian vector bundle with a unitary connection nabla, and D = (D_b) a smooth family of fibrewise elliptic self-adjoint operators on L^2(F_b, E|*{F_b}) whose kernels K_b have constant dimension k; P_b denotes the orthogonal projection onto K_b. The Berry connection on the bundle K -> B is nabla^K_X s = P_b nabla*{X^h} s for X in T_b B, with X^h the horizontal lift and s a section of K read as a family of sections of E over the fibres; this is unitary for the fibrewise L^2 product when horizontal transport preserves the fibre measure (Riemannian submersions with totally geodesic fibres, as in the homogeneous examples), and in general the unitary connection adds (1/2) k(X) with k the mean-curvature form of the fibres (Bismut; Berline-Getzler-Vergne, chapter 9), a term common to all the connections compared here, so that the statements below hold with that term included or with volume-preserving transport assumed (referee). For a fibre F_b, the compression map is

```
C_b : Gamma(F_b, End(E)) -> End(K_b),   C_b(X) = P_b X P_b,
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and in a unitary frame s_1, ..., s_k of K_b its matrix elements are C_b(X)_{ij} = int_{F_b} <s_i(y), X(y) s_j(y)> dy. Its adjoint for the L^2 pairing on Gamma(F_b, End(E)) with the Hilbert-Schmidt pairing on End(K_b) is C_b^*(Y) = rho_Y = sum_{ij} Y_{ij} s_i s_j^*, a smooth section of End(E) over F_b (the density of Y; the first version had the indices transposed, which changes nothing below since the two families of products coincide as sets).

Statements

Proposition 43.1 (the change of the Berry connection). Let nabla' = nabla + alpha with alpha in Omega^1(P, u(E)) vanishing on vertical vectors. Then the Berry connection of nabla' is nabla^K + beta, where beta_b(X) = C_b(alpha(X^h)|_{F_b}): the compression of the horizontal component of alpha, fibre by fibre, exactly. A fibrewise twisted Dirac operator uses only the vertical part of the coefficient connection, so such an alpha leaves the family D_b and its kernels unchanged. A vertical component of alpha changes D_b and moves the kernels, and a first-order motion of the kernels changes the Berry connection at first order through the second fundamental form (for a kernel line n(b) in C^2 over S^2 moved along the flow of a vector field xi, the Berry curvature changes by the Lie derivative along xi); such components are excluded here, and this is the effect at work in the collar connection of lemma 29 (referee).

Proof. With the kernels unchanged, nabla'*{X^h} s = nabla*{X^h} s + alpha(X^h) s, and P_b of the second term is C_b(alpha(X^h)) s. ■

Theorem 43.2 (which connections are Berry connections). The following are equivalent for the family (P, E, D):

1. Every unitary connection on K -> B is the Berry connection of nabla + alpha for some alpha in Omega^1(P, u(E)) vanishing on vertical vectors.
2. C_b : Gamma(F_b, u(E)) -> u(K_b) is surjective for every b in B.
3. For every b and every (one, hence every) unitary frame s_1, ..., s_k of K_b, the k^2 sections s_j s_i^* of End(E) over F_b are linearly independent.

When these hold, a preimage can be chosen smoothly: alpha(X^h) = C_b^*(C_b C_b^*)^{-1} beta_b(X) is a smooth u(E)-valued form on the horizontal distribution, extended by zero on vertical vectors.

Proof. (1) implies (2): a connection whose difference from nabla^K at b is a prescribed Y in u(K_b) must have Y in the image of C_b. (2) implies (3): C_b is surjective iff C_b^* is injective on u(K_b); C_b^* is complex linear with C_b^*(Y^*) = C_b^*(Y)^*, so its kernel in End(K_b) is closed under the adjoint and meets u(K_b) trivially exactly when it is zero, and C_b^*(Y) = sum Y_{ij} s_i s_j^* vanishes for a nonzero Y exactly when the sections s_i s_j^* are dependent. (3) implies (1): with C_b surjective, C_b C_b^* is a positive invertible operator on u(K_b) depending smoothly on b, so for a given beta in Omega^1(B, u(K)) the form alpha(X^h) = C_b^*(C_b C_b^*)^{-1} beta_b(X) is smooth, u(E)-valued, and C_b(alpha(X^h)) = beta_b(X); by Proposition 43.1 the Berry connection of nabla + alpha is nabla^K + beta. ■

Corollary 43.3 (scalar changes). Restricting alpha to scalar forms (alpha = i a (x) 1_E, a real), the attainable changes are the compressions of real functions, and every connection on K is attained by a scalar change iff the k^2 functions <s_i, s_j> on F_b are linearly independent. For the kernel H^0(CP^3, O(1)) of the Dolbeault-Dirac operator twisted by O(1), with s_i = z_i/|z| in the unitary frame, <s_i, s_j> = z-bar_i z_j/|z|^2, sixteen linearly independent functions, so the criterion holds and every connection on the rank-four kernel bundle is the Berry connection of an abelian change of the coefficient connection; in matrix form, Q(<w, B w>/<w, w>) = (B + tr B)/5 (lemma 10, the second-moment identity), a bijection of Hermitian matrices. For functions pulled back from the base S^4 of the twistor fibration, the compressed image is the real span of 1 and Gamma_1, …, Gamma_5, of dimension six (a real function from S^4 compresses to a Hermitian element commuting with j, Example 16.9, item 1, and Proposition 10.1 shows the span is attained), so scalar changes along the base attain only the connections whose difference from nabla^K lies in that span. Whether the complementary ten directions are reached by matrix-valued alpha of a restricted class, such as components between the summands of E, is a further application of criterion 3 to the core modes that has not been made; the first version of this corollary asserted it and read the collar connection of lemma 29 as an instance, which it is not (that connection acts at the wall, on the CP^3 copy, and moves the kernel).

Proof. The first sentence is Theorem 43.2 with the scalar subalgebra in place of Gamma(u(E)). The CP^3 statement is the second-moment identity Q(w^* B w/w^* w) = (B + tr B)/5 (Example 16.9, item 2, from the fourth-moment formula in the proof of Proposition 10.1), which maps Herm(4) onto itself with eigenvalues 1/5 and 1; the S^4 statement is Example 16.9, item 1, with Proposition 10.1. ■

Remark 43.4 (the test case of the corpus). On the Berger family of S^5 with the kernel of flux (1, 1, 2) the corpus engineered a connection with a prescribed Berry connection and found that all three coefficient lines keep a circle zero mode exactly when eta^2 is an integer; that calculation rests on stated spin lifts and was checked for its phase arithmetic only (historical/corpus/canonical_research.md, section 9, and research_checks, section 4.7). Theorem 43.2 says when such engineering is possible; it does not redo that calculation, and the integrality statement stays a record.

Scope

Constant-rank kernels, compact fibres, unitary connections with volume-preserving horizontal transport or the mean-curvature term; the Berry connection as the compressed horizontal derivative; changes of the connection with no vertical component, which leave the kernels fixed. The theorem is linear algebra fibre by fibre and a smoothness remark; its content for the project is the criterion of item 3, which separates the two sheets of lemma 29 for scalar changes: the wall copy (functions on CP^3) attains every connection, the core copy (functions on S^4) six directions of sixteen. The collar connection of lemma 29 is a vertical change at the wall and moves the kernel, so the question of T15, whether it is a choice or a construction, stays open; a theorem that covers it must include the motion of the kernels (referee).

Part 44

The retained algebra along the seesaw scaling

For STRATEGY T2c (CORPUS_SALVAGE items 3 and 7), following lemma 20 (Theorem 20.2: the centralizer of the Majorana block is C + H + M_3(C)), lemma 06 (the representation of Result 5 and the criterion

of Lemma 4) and the multiscale collapse of the corpus (historical/corpus/solidification.md, section 2: for F_t = tT + t^alpha V + S the bounded-commutator algebra is the commutant of T and of V). Written by Claude Fable 5.1 on 4 November 2026, by hand. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): the commutants and the spectrum recomputed and confirmed; amendments on the criterion (the bounded-commutator criterion agrees with the one of Theorem 20.2 for T_R and differs from it for V_nu, which satisfies order one on A_SM), on the conjugate sector (no condition), on y != 0, and on the role of S, applied below.

Outcome. The heavy Majorana block T_R selects the Standard Model algebra out of Pati-Salam (Theorem 20.2). A neutrino Dirac block V_nu that grows with the Majorana scale, as it must for a seesaw mass that stays finite (V_nu ~ t^{1/2}), is a second heavy operator, and the algebra retained by bounded commutators is its commutant inside C + H + M_3(C), which is C + M_3(C): the algebra of colour and electromagnetism, the same as after the heavy top. So, under the bounded-commutator criterion, the finite seesaw at fixed light mass is a limit in the broken algebra; the Standard Model algebra survives the heavy Majorana limit only with a bounded Dirac coupling, along which the light neutrino mass vanishes as 1/t. Under the order-one criterion of Theorem 20.2 the conclusion is different: V_nu is part of the Standard Model Dirac operator and satisfies order one on A_SM, so A_SM stays admissible at every alpha; the two criteria agree for T_R (both give A_SM) and differ for V_nu (referee). Read physically, an unbounded commutator is a gauge direction whose mass grows (lemma 20, Result 9), so the proposition says that a Dirac neutrino mass growing without bound breaks SU(2). In nine dimensions the Majorana block cannot be a bulk mass (a single Majorana field in dimension 8k + 1 has no scalar mass term), so in the interval parent it is wall data, and the statement above is about the wall operator.

Setting

One generation in the representation of lemma 06, Result 5, with H_p = (C^2 (x) F)_L + (C^2 (x) F)_R, F = C + C^3, the left action q (x) I_4 on p_L, Q_lambda (x) I_4 = diag(lambda, conj lambda) (x) I_4 on p_R, I_2 (x) diag(lambda, m) on the antiparticles, and J(p, pbar) = (K pbar, K p). The neutrino states are nu_L (p_L, upper weak component, lepton line), nu_R (p_R, upper component, lepton line) and their conjugates. Let

```
T_R = |nu_R><J nu_R| + |J nu_R><nu_R|   (the Majorana block, rank two, J-real, odd),
V_nu = y (|nu_L><nu_R| + |nu_R><nu_L|) + conjugate   (the neutrino Dirac block),
```

and S the remaining Yukawa blocks. The family of finite operators is D_t = t T_R + t^alpha V_nu + S with t -> infinity and 0 <= alpha < 1.

Statements

Proposition 44.1 (the retained algebra). The algebra of a in A_PS = H_R + H_L + M_4(C) with sup_t |||[D_t, rho(a)]|| < infinity is

```
A_PS cap {T_R}' cap {V_nu}'   for alpha > 0,   and   A_PS cap {T_R}' = C + H + M_3(C)   for alpha =
↪ 0.
```

For alpha > 0 it equals {(lambda, Q_lambda, m)} = C + M_3(C): the centralizer of the top block of lemma 06 and the commutant C_F of lemma 22 for the Standard Model data.

Proof. Boundedness of [t T_R + t^alpha V_nu + S, rho(a)] for all t forces [T_R, rho(a)] = 0 and, when 0 < alpha < 1, [V_nu, rho(a)] = 0 (compare the coefficients of t and t^alpha; at alpha = 1 only the commutant of T_R + V_nu would be forced). The first commutant is C + H + M_3(C) by Theorem 20.2 (rank-one block, kappa != 0), realized as a = (lambda, q, m) acting by lambda on nu_R. For the second, with y != 0, rho(a) V_nu nu_R = y rho(a) nu_L = y (q_11 nu_L + q_21 e_L) and V_nu rho(a) nu_R = lambda y nu_L, so q_11 = lambda and q_21 = 0; for q in H, q = [[alpha', beta], [-conj beta, conj alpha']] gives q = diag(lambda, conj lambda) = Q_lambda. The conjugate states give no condition: J nu_L and J nu_R both carry lambda, so the conjugate part of V_nu commutes with all of A_SM; and m is unconstrained. So a = (lambda, Q_lambda, m). ■

Proposition 44.2 (the light mass along the scaling). On the neutral block spanned by nu_L and nu_R with their conjugates, the spectrum of t T_R + t^alpha V_nu is plus and minus the roots mu_+- of mu^2 - t mu - t^{2 alpha} |y|^2 = 0 (the symmetric mass matrix [[0, s], [s, t]] of lemma 06, Proposition 6, with s = t^alpha y, whose characteristic polynomial on the four states is lambda^4 - (t^2 + 2|s|^2) lambda^2 + |s|^4), so the light mass is the singular value |mu_-| with

```
mu_-(t) = - 2 t^{2 alpha} |y|^2 / (t + sqrt(t^2 + 4 t^{2 alpha} |y|^2)) ~ - |y|^2 t^{2 alpha - 1}.
```

It tends to zero for alpha < 1/2, to the finite value -|y|^2 for alpha = 1/2 (the seesaw mass, the Schur complement -V_nu T_R^{-1} V_nu of the corpus), and diverges for alpha > 1/2. In one generation the other Yukawa blocks act on other species and leave the neutral block alone.

Proof. The 2 x 2 block [[0, s], [s, t]] has the displayed characteristic polynomial; the cancellation-free form of the small root is the corpus formula with m = t^alpha y (research_checks, section 2.6), and the three regimes are read off; the referee confirmed the eigenvalues at t = 10^4 for alpha = 1/4, 1/2, 3/4 against the asymptotics. ■

Corollary 44.3 (the seesaw and the algebra). Under the bounded-commutator criterion, a finite light neutrino mass at the heavy Majorana limit (alpha = 1/2) and the Standard Model algebra are obtained in two different limits of the same family: the algebra C + H + M_3(C) needs alpha = 0, along which the light mass is -|y|^2/t -> 0, and the finite mass needs alpha = 1/2, along which the retained algebra is C + M_3(C). The two coincide only at finite t, where no limit has been taken and, for a in A_SM, the commutators are bounded by t^{1/2} |y| up to a constant and the bounded term |||[S, rho(a)]||; physically that bound is the Dirac neutrino mass, of the order of the electroweak scale, so the statement concerns the strict limit and says that the seesaw limit at fixed light mass is a limit of the broken theory, in the sense of lemma 06's Proposition 5. Under the order-one criterion of Theorem 20.2 the Standard Model algebra is admissible at every alpha, and the corollary does not apply.

Remark 44.4 (nine dimensions). In dimension 8k + 1, hence in nine, a single Majorana field has a symmetric charge-conjugation bilinear and no Lorentz-invariant scalar mass; a Majorana mass needs an even number of real spinors or a background that breaks the nine-dimensional symmetry (Alvarez-Gaumé and Witten, Nucl. Phys. B 234 (1984) 269, section 4, as recorded in the corpus, connes_kk_research_state section 5.2). In the interval parent of lemma 22 every block of D_F, Yukawa and Majorana alike, is wall data, X = tan(l D_F), and that parent is five-dimensional; the nine-dimensional statement adds that in a nine-dimensional bulk the Majorana block could not be a bulk mass either, as lemma 20, item 1, already reads it (the connector is the Majorana block), and it agrees with lemma 06, Proposition 6, where a nonzero Majorana block obstructs the KO-5 reduction that carries the label 4 + 5. The scaling of Proposition 44.1 is then a scaling of the wall coupling, and Theorem 22.1 keeps the light pair in the window as long as l |D_F| < pi/2, which the growing Majorana entry violates first: the seesaw scale is the first datum that leaves the window.

Scope

One generation; the representation of lemma 06; the bounded-commutator criterion of the corpus for the retained algebra, which is the criterion behind the heavy projection of lemma 06 and agrees with that of Theorem 20.2 for T_R while differing from it for V_nu; the strict limit t -> infinity. Three generations replace y by the Dirac matrix Y_nu, and any nonzero entry of Y_nu forces q_11 = lambda and q_21 = 0, with the same conclusion. The lemma does not decide the sign s (lemma 06, Lemma 4, does: a generic Majorana block selects s = +1); it says which algebra the seesaw leaves. Direction: a seesaw that keeps the Standard Model algebra needs the Dirac coupling bounded and the light mass generated at a lower scale, or a second heavy neutral state pairing with nu_L, which lemma 06 lists as the escape from the KO-7-only conclusion.

Part 45

The Weyl crossed product and the two-sheet bimodule

For STRATEGY T16 (CORPUS_SALVAGE items 4 and 10), following lemma 16 (Theorem 16.7 and Corollary 16.8, the two-sheet bimodule) and the corpus constructions recorded as rs_N18 (the finite quotient of the Weyl crossed product) and rs_N6 (the 18-state orientable bimodule), historical/corpus/connes_kk_research_state.md, sections 6.5 and 6.13. Written by Claude Fable 5.1 on 4 November 2026, by hand; the crossed-product facts used are standard (evaluation at a fixed point, Green's imprimitivity for a finite orbit). Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): Proposition 45.1 and Theorem 45.2 correct with amendments (the word "complexification", the characters of Z_2, the intersection form of the complex algebra, the formal use of Corollary 16.8, the grading as a choice); two errors in Remark 45.3 corrected below (the number of irreducible representations; the route to H, which the corpus obtains at another orbit type).

Outcome. The two constructions the corpus left separate are one construction. The 18-state orientable KO-6 bimodule over C + M_2(C) + M_3(C) of section 6.5 is the two-sheet bimodule of Corollary 16.8 with the two sheets taken as the two orbit types of the Weyl group W = S_3 x Z_2 of SU(3) x SU(2) acting on T^6 = A x E: the fixed point, whose fibre of the crossed product is the group algebra and supplies C + M_2(C) and the grading, and a three-point orbit, whose fibre supplies M_3(C). So the first half of T16 is answered by what is already proved: the complex finite algebra with its real structure, a grading, order zero, order one and orientability come from the crossed product and the finite content of lemma 16. What remains open is the second half: the quaternionic real form H in place of M_2(C), which the corpus obtains from the binary dihedral lift Dic_3 at a different orbit type (Remark 45.3), and a Dirac operator from the geometry in place of the inserted D_v, for which the spin lift of the Weyl action and its obstruction rs_N20 return.

Setting

Let E be an elliptic curve, A = {(z_1, z_2, z_3) in E^3 : z_1 + z_2 + z_3 = 0} = E^2, and X = A x E = T^6, with W = S_3 x Z_2 acting by permutations of the z_i and by z -> -z on E; X/W = CP^2 x CP^1 (the corpus, section 2.5). Let B = C(X) x W be the crossed product, with the covariance relations u_g f u_g^{-1} = f o g^{-1}. Two orbit types enter: the fixed point x_0 = (0, 0) of W, and the three-point orbit O of x_1 = ((z, z, -2z), 0) with 3z != 0, whose stabilizer is H = S_2 x Z_2.

Statements

Proposition 45.1 (the two fibres of the crossed product). (i) Evaluation at the fixed point, f u_g -> f(x_0) u_g, is a surjective *-homomorphism B -> C*(W) = C^4 + M_2(C) + M_2(C). (ii) Restriction to the orbit O is a surjective *-homomorphism B -> C(O) x W = M_3(C*(H)) = M_3(C)^4, and composing with a character of H = S_2 x Z_2 gives B -> M_3(C). (iii) Selecting the trivial character and the standard two-dimensional representation of S_3 in (i), and one character in (ii), gives the surjection

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B -> (C + M_2(C)) + M_3(C),
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the complex algebra with M_2(C) in place of H (the quotient of the corpus, section 6.13; the complexification of the real A_F would be C^2 + M_2(C) + M_3(C)^2), with the fixed point carrying C + M_2(C) and the three-point orbit carrying M_3(C). Each selected representation of W = S_3 x Z_2 includes a character of the Z_2 factor.

Proof. (i) The evaluation map is well defined because W fixes x_0, so the covariance relation is preserved: u_g f u_g^{-1} -> f(g^{-1} x_0) = f(x_0). Its image is spanned by the u_g, hence is C*(W); for W = S_3 x Z_2, C*(S_3) = C + C + M_2(C) (the trivial, sign and standard representations) and C*(Z_2) = C + C, so C*(W) has six irreducible representations: four characters and the standard representation tensored with each character of Z_2. (ii) Restriction to the invariant closed set O is a *-homomorphism onto C(O) x W, and for a transitive action on a finite set G/H, Green's imprimitivity theorem gives C(G/H) x G = M_{|G/H|}(C*(H)) (here with finite groups it is the statement that the induced representation from H is a

Morita equivalence, concretely: functions on O act by diagonal matrices and u_g by permutation matrices twisted by H, and the algebra generated is M_3 (x) C*(H)); C(H) = C^4 for H = Z_2 x Z_2. A character of H gives the quotient M_3(C). (iii) is the direct sum of the two selected quotients; the two orbits are separated by an invariant function, so the sum map is surjective. ■

The corpus proved (iii) by a finite-matrix computation (rs_N18, generated dimension 1 + 4 + 9 = 14); the argument above is the structural one. The selection in (iii) is a choice among the six irreducible representations of C*(W) and the four characters of H; the others are discarded, and this lemma adds no criterion for the choice.

Theorem 45.2 (the two-sheet bimodule over the two orbits). Apply the finite content of Corollary 16.8 (Theorem 16.7 on finite even representations, which Remark 16.5 shows to be formal for any such pair) with the two selected representations of the crossed product as the two sheets, labelled by the two orbit types, and with

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V_X = C + C^2,  A_{F,X} = C + M_2(C) acting on it,  gamma_{V_X} = G = diag(1, -1, -1),  d_X = D_v =
↪  [[0, v^dagger], [v, 0]]  (v in C^2 nonzero);
V_Y = C^3,      A_{F,Y} = M_3(C),                          gamma_{V_Y} = 1,                    d_Y = 0.
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Then H_F = Hom(V_Y, V_X) + Hom(V_X, V_Y), of complex dimension 18, with J_F(x, y) = (y^dagger, x^dagger), Gamma_F(x, y) = (G x, -y G), and D_F(x, y) = (D_v x, y D_v), has the KO-6 signs J^2 = 1, J D = D J, J Gamma = -Gamma J, order zero, order one, nonzero one-forms, and the real orientability relation Gamma_F = pi(z c) c° with c = (G, 1) and z = (1, -1), since G lies in C + M_2(C) and 1 in M_3(C). This is, in Hilbert space, D, J and grading, the bimodule of the corpus, section 6.5 (rs_N6), stated there over the real algebra C + H + M_3(C). On minimal projections of the complex algebra C + M_2(C) + M_3(C) the intersection form is [[0, 0, 1], [0, 0, -1], [-1, 1, 0]], skew of rank 2 on a K_0 of rank 3, with kernel (1, 1, 0); the corpus form [[0, 0, 1], [0, 0, -2], [-1, 2, 0]] is the one of the real algebra, where the minimal projection of H is 1_H, of rank two (referee).

Proof. The data satisfy the hypotheses of Theorem 16.7 and Corollary 16.8: V_X is graded by G, which is even for the algebra C + M_2(C) (it is the central projection difference e_1 - e_2 of that algebra), D_v is odd for G and self-adjoint, V_Y is ungraded with d_Y = 0. Corollary 16.8 gives the signs, the order conditions and the orientability relation; its D_F is d_X (x) 1 + gamma (x) d_Y^c = D_v on the first factor, which is the operator of section 6.5 in the identification Hom(V_Y, V_X) = V_X (x) conj(V_Y). The grading G is a chosen central element: -G, or other sign assignments on the selected representations, are equally admissible. The intersection forms are computed on minimal projections as in Theorem 41.2. ■

Remark 45.3 (what the identification settles, and what it leaves). Settled: the complex finite algebra, its 18-state bimodule with the real structure, a grading that the fixed-point fibre allows, order zero, order one and orientability, all as the two-sheet construction of lemma 16 over the two orbit types of the Weyl action, with the sheets separated by an invariant function. The grading Gamma_F comes from the sheet sign z = (1, -1) and from the central element G of C + M_2(C); no spinor bundle on T^6 enters, so no spin lift of the Weyl action is used, and the obstruction rs_N20 (no equivariant ordinary spin lift at the transposition fixed point) is deferred to item (b), where a crossed-product Dirac operator on the spinors of T^6 needs W, or a cover of it, to act on the spinor bundle. Left open, in order: (a) the quaternionic real form H inside M_2(C). The corpus does not obtain it at the fixed point x_0 used here: at full fixed isotropy the lift of the inversion of E is central and complexifies the spin-sector algebra to C + C + M_2(C); the corpus obtains C + H at the orbit type (a, a, a), 3a = 0, with a point of E that is not 2-torsion (stabilizer S_3, orbit of size two), as a corner of M_2(C + H) (canonical section 10; referee). So the route to H needs a first sheet other than the one of Theorem 45.2, and the two halves of T16 as framed here use different first sheets; in either case Poincaré duality fails in the trace form at rank 2. (b) A Dirac operator from the geometry: D_v is inserted, and a crossed-product Dirac operator (the torus Dirac operator with the W-action on spinors) would have to be shown to compress to it on the selected orbit representations, which is the coupled-window problem of lemma 20, Result 7, in a new setting; the corpus (section 6.13) proves that no finite-rank projection gives a multiplicative compression of C(T^6), so the selected representations are quotients of the crossed product, as used here, and never spectral windows of multiplication operators. (c) The selection of the two representations in Proposition 45.1(iii) among the six irreducible representations of

C*(W) and the four characters of H.

Scope

The complex finite algebra and its bimodule; finite groups and finite orbits, where the crossed-product facts are elementary; the finite content of Corollary 16.8, applied to quotient representations rather than to spectral windows; the Dirac operator and the grading as inputs. The direction: T16 reduces to the questions (a), (b) and (c) of Remark 45.3, with (a) pointing to the orbit type (a, a, a) as the first sheet and (b) of the same kind as the connector problem of lemmas 20 and 42, now with a discrete group in place of the twistor fibration.

Part 46

The odd blocks of the quaternionic sheet: the bidoublet, its quaternionic half as the direction of the axis, and the up-down splitting

For STRATEGY T14, item (ii), and T0, following lemma 42 (Proposition 42.1 and the escape its referee recorded: Hom(W, V) = 1 + 5 + 10 contains T_p S^4 twice), lemma 16 (Example 16.9), lemma 10 (Proposition 10.1), lemma 28 (Propositions 28.19 and 28.20), lemma 29 (Proposition 29.8) and lemma 20 (Theorem 20.2). Written by Claude Fable 5.1 on 5 November 2026, by hand; the Clifford relations, the quaternionic structure, the commutant dimensions 16, 8 and 16, the K_p-invariance of the two four-dimensional spans, the spectrum of Proposition 46.3 and the mass formula of Proposition 46.2 were checked with numpy on random data (script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_46.md), who redid every computation (the Clifford structure and j, the dimensions, the convention of L_q, the mass formula on 2000 random pairs, the stabilizer dimensions, a Monte Carlo compression, the window spectrum): Propositions 46.1, 46.2 and Corollary 46.4 correct with amendments; Proposition 46.3 had an error in its hypothesis (a bounded operator in place of a bounded multiplication operator, with the counterexample B = v u^*); the Outcome and Reading had an error in the chain of stabilizers; one sentence identified the rank-one block with the top-only family of lemma 21, which the shared lepton block forbids; the escape by complex functions on the base was missing. All applied below, with the referee's sharper forms (the fibre-odd functions on CP^3, the determinant form of the mass formula, the stabilizers of the block) and the prior work it found.

Outcome. On the quaternionic sheet W = H^2 of the twistor window, with the grading axis p in S^4 of Proposition 28.19 and Example 16.9, the Hermitian blocks odd for the grading are Hom_C(L_p, L_p^perp), the complex bidoublet (2, 2) (x) C of the Pati-Salam pair Sp(1)_a x Sp(1)_b = SU(2)_L x SU(2)_R (Proposition 46.1). Its quaternionic half, the j-linear maps Hom_H(L_p, L_p^perp) = T_p S^4, is the span of the Gamma_a with a orthogonal to p, that is the odd part of the compression of the coordinates of S^4 (Proposition 10.1); its j-antilinear half, the span of the i Gamma_a Gamma_p, lies outside the real algebra M_2(H) of the base functions and inside the compression of the functions on CP^3. A Yukawa block X = q + i q' on a doublet (q, q' quaternions) gives the two members of the doublet the masses m_+-^2 = |q|^2 + |q'|^2 +- 2 |Im(conj(q) q')| (Proposition 46.2): equal masses exactly when X = z q is quaternionic up to a complex factor, as every block compressed from a real function on S^4 is; one massless member exactly when X is of rank one, with the two halves of equal norm; tensored with 1_V such a block gives the up-type quark and the Dirac neutrino of the generation the mass m_+ and the down-type quark and the charged lepton the mass zero. When the masses are split, the commutant of X^* X in Sp(1)_b is a circle: the up-down splitting fixes a maximal torus of SU(2)_R, which is the hypercharge circle of the connector of lemma 20 exactly when the connector's line is one of its eigenlines, and that eigenline condition is exactly the condition for the Yukawa block and the connector to leave a common circle unbroken (Corollary 46.4). In the window of lemma 29 all cross blocks between the core and the wall states are of the exponential order of lambda_M, and the cross block of a coordinate of S^4 is c_M Gamma_a, so the G-breaking blocks of the window operator that carry T_p S^4 are at most of the order of the bound on lambda_M, and their nonvanishing is not shown (Proposition 46.3). The reading: in Sp(2) = U(M_2(H)) the axis Gamma_p has stabilizer K_p and reduces M_2(H) + M_4(C) to A_PS (Example 16.9); the connector's line has stabilizer

Sp(1)_a x U(1)_b in K_p and the connector reduces A_PS to C + H + M_3(C) (Theorem 20.2); the Yukawa block has stabilizer the diagonal Sp(1) of K_p, the custodial SU(2)_{L+R}, when its masses are equal and a circle when they are split, a circle inside Sp(1)_a x U(1)_b exactly when the connector's line is an eigenline of X^* X; the Higgs doublet is the tangent of the orbit S^4 of the axis (lemma 42), and the up-down splitting, maximal for the top, is of the representation type of the compression of the real functions on CP^3 odd under the fibre involution j, which no real function on the base supplies through the odd part of its compression. With complex functions on the base and the projections of the grading the full bidoublet is reached: P_{L_p^perp} Q(x_a + i x_b) P_{L_p}, for a, b orthogonal to p and a != b, has rank one, as complex generation on the S^4 side gives M_4(C) (Example 16.9).

Setting

Let W = C^4 with the Hermitian Clifford generators Gamma_1, …, Gamma_5 of Cl(5), Gamma_a Gamma_b + Gamma_b Gamma_a = 2 delta_ab, and the antiunitary j with j^2 = -1 commuting with every Gamma_a (it exists since Gamma_1 … Gamma_5 = +-1 is real, so the conjugate representation is equivalent through a unitary C with C conj(C) = -1, the Gamma_a generating the real algebra M_2(H), and it is unique up to a phase, which does not affect Y -> j Y j^{-1}; referee), so that W = H^2 (Example 16.9: W is the kernel H^0(CP^3, O(1)) with the quaternionic structure of the twistor space, and Proposition 10.1 gives the compression Q(x_a) = Gamma_a/5 of the coordinates of S^4). The real algebra End_H(W) = M_2(H) is the real span of 1, the Gamma_a and the Gamma_a Gamma_b (a < b), of dimension 1 + 5 + 10 = 16; its unitary group is G = Sp(2), acting on W by the 4; under G the Gamma_a span the 5 (the traceless Hermitian elements of M_2(H)) and the Gamma_a Gamma_b the 10 = sp(2) (the anti-Hermitian elements). For p in S^4 put Gamma_p = p^a Gamma_a; Gamma_p^2 = 1, its eigenspaces L_p (+1) and L_p^perp (-1) are j-invariant, hence quaternionic lines, and p -> L_p identifies S^4 with HP^1. The stabilizer of p is K_p = Sp(1)_a x Sp(1)_b with Sp(1)_b = Sp(L_p) and Sp(1)_a = Sp(L_p^perp), in the notation of lemma 42, so that the bidoublet (2, 2) of K_p is Hom_H(L_p, L_p^perp) = T_p S^4 (the corpus note, section 6, and Proposition 42.2).

The second sheet V = C^4 carries the compression of the functions on CP^3, which generate M_4(C) (Example 16.9, item 2), and is identified with W as a G-representation (Proposition 28.19). The bimodule H_F = Hom(V, W) + Hom(W, V) = W (x) conj(V) + V (x) conj(W) is the sixteen states of one generation with their conjugates: W the two doublets of a lepton or a quark, conj(V) the lepton line and the three colours (Example 16.9). The axis p is the quaternionic grading of Proposition 28.19 and Example 16.9: Gamma_F acts as Gamma_p on the W factor, and the even algebra is H + H + M_4(C) = A_PS, acting on L_p, L_p^perp and V, the Pati-Salam algebra of lemma 20, with H_R on L_p and H_L on L_p^perp (the corpus note, section 5). A sheet-diagonal Dirac operator of Corollary 16.8 acts on H_F by a Hermitian block Y on the W factor, odd for Gamma_p, tensored with the identity of V: the Yukawa block, the same for the lepton and the three colours. With V ungraded the parent operator on V vanishes, so the sheet-diagonal operator on Hom(V, W) is Y (x) 1_V, with eigenvalues +-m_+ and +-m_-, each of multiplicity four, and the same on Hom(W, V); the doubling of lemma 16, Caveat 2, which would give (m_i^2 + m_j^2)^{1/2}, does not occur (referee).

Statements

Proposition 46.1 (the odd Hermitian blocks for the axis p). Let Odd_p = {Y in End_C(W) : Y^* = Y, Gamma_p Y = -Y Gamma_p}. Then:

1. Odd_p = {[[0, X^*], [X, 0]] : X in Hom_C(L_p, L_p^perp)}, of real dimension 8; as a representation of K_p it is Hom_C(L_p, L_p^perp) = (1, 2)^* (x) (2, 1) = (2, 2) (x)_R C, two real copies of the bidoublet.
2. The j-linear part Odd_p cap M_2(H) is the real span of the Gamma_a with a orthogonal to p, the (2, 2) inside the 5: the maps with X in Hom_H(L_p, L_p^perp) = T_p S^4. By Proposition 10.1 it is the odd part of the compression of the coordinates of S^4, Gamma_a = 5 Q(x_a).
3. The j-antilinear part, {Y in Odd_p : j Y j^{-1} = -Y}, is the real span of the i Gamma_a Gamma_p with a orthogonal to p, which is i times the (2, 2) inside the 10 = sp(2). It lies outside M_2(H), hence outside the compression of the real functions of S^4, and it is the compression of the real functions

on CP^3 that are odd under the fibre involution [w] -> [j w]: since j P_w j^{-1} = P_{jw} and the involution preserves the Fubini-Study measure, Q(f o j) = j Q(f) j^{-1}, so the j-odd real functions compress to j-antilinear Hermitian matrices and the j-even ones to M_2(H); for example i Gamma_a Gamma_p = Q(5 w^* i Gamma_a Gamma_p w / w^* w) (referee).

4. Under G, End_C(W) = 1 + 5 + 10, and under K_p: 1 -> (1, 1); 5 -> (1, 1) + (2, 2), spanned by Gamma_p and by the Gamma_a with a orthogonal to p; 10 -> (3, 1) + (1, 3) + (2, 2), spanned by the Gamma_a Gamma_b with a, b orthogonal to p, which form sp(1)_a + sp(1)_b, and by the Gamma_a Gamma_p. The even part End(L_p) + End(L_p^perp) is the complex span of 1, Gamma_p and the Gamma_a Gamma_b with a, b orthogonal to p; the odd part is the complex span of the two (2, 2).

Proof. (1) A Hermitian Y anticommuting with Gamma_p exchanges its eigenspaces, so Y = [[0, X^*], [X, 0]] with X : L_p -> L_p^perp arbitrary, and conversely; the real dimension is 2 dim_C Hom(C^2, C^2) = 8. Under K_p, Hom_C(L_p, L_p^perp) = L_p^* (x) L_p^perp = (1, 2) (x) (2, 1) = (2, 2) as a complex representation, the complexification of the real bidoublet. (2) j commutes with every Gamma_a, so Gamma_a lies in M_2(H); for a orthogonal to p, Gamma_a is Hermitian and anticommutes with Gamma_p; the four such Gamma_a are linearly independent, so they span a four-dimensional real subspace of the j-linear part of Odd_p, which is {[[0, X^*], [X, 0]] : X in Hom_H(L_p, L_p^perp)}, of real dimension 4, since the H-linear maps between two quaternionic lines form a copy of H. So the two spaces coincide, and Proposition 10.1 identifies them with the odd parts of the compressed coordinates. (3) Y -> j Y j^{-1} is a real-linear involution of Odd_p: (j Y j^{-1})^* = j Y^* j^{-1} for antiunitary j, and j commutes with Gamma_p. So Odd_p is the direct sum of its j-linear and j-antilinear parts, and the latter has real dimension 8 - 4 = 4. For a orthogonal to p, i Gamma_a Gamma_p is Hermitian, since (i Gamma_a Gamma_p)^* = -i Gamma_p Gamma_a = i Gamma_a Gamma_p; it anticommutes with Gamma_p, since Gamma_p Gamma_a Gamma_p = -Gamma_a; and it is j-antilinear, since j commutes with Gamma_a Gamma_p and anticommutes with i. The four are linearly independent, so they span the j-antilinear part. A nonzero j-antilinear element is outside End_H(W) = M_2(H), the real algebra generated by the compressed functions of S^4 (Example 16.9, item 1). For a real function f on CP^3, the covariance j P_w j^{-1} = P_{jw} and the invariance of the Fubini-Study measure under [w] -> [jw] give Q(f o j) = j Q(f) j^{-1}, so f odd under the involution has j Q(f) j^{-1} = -Q(f), and the formula of Example 16.9, item 2, applied to f = 5 w^* B w / w^* w with B = i Gamma_a Gamma_p, which is odd, gives Q(f) = B. (4) The 4 of Sp(2) = Spin(5) is symplectic, so End_C(W) = 4 (x) 4 = Lambda^2(4) + S^2(4) = (1 + 5) + 10, with the 5 the vector of SO(5) spanned by the Gamma_a and the 10 the adjoint spanned by the Gamma_a Gamma_b. Under K_p = Spin(4) the vector of SO(5) is 1 + (2, 2), the trivial summand being Gamma_p, and so(5) = so(4) + R^4 is (3, 1) + (1, 3) + (2, 2), the last spanned by the Gamma_a Gamma_p, which transform as the vector; Gamma_p commutes with 1, Gamma_p and the Gamma_a Gamma_b with a, b orthogonal to p and anticommutes with the Gamma_a and the Gamma_a Gamma_p with a orthogonal to p, which gives the even and odd parts, of complex dimensions 1 + 1 + 3 + 3 = 8 and 4 + 4 = 8. ■

Proposition 46.2 (the masses of a doublet). Identify L_p and L_p^perp with H = C^2 through j, so that an element X of Hom_C(L_p, L_p^perp) is X = L_q + i L_{q'} with q, q' in H acting by left multiplication, L_q the j-linear and i L_{q'} the j-antilinear part. Then the two singular values of X, the masses of the two members of the doublet under the Yukawa block [[0, X^*], [X, 0]] (x) 1_V, are

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m_+-^2 = |q|^2 + |q'|^2 +- 2 |Im(conj(q) q')|,
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with Im the imaginary part of the quaternion conj(q) q'. Consequently:

1. m_+ = m_- exactly when conj(q) q' is real, that is when X = z L_q for a complex number z and a quaternion q. In particular the odd part of the compression of any real function on S^4 is j-linear, q' = 0, and gives equal masses to the two members of every doublet.
2. m_- = 0 with m_+ > 0 exactly when q' = q u for a unit imaginary quaternion u (equivalently q' = u' q with u' = q u q^{-1}), that is X = L_q (1 + i L_u) = 2 L_q P_u with P_u = (1 + i L_u)/2 a projection of rank one: the pattern in which one member of the doublet is massless and the other is not is a block of rank one whose j-linear and j-antilinear halves have equal norm, in the Hilbert-Schmidt and in the operator norm. Tensored with 1_V it gives the up-type quark and the Dirac neutrino of the generation the mass m_+ and the down-type quark and the charged lepton the mass zero; the

top-only family of lemma 21, with the leptons massless, needs a lepton block different from the quark block, which a sheet-diagonal operator of Corollary 16.8 with V ungraded does not provide (referee).

3. For q, q' nonzero, with |q'| = t |q| and the quaternionic angle phi in [0, pi] given by Re(conj(q) q') = |q| |q'| cos phi, m_+-^2 = |q|^2 (1 + t^2 +- 2 t sin phi): the ratio of the two masses depends on the relative size of the two halves and on the angle between them, and on nothing else.

Proof. Write H = C + j C with C acting on the right, x = x_1 + j x_2. For q = a + j b with a, b in C, left multiplication is the matrix [[a, -conj(b)], [b, conj(a)]]; q -> L_q is a homomorphism of real algebras with L_{conj(q)} = L_q^*; L_q commutes with j (x_1, x_2) = (-conj(x_2), conj(x_1)), the right multiplication by j, and i L_{q'} anticommutes with it; the L_q and the i L_{q'} together span the eight real dimensions of M_2(C). Then X^* X = (L_{conj(q)} - i L_{conj(q')}) (L_q + i L_{q'}) = |q|^2 + |q'|^2 + i (L_{conj(q) q'} - L_{conj(q') q}) = |q|^2 + |q'|^2 + 2 i L_w, with w = Im(conj(q) q'), since conj(q') q = conj(conj(q) q') and the real parts cancel. Left multiplication by an imaginary quaternion w is a traceless anti-Hermitian matrix with eigenvalues +- i |w|, so 2 i L_w has eigenvalues -+ 2 |w| and the eigenvalues of X^* X are as displayed. Equivalently, det X = |q|^2 - |q'|^2 + 2 i q . q' with q . q' = Re(conj(q) q'), the complex-bilinear norm of q + i q' in H (x) C, so m_+ m_- = |det X| and m_+^2 + m_-^2 = 2 (|q|^2 + |q'|^2) (referee). (1) The masses coincide exactly when w = 0. If q = 0, X = i L_{q'}. If q != 0, conj(q) q' = s real gives q' = (s/|q|^2) q and X = (1 + i s/|q|^2) L_q; conversely X = z L_q has X^* X = |z|^2 |q|^2. The compression of a real function on S^4 is a Hermitian element of M_2(H) (Example 16.9, item 1), and Gamma_p is j-linear, so the odd part is j-linear. (2) m_- = 0 exactly when |q|^2 + |q'|^2 = 2 |w|; since |w| <= |conj(q) q'| = |q| |q'| <= (|q|^2 + |q'|^2)/2, this forces |q| = |q'| and Re(conj(q) q') = 0, so conj(q) q' = w with |w| = |q|^2 and q' = q w/|q|^2 = q u with u = w/|q|^2 a unit imaginary quaternion; then X = L_q + i L_q L_u = L_q (1 + i L_u), and i L_u is Hermitian with L_u^2 = -1, so (1 + i L_u)/2 is a projection of rank one. Conversely such an X has rank one. (3) Substitution, with |w|^2 = |q|^2 |q'|^2 - Re(conj(q) q')^2. ■

Prior work (found by the referee). The decomposition of the complex bidoublet Yukawa into its quaternionic and anti-quaternionic halves, and the up-down splitting by the second, are in print in bidoublet form: in the spectral Pati-Salam model of Chamseddine, Connes and van Suijlekom (arXiv:1304.8050, JHEP 11 (2013) 132, the section on the Dirac operator and inner fluctuations on H_R + H_L + M_4(C)) the right-left block after fluctuation is (k^u phi + k^d phi-tilde) (x) 1 in the unified case k^nu = k^u, k^e = k^d, with phi-tilde = sigma_2 conj(phi) sigma_2 = j phi j^{-1}, so that the two halves are (k^u + k^d)(phi + phi-tilde)/2 and (k^u - k^d)(phi - phi-tilde)/2; the general bidoublet mass matrices M_U = kappa h + kappa' e^{-i alpha} h-tilde, M_D = kappa' e^{i alpha} h + kappa h-tilde are those of the left-right symmetric models (Zhang, An, Ji and Mohapatra, arXiv:0712.4218, section III; Mohapatra and Senjanovic 1981); and the statement that the quaternionic block has a custodial SU(2)_{L+R} with m_t = m_b is the custodial-symmetry literature (Willenbrock, arXiv:hep-ph/0410370, section 3 and Exercise 3.5; Sikivie, Susskind, Voloshin and Zakharov 1980; Solberg, arXiv:1207.5194; Haber and O'Neil, arXiv:1011.6188). Proposition 46.2 is a restatement of these formulas in the variables (q, q') of the window; the closed formula in those variables, the identification of the second half with the fibre-odd functions on CP^3 and the stabilizer reading of Corollary 46.4 were not found in print. The identifiers are in paper/LITERATURE.md, section M.

Proposition 46.3 (cross blocks of the window operator). In the window W_M + V_M of Theorem 29.7, with u a unit core state and v a unit wall state, let B be the multiplication by a bounded measurable endomorphism field b of E, with ||B|| = ess sup |b| (for a general bounded operator the bound fails: B = v u^* has norm one and <v, B u> = 1; referee). Under the weighted estimates of the proof of Proposition 29.8 (with the cut-offs of its second reading), for every eta in (0, 1) there are M_eta and C_eta with

```
|<v, B u>| <= C_eta ||B|| exp(-(1 - eta)(M/2) int_e^{l - a} f dr),    M >= M_eta.
```

So every off-diagonal block of the compression of a bounded multiplication operator to the eight states is bounded by the exponential that bounds lambda_M in Proposition 29.8; no lower bound on either is proved. In the doubled operator, with W_M in the first copy of L^2(X, E) and V_M in the second, P_V B P_W is the cross block of the odd perturbation T_M -> T_M + B, the first-order term of the window block of D-hat for T_M + B; the higher-order terms of the reduction, of the form P_V B Q (Q D-hat Q - z)^{-1} Q B P_W, also tunnel through the bulk and are not estimated here, and the window stays isolated only for ||B|| below the gap sqrt 3 (referee). For B the multiplication by a coordinate x_a of S^4 pulled back

to X, the cross block is P_V x_a P_W = c_M Gamma_a under the identification W_M = V_M = 4 of Proposition 28.19, with |c_M| bounded as above; a window operator with off-diagonal block d = lambda + mu n^a Gamma_a (lambda >= 0 after a phase, mu complex, n a unit vector) has the eight eigenvalues +-(lambda^2 + |mu|^2 +- 2 lambda Re(mu))^{1/2}, that is +-|lambda + mu| on L_n and +-|lambda - mu| on L_n^perp, each with multiplicity two, on the quaternionic lines of the two copies of the 4; when lambda Re(mu) = 0 they coincide with multiplicity four.

Proof. The argument of the proof of Proposition 29.8, run with eta' = eta/2 and with its point r_* moved to the outer end of the region where the bulk cut-off is one, gives weights g, with g = 0 on r <= e', g' = (1 - eta')(M/2) f on [e', l - a'] and g constant beyond, and g-tilde, with g-tilde = 0 on r >= l - a', -g-tilde' = (1 - eta')(M/2) f on [e', l - a'] and g-tilde constant below, where e < e' and a < a' are as close to e and a as we wish and the cut-off transitions lie in [e, e'] and [l - a', l - a]. Then ||exp(g) u||*{L^2(r >= e')} <= C_eta, through the collar by Lemma 29.6 (1), and ||exp(g-tilde) v||*{L^2(r <= l - a')} <= C_eta, through the core by Lemma 29.3, where T_M^* has no low states; both hold for every unit vector of W_M and of V_M, since T_M^* T_M and T_M T_M^* equal lambda_M^2 there (Proposition 28.19). With I' = int_{e'}^{l - a'} f, choose e', a' with (1 - eta') I' >= (1 - eta) I, I = int_e^{l - a} f. On e' <= r <= l - a', g(r) + g-tilde(r) = (1 - eta')(M/2) I', a constant, so the integral of |u| |v| over that region is at most exp(-(1 - eta')(M/2) I') ||exp(g) u|| ||exp(g-tilde) v||; on r < e' the weight g-tilde equals (1 - eta')(M/2) I' and ||u|| = 1, and on r > l - a' the weight g equals the same and ||v|| = 1, so the two end regions contribute the same exponential; hence |<v, B u>| <= ||B|| int |u| |v| <= 3 C_eta^2 ||B|| exp(-(1 - eta')(M/2) I') <= 3 C_eta^2 ||B|| exp(-(1 - eta)(M/2) I) for C_eta >= 1, since B is a multiplication operator. The window projections P_W and P_V are G-equivariant, since T_M is, and the coordinates x_a span the 5; so a -> P_V x_a P_W is a G-map from the 5 to Hom(W_M, V_M) = End_C(4) = 1 + 5 + 10, and by Schur's lemma it is a multiple c_M of the unique equivariant embedding, a -> Gamma_a, with c_M complex (the Gamma_a transform under the same covering Sp(2) -> SO(5) as the x_a, up to the sign +-1 of O(5), which c_M absorbs); rephasing the identifications U_W, U_V of Proposition 28.19 multiplies lambda_M and c_M by the same phase, so arg(c_M conj(lambda_M)) is an invariant, the phase that enters Re(mu) once lambda_M >= 0; c_M != 0 is not shown (referee). For the spectrum, d^* d = (lambda + conj(mu) n . Gamma)(lambda + mu n . Gamma) = lambda^2 + |mu|^2 + 2 lambda Re(mu) n . Gamma, and n . Gamma has the eigenvalues +-1 on the quaternionic lines L_n and L_n^perp; d d^* has the same spectrum. ■

Corollary 46.4 (the hypercharge direction and the connector's line). Let X be the Yukawa block of a doublet with m_+ != m_-. The elements of Sp(1)_b = Sp(L_p) commuting with X^* X form a circle U(1)_X, the maximal torus of Sp(1)_b whose weight lines are the two mass eigenlines of X^* X in L_p; it is a maximal torus of the kind lemma 42 marked as an input, and it equals the U(1)_b of lemma 42, the stabilizer in Sp(1)_b of the connector's line Cw, exactly when Cw is one of its weight lines; under it T_p S^4 = (2, 2) restricts to (2)_{+-1}, one complex doublet of hypercharge 1/2, and the twistor fibre K_p/(Sp(1)_a x U(1)_b) = S^2 is the set of complex lines in L_p. For the connector of lemma 20 (Theorem 20.2), with the axis p taken as the point of S^4 under the quaternionic line of its marked line w, so that w lies in L_p = W_+ and carries the Majorana block, the Majorana state is a Dirac-mass eigenstate exactly when w is one of the two eigenlines of X^* X, that is a fixed point of U(1)_X on the fibre S^2. The stabilizer of X in K_p is the circle {(X b X^{-1}, b) : b in U(1)_X} for invertible X with m_+ != m_-, again a circle for a rank-one X, and the diagonal {(q b q^{-1}, b)} = Sp(1), the custodial SU(2)_{L+R}, for X = z L_q; the stabilizer of X and of the connector's line together contains a circle exactly when that line is an eigenline of X^* X, so the eigenline condition is exactly the condition for the Yukawa block and the connector to leave a common circle unbroken, the analogue of the electromagnetic U(1) (referee). For a j-linear Yukawa block, m_+ = m_-, every circle in Sp(1)_b commutes with X^* X and every line is an eigenline, so neither the torus nor the position of w on the fibre is determined by the masses.

Proof. X^* X is a Hermitian operator on L_p = C^2 with two distinct eigenvalues, so its commutant in U(L_p) = U(2) is the torus of its eigenbasis, whose intersection with Sp(1)_b = SU(2) is the circle U(1)_b with those weight lines; Proposition 42.2 gives the restriction of (2, 2) and the fibre. The Majorana block of lemma 20 sits on the state w (x) conj(v), on which the Dirac block acts as X^* X (x) 1 on the right-handed factor, so it is a mass eigenstate exactly when w is invariant under X^* X. For the stabilizers: (a, b) in K_p fixes X exactly when a X = X b, so b commutes with X^* X and a = X b X^{-1}; for split masses b runs

over U(1)_X and a lies in SU(L_p^perp), while for X = z L_q the condition is a = q b q^{-1} for every b in Sp(1); the V-phase does not enter, since the stabilizer of T_kappa contains the diagonal circle of Theorem 20.2 and X (x) 1_V is invariant under every unitary of V. The last statement is Proposition 46.2 (1). ■

Reading

The finite data of the twistor window carry three blocks, and their stabilizers in G = Sp(2), the unitary group of the base algebra M_2(H), are as follows:

1. the axis Gamma_p, a Hermitian involution in the 5, with stabilizer K_p = Sp(1)_a x Sp(1)_b: it reduces M_2(H) + M_4(C) to its commutant H + H + M_4(C) = A_PS, and its orbit is S^4 = G/K_p, whose tangent at p is the bidoublet (2, 2) (Example 16.9; Proposition 46.1);
2. the Yukawa block Y, odd for Gamma_p, in Hom_C(L_p, L_p^perp) = (2, 2) (x) C: its j-linear half is the direction of S^4 at p, the odd part of the compressed base coordinates; its j-antilinear half is the part no real function on the base supplies through the odd part of its compression (complex functions with the projections of the grading supply it), which splits the two masses of a doublet and thereby fixes a maximal torus of Sp(1)_b, the U(1)_b of the connector when the connector's line is one of its weight lines; the stabilizer of the block itself is the diagonal Sp(1) of K_p for equal masses and a circle for split ones (Propositions 46.1 and 46.2, Corollary 46.4);
3. the connector of lemma 20, the rank-one block marking a complex line w in L_p and a line v in V, with stabilizer Sp(1)_a x U(1)_b in K_p: it reduces A_PS to C + H + M_3(C) (Theorem 20.2), the orbit of its line Cw is CP^3 = G/H (Proposition 42.2), and the orbits of the connector have the moduli of Proposition 42.3.

The Higgs doublet of the corpus note is the tangent of the first orbit at the second block's foot: the variation of the axis, T_p S^4 = (2, 2) restricted by the U(1)_b of Corollary 46.4 to one complex doublet of hypercharge 1/2 (lemma 42). The heavy top of lemma 21 is the extreme case of the second block, Proposition 46.2 (2): the third generation, whose block in this scope is shared by the quark and the lepton doublets, has a Yukawa block close to rank one; with r = m_b/m_t, Proposition 46.2 gives ((1 - t^2)/(1 + t^2))^2 + (2 t cos phi/(1 + t^2))^2 = (2r/(1 + r^2))^2, so the two halves are equal in norm and in quadrature up to terms of order r, about 0.02 at the top scale; the odd part of the compression of a real function on the base would give m_t = m_b. Every G-breaking cross block of the window operator from a bounded multiplication operator, in the 5 or in the 10, is bounded by the exponential that bounds lambda_M (Proposition 46.3), and the block of a coordinate of S^4 is c_M Gamma_a, with c_M not shown to be nonzero; the Yukawa block of the bimodule is not a cross block of the window (Proposition 28.20 shows that the doubled window operator has zero compression to the bimodule, P_- DD P_- = 0): the two sheets supply the representation types, the quaternionic half from the real functions on S^4 and the full bidoublet from those on CP^3, and the block itself is an input of Corollary 16.8.

Scope

Finite linear algebra on the quaternionic sheet W = H^2 with the Clifford structure of Example 16.9; one generation, the Yukawa block tensored with the identity of V (the same block for the lepton and the three colours, as in a sheet-diagonal operator of Corollary 16.8); the grading axis as an input (Proposition 28.19); for Proposition 46.3, the window of Theorem 29.7 with the weighted estimates of Proposition 29.8. Three generations replace q and q' by 3 x 3 quaternionic matrices; if q^* q' is quaternion-Hermitian, X^* X is j-linear and each singular value has even multiplicity, the analogue of Proposition 46.2 (1) in one direction; the closed formula and the rank statement have no direct analogue and are not proved here (referee). Nothing here derives the Yukawa block, the Majorana line or a potential; the j-antilinear half is identified by representation type with the compression of functions on CP^3, not derived from one. The direction the result points to: a parent in which the Yukawa block is the compression of a function on the wall sheet, odd under the fibre involution, so that the up-down splitting, and with it the top's mass, is the fibre dependence of that function, against the alternative of complex functions on the base with the projections of the grading, which reach the same bidoublet and should be compared with it before either is adopted; and a potential that drives the block of the third generation to rank one. Leads from the referee: (1) a rank-one block X = lambda mu^* marks the line C mu in P(L_p) = S^2, a point of the twistor fibre over p, the same kind of datum as

the connector's line, and in the C-action of Theorem 20.2 the states on the connector's line are the up-type right-handed ones, so with the eigenline condition and the up-type assignment of the top, C mu = Cw: one marked point of CP^3 over p would carry both the Majorana line and the third-generation Yukawa block, a concrete target for a parent and for T12; (2) the custodial breaking is measured by |Im(conj(q) q')| = (m_+^2 - m_-^2)/4, a row for the dictionary; (3) by Theorem 29.9 and Proposition 28.8 the core compression of a real function on X has a j-antilinear half of norm at most C ||a||*{C^1} M^{-1/2}, so a block taken from a core compression has a bounded relative splitting unless its j-linear half is of the same order, and a heavy top at large M needs the wall sheet or the finite-M terms, whose j-breaking the numerics of lemma 28 showed; (4) at mu = lambda the window block of Proposition 46.3 is 2 lambda P*{L_n} with exact zero modes on the two copies of L_n^perp, so a G-breaking block in the 5 marks the quaternionic line that Proposition 28.19 leaves to be chosen, a lead for the axis as an input, with no dynamics behind it here; (5) the CKM matrix in the variables (q, q') with q^* q' quaternion-Hermitian as the custodial case.

Part 47

The Killing-form normalization on the Berger five-sphere and Connes' matching value

For STRATEGY T18 (the normalization test of the object at 9) and the parked job 04 (lemmas/README, the two Berger formulas), following lemma 03 (cos^2 theta_W = 5/8 at matching with the full module light; in its heavy-up family the photon and gluon coefficients keep their ratio) and the lead recorded from GitHub issue 1 of the repository (the round S^5 gives e^2/g_s^2 = 3/8, computed there by Schur's lemma; read by Claude Fable 5.1 on 5 November 2026). Written by Claude Fable 5.1 on 5 November 2026, by hand; the two sphere moments, the formula at eta = 1, 2, 1/2 and the charges of the coordinates in the exterior square of the spinor were checked with numpy (Monte Carlo with 2 x 10^6 points for the moments; script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_47_48.md), who redid the moments, the Berger form at four values of eta, the branchings and the traces: Proposition 47.1 correct; Corollary 47.2 and Remark 47.3 correct with amendments, applied below (the citation of the coupling formula to Weinberg 1983; the proof that the isometry group enlarges only at the round point; the reading of lemma 03; the scope of the eta-curve as a statement about the Einstein term alone, the round-point value being independent of any SO(6)-invariant source by Schur's lemma); the round-point value is implicit in print in holographic form, the charge unit being the superconformal R-charge (Barnes, Gorbatov, Intriligator and Wright 2005).

Outcome. On the Berger five-sphere, the round S^5 in C^3 with the Hopf fibre scaled by eta, the Kaluza-Klein kinetic form of the Killing vectors gives, for the SU(3) acting on C^3 and the central U(1) with charge c on the coordinates,

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e^2 / g_s^2 = (3 + eta^2) / (24 eta^2 c^2)    (Corollary 47.2).
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The two formulas of the parked job 04, 3(eta^2 + 3)/(8 eta^2) and 3(eta^2 + 3)/(32 eta^2), are the two charge units c = 1/3 and c = 2/3 on the coordinates; the second is the unit fixed by the spinor of SO(6) = SU(4) at the round point, where the SU(3)-singlet Killing spinor has charge 1 and the triplet -1/3, so that the coordinates, in the exterior square of the spinor, carry 2/3. With that unit the round point eta = 1 gives e^2/g_s^2 = 3/8, the value of Connes' finite triple at matching (sin^2 theta_W = 3/8 with g_2 = g_3, the 5/8 of lemma 03 being cos^2 theta_W). So the normalization test of T18 distinguishes the finite triple from the squashed spheres, and the round sphere, the one point of the Berger family where the isometry enlarges from U(3) to SO(6), gives the finite triple's value; at that point the value is independent of whatever holds the shape, since every SO(6)-invariant contribution to the gauge kinetic terms is proportional to the trace form by Schur's lemma. In the Einstein-term reduction the ratio decreases from infinity at eta -> 0 through 3/8 at eta = 1 to 3/32 at eta -> infinity, taking each value once; off the round point the curve depends on the source that supports the non-Einstein shape.

Setting

Let S^5 be the unit sphere in C^3, with the Hopf fibration S^5 -> CP^2 whose fibre at z is the circle e^{i alpha} z, with unit vertical vector i z. The Berger metric g_eta is the round metric on the horizontal space and eta^2 times the round metric on the vertical direction; its volume is V_eta = eta V_1. The group U(3) acts isometrically on (S^5, g_eta) for every eta > 0, through SU(3) and the central circle z -> e^{i alpha} z, since U(3) preserves the round metric and the field i z. At eta = 1 the isometry group is O(6), with identity component SO(6), whose Lie algebra so(6) = su(4) contains u(3); for eta != 1 it is U(3) x| Z_2, the Z_2 generated by complex conjugation, with identity component U(3): by O'Neill's formulas for the canonical variation of the Hopf fibration over CP^2 (Einstein constant 6), the Ricci tensor of g_eta has the eigenvalue 4 eta^2 on the unit vertical vector and 6 - 2 eta^2 on unit horizontal vectors, which differ unless eta = 1, so an isometry preserves the vertical line field, hence the unit vertical field up to sign, hence the round metric g_1 = g_eta + (eta^{-2} - 1) theta_eta (x) theta_eta with theta_eta the g_eta-dual of the unit vertical field; it then lies in O(6) and maps i z to +- i z, so it is complex linear or antilinear (referee). The same eigenvalues show that g_eta is Einstein only at eta = 1. For a Hermitian traceless T on C^3 the Killing vector of the one-parameter group e^{i t T} is K_T(z) = i T z, and for the central circle with charge c on the coordinates, z -> e^{i c alpha} z, it is K_c(z) = i c z.

The tree-level Kaluza-Klein couplings: the reduction of the Einstein-Hilbert term on a compact internal space K with Killing vectors K_a gives the gauge kinetic term with the matrix

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<K_a, K_b> = int_K g(K_a, K_b) dvol,
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so that for an orthogonal basis g_a^{-2} is proportional to <K_a, K_a>, with one overall constant for all generators (the ansatz of Witten, Nucl. Phys. B 186 (1981) 412, equation (3); the integral formula of Weinberg, Phys. Lett. B 125 (1983) 265, in the form of Barnes, Gorbatov, Intriligator and Wright, Nucl. Phys. B 732 (2006) 89, section 3, where the normalization point is stated: the Killing vectors normalized to canonical kinetic terms make the coupling the charge unit; the normalization of each generator on the fields is that charge unit, an input). Within this scope, the Einstein term as the only source of the gauge kinetic terms, the charge unit is the only input; a flux or another source supporting the background contributes further terms (Duff, Pope and Warner, Phys. Lett. B 130 (1983) 254: on an Einstein space the Freund-Rubin contribution is a multiple of the Einstein one through int g(nabla K, nabla K') = int Ric(K, K'), leaving every ratio unchanged). The colour generators are normalized by tr T^2 = 1/2 on C^3.

Statements

Proposition 47.1 (the form on the Berger sphere). For Hermitian traceless T, T' on C^3 and the central generator with charge c,

```
<K_T, K_T'>_eta = V_eta (3 + eta^2) tr(T T') / 12,    <K_c, K_c>_eta = V_eta eta^2 c^2,    <K_T,
↪ K_c>_eta = 0.
```

In particular, for tr T^2 = 1/2, <K_T, K_T>_eta = V_eta (3 + eta^2)/24.

Proof. At z, the vertical component of K_T is the real inner product of i T z with the unit vertical vector i z, which is <z, T z> (real, since T is Hermitian), and the horizontal part has squared norm |T z|^2 - <z, T z>^2; so g_eta(K_T, K_T) = |T z|^2 + (eta^2 - 1) <z, T z>^2, and g_eta(K_T, K_T') = Re <T z, T' z> + (eta^2 - 1) <z, T z> <z, T' z>. The moments of the uniform measure on the unit sphere of C^n are int |z_i|^2 = 1/n and int z_i conj(z_j) z_k conj(z_l) = (delta_ij delta_kl + delta_il delta_kj)/(n(n + 1)), so for n = 3

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int <T z, T' z> = tr(T T')/3,    int <z, T z> <z, T' z> = (tr T tr T' + tr(T T'))/12 = tr(T T')/12,
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the last for traceless T, T'. Hence <K_T, K_T'>_eta = V_eta [tr(T T')/3 + (eta^2 - 1) tr(T T')/12] = V_eta (3 + eta^2) tr(T T')/12. The central Killing vector is vertical with g_eta(K_c, K_c) = eta^2 c^2 |z|^2 = eta^2 c^2, and <K_T, K_c>_eta = c eta^2 int <z, T z> = c eta^2 tr(T)/3 = 0. ■

Corollary 47.2 (the coupling ratio). With g_s the coupling of the SU(3) generators normalized by tr T^2 = 1/2 and e the coupling of the central circle with charge c on the coordinates,

e^2 / g_s^2 = <K_T, K_T>_eta / <K_c, K_c>_eta = (3 + eta^2) / (24 eta^2 c^2).

1. c = 1/3, the unit in which the coordinates carry charge 1/3 and the colour-singlet cubic z_1 z_2 z_3 carries charge 1 (the singlet Killing spinor then carries 1/2), gives 3(eta^2 + 3)/(8 eta^2); c = 2/3 gives 3(eta^2 + 3)/(32 eta^2): the two formulas of the parked job 04, which belong to two charge units of one computation. In general the spinor triplet carries -c/2 and the singlet 3c/2.
2. At the round point the Killing spinors form the 4 of SU(4) = Spin(6), which restricts to U(3) as 3_q + 1_{-3q}; with the singlet at charge 1 the triplet has charge -1/3 (the charges of the down quark and the positron, up to the overall sign), Q = diag(-1/3, -1/3, -1/3, 1), and the coordinates of C^3, which span one of the two triplets of the vector 6 = Lambda^2(4) = 3_{2/3} + conj(3)_{-2/3} (which one depends on the orientation of the complex structure; c^2 does not), carry c = 2/3, the charge of the up quark. With this unit, e^2/g_s^2 = 3(eta^2 + 3)/(32 eta^2), equal to 3/8 at eta = 1.
3. At eta = 1 the same value follows from Schur's lemma alone: the form is ad-invariant under so(6) = su(4), which is simple, so it is proportional to the trace form in the 4, and tr Q^2 = 3/9 + 1 = 4/3 against tr T^2 = 1/2 gives e^2/g_s^2 = (1/2)/(4/3) = 3/8.
4. In the Einstein-term reduction, the function 3(eta^2 + 3)/(32 eta^2) = 3/32 + 9/(32 eta^2) decreases from infinity at eta -> 0 to 3/32 at eta -> infinity, so each value above 3/32 is taken at exactly one shape, and 3/8 at the round point only. Off the round point the metric is not Einstein, so the shape needs a supporting source whose contribution to the kinetic terms changes the curve: with a term lambda int Ric_eta(K, K) added, int Ric_eta(K_T, K_T) = V tr T^2 [(6 - 2 eta^2)/4 + eta^4/3] and int Ric_eta(K_c, K_c) = 4 V eta^4 c^2, the ratio stays 3/8 at eta = 1 for every lambda and falls below 3/32 at large eta for lambda = 2 (referee).

Proof. Items 1 and 2 are substitutions in Proposition 47.1 and the branching of the 4 and of Lambda^2(4) under U(3) in SU(4). Item 3: a U(3)-invariant symmetric form on so(6) that is also invariant under the rest of SO(6), which holds at eta = 1 since SO(6) acts isometrically, is ad(so(6))-invariant, and so(6) is simple. Item 4: the derivative of (eta^2 + 3)/eta^2 = 1 + 3/eta^2 is negative. ■

Remark 47.3 (the dictionary). Connes' finite triple fixes at matching g_3 = g_2 and sin^2 theta_W = 3/8 (lemma 03: cos^2 theta_W = 5/8 at matching with the full module light; in its heavy-up family the neutral-current angle stays 5/8 and the photon and gluon coefficients keep their ratio, so e^2/g_3^2 = 3/8 holds there too, while the charged-W/hypercharge matching ratio moves), so e^2 = (3/8) g_3^2 there; directly from the traces of lemma 03, Part B, per generation the colour trace is 2 and tr Q^2 = 8/3 + 2/3 + 2 = 16/3, so e^2/g_3^2 = 2/(16/3) = 3/8, which is also the SU(5) value, the lemma's Q being the electric charge on the first four entries of the 5 = (d, d, d, e^+, conj(nu)). The round S^5 gives the same ratio of the colour and electromagnetic couplings, at tree level, with the charge unit of its own Killing spinors. So the test proposed in T18, whether the object at 9 fixes the gauge normalization, distinguishes the finite triple from the squashed spheres, and the round sphere gives the finite triple's value; the round point is the point of the Berger family where the isometry algebra enlarges from su(3) + u(1) to su(4), an enhanced-symmetry point in the language of candidate 5 of T18, the matching value sits there, and there it is independent of the source holding the shape. In the Einstein-term reduction the values along the family lie in (3/32, infinity), and the direction of squashing that lowers the ratio is eta > 1, the fibre stretched. The value 3/8 depends only on the embedding u(3) in su(4) with the 4 = 3_{-1/3} + 1_1; the same embedding carries the B - L reading of Pati-Salam, where the 4 is 3_{1/3} + 1_{-1} and Q is -(B - L), and the superconformal R-symmetry of AdS_5 x S^5. At any enhanced point where su(3) + u(1) sits in a simple algebra g the ratio is tr_V T^2 / tr_V Q^2 for any representation V of g, so the matching value appears exactly when the embedding gives Q the SU(5) index, tr Q^2 = (8/3) tr T^2 (referee).

Prior work (found by the referee). The formula on the Berger sphere was not found in print (searches on squashed five-spheres and on the weak angle from Kaluza-Klein theory before 1992). The round-point value is implicit in holographic form: Barnes, Gorbatov, Intriligator and Wright (Nucl. Phys. B 732 (2006) 89, arXiv:hep-th/0507146, sections 3 and 6) give for IIB on S^5 the coefficients tau_SO(6) = N^2 and tau_RR = 4N^2/3, with the U(1)_R Killing vector of norm squared 4/9, which is the central generator with c = 2/3 (the holomorphic 3-form at charge 2); with tau_SU(3) = N^2/2 this gives tau_SU(3)/tau_RR = 3/8, and a field-theory check gives the same value from N = 4 super Yang-Mills. So the charge unit of Corollary 47.2 is the superconformal R-charge. Related: Ezawa, Koh and Rajpoot, Phys. Lett. B 147 (1984) 284, Pati-Salam

unification from an SU(4) x SU(2) isometry (the B - L reading); Hu and Shen, Phys. Lett. B 178 (1986) 373, the weak angle from a deformed three-sphere, the S^3 analogue of this family. The identifiers are in paper/LITERATURE.md, section F.

Scope

Tree-level Kaluza-Klein reduction of the Einstein-Hilbert term, with no other contribution to the gauge kinetic terms and no threshold corrections; the Berger family of U(3)-invariant metrics on S^5; the charge unit as an input, fixed here by the SU(4) spinor at the round point and continued along the family. The identification of the SU(3) acting on C^3 with colour and of the central circle with electromagnetism is the reading of the 9 end of T18 (a five-dimensional internal space with isometry su(3) + u(1)); the same embedding carries the B - L and the R-symmetry readings, and the value fixes the index of the embedding and leaves the reading open. Off the round point the Berger metric is not Einstein, so the shape needs a supporting source, whose contribution to the gauge kinetic terms is outside the statement; the value 3/8 at eta = 1 is unaffected by any SO(6)-invariant contribution, by Schur's lemma. The matter content, the running from the matching scale and the physical value of e^2/g_s^2 are outside the statement. The direction the result points to: the round point as the matching point of the dictionary, where both sides agree and the value is source-independent, and the squashing eta as the parameter that the finite triple does not have; for the other geometric families of T18 the question reduces to an index computation, whether the embedding of su(3) + u(1) at the enhanced point gives Q the SU(5) index, and the methods of Barnes, Gorbatov, Intriligator and Wright compute such ratios on other Sasaki-Einstein five-manifolds with flavour symmetry (referee).

Part 48

The spin lift of the Weyl action and the quaternions: the real Standard Model algebra from two orbit types

For STRATEGY T16 (second half: the real form H), following lemma 45 (Proposition 45.1, Theorem 45.2 and Remark 45.3 as corrected by its referee: the corpus obtains C + H at the orbit type (a, a, a) and not at the fixed point, where the lift of the inversion is central) and lemma 16 (Theorem 16.7, Corollary 16.8). Written by Claude Fable 5.1 on 5 November 2026, by hand, from lemma 45 and the representation theory of the binary dihedral group, without consulting the corpus construction (historical/corpus, canonical section 10, which the referee of lemma 45 cited for the same orbit type); the character table of Dic_3 and the Frobenius-Schur types were checked by hand. Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_47_48.md), who built an explicit Cl(6), the lifted groups with their element orders, the spinor decompositions, the real spans and their centres, the fibres of the spinorial crossed product and the bimodule axioms: Propositions 48.1 and 48.2 correct with amendments (one step of the proof; the displayed matrix of the standard representation; the lift group is also S_3 x Z_4; the spin structure on A is forced); Theorem 48.3 items 1 and 2 correct with amendments (the corner at O_1 against the quotient at O_2 is a choice; C + H is a real subalgebra, not a real form); item 3 had an error in its last sentence and overread the selection; the Outcome and Remark 48.4 misdescribed lemma 45's complex algebra; all applied below, with the referee's table of orbit types and its leads.

Outcome. The Weyl group W = S_3 x Z_2 of SU(3) x SU(2) acts on T^6 = A x E by isometries, and its lift to the spinors is the group W-tilde = (Dic_3 x Z_4)/Z_2 of order 24, isomorphic to S_3 x Z_4: the transpositions lift to elements of order four, which is the obstruction rs_N20 of the corpus, the inversion of E lifts to a central element of square -1, and the twisted copy of S_3 that pairs each odd permutation with the inversion lifts homomorphically (Proposition 48.1). The algebra that acts on the spinors at an orbit is the spinorial part of the real group algebra of the lifted stabilizer, the part where -1 acts as -1, each of whose irreducible representations occurs in the spinor fibre. At the orbit type (a, a, a), 3a = 0, with a point of E that is not 2-torsion, the stabilizer is S_3, its lift is Dic_3, and the spinorial part of R[Dic_3] = R + R + M_2(R) + C + H is C + H: the pair of characters in which the lifted transposition acts as +-i, and the faithful two-dimensional representation, which is quaternionic (Proposition 48.2). At the fixed

point the central lift of the inversion acts as +-i on every spinorial representation, so the spinorial part is C + C + M_2(C), with no quaternionic summand even over R (lemma 45 used the same point with the complex untwisted group algebra, from which it selected C + M_2(C)); at the three-point orbit the spinorial part is C + C, two conjugate pairs of characters, each pair a copy of M_3(C) after Green's imprimitivity. The spinorial condition fixes the algebra on each of the six orbit types, and the orbit type (a, a, a) is the only one whose stabilizer algebra has a quaternionic summand, as the three-point orbit is the only one with matrix size three. So the two-sheet bimodule of Theorem 45.2, with its first sheet moved from the fixed point to the orbit type (a, a, a), carries the real algebra C + H + M_3(C), the Standard Model algebra, the two orbit types being forced by that target (Theorem 48.3). What remains of T16: a Dirac operator from the geometry (item (b) of Remark 45.3), a real structure from the geometry in place of the adjoint flip of Corollary 16.8, and the choices listed in Remark 48.4.

Setting

Let E be an elliptic curve with its flat metric, A = {(z_1, z_2, z_3) in E^3 : z_1 + z_2 + z_3 = 0}, isometric to a flat four-torus with the metric induced from E^3, and X = A x E. The Weyl group W = S_3 x Z_2 acts by isometries: S_3 permutes the z_i, and the generator iota of Z_2 acts on E by z -> -z. The tangent space of A at any point is a (x) R^2, with a = {x in R^3 : sum x_i = 0} the standard real representation rho of S_3 and R^2 the tangent space of E; the tangent space of E is R^2. Fix the spin structure of X induced from the trivial spin structures of the factors (periodic spinors); the periodic spin structure is the only one on A invariant under S_3 (S_3 acts on A_2/2A_2 = F_2^2 as GL_2(F_2), with no nonzero invariant functional), and the four spin structures of E are invariant under the inversion and give the same lifted stabilizers (referee). The spinor bundle is X x Sigma_6 with Sigma_6 = C^8, and an affine isometry g with constant differential dg in SO(6) acts on spinors through either of the two lifts of dg to Spin(6) in Cl(6), which differ by the sign. Write B-tilde = C(X, R) x W-tilde for the real crossed product by the lift group W-tilde, the preimage of dW in Spin(6), acting on L^2(X, Sigma_6) with -1 in W-tilde acting as -1 on spinors; let B_spin be its quotient by the ideal generated by u_{-1} + 1, the part that acts on the spinors; since -1 is central in W-tilde and acts trivially on X, u_{-1} is central, B_spin is the direct summand B-tilde (1 - u_{-1})/2, and it is the twisted crossed product of C(X, R) by W for the cocycle of the extension W-tilde -> W (referee). For a finite group S-tilde with a central element -1 of order two, the spinorial part of its real group algebra R[S-tilde] is the quotient by u_{-1} + 1, the sum of the simple summands on which -1 acts as -1. The orbit types of W on X and the fibres of the complex crossed product are those of lemma 45, Proposition 45.1.

Statements

Proposition 48.1 (the lift of the Weyl action). (i) For a transposition tau in S_3, d tau = rho(tau) (x) 1 on a (x) R^2 is the rotation by pi in the two-plane n (x) R^2, with n the normal of the reflection rho(tau); both of its lifts to Spin(4) are the products +- f_1 f_2 of an orthonormal basis of that plane, of order four. For a three-cycle c, d c is the rotation by 2 pi/3 in two orthogonal two-planes, and one of its lifts has order three. The lift of S_3 to Spin(4) is the binary dihedral group Dic_3 = <a, x | a^6 = 1, x^2 = a^3, x a x^{-1} = a^{-1}> of order 12, and no homomorphism S_3 -> Spin(4) lifts d: this is the corpus statement rs_N20, either preimage of a transposition squares to -1. (ii) d iota = -1 on the tangent space of E lifts to iota-tilde = +- e_5 e_6 in Spin(2), with iota-tilde^2 = -1, which commutes with the lift of S_3. (iii) The lift group is W-tilde = (Dic_3 x Z_4)/Z_2, of order 24, with Z_4 generated by iota-tilde and the Z_2 identifying the two elements -1. It is isomorphic to S_3 x Z_4: the elements a^k x iota-tilde are involutions, since (x iota-tilde)^2 = x^2 iota-tilde^2 = 1, so the twisted copy {(sigma, iota^{sgn sigma})} of S_3 in W lifts homomorphically (tau iota is the rotation by pi of the four-dimensional subspace (n (x) R^2) + T E, whose lift squares to +1, as in the T^4/Z_2 orbifold), and W-tilde is the product of that lift with ; so rs_N20 holds for S_3 and for every subgroup of W that contains a transposition without the inversion (referee). The lifted stabilizers are Dic_3 at an orbit of stabilizer S_3, W-tilde at the fixed point, and (Z_4 x Z_4)/Z_2, of order 8, generated by the lifts of a transposition and of iota, at the three-point orbit of lemma 45.

Proof. (i) rho(tau) is a reflection of a, rho(tau) = 1 - 2 n n^T, so rho(tau) (x) 1 = 1 - 2 P with P the

orthogonal projection onto the two-plane n (x) R^2: the identity on the orthogonal complement and -1 on the plane, the rotation by pi in it. In Cl(4) the rotation by pi in the plane of the orthonormal vectors f_1, f_2 lifts to +- f_1 f_2, and (f_1 f_2)^2 = -f_1^2 f_2^2 = -1. rho(c) is the rotation by 2 pi/3 of a, so rho(c) (x) 1 is that rotation in each of the two planes a (x) e_1 and a (x) e_2; with orthonormal bases g_1, g_2 and h_1, h_2 of the two planes, the lift exp((pi/3)(g_1 g_2 + h_1 h_2)) has cube exp(pi g_1 g_2) exp(pi h_1 h_2) = (-1)(-1) = 1, of order three. The preimage of rho(S_3) (x) 1 in Spin(4) is a group of order 12 containing a cyclic subgroup of order 6 generated by a = (lift of c of order 3) times (-1), and an element x of order four with x^2 = -1 = a^3; x c-tilde x^{-1} has order three and lies above tau c tau^{-1} = c^{-1}, so it is the order-three lift c-tilde^{-1}, hence x a x^{-1} = x (-c-tilde) x^{-1} = -c-tilde^{-1} = a^{-1}; this is the presentation of Dic_3, which has a single involution, -1, and so differs from Z_2 x S_3 and from the abelian groups of order 12. A lift S_3 -> Spin(4) would send tau to an element of order two above d tau, and both candidates have order four. (ii) The rotation by pi of the plane of E lifts to +- e_5 e_6, of square -1; e_5 e_6 is an even element in the Clifford algebra of the E-directions and commutes with the even elements in the Clifford algebra of the A-directions. (iii) The lifts of S_3 and of Z_2 commute and share the element -1, which gives the central product; the stabilizers are read off from lemma 45 and from the fact that -1 lies in every lifted stabilizer. ■

Proposition 48.2 (the spinorial parts of the real group algebras). (i) The irreducible representations of Dic_3 are the four characters of its abelianization Z_4, x -> i^k and a -> (-1)^k for k = 0, 1, 2, 3, and two representations of dimension two, a -> diag(zeta, conj(zeta)) and x -> [[0, zeta^3], [1, 0]], that is [[0, -1], [1, 0]] for zeta a primitive sixth root of unity (faithful) and [[0, 1], [1, 0]] for zeta a primitive third root (the standard representation of S_3). Their Frobenius-Schur types are real for k = 0 and k = 2 and for the standard representation, complex for the conjugate pair k = 1, 3, and quaternionic for the faithful representation, so

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R[Dic_3] = R + R + M_2(R) + C + H,    of real dimension 1 + 1 + 4 + 2 + 4 = 12.
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The spinorial part, where -1 = x^2 = a^3 acts as -1, is C + H: the pair k = 1, 3, on which the lifted transposition x acts as +-i, and the faithful representation. Each spinorial irreducible representation of Dic_3, of W-tilde and of the lift of <tau, iota> occurs in Sigma_6 (Sigma_6 restricted to Dic_3 is 2(chi_1 + chi_3) + 2 V_f, with chi_1 + chi_3 and V_f in opposite chiralities of the A-directions), so the spinorial part is the algebra generated on the spinor fibres over the orbit (referee). (ii) At the fixed point, the spinorial irreducible representations of W-tilde are the spinorial representations of Dic_3 tensored with the two characters iota-tilde -> +-i, all of complex type, and the spinorial part of R[W-tilde] is C + C + M_2(C), with no quaternionic summand: in the description W-tilde = S_3 x Z_4 with -1 = iota-tilde^2, it is R[S_3] (x)_R C. (iii) At the three-point orbit, the lifted stabilizer (Z_4 x Z_4)/Z_2 has four spinorial characters, x -> +-i and iota-tilde -> +-i, forming two conjugate pairs, and the spinorial part of its real group algebra is C + C.

Proof. (i) The commutator subgroup of Dic_3 is <a^2>, of order three, since x a x^{-1} a^{-1} = a^{-2}, so the abelianization has order four and is generated by the image of x, with a mapping to x^2 = a^3; this gives the four characters, with a -> (-1)^k. The two-dimensional representations are induced from the characters a -> zeta of = Z_6 with zeta != +-1; zeta and conj(zeta) give the same representation, so there are two, and the sum of squares 4 x 1 + 2 x 4 = 12 shows the list is complete. The character k = 2 is the sign of S_3 and k = 0 the trivial one; k = 1, 3 are conjugate and take the value +-i, so their type is complex. The faithful representation lies in SU(2), with x^2 = -1, so its Frobenius-Schur indicator is -1 and its type is quaternionic; the standard representation of S_3 is real. The real group algebra is the sum over the irreducible real representations of the matrix algebras over their commutants, R, C or H, which gives the display; -1 acts as -1 exactly on the characters with k odd and on the faithful representation. (ii) W-tilde is a central product, so an irreducible representation of W-tilde is an irreducible representation of Dic_3 tensored with a character of = Z_4 agreeing on -1; the spinorial ones pair the spinorial representations of Dic_3 with iota-tilde -> +-i. A representation on which a central element acts as i or -i is of complex type, since a real or quaternionic structure would commute with the central element and conjugate i to -i. So the characters give C + C and the faithful representation tensored with +-i gives M_2(C). (iii) The group is abelian, generated by x and iota-tilde with x^2 = iota-tilde^2 = -1; its spinorial characters are the four sign choices, in the two conjugate pairs (i, i), (-i, -i) and (i, -i), (-i, i), each pair a copy of C in the real group

algebra. ■

Theorem 48.3 (the real two-sheet bimodule over the two orbit types). Let O_1 be an orbit of type (a, a, a), 3a = 0, with a point of E that is not 2-torsion, of size two with stabilizer S_3, and O_2 the three-point orbit of lemma 45. Then:

1. The fibre of B_spin at O_1 is M_2(C + H), and its corner at one point of the orbit is the real algebra C + H, with C the character pair of Proposition 48.2 (i) and H the faithful representation; the fibre at O_2 is M_3(C + C), and one conjugate pair of characters of the stabilizer gives the real algebra M_3(C). The first sheet uses the corner at one point of O_1 and the second the full quotient at O_2; this is a choice, like the choice of characters (with corners on both sheets the algebra would be C + H + C, with full fibres on both M_2(C) + M_2(H) + M_3(C)).

2. The data of Theorem 45.2 with V_X = C + C^2, the real algebra A_{F,X} = C + H acting by the character on C and by left multiplication on C^2 = H, the grading G = (1, -1), central in C + H, and d_X = D_v; V_Y = C^3 with A_{F,Y} = M_3(C), gamma_Y = 1, d_Y = 0, give through the finite content of Corollary 16.8 the 18-state bimodule H_F = Hom(V_Y, V_X) + Hom(V_X, V_Y) with the KO-6 signs, order zero, order one, nonzero one-forms and the real orientability relation, now over the real algebra

   A_F = C + H + M_3(C),

   the Standard Model algebra, with the intersection form on minimal projections [[0, 0, 1], [0, 0, -2], [-1, 2, 0]] of the corpus (rs_N6), skew of rank 2.

3. The spinorial condition fixes the algebra on each orbit type. The six orbit types of W on X, with the spinorial parts of the lifted stabilizer algebras and the fibres, are: trivial stabilizer, orbit of size 12, R, M_12(R); a transposition, size 6, C, M_6(C); the inversion, size 6, C, M_6(C); S_2 x Z_2 (O_2), size 3, C + C, M_3(C + C); S_3 (O_1), size 2, C + H, M_2(C + H); S_3 x Z_2 (the fixed points), size 1, C + C + M_2(C); there is no stabilizer of type Z_3 alone, since a three-cycle fixes only the diagonal points, which S_3 fixes (referee). So the orbit type (a, a, a) is the only one whose stabilizer algebra has a quaternionic summand, and there the spinorial part is exactly C + H; the three-point orbit is the only one with matrix size three. Given the target C + H + M_3(C), the two orbit types are forced. The module V_X = C + C^2 keeps one of the two conjugate characters, a choice of submodule of the spinor fibre chi_1 + chi_3 + V_f; on the whole fibre C acts by (lambda, conj(lambda)) and no choice is made. At O_2 the choice is between the two conjugate pairs of Proposition 48.2 (iii). At the fixed point the spinorial part is C + C + M_2(C), of complex type in every summand; lemma 45 used the same point with the untwisted complex group algebra, from which it selected C + M_2(C).

Proof. (1) Green's imprimitivity for the finite orbit, as in Proposition 45.1 (ii), holds with real coefficients and with the twisted action: C(O, R) x_{lift} W is M_{|O|} of the spinorial part of R[S-tilde] for the lifted stabilizer S-tilde, the functions on the orbit acting by diagonal matrices and the lifted group elements by permutation matrices twisted by S-tilde; the corner at a point is the stabilizer algebra. Proposition 48.2 gives the stabilizer algebras. (2) C + H is a real *-subalgebra of the complex algebra C + M_2(C) of Theorem 45.2, acting on the same V_X, whose complex span is C + M_2(C) (a real form of C + M_2(C) would have real dimension 5; C + H has 6); the grading G = (1, -1) lies in it and is central; Theorem 16.7 and Corollary 16.8 are formal for any finite even representation (Remark 16.5), and their conclusions, the signs, the order conditions, the one-forms and the orientability relation, hold for the real algebra as for its complexification, since they are stated for the represented operators. The intersection form over the real algebra, with the minimal projection of H of rank two, is the corpus form, as the referee of lemma 45 computed. (3) Proposition 48.2. ■

Remark 48.4 (what this settles and what it leaves). Settled: the real form H of the second summand, from the quaternionic type of the faithful representation of the binary dihedral lift, at the orbit type the corpus used, reached here from representation theory alone; the absence of a quaternionic summand at the fixed point in the spin sector; the forcing of the two orbit types by the target, since the spinorial condition fixes the algebra on each orbit type and the quaternionic summand occurs only at stabilizer S_3, which is the positive form of item (c) of Remark 45.3 for the orbit types. Left: item (b) of Remark 45.3, a Dirac operator

from the geometry in place of the inserted D_v, for which the torus Dirac operator with the W-tilde action on spinors is now in place (and the twisted S_3 acts without the cocycle), and the compression question of lemma 20, Result 7; a real structure from the geometry, since the J here is the adjoint flip of Corollary 16.8 between the two sheets, an input of the two-sheet construction (lemma 16, Caveat 4), and a derivation from the regular bimodule of the crossed product is open; the choices: the corner at O_1 against the quotient at O_2, one character or the whole fibre at O_1, the pair at O_2; and the physical identification of the orbit O_1, a pair of points of T^6 exchanged by the inversion of E at a 3-torsion configuration of A, against the fixed point of lemma 45. Leads from the referee: on the whole spinor fibre at O_1, Sigma_4 = chi_1 + chi_3 + V_f with C acting as (lambda, conj(lambda)) on the two characters and H on the doublet, in opposite chiralities of the A-directions, the action of C + H on the right-handed singlets and the left-handed doublet of one quark generation; with V_X = Sigma_4, G the A-chirality (1, 1, -1, -1) and V_Y = C^3, Corollary 16.8 gives a 24-state bimodule over C + H + M_3(C), the bimodule form of one quark generation, with all axioms (checked numerically by the referee) and intersection form [[0, 0, 2], [0, 0, -2], [-2, 2, 0]], of which the 18-state bimodule is the restriction to one character; the grading G of Theorem 48.3 is the A-chirality e_1 e_2 e_3 e_4 restricted to the stabilizer algebra, which lies in the image of R[Dic_3] on the spinors, so the grading is available from the geometry; a lepton sheet would need an orbit whose spinorial algebra is C, such as the corner at a point with stabilizer a transposition or the inversion, untested.

Scope

Flat T^6 with the periodic spin structure and the isometric action of W; real crossed products by the lift group, with the spinorial quotient; finite groups and finite orbits, where Green's imprimitivity is elementary; the Dirac operator D_v and the grading as inputs, as in lemma 45. The direction: with the lift group acting on the spinors of T^6, the compression of the torus Dirac operator to the two selected sheets is the next computation of T16, and the Frobenius-Schur type of a lifted stabilizer's spinorial representations is the rule that decides the real form of each summand, the rule that also decides the orthogonal, symplectic and unitary factors of orientifold gauge groups (Blum and Intriligator 1997, section 5, where the representations of Dic_3 are sorted by type with the spinorial ones pseudo-real or complex; paper/LITERATURE.md, section N).

Part 49

The quaternionic sheet as the spinor fibre of the four-torus: the chirality as the axis and the lifted transposition as the hypercharge

For STRATEGY T14 and T16, joining lemma 46 (the odd blocks of the sheet W = H^2 with axis p) and lemma 48 (the spin lift of the Weyl action and the real algebra C + H at the orbit type (a, a, a)), following lemma 16 (Example 16.9), lemma 42 (the isotropy representation of the marked point) and lemma 20 (Theorem 20.2). Proposition 49.5 is the lead that the referee of lemma 48 recorded and checked numerically (historical/reviews/2026-11-05_opus_check_47_48.md, New insights, items 2 and 3), written here with its proof. Written by Claude Fable 5.1 on 5 November 2026, by hand; the Clifford algebra of the four-torus with its chirality and quaternionic structure, the lifts of a transposition and of a three-cycle, their action on the two chiralities, the real dimension 6 of the algebra they generate, and the complex dimensions 2 and 0 of the Yukawa blocks invariant under the lifted transposition and under the whole lifted stabilizer were checked with numpy (script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_49.md), who rebuilt the Clifford algebra, the lifts, the invariant blocks, the stabilizers and the 24-state bimodule with all its axioms: Propositions 49.1, 49.2, 49.4 and 49.5 correct with amendments (the orientation that makes Sigma_R the line of H_R; the order-six lifts; the eigenline condition for lemma 42's torus; the unitary group {+-1} of R; a nonzero block in 49.5 and the condition for the restriction to Theorem 48.3), Proposition 49.3 with one phrase in error (the invariance group of the diagonal blocks is the circle of the lifted transposition on both chiralities, the algebra C_F of Chamseddine, Connes and Marcolli, and no block is invariant under the circle on Sigma_R alone), four sentences of the Outcome and Reading overread; all applied, with the referee's identifications in the spectral Standard Model and its leads.

Outcome. The spinor fibre Sigma_4 = Sigma_R + Sigma_L of the flat four-torus A of lemma 48, with the Clifford action of its tangent space and its chirality Gamma, is the quaternionic sheet W = H^2 of Example 16.9 with the axis p at Gamma: the chirality is the grading axis, the two chiralities are the quaternionic lines L_p and L_p^perp, Spin(4) is the Pati-Salam pair Sp(1)_a x Sp(1)_b, and the Clifford action of the tangent space of A is the bidoublet Hom_H(L_p, L_p^perp) = T_p S^4, the Higgs directions of lemmas 42 and 46 (Proposition 49.1). The lifted stabilizer Dic_3 of the orbit type (a, a, a) acts on one chirality, Sigma_R, through its abelianization Z_4 and on the other, Sigma_L, through its faithful representation, so the real algebra it generates is C + H inside the Pati-Salam H_R + H_L = End_H(Sigma_R) + End_H(Sigma_L), with C the Cartan subalgebra of H_R generated by the lifted transposition, which acts on Sigma_R as (i, -i) (Proposition 49.2): the circle of the lifted transposition is a maximal torus of Sp(1)_R, the T_3R circle, which is the U(1)_b that lemma 42 marked by hand exactly when the connector's line is one of its two weight lines, and the torus that Corollary 46.4 reads from the masses exactly when the mass eigenlines are its weight lines; and the whole fibre carries the action (lambda, conj(lambda)) of C on the two right-handed singlets and of H on the left-handed doublet of the spectral Standard Model. The Yukawa blocks commuting with the lifted transposition, acting on both chiralities, are exactly the diagonal ones, X = diag(y_1, y_2) in the weight bases, the Y_u, Y_d of that model, with quaternionic half (y_1 + conj(y_2))/2 and anti-quaternionic half (y_1 - conj(y_2))/(2i); commuting with that circle is the condition of Chamseddine, Connes and Marcolli that the finite Dirac operator commute with their subalgebra C_F, the photon condition; no nonzero block is invariant under the circle on Sigma_R alone, and none under the whole lifted stabilizer, since the three-cycles act freely on Sigma_L (Proposition 49.3). Along the orbit types of A the lifted stabilizer algebras are C + H at the nine S_3-fixed points, C on the mirror planes of the transpositions, where the generated algebra is the circle of the lifted transposition acting on both chiralities, the C_F of Chamseddine, Connes and Marcolli restricted to the first sheet and the unbroken circle of Corollary 46.4, and R elsewhere: unitary groups U(1) x SU(2), U(1) and {+-1} (Proposition 49.4). With the whole fibre as the first sheet, Corollary 16.8 gives a 24-state bimodule over C + H + M_3(C), the quark summand H_F^(3) of Chamseddine, Connes and Marcolli for one generation, with its grading the chirality of A and their printed pairing (Proposition 49.5).

Setting

Let A = a (x) R^2 be the flat four-torus of lemma 48, with tangent space R^4 = a (x) R^2 at every point, a the standard representation of S_3, and the periodic spin structure, the only one invariant under S_3 (lemma 48, Setting). Its spinor fibre is Sigma_4 = C^4 = Sigma_+ + Sigma_-, the two half-spin representations of Spin(4) = Sp(1)+ *x Sp(1)_-, each a quaternionic line H on which its Sp(1) acts by left multiplication and the other trivially; the quaternionic structure j, the right multiplication by the unit quaternion j on each line, commutes with Spin(4), with the Clifford multiplication c(v) by vectors v of R^4, and with the chirality Gamma = c(e_1) c(e_2) c(e_3) c(e_4), which is +1 on Sigma+* and -1 on Sigma_-. We orient A so that the lifts of order three of the three-cycles act trivially on Sigma_+; for the complex orientation of A inherited from E, with Gamma = c(e_1) c(e_2) c(e_3) c(e_4) in an oriented orthonormal basis, they act trivially on Sigma_- (the generator of the rotation of a three-cycle is an anti-self-dual two-form for that orientation), so the orientation used here is the opposite one, and with the complex orientation the axis is p = -e_5; with this convention Sigma_R = Sigma_+ = L_p, which carries Sp(1)_b and H_R as in lemmas 42 and 46 (referee). The real structure of the spin geometry of A, with J_M^2 = -1, J_M D = D J_M and J_M Gamma = Gamma J_M, anticommutes with the Hermitian c(v) and equals j Gamma up to a phase; it agrees with j up to sign on each chirality, so H_R + H_L and C + H are the same for both, while on the odd blocks the roles are exchanged, c(v) being J_M-antilinear and i c(v) Gamma J_M-linear; the words quaternionic and anti-quaternionic below refer to j, the structure of Example 16.9 (referee). The circle of the summand C below is the T_3R circle; lemma 42 calls it the hypercharge with Y = charge/2, which is exact on the Higgs, where B - L = 0, and the hypercharge of the spectral Standard Model is its unimodular combination with the U(1) of M_3(C). Put Gamma_a = c(e_a) for a = 1, ..., 4 and Gamma_5 = Gamma: five Hermitian operators with Gamma_a Gamma_b + Gamma_b Gamma_a = 2 delta_ab, commuting with j, whose real span together with the Gamma_a Gamma_b is End_H(Sigma_4) = M_2(H). The unitary group of End_H(Sigma_4) is Sp(2), containing Spin(4) as the stabilizer of Gamma_5; under Spin(4) the vector R^4 is the (2, 2) and acts, through c, as the odd Hermitian j-linear maps Sigma_+ -> Sigma_- and back. The

lifted stabilizer of the orbit type (a, a, a) is Dic_3 = <a, x> in Spin(4), x the lift of a transposition, of order four, and c-tilde = -a the lift of order three of a three-cycle (lemma 48, Proposition 48.1). Sigma_6 restricted to Dic_3 is twice Sigma_4 (lemma 48), so the whole fibre of T^6 carries two copies of everything below, with the spinors of E as multiplicity space, on which T E and the inversion act; c(e_5) and c(e_6) anticommute with the c(e_a) and commute with the chirality of A, outside the Cl(5) below.

Statements

Proposition 49.1 (the sheet as the spinor fibre). (Sigma_4, Gamma_1, …, Gamma_5, j) is the quaternionic sheet W = H^2 of Example 16.9 and lemma 46 with the axis p = e_5: Gamma_p is the chirality, L_p = Sigma_+ and L_p^perp = Sigma_-, the stabilizer K_p = Sp(1)_b x Sp(1)_a is Spin(4) = Sp(1)+ *x Sp(1)_-, and the bidoublet Hom_H(L_p, L_p^perp) = T_p S^4 of lemma 42 is the Clifford action of the tangent space of A: v -> c(v) restricted to Sigma+* is a Spin(4)-equivariant isomorphism R^4 -> Hom_H(Sigma_+, Sigma_-). The even algebra H + H of Example 16.9 for this axis is End_H(Sigma_+) + End_H(Sigma_-), the Pati-Salam pair acting on the two chiralities, and the odd Hermitian blocks of lemma 46 are the j-linear c(v) and the j-antilinear i c(v) Gamma.

Proof. Cl(4) acts on Sigma_4 irreducibly with c(v)^2 = |v|^2 and c(v)^* = c(v) for the Hermitian metric of the spinors, and the chirality Gamma anticommutes with every c(v), is Hermitian and squares to one; so Gamma_1, …, Gamma_5 generate Cl(5) on C^4, and Example 16.9 and lemma 46 apply with these generators. The real algebra End_H(Sigma_4) has real dimension 16 and contains the 16 linearly independent elements 1, Gamma_a, Gamma_a Gamma_b, so it is their real span, M_2(H); Spin(4) is the subgroup of Sp(2) = U(End_H(Sigma_4)) that commutes with Gamma_5, since the even Clifford algebra of R^4 is Spin(4)'s and the stabilizer of Gamma_5 in Sp(2) is the unitary group of the even algebra H + H, which is Sp(1) x Sp(1). The map v -> c(v)|*{Sigma+}* is Spin(4)-equivariant by the covariance of Clifford multiplication, injective, and both sides have real dimension four, Hom_H between two quaternionic lines being a copy of H. The rest is lemma 46, Proposition 46.1, with Gamma_p = Gamma. ■

The identification is the isomorphism of Cl(5)-modules with quaternionic structure that matches c(e_a) with Gamma_a for a = 1, …, 4 and Gamma with Gamma_5, unique up to a phase by Schur's lemma. Composing it with an element of Sp(2) carries the axis to any point of S^4: the axis is fixed by the four-torus, whose symmetry at a point is Spin(4), and it remains an input of the twistor window, whose symmetry is Sp(2) (referee).

Proposition 49.2 (the lifted stabilizer inside the sheet). (i) The lifts of order three of the three-cycles act trivially on one chirality, which we call Sigma_R, and by rotations of order three on the other, Sigma_L; the lifts of order six act as -1 on Sigma_R and with order six on Sigma_L; the lift x of a transposition acts on both with eigenvalues +-i, and the three transpositions give the same circle on Sigma_R up to sign. So Dic_3 -> Sp(1)_R has image the cyclic group Z_4 generated by x|*{Sigma_R}, and Dic_3 -> Sp(1)_L is the faithful embedding of the binary dihedral group in SU(2). (ii) The real algebra generated by Dic_3 on Sigma_4 is C + H, with C = R[x]/_{Sigma_R}] = span_R(1, x/*{Sigma_R}) acting on Sigma_R as (lambda, conj(lambda)) in the eigenbasis of x, and H = End_H(Sigma_L) the real span of the faithful representation: the part acting on the first sheet of the Standard Model subalgebra A_F = {(lambda, q_L, lambda, m)} of Chamseddine, Connes and Marcolli (arXiv:hep-th/0610241, Proposition 2.11 and Definition 2.14, where every embedding of C in H is conjugate to this one), inside the Pati-Salam H_R + H_L of Example 16.9, with C the Cartan subalgebra of H_R generated by the lifted transposition; in their model the same lambda also acts on the lepton line of the second sheet. (iii) Consequently the circle generated by the lifted transposition x is a maximal torus of Sp(1)_R, the same for the three transpositions. It is the U(1)_b of lemma 42 exactly when the connector's line Cw is one of its two weight lines, and it is the torus U(1)_X of Corollary 46.4 exactly when X^* X has distinct eigenvalues with the weight lines of x as eigenlines, as for the blocks of Proposition 49.3 with |y_1| != |y_2|; its two weight lines in Sigma_R are the two right-handed singlets on which C acts by lambda and conj(lambda).

Proof. (i) Spin(4) = Sp(1)+ *x Sp(1)_- acts on R^4 = H by v -> p v conj(q). Identify a with C by a complex structure, so that R^4 = a (x) R^2 = C (x) R^2 = C^2 = H; the differential of a three-cycle, the rotation by 2 pi/3 of a tensored with the identity of R^2, is the complex scalar e^{2 pi i/3} on C^2, a rotation by*

the same angle and in the same sense in two orthogonal planes, which is left (or right) multiplication by a unit complex number: an element of one Sp(1) factor only. Its lift c-tilde of order three therefore acts trivially on the chirality of the other factor, Sigma_R, and on Sigma_L by a rotation of order three; the lift -c-tilde acts as -1 on Sigma_R and with order six on Sigma_L. The differential of a transposition is the rotation by pi in the two-plane n (x) R^2 (lemma 48, Proposition 48.1), with eigenvalues (-1, -1, 1, 1). The lift x has x^2 = -1 (lemma 48), and -1 = (-1, -1) in Sp(1)+ x Sp(1)_-, so each component u of x satisfies u^2 = -1: both components are unit imaginary quaternions, with eigenvalues +-i on their chiralities. As a cross-check, an element of one factor alone acts on H by left or right multiplication by a unit quaternion, an isoclinic rotation with the same angle in two orthogonal planes, while d tau has the angles pi and 0 (for instance (i, i) acts as v -> i v i^{-1}, the identity on the complex line of 1 and i and -1 on that of j and k, a rotation by pi in one plane). (ii) The representation of Dic_3 on Sigma_R + Sigma_L is spinorial, -1 acting as -1, and faithful on the spinorial part C + H of R[Dic_3] (lemma 48, Proposition 48.2, where Sigma_R carries the character pair chi_1 + chi_3, since x acts as (i, -i) and a as -1, and Sigma_L carries the faithful representation V_f); so the image of R[Dic_3] is C + H, with the C acting on Sigma_R through span_R(1, x) = C as (lambda, conj(lambda)) in the eigenbasis of x, and the H acting on Sigma_L as the real span of the finite subgroup Dic_3 of Sp(1), which is all of H since Dic_3 contains 1, the imaginary unit u and an element outside the complex line of u. The even algebra of Example 16.9 for the axis Gamma is End_H(Sigma_R) + End_H(Sigma_L) = H_R + H_L (Proposition 49.1), and span_R(1, u') is a maximal commutative subalgebra of H_R, a Cartan subalgebra. (iii) The unitary group of the C of (ii) is the circle {cos t + u' sin t} in Sp(1)_R, the maximal torus with weight lines the eigenlines of u' = x|_{Sigma_R}; lemma 42 marks such a torus by hand, and Corollary 46.4 identifies it from a split Yukawa block when the connector's line is one of its weight lines. ■

Proposition 49.3 (the invariant Yukawa blocks). Identify Sigma_R and Sigma_L with H = C + j C so that x acts on both as left multiplication by i (possible by conjugation in Sp(1)_R and Sp(1)_L). The odd Hermitian blocks [[0, X^*], [X, 0]] with X in Hom_C(Sigma_R, Sigma_L) that commute with x are exactly those with X = diag(y_1, y_2) in the bases (1, j) of the two lines, a complex two-dimensional space: the blocks Y_u, Y_d of the spectral Standard Model. Commuting with x is commuting with the circle of x acting on both chiralities, whose real span on Sigma_4 is the image of the subalgebra C_F = {(lambda, lambda, 0)} with which Chamseddine, Connes and Marcolli require the finite Dirac operator to commute (arXiv:hep-th/0610241, Definition 2.20 and Remark 2.19: the photon remains massless). No nonzero odd block is invariant under the circle of x on Sigma_R alone, the circle of the summand C. In the variables of lemma 46, X = L_a + i L_c with a = (y_1 + conj(y_2))/2 and c = (y_1 - conj(y_2))/(2i), both complex numbers, and the mass formula of Proposition 46.2 reads |a|^2 + |c|^2 +- 2 |Im(conj(a) c)| = |y_1|^2, |y_2|^2; the quaternionic blocks commuting with x are those with y_2 = conj(y_1), equal masses with conjugate couplings. No nonzero odd block commutes with the whole of Dic_3, nor with the lift of order three of a three-cycle alone: a Yukawa block breaks the S_3 of the symmetric point, and on the mirror of tau, where the lifted stabilizer is , the blocks commuting with it are those above; the escapes are a fibre larger than Sigma_4 (the E factor) or another grading.

Proof. With x acting as L_i on both lines, X L_i = L_i X means that X preserves the eigenlines of L_i, which are C and j C, so X is diagonal in the basis (1, j); conversely every diagonal X commutes with L_i. For a, c complex, L_a = diag(a, conj(a)) and i L_c = diag(i c, i conj(c)) in that basis, so X = diag(a + i c, conj(a) + i conj(c)) and y_1 = a + i c, y_2 = conj(a) + i conj(c), which inverts as stated; the singular values of a diagonal matrix are |y_1|, |y_2|, and the formula of Proposition 46.2 with q = a, q' = c complex gives |a|^2 + |c|^2 +- 2 |Im(conj(a) c)| = |a +- i c|^2 in the appropriate order. The quaternionic case is c = 0, that is y_2 = conj(y_1). If X commutes with the lift c-tilde of a three-cycle, which acts as the identity on Sigma_R and without fixed vectors on Sigma_L, then X = c-tilde X maps into the fixed space of c-tilde on Sigma_L, which is zero. ■

Proposition 49.4 (the orbit types of A and the unbroken groups). The stabilizer in S_3 of a point of A is S_3 at the nine points (a, a, a) with 3a = 0; the group generated by one transposition on the mirror z_i = z_j of that transposition, off the nine points; and trivial elsewhere. The real algebras generated on Sigma_4 by the lifted stabilizers are C + H, C and R, with C = span_R(1, x) acting on both chiralities as (lambda, conj(lambda)) in the weight bases of x; their unitary groups are U(1) x SU(2), U(1) and {+-1},

the image of the kernel of Spin(4) -> SO(4). The algebra of the middle type is the restriction to Sigma_4 of the subalgebra C_F of Chamseddine, Connes and Marcolli (Proposition 49.3), and its circle is the common unbroken circle of Corollary 46.4: it is the stabilizer, in K_p = Spin(4), of every diagonal block diag(y_1, y_2) with |y_1| != |y_2| together with a weight line of x in Sigma_R. For a displacement delta of the point along the mirror plane of tau, the Clifford element c(delta) is a quaternionic odd block commuting with x, of the form L_a with a complex, that is y_2 = conj(y_1).

Proof. A permutation fixes (z_1, z_2, z_3) exactly when it permutes equal coordinates; with z_1 + z_2 + z_3 = 0, the three coordinates are equal exactly when 3 z_1 = 0, nine points of the 3-torsion of E, and two are equal on the mirror z_i = z_j. The lifted stabilizers are Dic_3, = Z_4 and the trivial group, whose spinorial real algebras on Sigma_4 are C + H (Proposition 49.2), span_R(1, x) = C acting on both chiralities by x = (i, -i) in its weight bases, and R; the unitary groups follow. The stabilizer in Sp(1)_R x Sp(1)_L of a diagonal block X = diag(y_1, y_2) with distinct moduli is the circle {(X b X^{-1}, b)} with b in the torus of x (Corollary 46.4), and X b X^{-1} = b for diagonal X and b in that torus, so the circle is {(b, b)}, the circle of x acting on both chiralities, which also fixes the weight lines. For delta in the fixed plane of d tau, x c(delta) x^{-1} = c(d tau delta) = c(delta), so c(delta) is a j-linear odd block commuting with x: by Proposition 49.3, L_a with c = 0. ■

Proposition 49.5 (the quark generation on the whole fibre). Apply the finite content of Corollary 16.8 with V_X = Sigma_4, A_{F,X} = C + H acting as in Proposition 49.2, gamma_X = Gamma (the chirality of A, +1 on Sigma_R and -1 on Sigma_L, with the orientation of the Setting; for the complex orientation gamma_X = -Gamma), d_X a nonzero odd Hermitian block of Proposition 49.3, and V_Y = C^3 with A_{F,Y} = M_3(C), gamma_Y = 1, d_Y = 0. Then H_F = Hom(V_Y, V_X) + Hom(V_X, V_Y) has complex dimension 24, carries the KO-6 signs, order zero, order one, nonzero one-forms and the real orientability relation of Corollary 16.8, over the real algebra C + H + M_3(C); it is the quark sector of one generation of the spectral Standard Model (the summand H_F^(3) of Chamseddine, Connes and Marcolli, arXiv:hep-th/0610241, Proposition 2.30, whose pairing they print as the matrix below; their nondegenerate pairing, Corollary 2.31, uses the lepton class as well), C acting on the two right-handed singlets by (lambda, conj(lambda)), H on the left-handed doublet, M_3(C) on colour, with the grading the chirality of A. Its intersection form on the minimal projections 1_C, 1_H, e_11 is [[0, 0, 2], [0, 0, -2], [-2, 2, 0]], skew of rank 2. The 18-state Hilbert space of Theorem 48.3, with its algebra, real structure and grading, is the sub-bimodule on one weight line of x in Sigma_R; D_F preserves it exactly when X vanishes on the other weight line (y_2 = 0 for the line of y_1), and then restricts to D_v with v = (y_1, 0) in the weight basis of Sigma_L.

Proof. As in Theorem 48.3 (2): the data are a finite even representation of the real algebra C + H, which is a real *-subalgebra of End_C(Sigma_4), with the central grading Gamma (1 on Sigma_R, where C acts, and -1 on Sigma_L, where H acts), so Theorem 16.7 and Corollary 16.8 apply formally (Remark 16.5), and their conclusions hold for the real algebra as for its complex span. The dimension is 2 x 4 x 3 = 24. The action is that of the spectral Standard Model on the quark sector: the bimodule Hom(C^3, Sigma_4) = Sigma_4 (x) conj(C^3) is a right-handed up-type and down-type singlet (the two weight lines of x, with lambda and conj(lambda)) and a left-handed doublet, each in three colours, with the conjugates in the other summand. The intersection form is <e, f> = tr(Gamma a_e) tr(b_f) - tr(b_e) tr(Gamma a_f) as in Theorem 48.3, with tr(Gamma 1_C) = 2, tr(Gamma 1_H) = -2, tr(b_{e_11}) = 1. The subspace on one weight line of x in Sigma_R, (line + Sigma_L) (x) conj(C^3) with its conjugate, has dimension 18 and is invariant under the algebra, J and Gamma_F; D_F maps Sigma_L to Sigma_R by X^*, whose image is all of Sigma_R when y_1 y_2 != 0, so D_F preserves the subspace exactly when X vanishes on the other weight line. ■

Reading

In the four-torus model the three inputs that the twistor picture left as data, the grading axis (Proposition 28.19, Example 16.9), the marked torus U(1)_b in Sp(1)_b (lemma 42) and the real form of the second summand (lemma 45), are supplied by the spinor fibre of A with its Weyl stabilizer: the axis is the chirality of A, fixed up to the orientation; the torus is the circle of the lifted transposition, which is lemma 42's U(1)_b when the connector's line is one of its two weight lines; and the quaternions are the real span of the faithful representation on the other chirality. The twistor window receives them through the isomorphism

of Proposition 49.1, which exists for every axis, so there the axis remains an input. The Higgs directions, Hom_H(L_p, L_p^perp) = T_p S^4 in lemmas 42 and 46, are the tangent directions of A, and the stabilizer algebras along the orbit types of A are C + H at the S_3-fixed points, with unitary group U(1) x SU(2); on the mirror planes span_R(1, x), the subalgebra C_F of Chamseddine, Connes and Marcolli on the first sheet, with which their finite Dirac operator must commute, whose unitary group is the circle that Corollary 46.4 found as the common stabilizer of a split Yukawa block and the connector's line; and R, with unitary group {+-1}, at generic points. As a reading of the electroweak pattern this differs from the spectral Standard Model in one respect: there A_F is fixed and C_F appears as a condition on D_F, while here the algebra changes along the strata. A displacement along a mirror gives a quaternionic, custodial block, equal masses with conjugate couplings; the anti-quaternionic half that splits the masses is i c(delta') Gamma, which is +- i times the Clifford multiplication by the three-form dual to delta'; every c(v) is j-linear, so no displacement of the point supplies it, and a constant three-form term of the Dirac operator on A, of the kind torsion or a flux adds, has exactly this form (referee). For delta, delta' in the mirror plane the general x-invariant block is c(delta) + i c(delta') Gamma, with m_+^2 + m_-^2 = 2 (|delta|^2 + |delta'|^2) and m_+^2 - m_-^2 = 4 |det(delta, delta')|: the up-down splitting is four times the area of the parallelogram spanned by the displacement and the dual of the three-form, and the masses are equal exactly when the two are parallel; the mirror directions are the neutral components of the bidoublet and the normal directions, whose blocks anticommute with x, the charged ones (referee). The crossed product of lemma 48 sees these orbit types as different quotients, so the motion of the point is a reading and the proved content is the list of stabilizer algebras. The whole fibre carries one quark generation (Proposition 49.5), and the lepton generation needs more than a sheet whose stabilizer algebra is C on a single line: with V_Y = C + C^3 Corollary 16.8 gives 32 states over the four-summand algebra C + H + C + M_3(C), and the identification of the two copies of C that gives A_F = {(lambda, q_L, lambda, m)} is, in Chamseddine, Connes and Marcolli, the order-one condition for an operator between particles and antiparticles (Proposition 2.11), a connecting operator that the sheet-diagonal D_F of Corollary 16.8 cannot supply; in the project that is the role of the connector of lemma 20, whose centralizer is C + H + M_3(C) (referee). The circle of C is also the complex structure of A: for each transposition the lift agrees on Sigma_R, up to sign, with half the Clifford action of the Kähler form of the complex structure that A inherits from E, so with Sigma_4 = Lambda^{0,*} (x) K^{1/2} the two right-handed singlets are K^{+-1/2}, the left doublet is Lambda^{0,1} (x) K^{1/2}, and C + H is the real span of the spin lift of the unitary group U(2) of that complex structure, of which Dic_3 is a finite subgroup spanning the same algebra; the Weyl action preserves the complex structure of E, and the two weight lines are exchanged by the lift of a reflection of T E (referee).

Scope

Finite linear algebra on the spinor fibre of the flat four-torus A with the periodic spin structure; the lifted stabilizer Dic_3 of lemma 48; the convention that Sigma_R is the chirality on which the three-cycles act trivially (which of the two half-spin representations this is depends on the orientation of A only, since each three-cycle has a single lift of order three; for the complex orientation of A inherited from E it is the chirality Gamma = -1, on which the Kähler form acts). The E factor of T^6 = A x E is outside the statements: its spinors double the fibre, Sigma_6 = Sigma_4 (x) Sigma_2, the inversion acts on Sigma_2 and is absent from the stabilizer at the orbit type (a, a, a), and the Clifford action of T E adds two generators outside the Cl(5) of Proposition 49.1. Proposition 49.4 proves stabilizer algebras, and the reading of a displacement as a Higgs field is a reading. The direction: the Dirac operator of the torus restricted to a point of a mirror, item (b) of Remark 45.3, with a constant three-form term whose dual lies in the mirror plane, as the candidate that supplies both halves of an x-invariant Yukawa block and derives the condition [D_F, C_F] = 0 from the geometry, the three mirrors being three candidates for the photon circle on Sigma_L; and the lepton generation through a connecting operator that identifies the two copies of C.

Part 50

The Majorana line and the top block: one point of the twistor fibre, the Higgs vector and the compressed height functions

For STRATEGY T12 and T14, taking up the first lead of the referee of lemma 46 (a rank-one Yukawa block marks a point of the twistor fibre, the same kind of datum as the connector's line), following lemma 20 (Theorem 20.2 and its second reading), lemma 42 (Proposition 42.2), lemma 46 (Propositions 46.1 and 46.2, Corollary 46.4), lemma 16 (Theorem 16.7, Corollary 16.8, Example 16.9), lemma 10 (Proposition 10.1), lemma 21 (Proposition 21.1), lemma 17 (Proposition 17.3), lemma 41 (Proposition 41.5), lemma 44 and lemma 49 (Propositions 49.3 and 49.4 and the three-form reading). Written by Claude Fable 5.1 on 5 November 2026, by hand; the Clifford structure and j, the isometry n -> Gamma_n w, the spectrum and the two halves of the block, the complex structure J_w and the action of the circle on it, the compression identity (Monte Carlo on CP^3 with 4 x 10^5 points), the order-one defects of the split block on M_4(C) and on C + M_3(C), the real dimension 20 of the centralizer with its forced embedding, and the three-form form of the anti-quaternionic half were checked with numpy (script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_50.md), who redid every computation by hand and in Python (400 random frames for both Cl(5) representations, exact fractions for the removal identities, the order-one defects, the centralizer dimensions, the Monte Carlo) and read the e-prints of the sources: two errors, in Proposition 50.1 (3) (with the universal block both cases keep 5/8 and 8/3, so the up-type assignment is an input) and in the closing sentence of Proposition 50.3 (the top-only block alone keeps order one on H_R + H_L + C + M_3(C); the connector is what reduces to A_SM), and sixteen amendments (the sign of delta' fixed by the definition of J_w, with no dependence on the orientation; Proposition 50.2 as Example 16.9, item 2, extended complex-linearly; the hypercharge -1/2 of the doublet in J_w; the general first-order criterion, Krajewski's rule; the prior work in Chamseddine-Connes-Marcolli, Proposition 3.5, and in the Standard Model vacuum of the spectral Pati-Salam model), all applied below with the referee's texts.

Outcome. On the quaternionic sheet W = H^2 with axis p, let the connector of Theorem 20.2 mark the complex line Cw in L_p and the line Cv in V, and let X be a Yukawa block of a doublet with one massless member (Proposition 46.2 (2), the heavy-top pattern). The eigenline condition of Corollary 46.4 says that the massive right-handed line of X is Cw or Cjw, and the up-type assignment of the heavy member selects Cw, the line on which the summand C of A_SM acts by lambda, the up-type right-handed line. Proposition 21.1 supports this choice for a quark-only block, which needs the split of Proposition 50.3; with the universal block both choices keep 5/8 and 8/3. Then X = c(h) P_w: the Clifford action of a vector h in T_p S^4, the Higgs directions of lemmas 42 and 46, composed with the projector on the connector's line, with |h| the mass (Proposition 50.1). One point [w] of the twistor fibre over p, one point of CP^3, carries the Majorana block (on w (x) conj(v)) and the right-handed states of the massive member (on w (x) conj(v^perp), the top), and the massive left-handed line is C c(h) w; the connector's circle U(1)_b acts on T_p S^4 by scalars in the complex structure J_w defined by the Majorana point. In J_w, (T_p S^4, J_w) is the complex doublet of Proposition 42.2 of hypercharge -1/2, the conjugate doublet through which u_R couples (-J_w gives +1/2), and h is a vector of it. The circle U(1)_h that the block and the connector leave unbroken fixes the J_w-line of h and rotates its complement with charge -2: the neutral and the charged components. The projector P_w is the compression of the height function of CP^3 at [w], 5 cos^2 d(., [w]) - 1, as the grading Gamma_p is the compression of the height function 5 cos d(., p) of S^4 at p (Proposition 50.2); so the block is Y = m {5 Q(n . x), Q(h_w)}, the anticommutator of a compressed coordinate of the base S^4 and the compression, to the same kernel, of a height function of CP^3, a function of the type that lemma 46 places outside the base algebra M_2(H), and its anti-quaternionic half is the three-form term of lemma 49 with dual -J_w h/2, the quarter turn of the Higgs vector. In the two-sheet data of Corollary 16.8 a Yukawa block that differs between the lepton line and the colours, Y_l (x) P_v + Y_q (x) (1 - P_v), keeps order one exactly when P_v commutes with the algebra of the second sheet: on the Pati-Salam M_4(C) the block is universal and the Dirac neutrino and the up quark, the charged lepton and the down quark, have equal masses, while on C + M_3(C), the second-sheet algebra of the connector, the lepton and quark Yukawas separate, with the lepton line forced to be the connector's line when colour is kept (Proposition 50.3); so the top-only family of lemma 21 keeps order one on the bimodule of Corollary 16.8 exactly when the second-sheet algebra lies in

the commutant of the lepton projector, and with the connector on the same line exactly on the connector's algebra. Its centralizer in A_SM, by the explicit-mass criterion of Proposition 17.3, is C + M_3(C) with the summand C embedded in H_L along the line C c(h) w: the algebra of the label 9 (lemma 06), the retained algebra of lemma 44 and the corpus row rs_F1, with the embedding now fixed by the Higgs vector (Proposition 50.4).

Setting

W = C^4 with the Hermitian Clifford generators Gamma_1, ..., Gamma_5 and the antiunitary j of lemma 46, so that W = H^2 and End_H(W) = M_2(H) with unitary group G = Sp(2); the axis p in S^4 with Gamma_p = p^a Gamma_a, the quaternionic lines L_p (eigenvalue +1) and L_p^perp (eigenvalue -1), the projector Pi_p = (1 + Gamma_p)/2, and the stabilizer K_p = Sp(1)_a x Sp(1)_b with Sp(1)_b = Sp(L_p), Sp(1)_a = Sp(L_p^perp). The tangent space T_p S^4 = {n in R^5 : n . p = 0}, and for n in T_p S^4 we write c(n) = n^a Gamma_a, the Clifford action of lemma 49, which is j-linear, Hermitian, odd for Gamma_p and maps L_p onto L_p^perp. The second sheet V = C^4 carries the compression of the functions on CP^3 (Example 16.9, item 2); the bimodule is H_F = Hom(V, W) + Hom(W, V) = W (x) conj(V) + V (x) conj(W), with J_F(x, y) = (y^*, x^*) and the grading Gamma_p on the W factor, the even algebra H_R + H_L + M_4(C) = A_PS on L_p, L_p^perp and V.

The connector of Theorem 20.2 is T_kappa x = kappa (w^* x v) v w^* = kappa <e, x> f with e = w v^* in Hom(V, W), f = e^* = J_F e, unit vectors w in L_p and v in V, and kappa != 0; its centralizer is A_SM = {(lambda, q, m)} = C + H + M_3(C), lambda acting on L_p as diag(lambda, conj(lambda)) in the basis (w, jw), q in H = End_H(L_p^perp), lambda on Cv and m in M_3(C) on v^perp. In the dictionary of Chamseddine, Connes and Marcolli (arXiv:hep-th/0610241, Definition 2.14, Proposition 2.11 for A_F = {(lambda, q_L, lambda, m)}, Remark 2.18 for the identification of the up arrow with nu and u and of the down arrow with e and d, and equation (TmapsMR) for the Majorana block on up_R (x) 1^0; paper/LITERATURE.md, section O), lambda acts on nu_R and u_R and conj(lambda) on e_R and d_R, and the lepton line of V is Cv; so the states of the line Cw (x) conj(V) are nu_R = e = w v^* on the lepton line and u_R = w c^*, c in v^perp, on the three colours, those of Cjw (x) conj(V) are e_R and d_R, and L_p^perp (x) conj(V) carries the left-handed doublets. The block S_T = [[0, T^*], [T, 0]] connects e with J_F e: the Majorana mass of nu_R (lemma 20, reading 1).

The compression Q is that of lemma 10, Q(f) = 4 int_{CP^3} f([w']) P_{w'} dmu([w']) with mu the Fubini-Study probability measure, extended complex-linearly to complex functions. So Q(1) = 1 and Q(5 x_a) = Gamma_a (Proposition 10.1, with x_a([w']) = w'^* Gamma_a w'/w'^* w'), and Example 16.9, item 2, gives Q(5 w'^* B w'/w'^* w') = B + tr(B) 1 for Hermitian B. This is the convention used in lemma 46, item 3; the functions on CP^3 are functions of the unit vector w' modulo phase, and d denotes the Fubini-Study distance, cos d([w'], [w]) = |<w', w>|/(|w'| |w|), as cos d(y, p) = y . p on S^4.

Statements

Proposition 50.1 (the rank-one block and the connector share one point of the twistor fibre). Let X in Hom_C(L_p, L_p^perp) be a Yukawa block with masses m_- = 0 < m_+ = m, so that X = m ell mu^* with unit vectors mu in L_p and ell in L_p^perp (Proposition 46.2 (2)), and let the connector mark the line Cw in L_p.

1. The map n -> c(n) w is a real-linear isometry of T_p S^4 = R^4 onto L_p^perp (as a real four-space); so ell = c(n) mu for a unique unit n in T_p S^4, and X = c(h) P_mu with h = m n, P_mu = mu mu^*. The Hermitian block is Y = {c(h), P_mu} = c(h) P_mu + P_mu c(h); its j-linear half is c(h)/2 and its j-antilinear half is {c(h), rho_mu}/2 with rho_mu = P_mu - P_{j mu}, both of operator norm m/2 and Hilbert-Schmidt norm m.
2. The following are equivalent: (a) Cw is an eigenline of X^* X (the eigenline condition of Corollary 46.4); (b) C mu = Cw or C mu = C jw; (c) the stabilizers of X and of the connector's line in K_p share a circle.
3. Under (b), the massive right-handed states are those of Cw (x) conj(V), nu_R and u_R, when C mu = Cw, and those of Cjw (x) conj(V), e_R and d_R, when C mu = Cjw. With the block universal on V

(Corollary 16.8, Y (x) 1_V), the first case gives the mass m to the up quark and to the Dirac neutrino, and the second to the down quark and to the charged lepton. In the scope of lemma 18 both cases keep cos^2 theta_NC = 5/8 and the ratio 8/3: the neutrino contributes nothing, and the down-type quark and the charged lepton together contribute what the up-type quark contributes (Proposition 21.1, last statement). For a quark-only block (Y_l = 0, available on the commutant of the lepton projector by Proposition 50.3), Proposition 21.1 keeps both values in the first case only; the second gives 59/92 and 46/15. We take the first case, the up-type assignment of the heavy member, as an input that Proposition 21.1 supports in this sense. In the first case, X = c(h) P_w: one point [w] of the twistor fibre P(L_p) = S^2 over p, one point of CP^3 = G/H, carries the Majorana block, on the line P_w Hom(V, W) P_v = C e, and the massive right-handed states, on P_w Hom(V, W); the massive left-handed line is C c(h) w in L_p^perp.

4. In that case the joint stabilizer of X and of the connector's line in K_p is the circle U(1)_h = {(a_theta, b_theta)}, with b_theta = e^{i theta} on w and e^{-i theta} on jw and a_theta = e^{i theta} on c(h) w and e^{-i theta} on j c(h) w; c(n) commutes with it. Let J_w be the complex structure of T_p S^4 defined by c(J_w n') w = i c(n') w. Then J_w is an orthogonal complex structure, Sp(1)_b acts on T_p S^4 = Hom_H(L_p, L_p^perp) by t -> t b^{-1}, and U(1)_b = {b_theta} acts on (T_p S^4, J_w) by the scalars e^{-i theta}: (T_p S^4, J_w) is the complex doublet (2)_{-1} of Proposition 42.2. With Y = charge/2 it has hypercharge -1/2: it is the conjugate doublet through which u_R couples. (T_p S^4, -J_w) is the doublet (2)_{+1} of hypercharge +1/2, and -J_w = J_{jw} is the complex structure of the antipodal point [jw] of the fibre. The map n' -> c(n') w is a J_w-complex-linear isomorphism onto L_p^perp that intertwines Sp(1)_a, and h is a vector of the doublet, with |h| = m. The joint circle U(1)_h acts on T_p S^4 by fixing the J_w-complex line span_R(h, J_w h) and by e^{-2i theta} on its J_w-orthogonal complement: the neutral and the charged components of the doublet. The blocks commuting with U(1)_h on both chiralities are the X' = c(n)(y_1 P_w + y_2 P_{jw}) with y_1, y_2 complex, the diagonal blocks of Proposition 49.3 with the circle of the lifted transposition replaced by U(1)_h, j-linear exactly when y_2 = conj(y_1).

Proof. (1) For n, n' in T_p S^4, c(n) c(n') + c(n') c(n) = 2 n . n', and c(n) w lies in L_p^perp since c(n) anticommutes with Gamma_p; so Re <c(n) w, c(n') w> = Re <w, c(n) c(n') w> = n . n' (the anticommutator is 2 n . n' times the identity and the commutator is anti-Hermitian, with purely imaginary expectation), which makes n -> c(n) w a real isometry from R^4 into the real four-space L_p^perp, hence onto. Write ell = c(n) mu for the unit n so obtained from mu; then X = m c(n) mu mu^* = c(h) P_mu, and Y = X + X^* = c(h) P_mu + P_mu c(h) since c(h) is Hermitian and P_mu is a projection. Put P_mu = (Pi_p + rho_mu)/2, where P_mu + P_{j mu} = Pi_p because mu and j mu are orthogonal and span L_p; j P_mu j^{-1} = P_{j mu}, so j rho_mu j^{-1} = -rho_mu and j Pi_p j^{-1} = Pi_p. Then Y = (c(h) Pi_p + Pi_p c(h))/2 + {c(h), rho_mu}/2 = c(h)/2 + {c(h), rho_mu}/2, since Pi_p c(h) = c(h)(1 - Pi_p); the first term is j-linear and the second j-antilinear. For the norms, c(h) has operator norm |h| = m and {c(h), rho_mu} = c(h) rho_mu + rho_mu c(h) with (c(h) rho_mu)^* (c(h) rho_mu) = rho_mu c(h)^2 rho_mu = m^2 rho_mu^2 = m^2 Pi_p, so both halves have operator norm m/2 and squared Hilbert-Schmidt norm (m^2/4) x 4 = m^2. (2) X^* X = m^2 P_mu has the eigenlines C mu and its orthogonal complement in L_p, which is C j mu; so Cw is an eigenline exactly when Cw = C mu or Cw = C j mu, that is C mu = Cw or C mu = C jw. The equivalence with (c) is Corollary 46.4. (3) The lambda-weight line of the summand C of A_SM is Cw by Theorem 20.2, so by the dictionary the states of Cw (x) conj(V) are nu_R (on the lepton line) and u_R (on v^perp), and those of Cjw (x) conj(V) are e_R and d_R; with Y (x) 1_V, the right-handed states of mass m are those of the massive line tensored with all of conj(V). Proposition 21.1 gives the removal values: u alone, (u, nu) and (d, e) give 5/8 and 8/3, and d alone gives 59/92 and 46/15. In the first case P_mu = P_w; the connector is kappa <e, .> f with e = w v^* = P_w e P_v, the unique element of P_w Hom(V, W) P_v up to scale; and X maps the line Cw onto C c(h) w. (4) For (a, b) in K_p, a X b^{-1} = X with X = m c(n) w w^* reads (a c(n) w)(b w)^* = c(n) w w^*, since w^* b^{-1} = (b w)^* for unitary b. An equality u_1 v_1^* = u_2 v_2^* of rank-one operators with unit vectors forces u_1 = e^{i theta} u_2 and v_1 = e^{i theta} v_2 for one phase. So b w = e^{i theta} w and a c(n) w = e^{i theta} c(n) w. Being j-linear, b and a act as e^{-i theta} on jw and on j c(n) w, which determines b in Sp(1)_b and a in Sp(1)_a for each theta. These elements fix the connector's line Cw too, and conversely the stabilizer of Cw in K_p is Sp(1)_a x U(1)_b, whose intersection with the stabilizer of

X is the displayed circle. For c(n): a_theta c(n) b_theta^{-1} w = e^{-i theta} a_theta c(n) w = c(n) w and a_theta c(n) b_theta^{-1} jw = e^{i theta} a_theta j c(n) w = e^{i theta} e^{-i theta} j c(n) w = c(n) jw, so c(n) commutes with U(1)_h. J_w is well defined because n' -> c(n') w is a real isometry onto L_p^perp and multiplication by i is an isometry of L_p^perp; J_w^2 = -1 and J_w is orthogonal, so it is an orthogonal complex structure on T_p S^4. Under (a, b) in K_p, the element t = c(n')|*{L_p} of Hom_H(L_p, L_p^perp) goes to a t b^{-1} (the action of Proposition 42.2), and for a = 1, b = b_theta: t b_theta^{-1} w = e^{-i theta} c(n') w = c(cos theta n' - sin theta J_w n') w, so U(1)_b acts on (T_p S^4, J_w) by the scalars e^{-i theta}, a complex doublet of charge -1 of U(1)_b in the normalization of lemma 42 (charge +1 with the opposite orientation of the circle), the (2)_{+-1} of Proposition 42.2. A block X' commutes with U(1)_h exactly when X' = c(n) A with A = c(n) X' in End_C(L_p) commuting with b_theta, since c(n) does; A commutes with b_theta exactly when it preserves the eigenlines Cw and Cjw, that is A = y_1 P_w + y_2 P*{jw}; and j X' j^{-1} = c(n)(conj(y_1) P_{jw} + conj(y_2) P_w) equals X' exactly when y_2 = conj(y_1), as in Proposition 49.3. ■

Remark (three generations). With a generation space C^3 and the quaternionic structure j (x) conj on W (x) C^3, the blocks commuting with U(1)_h are X' = c(n)(Y_u (x) P_w + Y_d (x) P_{jw}) with Y_u, Y_d in M_3(C), the up-type and down-type Yukawa matrices of the spectral Standard Model with a fixed Higgs direction n, the masses being the singular values of Y_u and Y_d and the j-linear ones those with Y_d = conj(Y_u); the top-only family of lemma 21 is Y_u = m_t P_g, Y_d = 0, for a generation line g, that is X' = c(h) (P_w (x) P_g).

Prior work (referee). The blocks c(n)(Y_u (x) P_w + Y_d (x) P_{jw}) of item 4 and of the remark are the Yukawa blocks of the spectral Standard Model. In Chamseddine, Connes and Marcolli (arXiv:hep-th/0610241, read in the e-print source), the discrete part of the inner fluctuation is a quaternion-valued Higgs field H (Proposition 3.5). The coupling of the up-arrow part is related to that of the down-arrow part by H -> jH, and Lemma 3.6 gives Tr((D_3^{(0,1)})^2) = 12 |1 + H|^2 Tr(Y_u^* Y_u + Y_d^* Y_d). In the variables of the sheet the quaternion 1 + H is c(n) up to normalization, and the operators D(Y) of Theorem 2.21 there are the case of a fixed n. The rank-one block c(h) P_w is the case Y_d = 0. The converse, the identification of P_w with the projector on the connector's line, and the reading of J_w were not found in those sources. That the points of the twistor fibre over p parametrize the orthogonal complex structures of T_p S^4 of one orientation is the twistor correspondence (Atiyah, Hitchin and Singer, Proc. R. Soc. Lond. A 362 (1978) 425-461, doi:10.1098/rspa.1978.0143). J_w is that complex structure at [w]: it is orthogonal, of one orientation class over the whole fibre, and J_{jw} = -J_w.

Proposition 50.2 (the projectors and the grading as compressions of height functions). In the convention of the Setting, for every B in End_C(C^4),

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Q(5 w'^* B w' / w'^* w') = B + tr(B) 1
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(Example 16.9, item 2, extended complex-linearly). Consequently:

1. 5 Q(p . x) = Gamma_p = 2 Pi_p - 1: the grading is the compression of the height function 5 cos d(., p) of S^4 at the axis (lemma 20, reading 2).
2. Q(h_w) = P_w for h_w([w']) = 5 |<w', w>|^2/|w'|^2 - 1 = 5 cos^2 d([w'], [w]) - 1: the projector on the connector's line is the compression of the height function of CP^3 at the marked point [w]; in the same way P_v = Q_V(h_v) on the second sheet, which is the same kernel with the same functions (Example 16.9, item 2).
3. The block of Proposition 50.1 is Y = m {5 Q(n . x), Q(h_w)}, and the Majorana block is supported on the line P_w Hom(V, W) P_v = Q(h_w) Hom(V, W) Q_V(h_v); the j-antilinear half of Y is i c(delta') Gamma_p with delta' = -J_w h/2, the sign being fixed by the definition of J_w (with J_{jw} = -J_w in place of J_w it changes; for a rank-one block c(h) P_mu without the eigenline condition, delta' = -J_mu h/2), the three-form term of lemma 49 whose dual is the quarter turn of the Higgs vector in the complex structure of the Majorana point.

Proof. The fourth moment of lemma 10, E[w_j w_k conj(w_i) conj(w_l)] = (delta_ij delta_kl + delta_jl delta_ki)/20 on the unit sphere of C^4, gives Q(5 w'^* B w')_{kl} = 20 sum_{ij} B_ij (delta_ij delta_kl + delta_jl delta_ki)/20 = tr(B) delta_kl + B_kl for every complex B. This is Example 16.9, item 2.

Equivalently, by Schur's lemma: the assignment B -> Q(f_B), f_B([w']) = 5 w'^* B w'/w'^* w', is complex linear and U(4)-equivariant, since f_{U B U^*}([U w']) = f_B([w']), the Fubini-Study measure is U(4)-invariant and the compression intertwines the action of U(4) on functions with its conjugation on the kernel; End_C(C^4) = C 1 + sl(4) under conjugation, with sl(4) irreducible, so by Schur's lemma Q(f_B) = alpha B_0 + beta tr(B) 1 with B_0 the traceless part. B = 1 gives f = 5 and Q(5) = 5, so beta = 5/4; B = Gamma_a gives f = 5 x_a and Q(5 x_a) = Gamma_a by Proposition 10.1 in the stated convention, so alpha = 1; hence Q(f_B) = B - (tr B/4) 1 + (5/4) tr(B) 1 = B + tr(B) 1. (As a check without Schur's lemma: with Q(f) = 4 int f P_{w'} dmu for the normalized measure, since int P_{w'} dmu = 1/4, the moments E|z_1|^4 = 1/10 and E|z_1 z_2|^2 = 1/20 on the unit sphere of C^4 give <w, Q(5 |<., w>|^2) w> = 2 and <jw, Q(5 |<., w>|^2) jw> = 1, and the U(3)-invariance under the stabilizer of w gives Q = 2 P_w + (1 - P_w) = P_w + 1.) (1) is B = Gamma_p. (2) is B = P_w, tr P_w = 1, minus the compression of the constant 1; the second sheet is a copy of the same kernel. (3) The first statement is Proposition 50.1 (1) with c(n) = 5 Q(n . x) (Proposition 10.1) and P_w = Q(h_w); the connector's line C e = C w v^* is P_w Hom(V, W) P_v. For the three-form: the j-antilinear odd Hermitian blocks are the real span of the i Gamma_a Gamma_p with a orthogonal to p (Proposition 46.1 (3)), so {c(h), rho_w}/2 = i c(delta') Gamma_p for a unique delta' in T_p S^4. Both sides are j-antilinear odd Hermitian blocks. Such a block Y satisfies Y jw = -j Y w, so the two sides agree when they agree on w. On w the left side gives (c(h) rho_w w + rho_w c(h) w)/2 = c(h) w/2, since rho_w vanishes on L_p^perp, and the right side gives i c(delta') w. So c(delta') w = -(i/2) c(h) w = c(-J_w h/2) w by the definition of J_w, and delta' = -J_w h/2 by the isometry of Proposition 50.1 (1). The same lines with mu in place of w give delta' = -J_mu h/2 for any rank-one block c(h) P_mu. The norm |delta'| = m/2 follows. As a reading: i rho_w = (1/2) sum_{a<b} omega_w(e_a, e_b) Gamma_a Gamma_b with omega_w(a, b) = <J_w a, b>, a two-form that is self-dual or anti-self-dual according to the orientation and acts on L_p alone. For a vector h and a two-form omega, {c(h), c(omega)} = 2 c(h wedge omega) with c(e_a wedge e_b wedge e_c) = Gamma_a Gamma_b Gamma_c. The Hodge dual of h wedge omega_w is, up to sign, the contraction of h with omega_w, the vector J_w h. This is the three-form term of lemma 49. With lemma 49's formulas for the block c(delta) + i c(delta') Gamma_p, delta = h/2 and delta' orthogonal to delta of the same length give m_+^2 + m_-^2 = 2 (|delta|^2 + |delta'|^2) = m^2 and m_+^2 - m_-^2 = 4 |delta wedge delta'| = m^2, with |delta wedge delta'| the area of the parallelogram, that is m_+ = m, m_- = 0, as it must. Lemma 49 states the formula for delta and delta' in the mirror plane, where the area is |det(delta, delta')|, and it holds for all delta, delta' in T_p S^4, being lemma 46's 4 |Im(conj(q) q')|. ■

Proposition 50.3 (the lepton and quark Yukawas separate exactly on the connector's algebra). In the data of Corollary 16.8 with V_X = W, A_{F,X} = A_X an even subalgebra containing Gamma_p (H_R + H_L or a subalgebra containing the grading), V_Y = V = C^4 ungraded with an algebra A_Y in M_4(C), let P_1, …, P_r be mutually orthogonal projections on V summing to 1 and Y_1, …, Y_r odd Hermitian blocks on W, and define

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D x = sum_i Y_i x P_i  (x in Hom(V, W)),    D y = sum_i P_i y Y_i  (y in Hom(W, V)).
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Then D is self-adjoint and odd, commutes with J_F, and satisfies order zero; its order-one defect is [[D, a], b°] x = sum_i [Y_i, a_X] x [b_Y, P_i] on Hom(V, W) and sum_i [P_i, a_Y] y [b_X, Y_i] on Hom(W, V). Hence:

1. (Universal block.) If all Y_i are equal, order one holds for every A_Y (Corollary 16.8).
2. (Split block.) If every P_i commutes with A_Y, order one holds for every choice of the Y_i.
3. (Converse.) If the Y_i are linearly independent, order one holds exactly when every P_i commutes with A_Y, since [Y_i, Gamma_p] = -2 Gamma_p Y_i are then linearly independent. For two terms, D = Y_1 (x) P_v + Y_q (x) (1 - P_v): order one holds exactly when Y_1 = Y_q or P_v commutes with A_Y. In general, with Gamma_p in A_X, order one holds exactly when D commutes with the right action x -> x b_Y of A_Y on Hom(V, W). By J_F the same then holds on Hom(W, V). Writing D x = sum_k E_k x Z_k with linearly independent E_k, order one holds exactly when every Z_k commutes with A_Y. Indeed [[D, a], b°] = [[D, b°], a] by order zero, and for a = Gamma_p the operator [D, b°] is odd and commutes with the grading, hence vanishes. This is Krajewski's decomposition of a first-order operator into a left-linear and a right-linear part, with the left-linear odd part zero because the grading lies in the left algebra (referee).

4. On the Pati-Salam sheet algebra A_Y = M_4(C) of Example 16.9 the block is universal: the lepton line and the three colours carry the same Y, so in each generation the Dirac neutrino and the up quark have the mass m_+ and the charged lepton and the down quark the mass m_- (Proposition 46.2), the lepton-quark unified Yukawas k^nu = k^u, k^e = k^d of the spectral Pati-Salam model. The commutant of P_v in M_4(C) is {lambda P_v + m : m in M_3(C) on v^perp} = C + M_3(C), the second-sheet part of the algebra A(T_kappa) of Theorem 20.2 for a connector on the line v; so the split block Y_l (x) P_v + Y_q (x) (1 - P_v) and the connector on v select the same second-sheet algebra, and on it both keep order one.
5. (One lepton line.) If a split block with lines P_{v'}, Y_l != Y_q, and a connector on the line v both keep order one on an algebra A_X + A_Y with A_Y containing End(v^perp) (colour unbroken), then Cv' = Cv.

So the top-only family of lemma 21, Y_l = 0 and Y_q = {c(h), P_w} (x) P_g, is a sheet-diagonal Dirac operator on the bimodule of Corollary 16.8 with order one exactly when the second-sheet algebra lies in the commutant C + M_3(C) of P_v. This operator lies outside the compressed form d_X (x) 1, which is what Corollary 16.8 gives for ungraded V. On the first sheet the family allows all of H_R + H_L, real dimension 28 in all. With the connector on the same line v added, order one of the sum holds exactly on A(T_kappa) = C + H + M_3(C) (Theorem 20.2; the defects of the two parts lie in different blocks). By (5), a connector and a split block with colour retained share the lepton line. The massive right-handed line is the Majorana line under the eigenline condition and the up-type assignment of Proposition 50.1 (3).

Proof. Self-adjointness: <x', D x> = sum_i tr(x'^* Y_i x P_i) = sum_i tr((Y_i x' P_i)^* x), using Y_i^* = Y_i, P_i^* = P_i = P_i^2 and the cyclicity of the trace; the same on the second summand. Oddness: the grading is Gamma_p (x) 1 on Hom(V, W) and -1 (x) Gamma_p^c on Hom(W, V) (Theorem 16.7), and the Y_i anticommute with Gamma_p. J_F D (x, y) = ((sum P_i y Y_i)^*, (sum Y_i x P_i)^*) = (sum Y_i y^* P_i, sum P_i x^* Y_i) = D (y^*, x^*) = D J_F (x, y). Order zero holds since the left actions (a_X on x, a_Y on y) commute with the right actions (b_Y on x, b_X on y). For order one, [D, a] x = sum_i (Y_i a_X - a_X Y_i) x P_i = sum_i [Y_i, a_X] x P_i, since a_X acts on the left of x and P_i on the right; b° acts by x -> x b_Y, so [[D, a], b°] x = sum_i [Y_i, a_X] (x b_Y P_i - x P_i b_Y) = sum_i [Y_i, a_X] x [b_Y, P_i]; the second summand is the same computation with the roles exchanged. (1) With Y_i = Y the sum is [Y, a_X] x [b_Y, sum P_i] = 0. (2) is immediate. (3) If order one holds, then for every a_X, b_Y the operator x -> sum_i [Y_i, a_X] x [b_Y, P_i] vanishes, that is sum_i [Y_i, a_X] (x) [b_Y, P_i]^t = 0 in End(W) (x) End(V); with a_X = Gamma_p the first factors -2 Gamma_p Y_i are linearly independent, so every [b_Y, P_i] = 0. For two terms with P_2 = 1 - P_1 the defect is [Y_l - Y_q, a_X] x [b_Y, P_v], which vanishes for all a_X, b_Y, x exactly when Y_l = Y_q (so that the first factor is 0 for every a_X) or [b_Y, P_v] = 0 for all b_Y, the first factor being nonzero for a_X = Gamma_p otherwise. (4) P_v does not commute with M_4(C), whose commutant is C 1, so the split block fails order one there by (3); the commutant of P_v is the block algebra End(Cv) + End(v^perp) = C + M_3(C), which is the second-sheet component of A(T_kappa) in Theorem 20.2, equation (6). (5) P_{v'} must commute with End(v^perp) + 0, so it preserves v^perp and restricts to it as a scalar; a rank-one projection cannot restrict to 1 on a three-space, so it restricts to 0, and v' lies in the orthogonal complement of v^perp, which is Cv. The last statement combines (4) and (5) with Proposition 50.1 (3). ■

Prior work (verified by the referee in the e-prints). The criterion is Krajewski's first-order rule (T. Krajewski, *Classification of finite spectral triples*, J. Geom. Phys. 28 (1998) 1-30, doi:10.1016/S0393-0440(97)00068-5, arXiv:hep-th/9701081). Section 3.4 there proves that the first-order operators on a bimodule over sums of matrix algebras are the sums of a left-linear and a right-linear operator, and that D = Delta + J Delta J^{-1} with Delta commuting with J pi(A) J^{-1}. Sections 4.1 and 4.2 show that the edges of a diagram are vertical or horizontal, with Delta_{ik}^{jl} = delta_{kl} M_{ij,k} (x) I_{n_k}. Proposition 50.3 is this rule in the two-sheet bimodule with the grading in the left algebra, the P_i being projections onto isotypic components of V under A_Y; on M_4(C) there is one right label and the block is universal. The Standard Model case is Theorem 2.21 of Chamseddine, Connes and Marcolli (arXiv:hep-th/0610241), with the subalgebra from Chamseddine and Connes (arXiv:0706.3688, section 4, Theorem 4.1). Chamseddine, Connes and van Suijlekom (arXiv:1304.8050, e-print read on 5 November 2026) give the tensor form of D_F with separate lepton and quark couplings in section 4, "Summary of tensor notation", and call the case k^nu = k^u, k^e

= k^d lepton and quark unification. In section 5 they note that the field Sigma_I^J, whose vacuum value is the projector on the lepton line, decouples in that case. Their Standard Model vacuum (section 8, equation SM-vev) is <phi_{a.}^b> = v delta_{a.}^{1.} delta_1^b, <Sigma_J^I> = u delta_1^I delta_J^1 and <Delta_{a.J}> = w delta_{a.}^{1.} delta_J^1. The Majorana field Delta and the Higgs bidoublet phi share the right-handed index 1., and Delta shares its Pati-Salam index with the lepton line of Sigma: these are the alignments of Propositions 50.1 and 50.3, taken there as the chosen vacuum, at which the Hessian in the Delta directions is nonnegative. The order-one forcing of the shared lepton line with colour retained (item 5) and the statement that order one on M_4(C) forces the unified Yukawas were not found stated in those papers; the latter follows from Krajewski's rule. Chamseddine, Connes and van Suijlekom introduce the unified case as a special case, also in arXiv:1507.08161, where the table of Pati-Salam scalars notes that Sigma decouples under quark-lepton coupling unification. See paper/LITERATURE.md, section P.

Proposition 50.4 (the centralizer of the top block is the algebra of the label 9). Let D_F = Y_q (x) (1 - P_v) + S_{T_kappa} on H_F, with Y_q = {c(h), P_w}, h != 0 (the top-only family: the lepton block zero, one generation), the connector on the lines w and v, and A_SM = {(lambda, q, m)} = A(T_kappa) of Theorem 20.2. Then

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{a in A_SM : [D_F, pi(a)] = 0} = {(lambda, lambda_h, m) : lambda in C, m in M_3(C)} = C + M_3(C),
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of real dimension 20, where lambda_h denotes lambda in the embedding C -> H_L = End_H(L_p^perp) whose weight line is C c(h) w, the massive left-handed line: q = lambda_h exactly when q (c(h) w) = lambda c(h) w. The connector commutes with all of A_SM, so the same holds for D_F with kappa = 0. The unitary group of the centralizer is U(1) x U(3), and under the unimodularity condition inherited from A_SM (Proposition 41.5, theta_lambda + 3 theta_c = 0) it acts on H_F, up to a finite group, as U(1)_em x SU(3) with the electric charges (2/3, -1/3, -1, 0): the gauge group of the label 9 (lemma 06, lemma 21, reading 3). By Proposition 17.3 there is no scalar fluctuation on the centralizer and the top mass is explicit there. The algebra coincides with the retained algebra of lemma 44 along the seesaw scaling (Proposition 44.1), which uses the same embedding when the block is universal (Proposition 50.3 (4)), and it is the corpus row rs_F1, the centralizer of a dominant top, with the embedding of C in H now fixed by the Higgs vector h.

Proof. For a = (lambda, q, m) in A_SM, pi(a) acts on Hom(V, W) by x -> a_W x with a_W = lambda P_w + conj(lambda) P_{jw} + q, q on L_p^perp, and on Hom(W, V) by y -> a_V y with a_V = lambda P_v + m. On Hom(W, V) the commutator is [1 - P_v, a_V] y Y_q = 0; the connector commutes with pi(a) by Theorem 20.2. So [D_F, pi(a)] = 0 exactly when [Y_q, a_W] = 0. On w: Y_q a_W w = lambda c(h) w and a_W Y_q w = q c(h) w. On jw: Y_q jw = c(h) P_w jw + P_w c(h) jw = 0 + P_w j c(h) w = 0, since j c(h) w lies in L_p^perp, and a_W jw is a multiple of jw, so both sides vanish. On u in L_p^perp: Y_q u = P_w c(h) u = <c(h) w, u> w, so a_W Y_q u = lambda <c(h) w, u> w and Y_q a_W u = <c(h) w, q u> w = <q^* c(h) w, u> w; these agree for all u exactly when q^* c(h) w = conj(lambda) c(h) w, which for the normal operator q (the left multiplication by a quaternion, q q^* = |q|^2) is the same as q c(h) w = lambda c(h) w. So the centralizer is the set of (lambda, q, m) with c(h) w a lambda-eigenvector of q; since q is j-linear, q then acts as conj(lambda) on j c(h) w, so q = diag(lambda, conj(lambda)) in the basis (c(h) w, j c(h) w) of L_p^perp, which is lambda in the embedding of C in H with that weight line, unique for each lambda; the real dimension is 2 + 18 = 20. The unitary group of C + M_3(C) is U(1) x U(3); the inherited unimodularity condition and the resulting charges are Proposition 41.5 (the lift u J u J^{-1} of the unitaries of A_SM restricted to the centralizer; Proposition 41.5 is stated for the embedding Q_lambda along nu_L and u_L, and lambda_h is its conjugate by an element of Sp(1)_a, an inner automorphism of A_SM that preserves the traces of the left action, so the condition theta_lambda + 3 theta_c = 0 and the charges transfer; recomputed for lambda_h they are (2/3, -1/3, -1, 0) on both chiralities). Proposition 17.3 applies to a subalgebra commuting with D_F. Lemma 44, Proposition 44.1, gives {(lambda, Q_lambda, m)} with Q_lambda the embedding along the nu_L line, which is c(h) w when the neutrino block is Y_q (x) P_v with the same h. The corpus row rs_F1 is the statement with the top block as a hypothesis; the embedding was left unspecified there. ■

Reading

The three finite data of the window, the axis, the connector and the Yukawa block (lemma 46, Reading), now stand in this relation: the connector marks the point [w] of the twistor fibre over the axis and the line v of the second sheet; under the eigenline condition and the up-type assignment, the top block is the Higgs vector h in T_p S^4, read in the complex structure J_w of the Majorana point, applied to the projector on the Majorana line; and both projectors and the grading are compressions of height functions at the marked points, P_w = Q(h_w), P_v = Q_V(h_v), Gamma_p = Q(5 p . x). The rank-one pattern, in which the quaternionic and anti-quaternionic halves have equal norm (Proposition 46.2 (2)), is the statement that the dual of the three-form term of lemma 49 is the quarter turn of the Higgs vector, delta' = -J_mu delta, in the complex structure of the massive right-handed point [mu] of the fibre, which is [w] under the eigenline condition. Conversely, every delta' orthogonal to delta and of the same length is -J_mu delta for exactly one [mu], so the rank-one blocks are the pairs (h, [mu]) (by lemma 49's formula, |delta| = |delta'| with delta orthogonal to delta' gives m_- = 0, hence rank one, hence c(h) P_mu). The custodial breaking |Im(conj(q) q')| = (m_+^2 - m_-^2)/4 of lemma 46 is here the area |delta wedge delta'|, which at fixed m_+^2 + m_-^2 is maximal exactly for the rank-one pattern. The top and the neutrinos share the lines. The right-handed states of both lie on the Majorana line w, and the left-handed ones on C c(h) w, which for the neutrinos is the neutral line of the embedding lambda_h of Proposition 50.4, and their Dirac line when the block is universal. The difference is the factor conj(V): the lepton line for the neutrinos, the three colours for the top. So the exchange of the top with the neutrinos of lemma 21 (Proposition 21.3) and lemma 34 (Proposition 5.2) can be read as a transposition of the colour triplet of one generation with the generation triplet of the lepton line, at fixed lines in W. In conj(V) (x) C^3 these are a 3 x 1 and a 1 x 3 block of a 4 x 3 array, and the transposition becomes a symmetry of the array only with four families, as in lemma 34. This is recorded as a reading. With the universal block the Dirac neutrino mass is the top mass, and the connector's kappa on the same line gives the neutral block [[0, m_t], [m_t, kappa]] on (nu_L, nu_R) in each generation, with light mass about m_t^2/kappa for kappa » m_t (lemma 44, Proposition 44.2, with y = m_t); for m_nu = 0.05 eV and m_t = 173 GeV this is kappa of about 6 x 10^14 GeV, of the order of the meeting point of lemma 26 (2.2 x 10^14 GeV). With the top mass run to that scale, about half its low-energy value in the Standard Model (recalled, not computed here), kappa is about 1.5 x 10^14 GeV, so the comparison holds within the factors of the running. With kappa equal on the three generations, the same relation gives m_c^2/kappa, about 3 x 10^-6 eV, for the second neutrino, far below the solar scale of about 9 x 10^-3 eV. So the remark concerns the heaviest neutrino, and the others need a hierarchical Majorana matrix. It is a remark under the hypothesis of the universal block, which the separation of Proposition 50.3 removes. The heavy top, by the explicit-mass criterion, leaves the algebra C + M_3(C) of the label 9 with the summand C embedded along its left-handed line: the same algebra that lemma 06 reaches from the strict heavy limit, that lemma 44 retains along the seesaw scaling and that the corpus assumed for the dominant top; and the top-only family of lemma 21 keeps order one on the bimodule of Corollary 16.8 only on a second-sheet algebra inside the commutant of a lepton projector, which the connector supplies with its own line; on the Pati-Salam M_4(C) the block is universal and the leptons are as massive as the quarks. For the dictionary note: the Yukawa block of the top is the Higgs vector times the Majorana projector; the grading and the two projectors are compressed height functions; order one on the Pati-Salam sheet is the lepton-quark unification of the Yukawas, and its relaxation to the connector's algebra is their separation.

Scope

Finite linear algebra on the quaternionic sheet with the Clifford structure of Example 16.9 and the connector of Theorem 20.2; one generation except in the remark after Proposition 50.1; the axis, the connector and the Yukawa block are inputs, as in lemmas 42 and 46; Proposition 50.2 holds in the convention of lemma 10, where Q(5 x_a) = Gamma_a, and the sign of delta' is fixed by the definition of J_w (it changes with J_{jw} = -J_w in place of J_w); the up-type assignment of Proposition 50.1 (3) is an input, which Proposition 21.1, in the scope of lemma 18 (the classical bosonic trace at fixed cutoff), supports for a quark-only block, while for the universal block the two cases give the same values; Proposition 50.3 concerns sheet-diagonal operators on the bimodule of Corollary 16.8 with projections on the second sheet, which for a split block lie outside the compressed form d_X (x) 1 of that corollary; its proof applies verbatim to three generations, with W (x) C^3 and Gamma_p (x) 1 in place of W and Gamma_p; Proposition 50.4 is the algebraic centralizer

by the criterion [D_F, a] = 0, and Counterexample 18.8 shows that mass decoupling in the spectral action does not select it dynamically. Nothing here derives the block, the connector or the axis, and the identity of Proposition 50.2 is Example 16.9, item 2, extended complex-linearly: at level one on CP^3 the contravariant symbol of B is 5 times its covariant symbol w'^* B w'/w'^* w' minus tr(B), so h_w is the contravariant symbol of P_w (Berezin's covariant and contravariant symbols; paper/LITERATURE.md, section P); in general, at level one on CP^{N-1}, Q((N + 1) sigma_B - tr(B)) = B with Q(f) = N int f P dmu (referee). The direction the result points to: a parent in which the Higgs vector h, the Majorana point [w] and the lepton line v are fields, h as the variation of the axis (lemma 42) and [w] as the variation of the point on the fibre, for which the fields Delta, Sigma and phi of the spectral Pati-Salam model (arXiv:1304.8050) are such a parent at the finite level, and a computation in its potential would test whether it forces the eigenline condition and the shared lepton line (referee); and, for item (b) of lemma 49, a Dirac operator on the four-torus at a point of a mirror whose three-form term is the quarter turn of the displacement in the complex structure of the lifted transposition, since that is the rank-one pattern of the heaviest quark.

Part 51

The Dirac operator of the four-torus with a Wilson line and a constant three-form: its lowest window is the Yukawa block, the photon condition is the invariance under the transposition, and the heaviest quark is the torsion equal to the quarter turn of the Wilson line

For STRATEGY T16 (item (b) of Remark 45.3: a Dirac operator from the geometry), T12 and T13, taking up the direction stated at the end of lemma 49 (a Dirac operator on the torus with a constant three-form term as the candidate that supplies both halves of an x-invariant Yukawa block and derives the condition [D_F, C_F] = 0 from the geometry) and the end of lemma 50 (the rank-one pattern as the three-form equal to the quarter turn of the Higgs vector), following lemma 48 (the torus A, its spin structure and the lifts of Proposition 48.1), lemma 49 (Propositions 49.1, 49.3 and 49.4), lemma 46 (Propositions 46.1 and 46.2), lemma 22 and lemma 25 (the finite Dirac operator as the lowest window of a parent with a constant connection), lemma 16 (Corollary 16.8) and lemma 17 (Proposition 17.3). Written by Claude Fable 5.1 on 5 November 2026, by hand; the three-form identity, the mass formula on 200 random pairs, the isolation of the window on 400 random pairs with the momenta enumerated in a box, the covariance x c(v) x^{-1} = c(d tau v), the commutation exactly for data in the mirror plane, the eigenvalues of the lift on the two chiralities, the rank-one case and the fixed space of a three-cycle were checked with numpy (script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_51_52.md), who rebuilt the Clifford set-up, tested the spectrum on 2000 pairs and the window on the actual lattice of A, computed the invariant blocks, the weight lines and the 24-state bimodule at the mirror, and read the e-prints of Agricola and of Friedman, Morgan and Witten: Propositions 51.1 and 51.2 correct with amendments (Agricola's normalization verified, the Wilson line as a gauge class), Proposition 51.3 with one error in item 2 (the heavier member is fixed by the sign of the oriented area of (A, eta) in the complex line of the Majorana point, which the orientation of A does not fix; the intrinsic form is |y_mu| = |A + J_mu eta| on each weight line) and amendments (the orientability relation fails at the mirror), and two errors in the Reading (the mass with a torsion along the Wilson line is (|A|^2 + |eta|^2)^{1/2}; A = Lambda_coroot (x) E = Hom(Lambda_weight, E)); all applied below with the referee's texts.

Outcome. On the flat four-torus A of lemma 48 with the periodic spin structure, the Dirac operator with a constant real one-form A (a flat U(1) connection, a Wilson line) and a constant real three-form H (a torsion or a flux term) is translation invariant, and on the sector of internal momentum k it is the constant block c(k + A) + i c(eta) Gamma on the spinor fibre Sigma_4, with eta the one-form dual to H up to the normalization of the three-form term; the constant spinors form its lowest window when |A| + |eta| is below half the first momentum, and there the operator is Y = c(A) + i c(eta) Gamma: a Yukawa block of lemma 46 with the Wilson line as its quaternionic, custodial half and the three-form as its anti-quaternionic half, of masses m_+-^2 = |A|^2 + |eta|^2 +- 2 |A wedge eta| (Proposition 51.1). The operator commutes with the lift of a transposition tau of the Weyl group exactly when A and eta lie in the mirror plane of tau, the tangent

plane of the mirror z_i = z_j, which is a complex line of A; then the window block commutes with the lifted transposition x on both chiralities, which is the condition [D_F, C_F] = 0 of Chamseddine, Connes and Marcolli (Proposition 49.3), so the photon condition is the Weyl invariance of the Wilson line and the torsion; the x-invariant blocks of Proposition 49.3 are exactly these; a nonzero Wilson line or torsion along a mirror is invariant under that transposition alone, so it chooses one of the three mirrors, and no nonzero data are invariant under a three-cycle, so the symmetric points carry no Yukawa block (Proposition 51.2). In the complex structure J of the mirror plane, inherited from the elliptic curve, the two members of the doublet have equal masses exactly when the torsion is parallel to the Wilson line, masses |A| + |eta| and ||A| - |eta|| exactly when it is perpendicular, and one massless member exactly when eta = +- J A: the heaviest quark is the generation whose torsion is the quarter turn of its Wilson line, and in that perpendicular case, for |eta| <= |A|, the Wilson line is the mean mass of the doublet and the torsion half its splitting, the up-type member being the heavier when <J_w A, eta> < 0; with the up-type member on the Majorana line w, the up and down Yukawa couplings are A + J_w eta and A - J_w eta in the complex line (P_tau, J_w), J_w = +-J according to which weight line of x is the Majorana line (Proposition 51.3). The finite data of Corollary 16.8 at the mirror stratum, with the algebra C of Proposition 49.4, this block and the colour sheet, are the quark sector with the algebra C + M_3(C) of lemma 50, Proposition 50.4, explicit masses and no one-forms: the label 9.

Setting

Let A = R^4/Lambda be the flat four-torus of lemma 48, A = a (x) R^2 with a the standard representation of S_3 and R^2 the tangent space of the elliptic curve E, with the periodic spin structure, the one invariant under S_3; its spinor bundle is A x Sigma_4, Sigma_4 = C^4 = Sigma_+ + Sigma_- with the Clifford action c(v) of v in R^4 (Hermitian, c(v)^2 = |v|^2), the chirality Gamma = c(e_1) c(e_2) c(e_3) c(e_4) for an oriented orthonormal basis, and the quaternionic structure j commuting with Spin(4), c and Gamma (lemma 49, Setting). By Proposition 49.1 (Sigma_4, c(e_1), ..., c(e_4), Gamma, j) is the quaternionic sheet W = H^2 of Example 16.9 with axis p at Gamma, L_p = Sigma_+, L_p^perp = Sigma_-, and T A = T_p S^4 through v -> c(v). The Dirac operator is D = -i sum_a c(e_a) partial_a on L^2(A, Sigma_4), self-adjoint on H^1, with kernel the constant spinors; its spectrum is the union over the dual lattice of the eigenvalues +-|k| of c(k) on the sectors e^{i k . z} Sigma_4, k in 2 pi Lambda^*. Let 2 pi/l be the length of the shortest nonzero vector of 2 pi Lambda^* (l the side for a cubic torus).

The data: a constant real one-form A = A_a e^a, the connection d + i A of the trivial line bundle, flat, with holonomy e^{i A(lambda)} around lambda in Lambda, a Wilson line, defined modulo 2 pi Lambda^* by the gauge transformations e^{i k . z}, which shift the momentum; and a constant real three-form H. The Dirac operator with these data is

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D_{A,H} = D + c(A) + s i c(H),
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where c(H) = sum_{a<b<c} H_{abc} c(e_a) c(e_b) c(e_c) is anti-Hermitian, so that s i c(H) is Hermitian for real s, and s is the constant of the chosen normalization of the three-form term. In Agricola's lectures (arXiv:math/0606705, section "The square of the Dirac operator and parallel spinors", read in the e-print by the referee) the spinor connection nabla^{s'}_X psi = nabla^g_X psi + s' (the contraction of T with X) . psi has torsion 4 s' T and Dirac operator D^{s'} = D^g + 3 s' T. In the Hermitian convention used here this is s = 3 s'. So s = 3/4 for the Dirac operator of the connection with torsion H (s' = 1/4), and s = 1/4 for the connection with torsion H/3 (s' = 1/12). The second is the rescaling Bismut used for his local index theorem, and on naturally reductive spaces it is Kostant's cubic Dirac operator. The coupling of a three-form flux to spinors gives another constant. The value of s enters none of the statements, provided s != 0. In four dimensions c(e_a) c(e_b) c(e_c) = +- Gamma c(e_d) for {a, b, c, d} = {1, 2, 3, 4} (for instance c(e_1) c(e_2) c(e_3) = Gamma c(e_4)), so for every constant three-form H there is a unique eta in R^4, eta = +- s *H with the sign fixed by the orientation conventions, such that

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s i c(H) = i c(eta) Gamma,
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and, for s != 0, every eta arises from a unique H. All statements below are in terms of eta. By Proposition 46.1, i c(eta) Gamma is the general j-antilinear odd Hermitian block and c(A) the general j-linear one.

The Weyl group S_3 acts on A by the permutations of the coordinates (z_1, z_2, z_3) of E^3 restricted to z_1 + z_2 + z_3 = 0; a transposition tau acts on T A = a (x) R^2 by rho(tau) (x) 1, the reflection of a in the hyperplane orthogonal to its normal n_tau, so d tau is the rotation by pi in the plane n_tau (x) R^2 with fixed plane P_tau = n_tau^perp (x) R^2, the tangent plane of the mirror {z_i = z_j}, which is a complex line for the complex structure J = 1 (x) J_E that A inherits from E; the lift of tau to Spin(4) is x = +- c(f_1) c(f_2) for an orthonormal basis of n_tau (x) R^2, of order four, acting on both chiralities with eigenvalues +-i (lemma 48, Proposition 48.1; lemma 49, Proposition 49.2), and tau-tilde psi(z) = x psi(tau^{-1} z) is its action on spinor fields, which is well defined for the periodic spin structure. A three-cycle acts by a rotation of order three of a tensored with the identity of R^2.

Statements

Proposition 51.1 (the lowest window of the torus with a Wilson line and a torsion). D_{A,H} commutes with the translations, and on the sector e^{i k . z} Sigma_4 it is the constant matrix

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Y_k = c(k + A) + i c(eta) Gamma,
```

an odd Hermitian block of lemma 46 with j-linear half c(k + A) and j-antilinear half i c(eta) Gamma, whose eigenvalues are +-m_+(k) and +-m_-(k), each once, with

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m_+-(k)^2 = |k + A|^2 + |eta|^2 +- 2 |(k + A) wedge eta|,
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|v wedge eta| = (|v|^2 |eta|^2 - (v . eta)^2)^{1/2}. If |A| + |eta| < pi/l, the constant spinors are the lowest spectral window of D_{A,H}: the four eigenvalues +-m_+(0), +-m_-(0) lie in [-(|A| + |eta|), |A| + |eta|] and every other eigenvalue has absolute value at least 2 pi/l - |A| - |eta|, which is larger. The compression of D_{A,H} to that window is exact and equals

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Y = c(A) + i c(eta) Gamma,
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the Yukawa block of lemma 46 on the sheet Sigma_4 = W of Proposition 49.1, with q <-> A and q' <-> eta in the variables of Proposition 46.2 and |Im(conj(q) q')| = |A wedge eta|. The tower above the window has the masses m_+-(k), k != 0, the Wilson line shifting the momenta and the torsion splitting each level into two.

Proof. The coefficients of D_{A,H} are constant, so it commutes with the translations z -> z + z_0 and preserves each sector e^{i k . z} Sigma_4, where -i partial_a acts as k_a; so D_{A,H} e^{i k . z} psi = e^{i k . z} (c(k + A) + s i c(H)) psi = e^{i k . z} Y_k psi. Y_k is Hermitian, odd for Gamma (c(v) anticommutes with Gamma, and i c(eta) Gamma anticommutes with Gamma), c(k + A) commutes with j and i c(eta) Gamma anticommutes with it. For the spectrum put v = k + A and X = Y_k restricted to L_p = Sigma_+, X : L_p -> L_p^perp; on L_p, Gamma = 1, and

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X^* X = (c(v) - i Gamma c(eta)) (c(v) + i c(eta) Gamma) = c(v)^2 + c(eta)^2 + i (c(v) c(eta) -
↪ c(eta) c(v)) Gamma = |v|^2 + |eta|^2 + 2 i c(v wedge eta),
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restricted to L_p, where c(v wedge eta) = (c(v) c(eta) - c(eta) c(v))/2 is the Clifford action of the two-form v wedge eta; we used Gamma c(eta) = -c(eta) Gamma and Gamma = 1 on L_p. The two-form v wedge eta is simple of norm |v wedge eta|, so c(v wedge eta) squares to -|v wedge eta|^2 and is traceless on each chirality (it is a multiple of c(f_1) c(f_2) for an orthonormal pair, which commutes with Gamma and has trace zero on each chirality), with eigenvalues +-i |v wedge eta| on L_p; hence X^* X has the eigenvalues m_+-^2 as displayed, and Y_k, with X^* X on L_p and X X^* on L_p^perp of the same spectrum, has the eigenvalues +-m_+, +-m_-, each once. For the window: m_+(0) <= |A| + |eta| by the triangle inequality for the operator norm of Y_0 = c(A) + i c(eta) Gamma, and for k != 0, m_-(k) >= |k + A| - |eta| >= 2 pi/l - |A| - |eta|, since m_-(k)^2 >= (|k + A| - |eta|)^2 (the smaller eigenvalue of X^* X is at least (|v| - |eta|)^2 because |v wedge eta| <= |v| |eta|) and |k| >= 2 pi/l; the inequality |A| + |eta| < 2 pi/l - |A| - |eta| is the hypothesis. The compression to the invariant subspace of the constant spinors is the restriction, Y_0 = Y. The identification with lemma 46 is Proposition 49.1, with the j-linear and j-antilinear halves matched by Proposition 46.1, and Proposition 46.2 gives the same masses. The map v -> c(v)|_{L_p} identifies R^4 isometrically with Hom_H(L_p, L_p^perp) = {L_q}, so q = q(A) and q' = q(eta) for one real-linear

isometry q : R^4 -> H. For quaternions, Re(conj(q) q') is the Euclidean inner product and |conj(q) q'| = |q| |q'|. Hence |Im(conj(q) q')|^2 = |q|^2 |q'|^2 - (q . q')^2 = |A|^2 |eta|^2 - (A . eta)^2 = |A wedge eta|^2. ■

Proposition 51.2 (the photon condition is the invariance under the transposition). Let tau be a transposition with mirror plane P_tau and lift x, and tau-tilde its action on spinor fields. Then

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tau-tilde D_{A,H} tau-tilde^{-1} = D + c(d tau A) + i c(d tau eta) Gamma,
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so D_{A,H} commutes with tau-tilde exactly when d tau A = A and d tau eta = eta, that is exactly when A and eta lie in P_tau. Since A is defined modulo 2 pi Lambda^*, D_{A,H} commutes with tau-tilde up to a gauge transformation e^{i k . z} exactly when d tau A - A lies in 2 pi Lambda^* and d tau eta = eta. Under the window hypothesis |A| < pi/l of Proposition 51.1, |d tau A - A| <= 2 |A| < 2 pi/l, which is below the length of every nonzero vector of 2 pi Lambda^*. So d tau A = A, and the two conditions agree. In that case the window block Y = c(A) + i c(eta) Gamma commutes with x on both chiralities, that is with the algebra C = span_R(1, x) of the mirror stratum (Proposition 49.4), which is the restriction to the first sheet of the subalgebra C_F of Chamseddine, Connes and Marcolli, so that [D_F, C_F] = 0, the condition that keeps the photon massless (Proposition 49.3), holds; conversely the blocks of Proposition 49.3 commuting with x are exactly the Y with A, eta in P_tau, a real four-dimensional family (two complex Yukawa couplings). A nonzero A, or a nonzero eta, in P_tau is fixed by tau and by no other element of S_3, since P_tau meets P_tau' in 0 for tau' != tau and a three-cycle fixes no nonzero vector of a (x) R^2; so a nonzero Wilson line or torsion breaks S_3 to the stabilizer of one mirror, and D_{A,H} commutes with the lift of the whole of S_3 exactly when A = eta = 0. So a first sheet at a symmetric point, whose algebra C + H requires invariance under the whole lifted stabilizer, carries no Yukawa block from these data, as Proposition 49.3 found for the invariant blocks. For v in the normal plane P_tau^perp = n_tau (x) R^2, x c(v) x^{-1} = -c(v): these are the components of the Higgs directions T A = T_p S^4 that anticommute with x, the charged ones of lemma 49.

Proof. tau is a linear isometry of R^4 preserving Lambda, so it acts on A, and tau-tilde preserves the periodic spin structure. The Dirac operator D commutes with every isometry lifted to the spinors, since c(d tau v) = x c(v) x^{-1} and partial transforms by d tau: tau-tilde D tau-tilde^{-1} = D. For the constant terms, tau-tilde c(A) tau-tilde^{-1} = x c(A) x^{-1} = c(d tau A), and tau-tilde i c(eta) Gamma tau-tilde^{-1} = i c(d tau eta) x Gamma x^{-1} = i c(d tau eta) Gamma, since x lies in Spin(4) and d tau has determinant one, so that x commutes with Gamma. The difference D_{A,H} - tau-tilde D_{A,H} tau-tilde^{-1} = c(A - d tau A) + i c(eta - d tau eta) Gamma vanishes exactly when both terms vanish, the j-linear and j-antilinear parts being independent (Proposition 46.1), that is when A and eta are fixed by d tau, whose fixed space is P_tau. The window block is the restriction of D_{A,H} to the constant spinors, on which tau-tilde acts as x; so it commutes with x. The real span of x on Sigma_4 with the identity is the algebra C of Proposition 49.4, and Proposition 49.3 identifies the commutation with x on both chiralities with the condition [D_F, C_F] = 0. For the converse count, the blocks commuting with x are X = diag(y_1, y_2) of complex dimension two (Proposition 49.3), real dimension four, and the Y with A, eta in P_tau form a real four-dimensional space of such blocks (A and eta in a two-plane; the map (A, eta) -> c(A) + i c(eta) Gamma is injective, by the injectivity of c and Proposition 46.1), so the two spaces coincide. P_tau = n_tau^perp (x) R^2 and P_tau' = n_tau'^perp (x) R^2 with n_tau, n_tau' linearly independent in the two-dimensional a, so n_tau^perp and n_tau'^perp are distinct lines of a and the planes meet in 0; a three-cycle acts on a by a rotation of order three, with no fixed vector, so its action on a (x) R^2 has none. For v in n_tau (x) R^2, d tau v = -v, and x c(v) x^{-1} = c(d tau v) = -c(v). ■

Proposition 51.3 (the rank-one pattern: the torsion as the quarter turn of the Wilson line). Let A and eta lie in the mirror plane P_tau, a complex line for the complex structure J of A inherited from E, and let phi in [0, pi] be the angle between them. Then m_+-^2 = |A|^2 + |eta|^2 +- 2 |A| |eta| sin phi, and:

1. m_+ = m_- exactly when eta is parallel to A (sin phi = 0), the custodial case of Proposition 46.2 (1): a torsion along the Wilson line gives equal masses to the two members of the doublet.
2. m_+ = |A| + |eta| and m_- = ||A| - |eta|| exactly when eta is perpendicular to A, that is eta in R J A. Then (m_+ + m_-)/2 = max(|A|, |eta|) and (m_+ - m_-)/2 = min(|A|, |eta|): for |eta| <= |A| the Wilson line is the mean mass of the doublet and the torsion half its splitting. Put the up-type

member on the Majorana line w, a weight line of x (Proposition 49.2 (iii); the up-type assignment of Proposition 50.1 (3) is an input). Then the up-type mass is |A + J_w eta| (item 4). So m_up = |A| + |eta| and m_down = |A| - |eta| when eta is a positive multiple of -J_w A, and the two masses are exchanged when eta is a positive multiple of J_w A. Which member is heavier is the sign of <J_w A, eta>, the oriented area of (A, eta) in the complex line (P_tau, J_w), a datum of the pair (A, eta).

3. m_- = 0 exactly when eta = +- J A: the heaviest quark, with its massless partner in this block, is the generation whose torsion dual is the quarter turn of its Wilson line in the complex structure of the elliptic curve; then m_+ = 2 |A| = 2 |eta|, and Y = {c(2A), P_mu} with mu the weight line of x whose complex structure is J_mu = -+J, so that eta = -J_mu A, lemma 50's delta' = -J_mu h/2 with h = 2A. It is the block c(h) P_w of lemma 50 exactly when the Majorana line w is mu, that is J_w = -+J for eta = +-J A.
4. For a weight line mu of x in Sigma_R, with complex structure J_mu = +-J, the coupling of Proposition 49.3 on mu has modulus |A + J_mu eta|. With the up-type member on the Majorana line w, y_u = A + J_w eta and y_d = A - J_w eta in the complex line (P_tau, J_w), up to a common phase and the conjugation of Proposition 49.3. In a complex coordinate z of P_tau in which J_w acts as i, y_u = z(A) + i z(eta) and conj(y_d) = z(A) - i z(eta). Hence |y_u|^2 = |A|^2 + |eta|^2 - 2 <J_w A, eta> and |y_d|^2 = |A|^2 + |eta|^2 + 2 <J_w A, eta>, with <J_w A, eta> = +-|A| |eta| sin phi. In the coordinate given by J these formulas hold when J_w = J, and they hold with i replaced by -i when J_w = -J. So the sign is fixed by which weight line of x is the Majorana line, an input, and no other convention enters.

The finite data of Corollary 16.8 with V_X = Sigma_4, A_{F,X} = C = span_R(1, x) (the mirror stratum, Proposition 49.4), d_X = Y with A, eta in P_tau, and V_Y = C^3 with M_3(C), d_Y = 0, are a 24-state bimodule over C + M_3(C) with the KO-6 signs, order zero and order one; every one-form vanishes, since C commutes with Y, so the masses are explicit (Proposition 17.3) and the algebra is the centralizer of the block (the centralizer of Y in C + H is span_R(1, x) for generic, rank-one and parallel data; checked by the referee), the algebra of the label 9 of lemma 50, Proposition 50.4; the orientability relation fails there: on the first summand every sum of products pi(a) b° lies in span_R(1, x) (x) M_3(C), which acts on the +i weight lines of x in Sigma_R and in Sigma_L by the same matrix, while Gamma_F = Gamma (x) 1 is 1 on the first and -1 on the second (referee).

Proof. In P_tau, |A wedge eta| = |A| |eta| sin phi, and the formula is Proposition 51.1. (1) sin phi = 0. (2) sin phi = 1 gives m_+-^2 = (|A| +- |eta|)^2; the perpendicular of A in the two-plane P_tau is R J A, J being orthogonal on P_tau; conversely m_+ = |A| + |eta| forces sin phi = 1 when A, eta are nonzero. The mass on the line w is |A + J_w eta| by item 4, which gives the assignment for each sign of <J_w A, eta>. (3) m_- = 0 exactly when |A|^2 + |eta|^2 = 2 |A| |eta| sin phi, which forces |A| = |eta| and sin phi = 1; then eta = +- J A, and Y = c(A) + i c(+- J A) Gamma; in lemma 50, Proposition 50.2 (3), the rank-one block c(h) P_w has j-linear half c(h)/2 and j-antilinear half i c(delta') Gamma with delta' = -J_w h/2, so A = h/2 and eta = delta' = -J_w A, that is J_w = -+ J, the sign according to which weight line of x carries the heavy member, since J_{jw} = -J_w (lemma 50, Proposition 50.2 (3)), and the two weight lines of x in Sigma_R are the two points of the twistor fibre with J_mu = J and J_mu = -J, since x|*{Sigma_R} = +-c(omega_J)/2 with omega_J the Kähler form of J (lemma 49, Reading; checked by the referee). (4) By Proposition 49.4 the Clifford element c(A) of a vector of the mirror plane is a block L_a commuting with x, with a complex, and A -> a(A) is a real-linear isometry P_tau -> C, complex linear for J_mu with mu the line on which x acts as i in the identification of Proposition 49.3, and i c(eta) Gamma = i L_c with c the image of eta; Proposition 49.3 gives y_1 = a + i c and conj(y_2) = a - i c, with |y_1|^2 and |y_2|^2 as displayed, Im(conj(a) c) being +- the area |A| |eta| sin phi. For the finite data, Theorem 16.7 applies formally to the real subalgebra C of End_C(Sigma_4) (Remark 16.5); the one-forms [D_F, pi(a)] are [Y, a_X] (x) 1 on Hom(V_Y, V_X) and vanish for a_X in span(1, x) by Proposition 51.2; Proposition 17.3 then gives the explicit masses; the centralizer statement is Proposition 50.4 with the summand C of A_SM embedded along the weight lines of x; Gamma acts as (1, -1) on the chiralities while x acts with eigenvalues +-i on both, so Gamma lies outside span_R(1, x), the hypothesis gamma*{V_X} in A_{F,X} of Corollary 16.8 fails, and the relation itself fails, since the products pi(a) b° act on the two +i weight lines by one matrix and Gamma_F by opposite signs. ■

Reading

Lemma 22 and lemma 25 read every finite Dirac operator as the lowest window of a derivative on an interval, unfolded to a circle with a constant connection; the present lemma is the four-torus form of that reading with the sheet of lemma 49 as the fibre: the custodial half of the Yukawa block is a Wilson line, the mass of a doublet without torsion is |A|, the internal momentum shifted by the holonomy, and a torsion along the Wilson line gives both members the mass (|A|^2 + |eta|^2)^{1/2} (the mechanism of Hosotani, recorded for the interval in lemma 40 and lemma 20, reading 3), and the up-down splitting, which no Wilson line and no displacement supplies, is a constant three-form, the torsion or flux term of the Dirac operator. Along a mirror of the Weyl group the two data make the two complex Yukawa couplings of a generation, y_u = A + i eta and conj(y_d) = A - i eta in the complex coordinate of the mirror, and the photon condition of Chamseddine, Connes and Marcolli, imposed there on D_F by hand, is the invariance of the Wilson line and the torsion under the transposition whose mirror carries them. The heaviest quark is the extreme in which the torsion is the quarter turn of the Wilson line; for the third generation in the perpendicular case the ratio |eta|/|A| = (m_t - m_b)/(m_t + m_b) is about 0.97 with the running masses at the scale m_t (m_t about 163 GeV, m_b about 2.8 GeV; 0.95 with m_t = 173 GeV and m_b = 4.2 GeV). For the second generation (m_c - m_s)/(m_c + m_s) is about 0.84 at a common scale (m_c/m_s about 11.8; 0.86 with m_c(m_c) = 1.27 GeV and m_s(2 GeV) = 0.093 GeV). The first generation has m_d > m_u, so in the perpendicular case its torsion has the opposite orientation relative to J_w A, with |eta|/|A| = (m_d - m_u)/(m_d + m_u) about 0.37. The lighter generations are further from the rank-one pattern. (Running values recalled, not computed here; ratios at a common scale are independent of the QCD running.) The symmetric points of A, where the algebra is C + H, carry no Yukawa block and the quark sector is massless with the full algebra; the mirror, where the algebra is C, carries the block with explicit masses, no one-forms and the algebra C + M_3(C) of the label 9: the two strata are the two ends of the 10 -> 9 reduction of the quark sector, as a reading, with the choice of mirror the choice of the photon circle among three (lemma 49). A lead, from the geometry of A: A = Lambda_coroot (x) E = Hom(Lambda_weight, E), the triples (z_1, z_2, z_3) of points of E with z_1 + z_2 + z_3 = 0. Its quotient by the Weyl group is the moduli space of semistable SU(3)-bundles on E, equivalently of flat SU(3)-connections on E. Sources: Friedman, Morgan and Witten, arXiv:hep-th/9701162, read in the e-print by the referee: section 2.1 (V = N_1 + … + N_n with N_1 (x) … (x) N_n = O, the moduli space P H^0(E, O(np)) = CP^{n-1}); section 3.1 ((T x T)/W for simply connected G); and section 4, "A note on flat connections" (Hom(L', E) = L' (x) E with L'' the weight lattice). For general G the quotient is a weighted projective space (Looijenga, Invent. Math. 38 (1976) 17-32, doi:10.1007/BF01390167). For SU(3) it is CP^2, the corpus quotient of lemma 45. A point of A is then a flat SU(3)-connection on E with holonomies in the maximal torus. A generic point has centralizer the maximal torus U(1)^2. A point of a mirror z_i = z_j off the nine symmetric points has centralizer S(U(2) x U(1)) = U(2). The nine symmetric points, z_1 = z_2 = z_3 with 3 z_1 = 0, are the pairs of holonomies in the centre Z_3 x Z_3, with centralizer SU(3); the Wilson line A of this lemma is a flat U(1)-connection on A, a different object, and whether the two can be identified through a duality of the torus is open. For three generations the block needs flavour-dependent data, for instance a Wilson line in a Cartan subalgebra of a flavour group with three weights q_g A and a flavour-dependent three-form; a torsion common to the three generations is excluded by the masses for any Wilson lines A_g. In every generation m_+^2 + m_-^2 = 2 (|A_g|^2 + |eta|^2) >= 2 |eta|^2, so the first generation bounds |eta| by about 4 MeV. The third has m_t^2 - m_b^2 = 4 |A_3 wedge eta| <= 4 |A_3| |eta| with |A_3|^2 <= (m_t^2 + m_b^2)/2, which needs |eta| of at least (m_t^2 - m_b^2)/(2 sqrt(2) (m_t^2 + m_b^2)^{1/2}), about 61 GeV. In the perpendicular case with |eta| <= |A_g| the splitting would be the common value 2 |eta| (referee).

Scope

The flat four-torus with the periodic spin structure and constant data A, H; the window statement needs |A| + |eta| < pi/l, and the compression is exact because the window is a reducing subspace, so no estimate is involved; the Wilson line is a gauge class modulo 2 pi Lambda^*, which under the window hypothesis changes nothing in Proposition 51.2; the constant s != 0 of the three-form term (verified in Agricola's lectures) and the sign of eta depend on conventions and enter no statement; the sign that does enter, which weight line of x is the Majorana line, is an input (Proposition 51.3); the equivariance statement uses the lift of

the transposition as an isometry of the whole torus, and the finite data at the mirror stratum are those of Corollary 16.8 with the algebra of Proposition 49.4, as an identification of data, the crossed product of lemma 48 seeing the orbit types as quotients; the one-generation form with one Wilson line and one three-form, the three-generation form being left as a direction; the E factor of T^6, whose spinors double the fibre and whose own Wilson line c(A_E) is even for the chirality of A, is outside the statements; the reading of the two strata as the ends of the 10 -> 9 reduction and the moduli-space lead are readings. Nothing here derives A or H from an action, and no potential selects eta = +- J A. Two remarks of the referee, in their scope: in heterotic flux compactifications the Dirac operator of the gauge fermions uses the connection with torsion one third (Kimura and Yi, arXiv:hep-th/0605247, abstract), the value s = 1/4 here, which would enter a flux-quantization condition relating |eta| to the integer fluxes of H over the three-cycles of A; and on the compact flat torus a constant H != 0 solves no leading-order NS-NS equation, since the trace of R_ij + 2 nabla_i nabla_j phi - (1/4) H_ikl H_j^kl = 0 integrates to (1/4) int |H|^2 = 0, so the direction below needs a source or another background (Lie groups with parallel torsion, where nabla^{+-1/4} are flat; brane or orientifold sources; higher orders), a remark not checked by a second model and leaving the statements, which take A and H as data, unchanged. The direction the result points to: a potential or a quantization condition on the torus that relates the three-form to the Wilson line (a flux quantization of H over three-cycles of A, with the Wilson line on the dual cycle); the E factor, where a Wilson line gives an even block, the candidate for the connecting operator of the lepton generation; and the identification of the point of A, a flat SU(3)-connection on E, with the Wilson line on A.

Part 52

The lepton generation in three views: the lepton line of the two sheets with the connector, the Majorana spinor of nine dimensions, and the partner of the flavour adjoint

For STRATEGY T16 (the lepton generation, the lead at the end of lemma 49: a connecting operator identifying the two copies of C), T12 (the scalar partners of lemma 21), T2c and T13 (the nine-dimensional reading and the masses as internal momenta), following lemma 20 (Theorems 20.1 and 20.2), lemma 16 (Theorem 16.7, Corollary 16.8), lemma 49 (Propositions 49.1, 49.2 and 49.5), lemma 50 (Proposition 50.3), lemma 51 (Proposition 51.1), lemma 41 (Lemma 41.1 and Theorem 41.2), lemma 42 (Proposition 42.2), lemma 06 (Lemma 4 and Proposition 7), lemma 44 (Remark 44.4) and lemma 21 (the SO(30) section and Proposition 21.3). The three views were named by the user on 5 November 2026 (plain Connes; partner to the scalars of the 24 with 6 positive, 6 negative and 12 neutral; the Dirac operator of a Kaluza-Klein reduction from nine dimensions). Written by Claude Fable 5.1 on 5 November 2026, by hand; the existence of the Majorana conjugation of Spin(1, 8) on an explicit Clifford representation, its commutation with the spin generators, the exchange of the two chiral eigenspaces, the commutant dimensions 32, 16 and 2 (of Spin(1, 3), Spin(5) and their product on C^16; the referee's values, the first draft recorded 4 for the first) that give the decomposition (2, 1) (x) 4 + (1, 2) (x) 4, and the lepton pairing were checked with numpy (script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_51_52.md), who rebuilt the 32-state bimodule and Cl(1, 8) with its Majorana and charge-conjugation matrices and read the e-print of Chamseddine, Connes and Marcolli: Propositions 52.1 and 52.3 correct with amendments (the dictionary needs the eigenline condition; the rank-one defect formula of the connector; the pairing nondegenerate over Q and equal to their Proposition 2.30 (3); the 3 against the 3-bar), Proposition 52.2 with one error in statement (iii) (a nine-dimensional Majorana field carries exactly one generation only on its self-conjugate sector, where the mass operator is i c(eta) Gamma with the equal masses |eta|; the sectors of nonzero momentum pair into a generation and a mirror, so the quantized momenta enter only there and the contrast with the quark masses falls) and one in its proof (the bilinear vanishes for anticommuting components), with amendments (two irreducible modules; e_R^c on w and nu_R^c on jw; the scope of the Majorana-mass statement; the factorizations); all applied below with the referee's texts.

Outcome. The lepton generation is the quark generation of Proposition 49.5 with the three colours replaced by one line, in three settings that agree on its content. In the two-sheet data with the first sheet the spinor

fibre Sigma_4 and the second sheet the lepton line Cv beside the colour C^3, the lepton sub-bimodule has eight states and is preserved by the algebra, the real structure, the grading, the split Yukawa block of Proposition 50.3 and the connector of Theorem 20.2, which lives entirely inside it; the four-summand algebra C + H + C' + M_3(C) of the two sheets is reduced by the connector's order-one condition to the Standard Model algebra {(lambda, q, lambda, m)} exactly when the connector's line is a weight line of the lifted transposition, the two copies of C being identified, and to R + H + M_3(C) otherwise; the trace-form pairing of the lepton module over C + H is [[0, 2], [-2, 0]], nondegenerate over Q, the restriction of the matrix printed by Chamseddine, Connes and Marcolli (Proposition 2.30), while over the three-summand algebra the lepton and quark modules together give the skew form of rank two of lemma 41 (Proposition 52.1). A Majorana spinor of nine-dimensional Minkowski space, sixteen real components, is under Spin(1, 3) x Spin(5) the complex space (2, 1) (x) 4: four left-handed Weyl spinors of the four-dimensional spacetime in the spinor fibre W = H^2 of the five-dimensional internal space, which is the sheet of Proposition 49.1; under Sp(1)_a x Sp(1)_b it is the left-handed doublet and the right-handed pair, under Sp(1)_a x U(1)_b the doublet of U(1)_b charge zero and the two singlets of U(1)_b charge +-1 (T_3R = +-1/2), one lepton generation with nu_R; its nine-dimensional mass bilinear vanishes, so its masses are the internal Dirac operator on the fibre, which on A x S^1 with the torsion of lemma 51 is c(k) + i c(eta) Gamma + k_5 Gamma on the sector of momentum (k, k_5), shifted by the spin structure, since a Majorana field carries no U(1) charge and admits no Wilson line. The self-conjugate sector carries exactly the generation, with the masses |eta| of both members, and the other sectors pair into a generation and a mirror with the masses of Proposition 51.1; a Majorana mass for nu_R alone is absent from Lorentz-invariant and constant bulk data, wall data in that scope, and selects the KO-7 factorization for one generation (Proposition 52.2). As representations of U(1)_em, three lepton generations with nu_R (24 states) and the complexified adjoint of the flavour SU(5) of the five light quarks (lemma 21) are isomorphic, with multiplicities 12, 6 and 6, the adjoint decomposing under SU(2)_u x SU(3)_d x U(1)_em as (3, 1)_0 + (1, 8)_0 + (1, 1)_0 + (2, 3-bar)_{+1} + (2-bar, 3)_{-1}, so that the charged states are (up flavour) x (down flavour) in number as the charged leptons are (chirality) x (generation), and the neutral ones span the flavour gauge algebra (Proposition 52.3).

Setting

The two-sheet data of Corollary 16.8 with V_X = Sigma_4, the spinor fibre of the four-torus A with the Clifford action, the chirality Gamma and the quaternionic structure j (Proposition 49.1: the sheet W = H^2 with axis p at Gamma, L_p = Sigma_R, L_p^perp = Sigma_L), A_{F,X} = C + H the algebra of the lifted stabilizer at a symmetric point (Proposition 49.2: C = span_R(1, x) acting on Sigma_R as (lambda, conj(lambda)) in the weight basis of the lifted transposition x (ordered (w, jw) when Cw is a weight line), H = End_H(Sigma_L)), and V_Y = V = Cv + C^3 = C^4 with A_{F,Y} = C' + M_3(C) = {lambda' P_v + m}, the second-sheet algebra of a connector on the line v (Theorem 20.2, Proposition 50.3 (4)). The bimodule is H_F = Hom(V, V_X) + Hom(V_X, V) = Sigma_4 (x) conj(V) + V (x) conj(Sigma_4), of complex dimension 32, with J_F(x, y) = (y^*, x^*) and Gamma_F = Gamma on the first summand and -Gamma^c on the second. The connector is T_kappa x = kappa <e, x> f with e = w v^*, f = e^*, w a unit vector of Sigma_R. The finite Dirac operators considered are D_F = D_Y + S_{T_kappa}, with D_Y x = Y_l x P_v + Y_q x P_v^perp and D_Y y = P_v y Y_l + P_v^perp y Y_q (Proposition 50.3). The dictionary of Chamseddine, Connes and Marcolli is as in lemma 50: lambda on nu_R and u_R, conj(lambda) on e_R and d_R, lambda on the antileptons, m on the antiquarks.

For the second view, Spin(1, 8) is the spin group of nine-dimensional Minkowski space R^{1,8} with signature (+, -, ..., -), its Dirac spinor is C^16, and the Majorana spinors are the fixed points of an antilinear C with C^2 = 1 commuting with the Clifford action (which exists in signature (1, 8), where t - s = -7 = 1 mod 8); M^4 x K_5 with K_5 a spin five-manifold has the structure group Spin(1, 3) x Spin(5) of its frame bundle at a point, Spin(5) = Sp(2) acting on the spinor fibre C^4 = H^2 of K_5 by the 4. For the third view, the five light quarks u, c, d, s, b of lemma 21 span the 5 of a flavour SU(5) with the electric charge Q = diag(2/3, 2/3, -1/3, -1/3, -1/3), whose commutant in SU(5) is S(U(2)_u x U(3)_d), and the 24 is the complexified adjoint sl(5, C) with the adjoint action.

Statements

Proposition 52.1 (the lepton line of the two sheets). (i) H_F = H_l + H_q with H_l = Sigma_4 (x) conj(Cv) + Cv (x) conj(Sigma_4), of complex dimension 8, and H_q = Sigma_4 (x) conj(C^3) + C^3 (x) conj(Sigma_4), of dimension 24 (the quark generation of Proposition 49.5); both are invariant under the left and the opposite actions of C + H + C' + M_3(C), under J_F and Gamma_F, under D_Y, and under the connector, whose vectors e and f lie in H_l; so every D_F above is block diagonal in H_l + H_q. In the dictionary, which item (ii) makes available when Cw is a weight line of x, H_l carries nu_R = w (x) conj(v) and e_R = jw (x) conj(v). These are the two weight lines of x tensored with the lepton line, w being the Majorana line. H_l also carries the left-handed doublet Sigma_L (x) conj(v) and the conjugates of all three. Here lambda, the eigenvalue of the summand C on w, acts on nu_R, conj(lambda) on e_R, q on the doublet and lambda' on the antileptons v (x) conj(Sigma_4).

(ii) The largest subalgebra of C + H + C' + M_3(C) on which D_F = D_Y + S_{T_kappa}, kappa != 0, satisfies order one is its intersection with A(T_kappa) of Theorem 20.2: it is the Standard Model algebra A_SM = {(lambda, q, lambda, m)}, in which the two copies of C are identified, exactly when the connector's line Cw is a weight line of x (the eigenline condition of Proposition 49.2 (iii)), and it is {(r, q, r, m) : r real} = R + H + M_3(C) otherwise. On A_SM the lepton module H_l is the summand H_F^(1) of one generation of the spectral Standard Model (Chamseddine, Connes and Marcolli, Proposition 2.30), with the Majorana block of the connector on nu_R and the Yukawa block Y_l of the leptons.

(iii) The trace-form pairing <p, p'> = Tr_{H}(Gamma_F p p'^o) of lemma 41 on H_l, for the minimal projections p_1 = (1_C, 0, 1_{C'}, 0), p_2 = (0, 1_H, 0, 0), p_3 = (0, 0, 0, e_11) of A_SM, is [[0, 2, 0], [-2, 0, 0], [0, 0, 0]]: over the lepton algebra C + H, the summands acting on H_l, the pairing [[0, 2], [-2, 0]] is nondegenerate over Q; its determinant is 4, so over Z it is not unimodular, and the trace-form Poincaré duality holds rationally for the lepton generation over the algebra that acts on it; the matrices on H_l and H_q are those printed by Chamseddine, Connes and Marcolli (arXiv:hep-th/0610241, Proposition 2.30 (3), per generation), and over the three summands their Corollary 2.31 obtains nondegeneracy from the pair of classes of H_F^(1) and H_F^(3), their Remark 2.32 adding that KO-homology is generated by two elements; on H_q it is [[0, 0, 2], [0, 0, -2], [-2, 2, 0]] (Proposition 49.5), and on H_F the sum [[0, 2, 2], [-2, 0, -2], [-2, 2, 0]], skew of rank two, the form of Theorem 41.2 at the Standard Model step up to the sign convention.

Proof. (i) The left actions are a_X (x) 1 on the first summand and a_Y (x) 1 on the second, the opposite actions 1 (x) (b_Y^*)^c and 1 (x) (b_X^*)^c (Theorem 16.7); P_v and P_v^perp commute with A_{F,Y} = C' + M_3(C), so Sigma_4 (x) conj(Cv) and Sigma_4 (x) conj(C^3) are invariant under the left action of A_{F,X} (which acts on the first factor) and under the opposite action of A_{F,Y} (which preserves Cv and C^3), and likewise for the second summand; J_F exchanges Sigma_4 (x) conj(Cv) with Cv (x) conj(Sigma_4), and Gamma_F acts on the first factors; D_Y x = Y_l x P_v + Y_q x P_v^perp maps x = xi (x) conj(v) to Y_l xi (x) conj(v), and x = xi (x) conj(c) to Y_q xi (x) conj(c) for c in C^3; e = w v^* lies in Sigma_4 (x) conj(Cv) and f = v w^* in Cv (x) conj(Sigma_4), and T_kappa and T_kappa^* have range in the lines C f and C e. The dictionary assignments are those of lemma 50, Setting, with Sigma_R = L_p the lambda-weight line of the summand C on w (Proposition 49.2 (ii)). (ii) By Theorem 20.1 and Theorem 20.2, D_Y satisfies order one on the whole of C + H + C' + M_3(C) (Proposition 50.3 (4)), and the defects of D_Y and of S_{T_kappa} lie in different blocks, so order one of the sum is order one of each. Write a = (a_c, q_a, lambda'*a, m_a) for an element of the four-summand algebra, with a_c its action on Sigma_R. The order-one defect of S*{T_kappa} on Hom(V, Sigma_4) -> Hom(Sigma_4, V) is the rank-one map x -> kappa <(a_c^* w - conj(lambda'_a) w) v^*, x> v (conj(lambda'_b) w - b_c^* w)^*, and the other block is its adjoint (checked by the referee on random data). On a *-subalgebra B with order one, a = b gives a_c^* w = conj(lambda'_a) w for every a in B, hence a_c w = lambda'_a w by a -> a^*. Conversely, these conditions make the defect vanish. So the largest *-subalgebra with order one is {a : a_c w = lambda'*a w}, the intersection with A(T_kappa) = {(lambda_1 on w, conj(lambda_1) on jw) + q + (lambda_1 on v) + m} of Theorem 20.2, the condition on v holding automatically in the four-summand algebra (Theorem 20.1 alone gives the inclusion in some A(R), and the rank-one form gives A(T_kappa) itself); an element (lambda on Sigma_R in the weight basis of x, q, lambda' on v, m) of the four-summand algebra lies there exactly

*when lambda' = lambda_1 and the action of C on Sigma_R, diag(lambda, conj(lambda)) in the weight basis of x, equals diag(lambda_1, conj(lambda_1)) in the basis (w, jw). If Cw is a weight line of x the two bases coincide up to order, and the condition is lambda' = lambda (if Cw is the weight line on which x acts as -i, the condition reads lambda' = conj(lambda_x) in the parameter lambda_x of the +i line; in the eigenvalue lambda of the summand C on w it reads lambda' = lambda in both cases, so the algebra is {(lambda on w, conj(lambda) on jw, q, lambda on v, m)}, which is A_SM with nu_R and u_R on Cw; the parameter of A_SM is the eigenvalue on the Majorana line, and the dictionary of item (i) uses it); if Cw is not a weight line, a matrix diagonal in both bases is scalar, so lambda = conj(lambda) = lambda_1 = lambda' is real. The identification with H_F^(1) is the dictionary, with the Majorana block S{*T_kappa} on the line C e = C nu_R (lemma 20, reading 1) and Y_l (x) P_v the lepton Yukawa block. (iii) On H_l the pairing is <p, p'> = tr(Gamma a_p) tr(b_{p'}) - tr(b_p) tr(Gamma a_{p'}), with a the first-sheet part and b the second-sheet part restricted to Cv, as in Theorem 48.3 and Proposition 49.5; tr(Gamma a_{p_1}) = tr(Gamma|Sigma_R) = 2, tr(b_{p_1}) = 1; tr(Gamma a_{p_2}) = -2, tr(b_{p_2}) = 0; a_{p_3} = 0 and b_{p_3} = 0 on Cv. The 2 x 2 block on (p_1, p_2) has determinant 4. On H_q the second-sheet traces are tr(b_{p_3}) = 1 and tr(b_{p_1}) = tr(b_{p_2}) = 0 (lambda P_v vanishes on C^3), which gives the form of Proposition 49.5; the sum is as displayed, with the sign convention of lemma 49, opposite to that of lemma 41. ■

Proposition 52.2 (the Majorana spinor of nine dimensions is a lepton generation). (i) The Dirac spinor C^16 of Spin(1, 8) restricted to Spin(1, 3) x Spin(5) is (2, 1) (x) 4 + (1, 2) (x) 4, where (2, 1) and (1, 2) are the two Weyl spinors of Spin(1, 3), conjugate to each other, and 4 is the spinor of Spin(5) = Sp(2), quaternionic. A Majorana conjugation C (antilinear, C^2 = 1, commuting with the Clifford action of R^{1,8}, hence with Spin(1, 8)) exists and exchanges the two summands; so the real Majorana spinor R^16 is, as a complex vector space with the action of Spin(1, 3) x Spin(5), the first summand (2, 1) (x) 4: four left-handed Weyl spinors of M^4 in the 4 of Sp(2).

(ii) On M^4 x K_5 the 4 is the spinor fibre W = H^2 of K_5 at a point, with Sp(2) its structure group; for K_5 = A x S^1 it is the sheet of Proposition 49.1 with c(e_5) = +-Gamma, the grading axis of the sheet being the circle direction. Under K_p = Sp(1)_a x Sp(1)_b it is (2, 1) + (1, 2), the left-handed doublet and the pair of right-handed singlets written as left-handed conjugates, and under Sp(1)_a x U(1)_b it is (2)_0 + (1)_{+1} + (1)_{-1}, the U(1)_b charges of Proposition 42.2 (T_3R = +-1/2 with Y = charge/2): one lepton generation with nu_R, the weak isospin Sp(1)_a and the T_3R circle, in the two-sheet data the quark generation is the same fibre tensored with the colour sheet C^3 (Proposition 49.5); in nine dimensions a colour triplet is a complex field, whose left-handed content (2, 1) (x) 4 (x) (3 + 3-bar) is a quark generation and its mirror.

(iii) The Lorentz-invariant mass bilinear of a Majorana spinor vanishes in dimension 8k + 1 (the corpus row rs_F3, Alvarez-Gaumé and Witten), so the four-dimensional masses of the generation are the eigenvalues of the internal Dirac operator on the fibre. A Majorana field carries no U(1) charge and admits no Wilson line, so on A x S^1 its internal momentum is a vector (k, k_5) of the dual lattice shifted by the spin structure. A single Majorana field admits the torsion of lemma 51, since the bilinear of Gamma^{abc} is antisymmetric in signature (1, 8) and the coupling survives. With it, the mass operator on the sector of momentum (k, k_5) is c(k) + i c(eta) Gamma + k_5 Gamma, with squared masses m_+-(k)^2 + k_5^2, m_+-(k)^2 = |k|^2 + |eta|^2 +- 2 |k wedge eta| (Proposition 51.1). The Majorana condition relates the sector (k, k_5) to (-k, -k_5). The self-conjugate sector (k, k_5) = 0 exists for the periodic spin structure on A x S^1. It carries one lepton generation, with mass operator i c(eta) Gamma: the Yukawa block of that generation is purely anti-quaternionic, and the two members of the doublet have the equal masses |eta| (Proposition 46.2 with q = 0). A pair of sectors (k, k_5), (-k, -k_5) with (k, k_5) != 0 carries an unconstrained complex field: a lepton generation and a mirror generation. These are four Dirac fermions, with the masses (m_+(k)^2 + k_5^2)^{1/2} and (m_-(k)^2 + k_5^2)^{1/2} each twice. On these levels the custodial half c(k) is a quantized internal momentum. The Lorentz-invariant bulk mass is absent (rs_F3). Constant three-form backgrounds with one leg on S^1 add to the self-conjugate sector the terms i c(beta) Gamma, beta a two-form on A; their mass bilinears omega(., i c(beta) Gamma .), with omega the Spin(5)-invariant form, give nu_R^c and e_R^c Majorana entries of equal modulus (checked by the referee). So a Majorana mass for nu_R

alone, the one that conserves the electric charge, is absent from Lorentz-invariant and constant bulk data. In that scope it is wall data in the sense of Remark 44.4 (the connector of Proposition 52.1), and with it the neutral sector of one generation admits only the KO-7 factorization (lemma 06, Lemma 4: the singular values of [[0, m_D], [m_D, M]] are distinct for M != 0). With Dirac masses only, the neutral singular values come in pairs and both factorizations exist (lemma 06, Proposition 7 with R = 0). With three generations and a general Majorana matrix, KO-5 exists exactly under the multiplicity condition of Proposition 7.

Proof. (i) Cl(1, 8) is the graded tensor product of Cl(1, 3) and Cl(0, 5), and each of its two irreducible complex modules, distinguished by the sign of the volume element, is C^16; the two restrict to equivalent representations of Spin(1, 8), and C^16 is the tensor product of the Dirac spinor C^4 of Spin(1, 3) and the spinor C^4 of Spin(5), on which the even subalgebras act factorwise; under Spin(1, 3) the Dirac spinor is (2, 1) + (1, 2), the eigenspaces of the chirality gamma_5 = gamma_0 gamma_1 gamma_2 gamma_3, which squares to -1 in this signature and commutes with Spin(5). The 4 of Spin(5) = Sp(2) is quaternionic, so conj(4) = 4, and conj(2, 1) = (1, 2); hence the complex conjugate of (2, 1) (x) 4 is (1, 2) (x) 4. In signature (1, 8) there is a matrix B with B conj(gamma_mu) B^{-1} = gamma_mu for all mu and B conj(B) = 1 (t - s = 1 mod 8; checked on the explicit representation: the solution space of the first condition is one-dimensional and B conj(B) = 1 on it), so C psi = B conj(psi) is an antilinear involution commuting with the gamma matrices and with the spin generators gamma_mu gamma_nu; C is antilinear, so it maps the i-eigenspace of gamma_5 to the -i-eigenspace, that is (2, 1) (x) 4 to (1, 2) (x) 4; the fixed points of C form a real form of C^16, and the projection of that real form onto the first summand is a real-linear isomorphism onto it (a fixed vector is psi + C psi with psi in the first summand, and psi -> psi + C psi is injective); the complex structure of the first summand and the action of Spin(1, 3) x Spin(5) are transported by it. (ii) The frame bundle of M^4 x K_5 reduces to Spin(1, 3) x Spin(5) and the nine-dimensional spinor bundle is the tensor product of the two spinor bundles; the spinor fibre of a five-manifold is C^4 with the structure group Sp(2) acting by the 4, and for A x S^1 the Clifford algebra of T A + R e_5 is generated by c(e_1), ..., c(e_4) and the chirality Gamma of A, since Gamma anticommutes with the c(e_a), squares to one and is Hermitian, so c(e_5) = +-Gamma, and the identification with the sheet is Proposition 49.1 with the axis p = e_5. The restrictions of the 4 are those of lemma 42 and lemma 46 (Proposition 42.2: (2, 2) and 4 = (2, 1) + (1, 2) under K_p, the charges of U(1)_b on the two lines of Sigma_R being +-1); the left-handed doublet is Sigma_L (x) (2, 1) and the right-handed singlets appear as left-handed conjugates on the two weight lines of Sigma_R tensored with (2, 1). The field on w, of U(1)_b charge +1, is e_R^c, and the field on jw, of charge -1, is nu_R^c. Their conjugates, the right-handed e_R and nu_R, lie in (1, 2) (x) conj(Sigma_R) on conj(w) and conj(jw). The quaternionic structure, conj(u) -> j u, identifies these with the lines jw and w (the latter up to sign) of the dictionary of lemma 50, with nu_R on w. (iii) The charge-conjugation matrix C with C Gamma^mu C^{-1} = Gamma^mu^T is symmetric in signature (1, 8), unique up to scale, and none exists with the opposite sign. So the scalar bilinear psi^T C psi vanishes identically for anticommuting components in 8k + 1. C Gamma^mu is symmetric, so the kinetic term survives, and C Gamma^{(2)} and C Gamma^{(3)} are antisymmetric (checked by the referee on an explicit representation) (the statement rs_F3 of the corpus, with reference there); a Majorana condition psi = C psi is incompatible with a U(1) action by phases unless the charge is zero, so the only momenta are those of the periodic or antiperiodic Fourier modes, k in 2 pi Lambda^* + shift and k_5 in (2 pi/L) (Z + shift); the operator on the sector e^{i(k . z + k_5 z_5)} is c(k) + k_5 c(e_5) + i c(eta) Gamma = c(k) + i c(eta) Gamma + k_5 Gamma up to the sign of c(e_5), whose square is Y_k^2 + k_5^2 since Y_k is odd for Gamma, and Proposition 51.1 gives the spectrum of Y_k; the mode -k is the conjugate of the mode k under C. The self-conjugate mode carries a real field, R^16 = (2, 1) (x) 4, one generation. A pair of distinct modes carries a complex field in C^16 = (2, 1) (x) 4 + (1, 2) (x) 4, eight left-handed Weyl fields: (2, 1) (x) Sigma_L and (1, 2) (x) Sigma_R make a generation, and (2, 1) (x) Sigma_R and (1, 2) (x) Sigma_L its mirror. The block Y_k is odd for Gamma, so it pairs the doublet of each with the singlets of the same set. The statements about the Majorana block and the factorizations are Remark 44.4 and lemma 06. ■

Proposition 52.3 (three lepton generations and the flavour adjoint). Under SU(2)_u x SU(3)_d x U(1)_em, the adjoint of the flavour SU(5) of the five light quarks decomposes as

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24 = (3, 1)_0 + (1, 8)_0 + (1, 1)_0 + (2, 3-bar)_{+1} + (2-bar, 3)_{-1},
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twelve neutral real components spanning the flavour gauge algebra su(2)_u + su(3)_d + u(1) (in sl(5, C) the neutral part has complex dimension 12, their complexification), six complex components of charge +1 (the up-antidown mesons) and their conjugates of charge -1. Three lepton generations with nu_R, as the 24-state module H_l^{(3)} = H_l (x) C^3 of Proposition 52.1, have the U(1)_em weights 0 with multiplicity 12 (nu_R, nu_L and their conjugates, three generations), +1 with multiplicity 6 and -1 with multiplicity 6 (e_R, e_L and their conjugates); so H_l^{(3)} and sl(5, C) are isomorphic as representations of U(1)_em, and as left-handed Weyl fields the three generations are six neutral and six charged fields, which pair with the six neutral and six charged complex scalars of the adjoint in chiral multiplets of equal charge, the slepton counts (6, 6, 12 neutral real) of lemma 21. The charged states number (up flavours) x (down flavours) = 2 x 3 on the adjoint side and (chiralities) x (generations) = 2 x 3 on the lepton side; an isomorphism of U(1)_em-modules can be chosen to map the generation index to the index of the 3 of SU(3)_d for the states e_L, e_R of charge -1, in (2-bar, 3)_{-1}, and of the 3-bar for their conjugates, and the pair (e_L, e_R) to the 2 of SU(2)_u, a choice recorded as a reading. The top is absent from the 5, as in lemma 21 (Proposition 21.4: the counts force it out).

Proof. With Q = diag(2/3, 2/3, -1/3, -1/3, -1/3), the adjoint of U(5) decomposes under U(2)_u x U(3)_d as the block matrices: the diagonal blocks gl(2) + gl(3), of charge 0 and real dimension 4 + 9 = 13 as a real Lie algebra, minus the trace for sl(5), 12 real, which is su(2) + su(3) + u(1) in the commutant S(U(2) x U(3)) of Q; the off-diagonal blocks Hom(C^3, C^2) = (2, 3-bar) with charge 2/3 - (-1/3) = 1 and its adjoint with charge -1, six complex components each. For the leptons, each generation contributes in H_l the states nu_R, nu_L (charge 0), e_R, e_L (charge -1) and their conjugates (charges 0, 0, +1, +1), by the dictionary; three generations give 12, 6, 6. As left-handed Weyl fields each generation is nu_L, nu_R^c (neutral), e_L (charge -1) and e_R^c (charge +1), four fields, twelve in all; a chiral multiplet pairs a complex scalar of charge q with a left-handed Weyl field of charge q, and a complex scalar of charge +1 is the conjugate of one of charge -1, so the six charged complex scalars pair with the three e_R^c (charge +1) and the three e_L (charge -1), and the six neutral complex scalars (twelve real) with the six neutral Weyl fields; these are the counts of lemma 21. The counts 2 x 3 on both sides are read off. ■

Reading

The three views give the same content, the sheet W tensored with a second factor: the lepton line Cv of the second sheet in the two-sheet data, and the left-handed Weyl spinor (2, 1) of spacetime in nine dimensions. In the flavour reading the 24 states of three generations have the electric charges of the adjoint of the five light quarks. The connector that the lepton generation needs, in lemma 49's reading, is the Majorana block itself: its order-one condition identifies the C of the right-handed singlets with the C of the antileptons exactly when the Majorana line is a weight line of the lifted transposition, the eigenline condition that lemmas 46, 49 and 50 met from three sides, and the identification yields the Standard Model algebra on the thirty-two states with the lepton module as the summand H_F^(1) and, over the lepton algebra C + H alone, a trace-form pairing nondegenerate over Q. This is the restriction to the lepton summand of the matrix of Chamseddine, Connes and Marcolli (Proposition 2.30). They note that a skew three-by-three form is degenerate and obtain nondegeneracy over the three summands from the pair of classes of the lepton and quark summands (Corollary 2.31, Remark 2.32). The lepton statement here is the same fact read over the algebra that acts on the lepton summand. The nine-dimensional view is the one the project's labels name: a single Majorana spinor of R^{1,8} on M^4 x K_5 is one lepton generation with nu_R, its weak isospin and T_3R being the stabilizer K_p of the fibre, its Dirac masses, on the self-conjugate sector that carries exactly one generation, the torsion |eta| for both members. The internal momenta, quantized because a Majorana field has no Wilson line, enter only the sectors that pair into a generation and a mirror. Its Majorana mass, in the scope of constant bulk data, is wall data that selects the KO-7 side (T2c). A colour triplet is a complex field, whose nine-dimensional fibre carries a quark generation and a mirror and admits the Wilson line of lemma 51. So in this reading the lepton generation is the one sector that a real nine-dimensional field carries alone, and the quark sector needs a mechanism that removes the mirror. Leads from the referee: the generator gamma_5 Gamma_A commutes with Spin(1, 3) x Spin(4)_A, preserves the real form and the bilinears of the kinetic term, of the momenta along A and of the torsion on A, and acts on the left-handed fields as -1 on the doublet and +1 on the singlets, the B - L of the generation, which with the U(1)_b charges gives the hypercharges -1/2, 0, +1 of l_L, nu_R^c, e_R^c; the S^1 momentum and the fluxes with

a leg on S^1 break it, so the circle direction, the grading axis of the sheet, carries exactly the B - L-violating terms; and a single lepton generation has an odd number of SU(2)_L doublets, Witten's global anomaly, cancelled in the Standard Model by the three colours of quark doublets, so the nine-dimensional Majorana singlet is consistent as a four-dimensional theory only together with the quark sector. For the third view, the adjoint of the flavour SU(5) is the representation of the mesons of the five light quarks, and the three lepton generations have exactly its electric charges, with the neutral leptons in the place of the flavour gauge algebra and the charged leptons in the place of the up-antidown mesons, the generation index matching the down flavours (d, s, b) and the two charged lepton states the up flavours (u, c): the content of the user's counting, recorded here as a representation statement under U(1)_em with the finer matching a reading; the quark partners sit in the (15, 3-bar) with the charge-4/3 sector of Proposition 21.4, and the top is outside both. Lead: the 24 of the leptons and the 24 of the adjoint are both the fibre of a line bundle tensored with a generation triplet, and the flavour group SU(2)_u x SU(3)_d acts on the adjoint while nothing in the finite geometry acts on the generation index; a three-generation finite geometry in which the generation space carries the 3-bar of a flavour SU(3)_d would make the reading a structure.

Scope

Proposition 52.1 is finite linear algebra in the data of Corollary 16.8 with the connector of Theorem 20.2, one generation, with the axis, the connector and the Yukawa blocks as inputs; the trace-form pairing is that of lemma 41 and Theorem 48.3, with the sign convention of lemma 49. Proposition 52.2 (i) is standard representation theory of Clifford algebras, with the Majorana conjugation checked on an explicit representation; (ii) identifies fibres at a point and does not construct a Kaluza-Klein vacuum, a gauge group or an action; (iii) uses rs_F3, checked by the referee on an explicit representation (C symmetric in signature (1, 8)), for the vanishing of the Lorentz-invariant bilinear, and the wall-data statement for the Majorana mass holds for Lorentz-invariant and constant bulk data, Proposition 51.1 for the spectrum and lemma 06 for the factorizations, and the statement about Wilson lines is the absence of a U(1) charge on a Majorana field, not a theorem about all couplings. Proposition 52.3 is a count of electric charges and dimensions; the matching of generations with down flavours and of the two charged lepton states with the up flavours is a choice among the isomorphisms of U(1)_em-modules, recorded as a reading, and nothing here is a supersymmetric theory, as lemma 21 states for its counts. The direction the result points to: a three-generation version of the two-sheet data in which the generation space is a representation of a flavour group, so that the third view becomes a statement about modules; the nine-dimensional Majorana field on M^4 x A x S^1 with the torsion of lemma 51 as the parent whose window is the lepton generation, with the masses |eta| of the self-conjugate sector and the momenta of the paired sectors compared with the charged-lepton masses; and the quark sector with a Wilson line in the U(1) of colour.

Part H (history of the field, to November 2026)

Noncommutative geometry in particle physics: a dated record to November 2026

Written by Claude Opus 5.5 on 4 November 2026, to help decide which results of this project are new and which articles to write. Companions: paper/LITERATURE.md (targeted priority searches) and the author table at a.rivero.nom.es/research/ncactors.html (authors by year to 2003).

Method. Every identifier below was checked in this session against INSPIRE, the arXiv API or Crossref (curl and jq in bash). No Python was used; the author table was counted with awk in bash; no script is kept. Year = first public appearance; journal data follow.

Labels. [R] passages read in this session; [L] read earlier and recorded in paper/LITERATURE.md (section given); [A] abstract read (and, where stated, the reference list held by INSPIRE or the publisher); [M] bibliographic data checked only. Statements about content rest on [R], [L] or [A]; the one-line descriptions of [M] entries are the standard ones and were not re-read.

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1. Origins (to 1991) beside the Kaluza-Klein programme

1.1 The Kaluza-Klein programme, 1968-1987

- •1968. R. Kerner, Ann. Inst. H. Poincaré A 9 (1968) 143 [M]: non-Abelian Kaluza-Klein. Kerner later co-authors the matrix geometry of 1.3.
- 1979-1981. N. S. Manton, Nucl. Phys. B 158 (1979) 141; D. B. Fairlie, Phys. Lett. B 82 (1979) 97; P. Forgács and N. S. Manton, Commun. Math. Phys. 72 (1980) 15; G. Chapline and N. S. Manton, Nucl. Phys. B 184 (1981) 391 [M]: the Higgs as extra-dimensional components of a gauge field. Y. Ne'eman, Phys. Lett. B 81 (1979) 190 [M]: the SU(2/1) superconnection.
- •1981. E. Witten, *Search for a realistic Kaluza-Klein theory*, Nucl. Phys. B 186 (1981) 412 [A]: "It is possible to obtain an SU(3) × SU(2) × U(1) gauge group, but the proper fermion quantum numbers are difficult to achieve."
- 1982-1983. S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. B 214 (1983) 491; C. Wetterich, Nucl. Phys. B 222 (1983) 20 [M]: elementary gauge backgrounds in six dimensions; reduction of Weyl spinors. Cited by Witten 1983 (refs. 13-14).
- •1983. E. Witten, *Fermion quantum numbers in Kaluza-Klein theory*, Shelter Island II (1985) 227, INSPIRE 193975 [R, preprint scan; L, "Sources read"]. Abstract: "In the absence of elementary gauge fields, a theorem by Atiyah and Hirzebruch states that the Dirac equation in 4+n dimensions always leads to vector-like fermion quantum numbers in four dimensions." Scope in the text: ground state M^4 × B with B compact; all gauge fields from isometries of the metric; the Dirac operator; for Rarita-Schwinger only homogeneous B ("a general impossibility theorem … is not proved"); spin 5/2 has a counterexample. Escapes named there: "the character-valued index also need not vanish in the presence of elementary gauge fields" (pseudo-realistic models in its section VI), and "start with a unified theory (perhaps a supergravity or superstring theory) that determines the original gauge group". Its ref. 16 (Witten, J. Diff. Geom. 17 (1982) 661) first noted the theorem's relevance. Theorem: M. Atiyah and F. Hirzebruch, *Spin-manifolds and group actions*, in Essays on Topology and Related Topics (1970) 18, doi:10.1007/978-3-642-49197-9_3 [M].
- •1986. M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Kaluza-Klein supergravity*, Phys. Rep. 130 (1986) 1 [opening sentence, quoted in the 2025 retrospective; full text not read]: "We do not yet know whether supergravity is a theory of the real world, nor whether the real world has more than four dimensions … However, our research in this area has convinced us that the only way to do supergravity is via Kaluza-Klein and that the only viable Kaluza-Klein theory is supergravity." The bulk treats S^7.
- •1987. D. Bailin and A. Love, *Kaluza-Klein theories*, Rep. Prog. Phys. 50 (1987) 1087 [A]: the spectrum "with particular emphasis on the spectrum of light fermions", anomaly constraints "when explicit higher-dimensional gauge fields are present", cosmology, and "Finally, an introductory account of Kaluza-Klein supergravity is given leading towards superstring theory."
- •1987. T. Appelquist, A. Chodos and P. G. O. Freund (eds.), *Modern Kaluza-Klein Theories*, Addison-Wesley [M]. INSPIRE records reprints of Kerner 1968, Chapline-Manton 1983, Witten 1981, Witten 1983 and Randjbar-Daemi-Salam-Strathdee; the introduction (pp. 1-47) was not read.
- Retrospectives. M. J. Duff, *Kaluza-Klein theory in perspective*, arXiv:hep-th/9410046 [R]: "the sorry state of affairs until the September 1984 superstring revolution"; the heterotic string "eschews the route of getting the Yang-Mills gauge group from the general coordinate group"; "Does this mean that the idea of non-abelian gauge fields from extra dimensions is dead? Actually, no." M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Kaluza-Klein supergravity 2025*, arXiv:2502.07710 [R]: the coset programme became "less and less tenable" for three stated reasons: the large negative cosmological constant, Witten's chirality result (cited as the 1981 paper), and the string models after Candelas-Horowitz-Strominger-Witten.

Assessment of the view under test. The texts read support it in part. Witten 1983 states the result with its hypotheses and names the escapes. The Bailin-Love abstract treats light fermions together with explicit higher-dimensional gauge fields, the escape route, and leads "towards superstring theory"; the heterotic string took that route. The retrospectives give chirality as one reason among three, and Duff 1994 denies

that the idea is dead. We found no text of the period that treats the obstruction as closing the subject. The open-problem sections of the 1986 and 1987 reviews and the 1987 introduction remain to be read in print.

1.2 Noncommutative geometry before the models

- 1980. A. Connes, C. R. Acad. Sci. Paris A 290 (1980) 599, arXiv:hep-th/0101093 [M].
- 1985. A. Connes, *Non-commutative differential geometry*, Publ. Math. IHES 62 (1985) 41, doi:10.1007/BF02698807 [M].
- 1988. A. Connes, *The action functional in noncommutative geometry*, Commun. Math. Phys. 117 (1988) 673 [A + INSPIRE references]: Dixmier trace = Wodzicki residue; Yang-Mills from the Dixmier trace.
- 1988, printed 1990. A. Connes, *Essay on physics and noncommutative geometry*, in *The Interface of Mathematics and Particle Physics* (Oxford 1988), OUP 1990, 9-48. **Unverified** (absent from INSPIRE; data from citing sources; text not read). Connes-Lott 1990 call their electroweak model "basically the same as for model II" of this essay, so the two-point model first appears here.

1.3 The first two-point and matrix models, 1988-1994

- Aug. 1988. J. Madore, *Kaluza-Klein aspects of noncommutative geometry*, Chester DGM proceedings (1989) 243, arXiv:1510.03915 [R]. "We can now generalize Kaluza-Klein theory by replacing C(F) by an associative algebra which is not necessarily commutative"; with M_n, "the internal laplacian has a finite spectrum and … the total momentum space remains of dimension 4. This is important for the renormalizability of the theory."
- 1988-1990. M. Dubois-Violette, R. Kerner and J. Madore: Class. Quant. Grav. 6 (1989) 1709 [M]; Phys. Lett. B 217 (1989) 485 [A + INSPIRE references]; J. Math. Phys. 31 (1990) 316 [M]; J. Math. Phys. 31 (1990) 323 [A + publisher references]. Derivation calculus on C^∞(V) ⊗ M_n: Maxwell's theory gives U(n) Yang-Mills plus adjoint scalars with a quartic potential and a massive phase. J. Madore, Phys. Rev. D 41 (1990) 3709 [A]: Kaluza-Klein with a semisimple internal algebra.
- Mar. 1990. A. Connes and J. Lott, *Particle models and noncommutative geometry*, Nucl. Phys. B Proc. Suppl. 18B (1991) 29 [R, author copy]. GWS, Peccei-Quinn and the Standard Model as pure Yang-Mills on M × {two points}; the Higgs is the connection along the discrete direction; Section VI gives m_H = √2 m_t (top-dominance limit, pure Dixmier trace) and "it is impossible to get GUTs by our methods". Then *The metric aspect of NCG*, Cargèse 1991, 53-93 [M].
- June 1990. R. Coquereaux, G. Esposito-Farèse and G. Vaillant, *Higgs fields as Yang-Mills fields and discrete symmetries*, Nucl. Phys. B 353 (1991) 689 [A + INSPIRE references]: Higgs fields from "the gauging of discrete directions". Mainz-Marseille sequels: R. Coquereaux, R. Häussling, N. A. Papadopoulos and F. Scheck, Int. J. Mod. Phys. A 7 (1992) 2809; R. Coquereaux, G. Esposito-Farèse and F. Scheck, Int. J. Mod. Phys. A 7 (1992) 6555 [M]. Coquereaux came from Kaluza-Klein: R. Coquereaux and A. Jadczyk, World Sci. Lect. Notes Phys. 16 (1988) [M]. Y. Ne'eman and S. Sternberg, Proc. Nat. Acad. Sci. 87 (1990) 7875 [M]: superconnections.
- 1992-1994. A. Dimakis, F. Müller-Hoissen and T. Striker, J. Phys. A 26 (1993) 1927 [M]; A. Dimakis and F. Müller-Hoissen, J. Phys. A 27 (1994) 3159, arXiv:hep-th/9401149 [M], and J. Math. Phys. 35 (1994) 6703, arXiv:hep-th/9404112 [L, §A]: calculus on finite sets and lattices; Connes-Lott models as approximations of higher-dimensional gauge theory. A. Sitarz, J. Geom. Phys. 15 (1995) 123, arXiv:hep-th/9210098 [M]: gauge theory on discrete groups.

1.4 Did the founders present the discrete direction as a way around the chirality obstruction? None of the texts read does so, and none cites Witten 1981 or 1983. The Kaluza-Klein reading is there from the start; the advantage claimed is finiteness; chirality enters by hand through the Z_2-grading of the finite Hilbert space.

- Connes-Lott 1990 [R]. The only Kaluza-Klein sentence (Section I): "The corresponding geometric space can be thought of as a spacetime of Kaluza-Klein type, where the fiber is formed of a two-point set." References: Connes (3), Dixmier, Ellis-Gaillard, Kasparov, Glashow, Peccei-Quinn, Salam, Weinberg (2), Wilczek.

- Connes 1988, CMP 117 [A]: INSPIRE's reference list cites Witten only for *Non-commutative geometry and string field theory*. The Essay was not accessible.
- Connes, *Noncommutative Geometry*, Academic Press 1994 (PDF, alainconnes.org) [R, full-text search]: "Kaluza" does not occur; Witten is cited once, for Morse theory. Chapter VI: "a double space-time", where "purely left-handed particles such as neutrinos live on the left-handed copy X_L".
- Madore 1988 [R]: Kaluza-Klein generalization motivated by the finite internal spectrum; spinors are built "in dimension 4 + n^2 − 1" with no discussion of chirality; no Witten among the references.
- Dubois-Violette-Kerner-Madore [A]: abstracts silent on Kaluza-Klein and chirality; the reference lists shown cite Manton 1979 and R. Kerner, L. Nikolova and V. Rizov, *A two-level Kaluza-Klein theory*, Lett. Math. Phys. 14 (1987) 333 [M], and no Witten.
- Coquereaux-Esposito-Farèse-Vaillant [A]: "The formalism presented here is therefore not a Kaluza-Klein-like approach to Yang-Mills theory, but is, in a sense, a discrete analog of it." INSPIRE's references cite Chapline-Manton 1981 and no Witten. Full text not read.

Later Kaluza-Klein readings (sources searched for "Witten", "chiral", "no-go"; none cites the obstruction): - G. Landi, Nguyen Ai Viet and K. C. Wali, Phys. Lett. B 326 (1994) 45, arXiv:hep-th/9402046 [M]; Nguyen Ai Viet and K. C. Wali, Int. J. Mod. Phys. A 11 (1996) 533, arXiv:hep-th/9412220 [R], and 2403, arXiv:hep-th/9508032 [L, §A]. First "discretization of the fifth dimension"; the contrast drawn is the infinite tower and its inconsistent truncation (citing M. J. Duff, C. N. Pope and K. S. Stelle, Phys. Lett. B 223 (1989) 386 [M]). - A. H. Chamseddine and A. Connes, arXiv:hep-th/9606001 [R, passage]: "our 'discrete Kaluza Klein' geometry M × F". T. Schücker, *Forces from Connes' geometry*, arXiv:hep-th/0111236, Lect. Notes Phys. 659 (2005) 285 [R, passage]: "Kaluza-Klein models whose fifth dimension is discrete". Martinetti-Wulkenhaar 2002 and Dai-Song 2001 [L, §A]. - Chamseddine-Connes, arXiv:0706.3690 [R]: "the advantages of both SO(10) and Kaluza-Klein unification, without paying the price of plethora of Higgs fields or the infinite tower of states". A. H. Chamseddine, arXiv:2511.05909 (2025) [R]: the mechanism is "different from Kaluza-Klein since it avoids a massive tower of states, and places the Higgs in the correct representation".

So the reading was made first by Madore (1988, matrix fibre) and Connes-Lott (1990, two-point fibre), called "a discrete analog" by Coquereaux-Esposito-Farèse-Vaillant (1990) and "a discretization of the fifth dimension" by Landi-Viet-Wali and Viet-Wali (1994). The nearest engagement with the obstruction's hypotheses that we found lies outside NCG: J. Baptista, JHEP 05 (2026) 008, arXiv:2506.09126 [L, §I] (non-Killing gauge fields with chiral couplings). Scope of this negative finding: the texts and reference lists above, plus INSPIRE full-text queries pairing "spectral triple" with "Atiyah-Hirzebruch" and "noncommutative geometry" with Witten's 1983 title (no NCG model-building paper among the hits).

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2. The 1990s

Zürich, Marseille, Mainz: the Connes-Lott model and its constraints. - 1992-1993. A. H. Chamseddine, G. Felder and J. Fröhlich, Phys. Lett. B 296 (1992) 109; Nucl. Phys. B 395 (1993) 672, arXiv:hep-ph/9209224; Commun. Math. Phys. 155 (1993) 205, arXiv:hep-th/9209044 [M]: GUTs and gravity (Einstein plus Brans-Dicke) in the Connes-Lott formalism. A. H. Chamseddine and J. Fröhlich, Phys. Rev. D 50 (1994) 2893, arXiv:hep-th/9304023 [M]: SO(10). - 1992-1995. D. Kastler, Rev. Math. Phys. 5 (1993) 477 and 8 (1996) 103; D. Kastler and T. Schücker, Rev. Math. Phys. 8 (1996) 205, arXiv:hep-th/9501077 [M]: detailed accounts of the model. T. Schücker and J.-M. Zylinski, *Connes' model building kit*, J. Geom. Phys. 16 (1995) 207, arXiv:hep-th/9312186 [M]. - 1993. D. Kastler, Commun. Math. Phys. 166 (1995) 633; W. Kalau and M. Walze, J. Geom. Phys. 16 (1995) 327, arXiv:gr-qc/9312031 [M]: the Wodzicki residue gives Einstein-Hilbert. - 1994-1997. B. Iochum and T. Schücker, *Yang-Mills-Higgs versus Connes-Lott*, Commun. Math. Phys. 178 (1996) 1, arXiv:hep-th/9501142; L. Carminati, B. Iochum and T. Schücker, J. Math. Phys. 38 (1997) 1269, arXiv:hep-th/9604169; B. Iochum, D. Kastler and T. Schücker, J. Math. Phys. 38 (1997) 4929, arXiv:hep-th/9607158 [M]: which models arise; mass constraints; the spectral action computed.

Zaragoza and Costa Rica. - 1992. J. C. Várilly and J. M. Gracia-Bondía, J. Geom. Phys. 12 (1993) 223, DFTUZ-92-20 [M]. 1996: C. P. Martín, J. M. Gracia-Bondía and J. C. Várilly, *The Standard Model as a noncommutative geometry: the low-energy regime*, Phys. Rep. 294 (1998) 363, arXiv:hep-th/9605001

[M]. - 1997. H. Figueroa, J. M. Gracia-Bondía, F. Lizzi and J. C. Várilly, J. Geom. Phys. 26 (1998) 329, arXiv:hep-th/9701179 [M]; J. M. Gracia-Bondía, B. Iochum and T. Schücker, *The Standard Model in NCG and fermion doubling*, Phys. Lett. B 416 (1998) 123, arXiv:hep-th/9709145 [M]. - 1996-2004, DFTUZ, A. Rivero [A]: arXiv:hep-th/9605006 (constraints on C + M_2 + M_3; a leptophobic Z'); hep-th/9804169; hep-th/9905021 and physics/0007027 (generations from ordering and discretization ambiguities: "three generations, and not four"); hep-th/0204238 (Poincaré duality for finite algebras, "to get continuum Dirac operators from discrete noncommutative manifolds"); physics/0409022. J. F. Cariñena, J. Clemente-Gallardo, E. Follana, J. M. Gracia-Bondía, A. Rivero and J. C. Várilly, J. Geom. Phys. 32 (1999) 79, arXiv:math/9802102 [M]: tangent groupoid. - 2001. J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser [M].

Naples: lattices, finite spaces, doubling. - 1993-1996. G. Bimonte, F. Lizzi and G. Sparano, Phys. Lett. B 341 (1994) 139, arXiv:hep-lat/9404007 [L, §A]; A. P. Balachandran et al., Nucl. Phys. B 418 (1994) 477, arXiv:hep-th/9310015, and J. Geom. Phys. 18 (1996) 163, arXiv:hep-th/9510217 [M]: finite (poset) approximations; G. Bimonte, E. Ercolessi, G. Landi, F. Lizzi and G. Sparano, J. Geom. Phys. 20 (1996) 318 and 329 [M]: continuum limits. - 1996-1997. F. Lizzi, G. Mangano, G. Miele and G. Sparano, Mod. Phys. Lett. A 11 (1996) 2561, arXiv:hep-th/9603095 [M]; Phys. Rev. D 55 (1997) 6357, arXiv:hep-th/9610035 [L, §I]: fermion doubling in L^2(S) ⊗ H_F.

Fuzzy spaces. J. Madore, *The fuzzy sphere*, Class. Quant. Grav. 9 (1992) 69 [M]; H. Grosse, C. Klimčík and P. Prešnajder, Commun. Math. Phys. 178 (1996) 507, arXiv:hep-th/9510083, and 180 (1996) 429, arXiv:hep-th/9602115 [M]. Books: J. Madore, LMS Lect. Note Ser. 257 (CUP, 2nd ed. 1999/2000) [M]; G. Landi, Lect. Notes Phys. Monogr. 51 (1997), arXiv:hep-th/9701078 [M].

Reality, axioms, spectral action. - 1995. A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. 36 (1995) 6194 [M]: the real structure J, the KO-dimension table mod 8, the first-order condition. - 1996. A. Connes, *Gravity coupled with matter and the foundation of NCG*, Commun. Math. Phys. 182 (1996) 155, arXiv:hep-th/9603053 [M]: axioms; the Standard Model with J. - 1996. A. H. Chamseddine and A. Connes, Phys. Rev. Lett. 77 (1996) 4868, arXiv:hep-th/9606056; *The spectral action principle*, Commun. Math. Phys. 186 (1997) 731, arXiv:hep-th/9606001 [M]: Tr f(D/Λ) yields gravity, Yang-Mills and Higgs with relations at the cut-off.

Classification and distances. T. Krajewski, J. Geom. Phys. 28 (1998) 1, arXiv:hep-th/9701081 [L, §I]: Krajewski diagrams. M. Paschke and A. Sitarz, J. Math. Phys. 39 (1998) 6191, arXiv:q-alg/9612029 [L, §I]. B. Iochum, T. Krajewski and P. Martinetti, J. Geom. Phys. 37 (2001) 100, arXiv:hep-th/9912217 [L, §A].

Hopf algebras (ramification). D. Kreimer, Adv. Theor. Math. Phys. 2 (1998) 303, arXiv:q-alg/9707029; A. Connes and D. Kreimer, Commun. Math. Phys. 199 (1998) 203, arXiv:hep-th/9808042; 210 (2000) 249, arXiv:hep-th/9912092; 216 (2001) 215, arXiv:hep-th/0003188 [M].

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3. 2003 to 2012

- 2003-2009. B. Iochum, T. Schücker and C. Stephan, J. Math. Phys. 45 (2004) 5003, arXiv:hep-th/0312276; J.-H. Jureit and C. Stephan, J. Math. Phys. 46 (2005) 043512, arXiv:hep-th/0501134; J.-H. Jureit, T. Schücker and C. Stephan, J. Math. Phys. 46 (2005) 072303, arXiv:hep-th/0503190; J.-H. Jureit and C. Stephan, J. Math. Phys. 49 (2008) 033502, arXiv:hep-th/0610040, and 50 (2009) 072301, arXiv:0901.3214; Comput. Phys. Commun. 178 (2008) 230, arXiv:hep-th/0503085 [M]: classification of irreducible almost-commutative geometries by Krajewski diagrams. C. Stephan, J. Phys. A 39 (2006) 9657, arXiv:hep-th/0509213, and sequels [M]: beyond the Standard Model.
- 31 Aug. 2006, same day. A. Connes, JHEP 11 (2006) 081, arXiv:hep-th/0608226 [L, §I]; J. W. Barrett, J. Math. Phys. 48 (2007) 012303, arXiv:hep-th/0608221 [A]. F of KO-dimension 6 and metric dimension 0; total 10 = 2 mod 8; doubling removed (Pfaffian in Connes; Lorentzian J-γ condition in Barrett); ν_R and see-saw. Two independent arrivals.
- Oct. 2006. A. H. Chamseddine, A. Connes and M. Marcolli, Adv. Theor. Math. Phys. 11 (2007) 991, arXiv:hep-th/0610241 [A, L §I]: full Lagrangian from the spectral action; "a Higgs mass around 170

GeV".

- 2007. A. H. Chamseddine and A. Connes, Phys. Rev. Lett. 99 (2007) 191601, arXiv:0706.3690 [R]; *Why the Standard Model*, J. Geom. Phys. 58 (2008) 38, arXiv:0706.3688 [A, L §I]: irreducible finite geometries of KO-dimension 6 have dim H_F = 2k^2; quaternion linearity gives k = 4 and A_F = C ⊕ H ⊕ M_3(C); "(the number of generations is still an input)".
- 2008. A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, AMS Colloquium Publ. 55, doi:10.1090/coll/055 [M]. A. Connes, *On the spectral characterization of manifolds*, J. Noncommut. Geom. 7 (2013) 1, arXiv:0810.2088 [M].
- Aug. 2008. Tevatron combination, arXiv:0808.0534 [A]: "We exclude at 95% C.L. a standard model Higgs boson of m_H = 170 GeV."
- 2009. C. Stephan, Phys. Rev. D 79 (2009) 065013, arXiv:0901.4676 [A]: a complex scalar coupling to ν_R, vector-like fermions and the Higgs, used to constrain the Higgs mass.
- 2009-2010. M. Marcolli and E. Pierpaoli, Adv. Theor. Math. Phys. 14 (2010) 1373, arXiv:0908.3683; W. Nelson, J. Ochoa and M. Sakellariadou, Phys. Rev. Lett. 105 (2010) 101602, arXiv:1005.4279 [M]: cosmology and astrophysical bounds from the spectral action.
- 2010. A. H. Chamseddine and A. Connes, Fortsch. Phys. 58 (2010) 553, arXiv:1004.0464 [M]; *Noncommutative geometric spaces with boundary*, J. Geom. Phys. 61 (2011) 317, arXiv:1008.3980 [L, §I].
- 2010-2012. L. Dąbrowski and G. Dossena, *Product of real spectral triples*, Int. J. Geom. Meth. Mod. Phys. 8 (2011) 1833, arXiv:1011.4456 [L, §D]; J. Boeijink and W. D. van Suijlekom, J. Geom. Phys. 61 (2011) 1122, arXiv:1008.5101 [M]; B. Ćaćić, Lett. Math. Phys. 100 (2012) 181, arXiv:1101.5908 [M]: reconstruction for almost-commutative triples.
- 2011. K. van den Dungen and W. D. van Suijlekom, *Electrodynamics from NCG*, J. Noncommut. Geom. 7 (2013) 433, arXiv:1103.2928 [A]: U(1) from a two-point space, which had "long been thought impossible"; review Rev. Math. Phys. 24 (2012) 1230004, arXiv:1204.0328 [M]. A. A. Andrianov, M. A. Kurkov and F. Lizzi, JHEP 10 (2011) 001, arXiv:1106.3263 [M]: Weyl anomaly and Higgs-dilaton potential.
- 2012. A. H. Chamseddine, J. Fröhlich, B. Schubnel and D. Wyler, J. Math. Phys. 54 (2013) 012302, arXiv:1201.2661 [A]: discrete extra dimensions give axion, dilaton and Higgs.
- July-Aug. 2012. ATLAS, Phys. Lett. B 716 (2012) 1, arXiv:1207.7214; CMS, Phys. Lett. B 716 (2012) 30, arXiv:1207.7235 [M]. A. H. Chamseddine and A. Connes, *Resilience of the spectral Standard Model*, JHEP 09 (2012) 104, arXiv:1208.1030 [A]: the real singlet σ, "already present in the spectral model and we wrongly neglected it", restores consistency with the low Higgs mass.
- Lorentzian. A. Strohmaier, J. Geom. Phys. 56 (2006) 175, arXiv:math-ph/0110001 [M]; M. Paschke and A. Sitarz, arXiv:math-ph/0611029 [M]; N. Franco, Rev. Math. Phys. 26 (2014) 1430007, arXiv:1210.6575 [M]; N. Franco and M. Eckstein, Class. Quant. Grav. 30 (2013) 135007, arXiv:1212.5171 [M]: causality; K. van den Dungen, M. Paschke and A. Rennie, J. Geom. Phys. 73 (2013) 37, arXiv:1207.2112 [M].

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4. 2013 to 2019

- Apr. 2013. A. H. Chamseddine, A. Connes and W. D. van Suijlekom, J. Geom. Phys. 73 (2013) 222, arXiv:1304.7583 [L, §I]; JHEP 11 (2013) 132, arXiv:1304.8050 [A]: without the first-order condition the classification gives SU(2)_R × SU(2)_L × SU(4), and the Standard Model when the vacuum restores the condition; JHEP 11 (2015) 011, arXiv:1507.08161 [M].
- Apr. 2013. A. Devastato, F. Lizzi and P. Martinetti, *Grand symmetry, spectral action and the Higgs mass*, JHEP 01 (2014) 042, arXiv:1304.0415 [L, §I]: σ from a larger algebra.
- 2012-2013. F. D'Andrea, F. Lizzi and P. Martinetti, *Spectral geometry with a cut-off*, J. Geom. Phys. 82 (2014) 18, arXiv:1305.2605 [L, §A]; F. D'Andrea, F. Lizzi and J. C. Várilly, Lett. Math. Phys. 103 (2013) 183, arXiv:1209.0108 [M]: fuzzy-sphere metrics. B. Ćaćić, *Real structures on almost-commutative spectral triples*, Lett. Math. Phys. 103 (2013) 793, arXiv:1209.4832 [M].
- 2014. L. Boyle and S. Farnsworth, New J. Phys. 16 (2014) 123027, arXiv:1401.5083 [A]: axioms recast through a graded algebra, extending to non-associative geometry and removing seven unwanted

terms; S. Farnsworth and L. Boyle, New J. Phys. 17 (2015) 023021, arXiv:1408.5367 [M]; later New J. Phys. 22 (2020) 073023, arXiv:1910.11888 [M] (Pati-Salam, "Jordan geometry").

- •2014. A. H. Chamseddine, A. Connes and V. Mukhanov, Phys. Rev. Lett. 114 (2015) 091302, arXiv:1409.2471 [L, §I]; JHEP 12 (2014) 098, arXiv:1411.0977 [M]: quanta of geometry. Ramification: A. H. Chamseddine and V. Mukhanov, *Mimetic dark matter*, JHEP 11 (2013) 135, arXiv:1308.5410 [M].
- Twisted triples. A. Connes and H. Moscovici, arXiv:math/0609703 [L, §I]; A. Devastato and P. Martinetti, Math. Phys. Anal. Geom. 20 (2017) 2, arXiv:1411.1320 [A]; G. Landi and P. Martinetti, Lett. Math. Phys. 106 (2016) 1499, arXiv:1601.00219 [M]; A. Devastato, S. Farnsworth, F. Lizzi and P. Martinetti, JHEP 03 (2018) 089, arXiv:1710.04965 [M]; P. Martinetti and D. Singh, J. Noncommut. Geom. 16 (2022) 513, arXiv:1907.02485 [M]. The twist bounds the vector terms of grand symmetry and makes the breaking dynamical; a Lorentzian action follows.
- •2015. W. D. van Suijlekom, *Noncommutative Geometry and Particle Physics*, Springer (2015), doi:10.1007/978-94-017-9162-5; **second edition** (2025), doi:10.1007/978-3-031-59120-4 [M, Crossref].
- 2015-2016. J. W. Barrett, J. Math. Phys. 56 (2015) 082301, arXiv:1502.05383; J. W. Barrett and L. Glaser, J. Phys. A 49 (2016) 245001, arXiv:1510.01377 [M]: fuzzy spaces as finite triples; Monte Carlo of random finite geometries.
- 2016-2019, Lorentzian and algebraic backgrounds. K. van den Dungen, Math. Phys. Anal. Geom. 19 (2016) 4, arXiv:1505.01939 [M]; N. Bizi, C. Brouder and F. Besnard, J. Math. Phys. 59 (2018) 062303, arXiv:1611.07062 [M]; F. Besnard and N. Bizi, J. Geom. Phys. 123 (2018) 292, arXiv:1611.07830 [M]; F. Besnard, J. Math. Phys. 60 (2019) 123506, arXiv:1902.09387 [A]: Dirac operators compatible with an "algebraic background" form "the configuration space of a noncommutative Kaluza-Klein theory", containing Z'_{B−L}, σ and flavour-changing fields.
- 2016-2018. S. Farnsworth, *The graded product of real spectral triples*, J. Math. Phys. 58 (2017) 023507, arXiv:1605.07035 [L, §D]; F. D'Andrea, M. A. Kurkov and F. Lizzi, Phys. Rev. D 94 (2016) 025030, arXiv:1605.03231 [M]: Wick rotation and doubling.
- Beyond the Standard Model. U. Aydemir, D. Minic, C. Sun and T. Takeuchi, Int. J. Mod. Phys. A 31 (2016) 1550223, arXiv:1509.01606 [M]; L. Dąbrowski, F. D'Andrea and A. Sitarz, Lett. Math. Phys. 108 (2018) 1323, arXiv:1703.05279 [M]: fermions as internal forms; A. Bochniak and A. Sitarz, Phys. Rev. D 97 (2018) 115029, arXiv:1804.09482 [M]; F. Besnard and C. Brouder, Phys. Rev. D 103 (2021) 035003, arXiv:2010.04960 [M].
- Reviews. F. Lizzi, PoS CORFU2017 (2018) 133, arXiv:1805.00411; A. Devastato, M. Kurkov and F. Lizzi, Int. J. Mod. Phys. A 34 (2019) 1930010, arXiv:1906.09583; M. Eckstein and B. Iochum, *Spectral Action in NCG*, SpringerBriefs Math. Phys. 27 (2018), arXiv:1902.05306; A. Connes, arXiv:1910.10407 [M].
- Higgs mass after 2012: the σ field of 2012, grand symmetry, the twist, the Pati-Salam completion and Besnard's background all supply the singlet that brings m_H to 125 GeV. In the abstracts read, no paper of this line reaches 125 GeV without it.

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5. 2020 to 2026

Truncations and operator systems. A. Connes and W. D. van Suijlekom, Commun. Math. Phys. 383 (2021) 2021, arXiv:2004.14115 [A, L §A]; arXiv:2111.02903 [M] (tolerance relations). W. D. van Suijlekom, J. Geom. Phys. 162 (2021) 104075, arXiv:2005.08544; L. Glaser and A. B. Stern, J. Math. Phys. 61 (2020) 033507, arXiv:1909.08054, and J. Geom. Phys. 159 (2021) 103921, arXiv:1912.09227; E.-M. Hekkelman, Lett. Math. Phys. 112 (2022) 20, arXiv:2111.13865; M. A. Rieffel, arXiv:2210.00387; R. Toyota, arXiv:2309.13469; M. Leimbach, arXiv:2409.16698; E.-M. Hekkelman and E. McDonald, arXiv:2412.00628; T. Camper, D. Giannakis and G. Hoefer, arXiv:2609.40189 [M]. M. Leimbach and W. D. van Suijlekom, Adv. Math. 439 (2024) 109496, arXiv:2302.07877 [L, §A]; V. Bhatt, S. Guin and P. Sharma, arXiv:2610.01132 [A] (noncommutative n-tori); M. Verhoeven, arXiv:2610.19791 [L, §K] (SU(2)). No "Gaunt-Rennie" paper was found; the Gaunt papers are J. W. Barrett and J. Gaunt, arXiv:1908.06796, and J. Gaunt, H. Nguyen and A. Schenkel, J. Phys. A 55 (2022) 474004, arXiv:2203.04817 [M].

Propinquity, convergence, collapse. F. Latrémolière, Adv. Math. 404 (2022) 108393, arXiv:1811.10843; Commun. Math. Phys. 388 (2021) 1049, arXiv:2102.03729 (fuzzy tori to quantum tori); Math. Ann. 389 (2024) 765, arXiv:2112.11000 [M]; C. Farsi, F. Latrémolière and J. Packer, Adv. Math. 437 (2024) 109442, arXiv:2301.00274 [M]; C. Farsi and F. Latrémolière, J. London Math. Soc. 114 (2026) e70628, arXiv:2404.00240 [L, §B, with the project's assessment of its Theorem 3.4]; F. Latrémolière, arXiv:2603.19128 [A]: continuity of Dirac spectra of almost-commutative manifolds when the metric (C^1) and D_F vary together. Classical collapse: B. Ammann and C. Bär, Ann. Global Anal. Geom. 16 (1998) 221, arXiv:math/9801091 [L, §B]; J. Lott, arXiv:math/0005009 [M]; S. Roos, arXiv:1707.00608 [M].

Fibrations. J. Kaad and W. D. van Suijlekom, arXiv:1610.02873 [M] and arXiv:1710.03182 [L, §K]; S. Brain, B. Mesland and W. D. van Suijlekom, J. Noncommut. Geom. 10 (2016) 135, arXiv:1306.1951 [M]; B. Ćaćić and B. Mesland, Commun. Math. Phys. 388 (2021) 107, arXiv:1912.04179 [M].

Quantum Riemannian geometry (ramification). E. J. Beggs and S. Majid, arXiv:1508.04808 [M]; *Quantum Riemannian Geometry*, Grundlehren 355 (2020), doi:10.1007/978-3-030-30294-8 [M]: bimodule connections and Levi-Civita theory on finite and noncommutative spaces.

Lorentzian. G. Nieuviarts, arXiv:2502.18105 and arXiv:2512.15450 [M]: Lorentz symmetry and time from twisted almost-commutative triples. T. Masson and G. Nieuviarts, J. Geom. Phys. 187 (2023) 104784, arXiv:2207.04466 [A]: sequences of spectral triples over AF inductive limits, Bratteli arrows lifted to Krajewski diagrams.

Spectral action. A. H. Chamseddine, A. Connes and W. D. van Suijlekom, Commun. Math. Phys. 373 (2019) 457, arXiv:1809.02944 (entropy) and Eur. Phys. J. ST 232 (2023) 3581, arXiv:2207.10901 (review) [M]; E.-M. Hekkelman, T. van Nuland and J. Reimann, Commun. Math. Phys. 407 (2026) 254, arXiv:2512.14581 [M]; Chamseddine 2025 (1.4).

Random NCG. C. I. Pérez-Sánchez, J. Noncommut. Geom. 16 (2022) 1137, arXiv:1912.13288; Ann. Henri Poincaré 22 (2021) 3095, arXiv:2007.10914 [M]; H. Hessam, M. Khalkhali and N. Pagliaroli, J. Phys. A 55 (2022) 335204, arXiv:2107.10333 [M]; M. Khalkhali, N. Pagliaroli and L. Verhoeven, J. Math. Phys. 66 (2025) 053502, arXiv:2405.05056 [M]; R. A. Iseppi, arXiv:2410.11823 [M] (BV for finite triples).

Kaluza-Klein readings. Besnard 2019; Ćaćić-Mesland 2021; Baptista 2026 (1.4); J. Aastrup and J. M. Grimstrup, arXiv:2504.03391 [A]: an almost-commutative triple in a semi-classical limit on a configuration space of connections, with doubled fermions. Fuzzy extra dimensions: P. Aschieri, J. Madore, P. Manousselis and G. Zoupanos, JHEP 04 (2004) 034, arXiv:hep-th/0310072; P. Aschieri, T. Grammatikopoulos, H. Steinacker and G. Zoupanos, JHEP 09 (2006) 026, arXiv:hep-th/0606021 [M].

Finite spectral triples as limits of geometries (the project's question). The searches of Section 9 found four kinds of result: (a) convergence of truncations or matrix approximations to a fixed manifold or quantum space (circle, tori, SU(2), quantum groups, fuzzy tori); (b) collapse of a fibre in the propinquity, with a nontrivial limit on ker D (Farsi-Latrémolière); (c) continuity of almost-commutative spectra in g and D_F (Latrémolière 2026); (d) emergence or inductive sequences of finite triples (Aastrup-Grimstrup; Masson-Nieuviarts). We found no paper that obtains a particle-physics finite triple (Connes-Lott, or the KO-6 F) as the limit or truncation of a continuous Kaluza-Klein internal geometry with chiral boundary conditions; paper/LITERATURE.md §A-§B reach the same conclusion for the interval.

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6. Ramifications outside particle physics

- Number theory. J.-B. Bost and A. Connes, Selecta Math. 1 (1995) 411, doi:10.1007/BF01589495; A. Connes, arXiv:math/9811068 (trace formula and zeta zeros); A. Connes and C. Consani, Enseign. Math. 69 (2023) 93, arXiv:2106.01715; A. Connes, C. Consani and H. Moscovici, *Zeta spectral triples*, EMS Ser. Lect. Math. (2026) 39, doi:10.4171/elm/37/3 [M].
- Renormalization and motives. Connes-Kreimer (Section 2); A. Connes and M. Marcolli, J. Geom. Phys. 56 (2006) 55, arXiv:hep-th/0504085 [M].

- Solid state. J. Bellissard, *K-theory of C*-algebras in solid state physics*, Lect. Notes Phys. (1986) 99, doi:10.1007/3-540-16777-3_74; J. Bellissard, A. van Elst and H. Schulz-Baldes, J. Math. Phys. 35 (1994) 5373, arXiv:cond-mat/9411052; E. Prodan and H. Schulz-Baldes, *Bulk and Boundary Invariants for Complex Topological Insulators*, Springer (2016) [M].
- Strings and Moyal spaces. A. Connes, M. R. Douglas and A. Schwarz, JHEP 02 (1998) 003, arXiv:hep-th/9711162; N. Seiberg and E. Witten, JHEP 09 (1999) 032, arXiv:hep-th/9908142; M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. 73 (2001) 977, arXiv:hep-th/0106048; H. Grosse and R. Wulkenhaar, Commun. Math. Phys. 256 (2005) 305, arXiv:hep-th/0401128, and 329 (2014) 1069, arXiv:1205.0465 [M].
- Matrix models. T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. D 55 (1997) 5112, arXiv:hep-th/9610043; N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B 498 (1997) 467, arXiv:hep-th/9612115; H. Steinacker, Class. Quant. Grav. 27 (2010) 133001, arXiv:1003.4134, and *Quantum Geometry, Matrix Theory, and Gravity*, CUP (2024), doi:10.1017/9781009440776 [M].

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7. Comparison with the author table (to 2003)

The table has 65 author rows (columns pre-1990 to 2003, arXiv codes) and excludes "purely mathematical authors" by its own statement. Post-2003 lines below rest on one checked paper each [M].

Continued, by line. - Standard-Model programme: Connes, Chamseddine (Sections 3-5); Fröhlich (2013 paper, Section 3). Schücker: classification with Jureit and Stephan to 2007, then mainly cosmology and lensing (Gen. Rel. Grav. 41 (2009) 67, arXiv:0712.1559). Iochum: spectral action with torsion and boundaries (Commun. Math. Phys. 310 (2012) 367, arXiv:1008.3630) and the 2018 book. Martinetti, Lizzi: grand symmetry, twists, distances, reviews. Sitarz, Dąbrowski: curved noncommutative tori, Standard-Model variants. Landi: twists with Martinetti. Paschke, Brouder, Rennie: Lorentzian triples; Rennie also boundaries. Várilly: Moyal planes, fuzzy-sphere metrics, reconstruction (arXiv:math/0610418). - Krajewski: from 2007 mainly group field theory (arXiv:1210.6257). - Moyal and matrix field theory: Grosse, Wulkenhaar ($\varphi\hat{}4$, 2005-2014); Steinacker (emergent gravity, IKKT); Aschieri, Castellani, Wess (twisted noncommutative gravity, arXiv:hep-th/0510059); Madore (fuzzy cosmologies with Burić); Ydri (fuzzy-sphere simulations); Balachandran (twisted statistics). - Algebraic lines: Majid (with Beggs); Dubois-Violette (exceptional Jordan algebra and the Standard Model, Nucl. Phys. B 912 (2016) 426, arXiv:1604.01247; with Todorov, arXiv:1806.09450); Kreimer (Hopf algebras, motives); Moscovici (twists, modular curvature); Goswami (Levi-Civita connections for spectral triples). - Changed topic, as far as the post-2005 records show: Coquereaux (graph quantum symmetries), Dimakis and Müller-Hoissen (integrable systems), Klimčík (integrable sigma models), Kerner, Gracia-Bondía, Figueroa, Requardt (emergent space), Lott (geometric analysis; his 2000 collapse paper is in Section 5); Rivero left in 2006. - Not checked: Atzmon, Bimonte, Brzeziński, Chakraborty, Dai and Song (13 codes in 2001), Drabant, Elsner, Frank, Frabetti, Gozzi, Grandjean, Hadfield, Hajac, Kalau, Karoubi, Kastler, Kauffman, Lord, Prešnajder, Recknagel, Rovelli, Scheck, Striker, Walze, Zumino.

Lines that stopped as particle-physics programmes. Derivation-based matrix Kaluza-Klein (Madore; Viet-Wali), which survives as fuzzy extra dimensions (Zoupanos and collaborators); discrete calculus for Connes-Lott models (Dimakis-Müller-Hoissen; Sitarz 1992-1995); Mainz-Marseille superconnection models; NCG applied to deconstruction (Dai-Song, one paper located).

Entered after 2003, absent from the table. Marcolli; Barrett; Stephan, Jureit, Squellari; van Suijlekom, van den Dungen, Boeijink; Ćaćić, Mesland, Kaad; Dossena; Sakellariadou, Nelson, Pierpaoli; Franco, Eckstein; Strohmaier (Lorentzian, preprint 2001, also absent); Kurkov, Devastato, D'Andrea, Filaci, Singh; Boyle, Farnsworth; Besnard, Bizi; Mukhanov; Aydemir, Minic, Sun, Takeuchi; Bochniak; Glaser, Stern, Hekkelman, Leimbach, Gaunt; Khalkhali, Pagliaroli, Hessam, Verhoeven, Pérez-Sánchez, Iseppi; Aastrup, Grimstrup; Masson, Nieuviarts; Latrémolière, Farsi (mathematical); Consani.

Pure-mathematics lines excluded by the table that matter here. - Manifolds with boundary: B. Iochum and C. Levy, arXiv:1001.3927; I. Forsyth, M. Goffeng, B. Mesland and A. Rennie, J. Noncommut. Geom. 13 (2019) 407, arXiv:1607.07143; C. Bär and W. Ballmann, Surv. Differ. Geom. 17 (2012) 1, arXiv:1101.1196, and Progr. Math. 319 (2016) 43, arXiv:1307.3021 [M]. - Fibrations and collapse: J.-M.

Bismut and J. Cheeger, J. Amer. Math. Soc. 2 (1989) 33 [M]; Ammann-Bär, Lott, Kaad-van Suijlekom, Farsi-Latrémolière (Section 5). - Callias-type index: C. Callias, Commun. Math. Phys. 62 (1978) 213; U. Bunke, Math. Ann. 303 (1995) 241, doi:10.1007/BF01460989 [M]; C. Kottke, arXiv:0909.5661 [L, §K].

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8. Relevance to this project

Kaluza-Klein reading of the two-point space; the Higgs as A_5. By 1.4 the reading is as old as the subject (Madore 1988; Connes-Lott 1990; Coquereaux-Esposito-Farèse-Vaillant 1990), the "discretized fifth dimension" is Landi-Viet-Wali and Viet-Wali 1994, and the Higgs-as-connection idea descends from Manton, Fairlie and Forgács-Manton, which the matrix and Marseille papers cite. paper/LITERATURE.md §A already quotes Connes-Lott §I. No founding text links the discrete fibre to the chirality obstruction, so a paper that states the obstruction in its scope and derives Connes-Lott from a chiral interval would make that link for the first time in print, within the searches of 1.4.

Truncations and windows. Section 5: truncation and convergence theorems exist for circles, tori, SU(2) and quantum groups, and collapse is treated by Farsi-Latrémolière; the chiral interval, whose minimal truncation is the algebra C ⊕ C, is absent (paper/LITERATURE.md §A, §K).

Real structure and KO-dimension. J and the mod-8 table (Connes 1995); KO-6 for F (Connes; Barrett, same day in 2006); products (Dąbrowski-Dossena; Farnsworth; Ćaćić). We found no discussion of total dimension 9 or 11 as alternatives to 10 (paper/LITERATURE.md §D).

Number of generations. Nothing in the Connes-Chamseddine line fixes it: *Why the Standard Model* says "the number of generations is still an input", and the 2007 PRL lists it among the "delicate points". The classification counts states per generation (2k^2 = 32); Krajewski diagrams and the Jureit-Stephan classification admit any multiplicity. The proposals located that derive three lie outside the spectral-triple axioms: the DFTUZ notes of 1999-2000 and the exceptional-Jordan line (Dubois-Violette 2016; Dubois-Violette-Todorov 2018; L. Boyle, J. Math. Phys. 67 (2026) 071701, arXiv:2006.16265 [M]). Whether these give three as a theorem was not checked.

Lorentzian and boundary versions. Lorentzian: Barrett 2007, Strohmaier, Franco-Eckstein, van den Dungen, Besnard-Bizi, twisted triples. Boundary: Chamseddine-Connes 2011 impose Π_−Ψ = 0 with a local chirality on the boundary of the manifold factor; Iochum-Levy, Forsyth-Goffeng-Mesland-Rennie and Bär-Ballmann give the mathematics. We found no spectral triple whose internal direction is a manifold with boundary (paper/LITERATURE.md §I, lemma 22).

Lines the project has not touched. Twisted triples and grand symmetry; the Boyle-Farnsworth graded and Jordan reformulations; Besnard's algebraic background, which already contains σ and Z'_{B−L}; random NCG; quantum Riemannian geometry; propinquity beyond the collapse paper; fuzzy extra dimensions (Aschieri-Madore-Manousselis-Zoupanos), which give finite internal spaces with truncated towers and should be read before any claim about truncated Kaluza-Klein towers; Moyal field theory; the number-theoretic and solid-state ramifications.

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9. Not covered, and identifiers left unverified

Not covered. Quantum groups and Hopf-Galois bundles; κ-Minkowski; noncommutative gravity of the Wess school; the local index formula and cyclic cohomology; Dixmier traces; loop-quantum-gravity spectral triples beyond one entry; Pati-Salam phenomenology after 2018; proceedings in general. Citation chains were not exhausted, and absence from these searches does not establish absence from the literature.

Read only through abstracts or retrospectives. Duff-Nilsson-Pope 1986 and Bailin-Love 1987 (open-problem sections); the Appelquist-Chodos-Freund introduction; the full texts of Coquereaux-Esposito-Farèse-Vaillant and Dubois-Violette-Kerner-Madore.

Unverified or partly verified. 1. A. Connes, *Essay on physics and noncommutative geometry* (OUP 1990): not on INSPIRE; data from citing sources only. 2. Journal data not found or not checked (arXiv identifiers checked): Connes-Moscovici math/0609703; Paschke-Sitarz math-ph/0611029; Beggs-Majid 1508.04808; Lott math/0005009; Kaad-van Suijlekom 1610.02873; Iochum-Levy 1001.3927. 3. "Gaunt-Rennie" (named in the request): no such paper found. 4. Madore's book, first edition (1995): not checked. 5. Volume of Bellissard's 1986 Lecture Notes in Physics paper: Crossref gives none (DOI checked).

Searches for Section 5 (4 Nov 2026). arXiv abstracts: "spectral truncation(s)"; "finite spectral triple"; "spectral triple" with "Kaluza-Klein", "fibration", "interval" and "boundary"; "almost-commutative" with "limit" or "spectral"; "operator system" or "Gromov-Hausdorff" with "spectral triple"; "noncommutative geometry" with "Kaluza-Klein" or "deconstruction"; "collapsing" with "Dirac operator". INSPIRE full text: as in 1.4. Two extended web searches (the two-point space and the chirality no-go; finite triples as Kaluza-Klein limits).

Part L (literature check and record of searches)

Literature check for claims A-I

Written by Claude Opus 5.5 on 3 November 2026, as recommendation 1 of historical/evaluation2026-11-03.md. Method: web searches (standard mode, extended mode where results were thin), arXiv abstract pages and full texts, Crossref and INSPIRE metadata for every journal reference below. Where an equation or section number is given, it was read in the full text (arXiv version unless stated). Items that could not be checked are marked **unverified** and collected at the end.

Verdicts: **KNOWN** (same statement in print), **PARTIAL** (a close or weaker version in print; the difference is stated), **NOT FOUND** (no source located after the searches listed).

A. Connes-Lott as the minimal spectral truncation of a chiral interval

Claim (lemma 05, Part 3; lemma 09, Theorem 9.6). The minimal nonzero grading-preserving spectral truncation of the Dirac operator on [0, l] with R(0) = 0, L(l) = 0 is unitarily equivalent to the Connes-Lott two-point triple (C + C, D = [[0, m], [m, 0]]) with |m| = pi/(2l); the compressed operator system P C([0, l]) P equals C + C; the truncated end states are at distance 2l/pi. The deconstructed chain of Part 1 has Connes-Lott as its N = 2 member.

Verdict: PARTIAL. The interpretation of the two-point space as a discrete or Kaluza-Klein fibre, the truncation machinery, the deconstruction picture and the lowest mixed-boundary mass pi/(2l) are all in print. We found no source that identifies the Connes-Lott triple as a spectral truncation of an interval Dirac operator with chiral boundary conditions, computes the compression P C([0, l]) P for such an interval, or relates |m| to the interval length.

Interpretation of the two-point space (in print).

- A. Connes and J. Lott, *Particle models and noncommutative geometry*, Nucl. Phys. B Proc. Suppl. 18B (1991) 29-47, doi:10.1016/0920-5632(91)90120-4; author's copy at math.berkeley.edu/~lott/partmodels.pdf. Section I: "The corresponding geometric space can be thought of as a spacetime of Kaluza-Klein type, where the fiber is formed of a two-point set." Section II.B, item (9): for the two-point space, d(p, p') = L, where L^{-2} is the largest eigenvalue of M*M. Section III (p. 36): the character space of the product is two copies of Z, "Thus the picture is of two copies of Z separated by a distance L. (As one loses information in passing to the character space, this picture should not be taken too literally.)", with (d_Z(p, p')^2 + L^2)^{1/2} <= d(p, p') <= d_Z(p, p') + L.
- N. A. Viet and K. C. Wali, *Noncommutative geometry and a discretized version of Kaluza-Klein theory with a finite field content*, Int. J. Mod. Phys. A 11 (1996) 533-551, doi:10.1142/S0217751X96000249, arXiv:hep-th/9412220; and *A discretized version of Kaluza-Klein theory with torsion and massive fields*,

Int. J. Mod. Phys. A 11 (1996) 2403-2418, doi:10.1142/S0217751X96001206, arXiv:hep-th/9508032. The second abstract: "We consider an internal space of two discrete points in the fifth dimension of the Kaluza-Klein theory." The "discrete fifth dimension" reading is therefore explicit in print since 1995.

- A. Dimakis and F. Müller-Hoissen, *Discrete differential calculus, graphs, topologies and gauge theory*, J. Math. Phys. 35 (1994) 6703-6735, doi:10.1063/1.530638, arXiv:hep-th/9404112. Abstract: the formalism "suggests that the [Connes-Lott two-point space] be regarded as an approximation of a manifold and thus opens a way to relate models with an 'internal' discrete space (à la Connes et al.) to models of dimensionally reduced gauge fields". Section III, Example 4: the universal differential algebra on two points has a Hasse diagram approximating the topology of S^2, linked to Manton's six-dimensional Yang-Mills reduction on S^2. Section VII: the Connes-Lott models should be understood "as approximations of higher-dimensional gauge theory models … The details have still to be worked out, however." This is the closest statement of the programme of lemma 05. The approximation there is topological (posets and Hasse diagrams) and the internal space is S^2; the passages we read contain no Dirac operator, boundary condition or mass-length relation.
- P. Martinetti and R. Wulkenhaar, *Discrete Kaluza-Klein from scalar fluctuations in noncommutative geometry*, J. Math. Phys. 43 (2002) 182-204, doi:10.1063/1.1418012, arXiv:hep-th/0104108. Section V, Theorem 4 and eq. (27): the distance between the same point on the two sheets is 1/||M|| (M the off-diagonal block of D_F), with a Pythagorean relation when [D_I, p] = 0. In the proof of Theorem 4 the authors embed the two sheets in a "virtual" five-manifold [0, 1] x M as a proof device and state that "the points between the sheets are not part of the geometry". Section VI: after scalar fluctuations the internal metric component becomes a function on M (the Higgs field); Section VII, Proposition 8, gives this component for the Standard Model. "Discrete Kaluza-Klein" refers to this Higgs-as-g_55 picture. Spectral truncation and boundary conditions do not occur.
- B. Iochum, T. Krajewski and P. Martinetti, *Distances in finite spaces from noncommutative geometry*, J. Geom. Phys. 37 (2001) 100-125, doi:10.1016/S0393-0440(00)00044-9: distances on finite spectral triples, including the two-point case.
- Matrix and discrete Kaluza-Klein precursors (bibliographic data verified; contents known from abstracts and secondary sources only): M. Dubois-Violette, R. Kerner and J. Madore, *Noncommutative differential geometry and new models of gauge theory*, J. Math. Phys. 31 (1990) 323 (Higgs fields as components of connections in matrix directions); J. Madore, *Kaluza-Klein aspects of noncommutative geometry* (1989 proceedings, posted as arXiv:1510.03915); J. Madore and J. Mourad, *Noncommutative Kaluza-Klein theory*, arXiv:hep-th/9601169; R. Coquereaux, G. Esposito-Farèse and G. Vaillant, *Higgs fields as Yang-Mills fields and discrete symmetries*, Nucl. Phys. B 353 (1991) 689 (**unverified** beyond citations); A. H. Chamseddine, J. Fröhlich, B. Schubnel and D. Wyler, *Dimensional reduction without continuous extra dimensions*, J. Math. Phys. 54 (2013) 012302, arXiv:1201.2661.

Spectral truncation (in print, for other spaces).

- A. Connes and W. D. van Suijlekom, *Spectral truncations in noncommutative geometry and operator systems*, Commun. Math. Phys. 383 (2021) 2021-2067, doi:10.1007/s00220-020-03825-x, arXiv:2004.14115. Section 1 (Introduction, pp. 2-3 of the arXiv version): "a natural spectral truncation is given simply by a spectral projection P onto eigenspaces of D … The natural truncation of the action of the *-algebra A to the Hilbert space PH is given by the space PAP. Since P does not commute with A, this is not an algebra anymore. Moreover in many interesting examples if one takes the C*-algebra generated by PAP one gets a non-informative full matrix algebra. However, not all is lost: the space PAP is a *-closed subspace in B(H), i.e. it is a so-called operator system". The same passage notes that symmetries of (A, D) induce symmetries of (PAP, PDP). The worked example is the circle (Toeplitz operator systems, Sections 4-5). Relation to lemma 05: the compression map Q of eq. (18) is non-multiplicative, as in this general picture, but for the chiral interval the minimal truncation has range exactly C + C, an algebra; this exceptional case is absent from the paper.
- F. D'Andrea, F. Lizzi and P. Martinetti, *Spectral geometry with a cut-off: topological and metric aspects*, J. Geom. Phys. 82 (2014) 18-45, doi:10.1016/j.geomphys.2014.03.014, arXiv:1305.2605. Cut-off by a projection on D and/or A; points are approximated by non-pure truncated states at finite distance; on the circle these are Fejér distributions; applications to the Moyal plane and the fuzzy sphere. The interval does not appear.

- Follow-ups on truncations (all circle, tori or groups): E.-M. Hekkelman, *Truncated geometry on the circle*, Lett. Math. Phys. 112 (2022), doi:10.1007/s11005-022-01514-5; L. Glaser and A. Stern, *Reconstructing manifolds from truncations of spectral triples*, J. Geom. Phys. 159 (2021) 103921; M. Leimbach and W. D. van Suijlekom, *Gromov-Hausdorff convergence of spectral truncations for tori*, Adv. Math. 439 (2024) 109496, arXiv:2302.07877.

Deconstruction and lattices.

- N. Arkani-Hamed, A. G. Cohen and H. Georgi, *(De)constructing dimensions*, Phys. Rev. Lett. 86 (2001) 4757-4761, doi:10.1103/PhysRevLett.86.4757, arXiv:hep-th/0104005.
- C. T. Hill, S. Pokorski and J. Wang, *Gauge invariant effective Lagrangian for Kaluza-Klein modes*, Phys. Rev. D 64 (2001) 105005, doi:10.1103/PhysRevD.64.105005, arXiv:hep-th/0104035. The abstract states that the chain of gauge groups linked by bifundamentals "matches the continuum compactified theory truncated in momentum space", and treats chiral fermions. This is the physics statement "deconstruction = momentum truncation of the KK tower".
- H.-C. Cheng, C. T. Hill, S. Pokorski and J. Wang, *The standard model in the latticized bulk*, Phys. Rev. D 64 (2001) 065007, arXiv:hep-th/0104179 (split-generation fermions on a transverse lattice).
- J. Dai and X.-C. Song, *Spontaneous symmetry broken condition in (de)constructing dimensions from noncommutative geometry*, ICFP 2001 proceedings, doi:10.1142/9789812777379_0044, arXiv:hep-ph/0105280. The only paper found that applies NCG (differential calculus on cyclic groups, in the Dimakis-Müller-Hoissen/Sitarz style) to the Arkani-Hamed-Cohen-Georgi model; it treats the bosonic vacuum, without spectral triples or fermion truncation.
- NCG distances on lattices: G. Bimonte, F. Lizzi and G. Sparano, *Distances on a lattice from noncommutative geometry*, Phys. Lett. B 341 (1994) 139-146, arXiv:hep-lat/9404007 (the anomalous lattice distances that lemma 05, Part 1 calls "the familiar lattice-distance anomaly"); E. Atzmon, *Distances on a one-dimensional lattice from noncommutative geometry*, Lett. Math. Phys. 37 (1996) 341-348, doi:10.1007/BF00343197.
- The lowest Dirac (or gauge) mass pi/(2l) for mixed boundary conditions on an interval is textbook interval physics; see C. Csáki et al., *Fermions on an interval* (reference in C), Section 3 and Appendix B.

What remains unmatched. The identification "minimal grading-preserving spectral truncation of the chiral interval = Connes-Lott with |m| = pi/(2l)", the computation P C([0, l]) P = C + C, the truncated end-state distance 2l/pi against the geodesic length l, and the chain family of Part 1 with an exact Connes-Lott member at N = 2 and Gromov-Hausdorff convergence to [0, l].

Searched: "two-point space" with "spectral truncation", "interval", "boundary conditions"; "discrete Kaluza-Klein"; "deconstruction" with "noncommutative geometry", "spectral triple"; "gauge-Higgs unification" with "Connes-Lott"; "spectral truncation" with "interval [0,1]" (extended mode); NCG plus orbifold S^1/Z_2.

B. The two limits of the massive chiral interval

Claim (lemma 09, Theorems 9.8-9.9, Corollary 9.10). As l -> infinity at fixed bulk mass, the light pair becomes two chiral wall modes and the rest of the spectrum obeys a one-dimensional Weyl law above the gap mu. As l -> 0 with mu(l) l ~ log(1/(m_f l)), a Connes-Lott pair of fixed mass m_f survives while the interval collapses; at fixed mu no light fermion survives.

Verdict: PARTIAL. The spectral physics is classical: wall zero modes, exponentially small mixing of the two boundary modes on a finite slab, and the resulting light mass ~ 2 mu e^{-mu l}. Collapse of spectral triples is an active topic in NCG, with a general result for collapse to a point. We found no source stating either limit as a theorem about families of finite or interval spectral triples, and no source for the tuned collapse mu(l) that keeps a Connes-Lott pair.

References.

- R. Jackiw and C. Rebbi, *Solitons with fermion number 1/2*, Phys. Rev. D 13 (1976) 3398-3409, doi:10.1103/PhysRevD.13.3398: the wall zero mode.
- D. B. Kaplan, *A method for simulating chiral fermions on the lattice*, Phys. Lett. B 288 (1992) 342-347, doi:10.1016/0370-2693(92)91112-M, arXiv:hep-lat/9206013: domain-wall fermions.
- Y. Shamir, *Chiral fermions from lattice boundaries*, Nucl. Phys. B 406 (1993) 90-106, doi:10.1016/0550-3213(93)90162-I, arXiv:hep-lat/9303005. Section 2: a Weyl mode bound to the boundary of a half-infinite fifth direction, profile b^s with |b| < 1 (eq. (14)). Section 3: with a finite fifth direction 0 <= s <= N, "a second chiral fermion appears on the new boundary at s = N … this fermion has the opposite chirality … Strictly speaking, there is a tiny mixing between the two chiral modes, which vanishes like (1 - M)^N." This is the lattice version of Lemma 9.5 and Theorem 9.8.
- C. G. Callan and J. A. Harvey (1985), see section H: the continuum domain-wall zero modes and anomaly inflow.
- C. Csáki, C. Grojean, J. Hubisz, Y. Shirman and J. Terning, *Fermions on an interval: quark and lepton masses without a Higgs*, Phys. Rev. D 70 (2004) 015012, doi:10.1103/PhysRevD.70.015012, arXiv:hep-ph/0310355. Section 2: boundary conditions from the variational principle; Section 4.1, eqs. (4.8)-(4.9): a bulk mass profile realizing Dirichlet-Neumann conditions, with a tan(m_n L) quantization; Appendix B: KK decompositions in flat space.
- C. Farsi and F. Latrémolière, *Collapse in noncommutative geometry and spectral continuity*, J. London Math. Soc. (2) 114 (2026) e70628, doi:10.1112/jlms.70628, arXiv:2404.00240. Collapse of the vertical part of spectral triples in the spectral propinquity, with spectral continuity; Section 4.1, Corollary 4.1: collapse of a spectral triple to a point gives a nontrivial limit acting on ker D (read in the published text). The published text (first published 3 July 2026, read on 3 Nov 2026 from the journal HTML) agrees with arXiv v3 in the statements and equations compared (copy-editing, and L_A corrected to L_B in Definition 2.13): Theorem 3.4 (Theorem 2.5 in v1) keeps the same six hypotheses, and the step at issue is printed as eq. (3.11), so Counterexample R1 of the assessment applies to the printed statement. Relation to lemma 09: Corollary 9.10 (fixed mu, condition (c), ker D = 0, no light sector) agrees with this picture. Theorem 9.9 keeps a nonzero light pair because the Dirac operator itself changes with l (mu = mu(l)); that family lies outside the metric-rescaling collapse treated there. **Assessed in detail:** historical/reviews/2026-11-03_farsi_latremoliere_assessment.md. Corollary 4.1 (a fixed triple scaled by 1/eps converges to (C, ker D, 0), and has no limit when ker D = 0) is sound. The printed hypotheses of Theorem 3.4 (Theorem 2.5 in v1) are insufficient; anticommutation of D_h and D_v, or a uniform relative bound, supplies a sufficient additional hypothesis; Counterexample R1 there, checked by hand by Claude and by Terra 6.1, satisfies the six listed hypotheses and contradicts Corollary 3.5. The anticommuting applications avoid this defect; Theorem 4.6 and Corollary 4.7 also need a nonzero vertical kernel for the displayed target to be a spectral triple. The published version keeps the statement without the hypothesis. Validated by Astra 6 (Astra validation, items 2 and 4). Our tuned collapse (Theorem 9.9) lies outside its hypotheses.
- B. Ammann and C. Bär, *The Dirac operator on nilmanifolds and collapsing circle bundles*, Ann. Global Anal. Geom. 16 (1998) 221-253, doi:10.1023/A:1006553302362, arXiv:math/9801091: in commutative geometry, collapse either sends all Dirac eigenvalues to infinity or keeps a convergent part, depending on the spin structure.

What remains unmatched. The statement of both limits for the interval spectral triple with condition (c) as theorems (Theorems 9.8-9.9), the formula c = log(2c/(m_f l)) + o(1) for the bulk mass needed to keep a fixed light mass, and the identification of the surviving pair with Connes-Lott at each l (Theorem 9.6).

Searched: domain-wall fermions on a finite slab (exponentially small mass); "collapse" with "spectral triple"; "decompactification" and "emergent dimension" with "almost-commutative"; Ammann-Bär collapse and its NCG follow-ups.

C. Towerless Yukawa textures

Claim (lemma 12). For bulk fields Q, U on [0, l] with bulk masses and a boundary Yukawa matrix X at x = l: an exact secular equation det(lambda^2 - A X B X^dagger) = 0 (Theorem 12.4); the one-flavour,

massless spectrum (n pi +- arctan|X|)/l (Example 12.5); singular-value bounds (Theorem 12.6); in the limit l -> 0 at fixed c and Y_hat = X/l the low spectrum converges to the finite triple with Yukawa block M_F = Phi_Q Y_hat Phi_U (Theorem 12.7); graded singular values of F X G (Lemma 12.8).

Verdict: PARTIAL. The physics is in print, in more elaborate (warped) form: exact KK determinant equations with brane Yukawa matrices, the effective Yukawa F(c_Q) Y F(c_u), masses from minors and mixings F(c_Qi)/F(c_Qj). The one-flavour flat spectrum is Csáki et al. eq. (5.6) (KNOWN). We found no NCG paper that derives the finite Dirac operator D_F, or its Yukawa texture, from localization on an extra dimension.

References.

- C. Csáki et al., *Fermions on an interval* (full data in B). Section 5.1, eqs. (5.4)-(5.6): flat interval, no bulk mass, boundary conditions psi_eL(0) = 0, chi_eR(0) = 0, psi_eL(L) = -M_D L psi_eR(L), chi_eR(L) = M_D L chi_eL(L), spectrum tan(m_n L) = M_D L. With X = M_D L and signed 4D masses, the roots are (n pi +- arctan|X|)/L, which is Example 12.5. Section 2, eqs. (2.8)-(2.9): general boundary conditions with matrices; the spectra are computed one field at a time (Section 7.1: "a much richer spectrum of possibilities for such mixings which we will however not deal with here"). Section 6.2.2, eq. (6.35): secular equation for bulk mass plus brane Majorana mass (warped). Appendix C.2, eqs. (C.9)-(C.14): the boundary parameter equals M_D L, 2 tanh(M_D L/2) or M_D L/(1 + M_D^2 L^2/4) depending on how the brane mass is regularized, so the X of lemma 12 is a boundary-condition parameter whose relation to a brane Lagrangian mass is scheme dependent.
- S. Casagrande, F. Goertz, U. Haisch, M. Neubert and T. Pfoh, *Flavor physics in the Randall-Sundrum model I*, JHEP 10 (2008) 094, doi:10.1088/1126-6708/2008/10/094, arXiv:0807.4937. Section 3.2: IR boundary conditions with the Yukawa matrices (eqs. (55), (63)) and the exact determinant condition for the KK masses, eq. (65), for N generations; eq. (71): profile normalization F(c); eq. (87): Y_eff = F(c_Q) Y F(c_u); eq. (90): zero-mode approximation det(m_n^2 - (v^2/2) Y_eff Y_eff^dagger) = 0; eqs. (94)-(96): masses as F(c_Q) F(c_u) times ratios of minors of Y; eq. (97): (U_q)_ij ~ F(c_Qi)/F(c_Qj). This is the warped analogue of Theorems 12.4, 12.7, Corollary 12.9 and Remark 12.10.
- Flat and warped localization: N. Arkani-Hamed and M. Schmaltz, *Hierarchies without symmetries from extra dimensions*, Phys. Rev. D 61 (2000) 033005, doi:10.1103/PhysRevD.61.033005; T. Gherghetta and A. Pomarol, *Bulk fields and supersymmetry in a slice of AdS*, Nucl. Phys. B 586 (2000) 141-162, doi:10.1016/S0550-3213(00)00392-8; Y. Grossman and M. Neubert, *Neutrino masses and mixings in non-factorizable geometry*, Phys. Lett. B 474 (2000) 361-371, doi:10.1016/S0370-2693(00)00054-X; S. J. Huber and Q. Shafi, *Fermion masses, mixings and proton decay in a Randall-Sundrum model*, Phys. Lett. B 498 (2001) 256-262, doi:10.1016/S0370-2693(00)01399-X; K. Agashe, G. Perez and A. Soni, *Flavor structure of warped extra dimension models*, Phys. Rev. D 71 (2005) 016002, doi:10.1103/PhysRevD.71.016002; D. E. Kaplan and T. M. P. Tait, *New tools for fermion masses from extra dimensions*, JHEP 11 (2001) 051, doi:10.1088/1126-6708/2001/11/051, arXiv:hep-ph/0110126 (flat compact extra dimension, zero modes localized by odd bulk masses, Higgs on a boundary; the flat-space setting closest to lemma 12).
- Graded matrices: C. D. Froggatt and H. B. Nielsen, *Hierarchy of quark masses, Cabibbo angles and CP violation*, Nucl. Phys. B 147 (1979) 277-298, doi:10.1016/0550-3213(79)90316-X; J. Demmel, M. Gu, S. Eisenstat, I. Slapničar, K. Veselić and Z. Drmač, *Computing the singular value decomposition with high relative accuracy*, Linear Algebra Appl. 299 (1999) 21-80 (singular values of D_1 X D_2 with well-conditioned X are determined to high relative accuracy; theorem numbers **unverified**). The leading-order "masses = products of profiles times ratios of minors" is Casagrande et al. eq. (96). We did not find the two-sided compound-matrix bound of Lemma 12.8 in this explicit form; its proof (Cauchy-Binet on the p-th compound) is routine.
- NCG side: Yukawa matrices enter D_F as free moduli. T. Krajewski (1998) and A. Connes (2006, Theorem 2.7) classify the admissible D_F; see section I. No derivation of these moduli from localization was found.

What remains unmatched. The flat-space exact secular equation for an arbitrary N_Q x N_U boundary matrix with bulk masses, stated with multiplicities; the two-sided bound of Theorem 12.6 (via concavity of

alpha, beta in lambda^2); the towerless limit l -> 0 at fixed c and Y_hat as a convergence theorem with error sigma^3 l^2 (1/L_Q^2 + 1/L_U^2) onto a finite even spectral triple; Corollary 12.3 on vectorlike residual theories; and the reading "D_F is the towerless remnant of internal momentum".

Searched: "noncommutative geometry" or "spectral triple" with "split fermions", "fermion localization", "Randall-Sundrum", "Yukawa overlap" (standard and extended); Csáki et al. full text; Casagrande et al. full text; graded-matrix SVD literature.

D. KO-dimension of products: 9, 10 or 11

Claim (lemmas 06, 07, 08). A finite KO-6 real triple is T_n x P_k with (s, n, k) = (+1, 7, 7) or (-1, 5, 1) iff a grading-reversing involution U commutes with A, A°, D and satisfies UJ = sJU (Theorem 06.2); J alone does not fix s; an unpaired nu_L forces the KO-7 reduction (No-go 06, with [U, Q] = 0); in the internal-first product with the interval, the bare reflection has s = -1 (Theorem 08.5).

Verdict: PARTIAL. Products of real spectral triples of all parities, including the doubled odd x odd product on H_1 x H_2 x C^2, the freedom J versus J gamma in even dimension, and the total KO-dimension 10 = 2 mod 8 of the Standard Model are in print. The converse direction, a criterion for when a given even KO-6 triple factors as (odd) x (odd point), was NOT FOUND, nor were the nu_L statement, the reflection sign, or any discussion of 9 or 11 as alternatives to 10.

References.

- L. Dąbrowski and G. Dossena, *Product of real spectral triples*, Int. J. Geom. Methods Mod. Phys. 8 (2011) 1833-1848, doi:10.1142/S021988781100597X, arXiv:1011.4456. Introduction: in even dimensions "there are always two real structure operators J, that differ by multiplication by the grading operator. None of them should be preferred as they are perfectly on the same footing." Section 4.1 (Tables 2-3) and 4.2 (Tables 4-5): even-even and even-odd products. Section 4.3, eq. (25) and Table 6: the odd-odd product H = (H_1 x H_2) x C^2, D = D_1 x 1 x sigma_1 + 1 x D_2 x sigma_2, chi = 1 x 1 x sigma_3, J = J_1 x J_2 x M K. Section 6: metric and KO-dimension need not agree mod 8. Lemma 06.1 already cites eq. (25) and Table 6 for its product convention. The paper treats the forward direction (building products); the question of when a given triple is a product is left open there.
- F. J. Vanhecke, *On the product of real spectral triples*, Lett. Math. Phys. 50 (1999) 157-162, doi:10.1023/A:1007690509512, arXiv:math-ph/9902029: the naive product J may violate the sign table; a corrected product real structure.
- F. J. Vanhecke, A. R. da Silva and C. Sigaud, *On the finite spectral triple of an almost-commutative geometry*, Braz. J. Phys. 42 (2012) 471-474, doi:10.1007/s13538-012-0101-2, arXiv:1011.4925. Section 2.1: with J = J_1 x J_2, a Minkowski factor of signature 2 or 6 forces an odd internal factor (sigma_2 in {1, 5} or {3, 7}) and "the chirality paradigm is lost"; Section 2.2: with J = J_1 x J_2 Omega_2 an even internal factor is allowed and sigma_2 = 6 gives total signature 0 (Barrett's case). This is the closest printed discussion of odd internal factors, framed as a consequence of the product convention.
- S. Farnsworth, *The graded product of real spectral triples*, J. Math. Phys. 58 (2017) 023507, doi:10.1063/1.4975410, arXiv:1605.07035: graded tensor product, commutative and associative, always a spectral triple.
- B. Ćaćić, *Real structures on almost-commutative spectral triples*, Lett. Math. Phys. 103 (2013) 793-816, doi:10.1007/s11005-013-0616-7, arXiv:1209.4832: real reconstruction theorem; compact oriented manifolds carry real (almost-)commutative triples of every KO-dimension.
- A. Connes, *Noncommutative geometry and the standard model with neutrino mixing*, JHEP 11 (2006) 081, doi:10.1088/1126-6708/2006/11/081, arXiv:hep-th/0608226, Section 1: "the KO-dimension of the product space M x F is in fact equal to 10 ~ 2 modulo 8. Of course the above 10 is very reminiscent of string theory".
- J. W. Barrett, *Lorentzian version of the noncommutative geometry of the standard model of particle physics*, J. Math. Phys. 48 (2007) 012303, doi:10.1063/1.2408400, arXiv:hep-th/0608221. Section 1: J^2 = 1 and J gamma = gamma J "hold only if the total signature is equal to zero"; Section 2: the internal space is "determined to be a real spectral triple of signature six".

- A. H. Chamseddine and A. Connes, *Why the Standard Model*, J. Geom. Phys. 58 (2008) 38-47 (section I): the finite geometry raises "the K-theoretic dimension from four to ten (modulo eight)".
- K. van den Dungen and W. D. van Suijlekom, *Electrodynamics from noncommutative geometry*, J. Noncommut. Geom. 7 (2013) 433-456, doi:10.4171/JNCG/122: the KO-6 two-point triple for electrodynamics, the natural reference for Counterexample 06.3.
- Background: the complex analogue of Theorem 06.2 (an even triple with a commuting, grading-reversing Clifford generator is an odd triple doubled) is the standard identification of odd Fredholm modules with Cl_1-graded ones in Kasparov theory; we did not locate a reference stating the real KO-6 version with the sign s of UJ = sJU.
- Terminology: "doubling" in lemma 06 (H = H_0 x C^2) differs from "fermion doubling" in the NCG literature (F. Lizzi, G. Mangano, G. Miele and G. Sparano, Phys. Rev. D 55 (1997) 6357, section I; resolved in Connes 2006 by the Pfaffian). A paper should use a different word.

What remains unmatched. Theorem 06.2 (criterion), Result 06.3 (both s for electrodynamics), the nu_L no-go under [U, Q] = 0, Theorem 08.5 (s = -1 for the bare reflection in the internal-first product), Counterexample 08.11 and Theorem 08.12.

Searched: "real spectral triple" with "factorization", "product of odd", "Clifford", "KO-dimension 6"; "9" and "11" with "noncommutative standard model"; full texts of Dąbrowski-Dossena, Vanhecke-da Silva-Sigaud, Connes 2006 and Barrett 2007.

E. Spectral action: the W mass relation, the top ratio, heavy-fermion weights

Claim (lemma 11; lemma 03). 8 M_W^2 = sum N_f m_f^2 at unification; m_t/M_W -> sqrt(8/3) when the top dominates; the bounds m_max/(2 sqrt 2) <= M_W <= sqrt(N_g) m_max; gauge kinetic terms weighted by sum_j f(m_j^2/Lambda^2) tr_{H_j} F^2.

Verdict: KNOWN for the mass relation and the top ratio; Proposition 11.1 and Corollary 11.2 are one-line consequences of it. **NOT FOUND** for the exact heavy-fermion weighting in this form.

References.

- A. H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007) 991-1089, doi:10.4310/ATMP.2007.v11.n6.a3, arXiv:hep-th/0610241. Eq. (1.5) (Introduction): sum over generations of m_e^2 + m_nu^2 + 3 m_d^2 + 3 m_u^2 = 8 M_W^2. Lemma 4.6, eq. (4.16), proved from the normalization (4.17) Tr(k* k ...) = 2 g^2. Section 5.4 ("The fermion-boson mass relation"): eq. (5.24) m_top = v y_t/sqrt 2 with v = 2M/g; eqs. (5.25)-(5.26): Y_2(S) = 4 g^2; eq. (5.27): neglecting all Yukawas but the top, y_t = (2/sqrt 3) g at unification; eq. (5.28): y_t ~ 0.597. With M_W = g v/2 this is m_t/M_W = sqrt(8/3), which follows at once from (5.24) and (5.27); CCM state the relation in the form y_t = (2/sqrt 3) g. Eq. (5.29): if the tau-neutrino Dirac Yukawa equals the top Yukawa, y_t ~ g, i.e. m_t/M_W = sqrt 2.
- A. H. Chamseddine and A. Connes, *Conceptual explanation for the algebra in the noncommutative approach to the standard model*, Phys. Rev. Lett. 99 (2007) 191601 (abstract: a mass relation linking the fermion masses to the W mass, used to predict the top mass).
- Heavy-fermion weights: CCM Section 3 expands Tr f(D_A/Lambda) with D_F entering polynomially through the coefficients a, b, c, d, e, and Section 5.3 places the Majorana mass at the scale Lambda (Lemma 5.2, eq. (5.20)). The resummed weighting sum_j f(m_j^2/Lambda^2) of lemma 03 Part A was not found. Checked: M. Eckstein and B. Iochum, *Spectral Action in Noncommutative Geometry*, SpringerBriefs in Mathematical Physics 27 (2018), doi:10.1007/978-3-319-94788-4, arXiv:1902.05306 (abstract and contents only); B. Iochum, C. Levy and D. Vassilevich, *Spectral action beyond the weak-field approximation*, Commun. Math. Phys. 316 (2012) 595-613, doi:10.1007/s00220-012-1587-8 (second order in the gauge field with full momentum dependence; abstract only). The physics counterpart is decoupling: T. Appelquist and J. Carazzone, *Infrared singularities and massive fields*, Phys. Rev. D 11 (1975) 2856-2861, doi:10.1103/PhysRevD.11.2856. The evaluation already classes lemma 03 as a routine computation; a paper should present it as such.

F. Weinberg angle from a squashed S^3 / lens space and from M^{pqr}

Claim (historical/corpus/canonical_research.md §14.3). For the Berger metric with fibre scaled by xi^2, K_W = V R^2 (2 + xi^2)/3, K_Y = V R^2 xi^2, hence cos^2 theta_W = 3 xi^2/(4 xi^2 + 2); the lens order p cancels; 5/8 needs xi^2 = 5/2.

Verdict: KNOWN (same formula, same parameterization).

- A. J. Niemi and S. Slizovskiy, *On relations between electroweak hierarchy problem, fluctuating three branes and general covariance*, arXiv:1004.0212 (no journal version found on INSPIRE; earlier titles "Hiding Higgs with brane" and "Embedding Weinberg-Salam in Kaluza-Klein"). Section IV, eqs. (77)-(78): internal metric dtheta^2 + sin^2 theta dpsi_-^2 + xi^2 (dpsi_+ - cos theta dpsi_-)^2, isometry SU(2)_L x U(1)_R for xi != 1; Section V, eqs. (94)-(95): q^2 = 3/(xi^2 + 2), q'^2 = 1/xi^2; Section VI: sin^2 theta_W = (xi^2 + 2)/(4 xi^2 + 2), i.e. cos^2 theta_W = 3 xi^2/(4 xi^2 + 2). They fix xi by a custodial-symmetry condition and quote sin^2 theta_W = 0.296 (that step, in Section VIII, was not read). The ratios 1/q^2 : 1/q'^2 = (2 + xi^2)/3 : xi^2 coincide with K_W : K_Y of §14.3. The cancellation of the lens order p in the common volume factor is immediate.
- E. Witten, *Search for a realistic Kaluza-Klein theory*, Nucl. Phys. B 186 (1981) 412-428, doi:10.1016/0550-3213(81)90021-3: eleven-dimensional supergravity on the seven-dimensional M^{pqr} with SU(3) x SU(2) x U(1) isometry; gauge couplings determined by the internal metric; the chirality obstruction. Read on 3 Nov 2026 from the PDF supplied by the user, and checked by Astra 6 (Astra validation, item 1). Eq. (3): gauge fields as components of the metric along Killing vectors. Pp. 416-417: seven is the minimum internal dimension with this symmetry, so eleven in total. P. 418: M^{pqr} = G/H with H = SU(2) x U(1) x U(1) and the deleted generator p lambda_8 + q T_3 + r Y; M^{001} = CP^2 x S^2 x S^1, M^{101} = S^5 x S^2, M^{011} = CP^2 x S^3. Pp. 421-423: "One would inevitably get vector-like rather than V - A weak interactions", for a reason that "depends only on the fact that the number of extra dimensions is odd": gamma_i -> -gamma_i fixes sigma_ij and flips Gamma_7, on an orientable manifold, with the gauge group acting by coordinate transformations. Escapes considered: non-orientable internal spaces (none suitable for this symmetry, p. 423) and torsion (left open, pp. 423, 426-427). Pp. 425-426: the three parameters of the invariant metric would determine the three gauge couplings. The paper computes no weak angle, so section F cites it for the method only.
- L. Castellani, R. D'Auria and P. Fré, *SU(3) x SU(2) x U(1) from D = 11 supergravity*, Nucl. Phys. B 239 (1984) 610-652 (INSPIRE): the Einstein metrics on M^{pqr}, with parameters depending on q/(3p). Content from abstract and secondary sources.

Added 5 November 2026, for lemma 47 (the Berger five-sphere), found by the Opus referee; bibliographic data verified by Claude Fable 5.1 on INSPIRE and arXiv.

- S. Weinberg, *Charges from extra dimensions*, Phys. Lett. B 125 (1983) 265-269, doi:10.1016/0370-2693(83)91281-9: the integral formula g_a^{-2} proportional to int g(K_a, K_a) for the Kaluza-Klein gauge couplings, with the normalization point at which the coupling becomes the charge unit. Known here through Barnes et al. below; the paper itself was not read.
- E. Barnes, E. Gorbatov, K. Intriligator and J. Wright, *Current correlators and AdS/CFT geometry*, Nucl. Phys. B 732 (2006) 89-117, arXiv:hep-th/0507146 (e-print read by the referee, sections 3 and 6): the formula (g^{-2}_{IJ})^{KK} proportional to int_Y g_ab K_I^a K_J^b vol(Y); for IIB on S^5, tau_SO(6) = N^2 and tau_RR = 4 N^2/3, with the U(1)_R Killing vector of norm squared 4/9, which is the central generator of lemma 47 with c = 2/3; hence tau_SU(3)/tau_RR = 3/8, the round-point value of Corollary 47.2 in holographic form.
- M. J. Duff, C. N. Pope and N. P. Warner, Phys. Lett. B 130 (1983) 254, doi:10.1016/0370-2693(83)91136-X: the flux contribution to the Killing gauge couplings on Freund-Rubin backgrounds (title from INSPIRE; content as reported in section 3 of Barnes et al.).

- Z. F. Ezawa, I. G. Koh and S. Rajpoot, *SU(4) x SU(2) unification in Kaluza-Klein supergravity*, Phys. Lett. B 147 (1984) 284-288, doi:10.1016/0370-2693(84)90117-5 (abstract): Pati-Salam partial unification from an SU(4) x SU(2) isometry, the B - L reading of the same embedding; whether it states the ratio was not checked.
- B. L. Hu and T. C. Shen, *Weak angle from Kaluza-Klein theory with deformed internal space*, Phys. Lett. B 178 (1986) 373-378, doi:10.1016/0370-2693(86)91396-1 (abstract): the weak angle from a deformed three-sphere as a function of the deformation, the S^3 analogue of the Berger family of lemma 47, beside Niemi-Slizovskiy above.
- The formula (3 + eta^2)/(24 eta^2 c^2) on the Berger S^5 was not found by the referee (INSPIRE titles on squashed five-spheres, Berger spheres, the weak angle and the coupling constants in Kaluza-Klein theory; web searches).

N. Real group algebras of the binary dihedral lift and the quaternions (lemma 48): prior work found by the Opus referee

Claim. Lemma 48, Proposition 48.2 and Theorem 48.3: the spinorial part of R[Dic_3] is C + H by the Frobenius-Schur types, and the two-sheet bimodule over the orbit types of the Weyl action carries C + H + M_3(C). **Verdict of the referee: the type decomposition is in print in the orientifold setting; no source deriving C + H from a binary dihedral spin lift in noncommutative geometry was found.** Bibliographic data verified by Claude Fable 5.1 on arXiv, 5 November 2026.

- J. D. Blum and K. Intriligator, *New phases of string theory and 6d RG fixed points via branes at orbifold singularities*, Nucl. Phys. B 506 (1997) 199-222, arXiv:hep-th/9705044 (e-print read by the referee, section 5 and appendix A): the irreducible representations of the finite subgroups of SU(2) sorted into real, pseudo-real and complex by the Frobenius-Schur indicator, the pseudo-real ones having the central element -1 represented as -1, and the breaking SO(N) -> prod SO x prod Sp x prod U by type; for Gamma_{D_5} = Dic_3 the real ones are the trivial, sign and standard representations, the pseudo-real one is the faithful two-dimensional representation and the complex pair has alpha^2 = -1, which is Proposition 48.2 (i), the spinorial ones giving H + C.
- J.-P. Serre, *Linear representations of finite groups*, Springer GTM 42, chapter 13: the real group algebra as a sum of matrix algebras over R, C and H by Frobenius-Schur type (textbook; not reopened).
- Noncommutative geometry: an extended web search on the Standard Model algebra with orbifold, crossed product, spin lift, binary dihedral and quaternions (titles and abstracts) found no source; the corpus construction recorded in lemma 45 is the only prior source known to the project.

G. The protected zero mode on S^5 (requested by the review)

Claim (historical/corpus/connes_kk_research_state.md, around the S^5 zero mode): for the nontrivial quaternionic line V -> S^5, dim ker D_{S^5, V} is odd (real mod-two index); for SU(3)-invariant data the invariant line is a zero mode; the text attributes the mechanism to Witten 1982, footnote 2.

Verdict: KNOWN, for the topological statement (a nontrivial SU(2) bundle on S^5 has odd mod-two index) and for the attribution. Footnote 2 of Witten 1982 (p. 327, read on 3 Nov 2026 from the PDF supplied by the user) reads: "Actually, in a special case one can easily find the zero mode. If one conformally compactifies S^4 x R to the five-sphere S^5, then on S^5 one can choose the instanton field to be invariant (up to a gauge transformation) under an SU(3) subgroup of the symmetry group O(6) of the five-sphere. The fermion zero mode is then the unique SU(3) invariant spinor field that can be defined." The wording of the corpus matches it.

- E. Witten, *An SU(2) anomaly*, Phys. Lett. B 117 (1982) 324-328, doi:10.1016/0370-2693(82)90728-6. As summarized by Wang, Wen and Witten (2019), Section 2: "The main result of [1] is that the fermion path integral for a fermion field on M in a given representation of SU(2) is phi-invariant if and only if the mod 2 index of the Dirac operator on Y – in that representation – vanishes. Here the mod 2 index is defined as the number of zero modes of the five-dimensional Dirac operator, reduced mod 2", where Y is the mapping torus of the gauge transformation phi (the mapping torus S^4 x S^1, with phi in the nontrivial class of pi_4(SU(2)) = Z_2). The original, read on 3 Nov 2026, gives the criterion for

other isospins on p. 328: the fermion integration is odd exactly when tr T^2, with tr T^2 = 1/2 on the doublet, is a half-integer.

- I. García-Etxebarria and M. Montero, *Dai-Freed anomalies in particle physics*, JHEP 08 (2019) 003, doi:10.1007/JHEP08(2019)003, arXiv:1808.00009. Section 3.1, eq. (3.4): Omega_5^Spin(BSU(2)) = Z_2, detected by the mod-two index; immediately after: "A simple example with non-vanishing mod-2 index was constructed in [2] [Witten 1982]. While S^5 itself is trivial in Omega_5^Spin …, there is a bundle over it such that the total space is no longer null-bordant in Omega_5^Spin(BSU(2)) = Z_2." Section 3.1.2: SU(3)/SU(2) = S^5, so in five dimensions every SU(3) bundle reduces to an SU(2) bundle.
- Y. Sato, *Majorana fermion zero modes and anomalies in string and M theories* (PhD thesis, Tokyo, 2024), arXiv:2404.18138, Appendix C: the five-dimensional configuration of Witten's argument (S^4 x R with pure-gauge ends) compactifies to S^5; G-bundles on S^5 are classified by pi_4(G), and for G = Sp(N) the mod-two index is an isomorphism onto KO^{-1}(pt) = Z_2.
- J. Wang, X.-G. Wen and E. Witten, *A new SU(2) anomaly*, J. Math. Phys. 60 (2019) 052301, doi:10.1063/1.5082852, arXiv:1810.00844: mod-two index on mapping tori, extended to non-spin manifolds. The corpus cites it for the parity of the index at isospin j = 2k + 1/2. That parity follows from the criterion of Witten 1982: tr T^2 = j(j + 1)(2j + 1)/3 equals (4k + 1)(2k + 1)(4k + 3)/6, a half-integer, at j = 2k + 1/2, and (4k + 3)(k + 1)(4k + 5)/3, an integer, at j = 2k + 3/2. We did not reread the 2019 paper for this statement.

Answer to the review's objection. Witten's original construction uses S^4 x R, with the specified asymptotic gauge fields; footnote 2 discusses its conformal compactification to S^5. The mapping-torus formulation uses S^4 x S^1, with the gauge bundle glued by the gauge transformation. Both that mapping torus (odd index for one doublet) and the nontrivial SU(2) bundle on S^5 (García-Etxebarria-Montero, sentence after eq. (3.4)) are nonzero in Omega_5^Spin(BSU(2)) = Z_2, so they represent the same generator; the mod-two index is a bordism invariant and is odd on both. Sato's Appendix C describes the S^5 bundle as the compactification of the S^4 x R configuration of Witten's argument. García-Etxebarria-Montero state the S^5 version and attribute the example to Witten 1982. What remains specific to the corpus is the identification of V = rho^*Q with this generator via the Hopf map from CP^3 data and the SU(3)-invariant zero-mode check.

H. Anomaly inflow as the five-dimensional parent of chiral boundary modes (requested by the review)

Relevance: claim B (chiral walls of Theorem 9.8) and Corollary 9.3 (equivariant index from boundary types).

Verdict: KNOWN as background; lemma 09 contains no gauge-field inflow, so these references supply context.

- C. G. Callan and J. A. Harvey, *Anomalies and fermion zero modes on strings and domain walls*, Nucl. Phys. B 250 (1985) 427-436, doi:10.1016/0550-3213(85)90489-4. Chiral zero modes of the Dirac equation on domain walls in 2n+1 dimensions and on strings in 2n+2 dimensions; their anomalies are cancelled by inflow from the bulk through an induced Chern-Simons (topological mass) term. Content from the abstract fragment and secondary sources; full text not read.
- S. G. Naculich, *Axionic strings: covariant anomalies and bosonization of chiral zero modes*, Nucl. Phys. B 296 (1988) 837-867, doi:10.1016/0550-3213(88)90400-2: the consistent/covariant anomaly bookkeeping in inflow.
- N. Arkani-Hamed, A. G. Cohen and H. Georgi, *Anomalies on orbifolds*, Phys. Lett. B 516 (2001) 395-402, doi:10.1016/S0370-2693(01)00946-7: anomalies of bulk fermions on S^1/Z_2 localized at the fixed points (content **unverified** beyond title and abstract-level knowledge).
- Kaplan (1992) and Shamir (1993), section B, built lattice chiral fermions on this mechanism.

For the corpus: Theorem 9.2 and Corollary 9.3 are the index-theoretic shadow of Callan-Harvey (boundary type fixes the chiral zero mode); an inflow statement would require bulk gauge fields with a Chern-Simons

term on the interval, which lemma 09 does not include.

I. The current Connes et al. framework (requested by the coordinator)

The corpus treats Connes-Lott 1990 as the obsolete layer. The following define the present framework. All bibliographic data verified (Crossref or arXiv) unless marked.

- **A. Connes, *Noncommutative geometry and the standard model with neutrino mixing*, JHEP 11 (2006) 081**, arXiv:hep-th/0608226. Reverses the grading on the antiparticle sector so that the finite space has KO-dimension 6, separating metric dimension (0) from KO-dimension. Theorem 2.7 classifies the Dirac operators D_F (Yukawa matrices, CKM and PMNS data, a Majorana mass); the fermion doubling problem is solved by the Pfaffian in the fermionic action. The total KO-dimension $10 = 2 \bmod 8$ is noted in Section 1.
- **J. W. Barrett, *Lorentzian version ...*, J. Math. Phys. 48 (2007) 012303**, arXiv:hep-th/0608221. Independently arrives at internal signature 6 from the Lorentzian requirement that physical fermions be simultaneous eigenvectors of J and gamma (total signature 0 mod 8), which removes the quadrupling of fermions.
- **A. H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007) 991-1089**, arXiv:hep-th/0610241. Section 2: the finite geometry, with A_F = C + H + M_3(C), Lemma 2.5 (the irreducible representations of the left-right bimodule algebra have total dimension 32 per generation), Section 2.4 (the order-one condition selecting A_F), Sections 2.6-2.7 (classification and moduli of D_F, i.e. the Yukawa parameters) and Section 2.8 (KO-dimension and Poincaré duality); N = 96 in Section 3.7.1; Section 3: the spectral action on M x F and its heat-kernel coefficients; Section 4: the full Lagrangian; Section 5: predictions at unification (gauge coupling relation, Higgs quartic, the mass relation (4.16)).
- **A. H. Chamseddine and A. Connes, *Conceptual explanation for the algebra in the noncommutative approach to the standard model*, Phys. Rev. Lett. 99 (2007) 191601**, doi:10.1103/PhysRevLett.99.191601, arXiv:0706.3690. Version 1 of arXiv:0706.3690 carried the title *A dress for SM the beggar*; the published version is this PRL (the coordinator's "PNAS 2007" does not match any record we found). Classifies irreducible finite geometries with reality and chirality and no fermion doubling, requiring total K-theoretic dimension 10.
- **A. H. Chamseddine and A. Connes, *Why the Standard Model*, J. Geom. Phys. 58 (2008) 38-47**, doi:10.1016/j.geomphys.2007.09.011, arXiv:0706.3688. Section 2: the order-zero condition; Proposition 2.5 (centre C: dim H = k^2) and Proposition 2.8 (centre C + C: dim H = 2k^2, A_C = M_k(C) + M_k(C)); Lemma 3.1: the KO-6 relation J gamma = -gamma J excludes centre C; eq. (3.1): with a quaternionic, nontrivially graded W of quaternionic dimension 2, A = M_2(H) + M_4(C), k = 4, dim H = 2k^2 = 32 per generation; Theorem 4.1: the order-one condition with an off-diagonal D selects the unique maximal subalgebra A_F = C + H + M_3(C). For the project this is the statement that D_F must connect the two components of the centre: the "two sheets" of the finite space are forced by the classification.
- **A. H. Chamseddine, A. Connes and W. D. van Suijlekom, *Inner fluctuations in noncommutative geometry without the first order condition*, J. Geom. Phys. 73 (2013) 222-234**, doi:10.1016/j.geomphys.2013.06.006, arXiv:1304.7583. Without the first-order condition, inner fluctuations acquire a quadratic term and form a semigroup depending only on A. Relevant to Amendment 2 of lemma 07 (the compressed two-point data fail the first-order condition).
- **A. H. Chamseddine, A. Connes and W. D. van Suijlekom, *Beyond the spectral Standard Model: emergence of Pati-Salam unification*, JHEP 11 (2013) 132**, doi:10.1007/JHEP11(2013)132, arXiv:1304.8050. Dropping the first-order condition, the classification gives an SU(2)_R x SU(2)_L x SU(4) model; the Standard Model is recovered when the vacuum restores the first-order condition.
- **A. Devastato, F. Lizzi and P. Martinetti, *Grand symmetry, spectral action and the Higgs mass*, JHEP 01 (2014) 042**, doi:10.1007/JHEP01(2014)042, arXiv:1304.0415. A larger "grand" algebra mixing gauge and spin degrees of freedom produces the extra scalar sigma that brings the Higgs

mass near 126 GeV; requiring an almost-commutative geometry breaks it to the Standard Model.

- **A. H. Chamseddine, A. Connes and V. Mukhanov, *Quanta of Geometry: Noncommutative Aspects*, arXiv:1409.2471** (journal data unverified; read from the LaTeX source, 3 Nov 2026). Clifford algebras C_kappa on n + 1 gammas with kappa = +-1; in dimension 4, C_+ = M_2(H) and C_- = M_4(C), "in perfect agreement with the algebraic constituents of the Standard Model"; two maps Y_+- : M -> S^4 of slashed form, with det e = Omega_+ + Omega_-. Bears on the twistor line of canonical section 4 and on current_framework.md.
- **J. Baptista, *Chiral interactions of fermions and massive gauge fields in Kaluza-Klein models*, JHEP 05 (2026) 008, arXiv:2506.09126** (found by the referee; introduction read from the LaTeX source). The no-go arguments against chiral fermions in Kaluza-Klein models (Witten 1983; Atiyah-Hirzebruch) assume that the gauge fields are linked to exact Killing fields; gauge fields linked to non-Killing fields are massive, can mix fermions of different masses, and can couple asymmetrically to the two chiralities, with examples on even-dimensional K = S^2 and T^2. Bears on the reading of no-go results in the paper and on T4.
- **Lemma 22: the finite Dirac operator as a boundary condition.** Known: the one-flavour formula (Csáki, Grojean, Hubisz, Shirman and Terning, section C here) and the gauge boundary conditions and the form M tan(M pi R) = -V (Csáki, Grojean, Murayama, Pilo and Terning, *Gauge theories on an interval: unitarity without a Higgs*, Phys. Rev. D 69 (2004) 055006, arXiv:hep-ph/0305237, read in the LaTeX source). Not found: the real structure with a finite triple as exact window, the wall algebra C_F, the average formula for the compression. **Searched** (arXiv API, 3 Nov 2026): "noncommutative geometry" with Higgsless; "spectral triple" with "boundary conditions", Higgs, interval; "spectral triple" with Yukawa, "boundary conditions"; "almost-commutative" with interval, boundary; "spectral triple" with "Kaluza-Klein", "boundary conditions": no hits. "noncommutative geometry" with "gauge-Higgs": one hit, arXiv:2511.05909, a survey, abstract only. Absence from these queries does not establish absence. The referee adds operator-system compression in Connes-van Suijlekom (arXiv:2004.14115, sections 1, 3), and A. Chamseddine and A. Connes, *Noncommutative geometric spaces with boundary: spectral action*, J. Geom. Phys. 61 (2011) 317, arXiv:1008.3980 (read in the LaTeX source: sections 3 to 5). There the boundary is that of the manifold in a product with a finite space, with the condition Pi_- Psi = 0, Pi_- = (1 - chi)/2, for a local chirality projector chi, and the spectral action acquires the extrinsic-curvature terms. The internal boundary with finite gradings of lemmas 22 and 23 is a different setting.
- **Lemma 25: the sum rule and the effective dimension.** Known: m_t = 2 M_W at tree level in flat gauge-Higgs models with suitable representations (Cacciapaglia, Csáki and Park, arXiv:hep-ph/0510366, section 5; Adachi and Maru, arXiv:1804.06012, Eq. (3.23)). Not found in the records inspected by the referee: the coefficient (3 - d_eff)/24 and its theta-function crossover. **Searched** by the referee: web-index queries for spectral action, heat kernels, compactification and gauge-Higgs mass relations, including a 2026 query, and the reference chain from Adachi, Kurahashi and Maru (arXiv:1404.4281). A forward-citation search, an author sweep and a broader database search remain open. This does not establish priority.
- **Twisted spectral triples:** A. Connes and H. Moscovici, *Type III and spectral triples*, arXiv:math/0609703 (journal data **unverified**); G. Landi and P. Martinetti, *On twisting real spectral triples by algebra automorphisms*, Lett. Math. Phys. 106 (2016) 1499-1530, doi:10.1007/s11005-016-0880-4, arXiv:1601.00219; A. Devastato and P. Martinetti, *Twisted spectral triple for the Standard Model and spontaneous breaking of the grand symmetry*, Math. Phys. Anal. Geom. 20 (2017) 2, doi:10.1007/s11040-016-9228-7, arXiv:1411.1320. The twist bounds the unbounded vector terms of the grand-symmetry model and introduces a twisted first-order condition; the breaking to the Standard Model becomes dynamical.
- **Classification of finite triples:** T. Krajewski, *Classification of finite spectral triples*, J. Geom. Phys. 28 (1998) 1-30, doi:10.1016/S0393-0440(97)00068-5, arXiv:hep-th/9701081 (Krajewski diagrams: vertices are chiral gauge multiplets, links are Yukawa couplings); M. Paschke and A. Sitarz, *Discrete spectral triples and their symmetries*, J. Math. Phys. 39 (1998) 6191-6205, doi:10.1063/1.532623, arXiv:q-alg/9612029 (classification of 0-dimensional triples over complex and real algebras). For the project, a Krajewski link is exactly the Yukawa block that lemma 12 derives as Phi_Q Y_hat Phi_U.
- **Fermion doubling:** F. Lizzi, G. Mangano, G. Miele and G. Sparano, *Fermion Hilbert space and*

fermion doubling in the noncommutative geometry approach to gauge theories, Phys. Rev. D 55 (1997) 6357-6366, doi:10.1103/PhysRevD.55.6357, arXiv:hep-th/9610035: unphysical degrees of freedom in H = L^2(S) x H_F, analogous to lattice doublers; the 2006-2007 KO-6 framework resolves them (Connes 2006, Barrett 2007).

- **Standard reference:** W. D. van Suijlekom, *Noncommutative Geometry and Particle Physics*, Mathematical Physics Studies, Springer (2015), doi:10.1007/978-94-017-9162-5 (a second edition exists; its data were **not verified**). Covers finite spectral triples and Krajewski diagrams, almost-commutative manifolds, the spectral action, and the electrodynamics and Standard Model examples.

J. Matching in the flat parent (lemma 26)

Claim. Lemma 26: one-loop vacuum with frequency-2 channels; the left-right difference of the couplings; the meeting point; the limit on a top in low multiplets. **Verdict: PARTIAL.** Found by the referee of lemma 26 (report, item 9), with forward citation chains incomplete. We have read none of these beyond what that report quotes.

- Y. Nomura, D. Smith and N. Weiner, *GUT breaking on the brane*, arXiv:hep-ph/0104041: differential logarithmic running, integer versus half-integer towers. M. Carena, T. Tait and C. Wagner, arXiv:hep-ph/0207056: boundary kinetic terms.
- Abe, Goto and Kawamura, PTEP 2018, 103B01: an interval Pati-Salam model with a compactification threshold of order 10^13 GeV, with other intermediate group and matter.
- G. Cacciapaglia, C. Csaki and S. C. Park, arXiv:hep-ph/0510366, section 5: representation-dependent enhancement, including a tree-level m_t = 2 M_W limit. Adachi and Maru, arXiv:1804.06012, Eq. (3.23): a limit approaching 2 M_W with brane mixing, and massless exotics.
- R. Contino, L. Da Rold and A. Pomarol, arXiv:hep-ph/0612048: 5 and 10 embeddings and coloured custodians. Y. Hosotani, S. Oda, T. Ohnuma and Y. Sakamura, arXiv:0806.0480: massless exotics, their lifting by wall masses, and theta_H = pi/2 in a warped model.
- C. S. Lim and N. Maru, arXiv:0904.0304: a finite one-loop violation of gauge-Yukawa universality.

Not found by that search: the classification with n_c, and the pair (M_KK, mu_*). Priority for those exact statements stays open.

K. Fibre collapse in a mapping cylinder and compression (lemma 28)

Claim. Lemma 28: a connected commutative algebra on a mapping cylinder gives a direct sum of two matrix algebras by compression to the two ends. **Verdict: NOT LOCATED.** Found by Astra 6 in a bounded search (report); novelty stays unestablished, and author-by-author citation searches were not done. Identifiers, authors and titles below were confirmed on arXiv by Claude Opus 5.5; the located passages are as reported by Astra 6.

- F. D'Andrea, F. Lizzi and P. Martinetti, *Spectral geometry with a cut-off*, arXiv:1305.2605, section 3.1 and 6: compression of commutative geometries.
- A. Connes and W. D. van Suijlekom, arXiv:2004.14115, section 1 and Proposition 4.2: operator systems from spectral truncation, with the warning on non-informative full matrix algebras, relevant to Counterexample 28.17.
- J. Medina, I. Huet, D. O'Connor and B. Dolan, arXiv:1208.0348: squashed fuzzy CP^3 as a fuzzy S^2 bundle over fuzzy S^4, with suppression of the fibre modes.
- M. Verhoeven, *Spectral truncations of SU(2) and the spectral localizer*, arXiv:2610.19791: full matrix envelopes in a recent truncation example.
- K. Uno and T. Inagaki, *Construction and geometric properties of the fuzzy CP^3 onion*, arXiv:2610.17807: radially coupled fuzzy CP^3 layers.

- C. Bär and W. Ballmann, arXiv:1307.3021: the boundary criterion of Theorem 28.9. C. Kottke, arXiv:0909.5661: the Callias index in families, for Proposition 28.11.
- I. Prokhorenkov and K. Richardson, *Perturbations of Dirac operators*, J. Geom. Phys. 57 (2006) 297-321, arXiv:math/0307251; M. Maridakis, arXiv:1510.07004, Definition 2.1 and Lemma 2.2, and arXiv:2110.11654, section 2: concentration of low modes of D + s A at the zero set of A on closed manifolds. Read by Terra 6.1 in arXiv HTML (report): the operator of lemma 28 is a concentrating pair, and Assumption 3 of the spectral separation theorem fails for the spin-3/2 mass.
- R. Parthasarathy, *Dirac operator and the discrete series*, Ann. of Math. 96 (1972) 1-30, doi:10.2307/1970892, bibliographic data from Crossref: the formula D^2 = Cas_G + R/8 - Cas_K(twist) on a symmetric space, used in lemma 29. Text not read here; the formula is tested on the slice blocks in Python.
- K.-D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature*, Ann. Global Anal. Geom. 4 (1986) 291-325, doi:10.1007/BF00128050, bibliographic data from Crossref: the limit (m + 1) R / (4 m) for odd complex dimension m, which is R/3 on CP^3, used as a check of the normalization in Lemma 29.5. Text not read here.
- Found by the Terra referee of lemma 29, identifiers checked on arXiv by Claude: J. Kaad and W. D. van Suijlekom, *Factorization of Dirac operators on almost-regular fibrations of spin^c manifolds*, arXiv:1710.03182, Proposition 18: a horizontal and vertical factorization with an explicit curvature contraction, under hypotheses that differ from the noncompact model of lemma 29; a close structural ancestor of Proposition 29.1. I. Prokhorenkov and K. Richardson, *Witten deformation and the equivariant index*, arXiv:math/0610160, section 3: equivariant localization and normal models. The referee reports that the search found general identities and did not establish whether the eight-state theorem has appeared elsewhere; it did not exhaust Bismut-Lebeau or the citation chains.
- Standard-mode web searches of 4 Nov 2026, by Claude Opus 5.5, for two statements of the 11D line: M^{111} as the hypercharge quotient of the 5-bar (11D/02, Theorem 2.1), and the SO(30) of the light quarks as the commutant of a real structure (11D/11). Neither search returned a source with the statement. The quotient with weights 2, -3 itself is in D. Fabbri et al., hep-th/9907219. These searches were thin and do not establish priority.

L. The eight-state window on the disc bundle (lemma 29): prior work found by the second Terra reading

Claim. Lemma 29, Theorem 29.7: eight eigenvalues in (-delta, delta) for the doubled operator on the disc bundle, with the S^4 core, the CP^3 wall, the spin-3/2 mass and the local boundary condition. **Verdict of the referee: NOT FOUND as a combined statement; the end-model spectra and the methods are in print.** Identified by GPT-6.1 Terra (report, 5 November 2026), who read the cited passages. The bibliographic data below were verified by Claude Fable 5.1 on 5 November 2026 on the publisher pages and arXiv abstracts (and, for Dolan, in the PDF: section 2(ii) on S^4 with the homogeneous instanton, equations (40) and (46) present); the passages themselves have not been reread in this project. By the rule on obscure references, any of these used in a paper is to be developed in full there, with theorem number, version and DOI.

- M. Ben Halima, *Spectrum of twisted Dirac operators on the complex projective space P^{2q+1}(C)*, Comment. Math. Univ. Carolin. 49 (2008) 437-445, DML-CZ 119734, Theorem 1: the spectrum of the Dirac operator on odd complex projective spaces twisted by powers of the tautological line, with scalar curvature 3; after the conversion m = -k and the factor 4 for scalar curvature 12 it reproduces Lemma 29.5 (zero and gap 5 on O(3), gap 3 on O(1)).
- B. P. Dolan, *The spectrum of the Dirac operator on coset spaces with homogeneous gauge fields*, JHEP 05 (2003) 018, arXiv:hep-th/0304037, section 2(ii), equations (40)-(46): the S^4 instanton spectrum of Proposition 29.2 (four zero modes at isospin 1, next squared value 5, untwisted spectrum from 4).
- M. Maridakis, arXiv:1510.07004, Lemma 2.2 (the zero-order cross-term identity of Lemma 29.4), and arXiv:2110.11654, sections 2 and 4-8 (normal-model localization and spectral separation; its Assump-

tion 3, direction-independent leading squared data, fails for (n . J)^2 here). Listed in section K; the referee's reading of the passages is new.

- J. Kaad and W. D. van Suijlekom, arXiv:1710.03182, Proposition 18: a horizontal-vertical factorization of Dirac operators with a curvature contraction, under different fibration hypotheses (cf. Proposition 29.1).
- J.-M. Bismut and G. Lebeau, *Complex immersions and Quillen metrics*, Publ. Math. IHES 74 (1991) 1-298, doi:10.1007/BF02699352, sections VIII-IX, equations (9.155)-(9.156): model-space projection and spectral-counting arguments of the kind used in Theorem 29.7, in the complex-immersion setting.
- D. Le Peutrec, *Small eigenvalues of the Neumann realization of the semiclassical Witten Laplacian*, Ann. Fac. Sci. Toulouse Math. 19 (2010) 735-809, doi:10.5802/afst.1265, sections 3-4: boundary localization and exponential estimates with a boundary condition.
- G. J. Parker, *Concentrating Dirac operators and generalized Seiberg-Witten equations*, arXiv:2307.00694, Theorem 1.1: exponential concentration and decay for Dirac operators D + eps^{-1} A with a zeroth-order perturbation vanishing on a subbundle, in the Seiberg-Witten and Z_2-harmonic-spinor context.
- H. Schulz-Baldes and T. Stoiber, *Spectral localization for semimetals and Callias operators*, arXiv:2203.15014, sections 3 and 5: IMS localization with explicit local models for isolated Dirac or Weyl points, and a spectral localization result for Callias operators.
- C. Kottke, arXiv:0909.5661, section 5: the families Callias index (already in section K for Proposition 28.11); the passage from a bounded potential at infinity to the linear normal model is to be matched explicitly.

M. The bidoublet Yukawa block, its quaternionic half and the up-down splitting (lemma 46): prior work found by the Opus referee

Claim. Lemma 46, Propositions 46.1 and 46.2 and Corollary 46.4: the odd Hermitian blocks of the quaternionic sheet are the complex bidoublet; the quaternionic half gives equal masses to the two members of a doublet, the anti-quaternionic half splits them, m_+-^2 = |q|^2 + |q'|^2 +- 2 |Im(conj(q) q')|; the custodial and hypercharge stabilizers. **Verdict of the referee: KNOWN in bidoublet form; the closed formula in the quaternion variables (q, q'), the identification of the second half with the fibre-odd functions on CP^3 and the stabilizer reading in the window were NOT FOUND.** Identified by Claude Opus 5.5 (report, 5 November 2026), who read the arXiv e-print sources where stated; the bibliographic data below were verified by Claude Fable 5.1 on the arXiv abstract pages on 5 November 2026.

- A. H. Chamseddine, A. Connes and W. D. van Suijlekom, *Beyond the spectral Standard Model: emergence of Pati-Salam unification*, JHEP 11 (2013) 132, arXiv:1304.8050 (e-print read by the referee): in the section on the Dirac operator and inner fluctuations on H_R + H_L + M_4(C), the right-left block after fluctuation is (k^nu phi + k^e phi-tilde) Sigma + (k^u phi + k^d phi-tilde)(1 - Sigma), with phi in (2_R, 2-bar_L, 1) and phi-tilde = sigma_2 conj(phi) sigma_2 = j phi j^{-1}; in the unified case k^nu = k^u, k^e = k^d it is (k^u phi + k^d phi-tilde) (x) 1, the block (x) 1_V of lemma 46, whose quaternionic and anti-quaternionic halves are (k^u + k^d)(phi + phi-tilde)/2 and (k^u - k^d)(phi - phi-tilde)/2 for real couplings.
- Y. Zhang, H. An, X. Ji and R. N. Mohapatra, *General CP violation in minimal left-right symmetric model and constraints on the right-handed scale*, Nucl. Phys. B 802 (2008) 247-279, arXiv:0712.4218 (e-print read by the referee, section III): the Yukawa Q-bar_L (h phi + h-tilde phi-tilde) Q_R and the mass matrices M_U = kappa h + kappa' e^{-i alpha} h-tilde, M_D = kappa' e^{i alpha} h + kappa h-tilde, the general bidoublet form of which Proposition 46.2 is a one-generation restatement. The original references, R. N. Mohapatra and G. Senjanovic, Phys. Rev. D 23 (1981) 165, were not checked.
- S. Willenbrock, *Symmetries of the Standard Model*, TASI 2004 lectures, arXiv:hep-ph/0410370 (e-print read by the referee, section 3): the Higgs as the matrix field Phi -> L Phi R^dagger under SU(2)_L x SU(2)_R with U(1)_Y inside SU(2)_R, the vacuum proportional to the identity preserving the custodial SU(2)_{L+R}, and Exercise 3.5, the t, b Yukawa couplings have a custodial symmetry in the limit m_t = m_b; the quaternionic Phi is the j-linear half of lemma 46 and the stabilizers of Corollary 46.4 are this breaking pattern. The lectures cite P. Sikivie, L. Susskind, M. B. Voloshin and

V. I. Zakharov, Nucl. Phys. B 173 (1980) 189, not checked directly.

- M. Aa. Solberg, *On the terms violating the custodial symmetry in multi-Higgs-doublet models*, J. Phys. G 40 (2013) 065001, arXiv:1207.5194 (abstract): custodially symmetric Higgs-quark couplings with aligned vacua give up-down mass degeneracy, which fails for misaligned vacua; in lemma 46's terms, misaligned vacua produce a j-antilinear half.
- H. E. Haber and D. O'Neil, *Basis-independent methods for the two-Higgs-doublet model III: the CP-conserving limit, custodial symmetry, and the oblique parameters S, T, U*, Phys. Rev. D 83 (2011) 055017, arXiv:1011.6188 (abstract): in the Standard Model the custodial symmetry is broken by the hypercharge gauge interactions and by the Yukawa couplings, while the Higgs potential preserves it.
- The identification of the rank-one 2 x 2 matrices with the null vectors of the complex-bilinear norm on C^4 = H (x) C (det X = |q|^2 - |q'|^2 + 2 i q . q') is classical two-spinor calculus; no identifier was checked.

O. The quaternionic sheet as a spinor fibre and the spectral Standard Model (lemma 49): prior work found by the Opus referee

Claim. Lemma 49: the spinor fibre of the four-torus is the quaternionic sheet; the lifted Weyl stabilizer spans C + H with the lifted transposition as the T_3R circle; the blocks commuting with that circle are the diagonal Yukawa blocks; the whole fibre carries the quark summand of one generation. **Verdict of the referee: the algebraic content as statements about the spectral Standard Model is in Chamseddine, Connes and Marcolli, and the two doublets as the two chiralities of Spin(4) spinors are in Baez and Huerta; the realization through the spin lift of the Weyl stabilizer on a four-torus was NOT FOUND.** Bibliographic data verified by Claude Fable 5.1 on arXiv, 5 November 2026.

- A. H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007) 991-1089, arXiv:hep-th/0610241 (e-print source of version 1 read by the referee; numbering from the source). Proposition 2.11: A_F = {(lambda, q_L, lambda, m)} inside A_LR = C + H_L + H_R + M_3(C), every embedding of C in H being conjugate to the chosen one; Definition 2.14 and the equation labelled matrixaction: lambda on the up line and conj(lambda) on the down line of the right-handed doublet; Remark 2.19 and Definition 2.20: D_F commutes with C_F = {(lambda, lambda, 0)}, so that the photon remains massless; Theorem 2.21 with the operators D(Y): the quark Yukawa block diag(Y_up, Y_down); Proposition 2.30 and Corollary 2.31: the quark summand H_F^(3), its pairing [[0, 0, 2], [0, 0, -2], [-2, 2, 0]] (times the number of generations) and the nondegenerate pairing from the lepton and quark classes together. These are the statements that Propositions 49.2 (ii), 49.3 and 49.5 realize on the torus.
- J. C. Baez and J. Huerta, *The algebra of grand unified theories*, Bull. Amer. Math. Soc. 47 (2010) 483-552, arXiv:0904.1556 (e-print source of version 2 read by the referee). Section 3.4, Theorem 4: Spin(4) = SU(2) x SU(2) on the Dirac spinors Lambda C^2, the left doublet on Lambda^1 C^2 and the right doublet on Lambda^0 + Lambda^2, inside the Pati-Salam representation Lambda C^2 (x) Lambda C^3; section 4, Theorems 8 and 9: the Standard Model group as the subgroup of Spin(10) preserving a complex structure, a volume form and a splitting R^4 + R^6. The two doublets as the two chiralities of a four-dimensional Euclidean spinor space are in print there; lemma 49 realizes the four-dimensional part on the torus A with its complex structure.
- A. H. Chamseddine, A. Connes and V. Mukhanov, *Quanta of geometry: noncommutative aspects*, arXiv:1409.2471 (section I above, not reopened): M_2(H) as the Clifford algebra of five gammas of square one, with the gammas internal and the four-manifold space-time.
- Orbifold GUTs, for the local symmetry depending on the fixed locus (abstract pages): Y. Kawamura, Prog. Theor. Phys. 105 (2001) 999, arXiv:hep-ph/0012125; A. Hebecker and J. March-Russell, Nucl. Phys. B 613 (2001) 3, arXiv:hep-ph/0106166. The mechanism differs: there the local group is the centralizer of the twist in the gauge group, here the algebra is the span of the lifted stabilizer.
- Searched without a hit by the referee (titles and abstracts): the Standard Model algebra with spin lift, finite subgroup, binary dihedral, torus orbifold and noncommutative geometry; right-handed fermions with the canonical bundle, Kähler surfaces and SU(2)_R. Hits with different constructions: arXiv:2008.01629, arXiv:0804.2102, arXiv:2508.02747 (excerpt only). The search does not establish

absence.

P. The top block as the Higgs vector times the Majorana projector, the compressed height functions and the quark-lepton split (lemma 50): prior work found by the Opus referee

Claim. Lemma 50: the rank-one Yukawa block with the eigenline condition is c(h) P_w; Q(5 w'^* B w'/w'^* w') = B + tr(B) on the level-one kernel of CP^3, so the projectors on the marked lines and the grading are compressions of height functions; a Yukawa block split along projections of the second sheet keeps order one exactly when the projections commute with the second-sheet algebra; the centralizer of the top-only operator is C + M_3(C). **Verdict of the referee (Claude Opus 5.5, 5 November 2026, e-prints read): the Yukawa block as a quaternionic Higgs times diag(Y_u, Y_d) is KNOWN (Chamseddine-Connes-Marcolli, Proposition 3.5 and Theorem 2.21); the projection criterion is KNOWN as Krajewski's first-order rule; the alignment of the Majorana line, the right-handed line of the Higgs bidoublet and the lepton line is KNOWN as the Standard Model vacuum of the spectral Pati-Salam model (arXiv:1304.8050, section 8); the compression identity is Example 16.9, item 2, a case of Berezin's symbols; NOT FOUND: J_w as the complex structure of the Majorana point with the hypercharge sign, the three-form dual -J_w h/2, and the order-one forcing of the shared lepton line (Proposition 50.3 (5)).** Bibliographic data verified by the referee on Crossref where stated.

- T. Krajewski, *Classification of finite spectral triples*, J. Geom. Phys. 28 (1998) 1-30, doi:10.1016/S0393-0440(97)00068-5, arXiv:hep-th/9701081 (e-print read by the referee): section 3.4, the first-order operators on a bimodule over sums of matrix algebras are sums of a left-linear and a right-linear operator, D = Delta + J Delta J^{-1}; sections 4.1 and 4.2, the edges of a diagram are vertical or horizontal, Delta_{ik}^{jl} = delta_{kl} M_{ij,k} (x) I_{n_k}. Proposition 50.3 is this rule in the two-sheet bimodule with the grading in the left algebra.
- A. H. Chamseddine, A. Connes and M. Marcolli, arXiv:hep-th/0610241 (e-print mc2.tex read by the referee, numbering recomputed from the counters): Definition 2.14 and equation (matrixaction) (lambda on the up arrow, conj(lambda) on the down arrow); Remark 2.18 (the up arrow is nu and u, the down arrow e and d); Proposition 2.11 (A_F = {(lambda, q_L, lambda, m)}); equation (TmapsMR) (the Majorana block on up_R (x) 1^0); Theorem 2.21 (the operators D(Y) with separate lepton and quark blocks and the condition [D, C_F] = 0 of Definition 2.20); Proposition 3.5 and Lemma 3.6 (the discrete inner fluctuation as a quaternion-valued Higgs H, the up and down couplings related by H -> jH, Tr((D_3^{(0,1)})^2) = 12 |1 + H|^2 Tr(Y_u^* Y_u + Y_d^* Y_d)): the blocks c(n)(Y_u (x) P_w + Y_d (x) P_{jw}) of Proposition 50.1 (4) with 1 + H the quaternion c(n) up to normalization.
- A. H. Chamseddine and A. Connes, *Why the Standard Model*, J. Geom. Phys. 58 (2008) 38-47, arXiv:0706.3688 (e-print read by the referee): section 4, Theorem 4.1, Lemma 4.2 and Theorem 4.3, the maximal subalgebra admitting off-diagonal Dirac operators with order one, Theorem 20.2 here.
- A. H. Chamseddine, A. Connes and W. D. van Suijlekom, *Beyond the spectral Standard Model: emergence of Pati-Salam unification*, JHEP 11 (2013) 132, arXiv:1304.8050 (e-print, a gzipped tar with patisalam2.tex and six figures, read on 5 November 2026 by Claude Fable 5.1 and by the referee): section 4, "Summary of tensor notation", the tensor form of D_F with k^nu, k^e on the lepton line and k^u, k^d on the colours, the sentence that this is the unique possibility consistent with the first-order condition on the subalgebra (citing *Why the Standard Model*), and "the special case of lepton and quark unification" k^nu = k^u, k^e = k^d; section 5, "Dirac operator and inner fluctuations on H_R + H_L + M_4(C)", the field Sigma_I^J in (1, 1, 1 + 15), whose vacuum value is the projector on the lepton line, decoupling in the unified case; section 3, the first-order condition with the Majorana mass restricting the Pati-Salam algebra to the subalgebra, and the initial Dirac operator satisfying first order relative to the subalgebra only; section 8, equation SM-vev, the Standard Model vacuum = v delta^{1.} delta_1, = u delta_1 delta^1, = w delta^{1.} delta_1, with the Majorana field Delta and the bidoublet phi sharing the right-handed index and Delta sharing its Pati-Salam index with the lepton line of Sigma, the alignments of Propositions 50.1 and 50.3 as the chosen vacuum, with a nonnegative Hessian in the Delta directions. Also A. H. Chamseddine, A. Connes and W. D. van Suijlekom, arXiv:1507.08161 (e-print read by the referee): the table of Pati-Salam scalars for a first-order Dirac operator, with Sigma decoupling under quark-lepton coupling unification.

- M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. R. Soc. Lond. A 362 (1978) 425-461, doi:10.1098/rspa.1978.0143 (Crossref; content not read): the twistor correspondence, the points of the fibre over p parametrizing the orthogonal complex structures of T_p S^4 of one orientation; J_w of Proposition 50.1 is that complex structure at [w], with J_{jw} = -J_w (computed by the referee).
- F. A. Berezin, *Covariant and contravariant symbols of operators*, Math. USSR-Izv. 6 (1972) 1117-1151, doi:10.1070/IM1972v006n05ABEH001913; *General concept of quantization*, Comm. Math. Phys. 40 (1975) 153-174, doi:10.1007/BF01609397 (bibliographic data verified on Crossref by the referee; content not read): the covariant and contravariant symbols. Proposition 50.2 is the level-one case on CP^3, which is Example 16.9, item 2, extended complex-linearly: the contravariant symbol of B is 5 times its covariant symbol minus tr(B), and in general at level one on CP^{N-1}, Q((N + 1) sigma_B - tr(B)) = B.
- F. Besnard, arXiv:2011.02708 (abstract only, found by the referee's search): a weak order-one condition that removes the Majorana mass in the Pati-Salam model.
- The custodial reading of the two halves and the bidoublet mass matrices: section M above (lemma 46). The retained algebra C + M_3(C) of a dominant top: the corpus row rs_F1 (historical/corpus/connes_kk_research_state.md, section 7.2), a conditional algebraic derivation with the embedding unspecified; Proposition 50.4 fixes it.
- Searches by the referee without a relevant hit (titles and abstracts, arXiv API): twistor and "spectral triple"; "Krajewski diagrams" and Yukawa; "Berezin transform", "projective space" and eigenvalues; twistor, noncommutative, Higgs and quaternion; "Pati-Salam", spectral, Yukawa and noncommutative. The Higgs-vector form tied to the twistor fibre and the compressed-height-function reading were not found.

Q. The torus Dirac operator with a Wilson line and a torsion as the Yukawa block (lemma 51)

Claim. Lemma 51: on the flat four-torus the Dirac operator with a constant connection and a constant three-form is c(k + A) + i c(eta) Gamma on each momentum sector, its lowest window is the Yukawa block of lemma 46 with the Wilson line as the custodial half and the torsion as the splitting, the photon condition is the invariance under a transposition of the Weyl group, and the rank-one pattern is the torsion equal to the quarter turn of the Wilson line. **Verdict (Claude Fable 5.1, 5 November 2026; refereed by Claude Opus 5.5, identifiers verified on Crossref and the arXiv API, e-prints of Agricola and Friedman-Morgan-Witten read): the two ingredients are classical (Wilson lines shifting internal momenta; Dirac operators with skew torsion); their reading as the two halves of a Yukawa block, the Weyl-invariance derivation of the photon condition and the quarter-turn characterization of the heaviest quark were NOT FOUND.**

- Y. Hosotani, *Dynamical mass generation by compact extra dimensions*, Phys. Lett. B 126 (1983) 309-313, doi:10.1016/0370-2693(83)90170-3 (title and pages verified in Crossref): fermion momenta shifted by the Wilson line, (n + q a)/R on a circle; Proposition 51.1 is the four-torus form with the torsion added. Gauge-Higgs unification and magnetized tori, from abstracts found by the referee: Y. Hosotani, S. Noda and Y. Sakamura, arXiv:hep-ph/0601241; K. Matsumoto and Y. Sakamura, arXiv:1602.01994 (Yukawa couplings on T^2/Z_N with magnetic fluxes and Wilson-line phases); D. Cremades, L. E. Ibáñez and F. Marchesano, arXiv:hep-th/0404229 (Yukawa couplings from magnetized tori); K. Benakli, arXiv:hep-th/9509115 (Scherk-Schwarz and Wilson lines). Fermion masses as momenta shifted by Wilson lines, and Yukawas from Wilson lines with fluxes on tori, are in print there.
- I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math. (Brno) 42 (2006) 5-84, arXiv:math/0606705 (e-print read by the referee): the spinor connection nabla^{s'} = nabla^g + s' (the contraction of T with X) has torsion 4 s' T and Dirac operator D^{s'} = D^g + 3 s' T, so 3/4 for the connection with torsion T and 1/4 for torsion T/3, Bismut's rescaling and, on naturally reductive spaces, Kostant's cubic Dirac operator; the constant s != 0 enters no statement of lemma 51. J.-M. Bismut, *A local index theorem for non-Kähler manifolds*, Math. Ann. 284 (1989) 681-699, doi:10.1007/BF01443359, and T. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. 6 (2002) 303-336, doi:10.4310/AJM.2002.v6.n2.a5, arXiv:math/0102142, for the same operators and their string origin (identifiers verified in Crossref

by the referee; abstracts). T. Kimura and P. Yi, arXiv:hep-th/0605247 (abstract): in heterotic flux compactifications the Dirac operator of the gauge fermions uses the connection with torsion one third, the normalization s = 1/4 of lemma 51, and a closed H != 0 forces a singular compactification.

- The identity c(e_a) c(e_b) c(e_c) = +- Gamma c(e_d) in four dimensions is standard Clifford algebra; checked numerically in lemma 51.
- Flat connections on an elliptic curve: E. Looijenga, *Root systems and elliptic curves*, Invent. Math. 38 (1976) 17-32, doi:10.1007/BF01390167 (Crossref gives 1976; Friedman, Morgan and Witten cite 1977); R. Friedman, J. Morgan and E. Witten, *Vector bundles and F theory*, Commun. Math. Phys. 187 (1997) 679-743, doi:10.1007/s002200050154, arXiv:hep-th/9701162, e-print read by the referee: section 2.1, the SU(n) case, unordered n-tuples of points of E with sum zero, the moduli P H^0(E, O(np)) = CP^{n-1}; section 3.1, (T x T)/W for simply connected G; section 4, Hom(L' ', E) = L' (x) E with L'' the weight lattice. So A = Lambda_coroot (x) E and A/S_3 = CP^2; recorded as a lead in lemma 51.
- For the photon condition and the Yukawa blocks of the spectral Standard Model: section O (lemma 49) and section P (lemma 50).

R. The lepton generation in three views (lemma 52)

Claim. Lemma 52: the lepton sub-bimodule with the connector identifying the two copies of C and its nondegenerate pairing over C + H; the Majorana spinor of nine dimensions as one lepton generation; three generations as the flavour adjoint under U(1)_em. **Verdict (Claude Fable 5.1, 5 November 2026; refereed by Claude Opus 5.5, identifiers verified, the e-print of Chamseddine-Connes-Marcolli read): the Clifford decomposition and the lepton summand with its pairing are in print; the identification of the connector's order-one condition with the identification of the two copies of C, under the eigenline condition, the quantized momenta of the Majorana field as the custodial half, and the isomorphism of U(1)_em-modules were NOT FOUND as stated.**

- A. H. Chamseddine, A. Connes and M. Marcolli, arXiv:hep-th/0610241 (section O above): Proposition 2.11 (the identification of the two copies of C by the order-one condition for an operator between particles and antiparticles, which Proposition 52.1 (ii) realizes with the connector of Theorem 20.2), Proposition 2.30 (3) prints the matrices [[0, 2, 0], [-2, 0, 0], [0, 0, 0]] on H_F^(1) and [[0, 0, 2], [0, 0, -2], [-2, 2, 0]] on H_F^(3) per generation, the same as Proposition 52.1 (iii); Corollary 2.31 obtains nondegeneracy over the three summands from the pair of classes, and Remark 2.32 reads it as KO-homology generated by two elements; the rational nondegeneracy over C + H of the lepton matrix is read off Proposition 2.30; Proposition 2.11 states A_F as unique up to an automorphism of A_LR (e-print read by the referee). The Majorana conjugation and the bilinear table of signature (1, 8) were checked by the referee on an explicit representation (C symmetric; C Gamma^{(3)} antisymmetric).
- Spinors in nine and ten dimensions: J. Polchinski, *String Theory*, vol. 2, appendix B (Majorana and Weyl conditions by dimension and signature; the decomposition of the 16 under SO(1,3) x SO(6)); M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vol. 2, chapter 15 (the Majorana-Weyl spinor of ten dimensions on M^4 x K as a left-handed Weyl spinor tensored with the internal spinor); from the record of this session, not reopened; the nine-dimensional case is the restriction to Spin(1,8) and was checked numerically in lemma 52.
- L. Alvarez-Gaumé and E. Witten, *Gravitational anomalies*, Nucl. Phys. B 234 (1984) 269-330, doi:10.1016/0550-3213(84)90066-X (Crossref): the Majorana mass bilinear by dimension (the corpus row rs_F3; content not reopened).
- A. Rivero, *An interpretation of scalars in SO(32)*, Eur. Phys. J. C 84 (2024) 1058, doi:10.1140/epjc/s10052-024-13368-3, arXiv:2407.05397, and *Supersymmetry with composite bosons*, hep-ph/0512065 (ledger rows d11_so32_comparison and l21_so30; arXiv API): the meson counts of the five light quarks and their match with the sleptons; Proposition 52.3 records the branching and the isomorphism of U(1)_em-modules.

Sources read on 4 November 2026

Read by Claude Opus 5.5 from PDFs supplied by the user (Banks-Casher, Dashen) and from the scan held by INSPIRE (Witten 1983).

- T. Banks and A. Casher, *Chiral symmetry breaking in confining theories*, Nucl. Phys. B 169 (1980) 103-125, doi:10.1016/0550-3213(80)90255-2, INSPIRE 143630. The paper works mainly with a second-order lattice formalism and a strong-coupling picture of trees. The spectral statement is a remark in Section 5, item (iv), p. 122: eq. (49), the integral of over spacetime equals the limit m -> 0 of the average of the integral of d mu rho(mu; A) m/(mu^2 + m^2), with rho the spectral function of D(A); eq. (50), the condition rho(mu; A) -> const as mu -> 0, on a finite fraction of the configurations. The paper states no factor pi, no division by the volume and no order of the limits m -> 0 and V -> infinity. Those belong to later statements, such as Giusti-Lüscher.
- R. Dashen, *Some features of chiral symmetry breaking*, Phys. Rev. D 3 (1971) 1879-1889, doi:10.1103/PhysRevD.3.1879, INSPIRE 61166. Section II.B, pp. 1881-1882: Theorem 1, the vacuum of H_0 + epsilon H' makes the expectation value of the rotated perturbation F(omega) stationary and at least a local minimum at omega = 0; Theorem 2, the same for the epsilon -> 0 limit of the vacuum. Whether omega = 0 is a global minimum is stated as a conjecture, with the existence of a global minimum of F noted separately. The ferromagnet in an external field is the worked example (eq. 15). This is the source for vacuum alignment; the stationarity calculation of lemma 17 is a separate derivation.
- E. Witten, *Fermion quantum numbers in Kaluza-Klein theory*, preprint Princeton, October 1983, 78 pages, in Shelter Island II (1985) 227-277, INSPIRE 193975, scan. Preprint page numbers. Pp. 4-5: bare masses of quarks and leptons are forbidden by gauge invariance, so they cannot have masses of order the Planck mass, and "we don't understand why some of the fermions are so much lighter than M_W". P. 7: if all gauge fields are part of the metric tensor, the Atiyah-Hirzebruch theorem gives vector-like quantum numbers for the Dirac operator on a compact B; with elementary gauge fields, complex representations are "possible but still subtle". P. 8, eq. (2): "We see that D^(n) is the mass operator, in effect. Its eigenvalues are observed in four dimensions as the particle masses. Its zero eigenvalues are the massless fermions." Pp. 43-44: the character-valued index is invariant under equivariant cobordism, so if M bounds a manifold to which the G action extends, it vanishes for any operator on M. This is the statement behind the balance on the boundary CP^3 of lemma 28: the summands of E restrict with indices 4, 0, 0, -4 and total zero. The hypotheses are a compact B without boundary and no mass profile; lemma 28 leaves them through the wall and through the Callias potential, with E as an elementary gauge bundle.

Items not verified in this session

- Castellani-D'Auria-Fré 1984 (section F): equation-level content not read.
- Banks-Casher 1980: read on 4 Nov 2026 from the PDF supplied by the user, in the passages below. The value pi rho(0) and the order of limits in the chiral limit note come from L. Giusti and M. Lüscher, JHEP 03 (2009) 013, arXiv:0812.3638, Section 2 (LaTeX source), which counts the modes of one flavour and states the order of limits lambda -> 0, m -> 0, V -> infinity. The original writes neither.
- Witten 1983: only the passages recorded below were read; the proofs of sections III to VI were not checked.
- Coquereaux-Esposito-Farèse-Vaillant 1991, Dubois-Violette-Kerner-Madore 1990 (section A): bibliographic data from secondary citations, content from secondary sources.
- Callan-Harvey 1985 and Arkani-Hamed-Cohen-Georgi 2001 (section H): full texts not read.
- Demmel et al. 1999: theorem numbers for the D_1 X D_2 results.
- Niemi-Slizovskiy, Section VIII (custodial fixing of xi).
- Connes-Moscovici journal reference; second edition of van Suijlekom's book.

Summary table

Claim	Verdict	Key references
A. Connes-Lott as minimal truncation of the chiral interval,	m	= pi/(2l)
B. l -> infinity (walls, 1D Weyl law) and l -> 0 with mu(l)	PARTIAL: wall modes and exponentially small mixing known; NCG statement and tuned collapse not found	Jackiw-Rebbi 1976; Kaplan 1992; Shamir 1993 §3; Csáki et al. 2004; Farsi-Latrémolière 2026 Cor. 4.1
C. Towerless textures Phi_Q Y_hat Phi_U, secular equation, one-flavour spectrum	PARTIAL: one-flavour spectrum KNOWN (Csáki et al. eq. (5.6)); multi-flavour warped analogue KNOWN (Casagrande et al. eqs. (65), (87), (90), (94)-(97)); NCG derivation of D_F not found	Csáki et al. 2004 §5.1, App. C.2; Casagrande et al. 2008 §3; Kaplan-Tait 2001; Arkani-Hamed-Schmaltz 2000
D. KO-6 = (odd) x (Clifford point) criterion; s from U; nu_L; reflection sign	PARTIAL: products, J vs J gamma, odd x odd product and 10 = 2 mod 8 known; factorization criterion and the 9/11 results not found	Dąbrowski-Dossena 2011 §4.3, eq. (25), Table 6; Vanhecke 1999; Vanhecke-da Silva-Sigaud 2012 §2; Farnsworth 2017; Connes 2006 §1; Barrett 2007
E. 8 M_W^2 = sum N_f m_f^2; m_t/M_W = sqrt(8/3); heavy weights	KNOWN (mass relation, top ratio); NOT FOUND (exact f(m^2/Lambda^2) weights)	CCM 2007 eq. (1.5), Lemma 4.6 eq. (4.16), §5.4 eqs. (5.24)-(5.29)
F. cos^2 theta_W = 3 xi^2/(4 xi^2 + 2)	KNOWN	Niemi-Slizovskiy 2010 eqs. (94)-(95), §VI; Witten 1981 (method)
G. Odd mod-two index on S^5	KNOWN (topology and footnote attribution)	García-Etxebarria-Montero 2019 §3.1; Witten 1982; Wang-Wen-Witten 2019
H. Anomaly inflow	KNOWN (background)	Callan-Harvey 1985; Naculich 1988; Arkani-Hamed-Cohen-Georgi 2001
I. Current framework	reference list	Connes 2006; Barrett 2007; CCM 2007; Chamseddine-Connes 2007, 2008; CCvS 2013 (two papers); Devastato-Lizzi-Martinetti 2014; Krajewski 1998; Paschke-Sitarz 1998; van Suijlekom 2015

Part R1 (Counterexample R1 to Farsi-Latrémolière, Theorem 3.4)

Counterexample R1: Theorem 3.4 of Farsi-Latrémolière needs a hypothesis it does not state

Source. C. Farsi and F. Latrémolière, *Collapse in noncommutative geometry and spectral continuity*, arXiv:2404.00240 (v1 30 Mar 2024; v2 and v3 Oct 2025, identical sources), J. London Math. Soc. (2) 114 (2026) e70628, doi:10.1112/jlms.70628. The theorem is Theorem 3.4 in v3 and in print, Theorem 2.5 in v1. Found by Claude Opus 5.5 on 3 November 2026 while reading the paper for lemma 09; confirmed analytically by Astra 6 (hypotheses (1) to (6) and compact resolvent checked one by one) and by Terra 6.1 (item 49 of its report on the main draft); the published text was compared word by word with v3 and keeps the statement and the defective step, printed as equation (3.11). Ledger row `fl_R1`. The main paper develops the piece used in its own text, by the rule that an obscure reference cannot carry a step of the argument.

The statement in question. For a metric spectral triple (A, H, D) with D = D_h + D_v, D_v self-adjoint with 0 isolated in its spectrum, D_eps = D_h + D_v/eps, a unital C*-subalgebra B of A and the projection p onto ker D_v, under six hypotheses ((1) ||[D_h, a]|| <= ||[D_eps, a]||; (2) ||[D_v, a]|| <= M eps ||[D_eps, a]||; (3) [D_v, b] = 0 for b in B; (4) [p, b] = 0 and [D_h, p] = 0; (5) (B, ker D_v, p D_h p) is a metric spectral triple; (6) a positive map E: A -> B fixing B with ||a - E(a)|| <= k ||[D_v, a]|| and ||[D_h, E(a)]|| <= ||[D_h, a]||), the theorem asserts that (A, H, D_eps) converges to (B, ker D_v, p D_h p) in the spectral propinquity as eps -> 0. Corollary 3.5 then gives convergence of spectra: Sp(p D_h p) is the set of limits of convergent sequences lambda_n in Sp(D_{eps_n}).

Counterexample. Let H have orthonormal basis e_0, e_1, e_2, ..., let A = B = C act by scalars, and set D_v e_n = n e_n and D_h e_n = -n^2 e_n. Then D_eps e_n = (n/eps - n^2) e_n has compact resolvent for every eps > 0, since |n/eps - n^2| -> infinity; D_v has gap 1 at 0; p is the projection onto C e_0 and commutes with D_h; hypotheses (1) to (6) hold trivially, all commutators with scalars being zero, with E = id and any M, k > 0; and (C, C e_0, 0) is a metric spectral triple. Its spectrum is {0}. Fix x > 0 and take eps_n = 1/(n + x/n) -> 0. Then D_{eps_n} e_n = x e_n, so the constant sequence x lies in Sp(D_{eps_n}) for every n, and Corollary 3.5 would put x in {0}. Hence the conclusion of Theorem 3.4 fails for this family.

The failing step. The proof's estimate (3.11),

```
||xi - p xi|| <= 4 sqrt(2 eps/delta) (||xi|| + ||D_v xi/eps||) <= 4 sqrt(2 eps/delta) DN_eps(xi),
```

needs ||D_v xi||/eps <= ||D_eps xi||, which is neither assumed nor proved. For xi = e_n at eps = eps_n the left side is 1 and the right side is 4 sqrt(2 eps_n) (1 + x) -> 0.

Sufficient repairs. Either hypothesis below makes the Fourier Lemma 3.3 of the paper unnecessary and improves the rate from O(sqrt eps) to O(eps):

1. D_h and D_v anticommute on a common core. Then ||D_eps xi||^2 = ||D_h xi||^2 + eps^{-2} ||D_v xi||^2 and ||xi - p xi|| <= ||D_v xi||/delta <= (eps/delta) ||D_eps xi||.
2. A relative bound ||D_v xi|| <= C eps (||xi|| + ||D_eps xi||). Then ||xi - p xi|| <= (C eps/delta)(||xi|| + ||D_eps xi||).

These are sufficient hypotheses; the counterexample shows the printed six are not. All applications in the paper (Corollary 4.1, Theorems 4.6, 5.5, 6.3) have anticommuting D_h and D_v and avoid the defect; Theorem 4.6 and Corollary 4.7 additionally need a nonzero vertical kernel for their displayed target to be a spectral triple (Astra 6, 3 November 2026). Corollary 4.1 (collapse of a fixed triple scaled by 1/eps to (C, ker D, 0), and to nothing when ker D = 0) has D_h = 0 and is sound.

Scope. This is a statement about one theorem of one paper. It leaves the paper's framework, its corollaries in the anticommuting setting and its applications intact. For the project, it is the reason lemma 15 proves its own collapse statements (Theorems 15.3 and 15.7) and the reason the main paper reaches the contradiction through Theorem 15.7 (the converse of Proposition G: along eps_n the eigenvalues 0 and x stay bounded, while a limit (C, C e_0, 0) would need one eigenvalue near 0 with a diverging gap) rather than through Corollary 3.5. The route through Theorem 15.7 was written by Claude Opus 5.5 and read by Claude Fable 5.1; no referee has checked that route as distinct from R1 itself.

Record. The full reading of the paper (its prerequisites, the structure of the proof, the v1 to v3 changes, the errors in the written proof of Theorem 6.3, the dependency audit) is in historical/reviews/2026-11-03_farsi_latremoliere_assessment.md; nothing there is needed for the statement above.

Part S1 (The staggered N-site chain has spectrum +-2 h_N sin((2r-1) pi/(4N-2)) and site distances |i-j|/h_N, and it converges to the chiral interval in spectrum, heat trace and Gromov-Hausdorff distance)

The staggered N-site chain has spectrum +-2 h_N sin((2r-1) pi/(4N-2)) and site distances |i-j|/h_N, and it converges to the chiral interval in spectrum, heat trace and Gromov-Hausdorff distance

Status. Theorem. Proved by Astra 6 (job 05, Parts 1 and 2); checked numerically by Claude Opus 5.5 (spectra to 1e-14, commutator norms and endpoint distances for l = 1 and N = 2, 3, 5, 9; lemma 05, section "Independent numerical check"). Ledger row `l05_chain`. Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_05_21_34.md): correct, every formula recomputed (spectra and commutator norms to N = 39, heat traces, the path distances), with the scope of item 3 as stated there. The chain is stated after Theorem A of paper/towerless_intervals.tex.

Setting. N >= 2, n = N - 1, A_N = C^N, H_N = span{L_1, …, L_{N-1}, R_2, …, R_N}, pi_N(a) = diag(a_1, …, a_{N-1}, a_2, …, a_N), Gamma_N = diag(I_n, -I_n), B_N = h_N (I_n - S_n) with S_n the lower shift, D_N = [[0, B_N], [B_N^*, 0]]. Spectral calibration theta_N = pi/(4N - 2), h_N = pi/(4 l sin theta_N), about N/l, which fixes the smallest positive eigenvalue at pi/(2l) for every N. Complex even triples, with no real structure or KO axiom imposed. The interval D_l is that of 05_connes_lott_minimal_truncation.md.

Statement.

1. spec(D_N) = {+-lambda_{N,r} : 1 <= r <= N - 1} with lambda_{N,r} = 2 h_N sin((2r - 1) theta_N), and K_N(t) = Tr exp(-t D_N^2) = 2 sum_r exp(-4 t h_N^2 sin^2((2r - 1) theta_N)).
2. ||[D_N, pi_N(a)]|| = h_N max_i |a_{i+1} - a_i|, so d_N(i, j) = |i - j|/h_N and d_N(1, N) = (4l(N - 1)/pi) sin(pi/(4N - 2)) -> l. At N = 2 the triple is exactly the Connes-Lott two-point space with h_2 = pi/(2l).
3. As N -> infinity, lambda_{N,r} -> (2r - 1) pi/(2l); K_N(t) -> K_l(t) = 2 sum_r exp(-t ((r - 1/2) pi/l)^2) uniformly on every [eps, T] (indeed on [eps, infinity)); and with x_i = (i - 1) l/(N - 1) the site metric converges to [0, l] in Gromov-Hausdorff distance, with distortion and mesh O(l/N); the Gromov-Hausdorff distance is at most l/(N - 1). Each eigenvalue increases to its limit, and the Hilbert spaces differ, so no resolvent convergence is claimed.
4. For t_N/l^2 -> 0 and N^2 t_N/l^2 -> infinity, K_N(t_N) ~ l/sqrt(pi t_N) and -2 t_N K_N'(t_N)/K_N(t_N) -> 1: a controlled one-dimensional heat window.

Scope. The geometric normalization h_N = (N - 1)/l gives d_N(1, N) = l exactly but mass 1/l at N = 2, so spectral and metric calibration differ at finite resolution. The one-component staggered path and the vertex-edge incidence Dirac operator have other spectra and distances; the incidence chain has a zero mode and same-grading endpoints and differs from Connes-Lott at N = 2. The endpoint chirality concerns the allowed boundary components and supplies no localized chiral zero modes.

Proof. Lemma 05, Parts 1 and 2; the proofs are there. In the basis L_1, R_2, L_2, …, R_N, D_N is a path adjacency matrix whose alternating signs a diagonal unitary removes; the bound lambda_{N,r} >= (2r - 1)/l gives a summable Gaussian majorant for the heat traces.

What it leaves open. At fixed rank the chain has heat dimension zero, and at fixed energy its collapse leaves no light fermions (05_nogo_bare_limits_fixed_rank.md). Stronger convergence (lead from the referee): the seminorm ||[D_N, pi_N(a)]|| is the Lipschitz constant for d_N, so by Kantorovich-Rubinstein duality the Connes metric on all states of C^N is the Monge-Kantorovich metric of d_N, and the same correspondence should give convergence of the full state spaces at distance O(l/N); convergence as spectral triples, with D_N and the Hilbert spaces, belongs to the spectral propinquity of Latremoliere, to be cited or developed in full by the rule on obscure references. The statements are uniform in l, as Part 4 needs.

Part S2 (The minimal grading-preserving spectral truncation of the chiral interval is the Connes-Lott two-point space, with |m| = pi/(2l) and truncated distance 2l/pi)

The minimal grading-preserving spectral truncation of the chiral interval is the Connes-Lott two-point space, with |m| = pi/(2l) and truncated distance 2l/pi

Status. Theorem. Proved by Astra 6 (job 05, Part 3); checked by Claude Opus 5.5 in Python (spectra, commutator norms, distances, truncation) and by Terra 6.1 by hand, as Theorem A of the main paper (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l05_truncation`. It is Theorem A of paper/towerless_intervals.tex. The truncation theorem with |m| = pi/(2l) was not found in print (paper/LITERATURE.md, section A).

Setting. H_l = L^2([0, l], C^2) with psi = (L, R), D_l = [[0, d/dx], [-d/dx, 0]] on {(L, R) in H^1 : R(0) = 0, L(l) = 0}, scalar multiplication by C^infinity([0, l]) and grading Gamma = diag(1, -1). Let m_1 = pi/(2l), P = 1_{[0,E]}(|D_l|) with m_1 <= E < 3 m_1, and S = P C([0, l]) P with its inherited operator-system structure (Connes-van Suijlekom, section 1).

Statement.

1. D_l is self-adjoint with compact resolvent, its spectrum is {+-(r - 1/2) pi/l : r = 1, 2, ...}, and its kernel is zero.
2. PH_l has the orthonormal chiral basis u_L = sqrt(2/l) (cos(pi x/(2l)), 0), u_R = sqrt(2/l) (0, sin(pi x/(2l))), in which P D_l P = m_1 sigma_1 and P Gamma P = sigma_3.
3. The compression is Q(f) = diag((2/l) int f cos^2(pi x/(2l)) dx, (2/l) int f sin^2(pi x/(2l)) dx), and its range is exactly C + C. So the truncation is unitarily equivalent to the Connes-Lott two-point space with |m| = m_1 = pi/(2l), any phase of m being supplied by a diagonal unitary.
4. The two states omega_0, omega_l of the compressed operator system are at Connes distance 1/m_1 = 2l/pi < l. Their pullbacks are the smeared measures (2/l) cos^2(pi x/(2l)) dx and (2/l) sin^2(pi x/(2l)) dx, at distance 4l/pi^2 in the original seminorm. Endpoint evaluations do not factor through Q, since Q(cos(3 pi x/l)) = 0.
5. The truncation is minimal: there is no zero mode and Gamma exchanges the eigenspaces at +-m_1, so every nonzero grading-preserving spectral subspace contains both.

Scope. Complex even data, with no real structure imposed. The construction supplies no KO-6 or Standard Model identification. With the inherited real structure the full diagonal algebra fails first order (08_counterexample_pointwise_reality_first_order.md).

Proof. Lemma 05, Part 3; the proof is there. Pointwise orthogonality of the L and R components makes Q(f) diagonal, and Q(1) = I, Q(cos(pi x/l)) = (1/2) sigma_3 span the diagonal algebra.

What it leaves open. This interval has no zero mode: a two-state sector at energies E « 1/l and light fermions after collapse are excluded for it (05_nogo_bare_limits_fixed_rank.md). The KO-6 and Standard Model identification is outside the construction.

Part S3 (At fixed rank, m -> 0 separates the two sheets without producing a heat dimension, and m -> infinity collapses the metric with an empty light sector)

At fixed rank, m -> 0 separates the two sheets without producing a heat dimension, and m -> infinity collapses the metric with an empty light sector

Status. No-go theorem in its scope (labelled Proposition in lemmas/README.md; ledger kind no_go). Proved by Astra 6 (job 05, Part 4); checked by Claude Opus 5.5 in Python (spectra) and by Terra 6.1 by hand, with one precision adopted in the lemma (historical/reviews/2026-11-03_terra_check_nogo_19.md).

Ledger row `l05_limits`. Stated, with No-go theorem 8 of lemma 08, as Proposition A.1 of paper/towerless_intervals.tex.

Setting. The chain D_N of lemma 05, Part 1, with the normalization h_N = pi/(4 l sin theta_N), and the interval D_l of Part 2, with m = m_1 = pi/(2l). Physical energies or heat times are held fixed unless stated otherwise.

Statement.

1. For fixed N, D_N = m T_N with T_N independent of m. As m -> 0, D_N -> 0, K_N(t) -> 2(N - 1) and d_N(i, j) -> infinity for i != j; every finite N has ultraviolet heat dimension zero, and the minimal truncation has K_2(t) = 2 exp(-t m^2).
2. A resolved one-dimensional window at fixed energy E needs 1/l « E « N/l. So, as l -> infinity, a continuum limit on a fixed positive energy window needs N/l -> infinity, equivalently N m -> infinity. The order of limits matters: lim_l lim_N K_N(t)/l = 1/sqrt(pi t), while lim_N lim_l K_N(t)/l = 0.
3. As m -> infinity all distances tend to zero, while every eigenvalue has absolute value at least m, so ||(D_N - i)^{-1}|| <= (m^2 + 1)^{-1/2} -> 0 and H_light = ker T_N = 0. The same holds for D_l. No fermion pair survives at fixed energy.
4. The two-state truncation exists only for m_1 <= E < 3 m_1, and a cutoff E « 1/l retains no modes; the first two positive levels have ratio 3.

Precision (Terra 6.1). N/l -> infinity gives uniform convergence of the dispersion on bounded energy windows and of normalized heat traces and counting densities. It does not by itself give convergence of the unnormalized difference of heat traces: for N = floor(l^(5/4)) that difference diverges.

Hypotheses and escapes. The bare chain and interval of lemma 05, with no bulk mass and no boundary coupling. The escapes for the empty light sector are a tuned bulk mass (lemma 09, Theorem 9.9; 09_two_limits.md) and a scaled boundary coupling (lemma 12, Theorem 12.7; 12_light_spectrum_towerless_limit.md).

Proof. Lemma 05, Part 4; the proof is there. Both limits follow from D_N = m T_N with T_N fixed and from |lambda| >= m.

What it leaves open. The result points to families in which the number of retained modes grows (N m -> infinity) or in which the light pair is held by localization or by a wall coupling; those are lemmas 09 and 12.

Part S4 (The electrodynamics triple admits doubling involutions of both signs with the same J, so J does not decide between 9 and 11)

The electrodynamics triple admits doubling involutions of both signs with the same J, so J does not decide between 9 and 11

Status. Counterexample. Proved by Astra 6 (job 06, Result 3); checked by Claude Opus 5.5 in Python (m = 0.7: KO-6 signs, both involutions, the descended signs) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_nogo_19.md). Ledger row `l06_counterexample`. Item 2 of Theorem D of paper/towerless_intervals.tex.

Setting. The electrodynamics triple of van den Dungen-van Suijlekom (section 4.1, equations (4.1)-(4.2)), in the basis (e_L, e_R, bar e_L, bar e_R) after a phase change compatible with A, J and gamma:

```
rho(a, b) = diag(a, a, b, b),   gamma = diag(1, -1, -1, 1),
J = [[0, I_2], [I_2, 0]] K,     D = m diag(sigma_1, sigma_1),   m >= 0.
```

It has the KO-6 signs.

Statement.

1. U_+ = diag(sigma_1, sigma_1) and U_- = diag(sigma_1, -sigma_1) both reverse the grading and commute with the algebra, the opposite algebra and D.

2. U_+ J = J U_+ and U_- J = -J U_-.
3. On H_0 = span(e_L, bar e_R) the descended triples are d_+ = m I_2, j_+ = sigma_1 K, of KO-degree seven, and d_- = diag(m, -m), j_- = [[0, 1], [-1, 0]] K, of KO-degree five.
4. This holds also at m = 0.

So the same triple, with the same J, admits both factorizations of 06_doubling_criterion.md.

Scope. The claim refuted is that the real structure J fixes a unique reduction. The statement concerns this finite triple.

Proof. Lemma 06, Result 3; the proof is there, by direct multiplication.

What it leaves open. Which further datum selects s. Spacetime charge conjugation and particle number do not (08_counterexample_both_signs_fixed_data.md, 08_particle_preservation_pairs_signs.md); the neutral-sector multiplicities restrict s = -1 (06_neutral_sector_criterion.md).

Part S5 (A finite KO-6 triple factors as T_7 x P_7 or T_5 x P_1 exactly when a grading-reversing involution U commutes with the algebra, its opposite and D, with UJ = JU or UJ = -JU)

A finite KO-6 triple factors as T_7 x P_7 or T_5 x P_1 exactly when a grading-reversing involution U commutes with the algebra, its opposite and D, with UJ = JU or UJ = -JU

Status. Theorem. Proved by Astra 6 (job 06, Result 2); checked by Claude Opus 5.5 in Python (both involutions on the electrodynamics triple) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_nogo_19.md). Ledger row `l06_factorization`. Item 1 of Theorem D of paper/towerless_intervals.tex; cited as "\supp{06}, Theorem 2" in paper/real_structure_note.tex. The criterion was not found in print (paper/LITERATURE.md, section D).

Setting. A finite KO-6 triple (A, H, D, J, gamma), signs (+, +, -), with order zero and first order, and a° = J a^* J^{-1}. P_k = (C, C, 0, K), k = 1 or 7, is the odd point (point algebra R for real algebras). Odd-times-odd products use H_1 (x) H_2 (x) C^2 with J = J_1 (x) J_2 (x) (BK) and B from Result 1 of lemma 06, in agreement with Dabrowski-Dossena (equation (25), Table 6). H_L and H_R are the gamma = +1 and -1 subspaces, D = [[0, M^*], [M, 0]], and C = J restricted to H_L.

Statement. For s = +1 or -1 the following are equivalent:

1. the triple is a product T_n x P_k with (s, n, k) = (+1, 7, 7) or (-1, 5, 1);
2. there is a complex-linear unitary U with U^2 = 1, U gamma = -gamma U, [U, a] = [U, a°] = [U, D] = 0 and UJ = sJU;
3. there is an A-bimodule unitary V: H_L -> H_R with V^\M = (V^\M)^* and (V^\C)^2 = s, where C and V^\C are antilinear.

Given U, the descended data are H_0 = H_L, d = (UD)|{H_L} and j = (UJ)/{H_L}, with d = d^*, j^2 = s and jd = s dj, so (j^2, jd/dj) is (+, +) for KO-7 and (-, -) for KO-5; order zero and first order descend. Replacing U by i gamma U exchanges [U, D] = 0 with {U, D} = 0 and preserves s. Equality of left and right module multiplicities is only part of the criterion: compatibility with D and J must also be checked.

Scope. Finite KO-6 triples. "Doubling" means this factorization of an even triple as an odd triple times a Clifford point; it is unrelated to the fermion doubling of the noncommutative Standard Model.

Proof. Lemma 06, Result 2; the proof is there. The identification (x, y) -> x + Uy of H_L + H_L with H puts a, D, gamma and J in the product form, with B = sigma_1 for (7, 7) and B = -i sigma_2 for (5, 1); conversely the product has U = 1 (x) sigma_1.

What it leaves open. The criterion does not say which sign a given triple selects: the electrodynamics triple admits both (06_counterexample_electrodynamics_both_signs.md). A factorization is a chosen removal of

a Clifford factor, and its identification with the dimension of a parent geometry is not supplied (lemma 06, Proposition 9).

Part S6 (The strict heavy-limit light spectrum of the Standard Model with one or three unpaired left neutrinos admits the KO-7 reduction and excludes the KO-5 reduction)

The strict heavy-limit light spectrum of the Standard Model with one or three unpaired left neutrinos admits the KO-7 reduction and excludes the KO-5 reduction

Status. Proposition (Propositions 5, 7 and 9 of lemma 06). Proved by Astra 6 (job 06); checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_nogo_19.md). Ledger row `106_nuL`, of kind no_go; the row also records No-go theorem 8, filed as 06_nogo_doubling.md. Item 2 of Theorem D of paper/towerless_intervals.tex.

Setting. The Standard Model representation of lemma 06, Result 5: F = C + C^3, H_p = (C^2 (x) F)_L + (C^2 (x) F)_R, H = H_p + H_pbar, J(p, pbar) = (K pbar, K p), gamma = diag(I_8, -I_8, -I_8, I_8), Dirac block S_p = [[0, Y^*], [Y, 0]] (CCM, Proposition 2.4 and Theorem 2.21). The strict projection of canonical section 3 removes nu_R, J nu_R and the complete up-type Dirac sector, restricts to B = {(lambda, Q_lambda, m)} = C + M_3(C), and keeps D_light = PSP on d_L, d_R, e_L, e_R, nu_L and their conjugates. In three generations, removing the three right-neutrino Majorana pairs and the top Dirac sector leaves 96 - 6 - 12 = 78 complex dimensions, and removing all up-type sectors leaves 54.

Statement.

1. In one generation, and in three generations under either removal, with the surviving left neutrinos unpaired and D_light = PSP, the light triple admits a KO-7 reduction and excludes KO-5.
2. The charged sectors (d and e, with any masses and with mixing within the species blocks) admit both reductions. On N = span(nu_L, J nu_L) both algebra actions are the character lambda and PSP|_N = 0, and gamma_N = sigma_3, J_N = sigma_1 K, U_N = sigma_1 is a KO-7 reduction; KO-5 is excluded by the neutral-sector criterion, and the descended spaces would have odd dimensions 9, 39 or 27.
3. Dropping the neutral sector, or retaining right neutrinos with purely Dirac masses, permits both reductions.
4. (Proposition 9) The heavy projection keeps KO-degree six, so the product with spacetime has degree 4 + 6 = 10 mod 8. A KO-5 reduction gives the formal representative 4 + 5 = 9; it exists for the charged truncation and for the pure-Dirac completion, which also admit KO-7. For the strict light spectrum only 4 + 7 = 11 is available among the proposed reductions. The odd reductions use J_{4+5} = (J_M gamma_M) (x) j_5 and J_{4+7} = J_M (x) j_7.

Scope. A statement about that finite triple; it concerns no parent manifold (ledger row). No finite-dimensional operation here changes heat dimension, so the loss of one dimension is at most a chosen Clifford-factor removal (Proposition 9).

Proof. Lemma 06, Results 5, 7 and 9; the proofs are there. Charged sectors reduce to copies of Result 3 by singular-value decomposition with bimodule unitaries; the neutral bimodule is inequivalent to the charged ones, so every permitted U preserves it, and Lemma 4 with M_N = 0 and odd r excludes KO-5.

What it leaves open. With particle preservation required, the strict light spectrum admits neither reduction (06_nogo_doubling.md, item 2). A proved metric collapse realizing the loss of one dimension is not supplied.

Part S7 (In a neutral KO-6 sector a KO-7 reduction always exists, and a KO-5 reduction exists exactly when every singular value of the mass matrix, zero included, has even multiplicity)

In a neutral KO-6 sector a KO-7 reduction always exists, and a KO-5 reduction exists exactly when every singular value of the mass matrix, zero included, has even multiplicity

Status. Lemma (Lemma 4 of lemma 06), with Propositions 6 and 7; ledger kind theorem. Proved by Astra 6 (job 06); checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand, with one amendment to Result 7 on the meaning of flavour mixing (historical/reviews/2026-11-03_terra_check_nogo_19.md). Ledger row `l06_neutral`. Item 2 of Theorem D of paper/towerless_intervals.tex; cited as "\supp{06}, Lemma 4 and Result 7" in paper/real_structure_note.tex.

Setting. A KO-6 sector H = H_+ + H_- with dim H_+ = dim H_- = r, on which both algebra actions are the same scalar character. In a basis of H_+ and its J-image, J = [[0, I], [I, 0]] K, and JD = DJ says M^T = M for the block M of D. The reductions are those of 06_doubling_criterion.md.

Statement.

1. (Lemma 4) A KO-7 reduction always exists. A KO-5 reduction exists exactly when every singular value of M, including zero, has even multiplicity. A single neutral Weyl state and its conjugate are a KO-7 doubling and no KO-5 doubling, even without a Dirac partner; so is a single Majorana pair, massive or massless.
2. (Proposition 6) For the reduced algebra B = C + M_3(C) of the strict heavy limit: with no neutral sector, or with nu_L, nu_R and a pure Dirac mass, both KO-5 and KO-7 exist; with only nu_L, J nu_L, or a single isolated Majorana pair, only KO-7; with a general Dirac-Majorana block, KO-7 always and KO-5 exactly under item 1. In one generation M_N = [[0, m], [m, R]] has singular values (sqrt(|R|^2 + 4|m|^2) +- |R|)/2, equal only at R = 0, so a nonzero right Majorana mass obstructs KO-5 in that sector and never KO-7.
3. (Proposition 7) With three generations and the algebra trivial on generation indices, retaining three right neutrinos gives M_N = [[0, Y_nu^T], [Y_nu, R]] with R^T = R. For R = 0 every singular value of Y_nu appears twice and both reductions exist; for general R, KO-7 always exists and KO-5 exists exactly when every singular value of this 6 x 6 matrix has even multiplicity. So nonzero Majorana masses are not an unconditional obstruction in several generations; the exact obstruction is the multiplicity condition.

Amendment (Terra 6.1). "Arbitrary flavor mixing" in Result 7 means arbitrary matrices within the displayed species blocks, with the heavy projection onto the specified top Dirac sector; mixing between inequivalent species is outside the displayed Dirac operator.

Proof. Lemma 06, Results 4, 6 and 7; the proofs are there. Takagi factorization, by a basis change preserving the form of J, reduces M to diag(mu_i) with mu_i >= 0, and V = I gives s = +1. For s = -1 the descended antiunitary j squares to -1 and commutes with d^2, so every eigenspace of d^2 has even dimension; conversely V = sigma_2 on pairs of equal entries satisfies the criterion with s = -1.

What it leaves open. The known escapes for KO-5 (Terra 6.1) are neutral-sector pairing and a symmetry enforcing even multiplicities. Generic Majorana masses exclude KO-5 (ledger row `l06_neutral`).

Part S8 (While the quaternions remain in the algebra no doubling exists for any D, and with particle preservation required an unpaired left neutrino prevents doubling)

While the quaternions remain in the algebra no doubling exists for any D, and with particle preservation required an unpaired left neutrino prevents doubling

Status. No-go theorem in its scope; two statements of lemma 06, No-go theorem 8 and the no-go theorem under the extra hypothesis [U, Q] = 0 in Result 6. Proved by Astra 6 (job 06); checked by Claude Opus 5.5 by hand (the dimension count) and by Terra 6.1 by hand, who listed the escapes below (historical/reviews/2026-11-03_terra_check_nogo_19.md). Ledger row `l06_nuL`; the particle-preserving form is also stated in Theorem 12 of lemma 08 (row `l08_pairing`). Item 2 of Theorem D of paper/towerless_intervals.tex.

Setting. The representation of lemma 06, Result 5, for g >= 1 generations, with the full algebra A = C + H + M_3(C); doubling involutions as in 06_doubling_criterion.md. Q is +1 on particles and -1 on antiparticles.

Statement.

1. (No-go theorem 8) With A = C + H + M_3(C), no involution satisfying the factorization criterion exists, for any admissible D; in particular at v = 0, when only the Majorana block remains. The central projection e_H = (0, 1_H, 0) has dim rho(e_H) H_+ = 8g and dim rho(e_H) H_- = 0, and an algebra-linear grading-reversing unitary would identify these spaces. Alternatively, with V_L = (3, 2)_{1/6} + (1, 2)_{-1/2} and V_R = (3, 1)_{2/3} + (3, 1)_{-1/3} + (1, 1)_{-1} + (1, 1)_0, the character [H_+] - [H_-] = I_G - conj(I_G) is nonzero, the coefficient of (3, 2)_{1/6} being g.
2. (No-go under [U, Q] = 0) An unpaired nu_L prevents doubling. Such a U maps particle states of positive grading to particle states of negative grading within the same bimodule, no particle nu_R exists, and the neutral involution of Result 5 maps nu_L to its conjugate, violating [U, Q] = 0.

Hypotheses. Item 1 needs H in the represented algebra with this representation. Item 2 needs [U, Q] = 0 and no particle bimodule of opposite grading matching nu_L.

Known escapes (Terra 6.1). For item 1: a change of algebra or representation, the even description, and boundary or defect realizations of chirality. For item 2: a matching particle bimodule of opposite grading.

Proof. Lemma 06, No-go theorem 8 and the no-go theorem in Result 6; the proofs are there.

What it leaves open. The obstruction points to the reduced algebra after the heavy projection, where doublings exist (06_heavy_limit_ko7_only.md), and to chirality carried by boundary conditions (lemmas 09 and 12). Physical chirality and doubling of the complete real triple are different tests (lemma 06).

Part S9 (Under pointwise reality the real structures of the chiral interval are diag(alpha, beta) K with signs (+, eps', +), they compress to the two-point truncation, and the reflection x -> l - x has sign s = eps' against them)

Under pointwise reality the real structures of the chiral interval are diag(alpha, beta) K with signs (+, eps', +), they compress to the two-point truncation, and the reflection x -> l - x has sign s = eps' against them

Status. Lemma (Lemmas 1, 2 and 3 of lemma 07), conditional on pointwise reality J f J^{-1} = conj(f). Proved by Claude Opus 5.5 (the Astra job stopped on the usage limit); checked by Claude Opus 5.5 by hand and by Astra 6 by hand in job 08 (lemma 08, Results 1 to 3), which confirmed them under pointwise reality and added the amendments below. Ledger rows `l07_real` (amended by `l08_ce_reality`) and `l07_reflection` (amended by `l08_product`). Item 3 of Theorem D of paper/towerless_intervals.tex.

Setting. I_l = [0, l], H = L^2(I_l, C^2), D = E d/dx with E = [[0, 1], [-1, 0]], Dom D = {psi in H^1 : R(0) = 0, L(l) = 0}, Gamma = diag(1, -1); D and Gamma are real. J is antiunitary with J f J^{-1} = conj(f) for f in C^infinity(I_l), J Dom D = Dom D, J^2 = eps, JD = eps' DJ, J Gamma = eps'' Gamma J.

Statement.

1. (Lemma 1) J = diag(alpha, beta) K with constant |alpha| = |beta| = 1, eps = eps'' = +1 and alpha = eps' beta. Both values of eps' occur and are exchanged by J -> J Gamma. With Connes' even convention eps' = +1 the chiral interval has KO-dimension 0; eps' = -1 is the extended convention 0_-. The chiral boundary conditions exclude every real structure of this form anticommuting with Gamma, while the bare line allows some (W = E, signs (-1, +1, -1)).
2. (Lemma 2) Every such J commutes with P = 1_{[0,E]}(|D|), and on the truncation span(u_L, u_R) of lemma 05 it induces J_2 = diag(alpha, beta) K, with D_2 = m_1 sigma_1, Gamma_2 = sigma_3 and signs (+1, eps', +1).
3. (Lemma 3) The reflection (sigma psi)(x) = a (R(l - x), L(l - x)) preserves Dom D, anticommutes with Gamma and commutes with constants; it commutes with D exactly when the phases on the two components agree, and sigma^2 = 1 forces a = +1 or -1. On the truncation sigma = a sigma_1, and sigma J = s J sigma with s = beta/alpha = eps'.

Amendments (job 08). Lemma 1 needs pointwise reality: without it J_ref = sigma_1 R_l K has the KO-6 signs (08_counterexample_pointwise_reality_first_order.md). Lemma 2 holds for the compressed operators; with the full compressed diagonal algebra and J_2, first order fails. The sign s = eps' is the sign against the interval's own J; in the internal-first KO-6 product the bare reflection has s = -1 for either eps' (08_product_reflection_sign.md). The reflection commutes with constants and generally not with the interval algebra, since sigma f sigma^{-1} = f(l - .).

Proof. Lemma 07, Lemmas 1 to 3, and lemma 08, Results 1 to 3; the proofs are there. Antilinearity with J f = conj(f) J gives J = W K for a unitary matrix function W, the derivative term forces W constant, and the two boundary conditions force W diagonal.

What it leaves open. The interval fixes s only as far as its real structure is fixed, so the 9 versus 11 ambiguity here is the J versus J gamma ambiguity of even real triples (Dabrowski-Dossena). Conjecture 07.4, that physical charge conjugation selects the sign in F_0 x interval, is false as stated (08_counterexample_both_signs_fixed_data.md).

Part S10 (A scalar-interval product with fixed D_F, J_F, Gamma_F and particle number admits particle-preserving involutions of both signs, so Conjecture 07.4 is false as stated)

A scalar-interval product with fixed D_F, J_F, Gamma_F and particle number admits particle-preserving involutions of both signs, so Conjecture 07.4 is false as stated

Status. Counterexample (Result 11 of lemma 08) to Conjecture 07.4. The conjecture is by Claude Opus 5.5 (lemma 07, Question 4); the counterexample is by Astra 6 (job 08), checked by Claude Opus 5.5 in Python (KO-6 signs, order zero and first order, both involutions admissible with s = -1 and s = +1) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger rows `l08_both_signs` and `conj07_4` (status refuted; refuted_by(conj07_4, l08_both_signs)). Item 5 of Theorem D of paper/towerless_intervals.tex.

The conjecture. For F = F_0 x (collapsing chiral interval) with the product conventions of lemma 06, the induced U = 1 (x) sigma has UJ = sJU with s fixed by eps' of the interval and the product grading twist, and the choice between 9 and 11 is fixed by whether J_M or J_M gamma_5 represents physical charge conjugation.

Setting. tau_i act on an internal particle/antiparticle space and sigma_i on the interval truncation, with

```
rho(a, b) = diag(a, b) (x) 1,   D_F = m tau_3 (x) sigma_1,   Gamma_F = tau_3 (x) sigma_3,
J_F = (tau_1 (x) sigma_3) K,    Q = tau_3 (x) 1,   m > 0.
```

This is Result 5 with D_0 = 0, Gamma_0 = tau_3, J_0 = tau_1 K; it has the standard KO-6 signs, and D_F commutes with the algebra, so first order holds.

Statement.

1. U_- = 1 (x) sigma_1 and U_+ = tau_3 (x) sigma_1 = Q U_- are grading-reversing unitary involutions commuting with D_F, both algebra actions and Q.
2. U_- J_F = -J_F U_- and U_+ J_F = J_F U_+.
3. The unitary W = diag(1, sigma_3) on the particle/antiparticle blocks carries the data to D_F' = m 1 (x) sigma_1, J_F' = (tau_1 (x) 1) K, preserving Gamma_F, Q and rho, with U_-' = diag(sigma_1, -sigma_1) and U_+' = diag(sigma_1, sigma_1): the electrodynamics counterexample of lemma 06 written as an interval product.
4. Tensoring with a fixed spacetime J_M leaves both possibilities available.

Scope. The claim refuted is that physical spacetime charge conjugation, together with particle number, selects the reduction. Selecting the bare reflection 1 (x) sigma is an additional restriction and gives s = -1 in the internal-first convention (08_product_reflection_sign.md).

Proof. Lemma 08, Result 11; the proof is there.

What it leaves open. What selects the lift of the L/R reflection. The neutral sector restricts s = -1 (06_neutral_sector_criterion.md), and Majorana terms break the sign pairing of 08_particle_preservation_pairs_signs.md.

Part S11 (Chiral boundary conditions alone do not force KO-dimension 0, and the two-point compression with its full diagonal algebra and inherited real structure fails first order)

Chiral boundary conditions alone do not force KO-dimension 0, and the two-point compression with its full diagonal algebra and inherited real structure fails first order

Status. Counterexample (Result 4 of lemma 08). Proved by Astra 6 (job 08); checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l08_ce_reality`; it amends `l07_real`. Cited after Theorem A of paper/towerless_intervals.tex ("\supp{08}") and in item 4 of Theorem D.

Setting. The chiral interval of lemma 07: D_I = E d/dx, R(0) = 0, L(l) = 0, Gamma_I = diag(1, -1), R_l psi(x) = psi(l - x). Its minimal truncation D_2 = m sigma_1, m = pi/(2l) > 0, Gamma_2 = sigma_3, with the compressed algebra A_2 = {diag(a, b)} (compression sends 1 to 1 and cos(pi x/l) to sigma_3/2).

Statement.

1. Without pointwise reality, the antiunitary J_ref = sigma K with sigma = sigma_1 R_l preserves the domain and has the standard KO-6 signs: J_ref^2 = 1, J_ref D_I = D_I J_ref, J_ref Gamma_I = -Gamma_I J_ref. It implements f -> conj(f)(l - .), the opposite action is still scalar multiplication, and order zero and first order hold.
2. With the inherited J_2 = diag(alpha, beta) K and e = diag(1, 0), one has e° = e and [[D_2, e], e°] = D_2 != 0, so the two-point data with the full diagonal algebra are not a real spectral triple satisfying first order when m > 0.
3. sigma_1 e sigma_1 = 1 - e, so the reflection is not an A_2-module map.

Scope. Item 1 violates exactly the pointwise reality hypothesis of Lemma 07.1. Items 2 and 3 concern the full compressed diagonal algebra. Restricting the interval algebra to scalar constants makes the compressed

interval a real triple and its reflection algebra-linear; lemma 08 adopts that restriction for its products as an additional assumption about the collapsed model.

Proof. Lemma 08, Result 4; the proof is there, by direct computation.

What it leaves open. The first-order failure has two escapes developed later: the inner fluctuations without order one of arXiv:1304.7583, which apply to these data as they stand (lemma 19; 19_fluctuations_two_point_data.md), and the doubled parent, where [[D_F, e], e°] = 0 at the price of doubling (lemma 16; 16_coarse_correspondence_order_one.md).

Part S12 (Literal collapse of the massless chiral interval in its product with a finite KO-6 triple leaves an empty light sector at every fixed cutoff)

Literal collapse of the massless chiral interval in its product with a finite KO-6 triple leaves an empty light sector at every fixed cutoff

Status. No-go theorem in its scope (No-go theorem 8 of lemma 08). Proved by Astra 6 (job 08); checked by Claude Opus 5.5 in Python (D_F^2 = D_0^2 (x) 1 + 1 (x) D_I^2 exactly) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l08_nogo`, depending on `l05_limits`. Part of Proposition A.1 of paper/towerless_intervals.tex.

Setting. The heavy-limit model with exactly the product Dirac operator of Result 5, D_F = D_0 (x) 1 + Gamma_0 (x) D_I, with the massless chiral interval or its minimal truncation and m = pi/(2l); a fixed energy cutoff Lambda.

Statement.

1. Anticommutation of D_0 and Gamma_0 gives D_F^2 = D_0^2 (x) 1 + 1 (x) D_I^2.
2. The chiral interval and its minimal truncation satisfy D_I^2 >= pi^2/(4l^2), so |D_F| >= pi/(2l).
3. Hence 1_{[0,Lambda]}(|D_F|) = 0 for l < pi/(2 Lambda): at every fixed cutoff the light subspace is zero for small l.
4. The same holds for the alternative product convention, by unitary equivalence.

Hypotheses and escapes. The product operator exactly as in Result 5, with a massless interval and no boundary coupling. The lemma states that retaining a light sector requires additional dynamics or a modified limiting prescription. The tuned bulk mass of lemma 09 (Theorem 9.9; 09_two_limits.md) and the scaled boundary coupling of lemma 12 (Theorem 12.7; 12_light_spectrum_towerless_limit.md) are such prescriptions.

Proof. Lemma 08, No-go theorem 8; the proof is there.

What it leaves open. The exact product supplies an algebraic model at each finite l, and neither 9 nor 11 is established by it as the dimension of a surviving heavy-W/Z limit (lemma 08, consequence for 9 versus 11).

Part S13 (If a particle-number involution Q commutes with D, Gamma, the algebra and its opposite and anticommutes with J, particle-preserving doubling involutions come in pairs U, QU of opposite sign)

If a particle-number involution Q commutes with D, Gamma, the algebra and its opposite and anticommutes with J, particle-preserving doubling involutions come in pairs U, QU of opposite sign

Status. Theorem (Result 12 of lemma 08). Proved by Astra 6 (job 08); checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l08_pairing`. Item 5 of Theorem D of paper/towerless_intervals.tex.

Setting. A standard KO-6 triple and a self-adjoint involution Q, +1 on particles and -1 on antiparticles, with [Q, D] = [Q, Gamma] = [Q, a] = [Q, a°] = 0 and QJ = -JQ. Admissible involutions U as in the factorization criterion (06_doubling_criterion.md).

Statement.

1. For an interval product with Q = Q_0 (x) 1, the bare reflection satisfies [1 (x) sigma, Q] = 0.
2. If an admissible U preserves particle number, then QU is another particle-preserving admissible involution with the opposite sign: from UJ = sJU, (QU)J = sQJU = -sJ(QU), and [QU, Q] = 0.
3. So particle preservation selects no sign in a particle-number-conserving sector.
4. Conversely, particle-preserving doubling requires the particle spaces of opposite grading to be isomorphic as bimodules, since U restricts to such an isomorphism. An unpaired left-handed neutrino with no matching right-handed particle in its bimodule therefore obstructs particle-preserving doubling.

Hypotheses. The assumption [Q, D] = 0 is essential to the pairing argument, and Majorana terms can violate it. The formal interval product builds in the missing partner, so it cannot represent the strict light spectrum without changing its particle content.

Proof. Lemma 08, Result 12; the proof is there: [Q, U] = 0 gives (QU)^2 = 1, and the remaining properties follow from those of Q and U.

What it leaves open. With Majorana terms, the sign is controlled by the neutral-sector multiplicities of lemma 06 (06_neutral_sector_criterion.md).

Part S14 (In the internal-first KO-6 product of a finite KO-6 triple with the chiral interval, the bare interval reflection has sign s = -1, formally 4 + 5 = 9, for either interval convention)

In the internal-first KO-6 product of a finite KO-6 triple with the chiral interval, the bare interval reflection has sign s = -1, formally 4 + 5 = 9, for either interval convention

Status. Theorem (Result 5 of lemma 08), with Proposition 6 on the grading and the lift of the reflection. Proved by Astra 6 (job 08); checked by Claude Opus 5.5 in Python (numpy: of the eight choices (t, a, b) exactly two give the KO-6 signs, both with s = -1) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger rows `l08_product` (it amends `l07_reflection`) and `l08_lift`. Item 3 of Theorem D of paper/towerless_intervals.tex.

Setting. F_0 = (A_0, H_0, D_0, J_0, Gamma_0) a finite real KO-6 triple; the interval or its scalar-algebra truncation, with J_I = diag(alpha, beta) K and J_I D_I = t D_I J_I (lemma 07); the internal-first

product D_F = D_0 (x) 1 + Gamma_0 (x) D_I, Gamma_F = Gamma_0 (x) Gamma_I, and the grading-twisted real structures J_{a,b} = (J_0 Gamma_0^a) (x) (J_I Gamma_I^b), a, b in {0, 1}.

Statement.

1. (Theorem 5) The standard KO-6 conditions force a = 0 and t(-1)^b = -1, and the induced reflection U = 1 (x) sigma satisfies U J_F = -J_F U. For t = +1 the required twist is b = 1, J_F = J_0 (x) (J_I Gamma_I), the formula of lemma 06; for t = -1 it is b = 0. Changing J_I to J_I Gamma_I while adjusting the twist leaves J_F unchanged up to phase.
2. A KO-0 factor leaves the KO-degree unchanged, so F_0 must already have degree six in this even-even construction; an odd internal triple times an even interval stays odd.
3. (Proposition 6) The interval grading alone cannot be the total KO-6 grading for such J; the product grading must be Gamma_0 (x) Gamma_I. The twisted lift U_c = Gamma_0^c (x) sigma has U_c J_F = (-1)^c t(-1)^b J_F U_c, and [U_1, D_F] = 0 iff D_0 = 0. Transport between equivalent product conventions preserves s, so keeping the bare symbol 1 (x) sigma after a change of convention can mean choosing a different involution.
4. Multiplying J_F by Gamma_F also flips s, and changes the signs to (-, -, -), outside the standard KO-6 convention.

Scope. The bare reflection 1 (x) sigma in the internal-first convention with the standard KO-6 signs. The general formula s = t(-1)^b holds, with t and b tied together by those signs.

Proof. Lemma 08, Results 5 and 6; the proofs are there. J_{a,b}^2 = (-1)^a, J_{a,b} (Gamma_0 (x) D_I) = -t(-1)^b (Gamma_0 (x) D_I) J_{a,b}, and sigma J_I = t J_I sigma, sigma Gamma_I = -Gamma_I sigma give U J_{a,b} = t(-1)^b J_{a,b} U.

What it leaves open. Selecting the bare reflection is an additional restriction: a product with fixed J_F also admits a particle-preserving involution of sign +1 (08_counterexample_both_signs_fixed_data.md). The literal collapse of the interval leaves no light fermions, so neither 9 nor 11 is established as the dimension of a surviving heavy limit (08_nogo_literal_collapse.md).

Part S15 (The spacetime grading twists J_M versus J_M gamma_M in the two odd reductions do not select the finite involution, since both reconstructions use the same J_M)

The spacetime grading twists J_M versus J_M gamma_M in the two odd reductions do not select the finite involution, since both reconstructions use the same J_M

Status. Proposition (Result 9 of lemma 08), with Proposition 10 on the identification of charge conjugation. Proved by Astra 6 (job 08); checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger rows `l08_spacetime` and `l08_chargeconj`. First sentence of item 5 of Theorem D of paper/towerless_intervals.tex.

Setting. A four-dimensional Euclidean spacetime triple with J_M^2 = -1, J_M D_M = D_M J_M, J_M gamma_M = gamma_M J_M; an odd reduction (d, j_s) with j_s^2 = s, j_s d = s d j_s of a finite KO-6 triple F = T_s x P_k with J_F = j_s (x) (B_s K), B_- = -i sigma_2, B_+ = sigma_1; D_{M x T_s} = D_M (x) 1 + gamma_M (x) d.

Statement.

1. (Proposition 9) The even-odd convention of lemma 06 requires J_{M x T_s} = (J_M gamma_M^{a_s}) (x) j_s with (-1)^{a_s} = s, so J_{4+5} = (J_M gamma_M) (x) j_5 and J_{4+7} = J_M (x) j_7; this real structure squares to -s, giving KO-1 for s = -1 and KO-3 for s = +1.

2. The unitary V = P_+^M (x) 1 (x) 1 + P_-^M (x) 1 (x) sigma_1 identifies (M x T_s) x P_k with M x F and sends (J_M gamma_M^{a_s}) (x) j_s (x) B_s K to J_M (x) J_F, the factor gamma_M^{a_s} cancelling because sigma_1 B_s = s B_s sigma_1. So both reconstructions use the same original J_M, and the different spacetime factors are required by the product rules.
3. (Proposition 10) In the conventions of Chamseddine-Connes-Marcolli (sections 3.1, 3.2, 3.5.2) and of van den Dungen-van Suijlekom (Example 2.1, Definition 2.3), the Euclidean charge conjugation is J_M, with signs (-, +, +), and J_M gamma_5 is the other even convention. Lorentzian charge conjugation, C_L = -i gamma_E^0 C_E in a common basis with C_L^2 = 1 and C_L gamma_5 = -gamma_5 C_L, is neither of the two Euclidean operators.

Scope. The identification of charge conjugation fixes the starting spacetime convention; by item 2 it does not fix the finite involution.

Proof. Lemma 08, Results 9 and 10; the proofs are there.

What it leaves open. The choice of the lift of the L/R reflection remains additional data (08_counterexample_both_signs_fixed_data.md).

Part S16 (The scalar-algebra truncated product F_0 x I_l is exactly T_5(m) x P_1, with d_5 = D_0 + m Gamma_0 and j_5 = J_0 Gamma_0)

The scalar-algebra truncated product F_0 x I_l is exactly T_5(m) x P_1, with d_5 = D_0 + m Gamma_0 and j_5 = J_0 Gamma_0

Status. Theorem (Result 7 of lemma 08). Proved by Astra 6 (job 08); checked by Claude Opus 5.5 in Python (numpy: R^* U D_F R = D_0 + m Gamma_0, R^* U J_F R = J_0 Gamma_0, KO-5 signs) and by Terra 6.1 by hand, who noted that it concerns the scalar-algebra truncation (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l08_reconcile`. Item 4 of Theorem D of paper/towerless_intervals.tex.

Setting. F_0 a finite real KO-6 triple as in Result 5; the minimal truncation of the chiral interval with its algebra restricted to scalars, and the phase chosen so that

```
D_F = D_0 (x) 1 + m Gamma_0 (x) sigma_1,   Gamma_F = Gamma_0 (x) sigma_3,   J_F = J_0 (x) (sigma_3
↪ K),
```

with U = 1 (x) sigma_1 and P_1 = (C, C, 0, K) (point algebra R for real algebras).

Statement.

1. F is isomorphic to T_5(m) x P_1, with T_5(m) = (A_0, H_0, d_5 = D_0 + m Gamma_0, j_5 = J_0 Gamma_0); d_5 is self-adjoint, j_5^2 = -1, j_5 d_5 = -d_5 j_5, [d_5, a] = [D_0, a], and order zero and first order descend.
2. The unitary R x = P_+^0 x (x) e_+ + P_-^0 x (x) e_- onto H_{F,+}, with the identification (x, y) -> Rx + URy, gives D_F = [[0, d_5], [d_5, 0]], Gamma_F = diag(1, -1) and J_F(x, y) = (-j_5 y, j_5 x): the odd-odd product with auxiliary matrix B = -i sigma_2.
3. More generally, any admissible involution of a standard KO-6 triple gives d = (UD)|{H+}, j = (UJ)|{H+} with j^2 = s, jd = s dj, so F = T_5 x P_1 for s = -1 and F = T_7 x P_7 for s = +1, with auxiliary matrices -i sigma_2 and sigma_1.
4. The two-dimensional space of the even KO-0 interval becomes the auxiliary doubling space after reorganizing the product, and the interval mass enters d_5; the point itself has a one-dimensional Hilbert space and zero Dirac operator.

Scope. The scalar-algebra truncation. With the full compressed two-sheet algebra the inherited data fail first order (08_counterexample_pointwise_reality_first_order.md). Because the point has zero Dirac operator, the distinction between P_1 and P_7 includes a choice of KO label.

Proof. Lemma 08, Result 7; the proof is there, by direct calculation of R^* U D_F R and R^* U J_F R.

What it leaves open. The reconciliation holds at each finite l. The literal collapse l -> 0 of this product leaves no light sector at fixed energy (08_nogo_literal_collapse.md), so retaining one needs additional dynamics or a modified limiting prescription.

Part S17 (On the massive chiral interval the index is +1, -1 or 0 for the boundary types (a), (b), (c), independent of the bulk mass and the length, and equivariantly it is [V_L] - [V_R])

On the massive chiral interval the index is +1, -1 or 0 for the boundary types (a), (b), (c), independent of the bulk mass and the length, and equivariantly it is [V_L] - [V_R]

Status. Theorem (Theorem 9.2 and Corollary 9.3). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 against a first-order discretization (N = 4000) and by Terra 6.1 by hand, effort max: Theorem 9.2 correct, Corollary 9.3 correct with the amendment below (report appended to lemma 09). Ledger row `l09_index`.

Setting. I = [0, l], H = L^2(I; C^2) with psi = (L, R), Gamma = diag(1, -1), A = -d/dx + M with M real, D_M psi = (A^dagger R, A L), that is (D psi)_L = R' + M R and (D psi)_R = -L' + M L. Boundary types: (a) R(0) = R(l) = 0; (b) L(0) = L(l) = 0; (c) R(0) = 0 and L(l) = 0, the chiral interval of lemma 05. For each type D_M is self-adjoint on the H^1 functions with that condition and has compact resolvent (Lemma 9.1, with the graph-norm estimate ||psi'|| <= ||D_M psi|| + |M| ||psi||).

Statement.

1. (Theorem 9.2) For every M in R and l > 0: in type (a), ker D = C (e^{Mx}, 0) and ind = dim ker_{Gamma=+1} - dim ker_{Gamma=-1} = +1; in type (b), ker D = C (0, e^{-Mx}) and ind = -1; in type (c), ker D = 0 and ind = 0.
2. (Corollary 9.3) Let a compact group G act on H_G = sum_rho V_rho (x) L^2(I; C^2), a sum of finitely many finite-dimensional representations V_rho, with boundary type tau(rho) and mass M_rho on each summand. The G-index of D in R(G) is the sum of [V_rho] over tau(rho) = a minus the sum of [V_rho] over tau(rho) = b, independent of every M_rho and of l.
3. Assigning (a) to the summands of V_L and (b) to those of V_R realizes I_G = [V_L] - [V_R]; vectorlike pairs of type (c) contribute 0.

Amendment (Terra 6.1). The direct sum must be finite. With infinitely many summands the operator need not be Fredholm: for l = 1, G = S^1, V_j(z) = z^j with j >= 2 and M_j = -j, all of type (c), the direct sum has no kernel and is not bounded below, so its R(G)-valued index is undefined.

Proof. Lemma 09, Theorem 9.2 and Corollary 9.3; the proofs are there. D psi = 0 forces L = alpha e^{Mx} and R = beta e^{-Mx}, and since exponentials never vanish a condition at either end kills the corresponding coefficient.

What it leaves open. Chirality is placed by the boundary conditions, as an input. The light spectrum of type (c) with a bulk mass is computed next (09_spectrum_with_bulk_mass.md), and boundary Yukawa couplings that deform types (a) + (b) into type (c) are lemma 12.

Part S18 (The light pair of the massive chiral interval truncates to the Connes-Lott two-point space with mass lambda_0 and distance 1/lambda_0)

The light pair of the massive chiral interval truncates to the Connes-Lott two-point space with mass lambda_0 and distance 1/lambda_0

Status. Theorem (Theorem 9.6). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 numerically (lemma 09) and by Terra 6.1 by hand, effort max: correct, with a correction to the adjacent remark on how the distance depends on l (report appended to lemma 09). Ledger row `l09_lightpair`, depending on `l05_truncation`. Theorem B of paper/towerless_intervals.tex.

Setting. Condition (c) with M = -mu and c = mu l > 1; P_0 the spectral projection onto +-lambda_0; u_L = (L_0, 0)/||L_0|| and u_R = (0, R_0)/||R_0|| with the profiles of Lemma 9.5 (09_spectrum_with_bulk_mass.md).

Statement.

1. P_0 D P_0 = lambda_0 sigma_1 and P_0 Gamma P_0 = sigma_3.
2. The compression of C(I) is Q(f) = diag(int f |u_L|^2, int f |u_R|^2), and its range is C + C, since the mean of x is below l/2 for |u_L|^2 and above l/2 for |u_R|^2.
3. The truncated end states are at Connes distance 1/lambda_0, and their pullbacks are smeared probability measures, distinct from endpoint evaluations.
4. The two-sheet distance is therefore 1/(fermion mass), as in the Standard Model finite geometry. It depends on l through lambda_0, and by Lemma 9.7, 1/lambda_0 > l for c > 1. At M = 0 it was 2l/pi (05_connes_lott_minimal_truncation.md).

Scope. Complex data. With the inherited real structure the full diagonal algebra fails first order (ledger statement of `l09_lightpair`; 08_counterexample_pointwise_reality_first_order.md).

Proof. Lemma 09, Theorem 9.6; the proof is there. D u_L = lambda_0 u_R, pointwise orthogonality of the L and R components makes Q(f) diagonal, and |||[lambda_0 sigma_1, diag(a, b)]|| = lambda_0 |a - b|.

What it leaves open. As whole triples the tuned family converges in the spectral propinquity to (C, C^2, m_f sigma_1) and stays at positive distance from the Connes-Lott triple (15_gap_criterion.md, Corollary 15.5). The compression and the whole triple are different objects, and both statements hold.

Part S19 (For the mixed condition with bulk mass -mu and mu l > 1, the chiral interval has exactly one light pair lambda_0 = mu sech(kappa_0 l), about 2 mu e^{-mu l}, below the bulk threshold mu)

For the mixed condition with bulk mass -mu and mu l > 1, the chiral interval has exactly one light pair lambda_0 = mu sech(kappa_0 l), about 2 mu e^{-mu l}, below the bulk threshold mu

Status. Theorem (Theorem 9.4, with Lemma 9.5). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 against a first-order discretization (N = 4000; table in lemma 09; at mu = 6, 2 mu e^{-mu l} = 0.029745 against lambda_0 = 0.029747) and by Terra 6.1 by hand, effort max: Theorem 9.4 correct with amendment (root indices, mu > 0, the case M = 0), and a gap in Lemma 9.5 closed by the referee (report appended to lemma 09). Ledger row `l09_lightpair`, shared with Theorem 9.6. Theorem B of paper/towerless_intervals.tex.

Setting. Condition (c) of lemma 09, R(0) = 0 and L(l) = 0, with D_M as in 09_index_from_boundary_conditions.md; c = mu l.

Statement. The spectrum is symmetric under lambda -> -lambda, all eigenvalues are simple, and 0 is not an eigenvalue.

1. M = -mu with mu > 0 and c > 1: exactly one positive eigenvalue lies below mu, lambda_0 = mu sech(kappa_0 l), with kappa_0 in (0, mu) the unique positive root of tanh(kappa l) = kappa/mu. The other positive eigenvalues are sqrt(mu^2 + k_n^2), n >= 1, with k_n the unique root of tan(kl) = k/mu in (n pi/l, (n + 1/2) pi/l).
2. M = -mu and c < 1: no eigenvalue in (0, mu]; the positive eigenvalues are sqrt(mu^2 + k_n^2), n >= 0, with k_0 in (0, pi/(2l)). At c = 1 the lowest eigenvalue is exactly mu.
3. M = +mu > 0: every positive eigenvalue is sqrt(mu^2 + k_n^2), k_n in ((n - 1/2) pi/l, n pi/l), n >= 1, so all exceed sqrt(mu^2 + (pi/(2l))^2).
4. M = 0: the spectrum is +-(n + 1/2) pi/l, n >= 0.
5. (Lemma 9.5) For c > 1, mu sech(mu l) <= lambda_0 <= 2 mu e^{-kappa_0 l}, and lambda_0 = 2 mu e^{-mu l} (1 + o(1)) as mu l -> infinity. The pair is spanned by L_0 = sinh(kappa_0 (l - x)) and R_0 = A L_0/lambda_0 = sinh(kappa_0 x) = L_0(l - x): mirror images with ||R_0|| = ||L_0||, decreasing away from their walls, localized on the scale 1/kappa_0 as kappa_0 l -> infinity. The localization is not uniform as c -> 1, where the profiles approach multiples of l - x and x.

Proof. Lemma 09, Theorem 9.4 and Lemma 9.5; the proofs are there. Eliminating R gives -L'' + M^2 L = lambda^2 L with L(l) = 0 and L'(0) = M L(0); the roots are counted by concavity of tanh t - t/c and by the monotonicity of tan t - t/c on each half-period. Remark 9.4a gives ||D_M psi||^2 = ||D_0 psi||^2 + M^2 ||psi||^2 + M (|R(l)|^2 + |L(0)|^2) on the domain, a one-line proof of item 3.

What it leaves open. The exponential sensitivity of lambda_0 to mu l is the localization mechanism for Yukawa hierarchies (lemma 09, final remark); its flavour structure is lemma 12. The truncation of the pair is 09_light_pair_is_connes_lott.md.

Part S20 (As l -> infinity at fixed mu the interval decompactifies with chiral wall modes and a one-dimensional Weyl law above mu; as l -> 0 at fixed light mass a unique mu(l) keeps the light pair while the tower diverges)

As l -> infinity at fixed mu the interval decompactifies with chiral wall modes and a one-dimensional Weyl law above mu; as l -> 0 at fixed light mass a unique mu(l) keeps the light pair while the tower diverges

Status. Theorem (Theorems 9.8 and 9.9), with Corollary 9.10. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 numerically (the secular equation and the collapse table for m_f = 1, down to l = 0.001) and by Terra 6.1 by hand, effort max: a gap in Theorem 9.8 closed by the referee, Theorem 9.9 correct, Corollary 9.10 wrong as first stated and corrected (report appended to lemma 09). Ledger row `l09_limits`. Theorem B of paper/towerless_intervals.tex. The ledger records Corollary 4.1 of Farsi-Latrémolière as prior work for the fixed-mu collapse, with the tuned collapse outside its hypotheses; the tuned collapse mu(l) was not found in print (lemma 09, Literature).

Setting. Condition (c) with M = -mu, lambda_0 and kappa_0 as in 09_spectrum_with_bulk_mass.md. For c > 1, s(c) is the positive root of tanh s = s/c and F(c) = (lambda_0 l)^2 = c^2 sech^2 s(c), strictly decreasing on (1, infinity) from 1 to 0 (Lemma 9.7).

Statement.

1. (Theorem 9.8, l -> infinity at fixed mu) lambda_0 <= 2 mu e^{-kappa_0 l} -> 0 with kappa_0 -> mu. The normalized profiles of L_0 near x = 0 and of R_0 near x = l converge, uniformly on compact sets of the wall distance y and in L^2([0, infinity)) after extension by zero, to sqrt(2 mu) e^{-mu y}: exact chiral zero modes of the two half-line operators, which are not zero modes at finite l. The rest

of the spectrum obeys #{n : lambda_n <= E} = (l/pi) sqrt(E^2 - mu^2) + O(1) for E > mu, a one-dimensional Weyl law (both signs double the leading term).

2. (Theorem 9.9, l -> 0 at fixed m_f > 0) For each l in (0, 1/m_f) there is a unique mu(l) > 1/l with lambda_0 = m_f. As l -> 0, c(l) = mu(l) l -> infinity with c = -W_{-1}(-m_f l/2) + o(1) = L + log L + o(1), L = log(2/(m_f l)); the next eigenvalue exceeds sqrt(mu(l)^2 + (pi/l)^2) -> infinity; the compression stays Connes-Lott with mass m_f while the interval collapses to a point in Gromov-Hausdorff distance (d_GH = l/2). As whole triples, the family converges in the spectral propinquity to (C, C^2, m_f sigma_1), at distance at least min(sqrt(2)/2, (1/m_f - l)/2) from the Connes-Lott triple.
3. (Corollary 9.10, l -> 0 at fixed mu) Eventually c < 1, every eigenvalue has absolute value greater than mu, and the smallest absolute value is (pi/(2l))(1 + o(1)). No light fermion survives; this extends lemma 05, Part 4 to M != 0.

Scope. The log log correction of item 2 is specific to this localization model. The readings 10 -> 11 and 10 -> 9 of the two limits are proposed interpretations (ledger statement of `l09_limits`).

Proof. Lemma 09, Theorems 9.8 and 9.9 and Corollary 9.10; the proofs are there. Item 2 applies Lemma 9.7 to F(c) = (m_f l)^2 in (0, 1), and Lemma 9.5 gives m_f = 2 (c/l) e^{-c} (1 + o(1)).

What it leaves open. Compatibility of the two readings with the KO factorizations of lemmas 06 and 08 is not established. The gauge sector of the collapsing family is lemmas 13 and 14, where bounded light fermions cannot match the Neumann-Dirichlet tower (14_nogo_bounded_light_fermions.md).

Part S21 (The level-one Berezin-Toeplitz compression on CP^3 sends the twistor coordinates x_a to Gamma_a/5, with Q(x_a)^2 = I/25 and Q(x_a^2) = I/5)

The level-one Berezin-Toeplitz compression on CP^3 sends the twistor coordinates x_a to Gamma_a/5, with Q(x_a)^2 = I/25 and Q(x_a^2) = I/5

Status. Proposition (Proposition 10.1). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 by Monte Carlo (2 x 10^6 samples; deviations at most 0.0009 and 0.0006, at a sampling error of about 10^-3) and by Terra 6.1 by hand, effort max: the three identities correct, and an index error in the displayed fourth moment of the proof corrected (report appended to lemma 11). Ledger row `l10_toeplitz`. It answers Part A of parked job 04, which the corpus had labelled a checked calculation without a script. Used in Proposition H of paper/towerless_intervals.tex ("\supp{10} and \supp{16}").

Setting. Gamma_1, …, Gamma_5 Hermitian generators of Cl(5) on C^4, Gamma_a Gamma_b + Gamma_b Gamma_a = 2 delta_ab. For [w] in CP^3, x_a([w]) = w^* Gamma_a w/w^* w, and the level-one Berezin-Toeplitz compression is

```
Q(f) = 4 int_{CP^3} f([w]) P_w d mu([w]),
```

with P_w the rank-one projector onto w and mu the Fubini-Study probability measure, so that Q(1) = I.

Statement.

1. Q(x_a) = Gamma_a/5.
2. Q(x_a)^2 = I/25.
3. Q(x_a^2) = I/5.
4. In particular Q is not multiplicative.
5. More generally (referee report appended to lemma 11), Q(w^* A w/w^* w) = (tr(A) I + A)/5 for every 4 x 4 matrix A; lemma 16 uses this for Hermitian A.

Proof. Lemma 10, Proposition 10.1; the proof is there. Lift to w uniform on S^7 in C^4; Wick's theorem gives E[w_j w_k conj(w_i) conj(w_l)] = (delta_ij delta_kl + delta_jl delta_ki)/20 and the sixth moment as the sum over the six matchings divided by 120; tr Gamma_a = 0 and Gamma_a^2 = I.

What it leaves open. These compressions generate M_4(C) at level one with no limit taken (16_compressed_algebras_noncommutative.md), and the real functions of S^4 generate M_2(H) (16_twistor_level_one_m2h_m4c.md). Part B of job 04, the Berger gauge normalizations, stays parked (lemmas/README.md).

Part S22 (Under the Chamseddine-Connes-Marcolli relation 8 M_W^2 = sum N_f m_f^2, the W mass satisfies m_max/(2 sqrt 2) <= M_W <= sqrt(N_g) m_max)

Under the Chamseddine-Connes-Marcolli relation 8 M_W^2 = sum N_f m_f^2, the W mass satisfies m_max/(2 sqrt 2) <= M_W <= sqrt(N_g) m_max

Status. Proposition (Proposition 11.1), conditional on the CCM relation at the matching scale (ledger hypothesis `ccm_relation`). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand, effort max: correct (report appended to lemma 11). Ledger row `l11_bounds`. Proposition E of paper/towerless_intervals.tex. CCM 2007 contains the relation (equation (1.5), Lemma 4.6, equation (4.16)); the proposition applies it to the heavy-W prescription of the corpus.

Setting. The almost-commutative Standard Model spectral action with the gauge normalization g^2 f_0/(2 pi^2) = 1/4 and the Higgs normalization by a = Tr(Y_nu^* Y_nu + Y_e^* Y_e + 3(Y_u^* Y_u + Y_d^* Y_d)), which give at the matching scale

```
8 M_W^2 = sum over generations of (m_nu^2 + m_e^2 + 3 m_u^2 + 3 m_d^2),
```

the m being Dirac masses (CCM, section 5; H the unrescaled Higgs field). Majorana masses do not enter a. N_g generations at fixed gauge coupling; m_max is the largest Dirac mass.

Statement.

1. m_max/(2 sqrt 2) <= M_W <= sqrt(N_g) m_max.
2. In particular M_W -> infinity if and only if m_max -> infinity.

Scope (referee). CCM derive the coefficient 8 for three generations. For N_g != 3 the coefficient 8 is an assumption beyond the cited normalization: replicating the representations N_g times with the same normalization gives sum N_f m_f^2 = (8 N_g/3) M_W^2. The neutrino terms are Dirac Yukawa parameters, not eigenmasses of the seesaw matrix. The relation belongs to the truncated asymptotic (heat-kernel) action at the matching scale, and small masses relative to Lambda alone do not make it exact for the finite-cutoff action.

Proof. Lemma 11, Proposition 11.1; the proof is there: every term has weight at least 1, so 8 M_W^2 >= m_max^2, and each generation has weights 1 + 1 + 3 + 3 = 8, so 8 M_W^2 <= 8 N_g m_max^2.

What it leaves open. At fixed cutoff, m_max <= eps Lambda implies M_W <= sqrt(N_g) eps Lambda, so a heavy-W limit needs Lambda -> infinity jointly or a calculation of the finite-cutoff action (lemma 11, amendment on the regime); lemma 18 carries out that calculation.

Part S23 (The heavy-W prescription with every fermion mass fixed is incompatible with the CCM relation, and a heavy-W limit forces m_max -> infinity)

The heavy-W prescription with every fermion mass fixed is incompatible with the CCM relation, and a heavy-W limit forces m_max -> infinity

Status. No-go theorem in its scope (Corollary 11.2), conditional on the CCM relation. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 by hand, by Terra 6.1 by hand (effort max; the conclusion as first

written was too strong and was corrected; report appended to lemma 11) and by Astra 6 by hand (lemma 20, Result 9, a consensus check). Ledger row `l11_nogo`, depending on `l11_bounds`. Proposition E of paper/towerless_intervals.tex.

Setting. The heavy-W prescription of canonical 14.1 and research_state 9: v -> infinity with y_f v fixed for every fermion and the gauge couplings fixed, so that the Yukawa couplings tend to zero. The CCM relation and the bounds of 11_heavy_w_mass_bounds.md, with N_g generations.

Statement.

1. Under the CCM relation the right-hand side stays bounded while M_W -> infinity, so the prescription is impossible in the spectral-action family.
2. A heavy-W/Z limit forces m_max -> infinity, with m_max >= M_W/sqrt(N_g).
3. Some fixed species is then unbounded along a subsequence; no single fixed species need diverge along the whole family. Example: M_W = t, theta(t) = (pi/4)(1 + sin t), m_e = 2 sqrt 2 t cos theta(t), m_nu = 2 sqrt 2 t sin theta(t), all other Dirac masses zero; then m_e^2 + m_nu^2 = 8 t^2 and each mass vanishes on an unbounded sequence.

Hypotheses and escapes (lemma 20, Result 9). The relation belongs to the stated normalization of the truncated asymptotic spectral action at matching, and concerns Dirac Yukawa masses, including the neutrino Dirac parameters, rather than physical seesaw eigenmasses. For N_g != 3 the coefficient 8 is an assumed normalization. Changing the action or the normalization, controlling the full finite-cutoff action, varying the particle content, or allowing Dirac masses to grow changes the problem.

Proof. Lemma 11, Corollary 11.2, and lemma 20, Result 9; the proofs are there. Both follow from the bounds of Proposition 11.1.

What it leaves open. Extrapolation to a heavy-mass limit at fixed cutoff needs a separate calculation, and the result directs attention to it (lemma 20, Result 9). Lemma 18 does it in the exact quadratic spectral action, where the top-only family gives M_W -> 0 at fixed cutoff (18_top_only_w_mass_at_fixed_cutoff.md).

Part S24 (If the up-type Dirac mass of one generation is the only diverging Dirac mass, the CCM relation gives m_t/M_W -> sqrt(8/3))

If the up-type Dirac mass of one generation is the only diverging Dirac mass, the CCM relation gives m_t/M_W -> sqrt(8/3)

Status. Proposition (Corollary 11.3), conditional on the CCM relation and on top dominance, which is a hypothesis. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand, effort max: the formula correct under the hypothesis, and the claim, as first written, that top dominance follows from the spectral action refuted and removed (report appended to lemma 11). Ledger row `l11_top`, depending on `l11_bounds`. Proposition E of paper/towerless_intervals.tex. The result restates CCM 2007, section 5.4, equation (5.27).

Setting. The CCM relation of 11_heavy_w_mass_bounds.md. The up-type Dirac mass m_t of one generation (three colours) diverges, and every other Dirac mass stays bounded.

Statement.

1. With R the weighted sum of the other Dirac masses squared, R = O(1) and 8 M_W^2 = 3 m_t^2 + R, so m_t^2 = (8/3) M_W^2 + O(1).
2. Hence m_t/M_W -> sqrt(8/3), about 1.633.
3. Under the hypothesis, the surviving SU(3) x U(1)_em theory is vectorlike and anomaly-free, each surviving charged Dirac fermion having identical left and right representations of the unbroken group.

Hypothesis. Top dominance is assumed. The relation alone allows, for example, y_e = 2g with every other Yukawa zero, which gives m_e = 2 sqrt 2 M_W and m_t = 0. With fixed Yukawas, y_t = y_nu = g gives m_t/M_W = sqrt 2, with both masses diverging as v -> infinity.

Proof. Lemma 11, Corollary 11.3; the proof is there, by item 1.

What it leaves open. The ratio holds in the asymptotic matching regime. At fixed cutoff the top-only family gives M_W^2 ~ Lambda^4/m_t^2 instead (18_top_only_w_mass_at_fixed_cutoff.md), and m_t/M_W -> sqrt(8/3) with all scales diverging needs Lambda -> infinity with m_t/Lambda -> 0 (18_joint_limit.md). The massless chiral interval assigns the top and the W to the same scale with a different order-one ratio (lemma 11, remark on the interval comparison).

Part S25 (With a boundary Yukawa matrix X between a doublet-type and a singlet-type field on the interval, D_X is self-adjoint, its kernel is ker X^dagger + ker X, and ind D_X = N_Q - N_U for every X; equivariantly the index is [rho_Q] - [rho_U])

With a boundary Yukawa matrix X between a doublet-type and a singlet-type field on the interval, D_X is self-adjoint, its kernel is ker X^dagger + ker X, and ind D_X = N_Q - N_U for every X; equivariantly the index is [rho_Q] - [rho_U]

Status. Theorem (Lemma 12.1, Theorem 12.2 and the index part of Corollary 12.3). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand, effort max: Lemma 12.1 and Theorem 12.2 correct, Corollary 12.3 correct with one amendment (report appended to lemma 12). Ledger row `l12_index`, depending on `l09_index`. Theorem C of paper/towerless_intervals.tex.

Setting. A doublet-type field Q in L^2(I; C^2) (x) C^{N_Q} and a singlet-type field U in L^2(I; C^2) (x) C^{N_U}, components (L, R), grading Gamma = diag(1, -1) on each, bulk operator (D psi)_L = R' + M R, (D psi)_R = -L' + M L with diagonal M_Q, M_U. For a complex N_Q x N_U matrix X, D_X acts on the H^1 functions with

```
Q_R(0) = 0,   U_L(0) = 0,   Q_R(1) = X U_R(1),   U_L(1) = -X^dagger Q_L(1).
```

At X = 0, Q has boundary type (a) and U type (b) of lemma 09. A compact group G acts unitarily by rho_Q, rho_U commuting with the bulk masses; the domain is invariant exactly under the stabilizer H_X = {g : rho_Q(g) X = X rho_U(g)}.

Statement.

1. (Lemma 12.1) D_X is self-adjoint with compact resolvent, and Gamma D_X = -D_X Gamma.
2. (Theorem 12.2) ker D_X = {(e^{M_Q x} a, 0; 0, 0) : X^dagger e^{M_Q l} a = 0} + {(0, 0; 0, e^{-M_U x} b) : X e^{-M_U l} b = 0}, so dim ker_{Gamma=+1} = N_Q - rank X, dim ker_{Gamma=-1} = N_U - rank X, and ind D_X = N_Q - N_U.
3. (Corollary 12.3, index) In R(H_X), ind_{H_X} D_X = [ker X^dagger] - [ker X] = [rho_Q] - [rho_U] restricted to H_X. At X = 0, H_X = G and the index is the chiral class [V_L] - [V_R] of Corollary 9.3.

Proof. Lemma 12, Lemma 12.1, Theorem 12.2 and Corollary 12.3; the proofs are there. The boundary conditions form a Lagrangian subspace for the boundary form; zero modes are exponentials on each field, and the conditions at l reduce to X U_R(l) = 0 and X^dagger Q_L(l) = 0; X maps (ker X)^perp isomorphically and H_X-equivariantly onto ran X = (ker X^dagger)^perp.

What it leaves open. The index N_Q - N_U is constant along X; what changes is the symmetry, reduced from G to H_X. Which zero modes lift is the no-go of 12_nogo_zero_modes_lift_only_vectorlike.md.

Part S26 (For M = F X G with graded positive diagonal F and G, the products sigma_1 … sigma_p are bounded on both sides through the trailing minors of X, so with nonzero trailing minors the masses are m_k = f_k g_k (Delta_{N-k+1}/Delta_{N-k}) (1 + O(eps)))

For M = F X G with graded positive diagonal F and G, the products sigma_1 … sigma_p are bounded on both sides through the trailing minors of X, so with nonzero trailing minors the masses are m_k = f_k g_k (Delta_{N-k+1}/Delta_{N-k}) (1 + O(eps))

Status. Lemma (Lemma 12.8), with Corollary 12.9. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (2000 random complex 3 x 3 cases with log-uniform gradings over 8 e-folds, 6000 inequalities, no violation) and by Terra 6.1 by hand, effort max: Lemma 12.8 correct, Corollary 12.9 wrong as first stated (without the minor condition) and corrected (report appended to lemma 12). Ledger row `l12_graded`. Item 5 of Theorem C of paper/towerless_intervals.tex ("\supp{12}").

Setting. F = diag(f_1 <= … <= f_N) and G = diag(g_1 <= … <= g_N) positive, X in M_N(C), M = F X G. For p = 1, …, N: T_p = {N - p + 1, …, N}, Delta_p = |det X_{T_p T_p}|, D_p = max over p-subsets I, J of |det X_{IJ}|, eps_p^F = f_{N-p}/f_{N-p+1} and eps_p^G = g_{N-p}/g_{N-p+1} (both 0 for p = N), K_p = binom(N, p) - 1, and f_I the product of f_i over i in I.

Statement.

1. (Lemma 12.8) f_{T_p} g_{T_p} Delta_p <= sigma_1 … sigma_p <= f_{T_p} g_{T_p} [Delta_p + D_p ((1 + K_p eps_p^F)(1 + K_p eps_p^G) - 1)].
2. (Corollary 12.9) If every trailing minor Delta_p is nonzero, put E_0 = E_N = 0 and E_p = (D_p/Delta_p)((1 + K_p eps_p^F)(1 + K_p eps_p^G) - 1) for 1 <= p < N. Then 1/(1 + E_{p-1}) <= sigma_p/(f_{N-p+1} g_{N-p+1} Delta_p/Delta_{p-1}) <= 1 + E_p. Indexing generations by k = N - p + 1, m_k = f_k g_k (Delta_{N-k+1}/Delta_{N-k}) (1 + O(eps)) as eps = max_p (eps_p^F + eps_p^G) -> 0, for fixed X and N.
3. With M_F = Phi_Q Y_hat Phi_U and anarchic Y_hat (all minors of order one), the masses follow phi(c_k^Q) phi(c_k^U). Since phi(c) = (2c)^{1/2} e^{-c} (1 + O(e^{-2c})), order-one differences of the c's give exponential hierarchies, m_k/m_{k+1} ~ exp(-(c_k^Q - c_{k+1}^Q) - (c_k^U - c_{k+1}^U)) up to factors of order one; the ratio also carries sqrt(c_k^Q c_k^U/(c_{k+1}^Q c_{k+1}^U)), bounded when adjacent c's differ by bounded amounts.

Scope. Without the minor condition the mass formula fails: F = G = diag(t, 1) and X = sigma_1 give Delta_1 = 0 while both singular values equal t. Uniform statements need uniform bounds on D_p/Delta_p.

Proof. Lemma 12, Lemma 12.8 and Corollary 12.9; the proofs are there. By Cauchy-Binet the p-th compound of M has entries f_I g_J det X_{IJ} and norm sigma_1 … sigma_p; its (T_p, T_p) entry gives the lower bound, and the sum of absolute entries the upper; dividing the bounds for p and p - 1 gives the corollary.

What it leaves open. The mixing pattern (12_mixing_upper_bound.md). The localization parameters c are free inputs of the construction.

Part S27 (The light eigenvalues of the boundary-Yukawa interval are the fixed points lambda = sigma_k(A^{1/2} X B^{1/2}), with two-sided bounds, and in the towerless limit the retained block is the Yukawa block Phi_Q Y_hat Phi_U)

The light eigenvalues of the boundary-Yukawa interval are the fixed points lambda = sigma_k(A^{1/2} X B^{1/2}), with two-sided bounds, and in the towerless limit the retained block is the Yukawa block Phi_Q Y_hat Phi_U

Status. Theorem (Theorems 12.6 and 12.7). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 numerically (secular roots on random samples; the errors decrease as l^2) and by Terra 6.1 by hand, effort max: Theorem 12.6 correct, Theorem 12.7 wrong as first stated (its fixed-energy list) and corrected (report appended to lemma 12). Ledger row `l12_towerless`. Theorem C of paper/towerless_intervals.tex.

Setting. The secular data of 12_secular_equation.md. F_Q = diag(f_{Q_i}), F_U = diag(f_{U_j}), Y_0 = F_Q X F_U; Lambda_Q = min_i Lambda_i^Q, Lambda_U = min_j Lambda_j^U, lambda_* = min(Lambda_Q, Lambda_U); for 0 < lambda < lambda_*, A and B are positive and Y(lambda) = A(lambda)^{1/2} X B(lambda)^{1/2}; singular values in decreasing order. Towerless scaling: the c's and Y_hat = X/l fixed, l -> 0; Phi_Q = diag(phi(c_i^Q)), Phi_U = diag(phi(c_j^U)), M_F = Phi_Q Y_hat Phi_U.

Statement.

1. (Theorem 12.6) lambda in (0, lambda_*) is an eigenvalue of multiplicity m iff exactly m indices k satisfy sigma_k(Y(lambda)) = lambda. Each equation lambda = sigma_k(Y(lambda)) has at most one solution lambda_k in (0, lambda_*), and one whenever 0 < sigma_k(Y_0) < lambda_*. Every solution satisfies sigma_k(Y_0) (1 - lambda_k^2/Lambda_Q^2 - lambda_k^2/Lambda_U^2) <= lambda_k <= sigma_k(Y_0).
2. (Theorem 12.7) For every E > 0 and all small l, the spectrum of D_X in [-E, E] consists of 0 with multiplicity N_Q + N_U - 2 rank Y_hat, and of +-lambda_k(l) for those k <= rank Y_hat with sigma_k(M_F) <= E, where 0 < sigma_k(M_F) - lambda_k(l) <= sigma_k(M_F)^3 l^2 (1/L_Q^2 + 1/L_U^2), with L_Q = l Lambda_Q and L_U = l Lambda_U depending only on the c's.
3. All eigenvalues outside zero and the light set have absolute value greater than lambda_* = min(L_Q, L_U)/l, which tends to infinity.
4. For E > ||M_F|| the light space has dimension N_Q + N_U and, after grading-preserving unitary identifications, its compressed Dirac matrices converge to D_F = [[0, M_F], [M_F^dagger, 0]] on H_F = C^{N_Q} + C^{N_U} with Gamma_F = diag(1, -1): up to unitaries on each chirality, the Yukawa block of the Chamseddine-Connes finite Dirac operator.

Scope. Calling the limit a spectral triple requires specifying an algebra and its representation; in the spectral propinquity the algebra of the limit is C (15_gap_criterion.md, Corollary 15.6). At fixed fermion masses X = l Y_hat -> 0, so the boundary condition tends to the chiral type (a) + (b) while the wall values of the normalized profiles grow like l^{-1/2}.

Proof. Lemma 12, Theorems 12.6 and 12.7; the proofs are there. Conjugation by A^{1/2} turns the secular equation into the eigenvalue equation of Y Y^dagger; sigma_k(Y(lambda)) is continuous and nonincreasing; concavity places alpha_i and beta_j above their chords, giving the lower bound; and Y_0 = M_F by item 2 of Theorem 12.4.

What it leaves open. The texture of D_F is localization data times an order-one boundary matrix; its mass hierarchies are 12_graded_singular_values.md and its mixing 12_mixing_upper_bound.md. The algebra and representation of a limit triple are inputs.

Part S28 (For two sectors sharing the doublet localization F, the left mixing matrix satisfies |V_ij| <= 3 K_u K_d f_i/f_j for i < j, independent of the right-hand profiles, with no universal lower bound)

For two sectors sharing the doublet localization F, the left mixing matrix satisfies |V_ij| <= 3 K_u K_d f_i/f_j for i < j, independent of the right-hand profiles, with no universal lower bound

Status. Proposition (the bound in Remark 12.10). Proved by Terra 6.1 in the referee report on lemma 12 (appended to lemma 12); the cofactor step that the report names was written out by Astra 6, who checked the proof by hand (historical/reviews/2026-11-03_astra_validation_corrections.md, item 10); checked numerically by Claude Opus 5.5 (numpy: 60000 inequalities on 20000 random complex 3 x 3 pairs, no violation; a numerical check of the bound, the proof being the referee's). Ledger row `l12_mixing_bound`; the numerical medians below are row `l12_mixing`, of status numerical, stated in paper/towerless_intervals.tex. The bound is cited there as "\supp{12}, from the referee report".

Setting. For each sector s = u, d: M_s = F X_s G_s with F = diag(f_1 <= f_2 <= f_3) and G_s = diag(g_1^(s) <= g_2^(s) <= g_3^(s)) positive, X_s invertible with nonzero trailing minors Delta_1^(s), Delta_2^(s), and D_p^(s) as in Lemma 12.8. K_s = (1 + sqrt3 D_1^(s)/Delta_1^(s)) (1 + sqrt3 D_2^(s)/Delta_2^(s)). U_s is the left singular-vector matrix with columns in increasing order of singular value, and V = U_u^dagger U_d.

Statement.

1. For i < j, |V_ij| <= 3 K_u K_d f_i/f_j.
2. The bound is independent of the right-hand localization profiles G_s.
3. There is no positive universal lower bound: identical up and down matrices give V = I, also when their minors are nonzero.

Numerical companion (Remark 12.10, numerical only). In 500 anarchic complex samples with c^Q = (3, 2, 0.5), the medians of |V_ij|/(phi(c_i^Q)/phi(c_j^Q)) were 1.02, 0.92 and 1.09 for (ij) = (12), (23) and (13): the left mixing follows the doublet localization, |V_ij| ~ phi(c_i^Q)/phi(c_j^Q), as in the split-fermion and warped-flavour literature. The medians depend on the sampling distribution.

Proof. Lemma 12, Remark 12.10 and the referee report appended there; the proof is there. With H_s = M_s M_s^dagger, Cauchy-Schwarz gives |U_i3| <= a f_i/f_3, the same argument for H^{-1} gives |U_i1| <= b f_1/f_i, cofactor identities for the unitary U bound the remaining off-diagonal entries, so |U_ij| <= (1 + a)(1 + b) min(f_i/f_j, f_j/f_i), and summing the three terms of V_ij gives the bound.

What it leaves open. The typical size |V_ij| ~ phi_i/phi_j is a numerical observation tied to its sampling distribution; no proof of it is claimed.

Part S29 (All zero modes of the boundary-Yukawa interval lift only if the residual theory is vectorlike, so without nu_R a massless left neutrino is forced for every X preserving electromagnetism)

All zero modes of the boundary-Yukawa interval lift only if the residual theory is vectorlike, so without nu_R a massless left neutrino is forced for every X preserving electromagnetism

Status. No-go theorem in its scope (Corollary 12.3). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand, effort max, with the amendment restricting the neutrino statement

to couplings preserving electromagnetism (report appended to lemma 12). Ledger row `l12_index`. Theorem C of paper/towerless_intervals.tex.

Setting. D_X, rho_Q, rho_U and the stabilizer H_X as in 12_boundary_yukawa_index.md. For the Standard Model, H = SU(3) x U(1)_em.

Statement.

1. D_X has no zero modes iff X is invertible.
2. This requires rho_Q and rho_U to be isomorphic as H_X-representations: the residual theory must be vectorlike.
3. For the Standard Model with H = SU(3) x U(1)_em, the quark sector satisfies this.
4. A lepton sector without nu_R does not: if X intertwines the electromagnetic representations (H contained in H_X) and there is no neutral right-handed lepton, the neutrino subspace lies in ker X^dagger, and a massless left-handed neutrino is forced. This agrees with the unpaired nu_L of lemma 06.

Hypotheses. Item 4 needs X to preserve electromagnetism and the absence of a neutral right-handed lepton.

Escape shown by the referee. Without electromagnetic invariance the neutrino statement can fail: for one lepton generation with zero bulk masses, Q_L = (nu_L, e_L), U_R = e_R and X = (1, 0)^T, the left kernel is spanned by e_L while nu_L is coupled to e_R; this X breaks electromagnetic charge.

Proof. Lemma 12, Corollary 12.3; the proof is there: the endpoint exponentials identify the chiral kernels equivariantly with ker X^dagger and ker X, and both vanish exactly when X is invertible, in which case X is an H_X-equivariant isomorphism.

What it leaves open. The obstruction sits in the neutral sector, as the obstruction to KO-5 does in lemma 06 (06_neutral_sector_criterion.md). Lemma 12 places chirality in the boundary types at X = 0 and does not derive them from a parent.

Part S30 (The boundary-Yukawa interval has the exact secular equation det(lambda^2 - A(lambda) X B(lambda) X^dagger) = 0 below the first poles, and for one massless flavour its spectrum (n pi +- arctan|X|)/l interpolates from types (a) + (b) to two copies of type (c))

The boundary-Yukawa interval has the exact secular equation det(lambda^2 - A(lambda) X B(lambda) X^dagger) = 0 below the first poles, and for one massless flavour its spectrum (n pi +- arctan|X|)/l interpolates from types (a) + (b) to two copies of type (c)

Status. Theorem (Theorem 12.4 and Example 12.5). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 against ODE shooting (scipy solve_ivp, rtol 1e-12; agreement to 6 digits) and by Terra 6.1 by hand, effort max: item 2 of Theorem 12.4 wrong as first stated on the common window and corrected, Example 12.5 correct with amendment (report appended to lemma 12). Ledger row `l12_secular`, depending on `l09_lightpair`. Theorem C of paper/towerless_intervals.tex. The one-flavour spectrum is equation (5.6) of Csaki-Grojean-Hubisz-Shirman-Terning (2004) (lemma 12, Literature).

Setting. D_X of 12_boundary_yukawa_index.md, with c_i^Q = -m_i^Q l and c_j^U = m_j^U l. For a field of mass m and lambda > 0, omega = (m^2 - lambda^2)^{1/2}, with omega coth(omega l) read as k cot(kl) for omega = ik and as 1/l for omega = 0; alpha_i(lambda) = omega coth(omega l) + m_i^Q, beta_j(lambda) = omega coth(omega l) - m_j^U, A = diag(alpha_i), B = diag(beta_j); W is the set of lambda > 0 with lambda^2 < m^2 + pi^2/l^2 for every bulk mass m of the system.

Statement.

1. For lambda in W, lambda is an eigenvalue of D_X iff det(lambda^2 - A X B X^dagger) = 0, with multiplicity dim ker(lambda^2 - A X B X^dagger).
2. alpha_i(0) = 2m/(1 - e^{-2ml}) with m = m_i^Q equals f_{Q_i}^2, the squared value at x = l of the normalized zero mode e^{mx}; likewise beta_j(0) = 2m/(e^{2ml} - 1) = f_{U_j}^2. In terms of c, f^2 = phi(c)^2/l with phi(c)^2 = 2c/(e^{2c} - 1) and phi(0) = 1.
3. On its own window 0 < lambda^2 < m^2 + pi^2/l^2, each alpha_i (and each beta_j) is strictly decreasing and concave in lambda^2, positive at 0 and tending to -infinity at the edge, so it vanishes exactly once, at the lowest positive eigenvalue Lambda_i^Q (respectively Lambda_j^U) of the single-field type-(c) problem. On the common window W it is still decreasing and concave but may stay positive: l = 1, M_Q = diag(0, 4) gives alpha_2 > 0 on all of W = (0, pi).
4. (Example 12.5) For N_Q = N_U = 1 and zero bulk mass, tan^2(lambda l) = |X|^2, and with theta = arctan|X| the positive spectrum is theta/l, (n pi - theta)/l and (n pi + theta)/l for n >= 1. At X = 0 zero has multiplicity two and the positive eigenvalues n pi/l have multiplicity two; theta = pi/2 is reached only as the limit |X| -> infinity, where each field becomes type (c) and the spectrum tends to two copies of (r - 1/2) pi/l. The light pair theta/l interpolates between the chiral zero modes of types (a) and (b) and the Connes-Lott mass pi/(2l) of lemma 05, with D acting as (theta/l) sigma_1 on a Gamma-graded two-dimensional space.

Proof. Lemma 12, Theorem 12.4 and Example 12.5; the proofs are there. Explicit solutions on each field and elimination give lambda^2 c = X B X^dagger A c, and the kernels of lambda^2 - C A and lambda^2 - A C have equal dimension for lambda != 0; monotonicity and concavity follow from x coth x = 1 + sum_{n >= 1} 2x^2/(x^2 + n^2 pi^2).

What it leaves open. The light spectrum and the towerless limit (12_light_spectrum_towerless_limit.md). Along the interpolation the index is constant and the symmetry changes (12_boundary_yukawa_index.md).

Part S31 (With the three-generation CCM trace and one massive colour triplet of mass m = pi/(2l), the spectral-action W mass is M_W = sqrt(3/8) m = sqrt(3/8) M_KK, with no interval-overlap correction)

With the three-generation CCM trace and one massive colour triplet of mass m = pi/(2l), the spectral-action W mass is M_W = sqrt(3/8) m = sqrt(3/8) M_KK, with no interval-overlap correction

Status. Corollary (Result 7 of lemma 13), proved under its stated hypotheses; the ledger lists the CCM relation as its hypothesis (`ccm_relation`). Proved by Astra 6 (job 13); checked by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md); the header of lemma 13 records that it reproduces the standard CCM relation. Ledger row `l13_ccm`, depending on `l13_mass_formula`. Item 3 of Proposition F of paper/towerless_intervals.tex.

Setting. The full three-generation Standard Model representation with its usual Higgs doublet; all charged fermion states retained in the spectral trace, including those of vanishing Yukawa mass; one colour-triplet Dirac mass at the vacuum identified with m = pi/(2l), every other Dirac mass zero; the formula of 13_spectral_action_vector_mass_formula.md at the matching scale.

Statement.

1. For T_1 = sigma_1/2, Q_11 = 12 (one generation of particles gives 2, antiparticles double it, three generations triple it) and K_11 = sum_f N_f m_f^2, so M_W^2 = (3/2) K_11/12 = (1/8) sum_f N_f m_f^2, the relation of CCM, Lemma 4.6, equation (4.16).
2. With K_11 = 3 m^2, M_W^2 = (3/8) m^2 and M_W/M_KK = sqrt(3/8).
3. No factor from the interval wavefunctions or from Wilson-line normalization multiplies this ratio:

||u_L|| = ||u_R|| = 1 and <u_L, D_I u_R> = m, and the factor 2/pi belongs to the coupling of a constant A_5 to the fixed link.

4. With other Dirac masses, M_W^2/M_KK^2 = (1/8)(3 + sum over the other f of N_f m_f^2/m^2); two massive coloured species of mass m give M_W/m = sqrt(3)/2.

Scope. The coefficient comes from Q_11 = 12; applied after deleting the other charged states it would no longer normalize the gauge kinetic term canonically. The ratio sqrt(3/8) belongs to the CCM completion; the minimal two-sheet data of the interval give sqrt(6) instead (13_counterexample_two_sheet_sqrt6.md).

Proof. Lemma 13, Result 7; the proof is there.

What it leaves open. The electroweak doublet assignment, its hypercharges and the charged fermion trace are supplied separately (Result 8: rho = 1 and M_W/M_Z = sqrt(5/8) with Q_2 = 12, Q_Y = 20). Hypothesis H needs a gauge-covariant reduction and a matching of representations, vacuum data and kinetic forms.

Part S32 (The minimal two-sheet spectral action gives a massless diagonal vector and a relative vector of mass sqrt(6) m, against the Neumann-Dirichlet tower mass m = pi/(2l))

The minimal two-sheet spectral action gives a massless diagonal vector and a relative vector of mass sqrt(6) m, against the Neumann-Dirichlet tower mass m = pi/(2l)

Status. Counterexample (Result 6 of lemma 13), with Lemma 4 on the tower mass. Proved by Astra 6 (job 13); checked by Claude Opus 5.5 by hand (the Neumann-Dirichlet mass and the eigenvalues 0 and 6 m^2) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger rows `l13_sqrt6` (depending on `l13_mass_formula` and `l13_kk_mass`) and `l13_kk_mass`. Item 3 of Proposition F of paper/towerless_intervals.tex.

Setting. H_F = (C_L^n + C_R^n) (x) C^N, V_0 = m [[0, I_n], [I_n, 0]] (x) I_N; both sheet groups gauged, U(1)_L x U(1)_R for n = 1 or SU(2)_L x SU(2)_R for n = 2; the action of 13_spectral_action_vector_mass_formula.md. Comparison (Lemma 4): the transverse-vector operator -d^2/dx^2 with f'(0) = 0, f(l) = 0 and no boundary kinetic terms has lowest mass M_KK = pi/(2l) = m; Dirichlet conditions at both ends give pi/l, and two Neumann conditions give a zero mode.

Statement.

1. For one normalized generator t with kappa = N tr(t^2), Q = kappa I_2 and K = 2 kappa m^2 [[1, -1], [-1, 1]], so M^2 = 3 m^2 [[1, -1], [-1, 1]], with eigenvalues 0 and 6 m^2: the diagonal combination is massless and the relative one has M_relative = sqrt(6) m, also for colour multiplicity N = 3. For SU(2) this applies to each t = sigma_a/2.
2. If only the diagonal group is gauged, every generator commutes with V_0 and all these vector masses vanish; neither choice realizes the assumed continuum mixed-boundary breaking automatically.
3. Adding S otherwise identical charged pairs with zero finite Dirac operator gives M_relative^2 = 6 N m^2/(N + S): the value depends on the full fermion trace, at fixed mass and colour multiplicity of the original pair.
4. Restricting the continuum field to its lowest mixed-boundary profile, A_mu = a_mu cos(mx), gives A_L = (8/(3 pi)) a, A_R = (4/(3 pi)) a and M_restricted^2 = 3 m^2 (c_L - c_R)^2/(c_L^2 + c_R^2) = (3/5) m^2, an additional restriction distinct from the unrestricted two-sheet theory.

Scope. The minimal two-sheet data at leading spectral-action order. The fermion boundary conditions do not derive the Neumann-Dirichlet problem (lemma 13, Results 1 and 4).

Proof. Lemma 13, Lemma 4 and Result 6; the proofs are there, by direct multiplication; the factor N cancels between scalar and gauge normalization.

What it leaves open. The CCM comparison (13_ccm_triplet_ratio.md), and a gauge-covariant reduction, in which the ratio 3/5 reappears (14_counterexample_three_fifths.md).

Part S33 (The fluctuated product of spacetime with the chiral interval compresses exactly to the lowest pair, with weighted sheet connections and link phi = m + iB, and the image phi + phi^* = 2m omits part of the Connes-Lott fluctuations)

The fluctuated product of spacetime with the chiral interval compresses exactly to the lowest pair, with weighted sheet connections and link phi = m + iB, and the image phi + phi^* = 2m omits part of the Connes-Lott fluctuations

Status. Theorem (Result 2 of lemma 13, with Result 1 on the fluctuations). Proved by Astra 6 (job 13); checked by Claude Opus 5.5 by hand (the compression integrals) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l13_compression`, depending on `l05_truncation`. Item 1 of Proposition F of paper/towerless_intervals.tex.

Setting. M a closed Riemannian spin four-manifold, Hermitian Euclidean gamma matrices, D_M = -i gamma^mu nabla_mu^S, E = [[0, 1], [-1, 0]], D = D_M (x) 1 + gamma_5 (x) E d/dx on L^2(S_M) (x) L^2([0, l], C^2) (x) C^n with R(y, 0) = 0 and L(y, l) = 0; the algebra acts by the same matrix multiplication on L and R; complex inner fluctuations D + omega, omega = sum a_j [D, b_j] = omega^*, with no real structure or extra bimodule. m = pi/(2l), u_L and u_R the lowest pair of lemma 05, P the projection onto their span (tensored with spacetime and gauge factors); w_L = (2/l) cos^2(mx), w_R = (2/l) sin^2(mx), k(x) = (1/l) sin(pi x/l).

Statement.

1. (Result 1) The fluctuated operators are exactly D_A = -i gamma^mu (nabla_mu^S + i A_mu) + gamma_5 E (d/dx + i A_5), with arbitrary smooth Hermitian n x n matrices A_mu(y, x), A_5(y, x): connections on the trivial U(n) bundle over M x I_l. The fermion conditions induce no Dirichlet, Neumann or endpoint-value conditions on A_mu or A_5, and every smooth gauge transformation preserves them.
2. (Result 2) Exactly, P D_A P = -i gamma^mu (nabla_mu^S + i diag(A_{L,mu}, A_{R,mu})) + gamma_5 [[0, phi], [phi^*, 0]], with A_{L,mu} = int w_L A_mu dx, A_{R,mu} = int w_R A_mu dx, phi = m I_n + iB and B = int k(x) A_5(y, x) dx = B^*.
3. The exact image consists of arbitrary Hermitian sheet connections and of links with phi + phi^* = 2m I_n.
4. The finite algebra M_n(C) + M_n(C) with D_F = m [[0, I], [I, 0]] admits an arbitrary complex matrix as total link, so not every Connes-Lott fluctuation arises by this compression; the fluctuation phi = 0, for example, has no continuum lift.

Scope. For SU(2), B is traceless, and the continuum A_5 is adjoint-valued, so an electroweak Higgs doublet needs a further identification. The real-triple assertion of lemma 08 (Theorem 5) restricts the interval algebra to constants and supplies no Standard Model real structure for the full compressed algebra.

Proof. Lemma 13, Results 1 and 2; the proofs are there. The upper-right matrix element of i E A_5 is (2i/l) int cos(mx) sin(mx) A_5 = iB; the lifts A_mu = (A_L + A_R)/2 + (A_L - A_R) cos(pi x/l) and A_5 = (pi/2) B show that the image is exact.

What it leaves open. The fixed compression is not gauge covariant (13_nogo_fixed_compression_not_covariant.md), and lemma 14 repairs that. Hypothesis H needs a gauge-covariant reduction and a matching of representations, vacuum data and kinetic forms (lemma 13, consequence).

Part S34 (Fixed spectral compression of the fluctuated chiral interval is not gauge covariant: a gauge-trivial A_5 with trivial Wilson line changes the compressed spectrum)

Fixed spectral compression of the fluctuated chiral interval is not gauge covariant: a gauge-trivial A_5 with trivial Wilson line changes the compressed spectrum

Status. No-go theorem in its scope (Result 3 of lemma 13). Proved by Astra 6 (job 13); checked by Claude Opus 5.5 by hand (the value B = -2a/(3 pi) with trivial Wilson line) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l13_not_covariant`. Item 2 of Proposition F of paper/towerless_intervals.tex.

Setting. The fixed (unfluctuated) projection P of 13_fixed_compression_gauge_fluctuations.md and its link phi[A_5]. U(1) potentials independent of y, with parallel transport W' = -i A_5 W, W(0) = 1, so that D_{I,A} = W D_I W^* and the complete interval spectrum is unchanged.

Statement.

1. The link phi[A_5] is not a functional of the Wilson line alone, and fixed compression does not descend to continuum gauge-equivalence classes.
2. For A_5 = a cos(2 pi x/l), W(l) = 1 while B = -2a/(3 pi), phi = m - 2ia/(3 pi) and |phi| = sqrt(m^2 + 4a^2/(9 pi^2)). The zero connection and this one are gauge equivalent through a transformation equal to 1 at both endpoints, yet their compressed internal eigenvalues differ.
3. The obstruction is that multiplication by an x-dependent gauge transformation generally fails to preserve PH: compression of multiplication is not multiplicative, and P g P generally fails to be unitary.
4. Already for constant A_5 = a, phi/m = 1 + i (4l/pi^2) a against W(l)^* = e^{ial}, so neither equality with the Wilson link nor a universal correction to the mass follows.

Hypotheses and escapes. The projection is fixed. The gauge-adapted transported frames u_L W and u_R W W(l)^* give the vertical matrix [[0, m W(l)^*], [m W(l), 0]], with phi^* phi = m^2 I, which omits general radial Higgs fluctuations. The covariant spectral projection of lemma 14 repairs the covariance defect (14_covariant_spectral_projection.md). Compression also differs from eliminating the omitted fermions, which adds -P D_A Q [Q (D_A - z) Q]^{-1} Q D_A P with Q = 1 - P.

Proof. Lemma 13, Result 3; the proof is there: B = (a/pi) int_0^pi sin t cos(2t) dt = -2a/(3 pi).

What it leaves open. The comparison of the covariant link with the finite fluctuations of the two-point data (lemma 19; 19_comparison_with_parent.md).

Part S35 (At leading spectral-action order the canonically normalized vector mass matrix is M^2 = (3/2) Q^{-1/2} K Q^{-1/2}, with Q the trace form of the generators and K the trace form of their commutators with the vacuum)

At leading spectral-action order the canonically normalized vector mass matrix is M^2 = (3/2) Q^{-1/2} K Q^{-1/2}, with Q the trace form of the generators and K the trace form of their commutators with the vacuum

Status. Proposition (Result 5 of lemma 13). Proved by Astra 6 (job 13); checked by Claude Opus 5.5 by hand (the 1 : 3 ratio of the a_4 gauge and scalar terms with the canonical normalization) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l13_mass_formula`. Item 3 of Proposition F of paper/towerless_intervals.tex ("\supp{13}"); lemma 18, Corollary 3, recovers it as the small-mass limit of the exact kernel.

Setting. A specified finite fermion space H_F with Hermitian represented gauge generators T_a; D = -i gamma^mu (nabla_mu^S + i A_mu) + gamma_5 V with V = V^* the total finite scalar matrix; the ordinary spectral trace, an even cutoff function with f_0 = f(0) > 0, the terms through order Lambda^0, and no additional kinetic terms. V_0 a chosen constant vacuum, Q_ab = Re Tr_{H_F}(T_a T_b), K_ab = Re Tr_{H_F}([T_a, V_0]^* [T_b, V_0]).

Statement.

1. The relevant Lagrangian is c Tr_{H_F}(F_{mu nu} F^{mu nu}) + 3c Tr_{H_F}(D_mu V D^mu V), with c = f_0/(24 pi^2).
2. On the faithfully represented gauge algebra, M^2 = (3/2) Q^{-1/2} K Q^{-1/2}.
3. Relative to the mixed-boundary continuum mass M_KK = m, M_j^2/M_KK^2 = (3/(2 m^2)) lambda_j(Q^{-1/2} K Q^{-1/2}).
4. The common coefficient f_0 cancels; the data needed are the represented generators, their full trace multiplicities and V_0.

Scope. The masses are conditional on the stated vacuum and refer to the leading spectral-action approximation. The bare finite Dirac matrix need not be the vacuum: determining V_0 requires the scalar potential, including its cutoff moments and curvature dependence.

Proof. Lemma 13, Result 5; the proof is there. It uses the heat coefficient a_4, which contains (1/(16 pi^2)) Tr(E^2/2 + Omega_{mu nu} Omega^{mu nu}/12) (CCM, Theorem 6.1, equation (6.10)), the Clifford traces tr_S(E_F^2) = 2 F^2, tr_S(E_V^2) = 4 D V D V, tr_S(Omega_F^2) = -4 F^2, and the change of variables A_can = 2 sqrt(c) Q^{1/2} A.

What it leaves open. The interval does not supply the generators and trace data: the minimal two-sheet model and the CCM trace give different answers (13_counterexample_two_sheet_sqrt6.md, 13_ccm_triplet_ratio.md). Near the cutoff the exact kernel of lemma 18 replaces the formula (18_exact_quadratic_kernel.md).

Part S36 (The gauge-covariant lowest-mode reduction of the massless mixed interval still gives M_spec^2 = (3/5) M_KK^2)

The gauge-covariant lowest-mode reduction of the massless mixed interval still gives M_spec^2 = (3/5) M_KK^2

Status. Counterexample (Result 6 of lemma 14). Proved by Astra 6 (job 14); checked by Claude Opus 5.5 by hand (agreement with lemma 13, Result 6: c_L = 8/(3 pi), c_R = 4/(3 pi)) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l14_three_fifths`. Cited after Proposition F of paper/towerless_intervals.tex.

Setting. Gauge group U(1), with d_x A_mu(0) = 0, A_mu(l) = 0 and the complementary A_5(0) = 0, d_x A_5(l) = 0; retained A_mu = a_mu(y) cos(kappa x) and A_5 = b(y) sin(kappa x), kappa = pi/(2l). One charge-one fermion with the mixed conditions and scalar bulk mass M; lambda its lowest positive vertical eigenvalue; c_L = int p_L^2 cos(kappa x) dx and c_R = int p_R^2 cos(kappa x) dx.

Statement.

1. The vector boundary problem selects cos(kappa x), with int (f')^2/int f^2 = kappa^2, so M_KK^2 = kappa^2.
2. The retained transformations epsilon = alpha(y) cos(kappa x) act by a -> a - d alpha, b -> b + kappa alpha, so the vector-scalar restriction is gauge closed.
3. The gauge-covariant light projection, with the Berry terms included, gives the quadratic action eta (c_L^2 + c_R^2) (da)^2 + 6 eta lambda^2 (c_L - c_R)^2 (a + db/kappa)^2, hence M_spec^2 = 3 lambda^2 (c_L - c_R)^2/(c_L^2 + c_R^2).
4. At M = 0, lambda = kappa, p_L = sqrt(2/l) cos(kappa x), p_R = sqrt(2/l) sin(kappa x), c_L = 8/(3 pi), c_R = 4/(3 pi), and M_spec^2 = 3 (8 - 4)^2/(8^2 + 4^2) M_KK^2 = (3/5) M_KK^2.

Scope. The one-fermion abelian mixed-boundary reduction at leading spectral-action order. The matrices diag(c_L, c_R) describe the coefficient of the retained vector in the induced connection, distinct from an endpoint U(1) charge assignment.

Proof. Lemma 14, Result 6; the proof is there. Transport W = exp(-(ib/kappa)(1 - cos kappa x)) gives Phi = lambda e^{ib/kappa} and the induced connections, from which D Phi = i Phi (c_L - c_R)(a + db/kappa) and F_L = c_L da, F_R = c_R da.

What it leaves open. Covariance repairs the defect of 13_nogo_fixed_compression_not_covariant.md and leaves the ratio 3/5 of the restricted computation of lemma 13 unchanged; a bulk mass can restore equality (14_matching_at_tuned_bulk_mass.md).

Part S37 (A gapped spectral projection of the fluctuated interval operator is gauge covariant and carries an induced Grassmannian connection on the light-fermion bundle, and the compressed operator is the Dirac operator of that bundle)

A gapped spectral projection of the fluctuated interval operator is gauge covariant and carries an induced Grassmannian connection on the light-fermion bundle, and the compressed operator is the Dirac operator of that bundle

Status. Theorem (Theorems 1 and 2 of lemma 14). Proved by Astra 6 (job 14); checked by Terra 6.1 by hand, with an amendment to Theorem 1 on the double-coset statement (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l14_covariant`. Cited after Proposition F of paper/towerless_intervals.tex ("\supp{14}").

Setting. I = [0, l], E = [[0, 1], [-1, 0]], Gamma = diag(1, -1); the vertical operator D_{A_5, M, B} = E (d/dx + i A_5) + [[0, M], [M, 0]] with Hermitian matrix functions A_5 and M and a self-adjoint boundary domain B; gauge transformations act in the same representation on both chiralities. A family smooth in y in the norm-resolvent sense, after local identification of domains, whose selected light cluster is separated from the rest of the spectrum and contains complete eigenspaces; P_A(y) its spectral projection. On a closed Riemannian spin four-manifold, D_A = -i gamma^mu (nabla_mu^S + d_mu + i A_mu) + gamma_5 D_{I,A_5}.

Statement.

1. (Theorem 1) P_{A^g} = G_g P_A G_g^dagger, provided the transformation includes A_5^g = g A_5 g^dagger + i (dg/dx) g^dagger, M^g = g M g^dagger and B^g = G_g B.
2. With transport dW/dx = -i A_5 W, W(0) = 1, the vertical spectrum is determined by W^dagger M W and W^dagger B. For a gauge-invariant bulk mass and R(0) = 0, L(l) = 0, the entire spectrum is independent of A_5. The open Wilson line transforms as W_l^g = g(l) W_l g(0)^dagger; with endpoint groups H_0, H_l acting independently, its class lies in H_l G / H_0.
3. (Theorem 2) With E_y = Ran P_A(y), the induced connection is nabla^E_mu = P_A (d_mu + i A_mu) P_A and the vertical endomorphism is V = P_A D_{I,A_5} P_A, and exactly P_A D_A P_A = -i gamma^mu (nabla_mu^S + nabla_mu^E) + gamma_5 V.
4. In a local orthonormal frame U(y), A_mu = U^dagger A_mu U - i U^dagger d_mu U and V_F = U^dagger D_{I,A_5} U, with curvature P_A (i F) P_A + P_A [nabla P_A, nabla P_A] P_A; under U^g = G_g U h^dagger the data transform covariantly.

Scope (amendment). The double-coset statement needs the endpoint transformations to act independently through H_0 and H_l, with extendability understood, and is a local pointwise classification unless the global topology is specified; periodic conditions require g(l) = g(0), and then the holonomy is classified by conjugation. If the gap closes, no fixed-rank smooth light bundle is asserted. The compression differs from eliminating the other fermions, which adds -P_A D_A P^perp [P^perp (D_A - z) P^perp]^{-1} P^perp D_A P_A.

Proof. Lemma 14, Theorems 1 and 2; the proofs are there. Conjugation of the resolvent in the contour formula for P_A proves covariance, W^dagger D W removes the connection, and the frame computation gives the connection and curvature.

What it leaves open. Covariance alone fixes neither the fluctuation image (14_link_fixed_singular_values.md) nor equality of the vector masses with the tower (14_matching_sum_rule.md).

Part S38 (For the mixed interval with gauge-invariant bulk mass, the covariant light link is lambda W_l^dagger, with fixed singular values, so it supplies only the unitary part of the two-sheet fluctuation space)

For the mixed interval with gauge-invariant bulk mass, the covariant light link is lambda W_l^dagger, with fixed singular values, so it supplies only the unitary part of the two-sheet fluctuation space

Status. Theorem (Theorem 3 of lemma 14). Proved by Astra 6 (job 14); checked by Terra 6.1 by hand, who found the global form as first written false and restricted the image statement to hold locally in spacetime (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l14_image`. Compared with the finite fluctuations in lemma 19 (19_comparison_with_parent.md).

Setting. D_{I,A_5} = [[0, d/dx + i A_5 + M], [-d/dx - i A_5 + M, 0]], M real, with R(0) = 0 and L(l) = 0 on C^n-valued fields; the parent algebra C^infinity(I, M_n(C)) acting identically on both chiralities. Real normalized lowest profiles p_L, p_R with (-d/dx + M) p_L = lambda p_R, (d/dx + M) p_R = lambda p_L; the endpoint-adapted frame U_L = p_L W, U_R = p_R W W_l^dagger; w_L = p_L^2, w_R = p_R^2, B_mu = W^dagger A_mu W - i W^dagger d_mu W.

Statement.

1. V_F = [[0, lambda W_l^dagger], [lambda W_l, 0]], A_{L,mu} = int w_L B_mu dx and A_{R,mu} = W_l (int w_R B_mu dx) W_l^dagger - i W_l d_mu W_l^dagger.
2. The range of the compression is A_F = M_n(C) + M_n(C) on H_F = C^n_L + C^n_R, Gamma_F = diag(1, -1). With its matrix product it gives a complex even finite triple with unitary group U(n)_L x U(n)_R, whose unrestricted inner fluctuations have an arbitrary complex total link.
3. The continuum construction produces exactly Phi^dagger Phi = Phi Phi^dagger = lambda^2 I_n for the link and, locally in spacetime, arbitrary pairs of Hermitian sheet connections: the fixed-singular-value restriction of that finite fluctuation space. The inherited local gauge action is h = diag(g(0), g(l)).
4. Compression is a representation on a unital *-subalgebra exactly when that subalgebra reduces P, since C(a^* a) - C(a)^* C(a) = P a^* (1 - P) a P. Here multiplication by a(x) reduces the light pair exactly when a(x) = W(x) b W(x)^dagger with b constant; its restriction is one diagonal M_n(C), commuting with V_F, with vanishing finite scalar one-forms.

Scope (correction). The link image holds locally in spacetime. Globally the unitary link must admit a path from the identity in the chosen trivialization: for M_4 = S^1 x S^3, n = 1 and Phi = lambda e^{i theta}, every globally smooth real A_5 gives zero winding, so this Phi is not attained. No real structure, real gauge-group prescription or Standard Model representation follows from the construction.

Proof. Lemma 14, Theorem 3; the proof is there. Gauge removal and the profile equations give V_F; the linear independence of w_L and w_R gives the range and the surjectivity on sheet connections.

What it leaves open. The radial direction of the link is absent here: in the commuting-bulk mixed model A_5 supplies only the unitary link, so there is no radial Higgs kinetic term to match, and boundary Yukawa matrices promoted to fields would need their own continuum dynamics (14_normalizations_not_fixed.md).

Part S39 (In the covariant mixed-interval reduction, M_spec = M_KK holds for some bulk mass 0 < M_* < 3 pi/(4l), at which the matched fermion sits at the tower scale)

In the covariant mixed-interval reduction, M_spec = M_KK holds for some bulk mass 0 < M_* < 3 pi/(4l), at which the matched fermion sits at the tower scale

Status. Proposition (Result 7 of lemma 14). Proved by Astra 6 (job 14); checked by Claude Opus 5.5 by hand (t = 3 pi/4: M = 3 pi/(4l), lambda^2/m^2 = 9/2, normalization l (1/2 + 1/(3 pi)), ratio 27/26, then the intermediate value theorem) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l14_tuned`, depending on `l09_lightpair`. Cited after Proposition F of paper/towerless_intervals.tex.

Setting. The gauge and fermion boundary conditions of 14_counterexample_three_fifths.md at fixed l, with the fermion bulk mass varying over 0 <= M <= 3 pi/(4l).

Statement.

1. At least one value 0 < M_* < 3 pi/(4l) satisfies M_spec(M_*) = M_KK.
2. An exact defining condition is 3 lambda(M_*)^2 (c_L(M_*) - c_R(M_*))^2 = (pi^2/(4l^2)) (c_L(M_*)^2 + c_R(M_*)^2).
3. No uniqueness is asserted.
4. At M = 3 pi/(4l): lambda^2 = 9 pi^2/(8l^2), c_L = 9/(3 pi + 2), c_R = 6/(3 pi + 2), and M_spec^2/M_KK^2 = 27/26 > 1; at M = 0 the ratio is 3/5 < 1.
5. So universal impossibility of mass matching is false, and automatic matching is false as well.

Scope. The matching bulk mass M_* > 0 leaves no light pair (lemma 09, Theorem 9.4, case 3), so the matched fermion has a mass of the order of the tower scale (lemma 14, header).

Proof. Lemma 14, Result 7; the proof is there. With t = kl in [pi/2, pi), M = -k cot t and lambda = k/sin t, with profiles proportional to sin(k(l - x)) and sin(kx); the lowest eigenvalue is simple, root and profiles depend continuously on M, and the intermediate value theorem applies.

What it leaves open. Covariance supplies no selection principle for M_* (lemma 14, consequence). With bounded light fermions no matching is possible as l -> 0 (14_nogo_bounded_light_fermions.md).

Part S40 (At leading spectral-action order, equality of the spectral-action vector masses with the Kaluza-Klein masses on a common profile space is an explicit trace-and-overlap sum rule)

At leading spectral-action order, equality of the spectral-action vector masses with the Kaluza-Klein masses on a common profile space is an explicit trace-and-overlap sum rule

Status. Theorem (Theorem 5 of lemma 14). Proved by Astra 6 (job 14); checked by Claude Opus 5.5 by hand (both mass matrices as quadratic forms) and by Terra 6.1 by hand, with an amendment requiring a common profile space (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l14_sum_rule`. Cited after Proposition F of paper/towerless_intervals.tex.

Setting. Five-dimensional Yang-Mills with no boundary kinetic or mass terms and positive invariant form <,>_5 including its couplings; retained transverse vectors A_mu = sum_a a_mu^a(y) F_a(x) with fixed Hermitian profiles F_a obeying the chosen gauge boundary conditions; G^(5)_ab = int <F_a, F_b>_5 dx, H^(5)_ab = int <F_a', F_b'>_5 dx. A light frame U_0 at the background, T_a = U_0^dagger F_a

U_0, V_0 = U_0^dagger D_I U_0; the literal spectral trace through a_4, f_0 > 0, no independent kinetic terms; Q and K as in 13_spectral_action_vector_mass_formula.md; eta = f_0/(24 pi^2).

Statement.

1. M_KK^2 = (G^(5))^{-1/2} H^(5) (G^(5))^{-1/2}, and on the subspace where Q > 0, M_spec^2 = (3/2) Q^{-1/2} K Q^{-1/2}.
2. Equality of the vector mass spectra is equivalent to spec((3/2) K, Q) = spec(H^(5), G^(5)); for a retained sector with KK mass matrix kappa^2 I it is the sum rule K = (2/3) kappa^2 Q. Equality of the quadratic actions in the same coordinates is stronger: G^(5) = 4 eta Q and H^(5) = 6 eta K.
3. For a chiral block V_0 = [[0, Y], [Y^dagger, 0]] and T_a = diag(t_{L,a}, t_{R,a}), with C_a = t_{L,a} Y - Y t_{R,a}: Q_ab = Re Tr(t_{L,a} t_{L,b} + t_{R,a} t_{R,b}) and K_ab = 2 Re Tr(C_a^dagger C_b).
4. For mixed pairs r of masses lambda_r, one gauge profile f with overlaps c_{L,r} = int |p_{L,r}|^2 f, c_{R,r} = int |p_{R,r}|^2 f and multiplicities tau_r, the one-vector condition is 3 sum_r tau_r lambda_r^2 (c_{L,r} - c_{R,r})^2 = kappa^2 sum_r tau_r (c_{L,r}^2 + c_{R,r}^2). If both sheet connections fluctuate independently, M_relative^2 = 6 sum_r tau_r lambda_r^2/sum_r tau_r, a comparison with an additional massless diagonal vector.

Scope (amendment). Both forms must be compared on one common profile space on which G^(5) and Q are positive, since a KK profile can be invisible to compression: for the massless mixed pair, f = cos(2 pi x/l) has positive norm while int w_L f = int w_R f = 0. The equalities concern the a_4 approximation and the chosen vacuum and compare quadratic transverse actions; a nonabelian truncation to nonconstant lowest profiles need not be closed under gauge transformations.

Proof. Lemma 14, Theorem 5; the proof is there, by expanding both actions and normalizing canonically, with a_can = 2 sqrt(eta) Q^{1/2} a.

What it leaves open. Covariance does not enforce the sum rule: it fails at zero bulk mass (14_counterexample_three_fifths.md) and holds at a tuned bulk mass (14_matching_at_tuned_bulk_mass.md).

Part S41 (At leading spectral-action order, bounded light fermions give M_spec^2 <= 6 C^2, so they cannot match the collapsing Neumann-Dirichlet tower mass pi/(2l))

At leading spectral-action order, bounded light fermions give M_spec^2 <= 6 C^2, so they cannot match the collapsing Neumann-Dirichlet tower mass pi/(2l)

Status. No-go theorem in its scope (No-go theorem 8 of lemma 14). Proved by Astra 6 (job 14); checked by Claude Opus 5.5 by hand (M_spec^2 <= 6 ||V_0||^2) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l14_nogo_collapse`. Cited after Proposition F of paper/towerless_intervals.tex.

Setting. A family of induced finite operators with ||V_0(l)|| <= C < infinity; the leading spectral action of 14_matching_sum_rule.md, with no additional vector-mass or kinetic terms. The retained gauge couplings may depend on l, and any finite multiplicities may be included.

Statement.

1. Every canonically normalized spectral-action vector mass satisfies M_spec^2 <= 6 C^2, also after restriction to any gauge-profile subspace; multiplicities cannot evade the bound because they enter both Hilbert-Schmidt norms.
2. Consequently M_spec^2 cannot equal M_KK^2 = pi^2/(4l^2) for arbitrarily small l, and matching any such vector requires ||V_0(l)|| >= pi/(2 sqrt6 l).
3. This applies to the fixed-light-mass collapse of lemma 09 (Theorem 9.9) and to the bounded-texture limit of lemma 14, Result 4.

Hypotheses. Leading (a_4) spectral-action order, no added bosonic terms, retained fermion masses bounded uniformly in l.

Escapes, as the lemma states them. Some retained fermion scale growing at least as 1/l, or a change of the bosonic action or of the stated approximation. Near the cutoff the leading formula is replaced by the exact kernel of lemma 18, from which the bound M^2 <= 6 ||V_0||^2 follows only through its asymptotic form (lemma 18, remark after Corollary 3).

Proof. Lemma 14, No-go theorem 8; the proof is there: ||[T, V_0]||_HS <= 2C ||T||_HS gives K <= 4 C^2 Q as quadratic forms, and the mass formula gives M_spec^2 <= 6 C^2 I.

What it leaves open. The result points to tower-scale fermions in the trace (14_matching_at_tuned_bulk_mass.md) or to a different bosonic action.

Part S42 (Matching one vector mass fixes neither the relative gauge couplings nor the scalar kinetic coefficients transverse to the gauge orbit)

Matching one vector mass fixes neither the relative gauge couplings nor the scalar kinetic coefficients transverse to the gauge orbit

Status. Proposition (Result 9 of lemma 14). Proved by Astra 6 (job 14); checked by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_08_13_14.md). Ledger row `l14_normalizations`.

Setting. A finite gauge kinetic form diagonal on two specified factors, Q restricted to g_s equal to q_s I for s = 1, 2, with fixed generator normalizations; five-dimensional couplings g_{5,s} and J_s = int f_s(x)^2 dx; eta = f_0/(24 pi^2). Scalar coordinates phi^i with induced connection A_mu = B_i(phi) d_mu phi^i at vanishing ambient A_mu, and H_i = d_i V + i [B_i, V].

Statement.

1. The leading finite spectral action predicts 1/g_{4,s}^2 = 4 eta q_s and g_{4,1}^2/g_{4,2}^2 = q_2/q_1, while the KK action gives 1/g_{4,s}^2 = J_s/g_{5,s}^2.
2. Rescaling g_{5,s}^{-2} multiplies G^(5)_s and H^(5)_s by the same factor and leaves every KK mass unchanged, so mass matching cannot determine g_{5,1}/g_{5,2}.
3. The spectral-action scalar metric, in the convention (1/2) Z_ij d phi^i d phi^j, is Z_ij^spec = 6 eta Re Tr(H_i H_j). Mass matching constrains it on the gauge-orbit directions: for the eaten scalar of Result 6, with G = l/(2 g_5^2), q = c_L^2 + c_R^2 and d = c_L - c_R, matching the gauge kinetic coefficient (4 eta q = G) and the vector mass (3 lambda^2 d^2 = kappa^2 q) gives Z_b^spec = 12 eta lambda^2 d^2/kappa^2 = G with no separate tuning.
4. A direction transverse to the gauge orbit is not tested by the mass equation. In the commuting-bulk mixed model A_5 supplies only the unitary link of Result 3, so there is no radial Higgs kinetic term to match.

Scope. A common simple parent group with a specified invariant form could remove the freedom in g_{5,1}/g_{5,2}; that is additional geometric input.

Proof. Lemma 14, Result 9; the proof is there, by canonical normalization of eta q_s F_s^2 and by D_mu V = H_i d_mu phi^i substituted in 3 eta Tr(D_mu V D^mu V).

What it leaves open. Boundary Yukawa matrices promoted to independent scalar fields need their continuum dynamics, including any boundary kinetic form, specified separately. Hypothesis H needs additional geometric relations (lemma 14, consequence).

Part S43 (Limits in the propinquity of commutative quantum compact metric spaces are commutative, and the number of connected components does not increase in the limit)

Limits in the propinquity of commutative quantum compact metric spaces are commutative, and the number of connected components does not increase in the limit

Status. Theorem (Theorem 15.2, Lemma R3 of the Farsi-Latrémolière assessment), with Lemma 15.1. Sketch by a Claude reading subagent; proof by Claude Opus 5.5 from the definitions in the LaTeX source of arXiv:2404.00240v3, Section 2; checked by Terra 6.1 by hand, who closed a gap in item 2 (historical/reviews/2026-11-03_terra_check_15_16.md). Ledger row `fl_R3`; Corollary 15.8 (row `l15_finite_limits`) depends on it. Item 3 of Proposition G of paper/towerless_intervals.tex, whose proof also uses Lemma 15.1.

Setting. Quantum compact metric spaces (A, L), quantum isometries, tunnels with their extent chi, and the propinquity Lambda, as in the Conventions of lemma 15 (Farsi-Latrémolière, Section 2). The spectral propinquity dominates Lambda up to the cap: Lambda_spec(T_1, T_2) >= min(sqrt(2)/2, Lambda(A_1, A_2)) (consequence (b)).

Statement.

1. (Lemma 15.1) For a tunnel (E, L_E, pi_1, pi_2) of extent chi and e = e^* with L_E(e) finite, ||pi_j e|| <= ||pi_k e|| + chi L_E(e) and ||e|| <= ||pi_j e|| + chi L_E(e) for {j, k} = {1, 2}; for a real polynomial p = sum p_m x^m, L_E(p(e)) <= L_E(e) sum_{m >= 1} |p_m| m ||e||^{m-1}.
2. (Theorem 15.2, item 1) If (A_n, L_n) -> (A, L) in the propinquity and every A_n is commutative, then A is commutative.
3. (Theorem 15.2, item 2) If every A_n = C(X_n) with X_n having at most k connected components, the spectrum of A has at most k connected components. A finite-dimensional limit is C^j with j <= k, and for k = 1 it is C.
4. Both statements hold for limits in Lambda_spec, by consequence (b).

Scope. One metric, the propinquity or the spectral propinquity, applied to the whole scalar algebra. The theorem is one-directional: noncommutative spaces may converge to commutative ones, as matrix algebras do to the sphere. Morita equivalence does not remove the restriction to the scalar algebra: C(K) and C(K, M_n) are Morita equivalent with different limits, and replacing the represented algebra by End(E) is a change of data. Truncations lie outside the scope (lemma 16).

Proof. Lemma 15, Lemma 15.1 and Theorem 15.2; the proofs are there. For item 1, lifts of two elements to the tunnel have a Lie product killed by one projection and of uniformly bounded L_E, so its image has norm at most chi_n times a bound. For item 2, polynomial approximations of step functions on the spectrum, together with Lemma 15.1, separate k + 1 clopen pieces and give a contradiction.

What it leaves open. We did not find Theorem 15.2 stated in arXiv:1302.4058, 1311.0104, 2404.00240 or 2603.19128; its proof is elementary and it may be folklore. Noncommutative finite data from commutative parents need truncations or a change of the represented algebra (lemma 15, "What this says about T0").

Part S44 (A sequence of metric spectral triples converges to (C, C^N, D_F) in the spectral propinquity iff the diameter tends to 0, the gap tends to infinity and the light-block error tends to 0, so the finite limits of connected commutative triples are exactly the triples (C, C^N, D_F))

A sequence of metric spectral triples converges to (C, C^N, D_F) in the spectral propinquity iff the diameter tends to 0, the gap tends to infinity and the light-block error tends to 0, so the finite limits of connected commutative triples are exactly the triples (C, C^N, D_F)

Status. Theorem (Theorem 15.7), with Corollary 15.8. Proved by Claude Opus 5.5; checked by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_15_16.md). Ledger rows `l15_converse` and `l15_finite_limits` (depending on `fl_R3`). Proposition G and its item 3 in paper/towerless_intervals.tex; the main paper reaches its contradiction with Theorem 3.4 of Farsi-Latrémolière through Theorem 15.7 (results/R1_counterexample_farsi_latremoliere.md).

Setting. T_n = (A_n, H_n, D_n) metric spectral triples and T_F = (C, C^N, D_F), with the conventions of lemma 15.

Statement.

1. (Theorem 15.7) If Lambda_spec(T_n, T_F) -> 0, then diam T_n -> 0.
2. For every R > ||D_F||, D_n has exactly N eigenvalues in [-R, R] for large n, so the gap g_n to the remaining spectrum tends to infinity (g_n is infinite if no spectrum remains).
3. There are unitaries W_n from the light space onto C^N with ||W_n D_n W_n^* - D_F|| -> 0 (at most 2 eps_n).
4. With Theorem 15.3: Lambda_spec(T_n, T_F) -> 0 iff diam T_n -> 0, g_n -> infinity and Delta_n -> 0.
5. (Corollary 15.8) If A_n = C(X_n) with X_n connected and the limit has finite-dimensional Hilbert space, the limit is (C, C^N, D_F), and the convergence is characterized by item 4. Every Hermitian D_F arises: on the circle of length l, -i d/dx + diag(lambda_i) on L^2(S^1_l; C^N) has the block diag(lambda_i) on constants and gap at least 2 pi/l - ||D_F||.

Scope. One metric, on the whole algebra. The construction in item 5 concerns ungraded triples; a grading or real structure imposes its own compatibility conditions. The class of metric triples excludes the Standard Model finite triple, where D_F commutes with the nonscalar subalgebra {(lambda, lambda, 0)}.

Proof. Lemma 15, Theorem 15.7 and Corollary 15.8; the proofs are there. The inner Leibniz inequality bounds the number of light eigenvalues by N through a Gram matrix; the separation of exp(i t D_n) concentrates the spectral measure of each lift within 2 eps_n of the corresponding eigenvalue of D_F; metric triples are faithful, and Theorem 15.2 gives the algebra C.

What it leaves open. Corollary 9.10 fits the converse (at fixed mu every eigenvalue diverges and no limit exists), and Theorem 9.8 lies outside the metric, since the diameter diverges. The truncation is a different object, which this metric does not see (lemma 15, "What this says about T0").

Part S45 (A metric triple on M_2(C) whose Dirac operator is the traceless projection commutes with every inner conjugation, so Proposition 15.10 needs order one)

A metric triple on M_2(C) whose Dirac operator is the traceless projection commutes with every inner conjugation, so Proposition 15.10 needs order one

Status. Counterexample (Counterexample 15.11). Proved by Terra 6.1 in the referee report on lemma 15 (historical/reviews/2026-11-03_terra_check_15_16.md); checked by Claude Opus 5.5 by hand. Ledger row `l15_symmetry_ce`. It refutes the first version of the T0 section of lemma 15 (row `t0_unbroken_unrestricted`, status refuted: "A metric spectral triple excludes every unbroken inner gauge symmetry").

Setting. H = M_2(C) with the Hilbert-Schmidt product, A = M_2(C) acting by left multiplication L_a, J x = x^*, and D x = x - (1/2) tr(x) I.

Statement.

1. Applying [D, L_a] to I gives a - (1/2) tr(a) I, so [D, L_a] = 0 forces a scalar, and the triple is metric.
2. Every conjugation x -> u x u^* = L_u J L_u J^{-1} commutes with D.
3. The triple fails order one.

Scope. The statement refuted is the unrestricted claim that metric triples exclude every unbroken inner gauge symmetry. Among finite real triples with order zero and order one the restricted claim holds (15_metric_class_no_unbroken_symmetry.md).

Proof. Lemma 15, Counterexample 15.11; the computation is there.

What it leaves open. Without order one, metric triples can carry unbroken inner conjugations, so frameworks that drop order one, such as the fluctuations of lemma 19, lie outside Proposition 15.10.

Part S46 (A metric spectral triple with a finite spectral block of gap g lies within 1/g^2 + max(delta/2, Delta) + Delta + sqrt(Delta) of (C, C^N, D_F) in the spectral propinquity, with no anticommutation hypothesis)

A metric spectral triple with a finite spectral block of gap g lies within 1/g^2 + max(delta/2, Delta) + Delta + sqrt(Delta) of (C, C^N, D_F) in the spectral propinquity, with no anticommutation hypothesis

Status. Theorem (Theorem 15.3, Lemma R2 of the Farsi-Latrémolière assessment), with Remark 15.4 and Corollaries 15.5 and 15.6. Proved by Claude Opus 5.5; the lower bound of the subagent sketch omitted the cap sqrt(2)/2 and was corrected. Checked by Terra 6.1 by hand, who also restored the cap in Remark 15.4 (historical/reviews/2026-11-03_terra_check_15_16.md). Ledger rows `l15_gap`, `fl_R2` (Corollary 15.5, depending on `l09_limits`) and `l15_yukawa` (Corollary 15.6). Proposition G of paper/towerless_intervals.tex, with its items 1 and 2.

Setting. T = (A, H, D) a metric spectral triple with delta = diam T; Sigma a finite subset of R, P = 1_Sigma(D) of rank N >= 1, Q = 1 - P, and g > 0 with |lambda| >= g on Sp(D) outside Sigma; D_F a Hermitian N x N matrix, W: PH -> C^N unitary (extended by 0), Delta = ||W D W^* - D_F||, T_F = (C, C^N, D_F). The spectral propinquity is capped at sqrt(2)/2 and has the units of a length.

Statement.

1. (Theorem 15.3) min(sqrt(2)/2, delta/2) <= Lambda_spec(T, T_F) <= 1/g^2 + max(delta/2, Delta) + Delta + sqrt(Delta); for Delta = 0 the upper bound reads delta/2 + 1/g^2. No anticommutation hypothesis enters.
2. (Remark 15.4) For T_eps = (A, H, D/eps) with 0 in Sp(D), Sigma = {0} and D_F = 0: min(sqrt(2)/2, eps delta_1/2) <= Lambda_spec <= eps delta_1/2 + eps^2/g_1^2, with delta_1 = diam T_1 and g_1 the gap of D. This is Corollary 4.1 of Farsi-Latrémolière with a sharp leading rate; Counterexample R1 violates the gap hypothesis (results/R1_counterexample_farsi_latremoliere.md).
3. (Corollary 15.5) For the tuned interval T_l of Theorem 9.9 and small l, l/2 <= Lambda_spec(T_l, (C, C^2, m_f sigma_1)) <= l/2 + l^2/(pi^2 + c^2), and for the Connes-Lott triple CL(m_f) = (C + C, C^2, m_f sigma_1) of diameter 1/m_f, Lambda_spec(T_l, CL(m_f)) >= min(sqrt(2)/2, (1/m_f - l)/2).
4. (Corollary 15.6) The towerless Yukawa family of Theorem 12.7 is a family of metric spectral triples of diameter l, and Lambda_spec(T_{X,l}, (C, C^{N_Q + N_U}, D_F)) = O(l) with D_F the Yukawa block of M_F = Phi_Q Y_hat Phi_U. In this metric the algebra of the limit is C.

Scope. One metric, applied to the whole algebra. The whole tuned triple converges to a point carrying a two-dimensional massive Dirac block while its compression at each l is the Connes-Lott triple (Theorem 9.6); both statements hold, since they concern different objects and the compression is not a quantum isometry.

Proof. Lemma 15, Theorem 15.3, Remark 15.4 and Corollaries 15.5 and 15.6; the proofs are there. The top tunnel is A + C with L_E(a, z) = max{L_T(a), ||a - z||/rho}, the module H + C^N carries DN_P = max{DN_T, DN_F, ||W xi - eta||/rho}, the Leibniz inequalities hold because the limit algebra acts by scalars, and ||Q xi|| <= ||D xi||/g bounds the rest.

What it leaves open. The converse is 15_converse_and_finite_limits.md. Convergence of J needs an extension of the pseudo-state formalism to antilinear functionals (15_grading_and_real_structure_descend.md).

Part S47 (For a symmetric spectral block the grading and the real structure of the parent descend to the block with their signs, and the tunnel of the gap criterion controls the grading)

For a symmetric spectral block the grading and the real structure of the parent descend to the block with their signs, and the tunnel of the gap criterion controls the grading

Status. Proposition (Proposition 15.9). Proved by Claude Opus 5.5; checked by Terra 6.1 by hand, with amendments incorporated (historical/reviews/2026-11-03_terra_check_15_16.md). Ledger row `l15_grading`. Item 4 of Proposition G of paper/towerless_intervals.tex.

Setting. Theorem 15.3 (15_gap_criterion.md) with Sigma = -Sigma. D anticommutes with a grading gamma and satisfies J D = eps' D J for an antiunitary J with J^2 = eps and J gamma = eps'' gamma J. D_B = W D W^*, and rho, c = 1/(1 + Delta) as in the proof of Theorem 15.3.

Statement.

1. P commutes with gamma and J.
2. gamma_F = W gamma W^* and J_F = W J W^* satisfy the same relations exactly with D_B, and with D_F up to defects bounded by 2 Delta.
3. For DN_T(omega) <= 1 and DN_P(xi, eta) <= 1, |<omega, gamma xi> - <W omega, gamma_F eta>| <= 1/g^2 + rho and |<omega, J xi> - <W omega, J_F eta>| <= 1/g^2 + rho.
4. For Delta > 0, W omega need not be a pseudo-state of T_F; the normalized c W omega is one, and adds at most Delta to these bounds.
5. So gamma added to the families converges along the same tunnels, and the KO-signs of the parent reach the block unchanged when Delta = 0. For the interval, P Gamma P = sigma_3 (Theorem 9.6).

Scope. The metric ignores gamma and J; the proposition shows they pass to the block. Composition with the antilinear J gives antilinear functionals, which the definition of the separation does not cover, so a statement of convergence for J needs an extension of the pseudo-state formalism that lemma 15 does not develop.

Proof. Lemma 15, Proposition 15.9; the proof is there: spectral projections of D for a symmetric set commute with gamma and J, because gamma and J map the lambda-eigenspace to the eigenspaces of -lambda and eps' lambda; then decompose along P and Q as in Theorem 15.3, using additivity of J_F.

What it leaves open. Convergence of the real structure J in a metric that sees antilinear functionals.

Part S48 (In a finite real triple with order zero and order one, a unitary u of the algebra whose inner automorphism u J u J^{-1} commutes with D commutes with D itself, so a metric such triple has no nontrivial unbroken inner gauge transformation)

In a finite real triple with order zero and order one, a unitary u of the algebra whose inner automorphism u J u J^{-1} commutes with D commutes with D itself, so a metric such triple has no nontrivial unbroken inner gauge transformation

Status. Proposition (Proposition 15.10). Proved by Terra 6.1 in the referee report on lemma 15 (historical/reviews/2026-11-03_terra_check_15_16.md); checked by Claude Opus 5.5 by hand. Ledger row `l15_first_order`.

Setting. (A, H, D, J) a finite-dimensional real triple with order zero, order one and J D = eps' D J; u a unitary in A and v = J u J^{-1}. A metric triple has ker L_T = R 1 for L_T(a) = ||[D, a]||.

Statement.

1. If D commutes with u v, then [D, u] = 0.
2. If the triple is moreover metric, u is scalar, and the inner gauge transformation is trivial.
3. Among finite real first-order triples, the metric class therefore has no nontrivial unbroken inner gauge transformation.

Scope. Order one is needed (15_counterexample_order_one_needed.md). A metric triple has no nonscalar element commuting with D, so the Standard Model finite triple lies outside the class: D_F commutes with the nonscalar subalgebra C_F = {(lambda, lambda, 0)}, which keeps the photon massless, and colour is unbroken (Chamseddine-Connes-Marcolli 2007, section 2.6). Its almost-commutative product lies outside the class as well, and A_F, a real algebra, needs a complexification for the complex framework.

Proof. Lemma 15, Proposition 15.10; the proof is there: from [D, uv] = 0, a = u^* [D, u] = -[D, v] v^*, which commutes with u by order one and order zero, so [D, u^n] = n a u^n by induction and boundedness of D forces a = 0.

What it leaves open. The theorems of lemma 15 apply to metric triples and therefore do not reach the Standard Model finite triple as a limit; lemma 16 treats truncations instead (lemma 15, "What this says about T0", item 2).

Part S49 (On the doubled parent H_0 (x) conj(H_0) with the graded flip J the compressed data satisfy order zero and order one for every generated algebra, noncommutative or not, and these two conditions are formal consequences of the tensor square)

On the doubled parent H_0 (x) conj(H_0) with the graded flip J the compressed data satisfy order zero and order one for every generated algebra, noncommutative or not, and these two conditions are formal consequences of the tensor square

Status. Theorem (Theorem 16.2), with Corollary 16.3 and Remark 16.5. Proved by Claude Opus 5.5; Remark 16.5 comes from the referee's assessment. Checked by Claude Opus 5.5 in Python (numpy on random finite data: all signs, order zero, order one) and by Terra 6.1 by hand, with amendments incorporated (historical/reviews/2026-11-03_terra_check_15_16.md). Ledger rows `l16_coarse`, `l16_truncation` and `l16_formal` (Remark 16.5, attempted by Terra 6.1 and checked by Claude Opus 5.5 by hand). Item 3 of Proposition H of paper/towerless_intervals.tex.

Setting. The coarse correspondence H = H_0 (x) conj(H_0) of A_0 (Connes, Noncommutative Geometry, chapter V), with the conjugate operators gamma^c, D^c and, for homogeneous xi, eta, pi(a) = a (x) 1, DD = D_0 (x) 1 + gamma_0 (x) D^c, Gamma = gamma_0 (x) gamma^c, J(xi (x) eta^c) = (-1)^{|xi||eta|} eta (x) xi^c. For the truncation, P = P_0 (x) conj(P_0) and the window V of 16_compressed_algebras_noncommutative.md.

Statement.

1. (Theorem 16.2) (A_0, H, DD, J, Gamma) is a real even spectral triple with J^2 = 1, J DD = DD J, J Gamma = Gamma J (KO-dimension 0). For even a, b with bounded commutators, a° = 1 (x) (a^*)^c, [pi(a), b°] = 0 and [[DD, pi(a)], b°] = 0. Also |||[DD, pi(a)]|| = |||[D_0, a]||, so a metric original triple gives a metric triple with the same Connes distance on A_0, and if the Weyl exponent p of D_0 exists, that of DD is 2p.
2. (Corollary 16.3) P commutes with DD, Gamma and J; it is a joint cutoff and need not be a spectral projection of DD. The truncation T_F = (A_F, V (x) conj(V), D_F, J_F, Gamma_F), D_F = d (x) 1 + gamma_V (x) d^c = Delta + J_F Delta J_F^{-1} with Delta = d (x) 1, has KO-dimension 0 and satisfies order zero and order one for every A_F, with dim H_F = k^2, exactly at each truncation. Example (a) gives (C + C, C^2 (x) C^2, m(sigma_1 (x) 1 + sigma_3 (x) sigma_1)), where [[D_F, e], e°] = 0 for e = diag(1, 0).
3. (Remark 16.5) For every finite even representation (A, V, d, gamma), V -> V (x) conj(V) and d -> d (x) 1 + gamma (x) d^c give a real bimodule with order zero and order one, using no prior finite real structure. These conditions follow from choosing the tensor square and say nothing specific about A_F.

Scope. Geometrically the data are a product-manifold Dirac operator on X x X with the restricted observable algebra C(X) (x) 1 and a J covering the flip, which differs from the pointwise spin real structure; on all of C(X x X) order zero generally fails again. The construction does not recover an arbitrary prescribed finite (H, D, J). T_F is usually not a metric triple, and its eigenvalues +-(lambda_i^2 + lambda_j^2)^{1/2} differ from the masses in d.

Proof. Lemma 16, Theorem 16.2, Corollary 16.3 and Remark 16.5; the proofs are there, by the graded-sign computation of J DD and by [DD, pi(a)] = [D_0, a] (x) 1 for even a.

What it leaves open. No dynamical reason for the doubled parent is given, and its Weyl exponent 2n against metric dimension n bears on the counting of 9, 10 and 11 without having been analyzed (lemma 16, Caveat 3). The KO-6 signs need a Clifford point or two sheets (16_ko6_by_clifford_point.md, 16_two_sheet_parent.md).

Part S50 (Compressing functions on a connected commutative space to a fixed spectral window gives a noncommutative finite algebra with no limit taken: M_2(C) + M_2(C) at the second level of the chiral interval and M_4(C) at level one on CP^3)

Compressing functions on a connected commutative space to a fixed spectral window gives a noncommutative finite algebra with no limit taken: M_2(C) + M_2(C) at the second level of the chiral interval and M_4(C) at level one on CP^3

Status. Proposition, proved by computation (Examples (b) and (c) of lemma 16). Proved by Claude Opus 5.5. Example (b) checked by Claude Opus 5.5 in Python (numpy quadrature with 23 test functions: dimensions 4 and 8, the two displayed compressions) and by Terra 6.1 by hand; example (c) checked by Terra 6.1 by hand, under Proposition 10.1 and example (c) (historical/reviews/2026-11-03_terra_check_15_16.md). Ledger rows `l16_interval_algebra` and `l10_generates_m4`; the latter, of kind counterexample, refutes an earlier T0 note (row `t0_note_overread`: compressions being operator systems would force the algebra to be produced by a limit). Item 1 of Proposition H of paper/towerless_intervals.tex.

Setting. An even triple (A_0, H_0, D_0, gamma_0), A_0 commuting with gamma_0 and D_0 of compact resolvent; Sigma = -Sigma finite, P_0 = 1_Sigma(D_0) of rank k; V = P_0 H_0 graded by gamma_V, d = D_0 restricted to V, Q(a) = P_0 a P_0, O_F = Q(A_0) and A_F = C^*(O_F) in B(V). O_F is an operator system, in general no algebra, and since Q(1) = 1, A_F is the bicommutant of O_F.

Statement.

1. (a) For the chiral interval with M = 0 and Sigma = {+-pi/(2l)}, O_F = C + C = A_F (lemma 05).
2. (b) For Sigma = {+-pi/(2l), +-3pi/(2l)}, with C_n = (1/l) int f cos(n pi x/l), Q(f) = (Q_L, Q_R) with Q_L = [[C_0 + C_1, C_1 + C_2], [C_1 + C_2, C_0 + C_3]] and Q_R = [[C_0 - C_1, C_1 - C_2], [C_1 - C_2, C_0 - C_3]]. O_F has dimension 4; each block ranges over all symmetric 2 x 2 matrices; and for f_1 = 1/2 + cos(pi x/l) + cos(2 pi x/l) + cos(3 pi x/l) and f_2 = 1/2 - cos(pi x/l) + cos(2 pi x/l) - cos(3 pi x/l), Q(f_1) = ([[1, 1], [1, 1]], 0) and Q(f_2) = (0, [[1, -1], [-1, 1]]). Hence A_F = M_2(C) + M_2(C), of dimension 8: the finite algebra of the interval is noncommutative at the second level.
3. (c) For the level-one Toeplitz compression on CP^3, Q(x_a) = Gamma_a/5 (Proposition 10.1) generate A_F = M_4(C) on V = C^4, the kernel of the Dolbeault-Dirac operator twisted by O(1), with d = 0 on V.

Scope. At fixed truncation; no limit is taken. The noncommutativity of A_F is modest, since generated algebras of compressions are in many examples full matrix algebras (Connes-van Suijlekom); the information lies in the operators commuting with all compressed functions, such as the grading in (b) and j in Example 16.9.

Proof. Lemma 16, Setting, examples (a) to (c); the computations are there, with Proposition 10.1 for (c) and the simplicity of M_2(C) for (b).

What it leaves open. With the inherited pointwise real structure these algebras fail order zero (16_nogo_inherited_reality_order_zero.md); the doubled parent restores it (16_coarse_correspondence_order_one.md).

Part S51 (The product of the truncated coarse triple with the Clifford point (C, C^2, 0, sigma_1 K, sigma_3) has the KO-6 signs, order zero, order one and dim H = 2 k^2, and fails finite orientability)

The product of the truncated coarse triple with the Clifford point (C, C^2, 0, sigma_1 K, sigma_3) has the KO-6 signs, order zero, order one and dim H = 2 k^2, and fails finite orientability

Status. Corollary (Corollary 16.4). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (numpy on random finite data: KO-6 signs) and by Terra 6.1 by hand, correct (historical/reviews/2026-11-03_terra_check_15_16.md). Ledger row `l16_truncation`, shared with Corollary 16.3. Item 3 of Proposition H of paper/towerless_intervals.tex.

Setting. The truncation T_F = (A_F, V (x) conj(V), D_F, J_F, Gamma_F) of Corollary 16.3 (16_coarse_correspondence_order_one.md), of dimension k^2, and the Clifford point (C, C^2, 0, sigma_1 K, sigma_3).

Statement.

1. The product (A_F, V (x) conj(V) (x) C^2, D_F (x) 1, J_F (x) sigma_1 K, Gamma_F (x) sigma_3) has signs (+, +, -), order zero and order one, and dim H = 2 k^2.
2. It fails the usual zero-dimensional orientability axiom: every zero-cycle built from the left and opposite actions acts as T (x) 1_2, while the grading is Gamma_F (x) sigma_3.
3. The pair (A_F, J) is reducible: the projection 1 (x) (1 + sigma_1)/2 commutes with A_F and J. With centre C, J gamma = -gamma J is impossible in the Chamseddine-Connes classification.

Scope. The Clifford point is an input, as the finite KO-6 factor F_0 is in Theorem D of the main paper. For k = 4, dim H = 32 is the count of the Chamseddine-Connes classification; in examples (b) and (c) this is a count and nothing more, since the classification's algebra M_2(H) + M_4(C) acts differently.

Proof. Lemma 16, Corollary 16.4; the proof is there: (sigma_1 K)^2 = 1 and sigma_1 K sigma_3 = -sigma_3 sigma_1 K, the D term is untouched, and a° is that of T_F tensored with 1.

What it leaves open. The irreducible form of the classification comes from a parent with two sheets (16_two_sheet_parent.md); a grading-reversing structure as in lemma 08 is a candidate for the KO-6 sign that lemma 16 has not tested (Caveat 4).

Part S52 (A grading inherited by compression from an even parent commutes with all compressed functions, so the grading Gamma_5 of the twistor-level-one data cannot come from the stated even parent)

A grading inherited by compression from an even parent commutes with all compressed functions, so the grading Gamma_5 of the twistor-level-one data cannot come from the stated even parent

Status. No-go theorem in its scope (the grading statement of Example 16.9). From the second referee report on lemma 16 by Terra 6.1 (historical/reviews/2026-11-03_terra_check_18_16b.md); checked by Claude Opus 5.5 by hand and by Astra 6 by hand (lemma 20, Result 11, a consensus check). Ledger row `l16_grading_not_even`. Stated in the discussion after Proposition H of paper/towerless_intervals.tex.

Setting. A parent grading gamma, a finite-rank projection P, and the retained observables of Example 16.9 (16_twistor_level_one_m2h_m4c.md), which include Q(x_a) = Gamma_a/5.

Statement. Under the simultaneous requirements

1. every parent observable considered is even, [gamma, a] = 0;
2. the finite grading is inherited as g = P gamma P on PH;
3. this compression is a grading, g^2 = 1;
4. the retained observables include Q(x_a) = Gamma_a/5,

the grading g = Gamma_5 is impossible. Indeed g^2 = 1 gives ((1 - P) gamma P)^* ((1 - P) gamma P) = 0, so [P, gamma] = 0 and [g, Q(a)] = P [gamma, a] P = 0 for every retained even observable, while [Gamma_5, Q(x_1)] = (2/5) Gamma_5 Gamma_1 != 0. In the specified Dolbeault kernel the inherited grading is already scalar.

Escapes (lemma 20, Result 11). The conclusion does not exclude Gamma_5 as additional finite data or as the grading of a prealgebra. Possible changes are compressing a graded prealgebra with odd observables, restricting to the algebra commuting with the proposed grading, or constructing an additional grading. Changing the window alone does not remove the contradiction while the four hypotheses are kept.

Proof. Lemma 16, Example 16.9, and lemma 20, Result 11; the proofs are there.

What it leaves open. A parent that yields the grading of the classification and nonzero mass data (lemma 16, Consequence for T0 and T0b).

Part S53 (With an inherited pointwise real structure J_0 a J_0^{-1} = a^*, compressed data satisfy order zero if and only if the generated algebra is commutative)

With an inherited pointwise real structure J_0 a J_0^{-1} = a^*, compressed data satisfy order zero if and only if the generated algebra is commutative

Status. No-go theorem in its scope (Proposition 16.1). Proved by Claude Opus 5.5; checked by Terra 6.1 by hand, with an amendment incorporated (historical/reviews/2026-11-03_terra_check_15_16.md), and by Astra 6 by hand (lemma 20, Result 10, a consensus check). Ledger row `l16_standard_nogo`. Item 2 of Proposition H of paper/towerless_intervals.tex; cited as "\supp{16}, Proposition 16.1" in paper/connes_kk_dictionary_note.tex.

Setting. The compression setting of 16_compressed_algebras_noncommutative.md, with a real structure J_0 on H_0 (antiunitary, J_0^2 = +-1, with the grading sign) such that J_0 a J_0^{-1} = a^* for all a in A_0 and J_0 D_0 = eps' D_0 J_0; J_F = J_0 restricted to V, which J_0 preserves because Sigma = -Sigma.

Statement.

1. J_F Q(a)^* J_F^{-1} = Q(a) for every a in A_0.
2. The data (A_F, V, d, J_F) satisfy order zero if and only if A_F is commutative.
3. The same holds for a product with a fixed finite factor that itself satisfies order zero, with the compatible product real structure; the commutativity then concerns the compressed function algebra, and the separate finite factor may be noncommutative.

Hypotheses (lemma 20, Result 10). Compression followed by algebra generation, with this inherited pointwise reality and a J_0-invariant window, on the standard correspondence L^2(A_0). Together with lemma 07, Amendment 2 (first order fails for C + C with the inherited J_2), this is the form of the obstruction.

Escapes (lemma 20, Result 10). The coarse correspondence supplies a separate opposite action and escapes the hypothesis, as Theorem 16.2 shows (16_coarse_correspondence_order_one.md). Other representations or real structures require their own order-zero calculation.

Proof. Lemma 16, Proposition 16.1, and lemma 20, Result 10; the proofs are there. Order zero would make Q(a) commute with Q(b)° = Q(b) for all a, b, so O_F and A_F would be commutative; conversely J_F

A_F J_F^{-1} = A_F.

What it leaves open. A noncommutative window needs a doubling of the parent, a second sheet, or another real structure; lemma 16 develops the first two (16_two_sheet_parent.md).

Part S54 (On the doubled parent of twisted CP^3 scaled by 1/l, compression and generation give the constant finite data A_F = M_4(C), dim H_F = 16, D_F = 0, while the whole triple converges to (C, C^16, 0))

On the doubled parent of twisted CP^3 scaled by 1/l, compression and generation give the constant finite data A_F = M_4(C), dim H_F = 16, D_F = 0, while the whole triple converges to (C, C^16, 0)

Status. Proposition, proved (Example 16.6 of lemma 16). From the referee's assessment of lemma 16 by Terra 6.1 (historical/reviews/2026-11-03_terra_check_15_16.md); checked by Claude Opus 5.5 by hand. Ledger row `l16_family`.

Setting. D_0 the Dolbeault-Dirac operator on CP^3 twisted by O(1), with V = ker D_0 = C^4; the doubled parent of 16_coarse_correspondence_order_one.md, scaled as DD_l = DD/l; the Clifford point of 16_ko6_by_clifford_point.md.

Statement.

1. DD_l has kernel V (x) conj(V) = C^4 (x) C^4 and a gap growing like 1/l; here P is a spectral projection.
2. Compression followed by algebra generation gives the constant finite data A_F = M_4(C), dim H_F = 16, D_F = 0.
3. The Clifford point gives dim H = 32 with the KO-6 signs.
4. Order one is automatic, since D_F = 0.
5. The whole triple converges to (C, C^16, 0) in the spectral propinquity, by Theorem 15.3.

Scope. Items 2 and 5 concern the truncation and the whole algebra respectively, consistently with lemma 15: the family exhibits both descriptions at once.

Proof. Lemma 16, Example 16.6; the argument is there.

What it leaves open. Nonzero mass data D_F and a parent that yields the grading are not produced (lemma 16, Consequence for T0 and T0b).

Part S55 (At twistor level one on two sheets, real functions on S^4 and on CP^3 generate M_2(H) + M_4(C) on E + E^* of dimension 32 with J(x, y) = (y^*, x^*), the algebra, bimodule and real structure of Chamseddine-Connes, as abstract finite data)

At twistor level one on two sheets, real functions on S^4 and on CP^3 generate M_2(H) + M_4(C) on E + E^* of dimension 32 with J(x, y) = (y^*, x^*), the algebra, bimodule and real structure of Chamseddine-Connes, as abstract finite data

Status. Proposition, proved by computation (Example 16.9 of lemma 16). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (numpy: real dimensions 16 and 32, j^2 = -1 commuting with the Gamma_a, the structure on k = 4) and by Terra 6.1 by hand in the second referee report on lemma

16, which found one error, corrected (historical/reviews/2026-11-03_terra_check_18_16b.md). Ledger row `l16_twistor`. Item 4 of Proposition H of paper/towerless_intervals.tex.

Setting. D_0 the Dolbeault-Dirac operator on CP^3 twisted by O(1), with kernel H^0(CP^3, O(1)) = C^4; j the antiunitary with j^2 = -1 commuting with the Gamma_a of Proposition 10.1, which makes C^4 a quaternionic space H^2. The space is used twice: as W, with the functions of S^4 pulled back along the twistor projection, and as V, with the functions of CP^3. Corollary 16.8 applies with the parent gradings, trivial on the kernels.

Statement.

1. A real function f on S^4 pulls back to a function invariant under w -> jw, so j Q(f) j^{-1} = Q(f), a Hermitian element of End_H(W). Since Q(x_a) = Gamma_a/5, the real algebra generated is End_H(W) = M_2(H), of real dimension 16.
2. For real functions on CP^3, Q(w^* B w/w^* w) = (B + tr(B) 1)/5 for Hermitian B, so Q is onto the Hermitian matrices and the real algebra generated is End_C(V) = M_4(C), of real dimension 32.
3. Hence A = M_2(H) + M_4(C) acts on H_F = Hom(V, W) + Hom(W, V) = E + E^\, *of dimension 32, with J(x, y) = (y^\, x^*)*: the algebra, bimodule and real structure of Chamseddine and Connes. Complex generation on the S^4 side would give M_4(C).

Scope. The inputs are the second sheet, the restriction of one algebra to base functions, real algebra generation, the four-dimensional window, the grading axis and d. The grading Gamma_F(xi, eta) = (Gamma_5 xi, -eta Gamma_5) is added after truncation as abstract data (16_nogo_grading_from_even_parent.md); the full algebra is then a graded prealgebra with even part H + H + M_4(C), on which the real orientability relation holds. The mass datum d = mu n^a Gamma_a, n orthogonal to the axis, is also abstract data. No operator connects the two pieces.

Proof. Lemma 16, Example 16.9; the computation is there, with Proposition 10.1.

What it leaves open. Four global sections are not turned into rank-four fibres over S^4, the two sheets are not derived from the fibration, and the Dirac operator connecting the two pieces, which the selection of C + H + M_3(C) requires, is not produced; the fibre-rank problem remains open (lemma 16, Example 16.9).

Part S56 (A two-sheet coarse parent gives the KO-6 signs with order zero and order one, and its truncation for equal dimensions and full algebras is the Chamseddine-Connes pair M_k(C) + M_k(C) with J(x, y) = (y^*, x^*))

A two-sheet coarse parent gives the KO-6 signs with order zero and order one, and its truncation for equal dimensions and full algebras is the Chamseddine-Connes pair M_k(C) + M_k(C) with J(x, y) = (y^*, x^*)

Status. Theorem (Theorem 16.7), with Corollary 16.8. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (numpy on random finite data: signs, order zero, order one, orientability) and by Terra 6.1 by hand in a separate referee report, which found one error in Corollary 16.8, corrected by the counterexample in item 4 (historical/reviews/2026-11-03_terra_check_18_16b.md). Ledger rows `l16_two_sheets` and `l16_ce_centre` (the referee's counterexample, checked by Claude Opus 5.5 by hand). Item 4 of Proposition H of paper/towerless_intervals.tex; Corollary 16.8 is cited in paper/connes_kk_dictionary_note.tex.

Setting. Even triples (A_X, H_X, D_X, gamma_X) and (A_Y, H_Y, D_Y, gamma_Y); H = (H_X (x) conj(H_Y)) + (H_Y (x) conj(H_X)); for even a in A_X, b in A_Y, pi(a, b) = (a (x) 1) + (b (x) 1), DD = (D_X (x) 1 + gamma_X (x) D_Y^c) + (D_Y (x) 1 + gamma_Y (x) D_X^c), Gamma = (gamma_X (x) gamma_Y^c) + (-gamma_Y (x) gamma_X^c), and J(xi (x) eta^c) = (-1)^{|xi||eta|} eta (x) xi^c mapping

each summand to the other. Symmetric windows V_X, V_Y of dimensions k_X, k_Y and even subalgebras A_{F,X} in B(V_X), A_{F,Y} in B(V_Y).

Statement.

1. (Theorem 16.7) J^2 = 1, J DD = DD J and J Gamma = -Gamma J, the KO-6 signs; (a, b)° = (1 (x) (b^\)^c) + (1 (x) (a^\)^c), so the summands are (A_X, A_Y)- and (A_Y, A_X)-bimodules, and order zero and order one hold.
2. (Corollary 16.8) The compression T_F = (A_{F,X} + A_{F,Y}, (V_X (x) conj(V_Y)) + (V_Y (x) conj(V_X)), D_F, J_F, Gamma_F) has the KO-6 signs, order zero, order one and dim H_F = 2 k_X k_Y. If gamma_{V_X} is in A_{F,X} and gamma_{V_Y} in A_{F,Y}, then with c = (gamma_{V_X}, gamma_{V_Y}) and z = (1, -1), Gamma_F = pi(zc) c°, a zero-cycle with coefficients in the algebra and its opposite, weaker than Gamma_F in pi(A_F).
3. For ungraded V_X, V_Y with k_X = k_Y = k and full generated algebras, (A_F, H_F, J_F) is exactly the irreducible pair with centre C + C of Chamseddine and Connes (arXiv:0706.3688, sections 2-3): A_C = M_k(C) + M_k(C) on itself with J(x, y) = (y^\, x^\).
4. D_F commutes with the two sheet projections, so the sheets are not connected. It need not commute with the whole centre: for V_X = C^2, gamma_X = sigma_3, d_X = sigma_1, A_{F,X} = C + C acting diagonally and V_Y = C, the central element (diag(1, 0), 0) does not commute with D_F.

Scope. The second sheet is an input, and the KO-6 sign comes from the block structure of two sheets. At the finite level the bimodule is Remark 2.9 of Chamseddine-Connes; lemma 16 adds its realization from parents and compressed functions.

Proof. Lemma 16, Theorem 16.7 and Corollary 16.8; the proofs are there, by the computation of Theorem 16.2 with D_Y in place of the second factor.

What it leaves open. An operator connecting the two sheets, and hence the selection of A_F = C + H + M_3(C), is absent from this parent; lemma 20 supplies one in another parent (20_interval_parent_connector.md). Lemma 16 establishes neither a construction of two sheets from one connected parent nor a no-go theorem against one (Caveat 4).

Part S57 (The unitaries of the quark sector commuting with the colour actions, the grading, J_F and D_F form G(M) = {(V_L, V_R) : V_L M = M V_R}, with G(0) = U(N_f) x U(N_f))

The unitaries of the quark sector commuting with the colour actions, the grading, J_F and D_F form G(M) = {(V_L, V_R) : V_L M = M V_R}, with G(0) = U(N_f) x U(N_f)

Status. Proposition (Proposition 17.5). Proved by Claude Opus 5.5; checked by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_17.md). Ledger row `l17_symmetry`.

Setting. N_f quark flavours, with q_L, q_R in C^{N_f} (x) C^3 and q-bar_L, q-bar_R = J q_L, J q_R; D_F acting on (q_L, q_R) by [[0, M], [M^*, 0]] (x) 1_3 and by the complex conjugate block on the antiparticles; M_3(C) acting on the antiparticles on the left and on the particles through its opposite action. G(M) is the group of unitaries on this sector commuting with both actions of M_3(C), gamma_F, J_F and D_F.

Statement.

1. G(M) = {(V_L, V_R) in U(N_f) x U(N_f) : V_L M = M V_R}, acting by V_L, V_R on q_L, q_R and by the complex conjugates on q-bar_L, q-bar_R.
2. G(0) = U(N_f) x U(N_f); G(m I) = U(N_f)_V for m != 0; G(M) = U(1)^{N_f}_V for M positive diagonal with distinct entries.
3. With m_u = m_d = 0 and m_s != 0 in one block, G contains U(2) x U(2).

4. If V is also required to commute with the electric charge, through lambda in B, then V_L and V_R are diagonal between u and d, and G(0) = U(1)^4 for two flavours.

Scope. The nonabelian flavour transformations are not colour gauge transformations, and at M = 0 the axial part is an ungauged symmetry of the reduced triple. If the central U(1) of U(3) is kept in the gauge group, it contains a common vector phase, which an imposed unimodularity condition can remove. The Lagrangian symmetry U(1)_A is anomalous in the quantum theory, which the classical data do not show.

Proof. Lemma 17, Proposition 17.5; the proof is there. The colour identity separates particles from antiparticles, the commutant of the colour matrix algebra is scalar on colour, the grading separates L and R, J_F fixes the antiparticle part, and commutation with D_F gives V_L M = M V_R.

What it leaves open. The dynamics of the chiral limit, which these classical data do not contain (17_condensate_hypotheses.md).

Part S58 (At finite volume and fixed background the regulated condensate tends to zero as m -> 0 apart from the zero-mode term, and the Banks-Casher form pi rho_0(0) needs a gauge measure, infinite volume, an order of limits and uniform control)

At finite volume and fixed background the regulated condensate tends to zero as m -> 0 apart from the zero-mode term, and the Banks-Casher form pi rho_0(0) needs a gauge measure, infinite volume, an order of limits and uniform control

Status. Proposition (Proposition 17.8). Proved by Claude Opus 5.5; checked by Terra 6.1 by hand, who found it wrong as first written by overreach and had it corrected (historical/reviews/2026-11-03_terra_check_17.md). Ledger row `117_condensate`.

Setting. D_M(A) with eigenvalues lambda_n and n_0 zero modes on M of volume V. For m > 0, c_V(m; A) = (1/V) sum_n m/(lambda_n^2 + m^2) = (1/V) d/dm log |det(D_M(A) + gamma_5 m)|, formal on a compact four-manifold and needing ultraviolet regularization, with subtraction of divergent terms including m Lambda^2, m^3 log Lambda and possibly curvature terms.

Statement.

1. With a subtraction compatible with the pairing +-lambda whose finite remainder is O(m), for example odd and analytic in m, at fixed A on compact M, c_V(m; A) = n_0/(V m) + O(m) as m -> 0. A subtraction merely regular at m = 0 may leave a constant. So at finite volume and fixed background no nonzero limit exists beyond the zero-mode term.
2. Let rho_m(lambda) = lim_{V -> infinity} <(1/V) sum_n delta(lambda - lambda_n)>*m for a measure* <.>m on gauge fields. Assume the family is equicontinuous at 0 with rho_m(0) -> rho_0(0), the tails and the ultraviolet subtraction are controlled uniformly in m, and the infinite-volume limit may be taken inside the integral. Then lim*{m -> 0} lim*{V -> infinity} = pi rho_0(0), the form of the Banks-Casher relation as stated by Giusti and Lüscher.
3. Equicontinuity alone is insufficient: rho_m(lambda) = (1 + (1/2) sin(1/m)) exp(-lambda^2) gives no limit.

Scope. The hypotheses of item 2 are those of the lemma; the original Banks-Casher paper has neither the factor pi nor an order of limits with the volume, and the sign and flavour normalization of a condensate depend on convention. The spectral triple supplies the operator D_M + gamma_5 m, the density of each fixed operator and the continuity of m -> 0 at fixed background; the gauge-averaged density, the regulated determinant or Pfaffian measure, the sequence of volumes and the order of limits need further input.

Proof. Lemma 17, Proposition 17.8; the proof is there: for a cutoff preserving the pairs, |det| = m^{n_0} times the product of (lambda^2 + m^2) over lambda > 0, and m/(lambda^2 + m^2) is pi times the Poisson kernel at scale m.

What it leaves open. A proof of existence of QCD with fermions would supply a measure without by itself giving rho_0(0) > 0; the points to check when it appears are whether m_q = 0 is included and whether infinite volume is reached (STRATEGY T10).

Part S59 (On a subalgebra commuting with D_F and with the involution, inner fluctuations have no scalar part, so the Yukawa block of the reduced triple is an explicit constant mass)

On a subalgebra commuting with D_F and with the involution, inner fluctuations have no scalar part, so the Yukawa block of the reduced triple is an explicit constant mass

Status. Proposition (Proposition 17.3), with Corollary 17.4. Proved by Claude Opus 5.5; checked by Terra 6.1 by hand, who found Corollary 17.4 wrong as first written by overreach and had it corrected (historical/reviews/2026-11-03_terra_check_17.md). Ledger row `117_no_higgs`.

Setting. The reduced triple of 17_reduced_action.md; a unital subalgebra B of A_F with [D_F, B] = 0, and U commuting with B and B°; inner fluctuations of D by C^infinity(M) (x) B.

Statement.

1. (Proposition 17.3) The inner fluctuation contains only Clifford one-forms, D_A commutes with UU, and D_A reduces to DD_A = D_M(A) + gamma_5 (x) d, with d constant on M.
2. If [D_F, B] != 0, the same reduction holds, and the fluctuation permits a finite scalar field d_A(x) in place of d.
3. (Corollary 17.4) In the representation of lemma 06, Result 5, with B = {(lambda, Q_lambda, m)} = C + M_3(C), u_L and u_R carry the same left character lambda and the same opposite colour action, d_L and d_R carry conj(lambda) and the same colour action, and e_L and e_R pair as well. So the kept mass blocks commute with B, in the strict projection of lemma 06 and also when only a vectorlike top pair is removed, and DD_A = D_M(A) + gamma_5 d with d the direct sum of the kept mass matrices: no Higgs field.

Scope (the two qualifications of Corollary 17.4). U(B) contains U(1) x U(3), so before unimodularity or an inherited restriction from the Standard Model gauge group, two independent abelian gauge fields act on the surviving charged fermions; the reading "photon and gluons" needs that restriction. A canonically normalized radial field couples to a fermion of fixed mass m_f with strength m_f/v, which tends to 0 as v -> infinity at fixed gauge coupling; under the CCM relation, keeping all Dirac masses fixed while M_W -> infinity is impossible (11_nogo_heavy_w_fixed_masses.md), the diverging mass need not be the top, and top dominance is a further hypothesis. Corollary 17.4 inherits the regime of the CCM relation.

Proof. Lemma 17, Proposition 17.3 and Corollary 17.4; the proofs are there: for a = f (x) alpha and b = g (x) beta, a [D, b] = f [D_M, g] (x) alpha beta + f g gamma_5 (x) alpha [D_F, beta], whose second term vanishes, and J A J^{-1} has finite coefficients in B°.

What it leaves open. The regime near the cutoff (STRATEGY T5b), treated in lemma 18.

Part S60 (A spacetime-dependent reduction of the fixed operator gives no kinetic term for Sigma: the spectral action of D is unchanged, and a kinetic term needs a new operator with m != 0, whose coefficient is N_c f(0) m^2/(4 pi^2))

A spacetime-dependent reduction of the fixed operator gives no kinetic term for Sigma: the spectral action of D is unchanged, and a kinetic term needs a new operator with m != 0, whose coefficient is N_c f(0) m^2/(4 pi^2)

Status. Proposition (Proposition 17.10). Proved by Terra 6.1 in the referee report on lemma 17 (historical/reviews/2026-11-03_terra_check_17.md); checked by Claude Opus 5.5 by hand (the spin trace evaluated with explicit Dirac matrices). Ledger row `l17_no_kinetic`.

Setting. U(x) a pointwise admissible involution on M, c^mu = [D_M, x^mu], B_mu = (U d_mu U) restricted to H_0, R defined pointwise as in Theorem 17.1 (17_reduced_action.md), P_- = (1 - gamma_5)/2. The quark sector of 17_set_of_reductions.md with N_c colours.

Statement.

1. R^{-1} UU D R = D_M + c^mu P_- B_mu + gamma_5 d(x), an operator that is generally not self-adjoint.
2. If Sigma(x) only changes coordinates for the fixed operator D, the spectral action of D does not depend on Sigma, and conjugating D by a smooth unitary preserves its spectrum; neither produces a two-derivative term.
3. A kinetic term needs an additional choice of operator. For D_new = D_M + gamma_5 d_Sigma(x) on flat M, with M = m I, Sigma^* = Sigma and Sigma^2 = 1, the heat expansion of Tr f(D_new^2/Lambda^2) contains (N_c f(0) m^2/(4 pi^2)) int tr(d_mu Sigma^* d^mu Sigma), tracing over both reduced blocks; the sign is positive when f(0) > 0.
4. At M = 0 this operator does not depend on Sigma.

Scope. On the even triple, D_M + gamma_5 S(x) with S built from Phi = m Sigma(x) is self-adjoint for every unitary Sigma, and its kinetic term is the Higgs-type term of an inner fluctuation, available when A_F contains flavour algebras acting on L and R, which brings flavour gauge fields with it. The coefficient is proportional to N_c m^2, as in constituent-quark estimates of f_pi^2 (quoted from memory in the lemma); here it is a classical spectral-action coefficient, and m is not derived.

Proof. Lemma 17, Proposition 17.10; the proof is there. B_mu is anti-Hermitian and P_- c^mu = c^mu P_+, which gives the adjoint; for the coefficient, D_new^2 = -(nabla^2 + E) with E containing -i gamma^mu gamma_5 d_mu d, and the a_4 term (1/(16 pi^2)) tr(E^2/2) gives (1/(8 pi^2)) tr_{H_0}(d_mu d d^mu d), the two blocks of d giving 2 N_c m^2 tr(d_mu Sigma d^mu Sigma).

What it leaves open. A kinetic term at m = 0 needs a quantum effective action (lemma 17).

Part S61 (The fermion action of M x F on its physical subspace equals (1/2) <jj psi, (D_M + gamma_5 d) psi> on an unconstrained spinor, and a paired Dirac sector gives one Dirac fermion of explicit mass for either sign s)

The fermion action of M x F on its physical subspace equals (1/2) <jj psi, (D_M + gamma_5 d) psi> on an unconstrained spinor, and a paired Dirac sector gives one Dirac fermion of explicit mass for either sign s

Status. Theorem (Theorem 17.1), with Corollary 17.2. Proved by Claude Opus 5.5. Theorem 17.1 checked by Claude Opus 5.5 in Python (numpy on a random finite model: the bilinear forms agree to about 1e-14 for both signs); both checked by Terra 6.1 by hand, with amendments incorporated (historical/reviews/2026-11-03_terra_check_17.md). Ledger rows `l17_action` (depending on `l08_reconcile`) and `l17_one_dirac`.

Setting. M a compact Riemannian spin 4-manifold with (D_M, J_M, gamma_5), J_M^2 = -1, J_M D_M = D_M J_M, J_M gamma_5 = gamma_5 J_M, gamma_5 D_M = -D_M gamma_5; F = (A_F, H_F, D_F, J_F, gamma_F) a finite real triple with the KO-6 signs; D = D_M (x) 1 + gamma_5 (x) D_F, J = J_M (x) J_F, Gamma = gamma_5 (x) gamma_F; the fermion action (1/2) <J xi, D xi> on Grassmann variables xi in H^+ = {Gamma = +1} (Chamseddine-Connes-Marcolli 2007). U an admissible involution of sign s (Theorem D), and on H_0 = {gamma_F = +1}, d = U D_F and j = U J_F, with j^2 = s and jd = s dj.

Statement.

1. (Theorem 17.1) R(phi (x) x) = phi_+ (x) x + phi_- (x) Ux is a unitary from L^2(S) (x) H_0 onto H^+, and <J R psi, D R psi'> = <jj psi, DD psi'> with DD = D_M (x) 1 + gamma_5 (x) d, jj = J_M (x) j for s = +1 and jj = gamma_5 J_M (x) j for s = -1, jj^2 = -s and jj DD = s DD jj. So the action is (1/2) <jj psi, DD psi> on an unconstrained Grassmann spinor psi.
2. DD^2 = D_M^2 (x) 1 + 1 (x) d^2, and if f(D^2/Lambda^2) is trace class, Tr f(D^2/Lambda^2) = 2 Tr f(DD^2/Lambda^2).
3. (Corollary 17.2) For H_0 = span(e_L, conj(e_R)), m > 0, and the two reductions (d_+, j_+) = (m I, sigma_1 K) and (d_-, j_-) = (diag(m, -m), [[0, 1], [-1, 0]] K) of lemma 06, the actions are S_+ = <J_M psi_2, (D_M + gamma_5 m) psi_1> and S_- = <gamma_5 J_M psi_2, (D_M + gamma_5 m) psi_1>, with S_-(psi_1, psi_2) = S_+(psi_1, gamma_5 psi_2).
4. With R_5 = exp(i pi gamma_5/4), R_5 (D_M + gamma_5 m) R_5 = D_M + i m, formally. So a paired sector gives the action of one Dirac fermion of explicit mass m for either s, up to an invertible change of Grassmann variables; at the classical level no anomaly or theta shift follows from changing s alone.

Scope. An unpaired reduction gives a Majorana-type action (1/2) <J_M psi, (D_M + gamma_5 m) psi>; the neutral sector of the strict light spectrum has this form with m = 0. "D = 9" is the formal KO-label 4 + 5, available only where an s = -1 involution exists, and the metric dimension of M x T is 4. The rotation R_5 is a formal chiral change of both independent variables; in a quantized theory its Jacobian must be included. Everything is at the level of the classical fermion operator.

Proof. Lemma 17, Theorem 17.1 and Corollary 17.2; the proofs are there. UU = 1 (x) U commutes with D, (1 + UU) R/sqrt(2) = 1 (x) r with r x = (x + Ux)/sqrt(2), so UU D R = R DD, and UU J R = R jj.

What it leaves open. Anomalies, the running of masses and confinement do not appear in these classical data (lemma 17, Caveat 1).

Part S62 (The admissible involutions of each sign on the colour-only quark sector are the Sigma in U(N_f) with Sigma^* M Hermitian, exactly the stationary points of Re tr(M Sigma^*), and at M = 0 they form the homogeneous space (U(N_f) x U(N_f))/U(N_f)_V)

The admissible involutions of each sign on the colour-only quark sector are the Sigma in U(N_f) with Sigma^* M Hermitian, exactly the stationary points of Re tr(M Sigma^*), and at M = 0 they form the homogeneous space (U(N_f) x U(N_f))/U(N_f)_V

Status. Theorem (Theorem 17.6), with Remark 17.7 on its scope. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (numpy: involution conditions, the formula for d, which this check found wrong in a first version, finite-difference criticality) and by Terra 6.1 by hand, with corrections incorporated, Remark 17.7 having overreached as first written (historical/reviews/2026-11-03_terra_check_17.md). Ledger row `l17_reductions`.

Setting. The quark sector of 17_chiral_symmetry_group.md, with B containing only the colour algebra; admissible involutions of sign s as in Theorem D; V_M(Sigma) = -Re tr(M Sigma^*) on U(N_f).

Statement. The admissible involutions of sign s are exactly the operators, tensored with 1_3, U_{Sigma,s} = [[0, Sigma], [Sigma^*, 0]] on (q_L, q_R) and s [[0, conj(Sigma)], [Sigma^T, 0]] on (q-bar_L, q-bar_R), with Sigma in U(N_f) and Sigma^* M Hermitian; this condition holds iff Sigma is a stationary point of V_M, saddles and maxima included. The reduced mass operator on H_0 = (q_L, q-bar_R) is d_{Sigma,s} = (Sigma M^* (x) 1_3) + (s conj(Sigma^* M) (x) 1_3), and G(0) acts by Sigma -> V_L Sigma V_R^*. Hence:

1. At M = 0 the set is all of U(N_f), for each s, with transitive action and stabilizer of Sigma = 1 the diagonal U(N_f).
2. At M = m I, m > 0, it is {Sigma : Sigma^2 = 1, Sigma^* = Sigma}, the disjoint union of the Grassmannians of C^{N_f}.
3. For M positive diagonal with distinct entries, it is the 2^{N_f} matrices diag(+-1, …, +-1).
4. For M invertible, the unique Sigma with Sigma^* M positive is the polar part of M; it is the unique global minimum of V_M and the unique Sigma for which the first block of d_{Sigma,s} is positive.

With the electric charge in B, the admissible set is the intersection with the commutant of the charge: diagonal phases at M = 0 for two differently charged flavours, and only signs +-1 for positive nonzero diagonal masses.

Scope (Remark 17.7). Item 1 is an equivariant statement, with the same stabilizer as in the symmetry breaking of QCD. The data contain no physical energy selecting among the stationary points; Dashen is cited for the vacuum-selection idea alone. Identifying a reduction with a condensate direction would need a map to a quantum order parameter, which lemma 17 does not have. A negative entry in the reduced mass block comes from choosing another reduction of one fixed even triple; a physical sign change at fixed fermion measure differs, with determinants at m and -m related by (-1)^{n_0}.

Proof. Lemma 17, Theorem 17.6; the proof is there. Along Sigma exp(itX), X Hermitian, the derivative of Re tr(M Sigma^\) *at t = 0 is Im tr(HX) with H = Sigma^* M, which vanishes for all X exactly when H is Hermitian; item 4 is von Neumann's trace inequality.

What it leaves open. The pions as dynamical objects need the QCD measure and a quantum effective action; in QCD the anomalous singlet axial direction also prevents identifying the full classical U(N_f) with the pion manifold without qualification.

Part S63 (As m_t -> infinity at fixed cutoff the broken vectors keep a finite nonzero kinetic form and a vanishing mass, so heavy fermions do not select the centralizer by mass decoupling)

As m_t -> infinity at fixed cutoff the broken vectors keep a finite nonzero kinetic form and a vanishing mass, so heavy fermions do not select the centralizer by mass decoupling

Status. Counterexample (Counterexample 8 of lemma 18). Proved by Astra 6 (job 18); checked by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_18_16b.md). Ledger row `l18_no_centralizer`. Item 4 of Proposition E.1 of paper/towerless_intervals.tex.

Setting. The data of Proposition 4 of lemma 18 (top-only, heat cutoff; 18_top_only_w_mass_at_fixed_cutoff.md) and the centralizer g_{V_0} = {X in g_SM : [X, V_0] = 0}, which for nonzero top mass is the represented unbroken algebra su(3) + u(1)_em.

Statement.

1. For fixed y, h_x(y) -> 0 as x -> infinity, so C_W(q^2) -> (Lambda^2/(4 pi^2)) (9/2) g(q^2/Lambda^2).
2. For y > 0, g(y) = 4y int_0^1 u(1 - u) exp(-y u(1 - u)) du > 0: a charged generator outside g_{V_0} keeps a finite nonzero quadratic kinetic form and a nonvanishing propagator at every fixed positive momentum, while its mass tends to zero.
3. The neutral kernel similarly leaves the positive matrix B_inf g(y), including the additional neutral direction.
4. These facts contradict decoupling of the broken vectors through diverging masses or vanishing propagators.
5. The narrower statement holds: if the connection is restricted to the centralizer from the outset, the finite mass blocks are invariant, the heat trace factorizes, the heavy block carries exp(-s m^2/Lambda^2) -> 0, and the restricted action converges to the light-block spectral action.

Scope. The top-only quadratic family at fixed Lambda. Item 5 proves decoupling of a finite fermion block under that restriction, without any dynamical elimination of the other gauge fields. Amendment to lemma 03, Part C: its transverse weights, including w_L(x, 0) = w(x), and its weak-angle formulas are correct under their conventions, must be accompanied by the mass coefficient with the quadratic-vertex subtraction, and do not give a centralizer-only endpoint.

Proof. Lemma 18, Counterexample 8; the proof is there, from the integration-by-parts form h_x(y) = 2 int_0^1 u [x + 2y(1 - u)] exp(-xu - yu(1 - u)) du.

What it leaves open. Lemma 18 supplies no mechanism selecting the centralizer; mass decoupling in this action does not do it.

Part S64 (For V_0 = m sigma_3 and T = sigma_1 the Taylor kinetic coefficient of the exact quadratic spectral action is negative for m^2 > Lambda^2, while the full form has a unique positive pole with M^2 ~ 4 m^2)

For V_0 = m sigma_3 and T = sigma_1 the Taylor kinetic coefficient of the exact quadratic spectral action is negative for m^2 > Lambda^2, while the full form has a unique positive pole with M^2 ~ 4 m^2

Status. Counterexample (Counterexample 7 of lemma 18). Proved by Astra 6 (job 18); checked by Claude Opus 5.5 in Python (direct heat trace at nonzero momentum against the kernel) and by Terra 6.1 by hand

(historical/reviews/2026-11-03_terra_check_18_16b.md). Ledger row `l18_wrong_sign`.

Setting. H_F = C^2, V_0 = m sigma_3, T = sigma_1, x = m^2/Lambda^2, with the spacetime and trace conventions of Theorem 1 of lemma 18 (18_exact_quadratic_kernel.md).

Statement.

1. For the heat cutoff, C_T(q^2) = (Tr(T^2) Lambda^2/(8 pi^2)) e^{-x} [(2 + 4x + y) H(y) - 2].
2. Its q^2 Taylor coefficient is proportional to e^{-x}(1 - x): it vanishes at x = 1 and is negative for x > 1, so the two-derivative truncation has a wrong-sign vector kinetic term. M_Taylor^2 = 6 m^2/(1 - x) is a positive mass only for x < 1.
3. The full quadratic form has a unique positive real timelike pole with positive residue, with 4x < M_pole^2/Lambda^2 < 4x + 2, so M_pole ~ 2m diverges at fixed cutoff.
4. For every f in class (4), the Taylor kinetic coefficient is proportional to f(x) + x f'(x) = d/dx [x f(x)], which is negative somewhere, since x f(x) > 0 for x > 0 and x f(x) -> 0 at infinity.

Scope. Arbitrary finite data; the top-only Standard Model family lies outside it. It distinguishes a wrong-sign Taylor truncation from a negative-residue pole of the full form factor. Absence of complex poles or other nonlocal problems is not proved in either model.

Proof. Lemma 18, Counterexample 7; the proof is there. The generator connects the eigenvalues m and -m, so d_ij^2 = 4 m^2 while both squared eigenvalues equal m^2; the pole equation has the form z - 2 + 2/H(-z) = beta of the neutral massive mode (equation (36) of lemma 18) with beta = 4x, and evaluating C_T at y = -4x and y = -4x - 2 gives the bounds.

What it leaves open. The exact mass conclusions of the top-only family therefore do not extend to arbitrary finite representations (lemma 18, after Corollary 10); each set of finite data needs its own computation.

Part S65 (The quadratic spectral action of a transverse vector about a constant finite background has an exact kernel with no expansion in m/Lambda, gauge invariant through the Goldstone orbit, whose small-mass limit gives the mass formula (3/2) Q^{-1/2} K Q^{-1/2} and the CCM relation)

The quadratic spectral action of a transverse vector about a constant finite background has an exact kernel with no expansion in m/Lambda, gauge invariant through the Goldstone orbit, whose small-mass limit gives the mass formula (3/2) Q^{-1/2} K Q^{-1/2} and the CCM relation

Status. Theorem (Theorem 1 of lemma 18), with Lemma 2 and Corollary 3. Proved by Astra 6 (job 18); checked by Claude Opus 5.5 in Python by computations independent of the kernel (a direct heat trace with constant a reproduces equation (2) to about 0.03%; a direct heat trace on Fourier modes reproduces the kernel at three values of (m, q) to 0.1-0.3%) and by Terra 6.1 by hand, correct with small amendments (historical/reviews/2026-11-03_terra_check_18_16b.md). Ledger row `l18_kernel`. Proposition E.1 of paper/towerless_intervals.tex.

Setting. D_A = -i gamma^mu (d_mu + i A_mu) + gamma_5 V_0 on R^4, V_0 = V_0^dagger with signed eigenvalues m_i; the literal trace on H_F including every multiplicity; A smooth and compactly supported; the vacuum-subtracted action. A_mu(q) = a_mu(q) T with T = T^dagger and q.a = 0; d_ij = m_i - m_j; I_t(a, b; q^2) = int_0^1 exp(-t[(1 - u)a + ub + u(1 - u)q^2]) du. Cutoff class (4): f(z) = int exp(-sz) d nu(s) with nu positive, 0 < f(0) < infinity and int (s^{-2} + s^2) d nu(s) < infinity.

Statement.

1. (Theorem 1) For S_t = Tr exp(-t D_A^2), S_t^(2) = int_q |a_T(q)|^2 C_{t,T}(q^2), with C_{t,T}(q^2) = (1/(8 pi^2)) sum_ij |T_ij|^2 [(2/t + d_ij^2 + q^2) I_t(m_i^2, m_j^2; q^2) -

(exp(-t m_i^2) + exp(-t m_j^2))/t]; for several generators |T_ij|^2 becomes Re((T_a)_ij (T_b)_ji). The Taylor coefficients C_0 and C_1 use a first divided difference and its mixed derivative, and the formulas apply after integration over t = s/Lambda^2 to every f in class (4).

2. (Lemma 2) For V(x) = exp(i pi(x) T) V_0 exp(-i pi(x) T), S[a, V] = S[B, V_0] with B_mu = a_mu + d_mu pi, so the nonzero mass coefficient C_0 is compatible with gauge invariance, without a derivative expansion.
3. (Corollary 3) If all finite eigenvalues are small relative to Lambda, C_0 = (f(0)/(8 pi^2)) K and C_1 = (f(0)/(12 pi^2)) Q, so M_Taylor^2 = (3/2) Q^{-1/2} K Q^{-1/2}, reproducing Proposition 13.5. For the full three-generation Standard Model, Q_11 = 12 and 8 M_W^2 = sum over generations of (m_nu^2 + m_e^2 + 3 m_u^2 + 3 m_d^2), the relations of CCM, equations (4.10) and (4.16).

Scope. Statements on R^4, or in the infinite-volume limit; on a finite torus the lattice sum replaces the Gaussian integral and holonomy can enter. The signed eigenvalues are needed, their squares alone being insufficient. The moment hypotheses of class (4) do not by themselves guarantee analytic continuation to every negative q^2. The kernel concerns the classical bosonic quadratic action about a prescribed constant background.

Proof. Lemma 18, Theorem 1, Lemma 2 and Corollary 3; the proofs are there. The second-order Duhamel expansion of exp(-t D_A^2), with the vertices (2p + q).a, gamma^{mu nu} q_mu a_nu and (m_j - m_i) gamma^mu gamma_5 a_mu and the quadratic vertex, gives the kernel after completing the momentum square.

What it leaves open. The positive Laplace class selects no finite stationary background along radial directions (18_nogo_radial_vacuum.md), so the masses computed from the kernel are conditional on the prescribed background.

Part S66 (With one common Dirac mass m for every species and no Majorana masses, the W and Z pole masses diverge at fixed cutoff, 2x < z_W < 2x + 2 and 16x/5 < z_Z < 16x/5 + 2, with M_W^2/M_Z^2 -> 5/8 at both ends and no light fermions)

With one common Dirac mass m for every species and no Majorana masses, the W and Z pole masses diverge at fixed cutoff, 2x < z_W < 2x + 2 and 16x/5 < z_Z < 16x/5 + 2, with M_W^2/M_Z^2 -> 5/8 at both ends and no light fermions

Status. Proposition (Proposition 14 of lemma 18). Sketch by Terra 6.1 in the referee report on lemma 18 (historical/reviews/2026-11-03_terra_check_18_16b.md); the ratio added by Claude Opus 5.5; complete proof by Astra 6, who checked it by hand and whose proof is written into the lemma from its validation report (historical/reviews/2026-11-03_astra_validation_corrections.md, item 11); checked by Claude Opus 5.5 in Python (the direct doublet kernel from equation (1), the pole equations, the ratio). Ledger row `l18_fully_massive`. Stated after Proposition E.1 of paper/towerless_intervals.tex and cited in paper/boundary_condition_note.tex ("\supp{18}").

Setting. Heat cutoff; three generations with right-handed neutrinos; no Majorana masses; one common Dirac mass m > 0 for every Standard Model species; x = m^2/Lambda^2; z_W and z_Z the charged and massive neutral poles in units of Lambda^2; H and g as in Proposition 4; v = (1, -1).

Statement.

1. The charged form factor is proportional to e^{-x} [g(y) + 2x H(y)], and the neutral particle-trace kernel to e^{-x} [diag(6, 10) g(y) + 12 x H(y) v v^T], whose massive factor has coefficient 12 x v^T diag(6, 10)^{-1} v = 16x/5.
2. Both massive pole equations have the form (2 + beta - z) H(-z) = 2, with beta_W = 2x and beta_Z = 16x/5, so 2x < z_W < 2x + 2 and 16x/5 < z_Z < 16x/5 + 2: both pole masses diverge with m at

fixed cutoff.

3. M_{W,pole}^2/M_{Z,pole}^2 -> 5/8 as x -> infinity, and also as x -> 0, where z_W ~ 3x and z_Z ~ 24x/5. In between the ratio differs: at x = 3, z_W = 7.4996, z_Z = 11.3746 and the ratio is 0.6593 (Python 3 with SciPy, not kept).
4. The kernel tends to zero at every fixed Euclidean momentum, and the charged Taylor kinetic coefficient is proportional to e^{-x} (1 - x/2), negative for x > 2.

Scope. A counterexample to any general no-go for diverging pole locations at fixed cutoff. The family has no light fermions.

Proof. Lemma 18, Proposition 14 and the complete proof written after it; the proofs are there. With F(z) = z - 2 + 2/H(-z), F(0) = 0 and F'(z) >= 1/2 for z >= 0, so each beta > 0 has a unique positive root, at which H(-z) > 1 gives beta < z < beta + 2.

What it leaves open. The physical quality of a limit with no light fermions is open (lemma 18, after Proposition 14).

Part S67 (The top-only Taylor result extends to nonnegative compactly supported cutoffs, and with several heavy doublets the Taylor W mass tends to zero whenever a complete massless doublet is retained)

The top-only Taylor result extends to nonnegative compactly supported cutoffs, and with several heavy doublets the Taylor W mass tends to zero whenever a complete massless doublet is retained

Status. Propositions 12 and 13 of lemma 18. Proved by Terra 6.1 in the referee report on lemma 18 (historical/reviews/2026-11-03_terra_check_18_16b.md); checked by Claude Opus 5.5 in Python (the Taylor ratio and the pole root at large x). Ledger row `l18_several`.

Setting. Proposition 12: nonnegative smooth compactly supported f with f(0) > 0, with the top-only data of 18_top_only_w_mass_at_fixed_cutoff.md. Proposition 13: the heat cutoff; P the total particle-trace weight of complete massless doublets; heavy doublet i of weight d_i > 0 with one Dirac member of mass m_i and one massless member, x_i = m_i^2/Lambda^2; g, h_x, A and w as in Proposition 4.

Statement.

1. (Proposition 12) The formulas of Proposition 4 hold with A_f(x) = (2/x) int_0^x z f(z) dz and w_f(x) = (3/x^3) int_0^x z^2 f(z) dz, so M_{W,Taylor}^2/Lambda^2 ~ (int_0^infinity z f(z) dz)/(f(0) x). This includes cutoffs flat near zero. For a nonanalytic cutoff there need be no analytic continuation defining the pole.
2. (Proposition 13) C_W(y) = (Lambda^2/(4 pi^2)) (P g(y) + sum_i d_i h_{x_i}(y)) and M_{W,Taylor}^2/Lambda^2 = (3/2) (sum_i d_i A(x_i))/(P + sum_i d_i w(x_i)).
3. For finitely many doublets with P > 0 and all x_i -> infinity, the Taylor mass tends to zero.
4. If P = 0 and all x_i = x, M_{W,Taylor}^2 ~ m^4/(2 Lambda^2), the form factor has a unique positive pole with M_{W,pole}^2 ~ m^2/2, and the whole unnormalized kernel tends to zero at every fixed Euclidean momentum.

Scope. For P = 0 the divergent Taylor and pole scales differ, and neither alone establishes a healthy limiting gauge theory.

Proof. Lemma 18, Propositions 12 and 13; the proofs are there. Proposition 12 approximates f, with its derivatives, by real combinations of e^{-nz}.

What it leaves open. In the families treated in lemma 18, a heavy W and Z at fixed cutoff is found only without a massless complete doublet, and the physical quality of such limits is open (lemma 18, consequence

for T5b; 18_fully_massive_family.md).

Part S68 (In the top-only heat-cutoff family, m_t, M_W and M_Z diverge together only jointly with Lambda -> infinity, and the limit m_t/M_W -> sqrt(8/3) needs m_t/Lambda -> 0)

In the top-only heat-cutoff family, m_t, M_W and M_Z diverge together only jointly with Lambda -> infinity, and the limit m_t/M_W -> sqrt(8/3) needs m_t/Lambda -> 0

Status. Corollary (Corollary 10 of lemma 18). Proved by Astra 6 (job 18); checked by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_18_16b.md). Ledger row `l18_joint`.

Setting. The background family and mass definitions of 18_top_only_w_mass_at_fixed_cutoff.md and 18_unbroken_coefficients_and_angles.md, with mu = m_t/Lambda.

Statement.

1. At fixed Lambda with mu -> infinity: m_t = mu Lambda -> infinity; M_{W,Taylor} and M_{W,pole} behave as Lambda/mu -> 0; M_{Z,Taylor} and M_{Z,pole} behave as sqrt(2/3) Lambda mu e^{-mu^2/2} -> 0; and m_t/M_W ~ mu^2 -> infinity.
2. At fixed 0 < mu < infinity with Lambda -> infinity, all four vector masses grow linearly with Lambda, as does m_t, and the ratios are functions of mu.
3. A controlled family with Lambda -> infinity, mu -> 0 and mu Lambda -> infinity has M_W, m_t -> infinity and m_t/M_W -> sqrt(8/3).

Scope. The top-only heat-cutoff Taylor family; other finite data and mass definitions need their own argument (ledger row). The obstruction has three scopes: the CCM identity belongs to the asymptotic truncation and cannot be extrapolated to mu -> infinity at fixed cutoff; the exact top-only quadratic action obstructs a fixed-cutoff heavy-W limit, for the heat cutoff under both mass definitions and for class (4) under the Taylor definition; and a family of actual nonzero stationary vacua meets the separate obstruction of No-go theorem 9 (18_nogo_radial_vacuum.md). Counterexample 7 excludes extending the exact mass conclusions to arbitrary finite representations.

Proof. Lemma 18, Corollary 10; the proof is there, from the limits of Proposition 4, Theorem 5 and Proposition 6 and the scaling of the dimensionless mass functions with Lambda.

What it leaves open. Neither centralizer-only decoupling nor a stationary Higgs vacuum follows from the joint scaling (lemma 18, consequence for T5b). In this family m_t/M_W differs from sqrt(8/3) for every x > 0 (18_weighted_sum_rule.md).

Part S69 (In the top-only limit the vanishing vector masses do not restore SU(2) x U(1) on the light space: the compressed generators on b_L fail the SU(2) relations, and the limiting coefficients have K_W != K_33 and K_3Y != 0)

In the top-only limit the vanishing vector masses do not restore SU(2) x U(1) on the light space: the compressed generators on b_L fail the SU(2) relations, and the limiting coefficients have K_W != K_33 and K_3Y != 0

Status. Proposition, proved (Remark 15 of lemma 18). From the referee report on lemma 18 by Terra 6.1 (historical/reviews/2026-11-03_terra_check_18_16b.md); checked by Claude Opus 5.5 by hand. Ledger row `l18_not_unbroken` (kind claim, status proved). Item 4 of Proposition E.1 of paper/towerless_intervals.tex.

Setting. The top-only family of 18_top_only_w_mass_at_fixed_cutoff.md in the limit x -> infinity; P_light the projection off the top mass block. The light space retains b_L after removing t_L.

Statement.

1. On b_L, P_light T_1 P_light = P_light T_2 P_light = 0 and P_light T_3 P_light = -1/2, which do not obey the SU(2) relations.
2. In the limit K_W = 9/2, K_33 = 21/4 and K_3Y = -1/4, while a linearly realized unbroken action would have equal weak coefficients and no mixing.
3. The remaining complete massless doublets carry the ordinary representation, and their contribution is gauge invariant.
4. If both members of every removed doublet are heavy, the retained light space can be invariant under the full group.
5. On the light space, the five other quarks and the leptons, SU(3) x U(1)_em acts with b_L paired with b_R.

Scope. "The heavy fermion leaves the loop" is shorthand for the suppression in this classical trace. A full limiting theory still needs control of the remaining fields, normalizations, interactions and vacuum.

Proof. Lemma 18, Remark 15; the argument is there, by compressing the generators and taking the limits of Proposition 6.

What it leaves open. The separate role of the top (lemma 21): among single charged species only an up-type quark can be removed while keeping cos^2 theta_NC = 5/8 and the ratio 8/3 (lemma 18, after Remark 15).

Part S70 (For cutoffs in the positive Laplace class and without an added potential or constraint, no finite c > 0 is stationary along a radial ray V = c V_* of the spectral action)

For cutoffs in the positive Laplace class and without an added potential or constraint, no finite c > 0 is stationary along a radial ray V = c V_* of the spectral action

Status. No-go theorem in its scope (No-go theorem 9 of lemma 18). Proved by Astra 6 (job 18); checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_18_16b.md). Ledger row `l18_radial`.

Setting. f in class (4): a Laplace transform of a nonzero positive measure nu with 0 < f(0) < infinity and int (s^{-2} + s^2) d nu < infinity. Scalar configurations V = c V_*, c > 0, V_* != 0, with A = 0; the literal spectral action on R^4, or on a finite flat torus before its positive momentum sum.

Statement.

1. The constant-background action density is U(c) = (Lambda^4/(4 pi^2)) int s^{-2} Tr_{H_F} exp(-s c^2 V_*^2/Lambda^2) d nu(s).
2. U'(c) = -(c Lambda^2/(2 pi^2)) int s^{-1} Tr_{H_F}(V_*^2 exp(-s c^2 V_*^2/Lambda^2)) d nu(s) < 0, strictly since V_* != 0 and nu is nonzero and positive.
3. So no finite c > 0 is stationary along this radial direction. In the top-only family the ordinary Higgs radial variation supplies the direction, and the masses of lemma 18 are conditional quadratic-background quantities.

Hypotheses. The positive Laplace class; the literal spectral action; no added scalar potential and no constraint fixing c; radial paths of the form c V_*.

Escapes. Proposition 11 of lemma 18 (18_radial_derivative_general_cutoff.md) gives them within real cutoffs: a nonnegative cutoff of compact support has a stationary plateau, sign-changing cutoffs can have

cancellations, and a radial path with an additional fixed noncommuting mass matrix need not have the form c V_*. A finite symmetry-breaking minimum of the polynomial asymptotic scalar potential cannot automatically be regarded as a minimum of this full action.

Proof. Lemma 18, No-go theorem 9; the proof is there, by differentiating U(c) under the integral.

What it leaves open. The result points to an added potential or constraint, a cutoff outside the class, or a quantum effective action as the source of a finite vacuum (lemma 18, Scope and the remark after the proof).

Part S71 (On flat R^4 the radial derivative of the spectral action is U'(c) = -(c Lambda^2/(2 pi^2)) sum_j lambda_j^2 int_{c^2 lambda_j^2/Lambda^2}^infinity f(z) dz, so nonnegative cutoffs of unbounded support have no finite stationary point and compactly supported ones have a plateau)

On flat R^4 the radial derivative of the spectral action is U'(c) = -(c Lambda^2/(2 pi^2)) sum_j lambda_j^2 int_{c^2 lambda_j^2/Lambda^2}^infinity f(z) dz, so nonnegative cutoffs of unbounded support have no finite stationary point and compactly supported ones have a plateau

Status. Proposition (Proposition 11 of lemma 18). Proved by Terra 6.1 in the referee report on lemma 18 (historical/reviews/2026-11-03_terra_check_18_16b.md); checked by Claude Opus 5.5 in Python (numerical derivative against the formula for a compact bump). Ledger row `l18_radial_general`.

Setting. A sufficiently integrable real cutoff f; flat R^4; radial configurations V = c V_*, c > 0, with lambda_j^2 the eigenvalues of V_*^2.

Statement.

1. U'(c) = -(c Lambda^2/(2 pi^2)) sum_j lambda_j^2 int_{c^2 lambda_j^2/Lambda^2}^infinity f(z) dz.
2. For nonnegative f, a finite c > 0 is stationary exactly when every tail associated with a nonzero lambda_j vanishes.
3. If f has nonzero support arbitrarily far out, there is no finite stationary point on the ray.
4. If f has compact support ending at R, there is a stationary plateau once every nonzero c^2 lambda_j^2/Lambda^2 >= R.
5. Sign-changing cutoffs can have cancellations.
6. A nonconstant cutoff flat near zero lies outside class (4), since a nonzero positive measure gives f'(0) < 0; formula (1) covers the nonnegative ones.

Scope. Radial paths of the form c V_*; a radial path with an additional fixed noncommuting mass matrix need not have that form.

Proof. Lemma 18, Proposition 11; the statement and its proof are there.

What it leaves open. The cases with cancellations (sign-changing f) and radial paths with a fixed noncommuting mass matrix are not computed; this proposition supplies the escapes listed in 18_nogo_radial_vacuum.md.

Part S72 (In the three-generation Standard Model with only the top massive, the W mass at fixed cutoff tends to zero as M_W^2 ~ Lambda^4/m_t^2 for both the Taylor and the pole definitions, with M_W^2 <= Lambda^2/e)

In the three-generation Standard Model with only the top massive, the W mass at fixed cutoff tends to zero as M_W^2 ~ Lambda^4/m_t^2 for both the Taylor and the pole definitions, with M_W^2 <= Lambda^2/e

Status. Proposition 4 and Theorem 5 of lemma 18. Proved by Astra 6 (job 18); checked by Claude Opus 5.5 in Python (closed forms, limits, the bound, the root of the pole equation) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_18_16b.md). Ledger row `l18_taylor`. Item 1 of Proposition E.1 of paper/towerless_intervals.tex.

Setting. The full three-generation Standard Model representation with charge conjugates; zero Majorana matrix; every Dirac mass zero except one up-type colour triplet of mass m = mu Lambda; x = mu^2, r = e^{-x}; heat cutoff f(z) = e^{-z}. H(y) = int_0^1 exp(-y u(1 - u)) du, g(y) = (2 + y) H(y) - 2, h_x(y) = (2 + x + y) int_0^1 exp(-xu - yu(1 - u)) du - 1 - e^{-x}, A(x) = 2[1 - (1 + x) e^{-x}]/x, w(x) = 3 int_0^1 u^2 e^{-xu} du.

Statement.

1. (Proposition 4) The charged-vector kernel is C_W(q^2) = (Lambda^2/(4 pi^2)) [(9/2) g(y) + (3/2) h_x(y)], y = q^2/Lambda^2, and K_W(x) = 9/2 + (3/2) w(x) > 0.
2. M_{W,Taylor}^2/Lambda^2 = 9 A(x)/(4 K_W(x)) = 3[1 - (1 + x) e^{-x}]/(x[3 + w(x)]) = (3/8) x - (23/128) x^2 + O(x^3), and ~ 1/x as x -> infinity. So m/M_{W,Taylor} -> sqrt(8/3) as mu -> 0 and ~ mu^2 as mu -> infinity.
3. For every f in class (4), M_{W,Taylor}^2 ~ (3/8) m^2 as x -> 0 and M_{W,Taylor}^2/Lambda^2 ~ (int s^{-2} d nu(s))/(f(0) x) as x -> infinity, so the constant 1/e of item 5 is specific to the heat cutoff.
4. (Theorem 5) For every x > 0, C_W(-M_{W,pole}^2) = 0 has exactly one solution with M_{W,pole}^2 > 0; it is simple, with positive residue, and z(x) = M_{W,pole}^2/Lambda^2 ~ (3/8) x as x -> 0 and ~ 1/x as x -> infinity.
5. Both mass definitions obey 0 < M_{W,Taylor}^2, M_{W,pole}^2 <= Lambda^2/e.

Scope. If the other Dirac masses are small and nonzero, the answer depends on them, and masses of surviving light doublets can leave a residual vector mass. No closed-form inversion of the pole equation, no classification of complex poles and no Lorentzian unitarity theorem is asserted; the pole statements extend to measures nu supported in a compact subset of (0, infinity), and for the rest of class (4) only the Euclidean and Taylor conclusions are established. The pole mass governs the linearized dispersion of the continued quadratic action, the Taylor mass its two-derivative truncation, and they need not coincide near the cutoff.

Proof. Lemma 18, Proposition 4 and Theorem 5; the proofs are there. The massive block has eigenvalues +-m, connected to the massless down states with d_ij^2 = m^2; h_x'(y) > 0 for y <= 0 gives monotonicity of F_x = (9/2) g + (3/2) h_x, and F_x' >= 3 with the mean-value theorem gives z(x) <= A(x)/2 <= 1/e.

What it leaves open. Within this family the results refute an infinitely heavy W, Z, t limit at fixed cutoff for these mass definitions. Other finite data (18_fully_massive_family.md), a different action or constraint, a joint limit with Lambda -> infinity (18_joint_limit.md) and a quantum effective theory lie outside them (lemma 18, Scope).

Part S73 (In the top-only heat-cutoff family the photon and gluon coefficients keep the ratio 8/3 and the neutral-current angle keeps cos^2 theta_NC = 5/8 for every m/Lambda, while M_Z tends to zero exponentially and the matching angle tends to 103/157)

In the top-only heat-cutoff family the photon and gluon coefficients keep the ratio 8/3 and the neutral-current angle keeps cos^2 theta_NC = 5/8 for every m/Lambda, while M_Z tends to zero exponentially and the matching angle tends to 103/157

Status. Proposition (Proposition 6 of lemma 18). Proved by Astra 6 (job 18); checked by Claude Opus 5.5 in Python (the neutral matrix recomputed from Standard Model charges; v^T B_inf^{-1} v = 8/27 with SymPy) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_18_16b.md). Ledger row `l18_angles`. Item 2 of Proposition E.1 of paper/towerless_intervals.tex.

Setting. The family of 18_top_only_w_mass_at_fixed_cutoff.md; colour generators with tr_3 t_a t_b = (1/2) delta_ab and Q = T_3 + Y; kappa_em, kappa_s the full-trace coefficients of F_em^2 and G_a^2; K the particle-trace neutral q^2 coefficient in the coordinates (A_3, A_Y); v = (1, -1).

Statement.

1. kappa_em = (1/(24 pi^2)) (16/3) (5 + r) and kappa_s = (1/(24 pi^2)) 2 (5 + r), so kappa_em/kappa_s = 8/3; both approach 5/6 of their all-massless values.
2. K_33 = 21/4 + (3/4 - 3x/8) r, K_YY = 103/12 + (17/12 - 3x/8) r, K_3Y = -1/4 + (1/4 + 3x/8) r. K, K_W and the unbroken coefficients are positive for every x >= 0, with K >= (5 - 3/(4e)) I.
3. With J_Z proportional to T_3 - sin^2 theta_NC Q after diagonalizing the neutral kinetic form, cos^2 theta_NC = (K_YY + K_3Y)/(K_33 + 2 K_3Y + K_YY) = 5/8 for every x.
4. The matching angle cos^2 theta_match = K_YY/(K_W + K_YY) equals 5/8 at x = 0 and tends to 103/157 as x -> infinity, so the two definitions cease to describe one common electroweak angle.
5. M_{Z,Taylor}^2/Lambda^2 = (9/4) x r v^T K^{-1} v ~ (2/3) x e^{-x}, and for x > 0 there is one positive massive neutral pole besides the photon, with M_{Z,pole}^2/Lambda^2 ~ (2/3) x e^{-x}; so M_W^2/M_Z^2 does not stay at 5/8.
6. For every f in class (4), with 1 -> f(0), r -> f(x) and x r -> -x f'(x), the ratio 8/3, the value 5/8 and the limit 103/157 persist.

Scope. At the endpoint both neutral masses vanish, so the distinguished massive neutral direction, and with it the angle, requires continuation from finite x.

Proof. Lemma 18, Proposition 6; the proof is there. Photon and colour generators commute with V_0; a Dirac species of mass m with neutral charges q_L, q_R contributes e^{-x} [q_L^a q_L^b + q_R^a q_R^b - (x/2)(q_L^a - q_R^a)(q_L^b - q_R^b)] to K; and K_33 + K_3Y = 5 + r, K_YY + K_3Y = (5/3)(5 + r).

What it leaves open. The separate role of the top, which keeps both values when removed, is lemma 21: among single charged species only an up-type quark has this property (18_light_space_not_electroweak.md).

Part S74 (For the heat cutoff and doublets with one massive member, 8 kappa M_{W,Taylor}^2 = sum_f N_f m_f^2 rho(m_f^2/Lambda^2) with rho(x) = 2[1 - (1 + x) e^{-x}]/x^2, so the CCM relation is the small-mass term of a weighted sum rule)

For the heat cutoff and doublets with one massive member, 8 kappa M_{W,Taylor}^2 = sum_f N_f m_f^2 rho(m_f^2/Lambda^2) with rho(x) = 2[1 - (1 + x) e^{-x}]/x^2, so the CCM relation is the small-mass term of a weighted sum rule

Status. Proposition, proved (Remark 16 of lemma 18). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (agreement with the closed Taylor mass to rounding, for x from 1e-3 to 1e3) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_paper_note.md, items 13 to 15). Ledger row `l18_sum_rule`. Stated after Proposition E.1 of paper/towerless_intervals.tex.

Setting. Heat cutoff; the mass-compatible doublet decomposition of Proposition 13 of lemma 18: P the particle-trace weight of complete massless doublets, heavy doublets i of weight d_i with one massive member of mass m_i, N_i = 2 d_i and P + sum_i d_i = 6; x_i = m_i^2/Lambda^2, w(x) = 3 int_0^1 u^2 e^{-xu} du, A(x) as in Proposition 4.

Statement.

1. 8 kappa M_{W,Taylor}^2 = sum_i N_i m_i^2 rho(x_i), with rho(x) = A(x)/x = 2[1 - (1 + x) e^{-x}]/x^2 and kappa = (P + sum_i d_i w(x_i))/6.
2. As every x_i -> 0, rho and kappa tend to 1, which is the CCM relation; its range is m « Lambda.
3. For the top alone, kappa = (3 + w)/4. Then m/M_W departs from sqrt(8/3) by 5% at m about 0.45 Lambda, and M_W has a unique maximum, M_W/Lambda = 0.5242 at x = 2.024 (referee, by a sign argument and Python).
4. In the top-only heat-cutoff family m_t/M_W differs from sqrt(8/3) for every x > 0, since 3 + w - 4 rho vanishes at zero and is increasing; so the limit with M_W and m_t diverging together and m_t/M_W -> sqrt(8/3) needs Lambda -> infinity faster than m_t (Corollary 10).

Scope. Item 4 is a necessity for that family: with one complete massless lepton doublet and one member of each other doublet of common mass m, the weighted sum equals the unweighted CCM sum at some x_* > 0 (referee). A massive diagonal contribution to the trace is exponentially suppressed above the cutoff, while the mixed loop with a massless partner has rho ~ 2/x^2 at large x, so a vacuum with no other mass leaves the W mass with no source.

Proof. Lemma 18, Remark 16; the derivation is there, from Proposition 13.

What it leaves open. Whether the classical action is meaningful for m > Lambda is a separate question, on which these formulas do not decide (lemma 18, Remark 16).

Part S75 (Along a gauge orbit the modulus of the link stays fixed for the covariant projection of the parent and for the finite data with the quadratic term, and grows at second order for the fixed compression and for the finite data with linear terms only)

Along a gauge orbit the modulus of the link stays fixed for the covariant projection of the parent and for the finite data with the quadratic term, and grows at second order for the fixed compression and for the finite data with linear terms only

Status. Proposition (Proposition 19.3), a parallel with no identification. Proved by Claude Opus 5.5; checked by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_nogo_19.md). Ledger row `l19_parent`.

Setting. The chiral interval at M = 0 with m = pi/(2l); the link of the fixed compression (lemma 13, Proposition 13.2; 13_fixed_compression_gauge_fluctuations.md), the link of the covariant projection (lemma 14, Theorem 3; 14_link_fixed_singular_values.md), and the two-point fluctuations of 19_fluctuations_two_point_data.md.

Statement.

1. With no real structure, the finite fluctuation is m(1 + phi_1), all of C, while the fixed compression of a gauge field gives only the line m + iB.
2. With pointwise reality, the parent C(I) is commutative with J f J^{-1} = conj(f), satisfies order one, and has A + J A J^{-1} = 0 for a scalar one-form, so the specified parent has no surviving self-adjoint scalar inner fluctuation, while the truncated data have the gauge-neutral radial parameter t.
3. With the reflection real structure, the parent satisfies order one, J_r D_{A_5} J_r^{-1} = D_{A_5 o R} with R the reflection, the symmetrized field A_5 + A_5 o R survives, and the fixed compression gives m + i B^s with B^s = 2B, the weight sin(pi x/l) being even about l/2.
4. Along a pure-gauge path the modulus of the link is constant for the covariant projection (lambda W_l^*, fixed singular value) and for the finite data with A_(2) (m); it is (m^2 + B^2)^{1/2} for the fixed compression and m(5 - 4 cos psi)^{1/2} for the finite data with J_r and linear terms only, both growing at second order.

Scope. A parallel, with no identification of variables: for constant unsymmetrized A_5 = a the transported link is m e^{ial} = m + i(pi a/2) + O(a^2), with Wilson phase al, against m + i(2a/pi) in the compression. Item 2 establishes neither a dynamical scalar field nor its action, and says nothing about parent fields in other models; compression is not multiplicative, so finite potentials have no canonical identification with parent potentials. The phase of a link depends on the frame, its modulus does not.

Proof. Lemma 19, Proposition 19.3; the proof is there, collecting lemma 14, Theorem 3, Corollary 19.2 and lemma 13, Proposition 13.2; for item 2, K (i E a) K^{-1} = -i E a for a real a.

What it leaves open. An exact statement relating a fluctuated parent and its truncation holds on gauge orbits by lemma 22, Corollary 22.6, and remains open for the nonunitary part (lemma 19, Reading).

Part S76 (The inner fluctuation without order one of the truncated two-point data factorizes: m |Phi|^2 sigma_1, gauge-neutral, with the pointwise real structure, and a link m Phi^2 of charge 2, whose quadratic term keeps the modulus fixed along a gauge orbit, with the reflection real structure)

The inner fluctuation without order one of the truncated two-point data factorizes: m |Phi|^2 sigma_1, gauge-neutral, with the pointwise real structure, and a link m Phi^2 of charge 2, whose quadratic term keeps the modulus fixed along a gauge orbit, with the reflection real structure

Status. Theorem (Theorem 19.1), with Corollary 19.2. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (numpy on random data: both factorizations to about 1e-15, gauge covariance, the orbit) and by Terra 6.1 by hand, who found item 3 of Corollary 19.2 and the Wilson phase after the table wrong as first written, corrected (historical/reviews/2026-11-03_terra_check_nogo_19.md). Ledger rows `l19_fluct` and `l19_gauge`. Cited in paper/towerless_intervals.tex and in paper/real_structure_note.tex ("\supp{19}").

Setting. A = C + C acting diagonally on H = C^2, D = m sigma_1 with m > 0, gamma = sigma_3. Two real structures, with x-hat = J x J^{-1}: J_0 = K (or sigma_3 K), inherited from pointwise reality on the interval, commuting with gamma; and J_r = sigma_1 K, inherited from the reflection real structure of lemma 08, Counterexample 4, with J_r^2 = 1, J_r D = D J_r and J_r gamma = -gamma J_r, the KO-6 signs. Both satisfy order zero and fail order one: [[D, e], e-hat] = m sigma_1 for J_0 and -m sigma_1 for J_r, with e = diag(1, 0). The fluctuations of arXiv:1304.7583: D' = D + A_(1) + A~*(1)* + *A*(2) with A_(1) = sum a_j [D, b_j] = m [[0, phi_1], [phi_2, 0]], A~*(1)* = *eps J A*(1) J^{-1}, A_(2) = sum a_j-hat [A_(1), b_j-hat]; gauge transformations U = u J u J^{-1}.

Statement.

1. (Theorem 19.1) For every universal one-form, J_0 gives D' = m [[0, |1 + phi_1|^2], [|1 + phi_2|^2, 0]] and J_r gives D' = m [[0, (1 + phi_1)(1 + conj(phi_2))], [(1 + phi_2)(1 + conj(phi_1)), 0]].
2. For a self-adjoint form, with Phi = 1 + phi_1: J_0 gives D' = m |Phi|^2 sigma_1, and J_r gives D' = m [[0, Phi^2], [conj(Phi)^2, 0]].
3. (Corollary 19.2) With u = diag(e^{i alpha}, e^{i beta}) and psi = alpha - beta, Phi -> e^{i psi} Phi. For J_0, U = 1 and the image of the self-adjoint forms is {t sigma_1 : t >= 0}, a real gauge-neutral field, with t = 0 reached at Phi = 0.
4. For J_r, U = diag(e^{i psi}, e^{-i psi}) and the link m Phi^2 goes to e^{2 i psi} m Phi^2: U(1) acts with charge 2, and the image is all of C, pointwise; a globally prescribed smooth link need not have a global smooth square root.
5. The gauge orbit of the zero form has Phi = e^{i psi} and gives D' = D for J_0 and the link m e^{2 i psi} for J_r. Without A_(2) the link would be m(2 cos psi - 1) for J_0, although U = 1, and m(2 e^{i psi} - 1) for J_r, of modulus m(5 - 4 cos psi)^{1/2}, off the orbit of D: the quadratic term restores gauge covariance.

Proof. Lemma 19, Theorem 19.1 and Corollary 19.2; the proofs are there, by direct computation with [X, b] = [[0, x(b_2 - b_1)], [y(b_1 - b_2), 0]] for off-diagonal X and diagonal b.

What it leaves open. The first-order failure of lemma 08, Counterexample 4 has this escape. In arXiv:1304.7583 the quadratic term generates no new scalar fields, and here too it depends on the same two parameters; relating the neutral radial parameter t to the singlet of the grand-symmetry model is an analogy that would need a map of algebras, representations, fields and gauge actions.

Part S77 (A block T: E -> E^* added to a sheet-diagonal finite Dirac operator keeps the KO-6 signs iff T = tau(T) and hT + Tg = 0, and has order one on B iff T pi_1(a) - pi_2(a) T intertwines the opposite actions; the maximal such B are the maximal A(R) over odd R with T = R + tau(R))

A block T: E -> E^* added to a sheet-diagonal finite Dirac operator keeps the KO-6 signs iff T = tau(T) and hT + Tg = 0, and has order one on B iff T pi_1(a) - pi_2(a) T intertwines the opposite actions; the maximal such B are the maximal A(R) over odd R with T = R + tau(R)

Status. Theorem (Result 1 of lemma 20). Proved by Astra 6 (job 20); checked by Claude Opus 5.5 in Python (the example with T different from R) and by hand (the averaging proof), and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_20_21.md). Ledger row `l20_blocks`. Item 1 of Proposition I of paper/towerless_intervals.tex.

Setting. E = Hom(V, W), E^* = Hom(W, V) with Hilbert-Schmidt products; c: E -> E^* the antiunitary of the real structure, J(x, y) = (c^{-1} y, cx), Gamma = diag(g, h) with h = -c g c^{-1} (in Example 16.9, cx = x^\, *gx = gamma_W x, hy = -y gamma_W). A^ev acts diagonally by pi_1, pi_2, with opposite actions rho_1, rho_2; the sheet-diagonal D_F already satisfies self-adjointness, oddness, J D_F = D_F J and order one on A^ev (Corollary 16.8). S_T = [[0, T^*], [T, 0]], D = D_F + S_T, tau(T) = c T^\ c.* A(R) = {a in A^ev : pi_2(a) R = R pi_1(a), pi_2(a^\) *R = R pi_1(a^\)*}, the commutation algebra of Chamseddine-Connes, section 4.

Statement.

1. D has the required signs and satisfies order one on a unital involutive subalgebra B of A^ev exactly when T = tau(T), hT + Tg = 0, and (T pi_1(a) - pi_2(a) T) rho_1(b) = rho_2(b) (T pi_1(a) - pi_2(a) T) for all a, b in B. In bases related by c, T = tau(T) says that the matrix of T is complex symmetric.
2. For fixed T, the maximal admissible algebras are the inclusion-maximal members of {A(R) : hR + Rg = 0, T = R + tau(R)}, so maximal dimension is reached by maximizing over these algebras.
3. The distinction between T and R is necessary: for ungraded W = V = C^2 (g = 1, h = -1), B = {(diag(alpha, beta), diag(alpha, beta))} and T e_11 = f_12, T e_12 = f_11, T is symmetric and odd and satisfies order one, but fails to intertwine the left actions when alpha != beta.

Proof. Lemma 20, Result 1; the proof is there. Averaging T over the compact unitary group of B gives R_0 intertwining the left actions and Q_0 = T - R_0 intertwining the opposite ones; tau exchanges the two conditions, and R = (R_0 + tau(Q_0))/2 satisfies T = R + tau(R).

What it leaves open. Which blocks a parent supplies: the rank-one case is 20_rank_one_block_classified_algebra.md and a parent is 20_interval_parent_connector.md.

Part S78 (Full scalar gluing f_Q(l) = f_U(l) on the one-flavour boundary-coupled interval gives a parent with signs (+, +, -), order zero and order one whose lowest window C + C, m sigma_1, sigma_1 K fails order one)

Full scalar gluing f_Q(l) = f_U(l) on the one-flavour boundary-coupled interval gives a parent with signs (+, +, -), order zero and order one whose lowest window C + C, m sigma_1, sigma_1 K fails order one

Status. Counterexample (Result 4 of lemma 20). Proved by Astra 6 (job 20); checked by Claude Opus 5.5 (the eigenvectors, J and the commutator by hand; the compression by quadrature in Python) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_20_21.md). Ledger row `l20_gluing_fails`. Item 5 of Proposition I of paper/towerless_intervals.tex.

Setting. Lemma 12 with N_Q = N_U = 1, both bulk masses zero and X = t > 0; theta = arctan t in (0, pi/2), m = theta/l. The full domain-preserving scalar algebra A_glue = {(f_Q, f_U) : f_Q(l) = f_U(l)} (Lemma 3 of lemma 20); Gamma(Q_L, Q_R; U_L, U_R) = (Q_L, -Q_R; U_L, -U_R) and J(Q_L, Q_R; U_L, U_R) = (conj U_R, -conj U_L; -conj Q_R, conj Q_L).

Statement.

1. The parent satisfies the signs (+, +, -), order zero and order one: the opposite algebra acts by the two scalar functions exchanged.
2. The lowest window is spanned by u_+ = l^{-1/2} (cos mx, 0; -sin mx, 0) and u_- = l^{-1/2} (0, sin mx; 0, cos mx), with the next absolute eigenvalue (pi - theta)/l.
3. The window gives A_fin = C + C, D_fin = m sigma_1, J_fin = sigma_1 K and Gamma_fin = sigma_3, and for p = diag(1, 0), p° = 1 - p and [[D_fin, p], p°] = -D_fin != 0: the finite data fail order one.
4. The projection onto the Q-field compresses to diag(1/2 + sin(2 theta)/(4 theta), 1/2 - sin(2 theta)/(4 theta)), which is not a projection: the coupled window does not preserve the original field decomposition.

Scope. The full scalar gluing algebra: agreement of scalar functions at a wall does not survive compression as the finite order-one restriction, even when the parent has all the required signs and order conditions. Constant multipliers selected by the domain, as in 20_interval_parent_connector.md, avoid the failure.

Proof. Lemma 20, Result 4; the proof is there. A nonnegative bump f_Q supported near zero with f_U = 0 satisfies the wall equality and separates the diagonal entries of the compression, by (1/l) int f_Q cos(2mx) dx > 0.

What it leaves open. The finite summands require an identification, such as continuation from the uncoupled zero modes (lemma 20, after Result 4).

Part S79 (The same-chirality analogue of the Horava-Witten walls is type (a) at both ends with opposite localization masses, and unbroken E_8 x E_8 allows no bilinear connector between (248, 1) and (1, 248))

The same-chirality analogue of the Horava-Witten walls is type (a) at both ends with opposite localization masses, and unbroken E_8 x E_8 allows no bilinear connector between (248, 1) and (1, 248)

Status. Proposition (Result 8 of lemma 20). Proved by Astra 6 (job 20); checked by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_20_21.md). Ledger row `l20_hw`. Cited in the discussion after Theorem D of paper/towerless_intervals.tex ("\supp{20}").

Setting. One common ten-dimensional chirality convention; the unbroken boundary gauge group E_8 x E_8; the boundary types (a), (b), (c) of lemma 09 and the pair of lemma 12 at X = 0.

Statement.

1. The same-chirality interval analogue is type (a) at both ends, or type (b) at both ends after reversing the chirality convention; two separately localized multiplets are two fields of that type with opposite signs of their localization masses.
2. Lemma 12 at X = 0 contains one type-(a) and one type-(b) field, so its displayed pair has opposite chiralities; type (c) produces opposite-chirality wall profiles.
3. On a space containing only the two same-chirality modes, an operator odd for that chirality grading has zero matrix elements between them.
4. Hom_{E_8 x E_8}((248, 1), (1, 248)) = 0, since the first factor acts trivially on the target and has no invariant vector in its adjoint representation; so these boundary multiplets supply no gauge-invariant bilinear connector of the interval mass-block type.

Scope. Horava-Witten impose Gamma_11 chi = chi on the boundary gaugino, and the boundary vector multiplets are intrinsic boundary fields (arXiv:hep-th/9603142, section 2). Labels based on outward normals change sign between the endpoints, which does not change the comparison in a fixed ten-dimensional convention.

Proof. Lemma 20, Result 8; the proof is there.

What it leaves open. Bulk propagation can define effective boundary-to-boundary interactions; identifying one with the finite T requires a specified field, representation and grading, possibly after compactification or symmetry breaking (lemma 20, after Result 8).

Part S80 (Two massless interval fields valued in E and E^* with wall coupling X = t S^* and constant multipliers B = A(S) give, in the lowest window, finite data with connecting block m S, m = arctan(t)/l, the KO-6 signs, order zero and order one)

Two massless interval fields valued in E and E^* with wall coupling X = t S^* and constant multipliers B = A(S) give, in the lowest window, finite data with connecting block m S, m = arctan(t)/l, the KO-6 signs, order zero and order one

Status. Theorem (Result 5 of lemma 20). Proved by Astra 6 (job 20); checked by Claude Opus 5.5 by hand (including the transport (21) and its compression to m(S + S^*)) and by Terra 6.1 by hand

(historical/reviews/2026-11-03_terra_check_20_21.md). Ledger row `120_parent`. Items 3 and 4 of Proposition I of paper/towerless_intervals.tex.

Setting. The Hom representation, c, g and h of 20_connecting_blocks.md; k = c g c^{-1}, h = -k; a unit vector e in E with ge = e, f = ce, S = |f><e| and X = t S^\: $E^\backslash$ -> E with t > 0. Massless interval fields Q valued in E and U valued in E^*, with the boundary conditions of lemma 12; constant elements of A^ev acting by pi_1 on Q and by pi_2 on U; Gamma-hat = (g Q_L, -g Q_R; k U_L, -k U_R) and J-hat = (c^{-1} U_R, -c^{-1} U_L; -c Q_R, c Q_L); n = dim_C E, theta = arctan t, m = theta/l, and m < Lambda_0 < (pi - theta)/l.

Statement.

1. The domain-preserving algebra of constant multipliers is B = A(S).
2. P = 1_{[-Lambda_0, Lambda_0]}(D_X) has rank 2n, and under an explicit identification PH = E + E^* the finite data are (B, E + E^*, [[0, m S^*], [m S, 0]], J, diag(g, -k)): self-adjoint and odd, with order zero, order one and the signs (+, +, -); the connecting block T = mS has complex rank one.
3. With t(l) = tan(ml), 0 < ml < pi/2, the finite mass stays exactly m while the next absolute eigenvalue pi/l - m tends to infinity.
4. The transport U(x) = exp(mxF), F rotating the coupled field channels, gives U^* D_X U = D_0 + m F (x) E_spin: a constant matrix connection with relative endpoint rotation angle theta = ml, whose compression to the uncoupled zero modes is m(S + S^*).
5. For E = Hom(C^4, H^2) and e = w v^* of Result 2, B = C + H + M_3(C), dim_C H_fin = 32, and the connecting block is the rank-one block of Result 2 with kappa = m. For E = E^* = C, B = C and the finite triple is (C, C^2, m sigma_1, sigma_1 K, sigma_3).

Scope. The Hom fibres, the constant matrix multipliers, the grading and the coupled channel are inputs; this parent is not built from scalar functions on the twistor parent. The usual zero-dimensional orientability axiom fails: the linked vectors e and f have identical left and opposite characters and opposite gradings. The block commutes with B, so it supplies no finite one-forms and no Higgs field.

Proof. Lemma 20, Result 5; the proof is there. The coupled channels have the eigenvectors of Result 4 with e and f inserted; the other channels are chiral zero modes of types (a) and (b) with next level pi/l; cX = X^* c^{-1} makes J-hat preserve the domain.

What it leaves open. Deriving these inputs from the commutative twistor parent, and one interval parent carrying both the Dirac Yukawa block of lemma 12 and this connector with order one on B; lemma 20 lists the latter as its next calculation (Readings, item 4).

Part S81 (In unchanged separate windows, scalar gluing on closed sets with empty interior leaves the compressed image unchanged, so marking points on the twistor windows selects no further algebra)

In unchanged separate windows, scalar gluing on closed sets with empty interior leaves the compressed image unchanged, so marking points on the twistor windows selects no further algebra

Status. Proposition (Result 7 of lemma 20); the ledger records it as a no-go (kind no_go) with its escapes. Proved by Astra 6 (job 20); checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand, with one amendment (historical/reviews/2026-11-03_terra_check_20_21.md). Ledger row `120_no_selection`. Item 5 of Proposition I of paper/towerless_intervals.tex ("\supp{20}").

Setting. Q_i(f) = P_i M_f P_i, with P_i projecting onto finite-dimensional spaces of smooth sections on compact manifolds M_i; closed subsets Z_i with empty interior; any scalar gluing condition that allows independent functions vanishing near Z_1 and Z_2, including equality at two marked points.

Statement.

1. Q_i(C_c^infinity(M_i Z_i)) = Q_i(C^infinity(M_i)).
2. Hence the image of the glued algebra under the unchanged separate compressions Q_1 + Q_2 equals the image of the unrestricted pair algebra.
3. In Example 16.9, with one observable algebra restricted to pullbacks from S^4, the generated real algebra in the original level-one windows remains M_2(H) + M_4(C), so scalar agreement at a base point, a point of CP^3 or a twistor fibre derives no further reduction.

Hypotheses. Unchanged separate windows. For a subset upstairs in twistor space, its image in S^4 must have empty interior, so that independent base test functions exist (referee amendment).

Escapes. A new coupled window, boundary degrees of freedom, matrix multipliers or a different observable representation lie outside the hypotheses. A marked base point does provide a quaternionic line through Pi_p = (1 + p_a Gamma_a)/2, and with a complex line inside it, a complex line in the other four-space and the rank-one map between the Hom vectors, the stabilizer calculation of Result 2 gives C + H + M_3(C); the grading 2 Pi_p - 1 and the rule selecting this connector are then additional finite data.

Proof. Lemma 20, Result 7; the proof is there: a linear functional separating the two images would be integration against a smooth density vanishing on a dense set, hence zero.

What it leaves open. An analytic parent for Example 16.9 needs an interface or interval collar, boundary trace spaces, self-adjoint coupling conditions, a compatible real structure and grading, and the spectral window of the coupled operator (lemma 20, after Result 7).

Part S82 (A rank-one connecting block T x = kappa K x K with K = v w^* has intertwining algebra C + H + M_3(C), of real dimension 24, the unique maximal-dimensional admissible algebra up to automorphism)

A rank-one connecting block T x = kappa K x K with K = v w^* has intertwining algebra C + H + M_3(C), of real dimension 24, the unique maximal-dimensional admissible algebra up to automorphism

Status. Theorem (Result 2 of lemma 20). Proved by Astra 6 (job 20); checked by Claude Opus 5.5 in Python (real dimensions 24 and 16 by linear algebra; signs and order conditions on random elements) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_20_21.md). Ledger row `120_rank_one`. Item 2 of Proposition I of paper/towerless_intervals.tex.

Setting. W = W_+ + W_- with dim_H W_+ = dim_H W_- = 1, V = C^4, the adjoint-flip convention of 20_connecting_blocks.md, so A^ev = H_+ + H_- + M_4(C). Unit vectors w in W_+ and v in V, kappa != 0, e = w v^* in E, f = v w^* = ce, and T_kappa x = kappa (w^* x v) v w^* = kappa |f><e| x.

Statement.

1. T_kappa has complex rank one and satisfies T = tau(T) and hT + Tg = 0; S_{T_kappa} has complex rank two.
2. A(T_kappa) = {(lambda + q, lambda + m) : lambda in C, q in H, m in M_3(C)}, the first lambda acting on W_+ as diag(lambda, conj(lambda)) in a basis starting with w, and the second on Cv.
3. It has real dimension 2 + 4 + 18 = 24, and up to an automorphism of A^ev it is the unique maximal-dimensional involutive subalgebra admitting a nonzero sheet-connecting Dirac block with order one; a block of rank two gives at most 4 + 4 + 8 = 16.
4. At kappa = 0 the block has rank zero and the admissible algebra has dimension 40 (ledger row).

Scope. The connecting condition is relative to the sheet projections of the original prealgebra, [S_T, e_1] != 0. On the selected algebra this block commutes with the algebra, so additional sheet-diagonal Dirac terms are needed for the usual nontrivial finite one-forms and Yukawa data.

Proof. Lemma 20, Result 2; the proof is there. The intertwining equations make Cw and Cv reducing subspaces with the same scalar action; the dimension argument follows Chamseddine-Connes, section 4, by compressing an intertwiner to irreducible spaces K: C^2 -> C^4 of rank one or two.

What it leaves open. Deriving the grading, the Hom fibres and the connector from the commutative twistor parent remains open (lemma 20, consequence for T0b and T11).

Part S83 (If the antisquarks of three generations are symmetric diquarks of the light quarks, their counts force (n_u, n_d) = (2, 3), with the top out, and leave three antitriplets of charge +4/3, half of a chiral-multiplet pairing and anomalous as superpartners)

If the antisquarks of three generations are symmetric diquarks of the light quarks, their counts force (n_u, n_d) = (2, 3), with the top out, and leave three antitriplets of charge +4/3, half of a chiral-multiplet pairing and anomalous as superpartners

Status. Proposition (Proposition 21.4), with Corollary 21.5. Stated by the user and Claude Fable 5.1 in conversation on 4 November 2026; checked by Claude Fable 5.1 by hand. Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_05_21_34.md): counts, uniqueness of (2, 3) and the SO(36) claim correct (exhaustive search to 60 and all eight chirality assignments in Python), two amendments (the group is SO(6(n_u + n_d)); the SU(3)^3 trace was missing) and one error, the window clause, which holds in reading (a) only; all applied. Ledger row `l21_exotic` (depending on `l21_so30`). Stated in the section on the charge-4/3 antitriplets of paper/real_structure_note.tex ("\supp{21}").

Setting. Let n_u up-type and n_d down-type light quarks in three colours with their conjugates, span the vector of SO(6(n_u + n_d)). The colour antitriplet of its adjoint contains the symmetric diquarks S^2(n_u + n_d) (x) 3-bar. Three generations of antisquarks need 6 states of charge -2/3 and 6 of charge +1/3 (21_so30_branching.md).

Statement.

1. The symmetric diquarks have n_d(n_d + 1)/2 states of charge -2/3, n_u n_d of charge +1/3 and n_u(n_u + 1)/2 of charge +4/3.
2. The antisquark counts 6 and 6 have the unique solution (n_u, n_d) = (2, 3) in nonnegative integers, so the top is out, and the charge +4/3 count is 3. With the top, (3, 3) gives (6, 9, 6), and the vector would need 36 = 30 + 6 real dimensions, against the 32 of SO(32), whose extra plane is colour-neutral.
3. (Corollary 21.5) A Dirac quark of charge 4/3 in each generation needs two complex scalars, so three generations need 6, and the three antitriplets are half of a chiral-multiplet pairing. Read as superpartners, their fermions are three left-handed Weyl fields in (3-bar, 4/3), SU(2) singlets, anomalous on their own: with T(3) = 1/2 and A(3) = 1 the SU(3)^3, SU(3)^2 U(1), U(1)^3 and gravitational U(1) traces are -3, 2, 64/3 and 12, and the meson sector (colour singlets of charges 0 and +-1, six of them charged) cancels none of them for any chirality of its fermions; any other chirality assignment of the three multiplets leaves the SU(3)^3 trace odd.
4. The readings compatible with the counting are (a) the three scalars have no fermion partner, as the Higgs doublet has none in the Standard Model, so supersymmetry is a property of the (6, 6) part alone; or (b) the top is included, which completes the exotic to 6 and breaks the +1/3 count to 9. In the window of the interval parent (lemma 38, Proposition 11.4, the shadow B_ij -> (m_i - m_j) B_ij, distinct masses) the same-flavour diquarks drop out: the counts become (3, 6, 1) in reading (a), with one state of charge +4/3, u and c antisymmetric in flavour with coefficient proportional to m_u - m_c, and (3, 9, 3) in reading (b), with uc, ut and ct. In both readings the window keeps 3 states of charge -2/3, so the match with the antisquarks holds in the parent and fails in the window.

Scope. The premise of the whole counting is that five light quarks generate every squark and slepton as a composite, scalars that the finite geometry of Connes et al. does not contain. Under that premise the corollary records a conflict between the count of three generations and the anomaly of the exotic sector.

Proof. Lemma 21, Proposition 21.4 and its proof: n_d(n_d + 1)/2 = 6 gives n_d = 3, then n_u n_d = 6 gives n_u = 2, and 2 x 3/2 = 3; a coloured quark adds 6 real dimensions, and 32 - 30 = 2. Corollary 21.5 is counting with the standard anomaly traces.

What it leaves open. A second independent reading. The selection of (2, 3) is a property of the parent, since the flavour-antisymmetric counts of the window and of s = -1 admit no solution of (6, 6); whether the wall condition that gives Lambda^2 V can be derived is the direction (lemma 21, Direction). The remark that the top cannot form the diquarks is from hep-ph/0512065, and the reading of the charge-4/3 states as an additional quark per generation, with the half-multiplet problem, is in the abstract of the EPJC paper (Rivero 2024) (lemma 21, Provenance). Identifying the states with physical scalar partners, in either reading, needs a field construction, spin and chirality assignments, interactions and dynamics.

Part S84 (For the five quarks other than the top, 30 = (5, 3) + (5-bar, 3-bar), and the adjoint of SO(30) contains in (15, 3-bar) and (24, 1) + (1, 1) exactly the charged scalar-partner counts of three generations, plus three triplets of charge +4/3 and one neutral state)

For the five quarks other than the top, 30 = (5, 3) + (5-bar, 3-bar), and the adjoint of SO(30) contains in (15, 3-bar) and (24, 1) + (1, 1) exactly the charged scalar-partner counts of three generations, plus three triplets of charge +4/3 and one neutral state

Status. Proposition (a representation and charge count, unnumbered in lemma 21; the ledger records it as kind claim). Computed by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (exact counting: 435 = 224 + 1 + 105 + 105, with 105 = 60 + 45 and 224 = 192 + 24 + 8) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_20_21.md). Ledger row `121_so30`, with prior work recorded as partial: hep-ph/0512065, Tables 1-2. Cited in paper/real_structure_note.tex ("\supp{21}"), in its section on the sign of an odd factorization and in its section on the charge-4/3 antitriplets.

Setting. SU(5) x SU(3) inside SO(30), acting on 30 = (5, 3) + (5-bar, 3-bar). On the 5, Q = diag(3/5, 3/5, -2/5, -2/5, -2/5) + (1/15) I_5, that is Q = 2/3 for u, c and Q = -1/3 for d, s, b, which includes the U(1) commuting with SU(5).

Statement.

1. The adjoint Lambda^2(30), of dimension 435, is (24, 8) + (24, 1) + (1, 8) + (1, 1) + (10, 6) + (15, 3-bar) + the conjugates of the last two.
2. The charges, with their numbers of states: on (15, 3-bar), -2/3: 6, +1/3: 6, +4/3: 3; on (10, 6), -2/3: 3, +1/3: 6, +4/3: 1; on (24, 1) + (1, 1), +1: 6, -1: 6, 0: 13.
3. Three generations of scalar partners in a supersymmetric Standard Model with nu_R need, for anti-squarks in the 3-bar, 6 of charge -2/3 and 6 of charge +1/3, and for sleptons 6 of charge +1, 6 of charge -1 and 12 neutral real states. So (15, 3-bar) and (24, 1) + (1, 1) contain exactly the charged counts, with three triplets of charge +4/3 and one neutral state left over.
4. SO(30) sits in SO(32), with 496 -> (435, 1) + (1, 1) + (30, 2) under SO(30) x SO(2).

Scope. These are representation and charge counts. Identifying the entries with physical scalar partners needs a field construction, spin and chirality assignments, interactions and dynamics. The counting of 15 diquarks and 25 mesons, with the charge +4/3 states and the remark that the top is unable to form them,

goes back to hep-ph/0512065, which finds neither SO(30) nor a role for SO(32); lemma 21 adds one group whose adjoint contains both families.

Proof. Lemma 21, section SO(30): the branching of the antisymmetric square of (5, 3) + (5-bar, 3-bar) under SU(5) x SU(3), with the charges read off from Q.

What it leaves open. The hint of a role for SO(32) in item 4 is not explored in lemma 21. Why the anti-squark counts exclude the top, and what the charge +4/3 states imply, is 21_counting_forces_top_out.md. The symmetry type of the massless light block and the realization of these counts on an interval are treated in paper/real_structure_note.tex.

Part S85 (With only an up-type Dirac mass the finite heat trace is 84 + 12 e^{-x}, and the 96 states of H_F split as 84 + 12 in two ways, by the top and by the neutrinos)

With only an up-type Dirac mass the finite heat trace is 84 + 12 e^{-x}, and the 96 states of H_F split as 84 + 12 in two ways, by the top and by the neutrinos

Status. Propositions (Propositions 21.2 and 21.3); the ledger records them as one row of kind claim. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (exact counting) and by Terra 6.1 by hand, effort max, who found one error in Proposition 21.2, corrected in the lemma (historical/reviews/2026-11-03_terra_check_20_21.md). Ledger row `121_counts`. The split 96 = 84 + 12 is stated in the introduction of paper/generations_note.tex ("\supp{21}").

Setting. Three generations with nu_R, so dim H_F = 96 (Chamseddine-Connes-Marcolli). The heat cutoff and the constant-background density of lemma 18, with only an up-type Dirac mass m, no Majorana mass, and x = m^2/Lambda^2. For the counts, four states per Dirac fermion and colour, with Dirac neutrinos.

Statement.

1. (Proposition 21.2) Tr exp(-V_0^2/Lambda^2) = 84 + 12 e^{-x}. So the coefficient of the vacuum density moves from 96 to 84 along the radial direction, with no finite nonzero stationary point on this free radial ray (No-go theorem 9 and Proposition 11 of lemma 18). At m = 0 the derivative is zero.
2. (Proposition 21.3) 96 = 84 + 12 as five quarks and all leptons (60 + 12 + 12) plus the top (12), and 96 = 84 + 12 as all electrically charged fermions (72 + 12) plus the neutrinos (12). The two families of 84 overlap in 72 states, and each is the other with the top and the neutrinos exchanged. Without nu_R the first family has 78 states and the second stays at 84.
3. In eleven dimensions the three-form C_3 has C(9, 3) = 84 on-shell components, and its dual C_6 has C(9, 6) = 84.

Scope. Item 1 concerns the literal flat-space action with no added potential or constraint. Compact support can give a plateau; sign-changing cutoffs, additional terms, constraints and paths with an additional fixed noncommuting mass lie outside it. The top-only family is the endpoint of this descending direction only, since a radial direction giving other species masses descends further. The 84 of H_F is a complex dimension and the count of C_3 a real bosonic polarization count, so an identification needs a map of representations and degrees of freedom.

Proof. Lemma 21, Propositions 21.2 and 21.3; the proofs are there. The top block has 3 colours, 2 chiralities and charge conjugates, so 12 states, each with V_0^2 = m^2; the rest is counting.

What it leaves open. Lemma 21 proposes, with no derivation, relating the first family to C_3 or to an M2 or M5 brane and the second to the dual object, with the exchange of the top and the neutrinos as the content of the duality on the Standard Model side; equality of the number is insufficient for this. What the eleven-dimensional line proves about this exchange is collected in paper/generations_note.tex.

Part S86 (In the exact quadratic spectral action, among single charged species only an up-type quark can be removed while keeping both cos^2 theta_NC = 5/8 and the photon/gluon ratio 8/3)

In the exact quadratic spectral action, among single charged species only an up-type quark can be removed while keeping both cos^2 theta_NC = 5/8 and the photon/gluon ratio 8/3

Status. Proposition (Proposition 21.1). Computed by Claude Opus 5.5 in exact arithmetic; checked by Claude Opus 5.5 in Python (exact fractions, all tables) and by Terra 6.1 by hand, effort max, whose one error in lemma 21 concerned Proposition 21.2 (historical/reviews/2026-11-03_terra_check_20_21.md). Ledger row `l21_removal` (depending on `l18_angles`). Cited in paper/real_structure_note.tex ("\supp{21}, for the classical bosonic trace with three generations") and used in the proof sketch of Proposition 1 of paper/generations_note.tex ("\supp{21}").

Setting. The exact quadratic action of lemma 18, classical bosonic trace, three generations. By equation (33) there, a Dirac species with charges q_L, q_R in the coordinates (T_3, Y) contributes to the neutral kinetic matrix K with weight e^{-x}. Its off-diagonal charge (T_3/2)(1, -1) is orthogonal to the photon direction (1, 1), so cos^2 theta_NC = (K_YY + K_3Y)/(K_33 + 2 K_3Y + K_YY) is fixed by the sums of T_3 Q and Y Q. Removing a species means that its mass tends to infinity at fixed cutoff, with the other species keeping the mass assignments of lemma 18.

Statement.

1. Per species, with colour multiplicity, the sums (sum T_3 Q, sum Y Q) on the particle trace and the (Q^2, colour) coefficients with charge conjugates and both chiralities are (1, 5/3; 16/3, 2) for an up-type quark, (1/2, 1/6; 4/3, 2) for a down-type quark, (1/2, 3/2; 4, 0) for a charged lepton, zero for a neutrino, and (6, 10; 32, 12) for three generations. So the ratios of the whole, 5/3 and 8/3, are those of an up-type quark alone.
2. Removing one species gives (cos^2 theta_NC, photon/gluon) = (5/8, 8/3) for an up-type quark, (59/92, 46/15) for a down-type quark and (17/28, 7/3) for a charged lepton. Only the up-type quark keeps both values, and for it they hold at every x, as in Proposition 6 of lemma 18.
3. Within one generation, the down-type quark and the charged lepton together contribute exactly what the up-type quark contributes, (1, 5/3; 16/3, 2).

Scope. That of lemma 18: classical bosonic trace, these definitions of the coefficients and of the angle, three generations. Other definitions of the weak angle do not stay at 5/8 (Proposition 6 of lemma 18). At the endpoint where the neutral mass vanishes, the angle is read as the limit from finite mass.

Proof. Lemma 21, Proposition 21.1; the proof is there: arithmetic with the Standard Model charges, Y = Q - T_3, and direct addition for item 3.

What it leaves open. The proposition says which species the action singles out at fixed cutoff, in the setting of lemma 18. The readings of lemma 21 built on the removal of the top (Seiberg duality with five flavours, branes, the odd labels) are proposals with no derivation behind them. The extension to blocks of 12 states, where the three neutrinos together also qualify, is Proposition 1 of paper/generations_note.tex, with lemma 33.

Part S87 (The compression of a constant observable to the interval window is the average Q_l(a) = sinc(l ad_{D_F})(pi(a)) of e^{ixD_F} pi(a) e^{-ixD_F} over abs(x) <= l, so compressed observables converge to pi(A_F) at rate l^2 while D_F, J_F and Gamma_F stay exact)

The compression of a constant observable to the interval window is the average Q_l(a) = sinc(l ad_{D_F})(pi(a)) of e^{ixD_F} pi(a) e^{-ixD_F} over abs(x) <= l, so compressed observables converge to pi(A_F) at rate l^2 while D_F, J_F and Gamma_F stay exact

Status. Theorem (Theorem 22.4). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (quadrature against the Schur form, to 2 x 10^-8; the l^2 scaling of the defects on one-generation Standard Model data; the bounds of items 4 and 6 on 30 random samples) and by Terra 6.1 by hand, who supplied the analytic example of discontinuity below (historical/reviews/2026-11-03_terra_check_22.md). Ledger row `122_compression` (depending on `122_window`). Theorem 4 of paper/boundary_condition_note.tex, with the generated algebra cited as "\supp{22}"; item 5 is cited in paper/connes_kk_dictionary_note.tex and paper/real_structure_note.tex ("\supp{22}").

Setting. The parent of 22_parent_real_triple_order_one.md, and Q_l(T) = W* T W for bounded T on H_l. For T(x) a bounded function with values in the even operators on H_F, acting pointwise and identically on both spin components, T also denotes the multiplication operator.

Statement.

1. Q_l(T) = l^-1 int_0^l cos(x ad_{D_F})(T(x)) dx and Q_l(E T) = -l^-1 int_0^l sin(x ad_{D_F})(T(x)) dx.
2. For constant a, Q_l(a) = sinc(l ad_{D_F})(pi(a)) = (2l)^-1 int_{-l}^{l} e^{ixD_F} pi(a) e^{-ixD_F} dx, with matrix elements pi(a)_{ij} sin(l (mu_i - mu_j))/(l (mu_i - mu_j)) in an eigenbasis of D_F.
3. Q_l(A_l) = Q_l(C^infinity([0, l], A_F)), which contains Q_l(a) for every a in A_F.
4. ||Q_l(a) - pi(a)|| <= (l^2/6) |||[D_F, [D_F, pi(a)]]|| <= (2/3) l^2 ||D_F||^2 ||a||.
5. Q_l([D_X, f]) = [D_F, Q_l(f)] for f in A_l.
6. With a' = Q_l(a) and b' = J_F Q_l(b)* J_F^-1, |||[a', b']|| <= (8/3) l^2 ||D_F||^2 ||a|| ||b||, and |||[[D_F, a'], b'] - [[D_F, pi(a)], pi°(b)]|| <= (16/3) l^2 ||D_F||^3 ||a|| ||b||.

Scope. The convergence concerns the maps Q_l on operator systems, in the setting of Connes and van Suijlekom (arXiv:2004.14115, sections 1 and 3). The generated algebra is discontinuous at l = 0: for M = [[0, 0, 1], [0, 0, 1], [1, 1, 0]], D = [[0, M], [M, 0]] and A_F = C^3 acting diagonally, the first two channel projections have [Q_l(P_1), Q_l(P_2)]_{12} = -l^2/3 + O(l^4) (referee). On one-generation Standard Model data a numerical closure (Python, tolerance 10^-7, no certified rank) gives complex dimension 15 for the algebra generated by pi(B) and 30 for Q_l(B) at l = 0.3.

Proof. Lemma 22, Theorem 22.4; the proof is there. With K = Gamma_F D_F, the even part of e^{-xK} T e^{xK} is cos(x ad_{D_F})(T); item 3 holds because a functional vanishing on A_l integrates against a smooth density vanishing on [0, l); item 4 is Taylor's formula with integral remainder.

What it leaves open. The finite order conditions hold in the limit exactly when they hold for the finite data, and these bounds do not control the generated algebra at l > 0. In the family of 22_family_ten_to_nine.md, l ||D_F(l)|| tends to Theta, and only the light block keeps an l^2 estimate.

Part S88 (With a fixed light block M_0 and a heavy block Theta/l, the limit l -> 0 keeps the light masses explicit with couplings vanishing as m l, and sends W and Z to pi/(2l) and the top to Theta/l, with m_t/M_W = 2 Theta/pi < 1 and M_W = M_Z)

With a fixed light block M_0 and a heavy block Theta/l, the limit l -> 0 keeps the light masses explicit with couplings vanishing as m l, and sends W and Z to pi/(2l) and the top to Theta/l, with m_t/M_W = 2 Theta/pi < 1 and M_W = M_Z

Status. Corollary (Corollary 22.9), with Proposition 22.8; the gauge spectrum is conditional on an assumed five-dimensional Yang-Mills action. Proved by Claude Opus 5.5; checked by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_22.md). Ledger row `l22_family` (depending on `l22_window`); the ledger records the corpus claim `rs_F4`, that at fixed g_2 the weak limit v -> infinity keeps m_f != 0 only with y_f ~ 1/v, as covered by this row. Stated as the strict family in the section on the gauge sector of paper/boundary_condition_note.tex; Proposition 22.8 is cited in paper/connes_kk_dictionary_note.tex and paper/real_structure_note.tex.

Setting. (Proposition 22.8) M^4 times the parent, with algebra of functions f(y, x) valued in A_F and f(y, l) in C_F; a flat product, a positive five-dimensional Yang-Mills kinetic form, the trivial background, unrestricted endpoint variation in the allowed directions, and no additional boundary kinetic or mass terms. (Corollary 22.9) D_F(l) with M(l) = M_0 (+) Theta/l, the heavy block on an orthogonal channel compatible with J_F and the grading, 0 < Theta < pi/2, for the top and if wanted for the Majorana block; l -> 0.

Statement.

1. (Proposition 22.8) The gauge one-forms take values in C_F at x = l. The directions in the Lie algebra of C_F have masses n pi/l, n >= 0, and the orthogonal directions (n - 1/2) pi/l, n >= 1. In the Standard Model data with nonzero Yukawas, M_W = M_Z = pi/(2l), and with the unimodularity condition of Chamseddine-Connes-Marcolli the massless modes are the photon and the gluons.
2. X(l) has the light block tan(l |M_0|) V_0* = l M_0* + O(l^3) and a heavy block with singular values tan(Theta): the light couplings vanish as m l at fixed mass m.
3. The heavy channel has mass Theta/l, the broken gauge directions pi/(2l), and the other levels diverge.
4. The massless gauge sector is the physical real gauge representation of C_F, with its kernel and any unimodularity condition. For the projection P_L onto the light block D_L, P_L Q_l(a) P_L = sinc(l ad_{D_L})(P_L pi(a) P_L), within (2/3) l^2 ||D_L||^2 ||pi(a)|| of P_L pi(a) P_L. The limiting light compression need not be a representation, and the full estimate of Theorem 22.4 does not apply, since l ||D_F(l)|| -> Theta.
5. With l = pi/(2 M_W), X_q ~ (pi/2) m_q/M_W, of the form of y_q = sqrt(2) m_q/v -> 0 at fixed mass. Identifying X_q with the four-dimensional Yukawa coupling is a hypothesis.
6. m_t/M_W = 2 Theta/pi < 1, and M_W/M_Z = 1.

Scope. Lowest level, no bulk mass, strict wall algebra. Item 6 does not reach m_t/M_W = sqrt(8/3), which belongs to the interior. The gauge boundary conditions are those of Higgsless models (Csaki, Grojean, Murayama, Pilo and Terning, 2004), and tan(m L) = M_D L is equation (5.6) of Csaki, Grojean, Hubisz, Shirman and Terning (2004); as far as the searches of lemma 22 went, what it adds is the combination with the real structure, the exact window, C_F, the compression formula and the fluctuations.

Proof. Lemma 22, Proposition 22.8 and Corollary 22.9; the proofs are there. The gauge spectrum is that of the interval in the gauge A_5 = 0, Neumann at 0 and Neumann or Dirichlet at l; the family follows from Theorems 22.1 and 22.4.

What it leaves open. Bulk masses for the top and a changed gauge wall condition are the directions beyond this family. In a separate wall-scalar model (lemma 22, numerical comparison in Python) M_W^2/M_Z^2 runs from 5/8 to 1, and in that flat tree-level toy the top and gauge requirements are incompatible. The

scaling family in which the tower opens at fixed energy is 30_window_and_tower_on_interval.md.

Part S89 (Every finite (H_F, D_F, J_F, Gamma_F) of signs (+, +, -) with l ||D_F|| < pi/2 is exactly the lowest spectral window of the pure derivative on an interval of length l, with the wall coupling X = tan(l D_F))

Every finite (H_F, D_F, J_F, Gamma_F) of signs (+, +, -) with l ||D_F|| < pi/2 is exactly the lowest spectral window of the pure derivative on an interval of length l, with the wall coupling X = tan(l D_F)

Status. Theorem (Theorem 22.1), with Corollary 22.2. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (eigenvectors by finite differences, boundary conditions, J and the norm of W, on random complex symmetric M with l ||M|| = 1.3) and by Terra 6.1 by hand, effort max, whose report on lemma 22 found two errors and one gap as first written, all corrected (historical/reviews/2026-11-03_terra_check_22.md). Ledger row `l22_window`. Theorem 1 of paper/boundary_condition_note.tex; stated in paper/connes_kk_dictionary_note.tex and in paper/real_structure_note.tex ("Theorem 22.1", "\supp{22}"); the input of Theorem 1 of paper/joint_limit_note.tex.

Setting. H_F = H_+ (+) H_-, Gamma_F = diag(1, -1), c : H_+ -> H_- antiunitary, J_F(a, b) = (c^-1 b, c a), and D_F = [[0, M*], [M, 0]] with M = c M* c, which is J_F D_F = D_F J_F. By the Takagi factorization M = V |M| with V = c V* c, and X = tan(l |M|) V* is the H_- -> H_+ block of tan(l D_F); conversely every X with c X = X* c^-1 arises from exactly one D_F. On H_l = L^2([0, l]; C^2) (x) (H_+ (+) H_-), with psi = (Q_L, Q_R; U_L, U_R), D_X = E d/dx, (E psi)_L = psi_R and (E psi)_R = -psi_L, on the domain Q_R(0) = 0, U_L(0) = 0, Q_R(l) = X U_R(l), U_L(l) = -X* Q_L(l); Gamma^ and J^ are as in the lemma.

Statement.

1. D_X is self-adjoint with compact resolvent, and J^^2 = 1, J^ D_X = D_X J^, J^ Gamma^ = -Gamma^ J^, Gamma^ D_X = -D_X Gamma^.
2. With R(x) = exp(x Gamma_F D_F), let W send (a, b) to the function with L-components l^{-1/2} R(x)(a, 0) and R-components l^{-1/2} R(x)(0, b). Then W is an isometry, and D_X W = W D_F, J^ W = W J_F, Gamma^ W = W Gamma_F.
3. With theta_k = l sigma_k for the singular values sigma_k of M, the signed spectrum of D_X is (n pi +- theta_k)/l, n in Z, and a zero singular value gives a zero mode of multiplicity two. So ran W is the spectral subspace of |D_X| <= ||D_F||, and the next level is at least (pi - l ||D_F||)/l.
4. (Corollary 22.2) D_X = R (D_0 + E Gamma_F D_F) R*, with D_0 the uncoupled operator, X = 0. After this unitary change of domain, D_F is a constant matrix connection, odd for the grading, with relative endpoint rotation R(l).

Scope. The finite data are inputs, so the existence of the window is close to a reparametrization, X = tan(l D_F). D_X has no mass term: a fermion mass is a momentum k with tan(k l) equal to a singular value of X, and in the finite description that condition is lost. One matrix X encodes the Yukawa and Majorana blocks together, and its entries need not keep the original block pattern.

Proof. Lemma 22, Theorem 22.1 and Corollary 22.2; the proofs are there. R(x)(a, 0) = (C(x) a, -S(x) a) and R(x)(0, b) = (S(x)* b, C'(x) b) give the wall conditions from X = tan(l |M|) V*, and constant unitaries on Q and U reduce X to diag(tan theta_k), which is Example 12.5 in each channel.

What it leaves open. The window identifies D_F, J_F and Gamma_F exactly and does not select A_F; the parent algebra is matrix-valued (22_parent_real_triple_order_one.md). Lemma 24 realizes the window on CP^3 x I with spectral walls (24_spectral_walls_exact_window.md), and the commutative-parent question T0 remains open.

Part S90 (The scalar inner fluctuations of the interval parent are pure gauge under its allowed endpoint gauge group and act on the moving window by the gauge orbit of D_F, which reproduces on that orbit the quadratic term of Chamseddine-Connes-van Suijlekom)

The scalar inner fluctuations of the interval parent are pure gauge under its allowed endpoint gauge group and act on the moving window by the gauge orbit of D_F, which reproduces on that orbit the quadratic term of Chamseddine-Connes-van Suijlekom

Status. Theorem (Theorem 22.5), with Corollary 22.6. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (on the two-point data with J = sigma_1 K, D + A_(1) + A~*(1)* + *A*(2) = U D U* to 4 x 10^-16, and the link of U_w* D U_w) and by Terra 6.1 by hand, who contributed the gauge equivalence under the allowed endpoint gauge group (historical/reviews/2026-11-03_terra_check_22.md). Ledger row `122_transport` (depending on `122_parent`). Theorem 6 of paper/boundary_condition_note.tex, together with Theorem 23.4; Corollary 22.6 is cited in paper/real_structure_note.tex ("\supp{22}, Corollary 22.6").

Setting. The parent of 22_parent_real_triple_order_one.md. Let h be smooth with values in the anti-Hermitian elements of A_F, u' = -h u, u(0) = 1, and w = u(l), a unitary of A_F.

Statement.

1. The self-adjoint one-forms of the parent are exactly the E pi(h).
2. D_X + E pi(h) = pi(u) D_{X_w} pi(u)*, with X_w = pi_+(w)* X pi_-(w), and the window of D_{X_w} carries pi(w)* D_F pi(w).
3. With A = E pi(h), D_X + A + J^ A J^^-1 has the window U_w* D_F U_w, with U_w = pi(w) J_F pi(w) J_F^-1, which commutes with J_F. With u~ = u w*, which lies in A_l since u~(l) = 1, and U~ = pi(u~) J_F pi(u~) J_F^-1, one has D_X + A + J^ A J^^-1 = U~ D_X U~*: every scalar connection in this model is gauge equivalent to the original parent under the allowed endpoint gauge group.
4. (Corollary 22.6) With v = w* and U = v J v J^-1, U D U* = D + A_(1) + A~*(1)* + *A*(2) (arXiv:1304.7583, section III, equations (10)-(11)). So on the gauge orbit the fluctuation of the parent, which has order one and no quadratic term, gives on the moving window exactly the fluctuated operator of that framework, quadratic term included, for finite data that fail order one. For the two-point data with J = sigma_1 K and w = diag(e^{i alpha}, e^{i beta}) the upper link is m e^{-2i(alpha - beta)}, the m e^{2 i psi} of Corollary 19.2.
5. The fixed compression is a different object: Q_l(E pi(h)) = -[D_F, l^-1 int_0^l x pi(h) dx] + O(l^3 ||D_F||^3 ||h||), which is -(l/2) [D_F, h] for constant h, while the first derivative of the transported operator is -l [D_F, h]. The difference is the contribution of the moving spectral projection.

Scope. Scalar fluctuations from the parent algebra A_l, with the moving window. This answers the first open point of lemma 19 on these gauge orbits. Theorem 22.5 and the covariant projection of lemma 14 share a transport principle.

Proof. Lemma 22, Theorem 22.5 and Corollary 22.6; the proofs are there. pi(u)* D pi(u) = D + E pi(u* u') cancels the connection, and at x = l the substitution psi = pi(u) phi gives the conditions with X_w.

What it leaves open. Since these fluctuations stay on the orbit, the singular values are fixed (22_nogo_radial_mode.md); connections odd for the grading change them (23_odd_fluctuations_radial_mode.md).

Part S91 (On one interval with fixed wall coupling, no bulk mass and even scalar inner fluctuations on the moving window, the singular values of the finite Dirac operator are fixed, so this parent has no radial Higgs mode)

On one interval with fixed wall coupling, no bulk mass and even scalar inner fluctuations on the moving window, the singular values of the finite Dirac operator are fixed, so this parent has no radial Higgs mode

Status. No-go theorem in its scope (No-go theorem 22.7). Proved by Claude Opus 5.5, with an independent endpoint-preserving proof by Terra 6.1 (historical/reviews/2026-11-03_terra_check_22.md); checked by Claude Opus 5.5 by hand, by Terra 6.1 by hand, and by Astra 6 by hand, whose recheck is No-go theorem 7 of lemma 24. Ledger row `l22_no_radial` (depending on `l22_transport`). Stated in the section on fluctuations of paper/boundary_condition_note.tex ("\supp{22}"); cited in paper/connes_kk_dictionary_note.tex and paper/real_structure_note.tex ("No-go theorem 22.7 of \supp{22}").

Setting. The parent of Theorem 22.3 on one interval: fixed X, no bulk masses, scalar inner fluctuations E h from the even algebra A_l with h* = -h, and the moving spectral window. For real fluctuations the conjugate one-form is included, under the order-zero representation.

Statement. Every fluctuated finite operator, pi(w)* D_F pi(w) or U_w* D_F U_w in the notation of 22_fluctuations_by_transport.md, has the same singular values as D_F. This is an exact finite-amplitude spectral statement, with no action and no loop expansion. In the usual Standard Model representation with nonzero Yukawa data, the normalized shifted Higgs quaternion runs through the unit quaternions, a three-dimensional gauge orbit, with no radial mode; degenerate data can have a smaller orbit.

Hypotheses. One interval, fixed X, no bulk masses, even scalar fluctuations, the moving window. The window has to be named: a fixed compression can change singular values at second order, as for D_F = m sigma_1 and h = i sigma_3, where Q_l(E h) = -((1 - cos(2 m l))/(2 m l)) sigma_2 (lemma 24, Result 7).

Escapes, as the lemma states them. The coupling X as a field on the wall, with its own kinetic term and potential; connections odd for the grading, as in Corollary 22.2 and lemma 23, which change the singular values; bulk masses that do not commute with the gauge action; two or more internal dimensions, where a connection has curvature, although curvature alone does not prove a radial mode; an additional composite or other degree of freedom; a fluctuation of the finite data alone, as with the radial parameter of lemma 19, which is not the moving window. A quantum effective action on the same configuration space does not by itself add a radial coordinate.

Proof. Lemma 22, No-go theorem 22.7, from Theorem 22.5: D_F and pi(w)* D_F pi(w) are unitarily equivalent, and the same holds with U_w.

What it leaves open. The result points to the wall, to odd directions or to more dimensions for the physical Higgs modulus, and at the 10 -> 9 endpoint it agrees with T10, where the Higgs is frozen. The odd escape is realized in 23_odd_fluctuations_radial_mode.md. The K x I_l parent of lemma 24 has tangential directions and tangential one-forms, so this theorem does not decide its full fluctuation space (lemma 24, Result 7).

Part S92 (On the interval parent a smooth A_F-valued function preserves the domain iff its wall value lies in the commutant C_F of D_F, and the parent is then a real even spectral triple with order zero and order one for any finite data with order zero)

On the interval parent a smooth A_F-valued function preserves the domain iff its wall value lies in the commutant C_F of D_F, and the parent is then a real even spectral triple with order zero and order one for any finite data with order zero

Status. Theorem (Theorem 22.3). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_22.md). Ledger row `l22_parent` (depending on `l22_window`). Theorem 3 of paper/boundary_condition_note.tex; cited in paper/real_structure_note.tex ("\supp{22}, Theorem 22.3").

Setting. The window of 22_finite_dirac_operator_as_window.md, with an even representation pi = pi_+ (+) pi_- of a unital finite-dimensional real or complex *-algebra A_F, faithful, unital and with order zero, and pi°(b) = J_F pi(b)* J_F^-1. Order one is not assumed. Let C_F = {a in A_F : [pi(a), D_F] = 0} and A_l = {f in C^infinity([0, l], A_F) : f(l) in C_F}, with f acting pointwise by pi_+(f(x)) on Q and pi_-(f(x)) on U.

Statement.

1. A function f in C^infinity([0, l], A_F) preserves Dom D_X if and only if f(l) lies in C_F.
2. (A_l, H_l, D_X, J^, Gamma^) is a real even spectral triple, with [D_X, pi(f)] = E pi(f') and J^ pi(g)* J^^-1 = pi°(g) pointwise.
3. It satisfies order zero and order one, with signs (+, +, -).

So A_F acts freely in the bulk, and the boundary condition restricts the observables at the wall to C_F. In the Standard Model data with nonzero Yukawa blocks, C_F is {(lambda, lambda, m)}, the algebra of the photon and the gluons; with vanishing Yukawa blocks the commutant is larger.

Scope. Order one holds in the parent for its pointwise algebra whatever the finite data; the finite condition appears only on the truncation, and arbitrary compressions need not preserve either order condition. This agrees with Counterexample 4 of lemma 20, whose finite data A_F = C + C, D_F = m sigma_1 fail order one while the parent satisfies it.

Proof. Lemma 22, Theorem 22.3; the proof is there. Applying f to the wall conditions gives [pi(f(l)), tan(l D_F)] = 0, equivalent to [pi(f(l)), D_F] = 0 since l ||D_F|| < pi/2 makes D_F a function of tan(l D_F); order one follows from [E pi(f'), pi°(g)] = E [pi(f'), pi°(g)] = 0 by finite order zero.

What it leaves open. The parent algebra is matrix-valued, so T0 for a commutative parent stays open, and the represented algebra is reached through compression, with no exact representation of A_F at l > 0 (22_compression_as_average.md). The parent with a bulk prealgebra and gradings at both walls is 23_aligned_walls_even_part.md.

Part S93 (With a bulk prealgebra P and a grading at each wall, a P-valued function preserves the domain iff it commutes with the grading at each wall, and aligned walls compress P to its even part, with the grading inherited from the parent operator, when Gamma_0 P Gamma_0 = P)

With a bulk prealgebra P and a grading at each wall, a P-valued function preserves the domain iff it commutes with the grading at each wall, and aligned walls compress P to its even part, with the grading inherited from the parent operator, when Gamma_0 P Gamma_0 = P

Status. Theorem (Theorem 23.2). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_23.md). Ledger row `l23_aligned`. Theorem 5 of paper/boundary_condition_note.tex.

Setting. D(Gamma_0, Gamma_1) as in 23_masses_as_principal_angles.md. P is a *-subalgebra of End(H) that need not commute with Gamma_0, and A(Gamma_0, Gamma_1) = {f in C^infinity([0, l], P) : [f(0), Gamma_0] = 0, [f(l), Gamma_1] = 0}, acting pointwise and identically on both spin components. E_0(Z) = (Z + Gamma_0 Z Gamma_0)/2 is the conditional expectation onto the even part.

Statement.

1. A function f in C^infinity([0, l], P) preserves the domain exactly when it lies in A(Gamma_0, Gamma_1). Then [D, f] = E f', and Gamma^ commutes with every f, including those with values odd for Gamma_0.
2. Under item 2 of Theorem 23.1, with W the isometry of Theorem 22.1, Q(T) = l^-1 int_0^l E_0(R(x)* T(x) R(x)) dx.
3. (Aligned walls) If Gamma_1 = Gamma_0, then D_F = 0, the window consists of constant L-values in H_+ and constant R-values in H_-, and Gamma^ restricts on it to Gamma_0. For an ambient constant a in P, Q(a) = E_0(a), and the image of the boundary algebra is E_0(P), by compactly supported profiles. If Gamma_0 P Gamma_0 = P, then E_0(P) is the even part of P, so the compressions of the prealgebra are exactly its even part, with the grading inherited from Gamma^. Without that hypothesis this can fail: for H = C^2, Gamma_0 = sigma_z and P generated by sigma_x + sigma_z, E_0(P) = span{1, sigma_z}, while the even part of P is C.
4. If P° = {J_1 b* J_1^-1 : b in P} commutes with P, with J_1 as in Theorem 23.1, the parent has order zero and order one for A(Gamma_0, Gamma_1).

Scope. Item 3 is the step that Example 16.9 had to add by hand; for M_2(H) + M_4(C) it gives H + H + M_4(C) (paper/boundary_condition_note.tex, after Theorem 5). It evades the inherited-grading obstruction of Result 11 of lemma 20 (16_nogo_grading_from_even_parent.md), because hypothesis 4 there fails: odd a give Q(a) = 0 here, while there the coordinates are retained as Q(x_a) = Gamma_a/5. So, for a matrix-valued parent, the prealgebra is the bulk algebra, the even algebra is what the window sees, and the grading is a boundary condition. By Remark 23.3, J_1 = Gamma_0 J incorporates the grading automorphism on the W-factor of the opposite representation, a structural analogy with the twisted triples of the grand symmetry, whose automorphism is different.

Proof. Lemma 23, Theorem 23.2; as in Theorems 22.3 and 22.4, using that the conditions at a wall put L- and R-values in the eigenspaces of Gamma_i; item 3 is the case R = 1, and item 4 is pointwise commutation.

What it leaves open. The commutative-parent and retained-coordinate questions. For a commutative parent, scalar functions compressed directly give R + R on the S^4 factor, and H + H needs the matrix algebra generated at the intermediate stage (24_two_stage_compression.md).

Part S94 (With P = M_2(H) + M_4(C), a connector at one wall and an assumed five-dimensional Yang-Mills action of one coupling, the lowest W and Z masses are sin^2(M_W l) = sin^2(2 theta)/2 and sin^2(M_Z l) = (4/5) sin^2(2 theta), with constants 2 and 16/5 at small theta and M_W^2/M_Z^2 -> 5/8 at both ends)

With P = M_2(H) + M_4(C), a connector at one wall and an assumed five-dimensional Yang-Mills action of one coupling, the lowest W and Z masses are sin^2(M_W l) = sin^2(2 theta)/2 and sin^2(M_Z l) = (4/5) sin^2(2 theta), with constants 2 and 16/5 at small theta and M_W^2/M_Z^2 -> 5/8 at both ends

Status. Theorem (Theorem 23.6), conditional on the assumed five-dimensional Yang-Mills action and wall conditions (ledger status conditional). Proved by Claude Opus 5.5, with the simpler form of the mass formulas and the proof of the global minimum at theta = pi/4 by Terra 6.1; checked by Claude Opus 5.5 in Python (the boundary problem on Sp(2) x SU(4) solved as a linear system on a grid in k l: the two lowest roots at four angles equal to the formulas to three digits, and the intersection of the stabilizers of dimension 9) and by Terra 6.1 by hand, with a second derivation through the principal angles between the Neumann subspaces (historical/reviews/2026-11-03_terra_check_23.md). Ledger row `123_wz`. Theorem 7 of paper/boundary_condition_note.tex; its small-angle limit is used in paper/joint_limit_note.tex.

Setting. P = M_2(H) (+) M_4(C) acting on E = Hom(V, W) on the left and on the right, with universal fermions of mass theta/l as in Example 23.5. Assumed: a five-dimensional Yang-Mills field valued in sp(2) (+) su(4) with the trace form Tr_W + Tr_V, so one coupling (g_3 = g_2 = sqrt(5/3) g_Y); at x = 0, Neumann for the stabilizer of p_0 and Dirichlet for its complement; at x = l, Neumann for the stabilizer of p_l and of a connector, the a with [a_W, Gamma_p] = 0, a_W w = lambda w and a_V v = lambda v for a line w in the +1 eigenspace of Gamma_p and a line v in V (Results 2 and 5 of lemma 20), and Dirichlet for the complement. The connector is prescribed gauge-breaking wall data.

Statement.

1. The modes massless for all theta are exactly nine, the photon and the eight gluons, with three additional electroweak zero modes at theta = 0 and theta = pi/2. The photon has 1/e^2 = (8/3)/g^2.
2. sin^2(M_W l) = sin^2(2 theta)/2 and sin^2(M_Z l) = (4/5) sin^2(2 theta); equivalently tan^2(M_W l) = sin^2(2 theta)/(1 + cos^2(2 theta)) and tan^2(M_Z l) = 4 sin^2(2 theta)/(1 + 4 cos^2(2 theta)).
3. As theta -> 0, with m = theta/l, M_W^2 -> 2 m^2 and M_Z^2 -> (16/5) m^2. As theta -> pi/2, M_W, M_Z -> 0 while m -> pi/(2l). In both limits M_W^2/M_Z^2 -> 5/8.
4. At theta = pi/4, M_W l = pi/4, tan(M_Z l) = 2 and M_W^2/M_Z^2 = ((pi/4)/arctan 2)^2 = 0.5032…, the global minimum, so the ratio stays in [0.5032, 0.625).
5. m/M_W runs from 1/sqrt(2) at theta -> 0 to infinity at theta -> pi/2, and equals sqrt(8/3) at theta = 1.0589.

Scope. The constants 2 and 16/5 are the Hilbert-Schmidt quotients ||[D_F, a]||^2/||a||^2 for universal mass, and the coefficients of the lower bounds for the poles of the fully massive family (18_fully_massive_family.md), whose small-mass pole coefficients are 3 and 24/5. The spectral-action Taylor mass for universal m is M_W^2 = 3 m^2, larger by the factor 3/2 of Proposition F of paper/towerless_intervals.tex, since this theorem assumes the Yang-Mills commutator norm alone. If the connector is instead a term in the fermion boundary operator, the parent order conditions are kept and the affected fermion tower changes, so the universal fermion formula must be recalculated for that channel. The limit theta -> pi/2 is a geometric form of heavy fermions with light vectors: at antipodal points both walls have the same stabilizer in sp(2).

Proof. Lemma 23, Theorem 23.6; the proof is there. In the gauge A_5 = 0 the generators split by the frequency of Ad F, 0 on S and 2 on the plane (A, O); the conditions at l in the frame rotated by Ad exp(theta

F) give the charged and neutral equations, and item 4 holds because [arcsin(a/sqrt 2)/arcsin(2a/sqrt 5)]^2 decreases with a = sin(2 theta).

What it leaves open. Whether the spectral action of the parent itself, with its tower, reproduces these formulas (lemma 23, open question 4); lemma 25 relates the two normalizations at leading order (25_sum_rule_effective_dimension.md). In the flat toy this family covers M_W^2/M_Z^2 in [0.5032, 0.625) and does not fit the measured top and weak ratios together (lemma 23, comparison in Python); the directions are bulk masses, warping, running, and combinations with a wall scalar.

Part S95 (A self-adjoint boundary condition for the derivative on an interval is a grading at each wall, and the finite Dirac operator is D_F = (1/2l) Gamma_0 log(Gamma_1 Gamma_0), with singular values the principal angles between the two gradings divided by l)

A self-adjoint boundary condition for the derivative on an interval is a grading at each wall, and the finite Dirac operator is D_F = (1/2l) Gamma_0 log(Gamma_1 Gamma_0), with singular values the principal angles between the two gradings divided by l

Status. Theorem (Theorem 23.1). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (random complex symmetric M on C^4 with l ||M|| = 1.25: the logarithm formula to 3 x 10^-15, and the principal angles over l against the singular values) and by Terra 6.1 by hand, effort max, whose report on lemma 23 found four errors and one gap as first written, all corrected (historical/reviews/2026-11-03_terra_check_23.md). Ledger row `l23_gradings` (depending on `l22_window`). Theorem 2 of paper/boundary_condition_note.tex; cited in paper/connes_kk_dictionary_note.tex.

Setting. H finite-dimensional, Gamma_0 and Gamma_1 self-adjoint involutions on H. On L^2([0, l]; C^2) (x) H, D = E d/dx with (E psi)_L = psi_R, (E psi)_R = -psi_L, on the domain Gamma_0 psi_L(0) = psi_L(0), Gamma_0 psi_R(0) = -psi_R(0), Gamma_1 psi_L(l) = psi_L(l), Gamma_1 psi_R(l) = -psi_R(l). Call it D(Gamma_0, Gamma_1), and let Gamma^ be +1 on psi_L and -1 on psi_R.

Statement.

1. D(Gamma_0, Gamma_1) is self-adjoint with compact resolvent, and it anticommutes with Gamma^.
2. Suppose the eigenspaces of Gamma_0 and Gamma_1 have equal dimensions and -1 is not an eigenvalue of Gamma_1 Gamma_0, and let theta_k in [0, pi/2) be the principal angles between the +1 eigenspaces. Then K = (1/2l) log(Gamma_1 Gamma_0), with the principal branch, is anti-Hermitian with Gamma_0 K Gamma_0 = -K; D_F = Gamma_0 K is self-adjoint, odd for Gamma_0, with singular values theta_k/l; and Gamma_1 = e^{lK} Gamma_0 e^{-lK}.
3. With X the H_- -> H_+ block of tan(l D_F), D(Gamma_0, Gamma_1) = D_X of lemma 22, so the spectral part of Theorem 22.1 gives the spectrum and the window, with no real structure needed. Conversely, every D_F odd for Gamma_0 with l ||D_F|| < pi/2 arises from exactly one Gamma_1.
4. An antiunitary J_1 with J_1^2 = -1 and J_1 Gamma_i J_1^-1 = -Gamma_i, i = 0, 1, defines by (J^ psi)_L = J_1 psi_R, (J^ psi)_R = -J_1 psi_L a real structure of signs (+, +, -). With J_F = Gamma_0 J_1 up to sign, J_F^2 = 1, J_F Gamma_0 = -Gamma_0 J_F and J_F D_F = D_F J_F, the signs of lemma 22, while J_F Gamma_1 J_F^-1 = -Gamma_0 Gamma_1 Gamma_0, the reflection of Gamma_1 through Gamma_0.

Scope. At theta_k = pi/2 the channel has type (c) of lemma 09, with no zero mode and lowest level pi/(2l), the Connes-Lott mass of Theorem A of paper/towerless_intervals.tex; there the principal logarithm and the isolated window cease to be unique. Everything finite is still an input.

Proof. Lemma 23, Theorem 23.1; the proof is there. U = Gamma_1 Gamma_0 is unitary with Gamma_0 U Gamma_0 = U^-1, its eigenvalues are e^{+-2 i theta_k} by the two-projection decomposition, and R(l) = e^{lK} carries H_+ and H_- to the graphs of -X* and X, which are the eigenspaces of Gamma_1.

What it leaves open. The parent algebra is matrix-valued (23_aligned_walls_even_part.md). The relative position of the two gradings becomes the scalar modulus in 23_odd_fluctuations_radial_mode.md, and on the unfolded circle D_F is a Wilson line (25_unfolding_to_circle.md).

Part S96 (In the flat massless family with this gauge content and frequency-one fermions, the one-loop potential has its unique global minimum at theta = pi/2 for every n_f > 0 and decreases strictly for n_f >= 10.8, with no interior minimum on the grids computed below that)

In the flat massless family with this gauge content and frequency-one fermions, the one-loop potential has its unique global minimum at theta = pi/2 for every n_f > 0 and decreases strictly for n_f >= 10.8, with no interior minimum on the grids computed below that

Status. No-go theorem in its scope (Proposition 23.8; the ledger records it as kind no_go). The analytic statements are by Terra 6.1, the referee of lemma 23 (historical/reviews/2026-11-03_terra_check_23.md); the potential and the scans are by Claude Opus 5.5. Checked by Claude Opus 5.5 in Python (grids and adaptive quadrature, with I = (3/2) zeta(3) also checked by the referee) and by Astra 6 by hand, whose Theorem 9 of lemma 24 proves the range left to numerics here. Ledger row `123_potential`. Cited, with lemma 24, in the section on the gauge sector of paper/boundary_condition_note.tex ("\supp{23} and \supp{24}").

Setting. From the exact spectra, the products over the towers relative to theta = 0, in Euclidean momentum x = p l, are 1 + sin^2(theta)/sinh^2(x) for a fermion channel, 1 + sin^2(2 theta)/(2 sinh^2 x) for the charged vectors and 1 + (4/5) sin^2(2 theta)/sinh^2 x for the neutral vector. With the usual one-loop formula, three polarizations per vector, n_f twice the number of physical Dirac species and s = sin(2 theta),

```
8 pi^2 l^4 V(theta) = int_0^infinity x^3 dx [3 log(1 + s^2/(2 sinh^2 x)) + (3/2) log(1 + (4/5)
↪  s^2/sinh^2 x) - n_f log(1 + sin^2(theta)/sinh^2 x)].
```

Statement.

1. For every n_f > 0, theta = pi/2 is the unique global minimum on [0, pi/2]: the gauge logarithms are nonnegative and vanish there, and the fermion term is minimized at sin^2(theta) = 1.
2. For n_f >= 10.8, V is strictly decreasing on the interior. With t = sin^2(theta) and J(z) = int_0^infinity x^3 dx/(sinh^2 x + z), dV/dt is, up to a positive factor, (1 - 2t)[6 J(2t(1 - t)) + (24/5) J((16/5) t(1 - t))] - n_f J(t).
3. For n_f in {1, 2, 3, 4, 6, 8, 10}, on grids of theta (Python), V has one interior maximum, at about 0.33 for n_f = 8, and no interior minimum.

Hypotheses. One interval, no bulk masses, fermions of frequency 1 under F, this gauge content, one loop, this condition at the connector wall. For bulk masses with c_Q + c_U != 0 the potential has angle-dependent ultraviolet divergences, and an earlier scan of general pairs is withdrawn; for c_U = -c_Q = c, with the flat gauge products kept as a separate tree-level assumption, a scan (Python) over c in {0, 1/2, 1, 3/2, 2, 3, 5} and n_f in {1, 3, 6, 8, 10, 16, 48} on 721 angles finds no interior minimum.

Escapes, as the lemma states them. A negative coefficient of sin^2(2 theta), as from fermions of frequency 2 under F, which the bimodule E = Hom(V, W) does not contain; general bulk masses with a gauge-invariant completion, or other gauge form factors, as with warping; a wall scalar with its own potential; tree-level terms at the walls. The leading flat form has the angular form alpha cos(h/f_pi) - beta sin^2(h/f_pi) of Agashe, Contino and Pomarol with spinorial SO(5) fermions, whose form factors can give an interior minimum, so these flat results do not exclude that construction.

Proof. Lemma 23, Proposition 23.8; the proof is there: for t <= 1/2 both arguments in the bracket are at least t and J decreases, and for t > 1/2 both contributions have the required sign.

What it leaves open. The result points away from a pure flat Wilson line, toward changed form factors or a wall scalar with its own potential. The range 0 < n_f < 10.8 is proved in 24_nogo_massless_one_loop_minimum.md, and frequency-2 channels give an interior minimum in 26_interior_minimum_frequency_two.md.

Part S97 (A scalar connection along the interval moves the second wall grading along its unitary orbit, and its odd part changes the principal angles, which supplies a radial mode; for M_2(H) the field space is S^4 and the Higgs modulus is half the geodesic distance between the points chosen by the two walls)

A scalar connection along the interval moves the second wall grading along its unitary orbit, and its odd part changes the principal angles, which supplies a radial mode; for M_2(H) the field space is S^4 and the Higgs modulus is half the geodesic distance between the points chosen by the two walls

Status. Theorem (Theorem 23.4), with Example 23.5. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (the universal connection on E (+) E*: J D = D J, Gamma_0 D = -D Gamma_0, all eigenvalues of modulus theta/l, J_1^2 = -1 and J_1 Gamma_1 = -Gamma_1 J_1, while J alone fails) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_23.md). Ledger row `123_radial`. Part of Theorem 6 of paper/boundary_condition_note.tex, with the odd escape and S^4 stated after it; cited in paper/connes_kk_dictionary_note.tex ("\supp{23}") and, as the odd escape, in paper/real_structure_note.tex.

Setting. D(Gamma_0, Gamma_1) with the prealgebra P of 23_aligned_walls_even_part.md; h smooth with anti-Hermitian values in P, u' = -h u, u(0) = 1, w = u(l). For Example 23.5: W = H^2 = C^4 with the Clifford generators Gamma_a of lemma 16 and Gamma_p = p_a Gamma_a for unit p in R^5; H = Hom(V, W) (+) Hom(W, V), J(x, y) = (y*, x*), Gamma_0(x, y) = (gamma x, -y gamma) with gamma = Gamma_{p_0}, and K(x, y) = (theta/l)(F x, y F), with F an odd anti-Hermitian element of M_2(H), F^2 = -1.

Statement.

1. D(Gamma_0, Gamma_1) + E h = u D(Gamma_0, w* Gamma_1 w) u*. An even connection gives w commuting with Gamma_0 and keeps the principal angles; if h has an odd part, the angles between Gamma_0 and w* Gamma_1 w change in general.
2. With H_i = Stab(Gamma_i) in the connected unitary group G of P, the scalar moduli of the parent are the double cosets H_1 G / H_0, the relative position of the two gradings, with the moving window.
3. (Example 23.5) The gradings of W with one-dimensional quaternionic +1 and -1 eigenspaces are the Gamma_p, forming S^4 = Sp(2)/(Sp(1) x Sp(1)); for two points at geodesic distance d all principal angles equal theta = d/2.
4. D_F = Gamma_0 K has all singular values theta/l, a universal Dirac mass for all colours. The second grading is Gamma_1(x, y) = (Gamma_p x, -y Gamma_{p'}), with p at distance 2 theta from p_0 and p' its mirror image through p_0, and J_1 = Gamma_0 J satisfies item 4 of Theorem 23.1, while J alone does not.
5. The coordinates (theta, q) on S^4, with F determined by the unit quaternion q, are polar coordinates of a Higgs quaternion: theta is radial, bounded by pi/2, and q gives three angular directions, undefined at the poles.

Scope. This lies outside No-go theorem 22.7, whose hypotheses include the even algebra. At theta = pi/2 the finite logarithm and the lowest window are degenerate. A canonically normalized field needs the scalar

kinetic term, which lemma 23 does not compute. The field space is the coset of the minimal composite Higgs model (Agashe, Contino and Pomarol, 2005), whose warped interval and additional fermion and boundary data do not identify the actions or potentials. The appearance of the S^4 of the twistor window is an observation, with no identification between the two roles of S^4.

Proof. Lemma 23, Theorem 23.4 and Example 23.5; the proofs are there. u* D u = D + E u* u' and u* E h u = E u* h u cancel the connection, and at l the substitution psi = u phi gives the grading w* Gamma_1 w. In the example Gamma_q Gamma_p = p . q + (p wedge q), with eigenvalues e^{+-i d}.

What it leaves open. The scalar kinetic normalization, and the relation between the two roles of S^4 (lemma 23, open questions 2 and 6). With bulk masses the light masses are about (tan(theta)/l) Phi_Q Phi_U by Theorem 12.6, before flavour data. Whether a potential selects an interior theta is treated in 23_nogo_one_loop_interior_minimum.md and in lemma 26.

Part S98 (For the heat cutoff and a universal angle theta, the spectral action of an interval channel is a Jacobi theta function, strictly decreasing on [0, pi/2] and stationary at theta = pi/2, the Connes-Lott value m = pi/(2l))

For the heat cutoff and a universal angle theta, the spectral action of an interval channel is a Jacobi theta function, strictly decreasing on [0, pi/2] and stationary at theta = pi/2, the Connes-Lott value m = pi/(2l)

Status. Proposition (Proposition 23.7). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (Poisson summation and the triple product to 10^-16, monotonicity on grids for s = 0.3, 1, 2, and the towerless value at s = 0.3 to five digits) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_23.md). Ledger row `l23_theta`. Proposition 8 of paper/boundary_condition_note.tex.

Setting. The channels of Example 23.5 with universal theta, whose eigenvalues are +-(n pi + theta)/l by Theorem 22.1; the heat cutoff; s = l Lambda and x = theta^2/s^2.

Statement.

1. Tr_channel exp(-D^2/Lambda^2) = 2 sum_{n in Z} exp(-(n pi + theta)^2/s^2) = (2 s/sqrt(pi)) theta_3(theta, exp(-s^2)).
2. It is strictly decreasing on [0, pi/2], with its maximum at theta = 0 and its minimum at theta = pi/2, where it is stationary, and it is symmetric under theta -> pi - theta.
3. In the towerless regime s -> 0 at fixed x it tends to 2 e^{-x}, the form of the vacuum density in lemma 18 and Proposition 21.2.

Scope. This family lies outside the hypotheses of No-go theorem 9 of lemma 18 (18_nogo_radial_vacuum.md), which concerns finite internal data and a free ray V = c V_*: here the radial direction is a compact orbit of relative gradings, and the tower, which the finite description omits, supplies a second branch with a stationary point at theta = pi/2. There m = pi/(2l) and M_W = M_Z = 0 by Theorem 23.6, with a degenerate lowest eigenspace, so the minimum of the literal action is a vacuum with massless vectors and is not the physical point.

Proof. Lemma 23, Proposition 23.7; the proof is there. Poisson summation and the Jacobi triple product write the sum as a constant times the product over n >= 1 of (1 + 2 q^{2n-1} cos(2 theta) + q^{4n-2}), q = exp(-s^2), and each factor is positive and strictly decreasing in theta on [0, pi/2].

What it leaves open. The sign of the term agrees with the fermion loop of the Hosotani mechanism, where gauge-boson loops enter with the opposite sign, so an interior minimum could come from that competition, which the classical spectral action does not contain. The one-loop potential is 23_nogo_one_loop_interior_minimum.md and 24_nogo_massless_one_loop_minimum.md.

Part S99 (Local wall gradings satisfying a principal-symbol condition give a self-adjoint elliptic realization preserved by every smooth scalar function, while the spectral walls are preserved only by scalar functions constant on CP^3 at each wall)

Local wall gradings satisfying a principal-symbol condition give a self-adjoint elliptic realization preserved by every smooth scalar function, while the spectral walls are preserved only by scalar functions constant on CP^3 at each wall

Status. Proposition (Result 2 of lemma 24). Proved by Astra 6 (gpt-6-astra, effort xhigh); checked by Terra 6.1 by hand, with an amendment on the ellipticity criterion (historical/reviews/2026-11-03_terra_check_24.md). Ledger row `l24_walls`, shared with Result 3. Stated in the Status section of paper/boundary_condition_note.tex ("\supp{24}").

Setting. The product differential expression D_r = E d_x + Z A/r of 24_spectral_walls_exact_window.md. For the local option, smooth self-adjoint bundle involutions G_0 at x = 0 and G_l at x = l, with G_i psi_L = psi_L and G_i psi_R = -psi_R at the wall, and [G_i, gamma_K] = 0, [G_i, sigma_A(y, xi)] = 0 for the tangential principal symbol sigma_A. For the spectral option, the conditions of Result 1.

Statement.

1. The local conditions give a self-adjoint elliptic realization with compact resolvent, and every smooth scalar function on K x I_l preserves its domain, with [D_r, f] = E d_x f + Z [A, f]/r bounded for each r > 0.
2. For the spectral option, the maximal smooth scalar *-algebra preserving the domain is A_APS = {f in C^infinity(K x I_l) : f(., 0) and f(., l) are constant on K}. The same holds for the subalgebra pulled back from S^4 x I_l.

Scope. The symbol condition is the sufficient criterion of Bär-Ballmann, Corollary 4.18 and Theorem 4.11 (referee). On the unaugmented Dolbeault bundle the Clifford representation is irreducible, so a fibre endomorphism commuting with every sigma_A is scalar, and a smooth involution of this form is +1 or -1; additional fibres and other local elliptic projectors lie outside this criterion (referee). An arbitrary bundle grading gives a pointwise isotropic condition, and self-adjointness on an H^1 domain with compact resolvent still needs an ellipticity check. A grading prescribed only on V leaves the boundary problem incomplete until a condition on V^perp is supplied.

Proof. Lemma 24, Result 2; the proof is there. For spectral walls, preservation means [M_f, chi_i] = 0, so [M_f, C_i] = 0 and [M_f, sgn A] = 0, hence [M_f, Pi] = 0. Then multiplication by f_i preserves H^0(K, O(1)); its sections have no common zero, so dbar f_i = 0, and f_i is constant on compact connected CP^3.

What it leaves open. The local option keeps all scalar functions, and by Result 3 its grading cannot compress to a nontrivial grading on the unchanged level-one window (24_local_grading_trivial_on_window.md). For a fixed smooth function with constant wall value, localization just inside a spectral wall compresses to that constant (lemma 28, Theorem 28.15), so recovering boundary functions needs a local wall.

Part S100 (On the unchanged level-one window of Example 16.9, the compression of a local multiplication grading G is a grading exactly when G commutes with the projection onto the window, and it is then +1 or -1)

On the unchanged level-one window of Example 16.9, the compression of a local multiplication grading G is a grading exactly when G commutes with the projection onto the window, and it is then +1 or -1

Status. Proposition (Result 3 of lemma 24). Proved by Astra 6 (gpt-6-astra, effort xhigh); checked by Claude Opus 5.5 in Python (tr(g^2) = 16/5 for the auxiliary field of item 3) and by Terra 6.1 by hand, with an amendment on marked points (historical/reviews/2026-11-03_terra_check_24.md). Ledger row `124_walls`, shared with Result 2. No paper cites it separately.

Setting. G is a self-adjoint bundle involution acting by multiplication, g = Pi G Pi restricted to V, and Q_K(f) = Pi M_f Pi restricted to V, with the level-one space V = C^4 and the scalar representation of Example 16.9 unchanged.

Statement.

1. 1_V - g^2 = ((1 - Pi) G Pi)* ((1 - Pi) G Pi), so g is a grading exactly when [G, Pi] = 0.
2. In that case [g, Q_K(f)] = 0 for every scalar f, and for the full CP^3 scalar algebra or for the S^4 pullback algebra this forces g = 1_V or g = -1_V. The unchanged four-section window then has trivial grading, and its generated algebra remains M_2(H) or M_4(C), according to the scalar algebra chosen.
3. An explicit local field fails reduction. On an added trivial C^4 bundle, G(y) = sum_a y_a Gamma_a^aux satisfies the symbol condition of Result 2, the tangential kernel has dimension 16, and g = (1/5) sum_a Gamma_a^section (x) Gamma_a^aux has tr(g^2) = 16/5, against 16 for a grading; moreover ||(1 - Pi) G Pi|| >= 2/sqrt(5), independently of r.

Scope. The restriction concerns the unchanged level-one space, a local multiplication grading, and its compression as an actual involution. Item 3 bounds boundary-trace mixing and does not identify the low spectral window of the coupled local problem. The natural field 2 Pi_y - 1 of the tautological line bundle marks no point, while the finite grading Gamma_p = 5 Q_K(p . y) uses a separately chosen point; a local field can encode a chosen point, as G_p(y) = p_a Gamma_a^aux does (referee).

Escapes, as the lemma states them. A different coupled window, additional fibres, or spectral boundary conditions.

Proof. Lemma 24, Result 3; the proof is there. Expanding Pi G^2 Pi = Pi with 1 = Pi + (1 - Pi) gives item 1. The compressed CP^3 functions generate M_4(C) and the compressed real S^4 functions M_2(H), whose complex-linear commutant on C^4 is C 1.

What it leaves open. Lemma 28 takes the first escape: on the disc bundle the window couples two ends, and the end grading is inherited from the chosen self-adjoint doubling (lemma 28, Proposition 28.19), with the quaternionic grading axis still an input.

Part S101 (For the flat massless one-loop determinant of lemma 23, theta = pi/2 is the unique global minimum for every n_f > 0; the potential decreases strictly for n_f >= 54/5, and for 0 < n_f < 54/5 its only interior stationary point is a strict maximum)

For the flat massless one-loop determinant of lemma 23, theta = pi/2 is the unique global minimum for every n_f > 0; the potential decreases strictly for n_f >= 54/5, and for 0 < n_f < 54/5 its only interior stationary point is a strict maximum

Status. No-go theorem in its scope (Result 9 of lemma 24, with Lemma 8; the ledger records it as kind no_go). Proved by Astra 6 (gpt-6-astra, effort xhigh), with no numerical computation in the proof; checked by Terra 6.1 by hand, who gave an independent proof of the key monotonicity without log-concavity or Prékopa's theorem (historical/reviews/2026-11-03_terra_check_24.md), and by Claude Opus 5.5 in Python (e and H on grids; one sign change of H - n_f for n_f in {1, 3, 6, 8, 10}, with the stationary angle 0.34 at n_f = 8). Ledger row `124_no_minimum`. Cited with lemma 23 in the section on the gauge sector of paper/boundary_condition_note.tex ("\supp{23} and \supp{24}"); paper/connes_kk_dictionary_note.tex uses its normalization of the potential ("In the normalization of \supp{24}").

Setting. Exactly the one-loop determinant of 23_nogo_one_loop_interior_minimum.md: a flat interval, zero bulk masses, frequency-one fermions, the stated charged and neutral gauge towers and multiplicities, and the prescribed connector wall condition, with no fermion connector modifying the fermion tower and no wall potential or other angle-dependent term. Put t = sin^2(theta), F(z) = int_0^infinity x^3 log(1 + z/sinh^2 x) dx, J = F', e(z) = -z J'(z)/J(z), and V(t) = 8 pi^2 l^4 V(theta) = 3 F(2t(1 - t)) + (3/2) F((16/5) t(1 - t)) - n_f F(t).

Statement.

1. (Lemma 8) 0 < e(z) < 1, and e is nondecreasing.
2. t = 1, that is theta = pi/2, is the unique global minimum.
3. If n_f >= 54/5, V is strictly decreasing on the interior.
4. If 0 < n_f < 54/5, there is exactly one interior stationary point, and it is a strict maximum.

Hypotheses. Those of the Setting, for theta in [0, pi/2]; at theta = pi/2 the determinant has its continuous value, while the principal-logarithm window is degenerate (referee).

Escapes, as the lemma states them. Frequency-two fermions, modified form factors through warping or other geometry, suitable bulk-mass sectors, wall potentials, and wall kinetic or interaction terms. The full effective action of the K x I_l parent, with its K-tower, has not been computed. By Result 10, opposite bulk masses c_U = -c_Q keep the potential finite with t = 1 as unique global minimum, the absence of other interior local minima at nonzero c staying numerical, while c_Q + c_U != 0 gives angle-dependent ultraviolet divergences that need a completion, a regulator and counterterms.

Proof. Lemma 24, Lemma 8 and Result 9; the proofs are there. Lemma 8 follows from concavity of u -> log J(e^u), by Prékopa's theorem (Theorem 6, p. 342 of the original paper) applied to a jointly log-concave integrand. With R_c(t) = (1 - 2t) J(c t(1 - t))/J(t), Lemma 8 makes R_c strictly decreasing for c >= 2, so H(t) = 6 R_2(t) + (24/5) R_{16/5}(t) decreases strictly from 54/5 to 0 on (0, 1/2), and V'(t) = J(t) [H(t) - n_f] there.

What it leaves open. The result points to the escapes. Pure frequency-two channels give an interior global minimum, with Lemma 8 entering through Lemma 26.1 (26_interior_minimum_frequency_two.md).

Part S102 (On CP^3 x I, nonlocal spectral walls that prescribe two gradings on the level-one kernel C^4 give exactly the two-gradings window of lemma 23, with every other mode above Delta_K/r, so the higher K-modes decouple as the radius shrinks)

On CP^3 x I, nonlocal spectral walls that prescribe two gradings on the level-one kernel C^4 give exactly the two-gradings window of lemma 23, with every other mode above Delta_K/r, so the higher K-modes decouple as the radius shrinks

Status. Theorem (Result 1 of lemma 24, Theorem 24.1 in the ledger), with Result 5, a theorem on the radius collapse. Proved by Astra 6 (gpt-6-astra, effort xhigh) and reproduced as delivered (prompt in historical/prompts/24_commutative_parent.md); checked by Terra 6.1 by hand, effort max, who found no error in the displayed equations and one gap, in the final assertion of Result 5, amended in the lemma (historical/reviews/2026-11-03_terra_check_24.md). Claude Opus 5.5 read Results 1 and 5 without a separate computation. Ledger row `l24_spectral`; Result 5 has no row of its own. Stated in the Status section of paper/boundary_condition_note.tex ("\supp{24}"), and used in the section on the cylinder of paper/joint_limit_note.tex ("the spectral walls of \supp{24}").

Setting. K = CP^3 with the Fubini-Study metric, H_K = L^2(K, Lambda^{0,*} (x) O(1)), A = sqrt(2) (dbar + dbar*), gamma_K = (-1)^deg, V = ker A = C^4 with projection Pi, and Delta_K > 0 the smallest nonzero |eigenvalue| of A; the radius r replaces A by A/r. On L^2(I_l; C^2 (x) H_K), D_r = E d_x + Z A/r and gamma^ = Z gamma_K. Gamma_0, Gamma_1 are self-adjoint involutions on V with two-dimensional eigenspaces and -1 not in spec(Gamma_1 Gamma_0), C_i = Pi Gamma_i Pi, and the walls are chi_0 psi(0) = psi(0), chi_l psi(l) = psi(l), with chi_0 = Z C_0 - S sgn A and chi_l = Z C_1 + S sgn A, sgn A = 0 on V.

Statement.

1. The H^1 realization of D_r is self-adjoint with compact resolvent and anticommutes with gamma^.
2. Pi (x) 1 reduces the operator, including its domain, and on this subspace the operator is exactly D(Gamma_0, Gamma_1) of lemma 23.
3. Every eigenvalue on the complement satisfies |lambda| >= Delta_K/r. With K_F = (1/2l) log(Gamma_1 Gamma_0) and D_F = Gamma_0 K_F, the window embedding W_l gives D_r W_l = W_l D_F and gamma^ W_l = W_l Gamma_0, exact when Delta_K/r > ||D_F||, and the next eigenvalue is at least min{Delta_K/r, (pi - l ||D_F||)/l}.
4. (Result 5) At fixed l, ||(D_r - i)^-1 - iota_Pi (D(Gamma_0, Gamma_1) - i)^-1 iota_Pi*|| <= r/Delta_K. For fixed D_F and Gamma_1(l) = e^{l Gamma_0 D_F} Gamma_0 e^{-l Gamma_0 D_F}, r -> 0 and l -> 0 decouple both towers and leave D_F, with no relation between the two rates.
5. (Result 5, as amended) For an x-independent T = Q_K(f), ||Phi_l(f) - E_0(T)|| <= l ||D_F|| ||T||, and Phi_l(f) = sinc(l ad_{D_F})(T) when T is even. For F(y, s) fixed on K x [0, 1] and f_l(y, x) = F(y, x/l), Phi_l(f_l) tends to E_0 Q_K(int_0^1 F(., s) ds).

Scope. On V^perp these are APS conditions with the inward-normal sign reversed between the walls, and they restrict scalar functions at the walls to constants (24_local_and_spectral_walls.md). The analytic data are those of Example 16.9; an operator on S^4 with the required kernel and compression map would have to be specified separately. Convergence of compression maps needs a common domain and a collapse map: with aligned walls, f_l = 6 (x/l)(1 - x/l) has zero endpoint values and Phi_l(f_l) = 1 (referee). The mixing bound (18) for common-domain perturbations needs, in addition, r (b_r + L) <= (1 - epsilon) Delta_K (referee).

Proof. Lemma 24, Results 1 and 5, with the referee amendments; the proofs are there. With B = S A/r, D_r = E (d_x + B); the chi_i are self-adjoint involutions anticommuting with E, so their positive eigenspaces are maximal isotropic, and integration by parts gives ||D_r q||^2 >= ||q'||^2 + (Delta_K^2/r^2) ||q||^2 on the complement. The resolvent is block diagonal with respect to Pi.

What it leaves open. The window is prescribed through the walls, and the derivation of a nontrivial local grading and of the desired algebra from one coupled scalar window remains open (lemma 24, consequence for T0). The joint scaling with r_eps = r_* eps^{-gamma} is Corollary 30.4 (30_window_and_tower_on_interval.md), and a window from a local operator on the disc bundle is 29_window_case_b.md.

Part S103 (The compression from the scalar cylinder to the finite window is Phi_l(f) = l^-1 int_0^l E_0(R(x)* T_f(x) R(x)) dx, independent of the radius, and with aligned walls it generates R + R on the S^4 factor, against H + H when M_2(H) is generated between the two stages)

The compression from the scalar cylinder to the finite window is Phi_l(f) = l^-1 int_0^l E_0(R(x)* T_f(x) R(x)) dx, independent of the radius, and with aligned walls it generates R + R on the S^4 factor, against H + H when M_2(H) is generated between the two stages

Status. Theorem (Result 4 of lemma 24). Proved by Astra 6 (gpt-6-astra, effort xhigh); checked by Claude Opus 5.5 in Python (E_p(Gamma_a) = p_a Gamma_p to 10^-15) and by Terra 6.1 by hand, with an amendment on the inputs of equation (16) (historical/reviews/2026-11-03_terra_check_24.md). Ledger row `l24_two_stage`. Stated in the Status section of paper/boundary_condition_note.tex ("\supp{24}").

Setting. The exact spectral construction of 24_spectral_walls_exact_window.md. For a scalar multiplier f(y, x), T_f(x) = Q_K(f(., x)) and E_0(T) = (T + Gamma_0 T Gamma_0)/2. Real-valued functions and real algebra generation are used for M_2(H).

Statement.

1. The direct compression is Phi_l(f) = l^-1 int_0^l E_0(R(x)* T_f(x) R(x)) dx, and it does not depend on r.
2. Q_K(C^infinity(S^4, R)) = span_R{1, Gamma_1, ..., Gamma_5}.
3. For aligned walls Gamma_1 = Gamma_0 = Gamma_p, E_p(Gamma_a) = p_a Gamma_p, so direct scalar compression on the S^4 factor has image span_R{1, Gamma_p}, which is the algebra R + R. Generating M_2(H) first and then compressing gives E_p(M_2(H)) = {a : [a, Gamma_p] = 0} = H + H.
4. On the CP^3 factor, real scalar compression is onto the Hermitian 4 x 4 matrices, and a 2 + 2 aligned grading generates M_2(C) + M_2(C). For the two-sheet coarse correspondence of Example 16.9, with Gamma_0(xi, eta) = (Gamma_p xi, -eta Gamma_p), direct aligned scalar compression generates R + R + M_4(C).
5. Endpoint restrictions do not change these finite-dimensional images, by the separation argument of Result 7 of lemma 20.

Scope. Equation (16) has the two Hom spaces and the two scalar algebras of Example 16.9 as inputs, and the endpoint argument proves equality of finite-dimensional linear images without identifying the parent algebras, norms, gauge groups or one-forms (referee). At the intermediate stage spectral walls still give scalar endpoint values, and enlarging them to the stabilizers of Gamma_0 and Gamma_1, as in lemma 23, is another change of the parent algebra.

Proof. Lemma 24, Result 4; the proof is there. Inserting the window embedding on both sides of the multiplication operator gives item 1; all compressed real S^4 functions are quaternionic Hermitian, a space of real dimension six spanned by the displayed matrices; Clifford multiplication gives E_p(Gamma_a) = p_a Gamma_p.

What it leaves open. In the second procedure the noncommuting products that generate the quaternionic

blocks are inserted between the two compressions, and a commutative parent producing H + H directly is open. Result 6 of lemma 24 records that the second sheet, the compatible reality, the quaternionic grading and the connector remain prescribed data. The disc bundle of lemma 28 gives a connected scalar parent whose prescribed windows generate M_2(H) + M_4(C) (28_prescribed_windows_m2h_m4c.md).

Part S104 (For the heat cutoff at leading order in small masses, a vector of constant transported profile on the interval has M^2 = ((3 - d_eff)/2) ||[D_F, T]||^2/||T||^2, with d_eff the spectral dimension of the unfolded circle at the cutoff, so that M_W^2 = ((3 - d_eff)/24) sum N_f m_f^2 and m_t/M_W runs from sqrt(8/3) to 2)

For the heat cutoff at leading order in small masses, a vector of constant transported profile on the interval has M^2 = ((3 - d_eff)/2) ||[D_F, T]||^2/||T||^2, with d_eff the spectral dimension of the unfolded circle at the cutoff, so that M_W^2 = ((3 - d_eff)/24) sum N_f m_f^2 and m_t/M_W runs from sqrt(8/3) to 2

Status. Theorem (Theorem 25.2), with Corollary 25.3. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (the kernel of lemma 18 summed over the tower with explicit 4 x 4 level blocks, independently of the unfolding: C_0/C_1 against 3/2 - <k^2> to six digits for s from 0.3 to 10, the top-only family and the neutral direction) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_25.md). Ledger rows `l25_mass` (depending on `l25_unfold`) and `l25_sum_rule`, with prior work recorded as partial: m_t = 2 M_W in flat gauge-Higgs models with suitable representations (hep-ph/0510366, section 5; arXiv:1804.06012, Eq. (3.23)), and the coefficient (3 - d_eff)/24 and its crossover not found in the records inspected. Theorem 9 of paper/boundary_condition_note.tex; cited in the table of paper/connes_kk_dictionary_note.tex and in paper/joint_limit_note.tex.

Setting. A_mu(y) = a_mu(y) T on R^4, with T even and constant along x in the transported frame, and D = -i gamma^mu (d_mu + i A_mu) + gamma_5 V, V = -i d/dx (x) 1 + 1 (x) D_F, whose eigenvalues are k_n + mu_i (25_unfolding_to_circle.md). The heat cutoff t = 1/Lambda^2, s = l Lambda, S(s) = sum_{n in Z} exp(-n^2 pi^2/s^2), and d_eff = d log S/d log s.

Statement.

1. With d_ij = mu_i - mu_j and I_t as in lemma 18, the kernel over the tower is C_t(q^2) = (1/(8 pi^2)) sum_n sum_ij |T_ij|^2 [(2/t + d_ij^2 + q^2) I_t((k_n + mu_i)^2, (k_n + mu_j)^2; q^2) - (e^{-t (k_n + mu_i)^2} + e^{-t (k_n + mu_j)^2})/t]; the sum converges, with no expansion in D_F.
2. At fixed s and for fixed nonzero T, M^2 = C_0/C_1 = ((3 - d_eff)/2) ||[D_F, T]||^2/||T||^2 + O(||D_F||^4/Lambda^2), in Hilbert-Schmidt norms. d_eff tends to 0 for s « 1 and to 1 for s » 1, and the factor (3 - d_eff)/2 is 1.5000, 1.4990, 1.3935, 1.1414, 1.0240, 1.0022, 1.0000 at s = 0.3, 1, 1.5, 2, 2.5, 3, 5.
3. (Corollary 25.3) For the three-generation Standard Model representation, with the literal particle and antiparticle trace, Dirac masses small compared with Lambda and 1/l, and T a charged generator of SU(2)_L, M_W^2 = ((3 - d_eff)/24) sum_f N_f m_f^2: 8 M_W^2 = sum N_f m_f^2 at d_eff = 0, the CCM relation, and 12 M_W^2 = sum N_f m_f^2 at d_eff = 1. For the top alone m_t/M_W = (8/(3 - d_eff))^{1/2}, from sqrt(8/3) = 1.633 to 2, that is y_t from 2g/sqrt(3) to sqrt(2) g, and M_W^2/M_Z^2 = 5/8 at all s to this order.
4. (Remark) For positive Laplace cutoffs f(z) = int e^{-tz} dnu(t) as in lemma 18, M^2 = ((3 - d_nu)/2) ||[D_F, T]||^2/||T||^2, with the same endpoints and a crossover that depends on nu.

Scope. A Taylor quotient for a prescribed constant-profile vector, at leading order in D_F, about a prescribed background and not a demonstrated vacuum; Taylor and pole masses differ in general, and a lowest physical gauge branch needs an admissible gauge field, a positive kinetic form and an isolated lowest branch. Flat profiles on one interval, no bulk masses, a matching scale and no running; the top-only statement counts all twelve doublets on the same interval, and already in the towerless top-only family M_W^2 = (3/8) m_t^2 - (23/128) m_t^4/Lambda^2. The value at d_eff = 1 is the one five-dimensional covariance requires on the unfolded circle, where A_5 = D_F is constant, and the unfolding removes the boundary heat-kernel terms for this operator and these generators. Vectors that are Dirichlet at a strict wall, as in Proposition 22.8, are a different boundary problem.

Proof. Lemma 25, Theorem 25.2 and Corollary 25.3; the proofs are there. Equation (1) of lemma 18 applies block by block, justified by Gaussian bounds. To second order in mu the bracket at q = 0 is d_ij^2 e^{-k^2} (1 - (2/3) k^2) and at order q^2 it is (2/3) e^{-k^2}; terms odd in k cancel between n and -n, and the sum gives 3/2 - <k^2> with <k^2> = (1/2) d log S/d log s. The corollary uses ||T||^2 proportional to Q_11 = 12 and ||[D_F, T]||^2 to K_11 = sum N_f m_f^2.

What it leaves open. In this constant-profile calculation the factor 3/2 of Proposition F of paper/towerless_intervals.tex is the ratio of the two endpoints. Priority for the coefficient and its crossover is open pending a wider search. The comparison with measured values in lemma 25 (formally d_eff = 1.27, or 1.06 with a running top mass) draws no conclusion, since no running from a matching scale has been computed; with bulk masses for the light fields the kinetic coefficient must be recomputed.

Part S105 (Unfolded, the interval parent is -i d/dx (x) 1 + 1 (x) D_F on a circle of length 2l, so its tower is D_F shifted by the momenta n pi/l, D_F is a constant connection, and an even generator constant in the transported frame becomes 1 (x) T)

Unfolded, the interval parent is -i d/dx (x) 1 + 1 (x) D_F on a circle of length 2l, so its tower is D_F shifted by the momenta n pi/l, D_F is a constant connection, and an even generator constant in the transported frame becomes 1 (x) T

Status. Proposition (Proposition 25.1). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (the level blocks of the interval against the shifted spectrum) and by Terra 6.1 by hand, effort max, whose report on lemma 25 found two errors and two gaps as first written, all corrected (historical/reviews/2026-11-03_terra_check_25.md). Ledger row `l25_unfold` (depending on `l22_window`). Stated before Theorem 9 of paper/boundary_condition_note.tex; the Wilson-line form is cited in paper/connes_kk_dictionary_note.tex ("the periodic circle of \supp{25}").

Setting. D_X = E d/dx is the parent of lemma 22, and D_0 + E Gamma_F D_F its form after the transport of Corollary 22.2, with D_0 the operator at X = 0. On the circle x in [-l, l] with periodic conditions, for psi in the domain of D_0, put Psi_Q(x) = Q_L(|x|) + i sgn(x) Q_R(|x|) and Psi_U(x) = U_R(|x|) - i sgn(x) U_L(|x|).

Statement.

1. After multiplication by 1/sqrt(2), psi -> (Psi_Q, Psi_U) is a unitary from the parent Hilbert space onto L^2(circle) (x) (H_+ (+) H_-); it maps the H^1 domain of D_0 onto the periodic H^1 domain and carries D_0 + E Gamma_F D_F to -i d/dx (x) 1 + 1 (x) D_F.
2. The signed spectrum is k_n + mu_i, with k_n = n pi/l, n in Z, and mu_i the eigenvalues of D_F. This gives a one-line proof of the spectrum in Theorem 22.1.
3. An even operator T = T_Q (+) T_U on H_F, constant in x in the transported frame and acting identically on both spin components, is carried to 1 (x) T.
4. D_F is a constant connection on the circle, with holonomy exp(2 i l D_F) up to the convention for parallel transport, and the two walls are the fixed points of the reflection that carries D_F to -D_F.

With U = exp(-i pi Gamma_0/4), the identity of lemma 23 reads Gamma_1 Gamma_0 = exp(2 l Gamma_0 D_F) = U exp(2 i l D_F) U*.

Scope. This is the usual orbifold description of an interval as a circle with a reflection. A generator constant after transport is R(x) T R(x)* in the original frame of D_X, which generally varies with x and has an odd bulk component; conversely, a constant generator in the original frame need not preserve the coupled wall if it is broken. So the fields used in this frame need an admissible enlarged gauge algebra, as the prealgebra P of lemma 23, and are generally not fields of the even algebra of lemma 22.

Proof. Lemma 25, Proposition 25.1; the proof is there. Since Q_R(0) = Q_R(l) = 0 and U_L(0) = U_L(l) = 0, the functions Psi are continuous and periodic; for x > 0, -i d/dx Psi_Q and -i d/dx Psi_U are the transforms of D_0 psi, and E Gamma_F D_F sends Psi_Q to M* Psi_U and Psi_U to M Psi_Q.

What it leaves open. In this form the exact quadratic kernel of lemma 18 applies termwise over the tower, which is 25_sum_rule_effective_dimension.md.

Part S106 (In the flat family with pure frequency-1 and frequency-2 channels at one loop, an interior global minimum exists for n_2 >= n_c = 2.8855..., at theta = pi/4 with M_KK = 4 M_W when the frequency-2 channels act alone, and in (pi/4, pi/2) when 0 < n_1 < n_* = (4/3)(4 n_2 - 54/5))

In the flat family with pure frequency-1 and frequency-2 channels at one loop, an interior global minimum exists for n_2 >= n_c = 2.8855..., at theta = pi/4 with M_KK = 4 M_W when the frequency-2 channels act alone, and in (pi/4, pi/2) when 0 < n_1 < n_* = (4/3)(4 n_2 - 54/5)

Status. Theorem (Theorem 26.2, with Lemma 26.1), with Proposition 26.2b and Corollary 26.3. Theorem 26.2 and Corollary 26.3 proved by Claude Opus 5.5, Proposition 26.2b by Terra 6.1, the referee. Theorem 26.2 checked by Claude Opus 5.5 in Python (11600 pairs, n_2 from 3 to 60 and n_1 from 0 to 199, on 20001 angles, no failure) and by Terra 6.1 by hand; Corollary 26.3 by Claude Opus 5.5 by hand and by Terra 6.1 in Python (independent roots and table rows); Proposition 26.2b by Claude Opus 5.5 by hand. The first report on lemma 26 found three errors and two gaps as first written, and the second one error in the summary, all corrected (historical/reviews/2026-11-03_terra_check_26.md, historical/reviews/2026-11-03_terra_check_27.md). Ledger rows `l26_classification` (depending on `l24_no_minimum`), `l26_no_small` and `l26_spectrum` (depending on `l26_classification` and `l23_wz`). Cited in the section on the gauge sector of paper/boundary_condition_note.tex ("\supp{26}").

Setting. The flat interval with the gauge content, couplings and connector wall of Theorem 23.6. A fermion channel is pure of frequency q when its two wall gradings differ by a rotation through q theta; its spectrum is +-(n pi + q theta)/l and its tower product 1 + sin^2(q theta)/sinh^2 x. E = Hom(V, W) is pure with q = 1. n_q >= 0 is twice the number of Dirac species in such channels, each colour counted once. With t = sin^2(theta), u = t(1 - t) and F, J as in Results 8 and 9 of lemma 24, assuming precisely these towers and multiplicities,

```
8 pi^2 l^4 V = 3 F(2u) + (3/2) F((16/5) u) - n_1 F(t) - n_2 F(4u).
```

Statement.

1. (Lemma 26.1) R(u) = [6 J(2u) + (24/5) J((16/5) u)]/(4 J(4u)) is strictly increasing on (0, 1/4], with R(0+) = 27/10 and R(1/4) = n_c = 2.8855033… (numerical quadrature).
2. (n_1 = 0) If n_2 <= 27/10, theta = 0 and pi/2 are the global minima, degenerate at n_2 = 27/10, and pi/4 is the maximum. If 27/10 < n_2 < n_c, there are exactly two global minima, at theta_* and pi/2 - theta_*, theta_* in (0, pi/4). If n_2 >= n_c, theta = pi/4 is the unique global minimum.

3. (n_1 > 0, n_2 >= n_c) If n_1 < n_*, there is exactly one stationary point in the open interval, the strict global minimum, in (pi/4, pi/2); if n_1 >= n_*, V decreases strictly and theta = pi/2 is the unique global minimum. The range n_1 > 0, n_2 < n_c is not classified.
4. (Proposition 26.2b) For n_1 > 0 and 0 < t < 1/2, V(1 - t) - V(t) = -n_1 [F(1 - t) - F(t)] < 0, so no point of (0, pi/4) is a global minimum; with item 2 this covers all integer counts.
5. (Corollary 26.3) At theta = pi/4, M_KK = pi/l = 4 M_W, m_1 = M_W, m_2 = 2 M_W, tan(M_Z l) = 2 and M_W^2/M_Z^2 = 0.5032. From pi/4 to pi/2, m_1/M_W increases from 1 to infinity and m_2/M_W decreases strictly from 2 to sqrt(2); M_KK >= 4 M_W for all theta in (0, pi/2).

Scope. One interval, flat profiles, this gauge content, one loop, no bulk masses, no wall terms. With M_W = 80.36 GeV the minimum at pi/4 has M_KK = 321 GeV, and a frequency-1 top with m_1/M_W = 2 needs theta = 1.13774, on the line n_1 = 2.71387 n_2 - 7.71725, with M_KK = 444 GeV (Python, recomputed by the referee). For n_1 > 0 and n_2 = 1, 2 a scan to n_1 = 200 (Python) finds no interior local minimum, which is not proved, and metastable minima at small angle in the cases left to scans are not excluded. A sector with tower product 1 + sin^2(2 theta)/(2 sinh^2 x), counted by n_d >= 27/5, also makes theta = pi/4 the unique global minimum at n_1 = 0.

Proof. Lemma 26, Lemma 26.1, Theorem 26.2, Proposition 26.2b and Corollary 26.3; the proofs are there. For n_1 = 0, dV/du = 4 J(4u) (R(u) - n_2); for n_1 > 0 and t > 1/2, dV/dt = J(t) [Psi(t) - n_1] with Psi increasing strictly from 0 to n_*. Lemma 26.1 uses Lemma 8 of lemma 24 with the strict covariance argument of the referee amendment there.

What it leaves open. Frequency-2 channels live in multiplets beyond E = (4, 4bar), such as the (10, 6) or the (5, 15), whose displayed channels carry light coloured states (26_wall_mass_light_coloured_state.md). The potential, the wall-mass spectrum, the wall kinetic term and the running of lemma 26 use different simplified spectra and have not been combined into one model (lemma 26, Outstanding), so T13 stays open; priority for the classification with n_c is open.

Part S107 (In the step-threshold model of the flat parent, differences of the inverse gauge couplings do not depend on a common bulk-fermion term, and the Sp(2) left-right difference alpha_R^-1 - alpha_2^-1 runs only logarithmically, with no power of the tower variable S)

In the step-threshold model of the flat parent, differences of the inverse gauge couplings do not depend on a common bulk-fermion term, and the Sp(2) left-right difference alpha_R^-1 - alpha_2^-1 runs only logarithmically, with no power of the tower variable S

Status. Propositions (Propositions 26.4 and 26.5). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (exact fractions for the coefficients, and the difference computed from equation (4) against equation (5) to 13 digits at S = 1000) and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_26.md, item 4). Ledger row `l26_lr`, with prior work recorded as partial: Nomura-Smith-Weiner hep-ph/0104041 (differential running) and Carena-Tait-Wagner hep-ph/0207056 (boundary terms). Cited in the section on the gauge sector of paper/boundary_condition_note.tex ("\supp{26}").

Setting. The conditions of Theorem 23.6 at theta = 0, S = mu/M_KK with M_KK = pi/l; a massive vector, with its fifth component and ghosts, contributes -(7/2) tr Q^2 per level. Above M_KK the model is

```
alpha_i^-1(mu) = alpha_i^-1(M_Z) - (b_i/2 pi) log(mu/M_Z) - (1/2 pi) [(B_i + F) I_n(S) + C_i
↪  I_h(S)],        (4)
```

with b = (41/10, -19/6, -7), B = (-21/10, -21/2, -21/2), C = (-49/5, 0, -7/2), I_n(S) the sum of log(S/n) over n <= S and I_h(S) the sum of log(S/(n - 1/2)) over n - 1/2 <= S. F = 8 for three complete massless

bulk generations of E on the integer tower, in the limit theta -> 0, and F = 0 with the fermions on the walls.

Statement.

1. (Proposition 26.4) In (4), the differences of the alpha_i^-1 are independent of F.
2. (Proposition 26.5) With alpha_R^-1 = (5/3) alpha_1^-1 - (2/3) alpha_3^-1, the coupling of SU(2)_R in 1/g_Y^2 = 1/g_R^2 + (2/3)/g_4^2, alpha_R^-1 - alpha_2^-1 = D_0 - (44/3)/(2 pi) log(mu/M_Z) + (7/pi) [I_h(S) - I_n(S)], with no power of S, and D_0 = 63.09 for the inputs (59.0, 29.6, 8.47).
3. For S » 1 the difference vanishes at log(mu_LR/M_Z) = (3/23) [2 pi D_0 + 7 log(pi) - 7 log(M_KK/M_Z)]. At M_KK = 321 GeV it is 61.69 at S = 1, against 60.15 from the Standard Model extrapolation alone, and it vanishes near 2 x 10^24 GeV; it vanishes above the Planck mass for M_KK below about 2 x 10^8 GeV. Without the tower it vanishes at 5 x 10^13 GeV.

Scope. One loop. The cancellation concerns the difference within Sp(2), and differences involving colour keep a power of S. Model (4) omits the frequency-2 multiplets used for the vacuum (a complete five-dimensional Dirac (10, 6) has level contributions (296/15, 24, 40/3), which are not a universal F) and the boundary kinetic counterterms generated by loops (Carena, Tait and Wagner). So (4) is the running of this spectrum, and of the parent only as far as this spectrum is the parent's.

Proof. Lemma 26, Propositions 26.4 and 26.5; the proofs are there. The zero modes give (5/3)(41/10) + (2/3)(7) + 19/6 = 44/3, the integer tower 14 and the half-integer tower -14, and I_h - I_n ~ (1/2) log S + (1/2) log(pi) for large S.

What it leaves open. This is the differential running of orbifold models (Nomura, Smith and Weiner). If only a hypercharge wall term and separate bulk couplings are admitted, a low tower needs a wall term of about 60 in alpha^-1, comparable with twice the bulk value, as an estimate (lemma 26, Two regimes). The meeting point of the three couplings in (4) is 26_meeting_point_numerical.md.

Part S108 (In the one-loop step-threshold model of the flat parent, without frequency-2 multiplets, the three gauge couplings meet at mu_* = 2.21 x 10^14 GeV with M_KK = 3.15 x 10^13 GeV, the only solution with S >= 1)

In the one-loop step-threshold model of the flat parent, without frequency-2 multiplets, the three gauge couplings meet at mu_* = 2.21 x 10^14 GeV with M_KK = 3.15 x 10^13 GeV, the only solution with S >= 1

Status. Numerical result (Result 26.6; Python, one loop), conditional on the spectrum of model (4) and its matching prescription; ledger status numerical. Computed by Claude Opus 5.5 in Python (nested root finding over a scan of 1401 values of M_KK, with input shifts); checked by Terra 6.1 in Python (root finding), with the uniqueness for S >= 1 proved by hand (historical/reviews/2026-11-03_terra_check_26.md, item 5). Ledger row `l26_meeting` (depending on `l26_lr`), with prior work recorded as partial: Abe-Goto-Kawamura, PTEP 2018 103B01, an interval Pati-Salam model near 10^13 GeV with other content. Cited in the section on the gauge sector of paper/boundary_condition_note.tex ("\supp{26}").

Setting. Model (4) of 26_left_right_difference_logarithmic.md, with the inputs of the steer note, no thresholds at the top or the W, and Standard Model running below M_KK.

Statement.

1. The two conditions alpha_1 = alpha_2 = alpha_3 in (4) have the solution M_KK = 3.15 x 10^13 GeV, mu_* = 2.21 x 10^14 GeV, S = 7.04, with alpha^-1 = 46.0 for F = 8 and 52.6 for F = 0.
2. It is the only solution with S >= 1: eliminating log(mu/M_Z) between the left-right equation and the equation for alpha_2 = alpha_3 leaves a continuous function of S whose derivative between thresholds is (315 N_h - 161 N_n)/(44 S) > 0, with N_n, N_h the numbers of levels below S (referee).

3. Shifts of 0.1 in alpha_1^-1 or alpha_2^-1 and of 0.15 in alpha_3^-1, one at a time, move M_KK within 2.9 to 3.4 x 10^13 GeV, and all three together within 2.66 to 3.70 x 10^13 GeV.

Scope. Larger effects are possible: one additional light colour-triplet Weyl fermion changes alpha_3^-1 by about -1.4 up to that scale, and the Standard Model gauge part of the two-loop terms by about (-0.20, -0.27, +0.39) (referee); finite wall terms and split thresholds are not included. So in this model the relation g_3^2 = g_2^2 = (5/3) g_Y^2 holds at one scale with no added thresholds, and that scale fixes the tower.

Proof. Lemma 26, Result 26.6: a numerical root of the two conditions in (4), with the uniqueness argument of item 2 written there.

What it leaves open. At this tower the small-angle relation M_W l = sqrt(2) theta requires theta = 5.7 x 10^-12, and by Proposition 26.2b no such angle is a global minimum of the potential of 26_interior_minimum_frequency_two.md at integer counts; metastable minima in the cases left to scans are not excluded (lemma 26, Two regimes). With the boundary values of Result 26.7, r = 1 gives a running top mass near 172 GeV at this tower, an illustration that predicts no spectrum. Priority for the pair (M_KK, mu_*) is open.

Part S109 (In the displayed charge-2/3 channel of the (10, 6) or the (5, 15), a pure frequency-2 channel carries an exact massless coloured exotic, and a wall mass of any finite positive size leaves a coloured state of charge 2/3 with 0 < m < M_W)

In the displayed charge-2/3 channel of the (10, 6) or the (5, 15), a pure frequency-2 channel carries an exact massless coloured exotic, and a wall mass of any finite positive size leaves a coloured state of charge 2/3 with 0 < m < M_W

Status. Proposition (Proposition 26.3c), with Proposition 26.3b. Proved by Claude Opus 5.5, with the self-adjoint boundary problem and the monotonicity of the root by Terra 6.1, the referee. Proposition 26.3b checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand (historical/reviews/2026-11-03_terra_check_26.md, item 3); Proposition 26.3c by Claude Opus 5.5 in Python (equation (3) against a direct 3 x 3 determinant of the boundary conditions, at theta = 0.3, pi/4, 1.0, 1.1377 and omega from 10^-3 to 10^4) and by Terra 6.1 by hand, with sympy on the determinant (historical/reviews/2026-11-03_terra_check_27.md, item 7). Ledger rows `l26_plane` (depending on `l23_wz`) and `l26_wall_mass` (depending on `l26_plane`), with prior work recorded as partial: Cacciapaglia-Csaki-Park hep-ph/0510366, Adachi-Maru 1804.06012, Hosotani-Oda-Ohnuma-Sakamura 0806.0480. Cited in the section on the gauge sector of paper/boundary_condition_note.tex ("\supp{26}").

Setting. The (10, 6) with F the generator J_45 of so(5); the frequency-2 planes (A^a, O^a), A = (L - R)/sqrt(2), O in (2, 2), S = (L + R)/sqrt(2). O^+ is the left-handed top, with (T_3L, R_3) = (+1/2, +1/2), and its partner A^+ contains the right-handed top R^+ with weight 1/sqrt(2), together with the triplet component L^+. phi = 2 theta, A' = cos(phi) A + sin(phi) O, O' = -sin(phi) A + cos(phi) O. At x = 0, psi_R is Neumann on S, A and Dirichlet on O, and psi_L has the complementary conditions. For the wall mass, u = (S + A')/sqrt(2), v = (S - A')/sqrt(2), and a wall Weyl field chi couples to g_u at x = l: H(f, g, chi) = (-g', f', w g_u(l)) on the domain f_u(l) = w chi, g_{O'}(l) = 0, f_v(l) = 0, with omega = w^2 l, a = m l and s = sin(2 theta).

Statement.

1. (Proposition 26.3b) If at x = l psi_R is Neumann on S and A' and Dirichlet on O', then (O^+, A^+) is a pure frequency-2 channel, and S_R is an exact massless coloured mode of charge 2/3. If instead psi_R is Dirichlet on O' and on S + A' and Neumann on S - A', the determinant is proportional to cos(a)(2 sin^2 a - sin^2 phi), so the angle-dependent tower obeys sin^2(m l) = sin^2(2 theta)/2, the equation of the charged vectors, and m_t = M_W at every theta, with an additional angle-independent

half-integer tower. In the (5, 15) the same holds with the chiralities exchanged, the plane containing U_R and (t_L + X)/sqrt(2), with X the exotic of Y = 7/6.

2. (Proposition 26.3c) The nonzero eigenvalues solve 2 a Delta sin(a) = omega cos(a) (Delta + sin^2 a), Delta = sin^2 a - s^2, where omega = 0 and omega -> infinity give the two cases of item 1. For every theta in (0, pi/2) and every omega in (0, infinity) there is an eigenvalue with 0 < m < M_W.
3. At theta = pi/4 the equation factorizes as cos(a) (omega cos(2a) - a sin(2a)) = 0: m = pi/(2l) = 2 M_W for every omega, and the other factor has a root in (0, pi/4) for every omega > 0, moving from 0 to M_W l = pi/4.
4. The eigenvalue m = 2 M_W occurs at omega = 4 u_W sin(2 u_W)/(1 + 2 cos(2 u_W)), u_W = M_W l; at theta = 1 this is omega = 2.04, with another state at 0.85 M_W (Python, referee).

Scope. These statements concern the displayed channel. A gauge-invariant completion needs conditions on complete wall multiplets, and the other modes of the multiplet also enter the potential; complete wall multiplets, other boundary mixings, bulk masses and the other towers can add still lighter states. By item 4 a higher eigenstate, with nonzero O_L and R_R components, can sit at 2 M_W, so identifying it with the top is not excluded; its physical identification and the couplings of the lighter state need analysis. E = (4, 4bar) has no frequency-2 channel and no such partner.

Proof. Lemma 26, Propositions 26.3b and 26.3c; the proofs are there. With f' = m g and g' = -m f, the conditions at 0 fix the profiles, and the three conditions at l give equation (3) as a determinant, used before any division so that it covers cos(2 theta) = 0. On 0 < a < u_W, omega = 2 a tan(a) (s^2 - sin^2 a)/(s^2 - 2 sin^2 a) increases strictly from 0 to infinity (referee).

What it leaves open. A top at 2 M_W in such a channel has a lighter partner whose couplings are not analyzed. Warping and larger representations are the standard directions (lemma 26, Two regimes), and the comparison is with the top-mass limits for low representations in flat gauge-Higgs models.

Part S110 (As Lambda -> infinity on the reduced action, the axial anomaly of a traceless flavour generator has only the electromagnetic term, with coefficient 2 N_c tr(T^3 Q^2) = 1 for two flavours, while at fixed cutoff a heavy top weights it as (4 e^{-x} - 1)/3 for the heat cutoff)

As Lambda -> infinity on the reduced action, the axial anomaly of a traceless flavour generator has only the electromagnetic term, with coefficient 2 N_c tr(T^3 Q^2) = 1 for two flavours, while at fixed cutoff a heavy top weights it as (4 e^{-x} - 1)/3 for the heat cutoff

Status. Corollary (Corollary 27.2), with Proposition 27.4. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (exact fractions for 2 N_c tr(T^3 Q^2) = 1 and tr T^3 = 0; the third-generation coefficient for the heat cutoff at x = 0, 0.1, 1, 5, 50, which is 1.0000, 0.8731, 0.1572, -0.3243, -0.3333) and by Terra 6.1 by hand, who corrected an error of interpretation in Proposition 27.4 (historical/reviews/2026-11-03_terra_check_27.md, items 2 and 4). Ledger row `127_anomaly` (depending on `127_trace`). Proposition 5 of paper/chiral_limit_note.tex, together with Theorem 27.1.

Setting. As in 27_regulated_axial_trace.md, with f continuous at 0 and Lambda -> infinity at fixed masses, the flavours in the same colour bundle. The normalization f(0) = 1 is the usual convention for a regulator, and the value that makes the Jacobian the anomaly. For Proposition 27.4: the doublet (t, b) with T = T^3, m_b/Lambda -> 0 and x = m_t^2/Lambda^2.

Statement.

1. (Corollary 27.2) If tr T = 0, then 2 Tr(gamma_5 T f(DD^2/Lambda^2)) -> f(0) 2 N_c tr(T Q^2) (e^2/8 pi^2) int F_em ^ F_em, with no gluonic and no gravitational term.

2. For two flavours with T = T^3 and Q = diag(2/3, -1/3), 2 N_c tr(T^3 Q^2) = 1, and the right side is f(0) (e^2/16 pi^2) int F_em,mu nu F~_em^mu nu.
3. A singlet T = 1 on N_f flavours gives in addition the gluonic term f(0) 2 N_f (g^2/8 pi^2) int tr_c(G ^ G) and the gravitational term -f(0) (N_c N_f/12) p_1(M).
4. For a smooth cutoff class in which the heat expansion can be used, the local form, with a smooth function chi on M inserted, follows at Lambda -> infinity from the a_4 coefficient (1/16 pi^2) tr(gamma_5 T (E^2/2 + Omega^2/12)).
5. (Proposition 27.4) At fixed cutoff the electromagnetic coefficient for T^3 on (t, b) is 2 N_c (1/2) (Q_t^2 f(x) - Q_b^2 f(0))/f(0). It equals 1 at x = 0; for the heat cutoff it is (4 e^{-x} - 1)/3, tending to -N_c Q_b^2 = -1/3 as x -> infinity, and the gluonic and gravitational regulated traces are proportional to f(x) - f(0).

Scope. The commuting subalgebra of Proposition 17.3. Item 5 is a trace in the fixed-cutoff family of lemma 18, and its weights are properties of the chosen regulator. An explicitly massive flavour breaks the axial symmetry, so this calculation alone does not establish a Wess-Zumino term in the low-energy theory; such a term needs the retained symmetry, its background fields or spurions, and an anomaly-matching prescription, as in D'Hoker and Farhi (named in the steer note and not read). Making f part of a finite-scale spectral action, as in lemmas 18 and 25, does not by itself identify the regulator of the fermion measure with that action or make these weights observable.

Proof. Lemma 27, Corollary 27.2 and Proposition 27.4; the proofs are there: Theorem 27.1 with the index theorem, where tr T = 0 removes the gluonic and gravitational terms, and the two weights f(x), f(0) for the heavy and light members of the doublet.

What it leaves open. With this result the algebraic coefficient quoted in the steer note, item 6, has the regulated trace and measure that it lacked. The measure on gauge fields, the quantum order parameter and the pion kinetic term remain to be supplied (lemma 27, consequence for T10).

Part S111 (On the reduced operator D_M(A) + gamma_5 d with masses commuting with the gauge field, the regulated axial trace is sum_f tau_f f(m_f^2/Lambda^2) ind_f at every cutoff, and the regulated mass insertion equals it)

On the reduced operator D_M(A) + gamma_5 d with masses commuting with the gauge field, the regulated axial trace is sum_f tau_f f(m_f^2/Lambda^2) ind_f at every cutoff, and the regulated mass insertion equals it

Status. Theorem (Theorem 27.1), with Proposition 27.3. Proved by Claude Opus 5.5, with the index theorem as a reference; checked by Claude Opus 5.5 in Python (Landau spectra on T^2 x T^2 with fluxes n_1, n_2, truncated to 2000 levels: Tr(gamma_5 exp(-t(D^2 + m^2))) = exp(-t m^2) n_1 n_2 to 15 digits at three parameter sets) and by Terra 6.1 by hand, effort max, whose report on lemma 27 found two errors of interpretation, in Propositions 27.4 and 27.5, both corrected (historical/reviews/2026-11-03_terra_check_27.md, items 1 and 3). Ledger row `l27_trace` (depending on `l17_action`). Proposition 5 of paper/chiral_limit_note.tex.

Setting. M a closed Riemannian spin 4-manifold. Following Corollary 17.2, DD = D_M(A) + gamma_5 d on L^2(S) (x) E, with d = diag(m_f), m_f >= 0, on E = C^{N_c} (x) C^{N_f}, and A the gauge connection of the commuting subalgebra, colour and electromagnetism, diagonal in flavour with charge Q_f on flavour f; for nontrivial backgrounds E is replaced by the bundles E_f. T is Hermitian on C^{N_f}, diagonal in the same basis, with eigenvalues tau_f. Then d, T and A commute, and DD^2 = D_M(A)^2 + d^2 by Theorem 17.1. The cutoff f makes f(DD^2/Lambda^2) trace class, for example in the positive Laplace class of lemma 18; gamma_5 = +1 on S^+, and ind_f = n_{f,+} - n_{f,-} is the index of D_f^+ on S^+ (x) E_f.

Statement.

1. (Theorem 27.1) For every Lambda > 0, Tr(gamma_5 T f(DD^2/Lambda^2)) = sum_f tau_f f(m_f^2/Lambda^2) ind_f, with no expansion in m/Lambda.
2. By the index theorem, ind_f = (1/8 pi^2) int tr_f(F ^ F) - (N_c/24) p_1(M), with tr_f(F ^ F) = N_c Q_f^2 e^2 F_em ^ F_em + g^2 tr_c(G ^ G), since G is traceless.
3. (Proposition 27.3) If m_f > 0 for all f, then Tr(f(DD^2/Lambda^2) d T DD^-1) = Tr(gamma_5 T f(DD^2/Lambda^2)). Since {gamma_5 T, D_M + gamma_5 d} = 2 d T, when the Jacobian and the composite mass insertion are given the same specified regularization this is the integrated Ward identity in a fixed background at every Lambda, with a common factor f(m_f^2/Lambda^2) on both sides.

Scope. The surviving zero modes contribute the signed count n_{f,+} - n_{f,-}, whereas the scalar density of Proposition 17.8 sees the unsigned count. The trace identity alone constructs neither an independently regulated measure nor a local current, and the local identity has a current-divergence term as well. T, d and A commute by hypothesis, which excludes T^3 together with the W, and off-diagonal d, for which DD^2 has a term gamma_5 [d, D_M]. The background is fixed; a measure on gauge fields and the order of limits are separate, as in Proposition 17.8. The regulated Jacobian by the spectrum of DD is one choice, gauge invariant under the commuting group, and reversing the chirality convention reverses the index.

Proof. Lemma 27, Theorem 27.1 and Proposition 27.3; the proofs are there. On block f, DD^2 = D_f^2 + m_f^2, and D_f anticommutes with gamma_5, so it maps each eigenspace of D_f^2 with eigenvalue mu > 0 isomorphically between the chiralities, and only ker D_f remains, with weight f(m_f^2/Lambda^2). For Proposition 27.3, DD^-1 = (D_M + gamma_5 d)(D_M^2 + d^2)^-1, the term with D_M has zero trace, and on ker D_f the factor m_f^2/m_f^2 is 1.

What it leaves open. A measure on gauge fields, the map to a quantum order parameter and the kinetic term of a pion field, which need a quantum effective action (lemma 27, consequence for T10). The limit Lambda -> infinity is 27_anomaly_coefficient.md.

Part S112 (For two massless flavours and a reduction Sigma : S^4 -> SU(2) in the nontrivial class of pi_4(SU(2)), the spectral action is unchanged and the Weyl Pfaffian of N_c doublets, transported along connections from 0 to the pure gauge Sigma d Sigma^*, has the relative sign (-1)^{N_c})

For two massless flavours and a reduction Sigma : S^4 -> SU(2) in the nontrivial class of pi_4(SU(2)), the spectral action is unchanged and the Weyl Pfaffian of N_c doublets, transported along connections from 0 to the pure gauge Sigma d Sigma^*, has the relative sign (-1)^{N_c}

Status. Proposition (Proposition 27.5), with Remark 27.6. Proved by Claude Opus 5.5, with the main result of Witten 1982 (Eq. (6), p. 325, read on 3 Nov 2026) as a reference; checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand, who corrected an error of interpretation in it (historical/reviews/2026-11-03_terra_check_27.md, items 5 and 6). Ledger row `l27_discrete` (depending on `l17_no_kinetic`). Proposition 6 of paper/chiral_limit_note.tex.

Setting. N_f = 2 and M = 0, with electromagnetism off, so that SU(2)_L x SU(2)_R is a symmetry of the reduced sector (Proposition 17.5); at M = 0 the admissible involutions are the U_Sigma of Theorem 17.6. For smooth Sigma(x), the pointwise reduction of Proposition 17.10 has at d = 0 the operator D_M + c^mu P_- B_mu, B_mu = Sigma d_mu Sigma*, in independent quark and antiquark variables. Sigma : S^4 -> SU(2) generates pi_4(SU(2)) = Z_2 on the round S^4, with colour background zero, so that D_M has no kernel. Pf_-(A_t) is the square root of Witten 1982 for one Weyl doublet, transported along A_t = t B, with the endpoint fibres compared by the gauge transformation.

Statement.

1. On the chirality selected by P_- the operator is the massless Dirac operator with the pure-gauge SU(2) connection B, and on the other chirality it has no flavour connection; the colour connection is unaffected, and the operator is generally not self-adjoint.
2. For N_c doublets of that chirality, Pf_-(B)^{N_c} = (-1)^{N_c} Pf_-(0)^{N_c}.
3. The spectral action of the fixed operator D does not depend on Sigma. The fixed classical form is unchanged as a form, while its expression in the fixed reduced variables acquires the connection.
4. (Remark 27.6) By Witten 1982 the sign is (-1) to the number of zero modes of the five-dimensional Dirac operator on S^4 x R, whose ends compactify to the nontrivial SU(2) bundle on S^5. For isospin j the parity is that of 2 tr T^2 = 2 j (j + 1)(2j + 1)/3, odd exactly at j = 2k + 1/2. So the discrete term has the parity of N_c times the mod-two index of the doublet on that bundle, the generator of Omega_5^Spin(BSU(2)) = Z_2.

Scope. A relative sign for the stated Pfaffian transport and endpoint identification, at one configuration. Pulling the original measure through the pointwise R_Sigma of Theorem 17.1 preserves the value of the integral, so the sign compares that pulled-back measure with the Pfaffian continued along connections, and for N_c odd they differ between the two classes of reductions. The coupling has the sign of the discrete term (-1)^{N_c [sigma]} of the N_f = 2 sigma model (Lee, Ohmori and Tachikawa, arXiv:2009.00033, section 2.5 and Eq. (2.29)), with [sigma] in the reduced spin bordism group of SU(2) in dimension four; the anomaly is a class in Omega_5^Spin(BSU(2)), and the two agree through the anomaly comparison while being distinct groups. For a nonzero colour background the integral is complex in general, and that case is not developed. Witten 1983 was not read.

Proof. Lemma 27, Proposition 27.5; the proof is there: the display counts N_c Weyl doublets, to which the main result of Witten 1982 applies, and the statements on the action are item 1 of Proposition 17.10.

What it leaves open. Matching the sign to a pion field needs the sigma model and its anomaly-matching assumptions, and a sum over Sigma needs a sigma model that is not constructed; Proposition 17.10 supplies neither a pion action nor its measure. The identification with the S^5 index of section 6.12 of the corpus research state concerns the bundle and the index: there the S^5 is an internal space, here the compactification of spacetime and an auxiliary flow, and no relation between the two spaces or their SU(2) groups is proved, so open question 6 of the steer note stays open for that part.

Part S113 (For the pull-back connection, the leading wall operator on the representation 4 has a one-dimensional kernel whose compression has fibre part -w(rho), w(rho) = 2(rho^2 + 2)^2/(9 + 2(rho^2 + 2)^2), which reaches the level-one value only as rho -> infinity, while the line-preserving collar connection gives the level-one value at every radius)

For the pull-back connection, the leading wall operator on the representation 4 has a one-dimensional kernel whose compression has fibre part -w(rho), w(rho) = 2(rho^2 + 2)^2/(9 + 2(rho^2 + 2)^2), which reaches the level-one value only as rho -> infinity, while the line-preserving collar connection gives the level-one value at every radius

Status. Proposition (Proposition 28.13), with Lemma 28.12. Proposition 28.13 proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (matrix elements, and the closed form to ten digits) and by Terra 6.1 by hand, who reproduced the matrix A(rho) in exact symbolic arithmetic from the bracket and Clifford data; the Terra report found three errors of statement and gave three amendments, applied (historical/reviews/2026-11-03_terra_check_28.md). Lemma 28.12 proved by Astra 6, job

28 (historical/reviews/2026-11-03_astra_job_28.md, Lemma 5), and checked by Claude Opus 5.5 in Python against the radial system. Ledger rows `l28_wall` (depending on `l28_mixing`) and `l28_mixing`. The function w(rho) enters case (a) of Theorems 2 and 4 of paper/joint_limit_note.tex.

Setting. The operator of 28_self_adjoint_local_wall.md, with the sign of Phi for which the right singular vector of T is at the zero section, and its left singular vector v at the wall, in the domain Z v = +v of T*. A wall profile exp(q(x - l)), q > 0, for T* = i Z d/dx + D_6 + i phi tau with phi > 0 has q = -phi Z tau at leading order, so the stable channels of v are (Z, tau) = (+, -3/2) and (+, -1/2) twice, mapping to the channels (-, -1/2) twice. Horizontal scale 1.

Statement.

1. (Lemma 28.12) On a product collar, normalized stable profiles for mass eigenvalues M q_i overlap by 2 sqrt(q_i q_j)/(q_i + q_j), independently of M, which is sqrt(3)/2 for q = 1/2, 3/2. So large mass keeps the coupling between the stable channels O(3) and O(1), which in the pull-back connection has order 1/rho.
2. In unitary bases of the channels, the compression of D_6(rho) with these overlaps is A(rho) = [[3/(2 rho), 1/rho, rho + 1/rho], [0, sqrt(2), -sqrt(2)]], with kernel spanned by c = (-(2/3)(rho^2 + 2), 1, 1).
3. For the mode c, tr(B Gamma_5)/tr B = -1 and tr(B t-hat)/tr B = -w(rho), for every rho; for horizontal scale h, rho^2 is replaced by rho^2/h^2. So w = 8/17 as rho -> 0, w = 2/3 at the Fubini-Study radius rho = 1, and w -> 1, the level-one value, as rho -> infinity. The nonzero singular values of A at rho = 1 are 1.863 and 2.789. With the opposite sign of Phi the fibre part is +w(rho).
4. For the connection preserving the line summands, the first row of A is (0, 1/rho, rho + 1/rho), the kernel is (1, 0, 0), so w = 1 for every rho, and the block on tau = -1/2 is invertible for every rho > 0, with D^2 = 3, 6 at rho = 1.

Scope. The kernel and the traces are exact for this finite block, the leading formal wall operator; the next correction is O(1/M) and does not change the limiting kernel. In lemma 28 the convergence of the boundary mode of T_M to it is numerical (Result 28.5, Python: the difference falls from 0.037 to 0.009 at wall radius 0.5 and from 0.008 to 0.002 near radius 1, between phi_0 = 16 and 32). The value 2/3 tests the overlap factor sqrt(3)/2, since without it w(1) would be 3/5.

Proof. Lemma 28, Proposition 28.13 and Lemma 28.12. The three stable channels all lie in Gamma_5 = -1 and have mutually orthogonal target spaces, so the cross terms vanish in the traces of Proposition 28.4.

What it leaves open. Lemma 29 proves the convergence under the kernel condition K(rho): the half-cylinder kernel lies within O(1/m) of the kernel of A(rho) (Proposition 29.10), and K(1) holds, with fibre part -2/3 (29_window_case_a.md). Other radii need K(rho).

Part S114 (Conjecture: for large mass the doubled operator on the disc bundle has an isolated window of eight states, one copy of the 4 of Sp(2) at the zero section and one at the boundary, and its compression maps tend to the two endpoint maps)

Conjecture: for large mass the doubled operator on the disc bundle has an isolated window of eight states, one copy of the 4 of Sp(2) at the zero section and one at the boundary, and its compression maps tend to the two endpoint maps

Status. Conjecture (Conjecture 28.6), proved in lemma 29 in case (b) for the metric class of Setting 29.0 (Theorem 29.7) and in case (a) at the Fubini-Study radius (Theorem 29.11), both refereed by Terra 6.1; open for other metrics, and for case (a) at other radii. Stated by Claude Opus 5.5, with Astra 6 also credited in lemmas/README.md; numerical support from Result 28.5 (Python, numpy and scipy, a radial solve whose script is not kept), rerun by Terra 6.1 in Python, with a separate test of A_2. A first version of these numerics had the wrong sign of the spin connection, and its numbers are withdrawn (lemma 28, Correction of

3 Nov 2026). Ledger rows `128_window` (status conjectural, amended by `129_window_b` and `129_window_a`), `128_numerics` (status numerical) and `128_criterion` (Lemma 28.14). Listed in the table of conjectures of lemmas/README.md; the proved cases are Theorem 2 of paper/joint_limit_note.tex.

Setting. Theorem 28.9 with the sign of Proposition 28.13, and either (a) the pull-back connection and wall radius rho, or (b) the collar connection of Proposition 28.13 and the Fubini-Study metric at the wall.

Statement (conjectured).

1. For large M the hypotheses of Lemma 28.14 hold, with the eight states being one copy of the 4 at the zero section in one Gamma-hat degree and one at the boundary in the other.
2. For a family of metrics and mass profiles with a uniform positive decay rate, the off-diagonal block is exponentially small in l.
3. The compression of a smooth function tends to (Q_B(a|*{S^4}), Q_Y(a*/{CP^3})), with Q_Y(x_a) = -Gamma_a/5 and Q_Y(y_T) a multiple of T, equal to the level-one value of Example 16.9 times w(rho) in case (a), and equal to it in case (b).

Lemma 28.14 (proved, Astra 6). An isometry U_M : C^8 -> Dom D-hat_M with ||D-hat_M U_M|| <= eps_M -> 0 and ||D-hat_M psi|| >= delta ||psi|| on the complement of its range, with delta independent of M, gives exactly eight eigenvalues in (-delta, delta) for large M, with the spectral space within O(eps_M/delta) of the range.

Numerical support (Result 28.5, Python). On the 4 at rho_inf = 1 the lowest singular value falls with l at the rate of the constant-coefficient exponent (0.877, 1.155, 1.464 measured against 0.856, 1.151, 1.476 at phi_0 = 3, 4, 5), and the next value is 1.86 to 1.87. On the 16, 20 and 40 the lowest values are near sqrt(5), sqrt(8), sqrt(12) at the core and above 1.9 at the wall, and the multiplicity is zero on the 1, 5, 10, 14 and 35. Both modes have Q(x_a) = -Gamma_a/5; the boundary fibre part approaches -w(rho), and -1.0000 with the collar connection, where the next value is sqrt(3). Small entries have a mesh floor and do not establish a zero continuum value.

Proof. The proved cases are in lemma 29 (29_window_case_b.md, 29_window_case_a.md, 29_tunnelling_exponentially_small.md, 29_compression_to_endpoint_maps.md). Lemma 28 named the missing steps: the normal reductions with their errors, the matching, and a coercive estimate uniform in the mass and in all representations of Sp(2).

What it leaves open. Other metrics in the class of Theorem 28.9, including the Bryant-Salamon metric; case (a) at radii other than 1, under the kernel condition K(rho); a lower bound on the tunnelling eigenvalue. The generated algebra at finite mass stays outside the conjecture (28_counterexample_generation_discontinuous.md).

Part S115 (Unital completely positive maps that converge in norm can generate M_4(C) for every positive parameter and M_2(H) at the limit, so generated algebras need not converge when compression maps do)

Unital completely positive maps that converge in norm can generate M_4(C) for every positive parameter and M_2(H) at the limit, so generated algebras need not converge when compression maps do

Status. Counterexample (Counterexample 28.17), to the claim that convergence of compression maps carries the generated algebras along. Constructed by Astra 6, job 28 (historical/reviews/2026-11-03_astra_job_28.md, Counterexample 9); checked by Claude Opus 5.5 by hand. The Terra report on lemma 28 does not cover it, and the ledger records it as refereed by Astra 6 or checked by Claude Opus 5.5. Ledger row `128_generation` (depending on `128_symmetry`). The caution is stated after Theorem 4 of paper/joint_limit_note.tex and in the section on the algebra of paper/connes_kk_dictionary_note.tex.

Setting. On the disc bundle of lemma 28, R_0 = Q_B E_fib, with E_fib the fibre average on CP^3, R_1 = Q_Y, and R_eps = (1 - eps) R_0 + eps R_1.

Statement.

1. The R_eps are unital completely positive, and ||R_eps - R_0|| <= 2 eps.
2. Real functions generate M_4(C) under R_eps for every eps > 0.
3. Real functions generate M_2(H) under R_0.

Scope. The correct limit statement concerns specified compression maps, and it gives no convergence of the generated algebras.

The exact test at finite mass (Proposition 28.16, Astra 6, checked by Claude Opus 5.5 in Python on the density at finite mass; ledger row `l28_symmetry`). For an equivariant copy of the 4, B(r) = F(r)* F(r) = diag(b_a(r) I_2, b_+(r), b_-(r)). All real scalar functions compress to operators commuting with the quaternionic antiunitary j if and only if b_+(r) = b_-(r) for almost every r; if in addition the 5 component is nonzero, the block generates M_2(H), and if both the 5 and the 10 components are nonzero, it generates M_4(C). A vanishing integral of tr(B(r) t-hat) alone does not decide this. At the limit the real linear image is Herm_2(H) + Herm_4(C), of real dimension 22, and the generated real algebra has dimension 48. At the sample points resolved numerically (Result 28.5, Python), the core mode has a nonzero density breaking the quaternionic structure, empirically of order lambda^2.

Proof. Lemma 28, Counterexample 28.17; the construction is in the Astra report, Counterexample 9.

What it leaves open. The result points to statements about bounded expressions in compressed observables: the quaternionic defect of the core block is O(M^{-1/2}) for products of bounded real observables, and exponentially small when j_E is parallel (30_quaternionic_defect.md). The exact generated algebra at any finite mass is not established.

Part S116 (The normal model Q = sigma . d_y + y . J with spin 3/2 has ker Q = 0 and ker Q^* = 3 + 1 with a gap, and over S^4 this gives exactly four core states, one copy of the 4 of Sp(2), the twistor spinors as sections of S)

The normal model Q = sigma . d_y + y . J with spin 3/2 has ker Q = 0 and ker Q^* = 3 + 1 with a gap, and over S^4 this gives exactly four core states, one copy of the 4 of Sp(2), the twistor spinors as sections of S

Status. Proposition (Proposition 28.11), with Proposition 28.3. Proposition 28.11 proved by Astra 6, job 28 (historical/reviews/2026-11-03_astra_job_28.md, Proposition 4), and checked by Claude Opus 5.5 in Python (in an oscillator basis: Q* Q starts at 1.02, and Q Q* has four near-zero values, with total spin 0 and three values near spin 1, then 1.02); Proposition 28.3 proved by Claude Opus 5.5 and checked by Astra 6 by hand. The Terra report on lemma 28 does not cover them, and the ledger records them as refereed by Astra 6 or checked by Claude Opus 5.5. Ledger rows `l28_normal` and `l28_core`. The fibre kernel (1, 3) (+) (1, 1) enters the proof of Theorem 2 of paper/joint_limit_note.tex.

Setting. Q = sigma . d_y + y . J on L^2(R^3; C^2 (x) C^4), with J Hermitian of spin 3/2 and the normal Sp(1) acting; the induced bundle over S^4 and the twisted Dirac operators there, with Cas_G(4) = 5/2, Cas_G(16) = 15/2, Cas_G(20) = 21/2, Cas_G(40) = 29/2 and Cas_K(j_a, j_b) = 2 j_a(j_a + 1) + 2 j_b(j_b + 1).

Statement.

1. ker Q = 0 and ker Q* = 3 (+) 1 under the normal Sp(1), with a positive gap above the zero modes; changing the sign of the mass exchanges the two kernels.

2. The induced bundle over S^4 is S^2(S) (+) 1, and the operator on Sigma (x) S^2(S) has exactly four zero modes, in one chirality. On the parallel inclusion S -> S (x) S^2(S) the twisted equation is a multiple of the twistor equation, so the kernel consists of harmonic spinors of the twisted operator, which are twistor spinors as sections of S.
3. (Proposition 28.3) Hom_K(4, S^- (x) (1, 3)) has dimension 1, through (1, 2), and Hom_K(4, S^+ (x) (1, 3)) = 0, in agreement with the index 4; every map in it annihilates (2, 1) in 4. On a representation V the twisted operator has D^2 = Cas_G(V) + 3/2 - Cas_K(tau), which is 0, 5, 8, 12 on the 4, 16, 20, 40 for the twist (1, 3), and 4 on the 4 for the untwisted part (1, 1).
4. Rescaling gives the localization scale r = O(M^{-1/2}) at the core and O(M^{-1}) at the wall.

Scope. By Proposition 28.4, a K-symmetric mode supported on (1, 2) has tr(B Gamma_5)/tr B = +-1 and tr(B t-hat) = 0, so the zero section gives Q(x_a) = +-Gamma_a/5 and nothing for the fibre-dependent part of a function: the core block is the quaternionic map Q_B of the base.

Proof. Lemma 28, Propositions 28.11 and 28.3; the proof of 28.11 is in the Astra report: Q* Q = -Laplacian + (y . J)^2 - sigma . J >= 0, with equality excluded since sum_i J_i^2 = 15/4, and the Callias index for the positive eigenbundles of n . J, of degrees 3 and 1.

What it leaves open. The core estimate uniform in the mass and in all representations of Sp(2) is Proposition 29.2 and Lemma 29.3 (29_window_case_b.md), which use the exact splitting of Proposition 29.1 in place of the spectral separation hypothesis of Maridakis, which fails for this mass.

Part S117 (On the disc bundle X of Lambda^2_- over S^4, whose smooth functions have independent values on the zero section S^4 and on the boundary CP^3, prescribed rank-eight windows at the two ends compress real functions to maps that generate M_2(H) + M_4(C) at every localization scale)

On the disc bundle X of Lambda^2_- over S^4, whose smooth functions have independent values on the zero section S^4 and on the boundary CP^3, prescribed rank-eight windows at the two ends compress real functions to maps that generate M_2(H) + M_4(C) at every localization scale

Status. Theorem (Theorem 28.21), with Proposition 28.8. Proved by Astra 6, job 28 (historical/reviews/2026-11-03_astra_job_28.md, Theorem 13 and Proposition 1; prompt in historical/prompts/28_two_walls_connected_parent.md); checked by Claude Opus 5.5 by hand. The Terra report on lemma 28 covers Proposition 28.13 and the numerics only, and the ledger records these two results as refereed by Astra 6 or checked by Claude Opus 5.5. Ledger rows `l28_prescribed` and `l28_endpoints`. Cited in the table of paper/connes_kk_dictionary_note.tex ("Theorems, refereed, lemmas 28, 29").

Setting. G = Sp(2), K = Sp(1)_a x Sp(1)_b and H = Sp(1)_a x U(1)_b, so that G/K = S^4, K/H = S^2 and G/H = CP^3. X is the disc bundle of Lambda^2_- over S^4, with zero section S^4 and boundary CP^3; away from the zero section X = (0, l] x CP^3, with radial coordinate x and S^2 fibres of radius rho(x). P_y = (1 + y_a Gamma_a)/2, p_z <= P_{pi(z)} is the tautological rank-one projector, and chi_0, chi_l are normalized radial profiles supported in disjoint eps-neighbourhoods of the two ends.

Statement.

1. (Proposition 28.8) Restriction C^infinity(X, R) -> C^infinity(S^4, R) (+) C^infinity(CP^3, R) is surjective, and in normal coordinates a(y, v) = a(y, 0) + O(|v|), uniformly on bounded subsets of C^1(X).
2. (Theorem 28.21) U_0 v = sqrt(2) chi_0 P_y v and U_l v = 2 chi_l p_z v are isometries into L^2(X; C^4) (+) L^2(X; C^4).

3. Real functions compressed by them generate M_2(H) + M_4(C) at every eps, and the compression differs from Q_B (+) Q_Y by at most C eps ||a||_{C^1}.

Scope. The window is an input, so this proves the connected-space mechanism without Conjecture 28.6. In Example 16.9 the second sheet and the restriction to S^4 were inputs, and X is a connected source for them. Allowing nonlocal smoothing perturbations, (1 - P) D (1 - P) + P d P makes such a window reducing, which enlarges the operator class and keeps the window as input. The choices of X, of E = S^3(S), of the hedgehog, of the wall condition and of the doubling stay inputs (lemma 28, balance against Proposition 6 of lemma 24).

Proof. Lemma 28, Proposition 28.8 and Theorem 28.21; the full proofs are in the Astra report, Proposition 1 and Theorem 13.

What it leaves open. A window from an actual local operator is Conjecture 28.6 (28_conjecture_window.md), proved in lemma 29 for case (b) and for case (a) at rho = 1 (29_window_case_b.md, 29_window_case_a.md), with the limit of the compression in 29_compression_to_endpoint_maps.md. The statement concerns compression maps, and generated algebras need not follow them (28_counterexample_generation_discontinuous.md). The connector, the quaternionic grading axis and the selection of the two Hom spaces stay inputs, and T0 stays open, with this as a candidate for its first half.

Part S118 (On the disc bundle with a connection metric, the doubled Callias-type operator built from T_M = D_X^E + i Phi_M, with a local chirality condition at the boundary, is self-adjoint, elliptic and odd with compact resolvent, every smooth scalar function preserves its domain, and no condition enters at the zero section)

On the disc bundle with a connection metric, the doubled Callias-type operator built from T_M = D_X^E + i Phi_M, with a local chirality condition at the boundary, is self-adjoint, elliptic and odd with compact resolvent, every smooth scalar function preserves its domain, and no condition enters at the zero section

Status. Theorem (Theorem 28.9). Proved by Astra 6, job 28 (historical/reviews/2026-11-03_astra_job_28.md, Theorem 2, using Bär-Ballmann, Corollary 4.18); checked by Claude Opus 5.5 by hand. The Terra report on lemma 28 does not cover it, and the ledger records it as refereed by Astra 6 or checked by Claude Opus 5.5. Ledger row `128_operator`. Cited in the section on the local operator of paper/joint_limit_note.tex ("\supp{28}, Theorem 28.9").

Setting. X as in 28_prescribed_windows_m2h_m4c.md, with the connection metric g = dr^2 + h(r)^2 pi* g_{S^4} + rho(r)^2 g_{S^2}, h smooth, even and positive at 0, rho odd and smooth with rho(r) = r + O(r^3), and a product near r = l. E = S^3(S), with S = G x_K (1, 2) and J the spin-3/2 action; Phi_M = M b(|v|^2) v . J with b smooth and positive; T_M = D_X^E + i Phi_M, D-hat_M = [[0, T_M*], [T_M, 0]] and Gamma-hat = diag(1, -1). At r = l, Z = -i c(dr), chi = Gamma-hat Z, and the condition is chi psi = -psi.

Statement.

1. D-hat_M on this H^1 domain is self-adjoint, elliptic and odd for Gamma-hat, with compact resolvent.
2. Every smooth scalar function on X preserves the domain.
3. No boundary condition enters at the zero section: once the outer condition is fixed, removing S^4 creates no further self-adjoint extension.

Scope. By Proposition 28.1, O(k) for k != 0 does not extend over X, while E restricts on CP^3 to O(3) (+) O(1) (+) O(-1) (+) O(-3), the summand O(3) giving the Dolbeault operator on O(1) of Example 16.9, and the hedgehog Phi(v) = phi(|v|) (v-hat . T) is smooth and G-equivariant with zero set exactly S^4. The wall condition reads u_even(l) = 0 for T and v_odd(l) = 0 for T*. Charge conjugation C = C_7 (x) j_E

commutes with T_M as a differential expression and exchanges the two local wall conditions, so it does not preserve the chosen domain (Proposition 28.18).

Proof. Lemma 28, Theorem 28.9; the proof is in the Astra report, Theorem 2, with the local elliptic boundary condition of Bär-Ballmann, Corollary 4.18.

What it leaves open. The spectrum of D-hat_M at large M: the four core states (28_four_core_states.md), the boundary mode (28_boundary_fibre_part.md), and the isolated eight-state window, Conjecture 28.6 (28_conjecture_window.md).

Part S119 (On the eight-state window, the compression of a C^1 function on the disc bundle converges to the pair of endpoint maps (Q_B(a|*{S^4}), Q_Y(a*/{CP^3})) with error O(M^{-1/2}), and real functions generate M_2(H) + M_4(C) under the limit map)

On the eight-state window, the compression of a C^1 function on the disc bundle converges to the pair of endpoint maps (Q_B(a|*{S^4}), Q_Y(a*/{CP^3})) with error O(M^{-1/2}), and real functions generate M_2(H) + M_4(C) under the limit map

Status. Theorem (Theorem 29.9). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (fibre and Gamma_5 parts of the wall mode in case (b), both -1.00000 at M = 8, 16, 32) and by Terra 6.1 by hand (historical/reviews/2026-11-04_terra_check_29.md). Ledger row `l29_compression` (depending on `l29_window_b`, `l28_compression` and `l28_endpoints`). Theorem 4 of paper/joint_limit_note.tex; cited in the section on the algebra of paper/connes_kk_dictionary_note.tex.

Setting. The window of Theorem 29.7 (case (b)) or Theorem 29.11 (case (a) at rho = 1, or at radius rho under K(rho)), with equivariant unitary identifications of the window spaces with the model copies of the 4; a in C^1(X).

Statement.

1. ||Q_M(a) - (Q_B(a|*{S^4}) (+) Q_Y(a*/{CP^3}))|| <= C ||a||_{C^1} M^{-1/2}.
2. Q_B(a) = int a(x) 2 P_x d mu(x), with P_x = (1 - x . Gamma)/2.
3. Q_Y(a) = int a(z) B(z) d mu(z). In case (b), B(z) = 4 p_z, the level-one map of Example 16.9. In case (a) at radius rho, B(z) = 2 ((1 + w) p_z + (1 - w) p_{jz}), with p_{jz} the quaternionic partner of p_z and w = w(rho), so Q_Y(x_a) = -Gamma_a/5, Q_Y(y_T) is w(rho) times the level-one value, and the signed fibre part is -w(rho).
4. Restriction to the two ends is surjective (Proposition 28.8), so real functions on X generate M_2(H) + M_4(C) under the limit map, by lemma 10 and Example 16.9, for every w(rho) > 0.

Scope. A statement about compression maps: generated algebras at finite mass stay outside it (Counterexample 28.17). The sign in P_x is the value tr(B Gamma_5)/tr B = -1 of Proposition 28.13.

Proof. Lemma 29, Theorem 29.9; the proof is there. It is Theorem 28.15 with eta_M = O(M^{-1/2}) from Theorems 29.7 and 29.11; the normal first moments are O(M^{-1/2}) at the core, by the Gaussian decay of the model, and O(M^{-1}) at the wall. At the core, F in Hom_K(4, (1, 2) (x) (1, 3)) kills (2, 1), so B = 2 P_x; at the wall the density is a sum over channels with the weights of Proposition 28.13.

What it leaves open. The quaternionic defect of the core block at finite mass (30_quaternionic_defect.md), the joint limit with the dilation (30_joint_limit_disc_bundle.md), and the connector of Proposition 28.20 and the quaternionic grading of Proposition 28.19, which stay inputs (lemma 29, Scope).

Part S120 (The singular value lambda_M that links the two ends of the eight-state window satisfies the exact identity lambda (<u, chi u> - <v, chi v>) = <v, c(d chi) u> and is exponentially small, |lambda_M| <= C exp(-(1 - eta)(M/2) int_e^{l-a} f dr))

The singular value lambda_M that links the two ends of the eight-state window satisfies the exact identity lambda (<u, chi u> - <v, chi v>) = <v, c(d chi) u> and is exponentially small, |lambda_M| <= C exp(-(1 - eta)(M/2) int_e^{l-a} f dr)

Status. Proposition (Proposition 29.8). Proved by Claude Opus 5.5; checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand (historical/reviews/2026-11-04_terra_check_29.md). Ledger row `l29_tunnelling` (depending on `l29_window_b` and `l28_tunnelling`). Proposition 3 of paper/joint_limit_note.tex. Read a second time by Terra 6.1 (5 Nov 2026, historical/reviews/2026-11-05_terra_check_29_second_reading.md): identity confirmed; the Agmon argument now keeps the partition weights in the transition region, and the statement for a family with l -> infinity is qualified by the uniformity it needs.

Setting. The window of Theorem 29.7 or 29.11. By Proposition 28.19, on one copy of the 4 in each Gamma-hat degree the window operator is d_M = [[0, conj(lambda_M) I_4], [lambda_M I_4, 0]] after equivariant unitary identifications, so the four singular values coincide; T_M u = lambda v and T_M* v = lambda u is a singular pair with unit vectors. chi is radial, equal to 1 near r = 0 and to 0 near r = l.

Statement.

1. lambda (<u, chi u> - <v, chi v>) = <v, c(d chi) u>.
2. For every eta in (0, 1) there are M_eta and C_eta with |lambda_M| <= C_eta exp(-(1 - eta)(M/2) int_e^{l - a} f dr) for M >= M_eta.
3. For a family with l -> infinity, M >= M_eta, and uniform bounds on the end geometry, the cutoffs, the connection and the positive bulk mass, the constants are uniform, so lambda is exponentially small in l at the rate (1 - eta)(M/2) inf f. This is the uniform decay rate required by Conjecture 28.6, as amended by the referee.

Scope. An upper bound: a lower bound on lambda is open. The factor 1/2 holds on every channel, since (n . J)^2 >= 1/4. The summands of E on CP^3 have indices 4, 0, 0, -4, of total zero, as equivariant cobordism invariance requires (Witten 1983, preprint pp. 43-44), and the window has Sp(2)-index 4 - 4 = 0, so index theory allows lambda != 0 (Remark 29.12).

Proof. Lemma 29, Proposition 29.8; the proof is there. Since chi u lies in Dom T_M, <T_M* v, chi u> = <v, T_M chi u> gives the identity. Agmon weights from each end, with the bulk bound of Lemma 29.4 and the wall bound of Lemma 29.6 or Proposition 29.10, bound the weighted norms of u and v, and the identity on a thin bulk shell gives the exponent.

What it leaves open. A lower bound on lambda, and the relation between lambda and M_W, which needs the vector sector (lemma 30, Remark 30.8). The joint limit of lemma 30 uses this bound to get m_eps <= C eps (30_joint_limit_disc_bundle.md).

Part S121 (With the pull-back connection throughout, Conjecture 28.6 holds at the Fubini-Study wall radius, where the kernel condition K(1) holds and the wall mode has fibre part -2/3, and at another radius it holds under the kernel condition K(rho))

With the pull-back connection throughout, Conjecture 28.6 holds at the Fubini-Study wall radius, where the kernel condition K(1) holds and the wall mode has fibre part -2/3, and at another radius it holds under the kernel condition K(rho)

Status. Theorem (Theorem 29.11), with Proposition 29.10. Proved by Claude Opus 5.5; checked by Claude Opus 5.5 by hand (Proposition 29.10) and in Python (the block norms and the 2 x 3 block on the 4), and by Terra 6.1 by hand, whose report on lemma 29 found two errors in case (a), the definition of A_eff and the sign of the fibre part, both corrected (historical/reviews/2026-11-04_terra_check_29.md). A fresh reading in the referee report on lemma 30 found one more sign and one missing step in Proposition 29.10; these are corrected (historical/reviews/2026-11-04_terra_check_30_note.md). The corrected text was read by Terra 6.1 a second time (historical/reviews/2026-11-05_terra_check_29_second_reading.md): consistent with the finite matrix of Proposition 28.13, with two steps added (the surjectivity of the complement block behind the Schur complement; the decay of the half-cylinder kernel behind the trial states) and the quantifiers of delta corrected; the derivation of that matrix from the geometry stays an obligation for a stand-alone paper. Ledger rows `l29_window_a` (depending on `l29_wall_a`) and `l29_wall_a`; the ledger records `l28_window` as amended by `l29_window_a`. Theorem 2 of paper/joint_limit_note.tex, case (a).

Setting. Setting 29.0 of 29_window_case_b.md, with the pull-back connection throughout. On the collar A = A_d + A_o, where A_d = D_line preserves the lines and A_o = A_E/rho has order zero, with [A_o, q] != 0; in Python the block of A_E between q = 3/2 and q = 1/2 has norm sqrt(3) and ||A_E|| = 2. P projects onto {sum_{q > 0} p_q g_q : Z g_q = -g_q}, p_q = sqrt(2 m q) exp(m q (x - l)), Y is the same with Z = +1, and F = Pi_Y (A_d + A_o) P. A_eff, with blocks F and F*, is identified with D_3(rho) (+) D_1(rho) + (sqrt(3)/2) B(rho) on sections over CP^3 of the lines q > 0, with B(rho) of norm sqrt(3)/rho; kappa_a(rho) > 0 is the gap of its nonzero spectrum, uniform in V. Condition K(rho): the kernel of A_eff(rho) is exactly one copy of the 4, in Z = +1.

Statement.

1. (Proposition 29.10) Under K(rho), the wall estimates of Lemma 29.6 hold with the constant 3 replaced by kappa_a(rho)^2 - O(1/m), and the half-cylinder kernel of T_M* is exactly one copy of the 4 for m large, within O(1/m) of (-(2/3)(rho^2 + 2), 1, 1), with fibre part tending to -w(rho) at rate O(1/m).
2. (Theorem 29.11) K(1) holds. For each fixed 0 < delta < min(2, kappa_a(1)) and M above a threshold depending on delta, the conclusion of Theorem 29.7 holds at the Fubini-Study radius, with the wall modes replaced by the kernel of Proposition 29.10 and the fibre part tending to -w(1) = -2/3.
3. kappa_a(1) >= sqrt(5) - 3/2 off the 4, and kappa_a(1) = 1.863 on the 4, where the squared values are (45 +- 3 sqrt(33))/8.
4. At another wall radius the same conclusion holds under K(rho).

Scope. At other radii only finitely many V can carry a kernel, those with |D_3(rho) (+) D_1(rho)| <= 3/(2 rho), by ellipticity. In Python, on the 16, 20 and 40 and for rho in [0.2, 4], the compressed block has smallest singular value at least 1.30. A check of K(rho) on all relevant V at a given radius needs a lower bound for D_line(rho) on all V, which lemma 29 does not write down away from rho = 1.

Proof. Lemma 29, Proposition 29.10 and Theorem 29.11; the proofs are there. T_d = i Z d/dx + A_d + i m q preserves the block structure, and eliminating the complement of P gives (1 + 2 ||A_o||/m) ||T_M w|| >= ||F w_1|| - (2 ||A_o||^2/m) ||w||; the Schur complement differs from F* by at most ||A_o||^2/(m/2 - ||A_o||). For K(1): off the 4, |D_3 (+) D_1| >= sqrt(5) while ||(sqrt(3)/2) B(1)|| = 3/2; on the 4, F* is A(1) of Proposition 28.13, with kernel (-2, 1, 1).

What it leaves open. K(rho) away from rho = 1 (an explicit list of the finitely many representations

that ellipticity allows at each radius), other metrics, and the derivation of the finite matrix A(rho) of Proposition 28.13 from the geometric data. The compression in case (a), with fibre part -w(rho), is 29_compression_to_endpoint_maps.md.

Part S122 (With the line-preserving collar connection and the Fubini-Study metric at the wall, and flat fibres near the zero section, the doubled operator on the disc bundle has, for each fixed 0 < delta < sqrt(3) and mass above a threshold depending on delta, exactly eight eigenvalues in (-delta, delta), one copy of the 4 of Sp(2) at each end, which is Conjecture 28.6 in case (b))

With the line-preserving collar connection and the Fubini-Study metric at the wall, and flat fibres near the zero section, the doubled operator on the disc bundle has, for each fixed 0 < delta < sqrt(3) and mass above a threshold depending on delta, exactly eight eigenvalues in (-delta, delta), one copy of the 4 of Sp(2) at each end, which is Conjecture 28.6 in case (b)

Status. Theorem (Theorem 29.7), with Propositions 29.1 and 29.2 and Lemmas 29.3 to 29.6. Proved by Claude Opus 5.5, without an Astra job (prompt in historical/prompts/29_window_coercive_estimate.md); checked by Terra 6.1 by hand, for whom case (b) stands with amendments, and who ran no computation (historical/reviews/2026-11-04_terra_check_29.md); read a second time by Terra 6.1 with the first reports withheld (historical/reviews/2026-11-05_terra_check_29_second_reading.md): two errors repaired (the quantifiers of delta, which the wording with delta = 0 contradicted; the sign of the Clifford commutator in the IMS step, the identity itself being right), amendments to Propositions 29.1, 29.2 and Lemmas 29.4 to 29.6 applied, no counterexample to the corrected theorem found; checked by Claude Opus 5.5 in Python (the identities {A_0, Z} = 0, [A_0, tau] = 0, {A_0, A_1} = 0 and A_0^2 against the Casimir formula on the 4, 16, 20, 40; the slice blocks against Lemma 29.5; and a direct radial solve under Setting 29.0 whose limit values were fixed before the run, giving on the 4 at M = 32 the values 1.7323, 1.9991, 2.4497 against sqrt(3), 2, sqrt(6)). Ledger rows `l29_window_b`, `l29_splitting`, `l29_model`, `l29_core`, `l29_bulk`, `l29_lines` and `l29_wall`; the ledger records `l28_window` as amended by `l29_window_b`. Theorem 2 of paper/joint_limit_note.tex, case (b); cited in paper/connes_kk_dictionary_note.tex.

Setting. (Setting 29.0) X, G, K, H and E = S^3(S) of lemma 28, Phi_M = eps M f(r) n . J, T_M = D_X^E + i Phi_M, and D-hat_M doubled, with the wall condition Z u = -u for T_M and Z v = +v for T_M*. (H1) g = dr^2 + pi* g_{S^4} + rho(r)^2 g_{S^2}, S^4 of radius 1, rho(r) = r on [0, 2e] and rho = 1 on the collar [l - 2a, l]. (H2) f(r) = r on [0, 2e], f = f_0 > 0 on the collar, f > 0 on (0, l]. (H3) The pull-back connection on r <= 2e, the connection preserving the four line summands on the collar, any invariant interpolation in between, in radial parallel gauge. (H4) eps = -gamma_b, the sign of Proposition 28.13.

Statement.

1. (Proposition 29.1) On the tube with flat fibres, T_M = T_0 + R with R Hermitian, |R| <= C_R |y|, and T_0* T_0 = D_H^2 + Q^* Q^ exactly, by K-equivariance of the fibre operator.
2. (Proposition 29.2) On the normal bundle, ker T_0 is one copy of the 4 and ker T_0* = 0, with ||T_0 u||^2 >= min(4, mu_1 M) ||u||^2 on the complement and the same for T_0*, for all representations (mu_1 = 1.02 in Python).
3. (Lemmas 29.3 and 29.4) Near the zero section ||T_M u|| >= (2 - o(1)) ||u|| off the model kernel, and ||T_M* v|| >= (2 - o(1)) ||v||; in the bulk ||T_M w||^2 >= (M^2 f_1^2/4 - C_Phi M) ||w||^2, independently of the representation.
4. (Lemmas 29.5 and 29.6) At the Fubini-Study metric D_k^2 = C_2(L) + 3/2 - (3/8) k^2: on O(3) the kernel is one copy of the 4, in Z = +1, with D_3^2 >= 5 off the 4; on O(1) there is no kernel, with

D_1^2 >= 3 on the 4 and >= 6 off it. On the collar ||T_M w||^2 >= min(m^2/4, 3) ||w||^2 on the stated domains and complement.

5. (Theorem 29.7) For each fixed 0 < delta < sqrt(3) there is M_delta such that for M >= M_delta, D-hat_M has exactly eight eigenvalues in (-delta, delta). The spectral subspace splits under Gamma-hat as W_M (+) V_M, each a copy of the 4, within O(M^{-1/2}) of the cut-off core modes in the domain of T_M and of the cut-off wall modes p (x) ker D_3 in the domain of T_M*. The gap is set by the wall at sqrt(3), with the core next at 2.

Scope. Spectral statements for a chosen metric family, with flat fibres near the zero section, for the sign of Proposition 28.13. Each estimate holds on all of L^2 of a fixed manifold, with boundary operators elliptic on CP^3 or S^4, so uniformity over the representations of Sp(2) needs no separate argument, and the spectral separation hypothesis of Maridakis, which fails for this mass, is never used. The window has Sp(2)-index 4 - 4 = 0 (Remark 29.12), so index theory allows lambda != 0, and the eight states come from the estimates.

Proof. Lemma 29, Theorem 29.7; the proof is there. It verifies Lemma 28.14 with a radial partition chi_c^2 + chi_b^2 + chi_w^2 = 1: the trial states have defect O(M^{-1/2}), and ||T_M u||^2 = sum_i ||T_M chi_i u||^2 - sum_i || |d chi_i| u ||^2 with Lemmas 29.3, 29.4 and 29.6 gives the bound (3 - o(1)) ||u||^2 on the complement.

What it leaves open. A stand-alone mathematics paper on this theorem: the second reading lists what it must display (conventions, the equivariant Callias kernel, domains, the SU(4)-to-Sp(2) eligibility, the repaired estimates with their cut-offs, the derivation of the matrix of Proposition 28.13, and a literature comparison; lemma 29, section "Second reading"), and the end-model spectra are in print (Ben Halima 2008; Dolan 2003). Other metrics in the class of Theorem 28.9, including product-collar modifications of Bryant-Salamon-type metrics and the complete Bryant-Salamon geometry, for which Remark 29.14 outlines a route that is not carried out; the connector of Proposition 28.20; the quaternionic grading of Proposition 28.19; any dynamics selecting M, l or rho. Case (a) is 29_window_case_a.md, and the off-diagonal block is 29_tunnelling_exponentially_small.md.

Part S123 (On the disc bundle with mass M_eps proportional to log(1/eps) and dilation R_eps = eps^{-alpha}, the operator keeps exactly eight eigenvalues, of size at most C eps, below delta eps^alpha, its compression converges to the endpoint maps, and its count at fixed energy has seven-dimensional Weyl growth C_X E^7 eps^{-7 alpha})

On the disc bundle with mass M_eps proportional to log(1/eps) and dilation R_eps = eps^{-alpha}, the operator keeps exactly eight eigenvalues, of size at most C eps, below delta eps^alpha, its compression converges to the endpoint maps, and its count at fixed energy has seven-dimensional Weyl growth C_X E^7 eps^{-7 alpha}

Status. Theorem (Theorem 30.5). Proved by Claude Opus 5.5, new in lemma 30 and resting on lemma 29; the dilation of Remark 29.13 is from the audit of 3 November 2026 by Astra 6. Checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand, whose corrections are applied and have not been read again (historical/reviews/2026-11-04_terra_check_30_note.md). Ledger row `l30_disc` (depending on `l29_window_b`, `l29_window_a`, `l29_tunnelling` and `l29_compression`). Theorem 5 of paper/joint_limit_note.tex; cited in the section on the massless limit of paper/connes_kk_dictionary_note.tex.

Setting. A shape as in Setting 29.0, with connection (b), or with connection (a) at rho = 1, and delta as in Theorem 29.7 or 29.11. G_0 = int_e^{l - a} f dr, 0 < alpha < beta < 1, E_* > 0, eta in (0, 1), M_eps

= (2 (1 - alpha)/((1 - eta) G_0)) log(1/eps), R_eps = eps^{-alpha} and D_eps = D-hat_{M_eps}/R_eps. By the dilation of Remark 29.13, exact under the canonical spinor identification and the L^2 unitary factor R^{-7/2}, D_eps is the operator of the metric R_eps^2 g with potential (M_eps/R_eps) f(s/R_eps) n . J in the physical radial coordinate s = R_eps r, of size M_eps/R_eps -> 0.

Statement. For eps small:

1. D_eps has exactly eight eigenvalues in (-delta eps^alpha, delta eps^alpha), equal to +-m_eps with multiplicity 4, or zero with multiplicity 8, where m_eps <= C eps. The next level is at most C log(1/eps) eps^alpha, so it tends to zero.
2. For E_eps = E_* eps^beta, N_eps(E_eps) = #{|D_eps| <= E_eps} = 8.
3. The compression converges to the endpoint maps, with error C ||a||_{C^1} (log 1/eps)^{-1/2} in shape coordinates.
4. For fixed E > 0, N_eps(E) ~ C_X E^7 eps^{-7 alpha}, with C_X = 64 omega_7 Vol_g(X)/(2 pi)^7 the Weyl constant of D-hat_0 on X, where omega_7 is the volume of the unit ball and 64 = 8 x 4 x 2 counts the spinors, E and the doubling.

Scope. Statements about operators, compression maps and spectra along a family given by hand, which no dynamics selects (Remark 30.8). Item 1 is an upper bound on m_eps, since a lower bound on lambda is unproved, and only the fermion window is treated, so the relation between lambda and M_W is unproved on the disc bundle. The constants of Proposition 29.8 are fixed once the shape and eta are fixed, so no uniformity in a varying shape is used. The pointed limit at fixed energy is flat at a fixed interior point and a half-space at a fixed boundary point, and nothing here identifies it with a complete Bryant-Salamon manifold or with a solution of field equations. A family with l_eps growing at fixed M has the same form, with the rate of Proposition 29.8.

Proof. Lemma 30, Theorem 30.5; the proof is there. Theorem 29.7 or 29.11 applies since M_eps -> infinity, Proposition 28.19 gives the multiplicity, and Proposition 29.8 gives lambda <= C eps^{1 - alpha}, hence m_eps = lambda/R_eps <= C eps; item 3 is Theorem 29.9. For item 4, D-hat_M = D-hat_0 + M V with V bounded by c = (3/2) sup f, so min-max gives N(D-hat_0, Lambda - c M) <= N(D-hat_M, Lambda) <= N(D-hat_0, Lambda + c M), and with Lambda = E R_eps, M_eps/Lambda -> 0 and the Weyl law for D-hat_0 apply.

What it leaves open. Remark 30.8 lists, in order: the vector sector on this family, through the kinetic matrix and mass Hessian H_eps u = M^2 K_eps u and then the momentum-dependent kernel, with the aim M_W, M_Z ~ c lambda/R at finite nonzero couplings; the gauge normalization, since a bulk Yang-Mills field with fixed microscopic coupling and constant profile has g_4^-2 = Vol(X)/g_11^2, so plain dilation gives g_4 proportional to R^{-7/2}, which must be separated from restoration at finite coupling; an action in the radius and the electroweak order parameter, where a toy potential a r^-p + b eps^2 r^q would select alpha = 2/(p + q) and none has been derived; the connector of Proposition 28.20 and the origin of the quaternionic grading of Proposition 28.19.

Part S124 (With bulk mass mu_eps = mu_* eps^alpha and length c_eps/mu_eps, c_eps = (1 - alpha) log(1/eps) + c_0, the light pair of the interval has mass m_eps ~ 2 mu_* e^{-c_0} eps, the gap is ~ mu_* eps^alpha, the localization width over the length tends to zero, and the opposite-wall coefficient is exactly (m/(mu + kappa))^2)

With bulk mass mu_eps = mu_* eps^alpha and length c_eps/mu_eps, c_eps = (1 - alpha) log(1/eps) + c_0, the light pair of the interval has mass m_eps ~ 2 mu_* e^{-c_0} eps, the gap is ~ mu_* eps^alpha, the localization width over the length tends to zero, and the opposite-wall coefficient is exactly (m/(mu + kappa))^2

Status. Proposition (Proposition 30.3). Proposed in the Astra 6 audit of commit b2ddf98 (historical/reviews/2026-11-04_outside_audit_assessment.md), with the proof redone by Claude Opus 5.5; checked by Claude Opus 5.5 in Python (exact secular roots, recomputed by Claude Sonnet 5.5: for mu_* = 1, alpha = 1/2, c_0 = 2 and eps = 10^-2, c = 4.3026 and m = 2.710 x 10^-3, against the asymptotic 2.707 x 10^-3) and by Terra 6.1 by hand (historical/reviews/2026-11-04_terra_check_30_note.md). Ledger row `l30_localized`. Stated in the section on the interval of paper/joint_limit_note.tex.

Setting. Bulk mass M = -mu with the mixed conditions of lemma 09, mu_* > 0, 0 < alpha < 1, mu_eps = mu_* eps^alpha and the length l^loc_eps = c_eps/mu_eps, c_eps = (1 - alpha) log(1/eps) + c_0, which differs from the l_eps of Theorem 30.1. s = kappa l is the root of s = c tanh s, and m = mu sech s is the light mass.

Statement.

1. m_eps ~ 2 mu_* exp(-c_0) eps, Delta_eps ~ mu_* eps^alpha, and (1/kappa)/l_eps = 1/s -> 0.
2. With the left profile L(x) proportional to exp(-kappa x) - r exp(kappa x), r = exp(-2 kappa l), the identities r = (m/(mu + kappa))^2 and mu - kappa = m^2/(mu + kappa) are exact.

Scope. The localization width and the length both grow, with a shrinking ratio. The finite two-point distance 1/m and the length l both diverge, with (1/m)/l ~ exp(c)/(2c), so a metric convergence statement would need its own comparison maps. The family is given by hand, and no dynamics selects it (Remark 30.8).

Proof. Lemma 30, Proposition 30.3; the proof is there. Lemma 9.5 gives m = 2 mu exp(-c) (1 + o(1)) and kappa/mu -> 1. The first oscillatory root has pi < k_1 l < 3 pi/2 and lambda_1^2 = mu^2 + k_1^2, so Delta/mu -> 1. From mu = kappa coth s and m = kappa/sinh s, m/(mu + kappa) = exp(-s), and m^2 = mu^2 - kappa^2.

What it leaves open. paper/joint_limit_note.tex presents this as the interval family closest to the geometry of the disc bundle, where localization comes from a Callias potential (30_joint_limit_disc_bundle.md). The vector sector along the family is not treated.

Part S125 (For the core copy of the 4 in the disc-bundle window, every bounded real observable compresses to within 2 eta_t of a quaternionic operator and products of k of them to within 2 k eta_t, and the defect eta_t is O(M^{-1/2}) in general and exponentially small in M when the quaternionic structure of E is parallel)

For the core copy of the 4 in the disc-bundle window, every bounded real observable compresses to within 2 eta_t of a quaternionic operator and products of k of them to within 2 k eta_t, and the defect eta_t is O(M^{-1/2}) in general and exponentially small in M when the quaternionic structure of E is parallel

Status. Propositions (Propositions 30.6 and 30.7). Proposition 30.6 was proposed in the Astra 6 audit of commit b2ddf98 (historical/reviews/2026-11-04_outside_audit_assessment.md), with the proof redone by Claude Opus 5.5; Proposition 30.7 is by Claude Opus 5.5, new in lemma 30. Both checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand, whose report found a missing hypothesis in the exponential bound, the parallel quaternionic structure, now added; the corrections have not been read again (historical/reviews/2026-11-04_terra_check_30_note.md). Ledger rows `l30_defect_bound` (depending on `l28_symmetry`) and `l30_defect` (depending on `l29_tunnelling`, `l28_conjugation` and `l30_defect_bound`). Proposition 6 of paper/joint_limit_note.tex; cited in paper/connes_kk_dictionary_note.tex ("up to a defect (\supp{30})").

Setting. The window of Theorem 29.7 or 29.11. For its copy of the 4 at the core, B(r) = F(r)* F(r) = diag(b_a I_2, b_+, b_-) as in Proposition 28.16, j is the quaternionic antiunitary commuting with Sp(2), P_H(A) = (A + j A j^-1)/2, Q is the normalized compression of this mode, and eta_t = int |b_+(r) - b_-(r)| dr / int tr B(r) dr.

Statement.

1. (Proposition 30.6) ||Q(f) - P_H Q(f)|| <= 2 eta_t for every real f with ||f||_infinity <= 1.
2. If X_1, ..., X_k are such compressions, ||X_1 ... X_k - P_H(X_1 ... X_k)|| <= 2 k eta_t.
3. (Proposition 30.7) eta_t = O(M^{-1/2}).
4. If the quaternionic structure j_E of E is parallel on all of X, as for the pull-back connection and for the real affine interpolation (1 - s(r)) nabla^pull + s(r) nabla^line, then for every eta in (0, 1) there are M_eta and C_eta with eta_t <= C_eta exp(-(1 - eta) M int_{2e}^{l - a} f dr) for M >= M_eta, the exponent of Proposition 29.8 doubled, with the integral starting at 2e.

Scope. Bounded expressions become quaternionic as eta_t -> 0, while an unrestricted generated algebra may use coefficients of size 1/eta_t and retain the other directions, as in Counterexample 28.17. The parallel hypothesis can fail for other invariant interpolations allowed by (H3). A bound in terms of lambda itself would need a lower bound on lambda, which is open. Under the added hypothesis the defect comes from the wall, which breaks C = C_7 (x) j_E (Proposition 28.18); for an operator with the same interior and no wall, on a complete manifold, Hom_G(4, ker T) carries a real structure, and if it is one-dimensional the core block is exactly quaternionic. The audit samples (Python, eta_t/lambda^2 between 0.5 and 0.9, rows reproduced by Claude Sonnet 5.5) vary the length at fixed mass with profiles outside Setting 29.0, so they are a qualitative comparison that does not test item 4.

Proof. Lemma 30, Propositions 30.6 and 30.7; the proofs are there. Schur's lemma writes Q(f) as an average of lambda(g)^-1 B(r) lambda(g), j exchanges b_+ and b_-, so ||B - P_H B|| = |b_+ - b_-|/2, and telescoping gives the product bound. For item 4, C commutes with T_M when j_E is parallel, F -> C F j^-1 is a real structure on the one-dimensional Hom_G(4, ker T_0), and zeta = u - C u satisfies T_M zeta = lambda nu in the interior with Pi(chi_c zeta) = 0, so the Agmon weights of Proposition 29.8 and a weighted Cauchy-Schwarz inequality give the doubled exponent.

What it leaves open. The exact generated algebra at finite mass, and a lower bound on lambda. The

wall block generates M_4(C), which needs the loss of the inherited symmetry at the wall or a different real correspondence (Proposition 28.18).

Part S126 (On the interval with D_F = eps D_* and length l_* eps^{-alpha}, the finite data stay exact in a window of r = dim H_F states below E_* eps^beta, while the tower gap closes as eps^alpha and the count at fixed energy grows as eps^{-alpha})

On the interval with D_F = eps D_* and length l_* eps^{-alpha}, the finite data stay exact in a window of r = dim H_F states below E_* eps^beta, while the tower gap closes as eps^alpha and the count at fixed energy grows as eps^{-alpha}

Status. Theorem (Theorem 30.1), with Corollaries 30.2 and 30.4. Proposed in an audit of commit b2ddf98 by Astra 6 (historical/reviews/2026-11-04_outside_audit_assessment.md), with the proofs redone by Claude Opus 5.5 against the source lemmas; checked by Claude Opus 5.5 by hand and by Terra 6.1 by hand, in a report on lemma 30 together with the joint-limit note, whose corrections are applied and have not been read again (historical/reviews/2026-11-04_terra_check_30_note.md). Ledger rows `l30_interval` (depending on `l22_family`), `l30_poles` and `l30_cylinder` (both depending on `l30_interval`). Theorem 1 of paper/joint_limit_note.tex, with the vector poles and the cylinder stated after it ("\supp{30}").

Setting. (H_F, D_*, J_F, Gamma_F) satisfying the hypotheses of Theorem 22.1, with A_F represented by pi as in Theorem 22.4 and r = dim H_F. For 0 < alpha < 1, l_* > 0 and small eps > 0, D_F(eps) = eps D_*, l_eps = l_* eps^{-alpha}, and D_eps is the interval operator D_X of Theorem 22.1 for these data. Delta_eps is the smallest element of spec |D_eps| outside spec |eps D_*|, and N_eps(E) = #{|D_eps| <= E}.

Statement. For eps small:

1. W_eps* D_eps W_eps = eps D_*, with J_F and Gamma_F exact.
2. Delta_eps = (pi/l_*) eps^alpha - eps ||D_*||.
3. ||Q_eps(a) - pi(a)|| <= (2/3) l_*^2 ||D_*||^2 eps^{2(1 - alpha)} ||a||.
4. For alpha < beta < 1, E_* > 0 and E_eps = E_* eps^beta, N_eps(E_eps) = r.
5. For fixed E > 0, N_eps(E) ~ (2 r l_* E/pi) eps^{-alpha}.
6. (Corollary 30.2) In the five-dimensional Yang-Mills toy of lemma 23, with universal mass m_eps = eps m_* and theta_eps = l_eps m_eps -> 0, M_W = arcsin(sin(2 theta_eps)/sqrt(2))/l_eps and M_Z = arcsin(2 sin(2 theta_eps)/sqrt(5))/l_eps, so M_W^2 = 2 m_eps^2 (1 + O(theta_eps^2)) and M_Z^2 = (16/5) m_eps^2 (1 + O(theta_eps^2)), and the vector poles follow the hierarchy.
7. (Corollary 30.4) On the cylinder of lemma 24, with finite data on V = C^4 with gradings of type (2, 2), so r = 4, and r_eps = r_* eps^{-gamma}, 0 < gamma < 1: for max(alpha, gamma) < beta < 1 the cutoff E_* eps^beta isolates the window, the next level being at least min((Delta_K/r_*) eps^gamma, (pi/l_*) eps^alpha - eps ||D_*||), and at fixed E the count is proportional to l_eps r_eps^6 E^7, a seven-dimensional Weyl growth, by Dirichlet and Neumann bracketing. For gamma = alpha all seven directions expand homothetically.

Scope. These families have the hierarchy m_weak « E_eps « Delta_KK -> 0, which alone does not make the count at fixed energy diverge (the signed spectrum 0, +-eps^alpha, +-1, +-2, … has it with a bounded count), so items 5 and 7 are proved separately. The finite data are supplied, the parent algebra is matrix-valued, and the wall data change with eps by hand; a fixed Majorana block does not scale as eps D_*, so an application to an electroweak Dirac block needs a reducing sector or a treatment of the heavy neutral sector. The constants 2 and 16/5 belong to the five-dimensional action, and the small-mass pole coefficients of the fully massive finite spectral action are 3 and 24/5; lemma 25 connects the two pairs at the level of the leading coefficient at fixed s = l Lambda, and a joint-limit use needs a remainder uniform in s, since s -> 0 along Lambda = E_eps and s -> infinity at fixed Lambda. Varying the cutoff of the action is an extra

prescription. In item 7 the window is prescribed by the spectral walls of lemma 24, and other values of r need an enlarged tangential bundle, which is not constructed.

Proof. Lemma 30, Theorem 30.1 and Corollaries 30.2 and 30.4; the proofs are there. l_eps ||D_F(eps)|| = l_* ||D_*|| eps^{1 - alpha} < pi/2 for small eps, so Theorems 22.1 and 22.4 (2) apply; eps ||D_*|| « eps^beta « eps^alpha; the signed spectrum (n pi +- theta_k)/l_eps consists of r arithmetic progressions of spacing pi/l_eps. Corollary 30.2 is Theorem 23.6, and Corollary 30.4 uses bound (5) of lemma 24.

What it leaves open. No dynamics selects these families (lemma 30, Remark 30.8). A window that comes from a local operator, with seven-dimensional growth, is the disc bundle (30_joint_limit_disc_bundle.md), and the localized interval family is 30_localized_family.md.

Part S127 (A scale-free potential with G(theta) = theta^4 near theta = 0 is stationary along every family m = eps, l = eps^{-alpha} with 0 < alpha < 1, so a flat direction allows a joint path without selecting it)

A scale-free potential with G(theta) = theta^4 near theta = 0 is stationary along every family m = eps, l = eps^{-alpha} with 0 < alpha < 1, so a flat direction allows a joint path without selecting it

Status. Counterexample, to the universal conclusion of the first version of Proposition 1.5: that every stationary family has constant angle and that a second scale is necessary for a joint family to exist. Constructed by Terra 6.1 in the referee report on the 11D line (historical/reviews/2026-11-04_terra_check_11d.md, section on Proposition 1.5), written out in the Referee proofs of lemma 31; it is part of the ledger row `d11_scale_free`, which Claude Opus 5.5 (the author of lemma 31) and Claude Sonnet 5.5 checked by hand. Stated in the section on the massless limit of paper/connes_kk_dictionary_note.tex ("a flat direction would select nothing (\supp{31})").

Setting. V(l, theta) = l^{-4} G(theta) with theta = l m at fixed m > 0, as in item 2 of 31_scale_free_potential.md.

Statement.

1. Let G(theta) = theta^4 on 0 <= theta < delta, with a C^1 extension to the angle domain. Then V(l, l m) = l^{-4} (l m)^4 = m^4 is independent of l on l m < delta.
2. For any 0 < alpha < 1, the family m = eps, l = eps^{-alpha} has theta = l m = eps^{1 - alpha} < delta for small eps, so it is stationary, l m -> 0, and by Theorem 23.6 M_W l ~ sqrt 2 l m -> 0.
3. An extension with G(theta) >= theta^4 on the rest of the fundamental interval gives V(l, l m) >= m^4, with equality on l m < delta, so the family is a flat family of minima.

Scope. The potential leaves the path undetermined and does not select it dynamically. The specified flat one-loop determinant of lemma 24 has no such flat direction near theta = 0 when n_f != 54/5 (item 3 of 31_scale_free_potential.md).

Proof. Lemma 31, Referee proofs, Counterexample (referee) to the universal conclusion of Proposition 1.5: substitution.

What it leaves open. The result points to an action that selects a joint path with finite gauge couplings, which remains to be constructed. A flat modulus, a tuned higher-order potential or additional dimensionful terms leave other families possible.

Part S128 (The chiral interval of length l is unitarily the antiperiodic circle of radius l/pi, with the grading as the reflection x -> -x, the walls as its two fixed points and the scalar functions as the reflection-even functions)

The chiral interval of length l is unitarily the antiperiodic circle of radius l/pi, with the grading as the reflection x -> -x, the walls as its two fixed points and the scalar functions as the reflection-even functions

Status. Proposition (Proposition 1.1). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author by hand, by Terra 6.1 by hand, who also checked the Weyl constant of the joint family, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_circle` (depending on `l05_truncation`). The first row of the table and the section on the grading of paper/connes_kk_dictionary_note.tex ("\supp{31}"); cited in the section on the interval parent of paper/real_structure_note.tex ("the reflection-even algebra of \supp{31}").

Setting. D_l(L, R) = (R', -L') on H_l = L^2([0, l], C^2), with R(0) = 0 and L(l) = 0, grading Gamma = diag(1, -1) and the scalar functions C^infinity([0, l]), as in lemma 05, eq. (8). S_R is the circle of length 2l, of radius R = l/pi, with coordinate x in [-l, l] and antiperiodic functions u(x + 2l) = -u(x).

Statement. The map (W(L, R))(x) = (L(abs(x)) + i sgn(x) R(abs(x)))/sqrt 2 is a unitary H_l -> L^2_antiperiodic(S_R), and

1. W D_l W^* = -i d/dx on its antiperiodic domain;
2. W Gamma W^* = P, the reflection (Pu)(x) = u(-x);
3. W f W^* is multiplication by f(abs(x)), so the algebra goes to the reflection-even functions;
4. the walls x = 0 and x = l go to the two fixed points x = 0 and x = l ~ -l of the reflection.

Hence spec D_l = {(k + 1/2)/R : k in Z}, each eigenvalue simple, which is (9) of lemma 05, and the Connes-Lott mass of the minimal truncation is m = pi/(2l) = 1/(2R).

Scope. The unfolded circle is scalar, and even extensions of arbitrary smooth functions on the interval need not be smooth at the fixed points (referee note). With finite data D_F supplied as in lemma 22, the parent unfolds to -i d/dx + D_F on the periodic circle of length 2l (25_unfolding_to_circle.md), and the wall angle interpolates between that circle (theta = 0) and the antiperiodic circle of this proposition (theta = pi/2).

Proof. Lemma 31, Proposition 1.1; the proof is there. The inverse is L(x) = (u(x) + u(-x))/sqrt 2, R(x) = (u(x) - u(-x))/(i sqrt 2) on [0, l]; continuity of u at 0 is the condition R(0) = 0, and the antiperiodic matching at l is the condition L(l) = 0.

What it leaves open. Proposal 1.3 of lemma 31, the circle as the M-theory circle, is a dictionary to be tested and has no file; its tests are the gauge couplings, which need g_5^2 ~ R along the family, and the selection of the path by an action (31_scale_free_potential.md). The joint family and the localized family that lemma 31 restates from the Astra 6 audit of 3 November 2026 (ledger rows `a11_joint_limit`, `a11_localized`) are 30_window_and_tower_on_interval.md and 30_localized_family.md. Which Witten spaces carry a spin structure whose fibre circle does not lift the circle action, as a fermion with both a fibre momentum and a base twist needs, is open.

Part S129 (Under the hypotheses of Theorem 9 of lemma 24, the angle-dependent one-loop potential, relative to theta = 0, has its minimum at theta = pi/2 with value -93 zeta(5) n_f/(512 pi^2 l^4), and near theta = 0 it is (3 zeta(3)/(16 pi^2))(54/5 - n_f) theta^2/l^4)

Under the hypotheses of Theorem 9 of lemma 24, the angle-dependent one-loop potential, relative to theta = 0, has its minimum at theta = pi/2 with value -93 zeta(5) n_f/(512 pi^2 l^4), and near theta = 0 it is (3 zeta(3)/(16 pi^2))(54/5 - n_f) theta^2/l^4

Status. Proposition (Proposition 1.4, corrected). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author in Python (historical/11D/checks_11d.py, sections F and F2: F(1) = 1.506786, J(0) = 1.803085, series against quadrature to 10^-6), by Terra 6.1 by hand, whose independent proof is written out in the Referee proofs of lemma 31, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). The first version read this potential as the whole energy of the walls and concluded that the walls attract; that reading is withdrawn (lemma 31, Correction of 3 Nov 2026). Ledger row `d11_casimir` (depending on `l24_no_minimum` and `l23_wz`). No paper cites it separately.

Setting. The flat interval with zero bulk masses, frequency-one fermions, the gauge towers and multiplicities of lemma 24, no wall potential and n_f > 0. With t = sin^2 theta and F(t) = int_0^infinity x^3 log(1 + t/sinh^2 x) dx, equation (22) of lemma 24 gives 8 pi^2 l^4 V(theta) = 3 F(2t(1 - t)) + (3/2) F((16/5) t(1 - t)) - n_f F(t).

Statement.

1. theta = pi/2 is the unique global minimum, with V(pi/2) - V(0) = -93 zeta(5) n_f/(512 pi^2 l^4), since F(1) = 93 zeta(5)/64.
2. Near theta = 0, V(theta) - V(0) = (3 zeta(3)/(16 pi^2))(54/5 - n_f) theta^2/l^4 + o(theta^2), since F'(0) = (3/2) zeta(3).
3. At theta = pi/2 the exact vector poles of lemma 23 give M_W = M_Z = 0 at fixed l, with the fermion at the tower scale pi/(2l).

Scope. The potential is relative to theta = 0. The theta-independent Casimir energy a_0/l^4 of the towers is not part of it, and the direction in which l moves at the angular minimum depends on the sign of a_0 - 93 zeta(5) n_f/(512 pi^2). The minimum over the angle at fixed l opens no dimension, and lemma 23 already notes that it is not the physical point.

Proof. Lemma 31, Proposition 1.4, and the independent proof in its Referee proofs: F is strictly increasing with F(0) = 0 and the gauge terms vanish at t = 1; log coth x = 2 sum over odd k of e^{-2kx}/k integrates termwise to F(1) = (3/2)(1 - 2^{-5}) zeta(5); and 1/sinh^2 x = 4 sum_n n e^{-2nx} gives F'(0).

What it leaves open. The coefficient a_0 was the open item of lemma 31; it is computed for the corpus content in lemma 37 (37_casimir_coefficient.md). The path along which l and theta move together is 31_scale_free_potential.md.

Part S130 (For a scale-free potential V = l^{-4} G(theta), a free angle gives a runaway of l at every angle with G != 0, and an angle tied to a mass, theta = l m, has stationary lengths l = theta_*/m at isolated roots of theta G' = 4 G, so M_W l is the same at every m)

For a scale-free potential V = l^{-4} G(theta), a free angle gives a runaway of l at every angle with G != 0, and an angle tied to a mass, theta = l m, has stationary lengths l = theta_*/m at isolated roots of theta G' = 4 G, so M_W l is the same at every m

Status. Proposition (Proposition 1.5, with the referee amendment; the ledger records it as conditional). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author by hand, by Terra 6.1 by hand, who found an error in the universal conclusion of the first version (31_counterexample_flat_direction.md) and analysed the displayed determinant, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_scale_free` (depending on `d11_casimir`). Stated in the section on the massless limit of paper/connes_kk_dictionary_note.tex ("\supp{31}").

Setting. V(l, theta) = l^{-4} G(theta) with G of class C^1: the form of every one-loop potential of a flat interval whose only scales are l and the wall angles (lemmas 23 and 24), with the angle-independent coefficient a_0 included in G.

Statement.

1. If the angle is free, V is monotone in l at every fixed theta with G(theta) != 0: at the minimizing angle the length runs to infinity if G(theta_min) > 0 and to zero if G(theta_min) < 0, and stationary points exist only on the set G = G' = 0.
2. If the angle is tied to a fixed mass, theta = l m (as for X = tan(l M) in lemma 22), the stationary lengths satisfy theta G'(theta) = 4 G(theta) at theta = l m. An isolated root theta_* gives l = theta_*/m, so M_W l and M_Z l, functions of theta alone by Theorem 23.6, are the same at every m: the path l ~ 1/M_W, alpha = 1 in the scaling of the joint family.
3. For the displayed flat one-loop determinant, G(theta) = a_0 + c theta^2 + o(theta^2) with c = (3 zeta(3)/(16 pi^2))(54/5 - n_f), and if n_f != 54/5 there is no sequence of positive stationary angles tending to zero, whether or not a_0 = 0; this covers every integer n_f.

Scope. Isolated stationary roots of a scale-free C^1 potential. A root interval, as for G = theta^4 near 0, makes every alpha < 1 stationary and selects none (31_counterexample_flat_direction.md), and the tuned case c = 0 needs its higher-order term. A joint family with alpha < 1 selected by the potential needs a second scale in the action: a bulk mass, the cutoff of the spectral action, curvature, flux or a wall term. Remark 10.7 of lemma 37 amends this last point: on the scale-free path the window can separate with a fixed small ratio if the light mass is exponentially small in a stabilized shape parameter.

Proof. Lemma 31, Proposition 1.5: d/dl [l^{-4} G(l m)] = l^{-5}(theta G'(theta) - 4 G(theta)) at theta = l m, and d/dl V = -4 l^{-5} G at a free angle. Item 3 is the referee's analysis in the Referee proofs of lemma 31: theta G' - 4 G = -4 a_0 - 2 c theta^2 + o(theta^2).

What it leaves open. A potential with a second scale that selects alpha < 1 with finite gauge couplings; the localized family of the audit, with G(l m, l mu) from the exact secular equations of lemma 09, is the first candidate. The modulus target V = a r^{-p} + b eps^2 r^q of the audit has q = -2 in the flat family, the borderline alpha = 1, so a term growing with l at fixed v must come from one of the second scales listed (lemma 37, Remark 10.8).

Part S131 (Abelian fluxes on CP^2 x CP^1 along hypercharge and B - L, with one line bundle per Standard Model field and the canonical spin^c shift, give no generation, with or without nu^c, and adding any real third charge does not change this)

Abelian fluxes on CP^2 x CP^1 along hypercharge and B - L, with one line bundle per Standard Model field and the canonical spin^c shift, give no generation, with or without nu^c, and adding any real third charge does not change this

Status. No-go theorem in its scope (Proposition 2.5, with the third-charge extension). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`), with a bounded search over a third integer charge in [-3, 3]^6 (Python, historical/11D/checks_11d.py, section D). Terra 6.1 gave an independent analytic proof of the exclusion, valid also for independent ten-dimensional chiralities, and extended it to an arbitrary real third charge; Claude Sonnet 5.5 checked the extension (rank of [y, b, p, w] over all sign patterns, exact fractions, not kept). Both proofs are written out in the Referee proofs of lemma 32 (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger rows `d11_flux_nogo`, `d11_flux_third_bounded` and `d11_flux_third_general` (the last attempted by Terra 6.1). No paper cites it separately.

Setting. Ten-dimensional Weyl fermions on M^4 x CP^2 x CP^1 with the Fubini-Study complex structure; each Standard Model Weyl field comes from the whole Dolbeault cohomology of one line bundle Lambda^{0,*} (x) K^{1/2} (x) N, with N the flux line bundle of its charges and K^{1/2} the formal canonical shift, and the four-dimensional chiral content is the index. A weak doublet needs O(1) or O(-3) on CP^1, so the CP^1 pairing has absolute value 2 on Q and on L.

Statement.

1. With fluxes only in the U(1) directions spanned by Y and B - L, for any flux quanta, no choice gives one generation, with or without nu^c: in the charges (6Y, 3(B - L)), c_L = -3 c_Q, so the CP^1 pairings satisfy abs(M_2(c_L)) = 3 abs(M_2(c_Q)), against 2 on both.
2. Adding any real third charge z to Y and B - L, with independently chosen source chiralities, gives no generation either; a changed common twist is the case z = F and is covered.
3. Under a common ten-dimensional chirality the pairing must also give u^c and d^c the same image, and 2j + 1 = (2, 1, 1, 2, 1, 1) on (Q, u^c, d^c, L, e^c, nu^c) must be linear in the fluxed charges (the lemma of required bidegrees); with independent chiralities these two conditions need not hold, and items 1 and 2 still do.

Hypotheses. Abelian fluxes, one ten-dimensional Weyl field per Standard Model field, the canonical formal spin^c shift, the Dolbeault index as the chiral content.

Escapes, as the lemma states them. Nonabelian bundles (an SU(2)_R doublet bundle), several ten-dimensional fields per four-dimensional field, a fourth charge direction, orbifolds and walls, and non-Kaluza-Klein (brane) states.

Proof. Lemma 32, Proposition 2.5, and the Referee proofs there. Item 1 is linearity of the pairing on the two doublets. For item 2, every vector of span(y, b) has its L entry equal to -3 times its Q entry, so the CP^1 pairing w lies outside it and span(y, b, z) = span(y, b, w); the linear form v -> 2 v_Q + v_u + v_d vanishes on y, b and the CP^2 pairing p, which forces the coefficient of w to vanish, and then the u^c and d^c entries give the L entry 15/2, against abs(p_L) = 3/2.

What it leaves open. The result points to Proposition 2.6 (32_one_generation_from_grading_flux.md), which uses the universal charge, the grading and B - L; its solution lies outside span(Y, B - L, z). Fourth charges and nonabelian bundles remain open.

Part S132 (With fluxed charges F (universal), chi (the grading) and B - L, the pairings P^(1) = c_0 chi + c_1 (B - L) and P^(2) = (3F + chi)/2 give one generation with nu^c as the exact index content on CP^2 x CP^1 for (c_0, c_1) = (-9/4, -3/4) or (-3/2, -3), and without chi or without B - L there is no solution)

With fluxed charges F (universal), chi (the grading) and B - L, the pairings P^(1) = c_0 chi + c_1 (B - L) and P^(2) = (3F + chi)/2 give one generation with nu^c as the exact index content on CP^2 x CP^1 for (c_0, c_1) = (-9/4, -3/4) or (-3/2, -3), and without chi or without B - L there is no solution

Status. Proposition (Proposition 2.6, with the referee amendments), conditional on a global gauge group, charge lattice, spin^c determinant and flux quantization, which are not specified. Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author in Python (historical/11D/checks_11d.py, section D2), by Terra 6.1 by hand, whose independent proof with the colour convention is written out in the Referee proofs of lemma 32, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_flux_fit` (depending on `d11_flux_nogo`). No paper cites it separately.

Setting. The construction of 32_nogo_flux_hypercharge_b_minus_l.md, with fluxed charges F = 1 on every field (a change of the common spin^c twist), chi = +1 on Q and L and -1 on u^c, d^c, e^c, nu^c, and B - L.

Statement.

1. For (c_0, c_1) = (-9/4, -3/4) or (-3/2, -3) the bidegrees are Q: (-4, 1); u^c, d^c: (1, 0); L: (-3, 1) or (0, 1); e^c, nu^c: (0, 0) or (-3, 0), all of even total Dolbeault degree, and the index content is one generation with nu^c.
2. Without chi, or without B - L, there is no solution, also for independent source chiralities.
3. With the colour action of the quotient of Theorem 2.1 the base is P(3-bar), and the quark bundles above carry the conjugate colour representations; replacing (c_0, c_1) by (9/4, 3/4) or (3/2, 3), with P^(2) unchanged, gives the labelled representations.

Scope. u^c and d^c see the same line bundle O(1, 0), and so do e^c and nu^c, so the flux treats them as SU(2)_R doublets, with hypercharge Y = T_3R + (B - L)/2 entering through T_3R, which carries no flux; chi equals the finite grading gamma_F on the particle basis. The flux examples establish a particular construction, and a general necessity for the grading flux is open (referee note).

Proof. Lemma 32, Proposition 2.6: substitution and the span test (Python, section D2). The referee's proof obtains the two failures from absolute values alone: without chi the doublet pairings force abs(P^(2)) = 2 or 0 on u^c, and without B - L the pairings of Q and L coincide while their colour pairings must be 5/2 and 3/2.

What it leaves open. The global flux data. The masses of u^c and d^c, which share one zero-mode profile, need data that break SU(2)_R, such as Higgs or wall couplings, and the nine-dimensional reduction of this index is open. The index calculation does not establish the physical interpretation of the grading as a gauge connection.

Part S133 (The circle bundle Y_{a,b} over CP^2 x CP^1 with Euler class a h_1 + b h_2, a != 0, has H^2 = Z + Z/gcd(a, b), b_3 = 0 and H^4 of order a^2, and is spin exactly when a is odd and b is even; for (3, 2), H^2 = Z, H^4 = Z/9, and the divisor classes of U and V go to 2 and -3)

The circle bundle Y_{a,b} over CP^2 x CP^1 with Euler class a h_1 + b h_2, a != 0, has H^2 = Z + Z/gcd(a, b), b_3 = 0 and H^4 of order a^2, and is spin exactly when a is odd and b is even; for (3, 2), H^2 = Z, H^4 = Z/9, and the divisor classes of U and V go to 2 and -3

Status. Proposition (Proposition 2.3). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author in Python (historical/11D/checks_11d.py, section C: Smith invariants and the spin condition), by Terra 6.1 by hand, whose independent proof is written out in the Referee proofs of lemma 32 (the cited Kreck-Stolz paper was unavailable to the referee, so no comparison with it is claimed), and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_topology`. No paper cites it separately.

Setting. B = CP^2 x CP^1 with H^*(B; Z) = Z[h_1, h_2]/(h_1^3, h_2^2), and Y_{a,b} -> B the circle bundle with Euler class e = a h_1 + b h_2, a != 0; g = gcd(a, b).

Statement.

1. H^2(Y) = Z + Z/g, and H^3(Y) = 0, so b_3(Y) = 0.
2. H^4(Y) = Z/g + Z/(a^2/g), a group of order a^2, cyclic when g = 1.
3. Y is spin exactly when a is odd and b is even.
4. For (a, b) = (3, 2): H^2 = Z, H^4 = Z/9, Y is spin, and the pullback H^2(B) = Z^2 -> H^2(Y) = Z is (x, y) -> 2x - 3y up to sign, so the divisor classes h_1 = [{l(U) = 0}] and h_2 = [{m(V) = 0}] go to 2 and -3, the values of 6Y on d^c and L.

The product member CP^2 x S^2 x S^1 of Witten's family (a = b = 0) is not spin.

Scope. The calculation is independent of the identification of Y_{3,2} with M^{111}.

Proof. Lemma 32, Proposition 2.3, and the Referee proofs there: the Gysin sequence, with the cup product H^2(B) -> H^4(B) of matrix [[a, 0], [b, a]], determinant a^2 and Smith invariants (g, a^2/g); w_2(Y) = pi^*(h_1 mod 2), since the vertical tangent line is trivial, and the kernel of the pullback on H^2(B; Z/2) is spanned by e mod 2.

What it leaves open. The reading of the Betti class, which pairs with the two divisors as 2 and -3, as hypercharge is Proposal 3.1 of lemma 39 and has no file.

Part S134 (Among the circle quotients Y_{a,b} of S^5 x S^3 with coprime weights only (3, 2) has a Calabi-Yau cone, and a free circle quotient of a moment level in a complex representation of dimension n has dimension 2n - 3, which is 7 only for n = 5: for SU(5) with hypercharge, the 5 and the 5-bar)

Among the circle quotients Y_{a,b} of S^5 x S^3 with coprime weights only (3, 2) has a Calabi-Yau cone, and a free circle quotient of a moment level in a complex representation of dimension n has dimension 2n - 3, which is 7 only for n = 5: for SU(5) with hypercharge, the 5 and the 5-bar

Status. Proposition (Proposition 2.2). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author in Python (historical/11D/checks_11d.py, section A), by Terra 6.1 by hand and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_unique` (depending on `d11_quotient`). Stated in the section on the string dualities of paper/connes_kk_dictionary_note.tex ("11 = 4 + (2 x 5 - 3)", "\supp{32}").

Setting. Y_{a,b} = (S^5 x S^3)/U(1), with weights b on C^3 and -a on C^2, gcd(a, b) = 1 and a, b > 0; and, for item 2, a complex representation of dimension n with a free circle action and its moment map mu.

Statement.

1. The cone over Y_{a,b} is Calabi-Yau only for (a, b) = (3, 2).
2. (S^{2n-1} cap mu^{-1}(0))/U(1) has dimension 2n - 3, which is 7 only for n = 5. Among representations of SU(5), with hypercharge as the circle, only the 5 and the 5-bar give a seven-dimensional quotient; the 10 gives 17 and the 5 + 10 gives 27.

So Witten's count reads 11 = 4 + (2 x 5 - 3), with the seven internal dimensions coming from the smallest representation of SU(5) and hypercharge as the circle.

Scope. Free circle quotients of moment levels in unit spheres; the Calabi-Yau condition of the cone as the vanishing of the sum of weights.

Proof. Lemma 32, Proposition 2.2: the Calabi-Yau condition is 3b - 2a = 0; the dimension is 2n - 1 for the sphere, minus one for the moment map and one for the circle; the irreducible representations of SU(5) of dimension 5 are the 5 and the 5-bar.

What it leaves open. The count fixes the representation. The eleven-dimensional vacuum AdS4 x Y_{3,2} as the v -> 0 endpoint, with hypercharge from the three-form, rests on Proposals 3.1 and 3.4 of lemma 39, which have no files.

Part S135 (The hypercharge quotient Y_{3,2} = (mu_Y^{-1}(0) cap S^9)/U(1)_Y of the 5-bar is a smooth simply connected seven-manifold, the circle bundle of O(3, 2) over CP^2 x CP^1, with a Calabi-Yau cone and effective group (U(3) x U(2))/U(1)_Y; its identification with M^{111} is conditional on the recalled fibration)

The hypercharge quotient Y_{3,2} = (mu_Y^{-1}(0) cap S^9)/U(1)_Y of the 5-bar is a smooth simply connected seven-manifold, the circle bundle of O(3, 2) over CP^2 x CP^1, with a Calabi-Yau cone and effective group (U(3) x U(2))/U(1)_Y; its identification with M^{111} is conditional on the recalled fibration

Status. Theorem (Theorem 2.1, with the referee amendments), conditional on the recalled description of M^{111} as the circle bundle of class 3h_1 + 2h_2 (Fabbri et al., hep-th/9907219, which the referee could not access). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author in Python (historical/11D/checks_11d.py, sections A and B: weights, the Calabi-Yau condition, Z_6), by Terra 6.1 by hand, whose independent proofs of freeness, of the dual action and of the Sasaki-Einstein normalization are written out in the Referee proofs of lemma 32, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_quotient`. Stated as an observation in the section on the string dualities of paper/connes_kk_dictionary_note.tex ("\supp{32}"), which records that a targeted search found no statement of the hypercharge reading.

Setting. 5-bar = C^3 + C^2 with the Standard Model action on d^c in (3-bar, 1)_{1/3} and L in (1, 2)_{-1/2}; U = d^c and V = L, so 6Y = 2 on U and -3 on V. mu_Y = (2 abs(U)^2 - 3 abs(V)^2)/6 is the moment map of hypercharge. Fabbri et al. write the cone over M^{111} as the Kähler quotient of C^5 with these weights, which are six times the hypercharges of the 5-bar.

Statement.

1. Y_{3,2} is a smooth, simply connected seven-manifold, the circle bundle of O(3, 2) over CP^2 x CP^1 up to orientation, and its cone is the Kähler quotient C^5 // U(1)_Y.
2. The cone is Calabi-Yau (Gorenstein, sum of weights 3 x 2 + 2 x (-3) = 0) because U(1)_Y lies in SU(5).
3. The group acting effectively is (U(3) x U(2))/U(1)_Y; its subgroup S(U(3) x U(2))/U(1)_Y is PSU(3) x SO(3), and the overall phase of U(5) is the Reeb (R-symmetry) circle. So the isometry group SU(3) x SU(2) x U(1) of the Witten space with Killing spinors is the centralizer of hypercharge in U(5) divided by hypercharge, and hypercharge itself is no longer an isometry.
4. The holomorphic functions are spanned by the monomials of degree (3k, 2k) in (U, V), which as functions transform as Sym^{3k}(3) x Sym^{2k}(2); for k = 1 they form the (10, 3) of Fabbri et al.

Scope. The identification with M^{111} uses the classification of principal circle bundles over CP^2 x CP^1 by their Euler class together with the recalled fibration. The natural quotient metric has transverse Ricci constant 10 on both factors and the Sasaki-Einstein metric needs 8, so the two agree only after the standard transverse homothety of factor 5/4; a Ricci-flat cone metric needs a separate argument. The action on holomorphic functions is dual to the action on U, and the convention must be fixed when the 10 and the 10-bar are identified.

Proof. Lemma 32, Theorem 2.1, and the Referee proofs there: on the level set abs(U)^2 = 3/5 and abs(V)^2 = 2/5, so the action (lambda^2 U, lambda^{-3} V) is free; the characters of T^2/U(1)_Y are generated by (3, 2); simple connectivity follows from the homotopy sequence of U(1) -> S^5 x S^3 -> Y_{3,2}.

What it leaves open. The spinor modes on Y_{3,2}, whose isometry group may lift to a double cover (32_z6_kernel_tensor_modes.md). The placement of hypercharge as the Betti vector of Y_{3,2} is Proposal 3.1 of lemma 39 and has no file.

Part S136 (Every monomial in U, V and every state of Lambda^*(5-bar) satisfies 2t + 3d + 6Y = 0 mod 6, the U(1)_Y-neutral monomials have zero triality and integer weak isospin, and no tensor Kaluza-Klein mode on Y_{3,2}, on an invariant background, is a colour triplet or a weak doublet)

Every monomial in U, V and every state of Lambda^*(5-bar) satisfies 2t + 3d + 6Y = 0 mod 6, the U(1)_Y-neutral monomials have zero triality and integer weak isospin, and no tensor Kaluza-Klein mode on Y_{3,2}, on an invariant background, is a colour triplet or a weak doublet

Status. Proposition (Proposition 2.4). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author in Python (historical/11D/checks_11d.py, section B), by Terra 6.1 by hand, who extended item 2 to all monomials in U, V and their conjugates (Referee proofs of lemma 32), and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_z6` (depending on `d11_quotient`). Stated in the section on the string dualities of paper/connes_kk_dictionary_note.tex ("no bosonic Kaluza-Klein mode is a triplet or a doublet", "\supp{32}").

Setting. t is the triality (1 for 3, -1 for 3-bar) and d = 2j mod 2; U and V are as in 32_witten_space_hypercharge_quotient.md, with (t, d, 6Y) = (-1, 0, 2) on U and (0, 1, -3) on V.

Statement.

1. Every monomial in U and V, and every state of the exterior algebra Lambda^*(5-bar), satisfies 2t + 3d + 6Y = 0 mod 6.
2. The U(1)_Y-neutral monomials in U, V and their conjugates have zero triality and d = 0.
3. Every tensor field on Y_{3,2} carries a representation of (U(3) x U(2))/U(1)_Y, on which SU(3) x SU(2) acts through PSU(3) x SO(3). So no tensor Kaluza-Klein mode on Y_{3,2}, on a background invariant under the displayed action, is a colour triplet or a weak doublet.

Item 1 is the statement that the Standard Model group is (SU(3) x SU(2) x U(1))/Z_6 = S(U(3) x U(2)); in the quotient realization the Z_6 is the kernel of SU(3) x SU(2) -> Isom(Y_{3,2}).

Scope. Tensor fields on a background invariant under the displayed action. Spinor modes are open, since the isometry group may lift to a double cover.

Proof. Lemma 32, Proposition 2.4, and the Referee proofs there: the congruence holds on U and V, is additive and is reversed by conjugation; neutrality 2 Delta_U = 3 Delta_V gives 3 divides Delta_U and 2 divides Delta_V; the effective group of Theorem 2.1 meets SU(3) x SU(2) in the centre Z_6.

What it leaves open. Whether fermionic Kaluza-Klein modes on Y_{3,2} can carry half-integer weak isospin or triality. States with fractional hypercharge are charged under the quotient circle and cannot be functions or tensors on Y_{3,2}, which points to states outside its Kaluza-Klein spectrum for the quarks and leptons (39_nogo_three_form_colour.md).

Part S137 (Among blocks of 12 states made of whole Dirac species, exactly one up-type quark and the three neutrinos together keep both spectral-action values cos^2 theta_NC = 5/8 and kappa_em/kappa_s = 8/3 on removal)

Among blocks of 12 states made of whole Dirac species, exactly one up-type quark and the three neutrinos together keep both spectral-action values cos^2 theta_NC = 5/8 and kappa_em/kappa_s = 8/3 on removal

Status. Proposition (Proposition 4.1). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand; checked by its author by hand, by Terra 6.1 by hand and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_blocks` (depending on `l21_removal`). Proposition 1 of paper/generations_note.tex, with the counting after it ("\supp{33}").

Setting. The 96 states of H_F, three generations with nu_R. A block of whole Dirac species contains all colours, both chiralities, particles and antiparticles: 12 states for a quark species and 4 for a lepton species. The totals (sum T_3 Q, sum Y Q; Q^2, colour) = (6, 10; 32, 12) are those of Proposition 21.1, in the quadratic spectral action of lemma 18, with cos^2 theta_NC = (sum Y Q)/(sum T_3 Q + sum Y Q).

Statement.

1. A 12-state block of whole species is one quark species or three lepton species.
2. Removing a quark keeps both values only for an up-type quark, whose contribution (1, 5/3; 16/3, 2) is proportional to the totals.
3. Three lepton species with k charged leptons give cos^2 theta_NC = (10 - 3k/2)/(16 - 2k), which is 5/8 only for k = 0.

So F1 = H_F minus the top block and F2 = H_F minus the neutrino block are the two kinds of solutions, a charged one and a neutral one.

Scope. The classical bosonic trace of lemma 21 with three generations, and blocks made of whole species only.

Proof. Lemma 33, Proposition 4.1: Proposition 21.1 for the quarks; each charged lepton contributes (1/2, 3/2; 4, 0), and neutrinos contribute zero to all four sums.

What it leaves open. A structure that exchanges the two blocks, which lemma 21 asked of the M2/M5 duality: the colour obstructions are 33_nogo_colour_obstructions.md and the placement in E8 is 33_e8_two_gradings_placement.md. At the 9 label the neutrino block decouples through zero charge and the top block through infinite mass (lemma 33, the reading at the 9 label).

Part S138 (Two commuting gradings of order three on e8, by SU(9) and by the family SU(3), place the left-handed 10 and H_u of a generation in the C_3 sector and (d^c, L), N and (D^c, H_d), nu^c in the C_6 and gravity sectors according to a sign t, so that the 96 states split as 45 + 45 + 6 for t = +1/2 and 33 + 33 + 30 for t = -1/2)

Two commuting gradings of order three on e8, by SU(9) and by the family SU(3), place the left-handed 10 and H_u of a generation in the C_3 sector and (d^c, L), N and (D^c, H_d), nu^c in the C_6 and gravity sectors according to a sign t, so that the 96 states split as 45 + 45 + 6 for t = +1/2 and 33 + 33 + 30 for t = -1/2

Status. Theorem (Lemma 4.A and Theorem 4.4, with the referee amendment). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand from branching rules; checked by its author by hand, by Terra 6.1 by hand, whose independent proof of the table and of the grades is written out in the Referee proofs of lemma 33, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_e8_grading`. Lemma 3 and Theorem 4 of paper/generations_note.tex ("The proofs are in \supp{33}").

Setting. e8 = sl(9) + Lambda^3 C^9 + Lambda^6 C^9 (80 + 84 + 84), the grading b = 0, 1, 2 of an inner automorphism of order three; the names gravity, C_3 and C_6 for the three pieces follow West's level decomposition of e8(8), recalled. The family grading a = 0, 1, 2 is given by the centre s_f of the SU(3) of (SU(3) x E6)/Z_3 in E8, with C^9 = (3, 1)_{-2} + (1, 6)_1 under SU(3) x SU(6) x U(1). SU(2)_I is the commutant of the Georgi-Glashow SU(5) in E6, with Cartan T_3, and t = +-1/2.

Statement.

1. (Lemma 4.A) s_f commutes with the SU(9) grading, its centralizer is (SU(3) x E6)/Z_3, and b = a + 2 T_3 mod 3; in the row a = 1 the pieces have dimensions 18 (b = 0), 45 (b = 1) and 18 (b = 2).
2. (Theorem 4.4) For the generation 16 in the 27 of family grade a = 1, in the left-handed basis, the 10 = (Q, u^c, e^c) and the 5 = (D, H_u) lie in g_{1,1}, the C_3 sector; (d^c, L) and N lie in g_{1,1+2t}, and (D^c, H_d) and nu^c in g_{1,1-2t}; antiparticles lie in g_{2,-b}. The two signs of t differ by the Weyl reflection of SU(2)_I.
3. With conjugates, the top block has (C_3, C_6, gravity) = (6, 6, 0) and the neutrino block (3, 3, 6), for both signs of t, and the 96 states of H_F split as 45 + 45 + 6 for t = +1/2 and 33 + 33 + 30 for t = -1/2.

Scope. t_L and t^c lie in b = 1 in the left-handed basis, and t^c is an antitop field, so the two chiralities of the top particle lie in different sectors. The names C_3 and C_6 express an algebraic grading, and a physical M2/M5 identification requires a field construction. Selecting t = +1/2, where nu^c is the only state in the gravity sector, is a proposal.

Proof. Lemma 33, Lemma 4.A and Theorem 4.4, and the Referee proofs there: Lambda^3 of (3, 1)_{-2} + (1, 6)_1 is (1, 1)_{-6} + (3-bar, 6)_{-3} + (3, 15)_0 + (1, 20)_3; the column a = 0 sums to 8 + 78 and the row a = 1 to 3 x 27; in the E6 charge normalization, T_3 = (psi + chi)/8 vanishes on the 10 and the 5 and is +-1/2 on the doublet members.

What it leaves open. A finite real spectral triple on the graded e8 with D_F = ad(Higgs), its order zero and order one, and its comparison with C + H + M_3(C); the E11 level decomposition beyond E8. The couplings that the gradings allow are 33_yukawa_grades.md, and the exchange of the two blocks within this structure is excluded by 33_nogo_graded_e8_exchange.md.

Part S139 (Under colour F1 = 10 (3 + 3-bar) + 24 (1) and F2 = 12 (3 + 3-bar) + 12 (1), so no colour-commuting unitary exchanges the two families of 84, and for every homomorphism SU(3) -> SU(9) the exterior cube Lambda^3 C^9 has the colour content of neither family)

Under colour F1 = 10 (3 + 3-bar) + 24 (1) and F2 = 12 (3 + 3-bar) + 12 (1), so no colour-commuting unitary exchanges the two families of 84, and for every homomorphism SU(3) -> SU(9) the exterior cube Lambda^3 C^9 has the colour content of neither family

Status. No-go theorem in its scope (Propositions 4.2(a) and 4.3, with the referee amendments). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand; checked by its author by hand, by Terra 6.1 by hand, whose independent proofs are written out in the Referee proofs of lemma 33, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). The first version of the "exactly" clause of Proposition 4.3 omitted the trivial representation 9 (1), whose cube is 84 (1); the referee's counterexample is recorded in lemma 33, and the statement below includes that case. Ledger row `d11_colour_families` (depending on `d11_colour84`). Items (a) and (c) of Proposition 2 of paper/generations_note.tex, with Proposition 4.2(b) cited after it ("\supp{33}").

Setting. F1 = H_F minus the top block T and F2 = H_F minus the neutrino block N, with colour acting through A_F; a complex nine-dimensional representation C^9 of SU(3), for any homomorphism SU(3) -> SU(9).

Statement.

1. (Proposition 4.2(a)) As colour representations F1 = 10 (3 + 3-bar) + 24 (1) and F2 = 12 (3 + 3-bar) + 12 (1). No unitary of H_F commuting with colour maps F1 onto F2, and no automorphism of the finite spectral triple of the Standard Model does.
2. (Proposition 4.2(b), in the scope of the referee note) A bijection of the top block onto the neutrino block that preserves the tensor factors, identifying the colour factor of the top with the family factor of the neutrinos, exchanges the colour index (r, g, b) with the generation index (e, mu, tau). Linearity in each index alone does not imply this, since a unitary may entangle the three-dimensional index with the index of chirality and conjugation.
3. (Proposition 4.3) Lambda^3 C^9 is the colour content of neither family. It contains only 1, 3 and 3-bar exactly when C^9 is 9 (1), 3 + 6 (1) or 3-bar + 6 (1), and it is then 84 (1), 15 (3) + 6 (3-bar) + 21 (1) or its conjugate.

Scope. The full exterior cube of a complex nine-dimensional linear representation, and the stated colour action on F1 and F2. With the real case, 39_nogo_three_form_colour.md, this excludes the colour-equivariant route through Lambda^3 of a linear nine-dimensional space.

Escapes, as the lemma states them. Other dimensions, non-equivariant identifications, nonlinear or brane constructions, and maps that do not preserve the colour action.

Proof. Lemma 33, Propositions 4.2 and 4.3, and the Referee proofs there: an intertwiner preserves isotypic components, with 24 singlets against 12; the irreducibles of dimension at most nine are 1, 3, 3-bar, 6, 6-bar and 8, and the case analysis finds a 6, 6-bar, 8, 10, 10-bar, 15 or 15-bar in every case except 3 + 6 (1), its conjugate and 9 (1), which have unequal numbers of 3 and 3-bar or no triplet.

What it leaves open. The result points to structures in which colour and generation can be exchanged: the colour-family transposition of four families in E8 (34_colour_family_transposition.md) and the flip of a four-family finite geometry (34_four_family_flip.md).

Part S140 (In the aligned three-family embedding in e8 with its two gradings, no complex automorphism preserving or inverting the family grading maps the top block onto the neutrino block, and the Chevalley involution maps each block to its conjugate)

In the aligned three-family embedding in e8 with its two gradings, no complex automorphism preserving or inverting the family grading maps the top block onto the neutrino block, and the Chevalley involution maps each block to its conjugate

Status. No-go theorem in its scope (Proposition 4.6, with the referee amendment). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand; checked by its author by hand, by Terra 6.1 by hand, whose independent proof, with the block argument by Schur's lemma, is written out in the Referee proofs of lemma 33, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_m2m5_is_j` (depending on `d11_e8_grading`). Proposition 6 of paper/generations_note.tex, with the block argument as the proof of its item (3).

Setting. The two gradings of 33_e8_two_gradings_placement.md, with the aligned embedding 3_f (x) 16 of three families in g_1 = 3_f (x) 27; T and N are the top and neutrino blocks.

Statement.

1. No automorphism of e8 preserving the family grading maps the C_3 sector g_{1,1} (dimension 45) onto the C_6 sector g_{1,2} (dimension 18).
2. The Chevalley involution theta, the element -1 of W(E8), maps g_{a,b} to g_{-a,-b} and each placed state to its conjugate, so it preserves F1 and F2. It is complex-linear; with the coefficient conjugation kappa of a real Chevalley basis, J = theta kappa is antilinear with J^2 = 1 and J gamma = -gamma J for the grading +1, -1 on family grades 1, 2.
3. No complex automorphism phi with phi(s_f) = s_f^{+-1} maps T onto N.
4. The Weyl reflection of SU(2)_I fixes s_f and replaces b by 2a - b, so on the matter it fixes the C_3 sector and exchanges the C_6 and gravity sectors.

Scope. Complex-linear automorphisms of the specified graded e8, the aligned 3_f (x) 16 embedding and the blocks T and N. The identification of theta, or of J, with the M2/M5 exchange is a proposal, and the order conditions and J D = D J of a finite triple remain to be established.

Escapes, as the lemma states them. Automorphisms that neither preserve nor invert the family grading, such as the colour-family exchanges inside the SU(3)^4 subgroup of E8; other embeddings or state spaces; dualities not represented by such an automorphism.

Proof. Lemma 33, Proposition 4.6, and the Referee proofs there. For item 3, phi preserves g_0 = sl_3 + e_6 and its two simple ideals, so by Schur's lemma its restriction to g_1 = 3_f (x) 27 is a multiple of some A (x) B; it maps a subspace supported on one family line to a subspace supported on one family line, while the grade-one part of N spans 3_f.

What it leaves open. The result points to automorphisms outside the pair of gradings. For aligned families 34_nogo_families_not_isometric.md also excludes every element of W(E8) mapping F1 onto F2, and with four families the transposition of 34_colour_family_transposition.md exchanges the blocks.

Part S141 (If the cubic couplings come from the structure constants of e8 and both gradings are symmetries, the up-type Yukawa couplings have all three fields in the C_3 sector, and the down-type, charged-lepton and Dirac-neutrino couplings take one field from each sector)

If the cubic couplings come from the structure constants of e8 and both gradings are symmetries, the up-type Yukawa couplings have all three fields in the C_3 sector, and the down-type, charged-lepton and Dirac-neutrino couplings take one field from each sector

Status. Proposition (Proposition 4.5, with the referee amendment), conditional on the two gradings being symmetries and on the cubic couplings being structure constants of e8, as in heterotic and Hořava-Witten compactifications. Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand; checked by its author by hand, by Terra 6.1 by hand, whose independent proof with the family tensor is written out in the Referee proofs of lemma 33, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_yukawa_brackets` (depending on `d11_e8_grading`). Proposition 5 of paper/generations_note.tex.

Setting. The placement of 33_e8_two_gradings_placement.md, with H_u in the 5 in g_{1,1} and H_d in the 5-bar of the 10 of E6, in g_{1,1-2t}; a coupling needs grades summing to (0, 0) mod 3.

Statement. For either sign t:

1. Q u^c H_u has all three fields in the C_3 sector.
2. Q d^c H_d, L e^c H_d and L nu^c H_u have one field in each of the three sectors.
3. In Dirac form the top mass is ad() : g_{1,1} -> g_{2,2}, the wedge Lambda^3 -> Lambda^6 with the Higgs trivector, and the Dirac-neutrino mass is a bracket through the gravity sector, Lambda^6 -> sl(9) for t = +1/2 and sl(9) -> Lambda^3 for t = -1/2.

Scope. These are necessary grade conditions; a particular bracket or four-dimensional coupling may still vanish, and they establish neither a physical top mass nor its family hierarchy. In the literal 3_f (x) 27 realization with exact SU(3)_f the family tensor is epsilon_{ijk}, and Y_{ij} = epsilon_{ijk} h_k has Y^* Y = norm(h)^2 I - h h^*, two equal nonzero singular values and zero on h, so the exact one-Higgs family tensor gives no isolated top mass.

Proof. Lemma 33, Proposition 4.5, and the Referee proofs there: the b-grades are (1, 1, 1) for the up type and {1, 1 + 2t, 1 - 2t} = {1, 2, 0} for the others, with family grades 1, 1, 1; the Dirac masses pair a left-handed field with the conjugate of the right-handed one through the Killing form between g_{a,b} and g_{-a,-b}.

What it leaves open. The reading that the top takes its mass from the metric-independent C_3 bracket is a proposal of lemma 33. A compactification in which family labels are cohomology labels and couplings are overlap integrals may evade the triplet-tensor restriction, and its mode functions and intersections must be supplied before the physical Yukawa matrix is identified; the M-grading survives the standard embedding, with s_M = s_f exp(2 pi i (2/3) T_3).

Part S142 (With four families, M = -P, where P exchanges the colour and family coordinates and the two weak coordinates, is an involution in W(E8) that preserves SU(2)_L and the 128 roots and exchanges the up-type quark of family 0 with the neutrinos of families 1, 2 and 3)

With four families, M = -P, where P exchanges the colour and family coordinates and the two weak coordinates, is an involution in W(E8) that preserves SU(2)_L and the 128 roots and exchanges the up-type quark of family 0 with the neutrinos of families 1, 2 and 3

Status. Proposition (Proposition 5.2, with the referee clarification). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand; checked by its author by hand, by Terra 6.1 by hand (Referee proofs of lemma 34) and by Claude Sonnet 5.5 in Python (M = -P on all 240 roots) (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_transposition`. Proposition 9 of paper/generations_note.tex.

Setting. The 128 half-integer roots of E8 as four families with nu^c and their conjugates, in the coordinates (chi | w | phi) of 34_nogo_families_not_isometric.md; family 0 has the sign pattern (+++), the families 1, 2, 3 and the quark colours have two and one minus signs, and the lepton colour is the pattern -(+++).

Statement. M = -P is an involution in W(E8) that preserves the 128 roots and SU(2)_L, reverses T_3 of SU(2)_R, and acts on the up-type and down-type states as the transposition of the Pati-Salam colours (r, g, b, lepton) with the families (1, 2, 3, 0), the colour c going with the family -c:

1. the up-type quark of family 0 (t_L and t^c, three colours, with conjugates) goes onto the neutrinos of families 1, 2, 3 (nu_L and the conjugate of nu^c), and back;
2. the down-type quark of family 0 goes onto the charged leptons of families 1, 2, 3;
3. nu_L and e_L of family 0 are fixed, nu^c and e^c of family 0 go to their negatives, so these species blocks are fixed setwise, and the quarks of families 1, 2, 3 go to quarks of families 1, 2, 3 with colour and family transposed.

So the sets 128 - T(0) and 128 - N(1, 2, 3), of 116 roots each, are exchanged.

Scope. A Weyl element acts on an E8 holonomy by conjugation, so it exchanges colour and family physically only relative to a structure that tells them apart, such as this placement of H_F (lemma 36, section 5, item 2). The search of lemma 34 found further signed permutations with the same effect on the two blocks, and M = J P with J = -1 the Chevalley involution of 33_nogo_graded_e8_exchange.md.

Proof. Lemma 34, Proposition 5.2, and the Referee proofs there: P is an even permutation of coordinates and -1 lies in W(E8); M maps (chi | w_1 w_2 | phi) to (-phi | -w_2 -w_1 | -chi), so t_L(c) = (c | -+ | +++) goes to nu_L(-c) and t^c to -nu^c(-c).

What it leaves open. A finite spectral triple, or a Hořava-Witten configuration, with four families in which M is a symmetry and the fourth family decouples in a controlled way, so that the three-family statement of lemma 21 follows as a limit; whether the fourth family can be made heavy by the mechanism of the top. The readings of lemma 34, three families as the obstruction and M as the exchange of the two heavy directions of the algebra chain, are proposals.

Part S143 (In a four-family finite geometry with KO-6 signs, the flip of colour and family through a unitary identification is a unitary involution commuting with gamma and J that conjugates the top Dirac block into the Dirac blocks of the neutrinos of families 1, 2 and 3, so the two heavy limits have conjugate light spaces of 116 states)

In a four-family finite geometry with KO-6 signs, the flip of colour and family through a unitary identification is a unitary involution commuting with gamma and J that conjugates the top Dirac block into the Dirac blocks of the neutrinos of families 1, 2 and 3, so the two heavy limits have conjugate light spaces of 116 states

Status. Proposition (Proposition 5.3, with the referee amendment that the identification iota be unitary, or unitary up to a common scalar). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand; checked by its author by hand, by Terra 6.1 by hand, whose example iota(ell) = 2 e_0 shows that an arbitrary isomorphism does not make F unitary (Referee proofs of lemma 34), and by Claude Sonnet 5.5 in Python (unitarity, F^2, commutation, conjugation of the blocks, F against M) (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_flip` (depending on `d11_transposition`). Proposition 10 of paper/generations_note.tex.

Setting. H_f = (C^2_L + C^2_R) (x) C^4_c (x) C^4_f and H = H_f + conj(H_f), with J the antilinear exchange of the two summands and gamma = +1 on C^2_L, -1 on C^2_R in H_f and its negative on conj(H_f), the signs of KO-dimension six. iota: C^4_c -> C^4_f is unitary, sends the lepton line ell to e_0 and C^3 onto the span of e_1, e_2, e_3, and F(x (x) y) = iota^{-1}(y) (x) iota(x) on C^4_c (x) C^4_f, extended by the identity on the chirality factor and to conj(H_f) by J F J^{-1}.

Statement.

1. F is unitary, F^2 = 1, and F commutes with gamma and J.
2. For T_top = Y (x) P_q (x) P_0 and T_nu = Y (x) P_ell (x) (1 - P_0), with Y the map of the up state of C^2_L to the up state of C^2_R and P_ell, P_q = 1 - P_ell, P_0 the lepton, quark and family-0 projectors, F T_top F^{-1} = T_nu; likewise for the down-type block of family 0 and the charged leptons of families 1, 2, 3.
3. For D commuting with F, D + t T_top and D + t T_nu are conjugate by F for every t, and their heavy limits have conjugate light spaces of 116 states. The light algebras correspond when the algebra is F-invariant; the Pati-Salam and Standard Model algebras act on C^4_c alone and are not F-invariant.

Scope. F equals M of 34_colour_family_transposition.md on the left-handed states and M followed by the particle-antiparticle exchange on the right-handed ones, where M reverses gamma on 64 states, so F lies outside W(E8). F maps the top Yukawa block to Dirac blocks of neutrinos, and it does not map the Majorana block to a Dirac block, since the Majorana block joins H_f to conj(H_f).

Proof. Lemma 34, Proposition 5.3: F acts on C^4_c (x) C^4_f only, conjugating a (x) b to iota^{-1} b iota (x) iota a iota^{-1}, so P_q (x) P_0 goes to P_ell (x) (1 - P_0); the two Dirac blocks have rank 12 and kernels of dimension 128 - 12 = 116.

What it leaves open. A finite geometry whose generation space is C^4 with an algebra exchangeable with M_4(C), and a Dirac operator compatible with the exchange (reading 2 of lemma 34, a proposal); a Pati-Salam finite geometry on C^4_colour (x) C^4_family in which the flip exchanges the sigma field with the top Yukawa direction (reading 4).

Part S144 (With four families, no isometry of R^8 maps the 44 roots of the fourth family with the top block onto the 44 roots of the fourth family with the neutrino block: up to family permutations their degree multisets are {10^16, 12^12, 13^16} and {10^24, 13^16, 16^4})

With four families, no isometry of R^8 maps the 44 roots of the fourth family with the top block onto the 44 roots of the fourth family with the neutrino block: up to family permutations their degree multisets are {10^16, 12^12, 13^16} and {10^24, 13^16, 16^4}

Status. No-go theorem in its scope (the strengthening in Reading 3b (b) of lemma 34, a computer-assisted statement inside a reading; the ledger records it as a proposition). Lemma 34 credits the computation to Claude Sonnet 5.5; the ledger records Claude Opus 5.5 as author and the Sonnet computation as the check (Python, degree multisets over W(D8) and family permutations, exact integers, not kept). Ledger row `d11_44_sets` (depending on `d11_no_isometry`). Proposition 11 of paper/generations_note.tex ("Python, exact integers").

Setting. The 128 roots of four families (34_colour_family_transposition.md); S_T is the fourth family together with the top block T and S_N the fourth family together with the neutrino block N, 44 roots each; d counts the roots of the set at inner product +1 with a given root.

Statement. Up to family permutations the degree multisets of S_T and S_N are {10^16, 12^12, 13^16} and {10^24, 13^16, 16^4}, so no isometry of R^8 maps S_T onto S_N.

Scope. One root per state, four family slots, Euclidean isometries acting on the roots.

Escapes. Those of 34_nogo_families_not_isometric.md: maps not linear on the roots, larger groups, and states spread over several roots, which heterotic compactification uses.

Proof. Lemma 34, Reading 3b (b), the strengthening: exhaustive computation of the degrees, which are invariant under isometries; both sets have the same size and the multisets differ.

What it leaves open. With 34_nogo_colour_not_in_spin9.md this leaves the fit 44 = 32 + 12 as a count of states only. States spread over several roots, as zero modes of an SU(4) bundle are, lie outside the hypotheses (Proposal 5.5 of lemma 34, which has no file).

Part S145 (The colour SU(3) of the root table lies in no Spin(9) in SO(16) with 128 -> 44 + 84, so the split 44 + 84 of the four-family spinor is not colour-covariant and the fit 44 = 32 + 12 is numerical)

The colour SU(3) of the root table lies in no Spin(9) in SO(16) with 128 -> 44 + 84, so the split 44 + 84 of the four-family spinor is not colour-covariant and the fit 44 = 32 + 12 is numerical

Status. No-go theorem in its scope (Reading 3b, part (b), of lemma 34: a proved statement inside a reading, after an observation of the user in part (a)). Proved by Claude Opus 5.5 on 4 November 2026 by hand; checked by its author by hand and by Claude Sonnet 5.5 by hand. Ledger row `d11_spin9_colour` (depending on `d11_colour84`). Proposition 12 of paper/generations_note.tex, with the remark after it on the so(9) of e8.

Setting. Four families: the 128 half-integer roots of E8, the chiral spinor of SO(16), with colour the regular SU(3) of the root table, under which the vector 16 of SO(16) is 3 + 3-bar + 10 (1). Spin(9) is

embedded in SO(16) through its spinor, in the class with 128 -> 44 + 84, the graviton and the three-form of eleven-dimensional supergravity (recalled; the other chirality of the embedding gives an irreducible 128).

Statement.

1. Under such a Spin(9) the vector 16 of SO(16) is the spinor of Spin(9), which restricted to an SU(3) inside Spin(9) is 16 (1), 2 (3 + 3-bar) + 4 (1) or 8 + 8, by the three real nine-dimensional representations of 39_nogo_three_form_colour.md. So colour lies in no such Spin(9).
2. The stabilizer of the split 44 + 84 in so(16) is spin(9), so the split is not colour-covariant, and no set of 84 roots spans the 84.
3. Colour, with 248 = 8 + 78 (1) + 27 (3 + 3-bar), lies in no so(9) of e8 with 248 = 36 + 44 + 84 + 84, that of 33_e8_two_gradings_placement.md included, since the 44 contains a 6 or a 27 under any SU(3) in such an so(9).

Scope. The regular colour SU(3) of the root table, in this class of Spin(9).

Escapes, as the lemma states them. Other colour embeddings; identifications that are not equivariant.

Proof. Lemma 34, Reading 3b (b): the restrictions in item 1 differ from 3 + 3-bar + 10 (1); a subalgebra between spin(9) and so(16) would be a submodule of 36 + 84, and so(16) is irreducible on the 128; a partition of roots has the rank-8 Cartan subalgebra in its stabilizer.

What it leaves open. With four families 128 - 84 = 44 = 32 + 12, so removing either family of 84 leaves the fourth family with the top block or with the neutrino block (part (a)); M of 34_colour_family_transposition.md carries neither set of 44 onto the other, and no isometry does (34_nogo_44_sets_not_isometric.md). How the 96 states of the three-family frame sit against the Spin(9) conjugate to the so(9) of e8 (34_spin9_conjugate_so9.md) has not been worked out.

Part S146 (In the aligned three-family embedding in the roots of E8, the two families of 84 have equal numbers of pairs at each inner product, and no isometry of R^8 maps F1 onto F2, so no element of W(E8) does and no isometry of H_F maps the top block onto the neutrino block)

In the aligned three-family embedding in the roots of E8, the two families of 84 have equal numbers of pairs at each inner product, and no isometry of R^8 maps F1 onto F2, so no element of W(E8) does and no isometry of H_F maps the top block onto the neutrino block

Status. No-go theorem in its scope (Proposition 5.1). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand, with an exploratory search in Python (exact fractions; rerun on 4 Nov 2026, not kept); checked by its author by hand, by Terra 6.1 by hand, whose independent neighbour table is written out in the Referee proofs of lemma 34, and by Claude Sonnet 5.5 in Python (the 96 roots rebuilt, degrees, pair counts and the 12 slot choices) (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_no_isometry` (depending on `d11_blocks` and `l21_counts`). Proposition 7 of paper/generations_note.tex, with a proof sketch ("\supp{34}").

Setting. The roots of E8 in R^8, with the 128 half-integer roots as the chiral spinor (16, 4) + (16-bar, 4-bar) of SO(10) x SU(4) in SO(16); a root, written (chi | w | phi), carries the signs of the three colour, two weak and three family coordinates. H_F is the 96 roots of three aligned family slots, T the 12 roots of the up-type quark of one family f_t with conjugates, N the 12 roots of the three neutrinos with conjugates, F1 = H_F - T and F2 = H_F - N. d_S(v) counts the roots of S at inner product +1 with v.

Statement.

1. F1 and F2 both have 42 pairs at inner product -2, 816 at -1, 1812 at 0 and 816 at +1.

2. d = 22 on every root of H_F. In F2, d = 20 on the 36 up-type quark roots and 19 on the 48 others. In F1, d = 16 on the 4 neutrino roots of the top's family, 22 on the other 8 neutrino roots, 20 on the 24 remaining up-type quark roots and 19 on the 48 others.
3. No isometry of R^8 maps F1 onto F2.

Scope. The roots of H_F through this embedding, unique up to W(E8) for the aligned (16, 3) + (16-bar, 3-bar); complete species blocks; Euclidean isometries acting on the roots. The neutrino of the top's own family carries the obstruction, as the only state whose neighbourhood changes by six.

Escapes, as the lemma states them. Maps that are not linear on the roots (bijections of states with no geometric meaning), other realizations of H_F in a larger group, and a fourth family (34_colour_family_transposition.md).

Proof. Lemma 34, Proposition 5.1, and the neighbour table of its Referee proofs: two half-integer roots are at +1 when their signs agree in six of the eight places; both sets are closed under negation with centroid zero, so an isometry would fix the origin and preserve d, and F2 has no root with d = 16 or 22.

What it leaves open. The three-slot assignments, aligned or misaligned, are 34_nogo_three_slot_assignments.md. States spread over several roots, as zero modes of an SU(4) bundle are (Proposal 5.5 of lemma 34, which has no file), lie outside the hypotheses.

Part S147 (For none of the 192 assignments of the up-type quarks, down-type quarks, charged leptons and neutrinos to three of the four family slots, aligned or misaligned, does an isometry of R^8 map S - T onto S - N)

For none of the 192 assignments of the up-type quarks, down-type quarks, charged leptons and neutrinos to three of the four family slots, aligned or misaligned, does an isometry of R^8 map S - T onto S - N

Status. No-go theorem in its scope (Proposition 5.4, with the correction of its neighbour counts). Found by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by exhaustive computation (Python, exact integers, rerun twice on 4 Nov 2026, not kept). Terra 6.1 gave an analytic proof for all 192 assignments without enumeration, written out in the Referee proofs of lemma 34, and showed that the universal neighbour counts of the first version are false for misaligned slots, for instance A = C = {0, 1, 2} and B = D = {0, 1, 3}; Claude Sonnet 5.5 reran all 192 configurations in Python (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). The conclusion holds on rerun, and the correction is recorded in lemma 37, section on corrections. Ledger row `d11_misaligned` (depending on `d11_no_isometry`). Proposition 8 of paper/generations_note.tex, with the corrected degrees after it ("\supp{34}").

Setting. The roots of 34_nogo_families_not_isometric.md, with the top in slot 0 and the up-type quarks, down-type quarks, charged leptons and neutrinos each in three of the four slots, independently, in slot sets A (containing 0), B, C and D. This includes the aligned families and the configurations of the 9 label, where SU(2)_L is off and does not tie the slots of up-type and down-type quarks. S is the configuration, T the top block and N the neutrinos, each with conjugates.

Statement.

1. S - N has degrees d in {13, 17, 19, 20}: 20 on the up-type quarks, 17 + 2 1_C(f) on the down-type quarks of slot f and 13 + 6 1_B(f) on the charged leptons of slot f, so only 19 and 20 occur in the 48 configurations with B = C.
2. Each neutrino root of slot f in S - T has d = 16 + 6 (1_A(f) - 1_{0}(f)), which is 16 or 22.
3. So for none of the 3 x 4^3 = 192 configurations does an isometry of R^8 map S - T onto S - N.

Scope. One root slot per species, whole colour multiplets, the specific half-root realization and Euclidean isometries. So four slots cannot be reduced to three while keeping the exchange as an isometry of the roots.

Escapes, as the lemma states them. General zero modes of a non-split SU(4) bundle, which are sections involving all four slots, and maps without an isometric action on the roots.

Proof. Lemma 34, Proposition 5.4, and the Referee proofs there: the neighbour table per slot gives the degrees above; both sets are closed under negation, so an isometry would preserve degrees, and the values 16 and 22 are absent from S - N.

What it leaves open. Proposal 5.5 of lemma 34, which has no file: in heterotic compactifications with SO(10) x SU(4) in E8 the families are zero modes of a rank-four bundle, with net number abs(c_3(V))/2 by the index. For split bundles with n_i >= 0 and sum 3 the degree multisets still differ in all 40 cases checked (second check), and a non-split bundle with index 3 is the case to compute; exactly three 16 zero modes also needs H^1(V^*) = 0, since the index counts net chirality.

Part S148 (The so(9) fixed by the Chevalley involution on sl(9) is conjugate in E8 to the Spin(9) of the four-family count, with k = so(9) + 84 and p = 44 + 84, the 44 inside the gravity sector, and both are SO(9) inside E8)

The so(9) fixed by the Chevalley involution on sl(9) is conjugate in E8 to the Spin(9) of the four-family count, with k = so(9) + 84 and p = 44 + 84, the 44 inside the gravity sector, and both are SO(9) inside E8

Status. Proposition (Proposition 5.6 of lemma 34, stated in Reading 3b (c)). Stated by Claude Sonnet 5.5 on 4 November 2026; proved and refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_05_21_34.md), with the weight-multiset branchings and Dynkin indices computed in Python with exact fractions (scripts not kept). Ledger row `d11_spin9_conjugate`. Stated after Proposition 12 of paper/generations_note.tex.

Setting. h a Cartan subalgebra of e8 inside the regular sl(9) of lemma 33, theta the Chevalley involution for h (-1 on h, e_alpha -> -e_{-alpha}); sigma the involution of the root table of lemma 34, +1 on the integer roots and -1 on the 128 half-integer roots, with fixed algebra so(16)_sigma; the Spin(9) of Reading 3 is the image of Spin(9) under its spinor 16 inside so(16)_sigma, in the class with 128 -> 44 + 84.

Statement.

1. theta preserves sl(9), acts on it by an outer involution, and its fixed algebra so(9)_theta is conjugate in SL(9) to the so(9) of lemma 33.
2. k_theta = so(16) as an algebra; as so(9)_theta-modules k_theta = 36 + 84 and p_theta = 44 + 84, with 44 = p_theta ∩ sl(9) in the gravity sector and the two 84 the graphs x ± theta x of Λ^3 C^9, each meeting the C_3 and C_6 sectors equally.
3. so(9)_theta acts on the vector 16 of k_theta by the spinor.
4. so(9)_theta is conjugate in E8 to the Spin(9) of Reading 3; under both, 248 = 36 + 44 + 84 + 84.
5. Inside E8 both connected subgroups are SO(9); the other spinor class, with 248 = 36 + 84 + 128, is Spin(9).

Scope. The two specific subalgebras; no claim that the decomposition 36 + 44 + 84 + 84 alone is one conjugacy class. The two placements of the top (C_3 sector of the SU(9) grading; with the fourth family in the 44 of Spin(9)) use two frames for conjugate subgroups, and no identification between the sectors is claimed.

Proof. Lemma 34, Proposition 5.6; the proof is there: -1 lies outside W(A_8), so theta is outer on sl(9); the fixed algebra of theta has dimension 120 and is so(16); Λ^2(16) = 36 + 84 forces the spinor; involutions of e8 with a 120-dimensional fixed algebra form one class, and the spinor embeddings with 128 -> 44 + 84

one class under Spin(16)/Z_2, since det A = -1 would exchange the half-spinors; the tensor modules 36, 44, 84 see no centre.

What it leaves open. A second independent reading. How the 96 states of the three-family frame sit against the conjugated Spin(9): for theta itself the projection of H_F to the 128 has 3 dimensions in the 44 and 45 in the 84 for t = +1/2, so the top meets only the three-form, against the numerical count of Reading 3b (a); the Spin(8) x Spin(8) frame of the commuting pair sigma, theta is where to compute it (lemma 34, What remains open).

Part S149 (On each eigenspace of d^2 in H_+, of dimension n, the unitaries commuting with the antiunitary j of an odd factorization, j^2 = s, form O(n) for s = +1 and USp(n) for s = -1, in which case n is even)

On each eigenspace of d^2 in H_+, of dimension n, the unitaries commuting with the antiunitary j of an odd factorization, j^2 = s, form O(n) for s = +1 and USp(n) for s = -1, in which case n is even

Status. Proposition (Proposition 8.1, with the referee clarification). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author by hand, by Terra 6.1 by hand, whose independent proof is written out in the Referee proofs of lemma 35, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_orientifold_sign` (depending on `l06_factorization`). Proposition 8.1 in the section on the sign of an odd factorization of paper/real_structure_note.tex ("\supp{35}"), and stated in the section on the real structure of paper/connes_kk_dictionary_note.tex ("\supp{35}").

Setting. A KO-6 finite triple factored as in lemma 06 through a grading-reversing unitary involution U, commuting with the algebra and with D, with U J = s J U: s = -1 gives KO-5 (label 9) and s = +1 gives KO-7 (label 11). The antiunitary j induced on H_+ has j^2 = s and commutes with d^2 = D^2 restricted to H_+.

Statement. Let E be an eigenspace of d^2 in H_+, of dimension n; j preserves E because it commutes with d^2.

1. If s = +1, j is a real structure on E, and the unitaries of E commuting with j form O(n).
2. If s = -1, j is a quaternionic structure on E, n is even, and the unitaries of E commuting with j form USp(n).

Scope. The commutant is defined by j and the eigenspace of d^2 alone, and it need not commute with d, with the represented algebra or with other physical structures: for j the complex conjugation of C^2 and d = diag(1, -1), a generic rotation in O(2) does not commute with d. The symmetries of the pair (j, d) are smaller, O(n_+) x O(n_-) for s = +1 and U(n/2) for s = -1, where j swaps the eigenspaces of d at +mu and -mu (second check). O(n) is disconnected, with identity component SO(n) for n >= 2. The even dimension for s = -1 is the Kramers degeneracy already stated in lemma 06.

Proof. Lemma 35, Proposition 8.1, and the Referee proofs there: for j^2 = 1 the fixed points form a real subspace E_R with real inner product, and a real orthonormal basis of E_R identifies j with complex conjugation, whose unitary commutant is O(n); for j^2 = -1, v and jv are orthogonal, and induction gives a quaternionic orthonormal basis, so the commutant is USp(n).

What it leaves open. The reading of the two groups as the Chan-Paton groups of O^- and O^+ orientifold planes, with s = +1 selected by the seesaw sector (STRATEGY T2c), is a proposal of lemma 35, and it needs an explicit Chan-Paton representation compatible with the rest of the data. On the massless light block of the five light quarks the s = +1 commutant is the SO(30) of lemma 21 (38_so30_commutant_of_j.md).

The relation between the window's degenerate levels in the bulk and the U(k) of k D8-branes in the bulk is open.

Part S150 (In the parent of lemma 22, k channels at a common angle theta give the signed eigenvalue theta/l multiplicity k for 0 < theta < pi/2 and 2k at the ends, and the lowest eigenvalue of abs(D) multiplicity 2k in the interior, 2k at theta = 0 and 4k at theta = pi/2)

In the parent of lemma 22, k channels at a common angle theta give the signed eigenvalue theta/l multiplicity k for 0 < theta < pi/2 and 2k at the ends, and the lowest eigenvalue of abs(D) multiplicity 2k in the interior, 2k at theta = 0 and 4k at theta = pi/2

Status. Proposition (Proposition 8.2, with the referee clarification on abs(D)). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) from Proposition 7.1; checked by its author by hand, by Terra 6.1 by hand (Referee proofs of lemma 35) and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_wall_multiplicity` (depending on `d11_angle_interval`). No paper cites it separately.

Setting. The parent of Theorem 22.1 with k independent channels at the same angle theta = l sigma, each with the signed spectrum {(n pi +- theta)/l : n in Z} of 40_channel_angle_interval.md.

Statement.

1. The signed eigenvalue theta/l has multiplicity k for 0 < theta < pi/2, and 2k at the ends: at theta = 0 as the zero mode of multiplicity two per channel, and at theta = pi/2 as (0 + pi/2)/l = (pi - pi/2)/l.
2. The lowest eigenvalue of abs(D) has multiplicity 2k in the interior (the values +-theta/l, k each), 2k at theta = 0, and 4k at theta = pi/2 (the values +-pi/(2l), 2k each).

Scope. At theta = 0 the two interior signed eigenvalues coalesce, and at theta = pi/2 the next level meets the window. The signed count alone does not establish a doubling of independent physical Chan-Paton labels or an orientifold projection.

Proof. Lemma 35, Proposition 8.2, and the Referee proofs there: in the interior only n = 0 in the positive progression gives +theta/l, since (n pi - theta)/l = theta/l forces theta = n pi/2; at the ends the two progressions coincide.

What it leaves open. The reading that k D8-branes on an orientifold plane carry 2k Chan-Paton labels, reproduced here with the ends of the angle interval as the walls, is a proposal, and so is a parent whose two walls carry opposite signs s, one O8^- and one O8^+. The wall bookkeeping of Remark 8.3, conditional on recalled type I' rules and recorded in the ledger as conjectural (`d11_e_walls`), has no file.

Part S151 (The two del Pezzo surfaces of degree eight have different lattices orthogonal to the canonical class: P^1 x P^1 has a class of square -2 there and dP_1 has none, so the empty far wall with E_1 cannot be assigned to either surface indifferently)

The two del Pezzo surfaces of degree eight have different lattices orthogonal to the canonical class: P^1 x P^1 has a class of square -2 there and dP_1 has none, so the empty far wall with E_1 cannot be assigned to either surface indifferently

Status. Counterexample, to the unqualified identification "dP_1 or P^1 x P^1" for the empty far wall in section 5, item 1, of lemma 36, a recalled proposal (ledger row `d11_dp_walls`, kind conjecture, with the referee amendment). Constructed by Terra 6.1 in the referee report on the 11D line (historical/reviews/2026-11-04_terra_check_11d.md, section on the del Pezzo wall correspondence), written out in the Referee proofs of lemma 36; the ledger row records checks by hand by Terra 6.1 and Claude Sonnet 5.5. No paper cites it.

Setting. dP_1 = F_1 with H^2 = 1, E^2 = -1, H . E = 0 and K = -3H + E; P^1 x P^1 with ruling classes F_1, F_2, F_1^2 = F_2^2 = 0, F_1 . F_2 = 1 and K = -2F_1 - 2F_2; dP_2 with exceptional classes E_1, E_2 and K = -3H + E_1 + E_2.

Statement.

1. On dP_1 a class D = aH + bE orthogonal to K has b = -3a and D^2 = -8 a^2, so there is no integral class of square -2.
2. On P^1 x P^1 the class F_1 - F_2 is orthogonal to K and has square -2.
3. On dP_2 the class E_1 - E_2 is orthogonal to K with square -2, and the orthogonal lattice has rank two, consistent with an A_1 factor and an additional abelian direction.

Scope. In the usual local engineering (recalled), P^1 x P^1 gives E_1 and dP_1 gives the distinct theory E-tilde_1. The computation constructs neither the five-dimensional field theory nor the type I' duality, and the map from branes to geometry stays conditional on the cited dualities, whose primary sources were unavailable to the referee.

Proof. Lemma 36, Referee proofs, Counterexample (referee) to the unqualified E_1 surface identification: D . K = -3a - b = 0 and D^2 = a^2 - b^2.

What it leaves open. The local dP_2 of the wall with the top's brane, and its relation to the Witten base CP^2 x CP^1; the occurrence of CP^2 and CP^1 in that six-dimensional base does not by itself identify the compactification with a local dP_2.

Part S152 (Given the type I' slope rule and sixteen D8-branes at x_k, the coupling function is z(x) = z(0) + c sum_k min(x/2, x_k - x/2), so z(L) = z(0) - c (8L - sum_k x_k), and infinite coupling at the far wall fixes z(0) by the sum of the brane positions)

Given the type I' slope rule and sixteen D8-branes at x_k, the coupling function is z(x) = z(0) + c sum_k min(x/2, x_k - x/2), so z(L) = z(0) - c (8L - sum_k x_k), and infinite coupling at the far wall fixes z(0) by the sum of the brane positions

Status. Proposition (Proposition 9.1, with the referee clarification and the second check), conditional on the slope rule of Polchinski and Witten (hep-th/9510169), which the second check and lemma 37 confirmed against the source. Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked

by its author by hand, by Terra 6.1 by hand (Referee proofs of lemma 36) and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). The first version stated the slope rule for e^{-phi}; the piecewise-linear function is z, with e^phi = z^{-5/6}. Ledger row `d11_dilaton_spectral` (depending on `d11_angle_interval`, `d11_angle_map` and `d11_e_walls`). No paper cites it separately; paper/connes_kk_dictionary_note.tex states the correction of its dictionary ("\supp{37}").

Setting. Type I' on [0, L] in the conformal coordinate x, with O8^- planes at the ends and sixteen D8-branes at x_1, ..., x_16. e^phi = z^{-5/6}, with z continuous and piecewise linear of slope c (8 - N(x)), N(x) the number of branes to the left of x and c > 0; the coupling diverges where z vanishes.

Statement.

1. z(x) = z(0) + c sum_k min(x/2, x_k - x/2) for x in [0, L].
2. z(L) = z(0) - c (8L - sum_k x_k), exactly in the positions.
3. z is concave, so z(0) > 0 and z(L) = 0 give z > 0 on [0, L).
4. With the dictionary x_k = (2L/pi) theta_k, theta_k = l m_k, z is z(0) plus c Tr g_x(l abs(D_F)) over the sixteen channels, g_x(theta) = min(x/2, (2L/pi) theta - x/2), and z(L) = 0 holds exactly when z(0) = (2cL/pi)(4 pi - l sum_k m_k).

Scope. The dictionary of item 4 is exact only for constant z (8 + 8), and asymptotically for large R; in general theta_k = (pi/2) W(x_k)/W(L) with W = int z^{-1/3} dx (37_angle_to_position_map.md), so the coupling function is a trace of a function of abs(D_F) only implicitly. The trace is over sixteen selected channels, and the physical D_F, with its chirality, conjugation, colour and generation multiplicities, enters only through a specified projection onto them. In the profile 15 + 1 the endpoint values 7cL and 8cL stand, and theta_t = pi/4 gives 7.358 cL in place of 7.5 cL.

Proof. Lemma 36, Proposition 9.1, and the Referee proofs there: int_0^x N(y) dy = sum_k (x - x_k)+, *and x/2 - (x - x_k)+* = min(x/2, x_k - x/2); the slopes are nonincreasing, so z is concave.

What it leaves open. Whether the type I' coupling, as a function of D_F, supplies the second scale of Proposition 1.5 in the flat family; in the scope of lemma 37 the dilaton gradient does not (37_nogo_dilaton_gradient.md). The profile alone derives no stabilizing term for the interval modulus.

Part S153 (Under U(15), the adjoint of SO(30) is Lambda^2 V + Lambda^2 V-bar + V (x) V-bar and that of USp(30) is S^2 V + S^2 V-bar + V (x) V-bar, and the colour antitriplets of Lambda^2 V and S^2 V carry the charge counts (6, 6, 3) and (3, 6, 1))

Under U(15), the adjoint of SO(30) is Lambda^2 V + Lambda^2 V-bar + V (x) V-bar and that of USp(30) is S^2 V + S^2 V-bar + V (x) V-bar, and the colour antitriplets of Lambda^2 V and S^2 V carry the charge counts (6, 6, 3) and (3, 6, 1)

Status. Proposition (Proposition 9.2, with the referee amendments), conditional on the recalled orientifold projections for its reading as strings. Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) from representation theory; checked by its author by hand, by Terra 6.1 by hand, whose independent proof of items 2 and 3 is written out in the Referee proofs of lemma 36, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_diquark_strings` (depending on `l21_so30` and `d11_wilson_top`). Cited in the section on the sign of an odd factorization of paper/real_structure_note.tex ("\supp{36}, Proposition 9.2") for the decomposition behind its table of counts.

Setting. V = C^5 (x) C^3, five light flavours in three colours, with Q = 2/3 on u, c and -1/3 on d, s, b; the 30 of lemma 21 is V + V-bar. Fifteen D8-branes on an O8 plane carry the Chan-Paton space 30, with

SO(30) on an O8^- plane, USp(30) on an O8^+ plane and U(15) off the plane (recalled).

Statement.

1. The adjoint of SO(30) is Lambda^2 V + Lambda^2 V-bar + V (x) V-bar and that of USp(30) is S^2 V + S^2 V-bar + V (x) V-bar; the first two summands are the strings between the stack and its image, and V (x) V-bar the strings with both ends on the stack.
2. Lambda^2 V = (S^2 5, 3-bar) + (Lambda^2 5, 6) and S^2 V = (S^2 5, 6) + (Lambda^2 5, 3-bar), of dimensions 105 and 120.
3. In the colour antitriplet Lambda^2 V has the charge counts (-2/3: 6, +1/3: 6, +4/3: 3) of lemma 21, and S^2 V has (-2/3: 3, +1/3: 6, +4/3: 1).
4. With the angles theta_k = l m_k, a string from channel i to the image of channel j has the spectrum (n pi +- (theta_i + theta_j))/l, and from channel i to channel j (n pi +- (theta_i - theta_j))/l.
5. A single brane has no string to its own image on an O8^- plane, since Lambda^2 of a line vanishes, and one on an O8^+ plane.

Scope. Items 4 and 5 concern the field-theory Kaluza-Klein momenta of the specified gauge-field channels for the assumed Wilson-line action; oscillator states, the strings between the top brane and the other branes, the orientifold projections and global tadpole conditions are outside them, and identifying the Chan-Paton modes with composites is an additional construction. The window of the interval parent sees the counts (3, 6, 1) also at s = +1, through the commutator with the masses (38_window_shadow_of_connection.md).

Proof. Lemma 36, Proposition 9.2, and the Referee proofs there: the swap of V (x) V is the product of the swaps of the two factors, so Lambda^2(A (x) B) = (S^2 A (x) Lambda^2 B) + (Lambda^2 A (x) S^2 B); splitting 5 = 2 + 3 by charge gives the counts; 2 x 105 + 225 = 435 and 2 x 120 + 225 = 465.

What it leaves open. The reading that the seesaw sign of Proposition 8.1 and these counts select the same orientifold type, O8^-, is a proposal, qualified by lemma 38: parent scalars with the counts (6, 6, 3) need M_15(C) at both walls (38_interval_scalar_zero_modes.md). The spin of these states, one real scalar per nine-dimensional vector multiplet, is the gap shared with Proposition 6.1 of the 11D line.

Part S154 (For an antiunitary J exchanging V and V-bar with J^2 = eps and a Hermitian X with J X = eps' X J, the particle-antiparticle block obeys B = eps eps' B^T, so in the labels 9, 10 and 11 a J-compatible operator has no nonzero antisymmetric block within one multiplet)

For an antiunitary J exchanging V and V-bar with J^2 = eps and a Hermitian X with J X = eps' X J, the particle-antiparticle block obeys B = eps eps' B^T, so in the labels 9, 10 and 11 a J-compatible operator has no nonzero antisymmetric block within one multiplet

Status. No-go theorem in its scope (Proposition 9.3, with the referee amendment). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) from linear algebra and Connes's sign table; checked by its author by hand, by Terra 6.1 by hand, whose independent proof is written out in the Referee proofs of lemma 36, and by Claude Sonnet 5.5 by hand, who showed that B ranges over exactly S^2 V for eps eps' = +1 (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_real_structure_diquarks` (depending on `d11_diquark_strings`, `l06_factorization`, `l21_so30` and `d11_wilson_top`). Proposition 9.3 in the section on J-compatible operators of paper/real_structure_note.tex ("\supp{36}"); stated in the section on the real structure of paper/connes_kk_dictionary_note.tex.

Setting. E = V + V-bar with V as in 36_diquark_counts_stack_image.md; an antiunitary J on E mapping V onto V-bar and V-bar onto V, with J^2 = eps; a Hermitian X on E with J X = eps' X J, the condition on

inner fluctuations in a KO-dimension with signs (eps, eps'); B is the block of X from V-bar to V, an element of V (x) V.

Statement.

1. B = eps eps' B^T: B lies in S^2 V if eps eps' = +1 and in Lambda^2 V if eps eps' = -1.
2. In the KO-dimensions 5, 6 and 7 of the finite part for the labels 9, 10 and 11, (eps, eps') = (-1, -1), (+1, +1) and (+1, +1), so the diquark block of every J-compatible fluctuation lies in S^2 V, whose colour antitriplet (Lambda^2 5, 3-bar) has the counts (3, 6, 1), and it cannot carry the counts (6, 6, 3) of lemma 21.
3. For a connection on a circle with complex conjugation as real structure (KO-dimension 1, (eps, eps') = (+1, -1)) the block lies in Lambda^2 V, with the counts (6, 6, 3); the SO(32) Wilson line is of this type.

Scope. Hermitian J-compatible operators between one multiplet and its conjugate, with the stated signs. The signs permit all of S^2 V, and the algebra, grading and order conditions of a triple select a subspace; with D = 0 every inner one-form vanishes. Item 2 is the finite form of the symmetry of a Majorana mass matrix and of the Fermi-statistics constraint on a Lorentz-scalar bilinear of one Weyl multiplet without derivatives.

Escapes, as the lemma states them. Connection components along a circle or an interval (item 3); blocks between two distinct multiplets, which J relates only by B_21 = eps eps' B_12^T; derivative couplings; other real structures or signs.

Proof. Lemma 36, Proposition 9.3, and the Referee proofs there: with J(v, w) = (eps K^{-1} w, K v) and X = [[A, B], [B*, D]], the V -> V component of J X = eps' X J gives eps K^{-1} B* = eps' B K, so B = eps eps' B^T in coordinates where K is conjugation.

What it leaves open. The result points to the interval parent of lemma 22 and its gauge field: the interval component of an so(30) connection carries Lambda^2 V (38_interval_connection_so30.md), and its massless scalars need conditions at the walls (38_interval_scalar_zero_modes.md). Whether the (15, 3-bar) of the EPJC paper can be read as bilinears of two distinct multiplets is open.

Part S155 (Under SU(16) x U(1) the complex adjoint of SO(32) is 120 + 120-bar + 255 + 1, and under SU(5) x SU(3) the 105 in the 120 is (15, 3-bar) + (10, 6), whose antitriplet has the charge counts (6, 6, 3))

Under SU(16) x U(1) the complex adjoint of SO(32) is 120 + 120-bar + 255 + 1, and under SU(5) x SU(3) the 105 in the 120 is (15, 3-bar) + (10, 6), whose antitriplet has the charge counts (6, 6, 3)

Status. Proposition (section 1 of lemma 36, the referee algebra). Stated by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author by hand, by Terra 6.1 by hand, whose independent proof is written out in the Referee proofs of lemma 36, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger rows `d11_so32_algebra` (depending on `d11_diquark_strings`) and `d11_so32_comparison`, the comparison with the EPJC paper read from its LaTeX source by Claude Sonnet 5.5, with prior work recorded as known. Cited in the introduction of paper/real_structure_note.tex ("\supp{36}, section 1") and in the introduction of paper/generations_note.tex ("\supp{36}, Section 1").

Setting. 32_C = 16_{+1} + 16-bar_{-1} under U(16) in SO(32); 16 = 15 + 1 under SU(15), and 15 = 5 (x) 3 under SU(5) x SU(3).

Statement.

1. so(32)_C = Lambda^2 32 = 120_{+2} + 120-bar_{-2} + 255_0 + 1_0.
2. Under SU(15), 120 = 105 + 15, and 105 = Lambda^2(5 (x) 3) = (15, 3-bar) + (10, 6), with the antitriplet charge counts of 36_diquark_counts_stack_image.md.
3. 15 (x) 15-bar = (24, 8) + (24, 1) + (1, 8) + (1, 1).

Scope. Lie-algebra shorthand; global embeddings involve finite central quotients. Matching representations does not identify physical composites, and assigning the colour-neutral SO(2) plane to a top brane is a label of the proposed dictionary.

Proof. Lemma 36, Referee proofs, the branching of section 1: Lambda^2(16 + 16-bar) = Lambda^2 16 + Lambda^2 16-bar + 16 (x) 16-bar, with the plethysm of Proposition 9.2.

What it leaves open. The comparison of section 1 with A. Rivero, Eur. Phys. J. C 84 (2024) 1058 (arXiv:2407.05397): the paper keeps (1, 24, 1), (1, 15, 3-bar) and (1, 15-bar, 3), favours the top as the non-binding quark without deriving it, with postulates that make that quark up-type, has colour singlets only for mesons, no SO(2) for the top and no D-branes, and its abstract specifies a chiral +4/3 quark per generation. Its selection of an up-type quark and that of lemma 21 by the spectral-action values are independent arguments with the same outcome. The binding mechanism and the fermion partner of the three scalars of charge +4/3 are open there.

Part S156 (The Wilson-line angle of a type I' D8-brane at x_k is theta_k = (pi/2) W(x_k)/W(L) with W(x) = int_0^x z^{-1/3} dy, so the linear dictionary x_k = (2L/pi) theta_k holds as a function of every position exactly for constant z, and asymptotically for large R)

The Wilson-line angle of a type I' D8-brane at x_k is theta_k = (pi/2) W(x_k)/W(L) with W(x) = int_0^x z^{-1/3} dy, so the linear dictionary x_k = (2L/pi) theta_k holds as a function of every position exactly for constant z, and asymptotically for large R

Status. Proposition (Proposition 10.1). Proved by Claude Opus 5.5 on 4 November 2026 from the arXiv LaTeX of Polchinski and Witten (hep-th/9510169, section 3); checked by its author by hand and read independently by Claude Sonnet 5.5 by hand (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md), whose amendments were rechecked by Claude Opus 5.5 and applied. Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): the map and the radius formula correct, with the qualification that a particular configuration can satisfy the linear dictionary at its brane positions with z nonconstant (fifteen branes at 0 and one at L), and the derivation from brane-to-brane mass differences; applied. Ledger row `d11_angle_map`. The first of the three corrections in the section on the string dualities of paper/connes_kk_dictionary_note.tex ("\supp{37}").

Setting. Type I' on [0, L] in conformal gauge, g = Omega^2(x) eta with Omega = C z^{-1/6}, e^phi = z^{-5/6} and z piecewise linear of slope 8 - N(x), N(x) the number of D8-branes to the left of x; this is the slope rule recalled in lemmas 35 and 36, confirmed here against the source. L = 2 pi and alpha' = 2, in the units of Polchinski and Witten.

Statement.

1. The Wilson-line eigenvalue of a brane at x_k is lambda_k = W(x_k)/(2 W(L)), that is theta_k = pi lambda_k = (pi/2) W(x_k)/W(L).
2. R^2 = (3/4) int z^{1/3} dx/(int z^{-1/3} dx)^2.
3. Hence x_k = (2L/pi) theta_k holds as a function of every position exactly when z is constant (8 + 8), and asymptotically for large R; fifteen branes at 0 and one at L satisfy it at their positions with slope -7 in the bulk.

Scope. The map from angles to positions depends on the whole configuration. Items 1 and 2 of Proposition 9.1 of lemma 36 hold as statements about z in the positions, and its reading as a trace of a function of abs(D_F) holds only implicitly, through this map (36_dilaton_profile_as_trace.md).

Proof. Lemma 37, Proposition 10.1: the string stretched from the wall to x_k has mass (1/(2 pi alpha')) int_0^{x_k} Omega^2 dx, with Omega^2 proportional to z^{-1/3}, a string winding from 0 to L and back has mass 1/R, and the ratio gives lambda_k; the formula for R is equation (rad) of Polchinski and Witten, reduced as in their equation (dil).

What it leaves open. An explicit spectral formula for the coupling function, which with this map is a trace of a function of abs(D_F) only implicitly. The readings of lemma 40 that used the linear dictionary, such as the brane "at distance proportional to a" in the reading of Proposition 7.2, hold only at constant dilaton (lemma 37, section on corrections).

Part S157 (The angle-independent one-loop coefficient of the flat interval is 8 pi^2 a_0 = (3 zeta(5)/16)(4 n_D' - 3 (n_i - (15/16) n_h) - n_s - 5 n_g), positive for the corpus content, so at v = 0 the interval runs to large length)

The angle-independent one-loop coefficient of the flat interval is 8 pi^2 a_0 = (3 zeta(5)/16)(4 n_D' - 3 (n_i - (15/16) n_h) - n_s - 5 n_g), positive for the corpus content, so at v = 0 the interval runs to large length

Status. Proposition (the coefficient a_0 of lemma 37, unnumbered, with its proof). Proved by Claude Opus 5.5 on 4 November 2026; read independently by Claude Sonnet 5.5 (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md), who found that the first version counted 21 gauge generators with integer towers against the 25 of sp(2) + su(4) used in lemmas 23, 24 and 26. The near-cancellation 64 - 63 and the angle theta_* = 0.202 that followed from the wrong count are withdrawn (lemma 37, Correction of 4 Nov 2026). Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): A_0 = 32.9184 for the three-generation content recomputed; positivity and the runaway direction depend on the displayed field content and on the tuning of the bulk energy and the wall tensions. Part of the ledger row `d11_gradient`, with Propositions 10.5 and 10.6. The displayed formula for a_0 in the section on the massless limit of paper/connes_kk_dictionary_note.tex ("\supp{37}"), which calls it new there.

Setting. 8 pi^2 l^4 V = A_0 + 3 F(2u) + (3/2) F((16/5) u) - n_f F(t), with t = sin^2 theta and u = t(1 - t): the potential of lemma 24 with its theta-independent part A_0 = 8 pi^2 a_0 restored. n_D' = 2 N_D counts twice all Dirac species of any frequency, n_i and n_h are the five-dimensional gauge generators with integer and half-integer towers at theta = 0, n_s the real bulk scalars with integer towers, and n_g = 1 if the five-dimensional graviton is included. The bulk vacuum energy and the wall tensions, which carry the terms of order Lambda^5 l and Lambda^4, are assumed tuned away.

Statement.

1. A_0 = (3 zeta(5)/16)(4 n_D' - 3 (n_i - (15/16) n_h) - n_s - 5 n_g).
2. For sp(2) + su(4), n_i = 16 (8 gluons, 3 W_L, one hypercharge combination with Neumann conditions at both walls, and the 4 odd generators with Dirichlet conditions at both, whose interval components are the Higgs quaternion) and n_h = 9 (2 R^+-, 6 X and one neutral direction), so n_i - (15/16) n_h = 121/16.
3. For three generations, with 24 Dirac species, the bracket is 192 - 22.6875 = 169.3125 before gravity and bulk scalars, and A_0 = 32.92; for one generation the bracket is 41.3125. So a_0 > 0, and at v = 0 the length runs to infinity, the direction H requires; this is the first half, a r^{-p} with a > 0 and p = 4, of the modulus target of the Astra 6 audit.

Scope. A real degree of freedom on a periodic circle of length 2l has V = -3 zeta(5)/(4 pi^2 (2l)^4), an interval

tower n >= 0 half of it, and a half-integer tower the factor -15/16 relative to an integer one; fermions enter with the opposite sign and four degrees of freedom per unit of n_f.

Proof. Lemma 37, the paragraph on the coefficient a_0, with its proof; the check is that the shift to theta = pi/2 multiplies a tower by -15/16, which reproduces F(1) = 93 zeta(5)/64 of Proposition 1.4 (31_one_loop_endpoint_value.md).

What it leaves open. The dilaton gradient can act as the second scale only if a_0 = 0, which would need a Bose-Fermi balance at theta = 0 that is not a supersymmetry there, with about 169 further real bosonic degrees of freedom for three generations (37_nogo_dilaton_gradient.md).

Part S158 (In the flat one-loop family with a_0 restored and the angle tied to a mass, top-only content has G > 0 on [0, pi/2] and no stationary point in the window, while a universal mass gives one stable stationary angle, theta_* = 1.009 for three generations and 1.031 for one)

In the flat one-loop family with a_0 restored and the angle tied to a mass, top-only content has G > 0 on [0, pi/2] and no stationary point in the window, while a universal mass gives one stable stationary angle, theta_* = 1.009 for three generations and 1.031 for one

Status. Numerical (Python: numpy, scipy, mpmath; scripts not kept), conditional on the coefficient a_0 of 37_casimir_coefficient.md and on the content of lemmas 23 and 24. Computed by Claude Opus 5.5 on 4 November 2026 (lemma 37, paragraph on the flat family with the angle tied); read independently by Claude Sonnet 5.5 in the reading of lemmas 37 and 38 (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md), after which the generator count of a_0 was corrected; Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): the sample values reproduced; the statement on the periodically continued potential corrected (first roots near 8.15 and 8.40, then 11.17, 11.71, …; the scan proves no global count); the minimum of the localized interaction term at c = 1 shown to be exact. No ledger row of its own. Stated in the section on the massless limit of paper/connes_kk_dictionary_note.tex ("(Python)").

Setting. G(theta) = A_0 + 3 F(2u) + (3/2) F((16/5) u) - n_f F(t), with t = sin^2 theta, u = t(1 - t), A_0 = 32.92 for three generations (37_casimir_coefficient.md), and theta = l m, as in item 2 of 31_scale_free_potential.md, with stationary lengths at the roots of theta G' = 4 G.

Statement.

1. For the top alone, n_f = 6: the small-angle coefficient is proportional to 54/5 - 6 > 0 and G > 0 on [0, pi/2], with minimum 23.88, so V has no stationary point with theta < pi/2; l runs until theta leaves the fundamental domain of Proposition 7.1, where the top joins the tower, and the periodically continued potential has its first stationary points far outside the window, near theta = 8.15 and 8.40, with further roots near 11.17, 11.71, 14.27, …, as periodicity with an interval where G' > 0 requires.
2. For a universal mass, n_f = 48 at three generations, G changes sign and there is one stable stationary point, theta_* = 1.009, with M_W/m = 0.685; at one generation, n_f = 16, theta_* = 1.031 and M_W/m = 0.653.
3. Adding a graviton does not change this classification.

Scope. The stationary angles are of order one, so these vacua do not separate the window, and top-only content has no vacuum in the window in this scope.

Proof. Lemma 37, paragraph on the flat family with the angle tied: numerical evaluation of G and of the roots of theta G' - 4 G.

What it leaves open. A separated window on the scale-free path needs a light mass exponentially small in a stabilized shape parameter (Remark 10.7), and that stabilization at a large value is a target: the one-loop interaction term of one localized fermion channel has its unique minimum at c = mu l = 1 exactly (d_c log Q = 2(1 - c)/w), where the light pair merges with the tower (lemma 37, paragraph on a localized channel).

Part S159 (For a potential with a regular joint expansion in theta and alpha'/l^2 around the flat family, the type I' dilaton gradient supplies no second scale for Proposition 1.5: its relative variation is bounded by 6/R^2 along an opening path, and it can matter only if a_0 = 0, and then only with b = 0 when a_1 = 0)

For a potential with a regular joint expansion in theta and alpha'/l^2 around the flat family, the type I' dilaton gradient supplies no second scale for Proposition 1.5: its relative variation is bounded by 6/R^2 along an opening path, and it can matter only if a_0 = 0, and then only with b = 0 when a_1 = 0

Status. No-go theorem in its scope (Propositions 10.5 and 10.6). Proved by Claude Opus 5.5 on 4 November 2026; checked by its author by hand and read independently by Claude Sonnet 5.5 by hand (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md), whose amendments were rechecked by Claude Opus 5.5 and applied. Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): Proposition 10.5 correct as an upper bound, with its consequence for the potential a hypothesis (regular expansion); Proposition 10.6, item 3, had an error, since vanishing at theta = 0 alone does not exclude a joint family: G_0 = theta^6, G_1 = theta^2 gives V = m^6 l^2 + alpha' m^2/l^4 with a minimum at l^6 = 2 alpha'/m^4 and l m -> 0; the condition b != 0 is now stated. With the corrections applied, this negative result has two independent readings in its stated scope. Ledger row `d11_gradient`. Stated in the section on the string dualities of paper/connes_kk_dictionary_note.tex ("it can act as the second scale only if the coefficient in Eq. (a0) vanishes").

Setting. The profile of 37_angle_to_position_map.md with L = 2 pi and alpha' = 2, where z is B plus a function of size at most 8L = 16 pi; a joint family R_eps ~ eps^{-alpha}; and V = l^{-4} [G_0(theta) + (alpha'/l^2) G_1(theta) + …], theta = l m, with G_0 = a_0 + b theta^2 + … near 0.

Statement.

1. (Proposition 10.5) With B = min h, for large R, R^2 = 3B/(8 pi) + O(1), and the relative variation of the profile across the interval is at most 16 pi/B ~ 6/R^2, an upper bound (zero in the chamber 8 + 8); along the family it falls as eps^{2 alpha}, the map of Proposition 10.1 linearizes and the differences of the dilaton tend to zero. That its corrections to the potential are powers of alpha'/l^2 is the hypothesis of a regular expansion, which small variation alone does not give.
2. (Proposition 10.6, item 1) If a_0 != 0 and the joint expansion with its derivatives is controlled near (theta, s) = (0, 0), the leading balance at small m is the scale-free one of Proposition 1.5, alpha = 1 or a runaway, with no stationary joint branch with theta, s -> 0.
3. (item 2) If a_0 = 0, G_1(0) = a_1 > 0 and b < 0, the truncated potential has a minimum at l^4 = 3 a_1 alpha'/(abs(b) m^2), so alpha = 1/2, and 1/l is the geometric mean of m and the string scale up to constants; with regular higher terms this is the leading asymptotic.
4. (item 3) If supersymmetry is exact at theta = 0, then V = m^2 l^{-2} H(theta, alpha'/l^2); if b != 0 and the higher terms are regular, l is fixed by H independently of m or runs away, and no joint family results, unless the stationarity condition has a root interval, a flat direction as in 31_counterexample_flat_direction.md. If b = 0 a joint family can result (the counterexample above).

Hypotheses. A potential with a regular joint expansion in theta and alpha'/l^2 around the flat family with its towers, and the frame identification relating l to R_h; the corpus content, with a_0 > 0

(37_casimir_coefficient.md); the dilaton gradient, as distinct from a dilaton potential. In supersymmetric type I' there is no potential at all, since the D8 positions, B and C are flat directions.

Escapes, as the lemma states them. A dilaton potential from string tree level, in non-supersymmetric strings or with tadpoles, for which a term in positive powers of 1/g could grow with l at fixed v through Proposition 10.4; a different frame relation between l and the four-dimensional Planck mass; wall-localized dynamics; the other chambers of the moduli space, where 8 + 8 removes the Romans mass and E8 x E8 is available.

Proof. Lemma 37, Propositions 10.5 and 10.6: insert z = B + O(16 pi) in the formula for R; differentiate, with V'' = 8 abs(b) m^2/l^4 > 0 at the stationary point of item 3; V even in theta and vanishing at theta = 0 for every l gives V = theta^2 l^{-4} H.

What it leaves open. The result points to three targets, none derived: a bulk energy depending on the electroweak order parameter (Remark 10.8: V = a/l^4 + b v^2 M^3 l gives alpha = 2/5 if the bulk energy vanishes at v = 0); a cancellation a_0 = 0 without supersymmetry at theta = 0; or a fixed shape with an exponentially small ratio, as on the disc bundle of lemma 29 (Remark 10.7).

Part S160 (In the chamber 15 + 1 with the top's brane on the far wall, the D0 states massless there at infinite coupling satisfy M_D0 R = R^2 - R_*^2, so W bosons of this kind are massless at the string-scale radius and grow heavier as R grows, and a neutral Z cannot be of this kind)

In the chamber 15 + 1 with the top's brane on the far wall, the D0 states massless there at infinite coupling satisfy M_D0 R = R^2 - R_*^2, so W bosons of this kind are massless at the string-scale radius and grow heavier as R grows, and a neutral Z cannot be of this kind

Status. No-go theorem in its scope (the section on the far-wall electroweak group of lemma 37, unnumbered, with no ledger row of its own). Derived by Claude Opus 5.5 on 4 November 2026 from equations (windingmass) and (D0mass) of Bergman, Gaberdiel and Lifschytz (hep-th/9711098, section 3, read from the arXiv LaTeX); read independently by Claude Sonnet 5.5 (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md); refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): equation confirmed as the full D0 mass in the heterotic frame; the enhancement states with heterotic winding +-1 contain half a D0-brane, so for them M_{W,h} R_h = (1/2)(R_h^2 - 7/4), with the same growth; the identification with the W stays conditional. Stated in the section on the string dualities of paper/connes_kk_dictionary_note.tex among the negative statements in the scope of that dictionary.

Setting. The chamber 15 + 1 of 37_radius_fifteen_plus_one.md with n = 1 brane, the top's, at the strong far wall, placed at x^9 = 0 by reflecting the interval; N' = 8 - n, A = (B + 2 pi N')^{2/3}, b = B^{2/3}, alpha' = 2. Remark 8.3 of lemma 35 places SU(2) x U(1) at this wall at infinite coupling, conditional on the recalled type I' rules.

Statement.

1. M_D0 R = (N'/2) b/(A - b) = R^2 - R_*^2, which vanishes at B = 0, where R = R_*, and grows like R^2; for the enhancement states with heterotic winding +-1, which contain half a D0-brane, M_{W,h} R_h = (1/2)(R_h^2 - 7/4).
2. Only W^+ and W^- can be such states: a neutral Z is a Cartan combination, and the rank of a nine-dimensional theory with sixteen supercharges is fixed, so no massive neutral vector arises this way.
3. If the W^+- are these D0 states, they are massless at the string-scale radius with the tower fixed, and M_W l rises with l, the direction opposite to l ~ 1/M_W. So on one path, in this scope, the dictionary

can place the electroweak group at the far wall, or the opening of H on the circle of the SO(32) Wilson line, and not both.

Scope. The chamber 15 + 1, the classical background and the identification of the W with the D0 states at the wall. This is the fixed-tower path of the Astra 6 audit (its section 2.2), on which masslessness opens no dimension.

Escapes, as the lemma states them. Other chambers; the W as a Wilson-line string between D8-branes, as in lemma 23; the top's brane in the bulk, with the far wall giving E_1.

Proof. Lemma 37, section on the far-wall electroweak group: the winding mass and the D0 mass of Bergman, Gaberdiel and Lifschytz combined, with R_* of 37_radius_fifteen_plus_one.md.

What it leaves open. The result points to the escapes above. The physical status of the far-wall group itself rests on the recalled enhancement rule (Remark 8.3, ledger row `d11_e_walls`, conjectural), which has no file.

Part S161 (On a type I' segment with N != 8 branes to its left the proper length is linear in the inverse coupling, so the classical profile of the chamber 15 + 1 has no region that is both strongly coupled and resolved at the formal local scale g^{1/3} l_s)

On a type I' segment with N != 8 branes to its left the proper length is linear in the inverse coupling, so the classical profile of the chamber 15 + 1 has no region that is both strongly coupled and resolved at the formal local scale g^{1/3} l_s

Status. No-go theorem in its scope (Proposition 10.4), the classical profile extrapolated to where the coupling is of order one. Proved by Claude Opus 5.5 on 4 November 2026 from the type I' profile of Polchinski and Witten (hep-th/9510169, read from the arXiv LaTeX); checked by its author by hand and read independently by Claude Sonnet 5.5 by hand (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md); refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): coefficient and bound confirmed, with the orientation (absolute value), the exclusion of N = 8 and the formal character of the eleven-dimensional scale made explicit; applied. Ledger row `d11_romans`. The third of the three corrections in the section on the string dualities of paper/connes_kk_dictionary_note.tex ("\supp{37}").

Setting. Type I' as in 37_angle_to_position_map.md; a segment with N branes to its left, with Romans mass F_0 = (8 - N)/(2 pi l_s) in the standard normalization, and C_L = 8 pi/5.

Statement.

1. Between points of couplings e^{phi_1} and e^{phi_2}, L_p = (C_L l_s/abs(8 - N)) abs(e^{-phi_2} - e^{-phi_1}), with l_s^2 = alpha'_{I'} and abs(F_0) = abs(8 - N)/(2 pi l_s); for N = 8, F_0 = 0 and the formula does not apply.
2. In the configuration 15 + 1, g(0) L_p <= C_L l_s/7 for the total length.
3. Near an infinite-coupling wall the coupling is g = C_L l_s/(abs(8 - N) s) at proper distance s, and the region g >= g_0 has length (3/4)(C_L/abs(8 - N)) g_0^{-4/3} in the formal local scale l_11 = g^{1/3} l_s, 0.54 g_0^{-4/3} for N = 15.

Hence no region of the classical 15 + 1 background is both strongly coupled and resolved at that scale; the same holds wherever N != 8. A standard eleven-dimensional lift has not been established for this massive background, and the source presents its M-theory conclusion as an expectation.

Scope. The classical profile, which is not controlled where g is of order one, so this is a statement about the extrapolated background. It agrees with Bergman, Gaberdiel and Lifschytz, who expect neither an M-theory interpretation nor a matrix theory for these backgrounds and state that only eight D8-branes with

an orientifold plane are known to arise in M-theory (their N is 8 - N here).

Escapes, as the lemma states them. The other chambers of the moduli space, where 8 + 8 removes the Romans mass and E8 x E8 is available; a controlled description at coupling of order one.

Proof. Lemma 37, Proposition 10.4: on the segment Omega dx = C z^{-1/6} dz/(a C (8 - N)), which integrates to a multiple of z^{5/6} = e^{-phi}.

What it leaves open. The result points to the other chambers, without the Romans mass, and to other completions. The far-wall statement that rests on the same chamber is 37_nogo_far_wall_electroweak.md.

Part S162 (With fifteen D8-branes at x = 0, one at x_t and infinite coupling at the far wall, theta_t = (pi/2) 8 (1 - rho)/(8 - rho) and R_*^2 = 14 (8 - rho^2)/(8 - rho)^2 with rho = (z(x_t)/z(0))^{2/3}, so infinite coupling sits at the smallest radius of the chamber, at the string scale)

With fifteen D8-branes at x = 0, one at x_t and infinite coupling at the far wall, theta_t = (pi/2) 8 (1 - rho)/(8 - rho) and R_*^2 = 14 (8 - rho^2)/(8 - rho)^2 with rho = (z(x_t)/z(0))^{2/3}, so infinite coupling sits at the smallest radius of the chamber, at the string scale

Status. Proposition (Proposition 10.2), with no ledger row of its own. Proved by Claude Opus 5.5 on 4 November 2026 from the formulas of Polchinski and Witten (hep-th/9510169, section 3, read from the arXiv LaTeX); Python (numpy, scipy, mpmath, script not kept) agrees at x_t/L = 0.5, with theta_t = 0.5859 and R^2 = 1.9652, against 0.7854 for the linear dictionary. Read independently by Claude Sonnet 5.5 (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md); refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): formulas, endpoint values and the bound at fixed Wilson angle confirmed (SciPy: theta_t = 0.585918469, R_h^2 = 1.965216392 at x_t/L = 1/2), with the units and the bound restated as below. Used in the section on the string dualities of paper/connes_kk_dictionary_note.tex for the far-wall states massless at the string-scale radius ("\supp{37}").

Setting. Type I' as in 37_angle_to_position_map.md, alpha' = 2, in the chamber 15 + 1: fifteen branes at x = 0 and one at x_t, with z(L) = 0, so that the slopes on the two segments are -7 and -8 and z(x_t) = 8 (L - x_t); rho = (z(x_t)/z(0))^{2/3} lies in [0, 1].

Statement.

1. theta_t = (pi/2) 8 (1 - rho)/(8 - rho) and R_*^2 = 14 (8 - rho^2)/(8 - rho)^2.
2. R_* runs from sqrt 2, for the top's brane on the stack, to sqrt 7/2, for the top's brane on the far wall, which are the values (1/2) abs(n - 8)^{1/2} of Polchinski and Witten at n = 16 and n = 15; in heterotic string units R_*/sqrt(alpha'*h) lies in [sqrt(7/8), 1], and R*^2 = 2 - (theta/pi)^2.*
3. At each fixed Wilson angle a nonnegative far-wall value of the normalized profile gives R_h >= R_*(theta), by R_h^2 = 2 - a^2 + 28 E/(8A - T - 7E) with A, T, E the 2/3 powers of the profile at 0, x_t and L and a = theta/pi; so infinite coupling sits at the smallest radius, at the string scale, and large R is large B, the weakly coupled region, as Polchinski and Witten state.

Scope. The classical profile of the chamber 15 + 1 with infinite coupling at the far wall.

Proof. Lemma 37, Proposition 10.2: integrate z^{1/3} and z^{-1/3} on the two segments, with slopes -7 and -8, and use z(x_t) = 8 (L - x_t) in the formulas of 37_angle_to_position_map.md.

What it leaves open. With the top's brane on the far wall, the D0 states that become massless there grow heavier as R grows (37_nogo_far_wall_electroweak.md), so on one path the dictionary has to choose

between the far-wall electroweak group and the opening of the circle. The other chambers, without the Romans mass, are outside this computation.

Part S163 (If the tower of the SO(32) Wilson-line channel is identified with the type I' winding tower, the length l of the interval of H and the proper length L_p of the type I' interval satisfy l L_p = pi^2 alpha' at constant dilaton, and l -> infinity at fixed heterotic coupling shrinks the type I' interval and ends at ten-dimensional SO(32))

If the tower of the SO(32) Wilson-line channel is identified with the type I' winding tower, the length l of the interval of H and the proper length L_p of the type I' interval satisfy l L_p = pi^2 alpha' at constant dilaton, and l -> infinity at fixed heterotic coupling shrinks the type I' interval and ends at ten-dimensional SO(32)

Status. Proposition conditional on the string identification (Proposition 10.3), with no ledger row of its own, read from the sources: Polchinski and Witten (hep-th/9510169: "a heterotic Kaluza-Klein state which is dual to a Type I' winding state") and Bergman, Gaberdiel and Lifschytz (hep-th/9711098) for the heterotic metric. Stated by Claude Opus 5.5 on 4 November 2026; read independently by Claude Sonnet 5.5 (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md); refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): the identification of the interval tower with the string winding states needs the proposed string embedding (the source identifies one heterotic BPS state with one type I' winding state, and Wilson-line charges shift the dictionary), the flat relations hold at constant dilaton with the stated frame identifications and the source restricts them to eight branes at each wall, and equality of lengths at a self-dual value does not identify the walls; the proposition is now stated as conditional. The second of the three corrections in the section on the string dualities of paper/connes_kk_dictionary_note.tex ("\supp{37}").

Setting. The tower n/R of the SO(32) Wilson-line channel (Proposition 6.1 of the 11D line, ledger row `d11_wilson_top`); type I' with constant dilaton; R_I the type I radius, l = pi R_I measured in the type I' metric, and lambda_h the heterotic coupling.

Statement.

1. L_p R_I = pi alpha' on the type I side, with alpha' that of type I', so l L_p = pi^2 alpha'.
2. For the heterotic radius in the heterotic metric, l L_p = pi^2 alpha' sqrt(lambda_h).
3. The walls of H, at distance l, and the O8 planes, at distance L_p, coincide only at the self-dual length. At fixed heterotic coupling, l -> infinity shrinks the type I' interval and decompactifies the SO(32) circle, and since the type I' coupling falls as 1/(sqrt(lambda_h) R_h) the endpoint of the limit is the ten-dimensional SO(32) theory.

Scope. Conditional on the identification of the tower with the type I' winding states; the constant-dilaton point for the factor pi^2 alpha', where in the chamber 15 + 1 the applicable relation carries the Weyl factor, 1/R_h = (D/(pi alpha'_{I'})) int Omega^2 dx with L_p = int Omega dx. Recalled, and supported in the two sources only by remarks at the constant-dilaton point: the eleven-dimensional limit of the rank-17 moduli space is M-theory on an interval times a circle with E8 x E8 at the walls, which Remark 8.3 of lemma 35 places out of reach with colour-respecting moves; if so, label 11 of section 6b of lemma 40 is not reached by theta -> 0, l -> infinity in the chamber 15 + 1.

Proof. Lemma 37, Proposition 10.3: T-duality of the winding tower at constant dilaton, with the factor sqrt(lambda_h) from Bergman, Gaberdiel and Lifschytz.

What it leaves open. The reading "the interval opens" at label 11 in section 6b of lemma 40 has to be restated. The other chambers of the moduli space, where 8 + 8 removes the Romans mass and E8 x E8 is available, are outside this statement.

Part S164 (With the real algebra M_30(R) on the massless light block and wall algebras containing 1, the self-adjoint one-forms of the interval parent are exactly E a_x with a_x an so(30)-valued function free at the walls, whose block between particles and antiparticles lies in Lambda^2 V)

With the real algebra M_30(R) on the massless light block and wall algebras containing 1, the self-adjoint one-forms of the interval parent are exactly E a_x with a_x an so(30)-valued function free at the walls, whose block between particles and antiparticles lies in Lambda^2 V

Status. Proposition (Proposition 11.2). Proved by Claude Opus 5.5 on 4 November 2026; checked by its author by hand and read independently by Claude Sonnet 5.5 by hand (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md), whose amendments are applied. Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): domain preservation, order zero, order one and the one-form construction confirmed at D_F = 0; the representation is nonunital on the doubled space (pi(1) is the projection onto H_+), so Theorem 22.3 does not apply to it as stated and a unital completion, such as f (+) r 1, must be specified where unitality is used; applied. Ledger row `d11_so30_block` (with Proposition 11.1). Proposition 11.2 in the section on the interval parent of paper/real_structure_note.tex ("\supp{38}").

Setting. The interval parent of lemma 22, D_X = E d/dx on L^2([0, l]; C^2) (x) (H_+ + H_-), with the light block of 38_so30_commutant_of_j.md, D_F = 0 on it (so X = 0) and the wall conditions Q_R(0) = U_L(0) = 0 and their counterparts at l. C_0 and C_l are subalgebras of A = M_30(R), the real elements of End H_+, both containing 1; A_l = {f in C^infinity([0, l], A) : f(0) in C_0, f(l) in C_l} acts by pi(f) = f on the H_+ components and 0 on the H_- components, a nonunital representation on the doubled space; pi~(f, r) = f (+) r 1 is a unital completion.

Statement.

1. pi(f) preserves the domain for every choice of C_0 and C_l, and order zero and order one hold on this block.
2. The self-adjoint one-forms sum pi(f_i) [D_X, pi(g_i)] are exactly E a_x on the H_+ components, with a_x in C^infinity([0, l], so(30)) and no condition at the walls, and the fluctuated operator is D_A = E (d/dx + a_x on H_+ and U a_x U^{-1} on H_-).
3. J-hat A J-hat^{-1} = E U a_x U^{-1} on H_-, with sign +, as KO-dimension six requires. The block of a_x between V-bar and V is antisymmetric, and the block of the Hermitian operator E a on C^2 (x) E is symmetric, as Proposition 9.3 requires, the factor E carrying the other sign: the interval component evades Proposition 9.3 through the escape of its item 3.

Scope. For D_F != 0 on this block, order one fails in the finite window unless the mass block M vanishes (a = b = 1 gives the blocks -M^* and -M); the corner representation keeps order one by the pointwise calculation, its endpoint corner is zero for invertible M, and the unital completion has the endpoint condition f(l) = r(l) 1; so this algebra describes the massless block, and the masses need a treatment without order one, as in lemma 19.

Proof. Lemma 38, Proposition 11.2: the wall conditions are preserved by any operator acting on the H_+ index; [D_X, pi(g)] = E g' on the H_+ components; a given skew T is attained with f = 1 and g = int_0^x (T - tau), where tau has support inside (0, l) and the same integral as T, and tau itself is a finite sum of products f_i g_i'.

What it leaves open. Which wall algebras make these components massless scalars is 38_interval_scalar_zero_modes.md, and whether a boundary condition derives them is open; at nonzero masses, the quadratic terms of lemma 19 for this algebra.

Part S165 (Under a five-dimensional Yang-Mills action the massless four-dimensional scalars from the interval component of an so(30) connection form the orthogonal complement of k_0 + k_l, so lemma 22's parent has none, wall algebras M_15(C) give exactly Lambda^2 V + Lambda^2 V-bar, and a generator has a massless vector or a massless scalar, never both)

Under a five-dimensional Yang-Mills action the massless four-dimensional scalars from the interval component of an so(30) connection form the orthogonal complement of k_0 + k_l, so lemma 22's parent has none, wall algebras M_15(C) give exactly Lambda^2 V + Lambda^2 V-bar, and a generator has a massless vector or a massless scalar, never both

Status. Theorem (Theorem 11.3, with the amendments of the independent reading). Proved by Claude Opus 5.5 on 4 November 2026; checked by its author by hand and read independently by Claude Sonnet 5.5 by hand (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md), whose amendments are applied. Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): the local zero-mode calculation correct once the gauge action and the Yang-Mills boundary variational problem are specified, which the nonunital corner representation does not do by itself (pi(u) J-hat pi(u) J-hat^{-1} = 0); the counts are 210 real scalar and 225 real vector components for two U(15) walls, tree-level field counts; the double coset needs det g(0) = det g(l) and the component qualification; applied. Ledger row `d11_so30_zero`. Theorem 11.3 in the section on the interval parent of paper/real_structure_note.tex ("\supp{38}"); stated in the section on the Higgs field of paper/connes_kk_dictionary_note.tex ("\supp{38}"), whose table lists it with one reading.

Setting. The algebra A_l of 38_interval_connection_so30.md, with wall algebras C_0, C_l of skew parts k_0, k_l, a five-dimensional Yang-Mills action for the fluctuation on a flat product with M^4, as in Proposition 22.8, at trivial background with unrestricted endpoint variations and no wall terms, and the usual gauge action, supplied by a unital completion of pi.

Statement.

1. The gauge group is {g : I -> O(30), g(0) in K_0, g(l) in K_l}, with det g(0) = det g(l); the holonomy of a_x transforms as P -> g(l) P g(0)^{-1}, so the moduli are the double cosets of SO(30) under K_l on the left and K_0 on the right, the massless four-dimensional scalars from a_x form the orthogonal complement of k_0 + k_l in so(30), and the vector zero modes form k_0 cap k_l.
2. In lemma 22's algebra C_0 = A, so k_0 = so(30), and there are no such scalars, for any algebra of this kind.
3. If C_0 = C_l = M_15(C), the massless scalars are exactly Lambda^2 V + Lambda^2 V-bar, 210 real components with colour-antitriplet counts (6, 6, 3), and the massless vectors are u(15), 225 real components.
4. A generator in k_0 cap k_l has a massless vector and no massless scalar, one in the orthogonal complement of k_0 + k_l has a massless scalar and no massless vector, and all others have neither. For aligned walls the towers are n pi/l in the first two cases and (n - 1/2) pi/l for a generator in exactly one of k_0, k_l; for k_0 spanned by T_1 and k_l by cos(phi) T_1 + sin(phi) T_2 they move to k l = n pi +- phi, the mechanism of lemma 23.
5. With colour unbroken at both walls the (1, 8) of Lambda^2(30) is a vector, so Lambda^2(30) cannot consist entirely of massless scalars; in item 3 the (24, 8) + (24, 1) + (1, 8) + (1, 1) are vectors and the

scalars are (15, 3-bar) + (10, 6) and conjugates.

Scope. Free five-dimensional Yang-Mills at trivial background, with the gauge action from a unital completion; tree-level field counts, not a quantum massless spectrum; for nonaligned walls the zero-mode statements are the subspaces above, and a nontrivial holonomy can change them. The wall algebra M_15(C) of item 3 is an input, since every f preserves the domain at D_F = 0; it says that the walls conserve quark number and the bulk does not. On a circle without walls every generator has both a massless vector and a massless scalar, and all of Lambda^2(30) appears.

Escapes, as the lemma states them. A circle without walls, nontrivial holonomy, brane-localized terms, further internal dimensions.

Proof. Lemma 38, Theorem 11.3: a gauge parameter with xi(0) in k_0 and xi(l) in k_l shifts int a_x by xi(l) - xi(0), so the linear moduli are so(30)/(k_0 + k_l); A_mu has Neumann or Dirichlet conditions at each wall by membership in k_0 or k_l, and a_x the opposite.

What it leaves open. Whether a wall term that conserves quark number derives M_15(C) at the walls; scalars in the slepton representations (24, 1) + (1, 1), which are vectors on an interval and need components along further internal directions, such as the base of lemma 32; the colour sextets (10, 6), which nothing removes in this construction.

Part S166 (For s = +1 the skew-Hermitian operators on the 30-dimensional massless block of the five light quarks commuting with j form so(30) = u(15) + Lambda^2 V, with u(15) the commutant of quark number, and for s = -1 they form usp(30) = u(15) + S^2 V; the colour antitriplets carry the counts (6, 6, 3) and (3, 6, 1))

For s = +1 the skew-Hermitian operators on the 30-dimensional massless block of the five light quarks commuting with j form so(30) = u(15) + Lambda^2 V, with u(15) the commutant of quark number, and for s = -1 they form usp(30) = u(15) + S^2 V; the colour antitriplets carry the counts (6, 6, 3) and (3, 6, 1)

Status. Proposition (Proposition 11.1). Proved by Claude Opus 5.5 on 4 November 2026; checked by its author by hand and read independently by Claude Sonnet 5.5 by hand (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md), whose amendments are applied. Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): correct, with the decomposition stated as one of real vector spaces, 435 = 225 + 210 for so(30) and 465 = 225 + 240 for usp(30), not a direct sum of Lie algebras. Ledger row `d11_so30_block` (with Proposition 11.2). Proposition 1 of paper/connes_kk_dictionary_note.tex ("The proof is linear algebra (\supp{38})"), and Proposition 11.1 with its table of counts in the section on the sign of an odd factorization of paper/real_structure_note.tex.

Setting. V = C^5 (x) C^3 with Q = 2/3 on u, c and -1/3 on d, s, b. The chirality +1 subspace of the five light quarks with colour is H_+ = {q_L} + {conj(q_R)} = V + V-bar, of dimension 30. U is unitary and grading-reversing, with U q_L = q_R, U conj(q_R) = s conj(q_L) and U J_F = s J_F U; c is J_F restricted to H_+, and j = U^{-1} c is an antiunitary on H_+ with j^2 = s. Here U only defines j: the representation used in lemma 38 does not commute with U, so this is not a doubling in the sense of lemma 06.

Statement.

1. For s = +1, in the basis V + V-bar with j(v, w) = (K^{-1} w, K v), the skew-Hermitian a with j a j^{-1} = a are a = [[alpha, B], [conj(B), conj(alpha)]] with alpha^* = -alpha and B^T = -B; they form so(30) = u(15) + (Lambda^2 V)_R as real vector spaces (435 = 225 + 210; not a direct sum of Lie algebras), and the Hermitian generator H = i a satisfies j H j^{-1} = -H.
2. For s = -1 the same computation gives usp(30) = u(15) + (S^2 V)_R, 465 = 225 + 240.

3. In the colour antitriplet, with charges (-2/3, +1/3, +4/3), Lambda^2 V contributes (S^2 5, 3-bar) with the counts (6, 6, 3) and S^2 V contributes (Lambda^2 5, 3-bar) with (3, 6, 1); the colour sextets are (Lambda^2 5, 6) and (S^2 5, 6).

Scope. The massless light block. The group statement is Proposition 8.1 of lemma 35 (35_orientifold_sign_commutant.md). So the SO(30) of lemma 21 is the symmetry of this block for s = +1, the sign that Proposition 8.1 and STRATEGY T2c associate with label 11 and with the seesaw, and the counts of lemma 21 select the same sign.

Proof. Lemma 38, Proposition 11.1: j a j^{-1} = a gives the block form, and a^* = -a gives alpha^* = -alpha and B^* = -conj(B), that is B^T = -B; then j (i a) j^{-1} = -i a.

What it leaves open. The Lambda^2 V block as a connection along the interval is 38_interval_connection_so30.md, its massless scalars are 38_interval_scalar_zero_modes.md, and its image in the finite window is 38_window_shadow_of_connection.md. The relation to the top through the (30, 2) of SO(32), with the second factor as the top's SO(2), is open.

Part S167 (The interval component of an so(30) connection compresses to the finite window through its commutator with the masses, B_ij -> -((1 - cos(l (m_i - m_j)))/(l (m_i - m_j))) B_ij, which maps the (S^2 5, 3-bar) of Lambda^2 V onto the (Lambda^2 5, 3-bar) of S^2 V, with counts (3, 6, 1), and vanishes for a universal mass)

The interval component of an so(30) connection compresses to the finite window through its commutator with the masses, B_ij -> -((1 - cos(l (m_i - m_j)))/(l (m_i - m_j))) B_ij, which maps the (S^2 5, 3-bar) of Lambda^2 V onto the (Lambda^2 5, 3-bar) of S^2 V, with counts (3, 6, 1), and vanishes for a universal mass

Status. Proposition (Proposition 11.4). Proved by Claude Opus 5.5 on 4 November 2026; checked by its author in Python on a small model (V = C^3: reality, symmetry, the blocks of item 1 and their squares) and read independently by Claude Sonnet 5.5, who also checked the compression by direct quadrature of W^* (E a_tot) W against Theorem 22.4, with five flavours in three colours (Python, to 10^-15) (historical/reviews/2026-11-04_sonnet_check_11d_10_11.md). Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report in historical/reviews/2026-11-05_terra_check_37_38.md): the matrix identities and the compression formula confirmed for a bounded constant perturbation under the window condition of Theorem 22.1; the identification of the perturbation with an inner one-form of the nonunital corner algebra at nonzero mass fails, since for invertible M every one-form coefficient vanishes at l, so the realization as an inner fluctuation needs a unital completion; the map of item 3 is SU(5)-equivariant only with the masses as a spurion; applied. Ledger row `d11_shadow`. The displayed shadow formula in the section on the real structure of paper/connes_kk_dictionary_note.tex ("\supp{38}"), and Proposition 11.4 in the section on the compression to the window of paper/real_structure_note.tex.

Setting. s = +1; a bounded constant perturbation under the window condition of Theorem 22.1; the mass block is M = U Lambda with Lambda real for j and symmetric, Lambda = diag(m_i) on flavours on both V and V-bar; a_tot is constant, equal to a in so(30) on H_+ and to U a U^{-1} on H_-.

Statement.

1. [D_F, a_tot] has the off-diagonal blocks U S and S U^{-1}, with S = [Lambda, a] real and symmetric, and [D_F, [D_F, a_tot]] is [Lambda, S] on H_+ and U [Lambda, S] U^{-1} on H_-.

2. By Theorem 22.4 the compression of E a_tot to the window is -l^{-1} int_0^l sin(x ad_{D_F})(a_tot) dx, which sends B_ij to -((1 - cos(l (m_i - m_j)))/(l (m_i - m_j))) B_ij, that is -(l/2)(m_i - m_j) B_ij to leading order, as a change of Lambda; the result is symmetric, as Proposition 9.3 requires of a finite J-compatible fluctuation.
3. For distinct masses B -> (m_i - m_j) B_ij maps (S^2 5, 3-bar) in Lambda^2 V onto (Lambda^2 5, 3-bar) in S^2 V as vector spaces (equivariantly only with the mass matrix as a spurion), with kernel the five same-flavour diquarks, five colour-antitriplet multiplets, so (6, 6, 3) - (3, 0, 2) = (3, 6, 1), and maps (Lambda^2 5, 6) into (S^2 5, 6). With the wall condition imposed on the gauge unitaries, k_l is the commutant of Lambda in so(30); imposed on pi(f) itself, as in lemma 22, k_l = 0 for invertible M.
4. For a universal mass Lambda = m the compression vanishes: S = 0, and M = m U is invariant under SO(30).

Scope. First order, the fixed compression and constant a_tot. The compression is an operator identity; a nonzero constant a_tot is not an inner one-form of the corner algebra of lemma 22 at invertible M, so its realization as an inner fluctuation needs a unital completion and the construction of the real one-form there (lemma 38, Reading). The shadow is tangent to the O(30) orbit of Lambda, so it gives no radial mode, in agreement with No-go theorem 22.7 (22_nogo_radial_mode.md), and a generator commuting with the masses has no shadow. So the counts (6, 6, 3) belong to the parent and (3, 6, 1) to its window, which reconciles Propositions 9.2 and 9.3 of lemma 36.

Proof. Lemma 38, Proposition 11.4: item 1 by direct multiplication; item 2 from sin(x ad)(a_tot) = (sin(x delta)/delta) ad(a_tot) on B_ij with delta = m_i - m_j; item 3 since (m_i - m_j) B_ij is antisymmetric in flavour when B_ij is symmetric in flavour.

What it leaves open. The moving window of Corollary 22.6, which gives -l [D_F, h] and a quadratic term (lemma 19); nonzero masses, where order one fails on the block; the relation to the top through the (30, 2) of SO(32).

Part S168 (As SU(3) x SU(2) x U(1)_Y representations, Lambda^1 + Lambda^3 + Lambda^5 of the 5-bar is one generation with nu^c, the 16, and Lambda^0 + Lambda^2 + Lambda^4 is its conjugate, with the charge X = 5 - 2k on Lambda^k)

As SU(3) x SU(2) x U(1)_Y representations, Lambda^1 + Lambda^3 + Lambda^5 of the 5-bar is one generation with nu^c, the 16, and Lambda^0 + Lambda^2 + Lambda^4 is its conjugate, with the charge X = 5 - 2k on Lambda^k

Status. Proposition (Proposition 3.2). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by enumeration; the identification of the 16 of SO(10) with the odd forms on C^5 is standard, and the ledger records it as known prior work. Checked by its author in Python (historical/11D/checks_11d.py, section B), by Terra 6.1 by hand, whose independent proof is written out in the Referee proofs of lemma 39, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_fock`. No paper cites it separately.

Setting. 5-bar = U + V with U = (3-bar, 1)_{1/3} and V = (1, 2)_{-1/2}, with 6Y = 2 on U and -3 on V, as in the quotient of 32_witten_space_hypercharge_quotient.md.

Statement.

1. Lambda^1(5-bar) = d^c + L; Lambda^3(5-bar) = Lambda^3 U + Lambda^2 U (x) V + U (x) Lambda^2 V = e^c + Q + u^c, with 6Y = 6, 1 and -4; Lambda^5(5-bar) = nu^c, since the determinant charge is 3 x 2 + 2 x (-3) = 0.
2. The odd degrees have dimension 5 + 10 + 1 = 16, the even degrees are their conjugates, and the 32 states are the finite Hilbert space of one generation with nu^c (dim H_F = 96 for three).

3. The charge of SO(10)/SU(5) is X = 5 - 2k on Lambda^k, so 3, -1 and -5 on the odd degrees.

Scope. The representation and charge count is the standard spinor construction. With the divisor classes of Proposition 2.3 added formally with the weights 2 and -3, a state of Lambda^k carries the total class of k cycles, which is its 6Y (for Q = U^1 U^2 V^1, 2 + 2 - 3 = 1); the identification of these classes with wrapped-brane states is a proposal.

Proof. Lemma 39, Proposition 3.2, and the Referee proofs there: Lambda^3 3-bar = 1, Lambda^2 3-bar = 3 and Lambda^2 2 = 1, and Lambda^k(5-bar) is dual to Lambda^{5-k}(5-bar) when the determinant is trivial.

What it leaves open. Conjecture 3.3 of lemma 39, a brane reading with no file: at N = 1 one generation and its conjugate are the exterior algebra of five fermionic zero modes of fivebranes wrapped on the baryonic cycles {U^i = 0} and {V^A = 0}, with hypercharge the homology class and X the number of fivebranes. It needs the fermionic statistics of the wrapped states, the regime N = 1 and three generations, and it rests on Proposal 3.1, which reads hypercharge as the Betti vector of Y_{3,2}.

Part S169 (On the brane coordinates (U, V) with Betti charges (2, -3), the point U = 0, V = (v, 0) is fixed by SU(3) and by T_3 + Y and moved by T_3 - Y, and the Fayet-Iliopoulos quotient at zeta != 0 has residual Z_3 or Z_2 stabilizers along CP^1 or CP^2, so it is a partial orbifold resolution)

On the brane coordinates (U, V) with Betti charges (2, -3), the point U = 0, V = (v, 0) is fixed by SU(3) and by T_3 + Y and moved by T_3 - Y, and the Fayet-Iliopoulos quotient at zeta != 0 has residual Z_3 or Z_2 stabilizers along CP^1 or CP^2, so it is a partial orbifold resolution

Status. Proposition (the proved part of Proposal 3.4 of lemma 39, with the referee amendment on the Fayet-Iliopoulos quotient), conditional on the proposed action of the Betti U(1) of Proposal 3.1 on the brane coordinates. Computed by Claude Opus 5.5 (branch 11D; ledger agent `claude`; Python, historical/11D/checks_11d.py, section B2); checked by Terra 6.1 by hand, whose proof of the stabilizer and of the residual stabilizers is written out in the Referee proofs of lemma 39, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_higgs_m2`, of kind conjecture, since the row covers the whole proposal. No paper cites it.

Setting. The cone C^5 // U(1)_Y of 32_witten_space_hypercharge_quotient.md resolved by a Fayet-Iliopoulos parameter, 2 abs(U)^2 - 3 abs(V)^2 = zeta, with the action lambda (U, V) = (lambda^2 U, lambda^{-3} V); T_3 = diag(1/2, -1/2) on V, Y(V) = -1/2 and Q = T_3 + Y.

Statement.

1. The value V = (v, 0), U = 0 on the doublet line is fixed by SU(3) and by T_3 + Y, and moved by T_3 - Y; in the quotient geometry, where U(1)_Y is divided out, the same point is fixed by T_3. Q gives U the charge +1/3 of d^c and V the charges (0, -1) of (nu, e).
2. For zeta < 0 every point has V != 0; points with U != 0 have trivial stabilizer, and the locus U = 0 has stabilizer Z_3, acting faithfully on the three transverse U coordinates, with coarse exceptional locus CP^1 and transverse singularity C^3/Z_3.
3. For zeta > 0 the locus V = 0 has stabilizer Z_2, acting by -1 on the two transverse V coordinates, with coarse exceptional locus CP^2 and transverse singularity C^2/Z_2.

Scope. Identifying the action of the Betti U(1) with the electroweak one is part of Proposal 3.4, and the quotient needs further geometric data to be a smooth resolution.

Proof. Lemma 39, Referee proofs, the proof of the proved part of Proposal 3.4: (T_3 + Y) V = (1/2 - 1/2) V = 0 on the first component, and on the level sets the stabilizers are lambda^3 = 1 on U = 0 and lambda^2 = 1 on V = 0.

What it leaves open. Proposal 3.4 itself, the Higgs as the position of the M2-brane on the doublet line with the group restored at the apex as zeta -> 0, is a proposal with no file, and the relation between M_W and zeta is the next computation. At fixed N the Kaluza-Klein scale of AdS4 x Y_{3,2} does not depend on zeta, so zeta -> 0 restores the group at a fixed tower; a joint family needs N -> infinity, against the N = 1 that Proposal 3.1 needs for the baryons to be d^c and L.

Part S170 (AdS5 x CP^2 x S^2 with G_4 = f_1 vol(CP^2) + f_2 J wedge vol(S^2) solves the equations of eleven-dimensional supergravity, with Ricci components -(f_1^2 + 2 f_2^2)/6, f_1^2/3 + f_2^2/6 and 2 f_2^2/3 - f_1^2/6, and round factors exist exactly when abs(f_1) < 2 abs(f_2))

AdS5 x CP^2 x S^2 with G_4 = f_1 vol(CP^2) + f_2 J wedge vol(S^2) solves the equations of eleven-dimensional supergravity, with Ricci components -(f_1^2 + 2 f_2^2)/6, f_1^2/3 + f_2^2/6 and 2 f_2^2/3 - f_1^2/6, and round factors exist exactly when abs(f_1) < 2 abs(f_2)

Status. Proposition (Proposition 3.5). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author in Python (historical/11D/checks_11d.py, section G: all frame components of the flux bilinear), by Terra 6.1 by hand, whose independent proof is written out in the Referee proofs of lemma 39, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_ads5`. No paper cites it.

Setting. The product metric on AdS5 x CP^2 x S^2, with J the Kähler form of CP^2, f_1 and f_2 the orthonormal-frame components of G_4, and the Einstein equation R_MN = (G_MPQR G_N^PQR - g_MN G^2/12)/12.

Statement.

1. dG_4 = 0; the Hodge dual of G_4 is vol(AdS5) wedge (f_1 vol(S^2) + f_2 J), which is closed; and G_4 wedge G_4 = 0, an eight-form on the six-dimensional factor, so the form equation holds.
2. R_AdS = -(f_1^2 + 2 f_2^2)/6, R_CP2 = f_1^2/3 + f_2^2/6 and R_S2 = 2 f_2^2/3 - f_1^2/6, with zero off the diagonal.
3. Positive curvature of the S^2 factor is equivalent to abs(f_1) < 2 abs(f_2), which also makes the CP^2 curvature positive and the AdS curvature negative, and the radii follow from the three Einstein constants.

Scope. A classical local solution with scaled Einstein factors; "round factors" means the scaled Fubini-Study metric on CP^2 and the round metric on S^2. Flux quantization, global consistency, supersymmetry and stability are not checked. The product with flux on KE_4 x S^2 appears in Pope and van Nieuwenhuizen (1989), and supersymmetric AdS5 solutions fibre the S^2 over the Kähler-Einstein base (Gauntlett, Martelli, Sparks and Waldram, hep-th/0402153); both are recalled.

Proof. Lemma 39, Proposition 3.5, and the Referee proofs there: in a frame with J = e^{12} + e^{34}, G^2 = 24 (f_1^2 + 2 f_2^2), and the contraction G_aPQR G_a^PQR is 6 (f_1^2 + f_2^2) in a CP^2 direction, 12 f_2^2 in an S^2 direction and zero in an AdS direction; the forms involved are parallel.

What it leaves open. The flux quantization, supersymmetry and stability of the solution. Its reading as the magnetic completion of the base CP^2 x S^2, against the electric completion AdS4 x Y_{3,2}, with the M2/M5 exchange acting on hypercharge as electric-magnetic duality, is Proposals 3.6 and 3.7 of lemma 39, which have no files.

Part S171 (For every homomorphism SU(3) -> SO(9), Lambda^3 R^9 (x) C and Lambda^6 R^9 (x) C have the colour content of neither family of 84, and on Y_{3,2} no tensor Kaluza-Klein mode of C_3 or of the metric is a colour triplet, so the coloured states of the 84 are not modes of C_3 there)

For every homomorphism SU(3) -> SO(9), Lambda^3 R^9 (x) C and Lambda^6 R^9 (x) C have the colour content of neither family of 84, and on Y_{3,2} no tensor Kaluza-Klein mode of C_3 or of the metric is a colour triplet, so the coloured states of the 84 are not modes of C_3 there

Status. No-go theorem in its scope (Proposition 3.8 and Corollary 3.9). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`); checked by its author in Python (historical/11D/checks_11d.py, section E: weights of Lambda^3) and by hand, by Terra 6.1 by hand, whose independent proofs are written out in the Referee proofs of lemma 39, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger rows `d11_colour84` (depending on `l21_counts`) and `d11_kk_triality` (depending on `d11_z6` and `d11_colour84`). Item (b) of Proposition 2 of paper/generations_note.tex, whose proof sketch cites the 11D files 03 and 04 (lemmas 39 and 33).

Setting. The two families of 84 of lemma 21, with colour contents 10 (3 + 3-bar) + 24 (1) and 12 (3 + 3-bar) + 12 (1) (33_nogo_colour_obstructions.md); a real nine-dimensional representation of SU(3) on the nine transverse directions of the eleven-dimensional little group; and Y_{3,2} in the quotient realization of lemma 32.

Statement.

1. (Proposition 3.8) The complexification of a real nine-dimensional representation is 9 (1), 3 + 3-bar + 3 (1) or 8 + 1, and its Lambda^3 is then only singlets, contains a 6-bar, or contains Lambda^3 8 = 1 + 8 + 10 + 10-bar + 27. So Lambda^3 R^9 (x) C, and Lambda^6 R^9 (x) C, isomorphic to it under SO(9), is the colour content of neither family.
2. (Corollary 3.9) On a background invariant under the displayed action, every tensor Kaluza-Klein mode of C_3 or of the metric on Y_{3,2} carries a representation with trivial colour centre, so none is a colour triplet, and the 60 coloured states of the first family and the 72 of the second are not such modes.

Scope. A linear colour action on the nine transverse directions; tensor modes on the stated quotient with the stated action, which needs only to preserve the background metric and flux. Together the two statements exclude the two direct routes, the little group and the tensor Kaluza-Klein tower, for a colour-equivariant identification.

Escapes, as the lemma states them. Colour acting on the internal space, outside the little group; a nonlinear or brane realization; spinor modes; a map that is not colour-equivariant, in which case the equality of the two 84s is an equality of numbers only.

Proof. Lemma 39, Proposition 3.8 and Corollary 3.9, and the Referee proofs there: the irreducibles of dimension at most nine are 1, 3, 3-bar, 6, 6-bar and 8, and a self-conjugate complexification excludes a sextet and two triplet pairs; Lambda^2 3 (x) 3-bar = 6-bar + 3, and Lambda^2 8 contains a 10. For the corollary, (omega I_3, I_2) is the quotient-circle element at lambda = e^{4 pi i/3}, so it acts trivially on every tensor bundle.

What it leaves open. The route left open is the brane one: wrapped M5 and M2 states, which carry triality through their hypercharge class (item 1 of 32_z6_kernel_tensor_modes.md). The complex case, Lambda^3 C^9, is 33_nogo_colour_obstructions.md.

Part S172 (The signed spectrum {(n pi +- theta)/l} of one channel of the interval parent depends only on theta modulo theta -> -theta and theta -> theta + pi, with fundamental domain [0, pi/2], whose ends are the periodic spectrum with a double zero mode and the antiperiodic spectrum with every value doubled)

The signed spectrum {(n pi +- theta)/l} of one channel of the interval parent depends only on theta modulo theta -> -theta and theta -> theta + pi, with fundamental domain [0, pi/2], whose ends are the periodic spectrum with a double zero mode and the antiperiodic spectrum with every value doubled

Status. Proposition (Proposition 7.1, with the referee clarification). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand; checked by its author by hand, by Terra 6.1 by hand, whose independent proof is written out in the Referee proofs of lemma 40, and by Claude Sonnet 5.5 by hand (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_angle_interval` (depending on `l22_window` and `l23_wz`). Stated in the section on the finite Dirac operator of paper/connes_kk_dictionary_note.tex ("\supp{40}").

Setting. The parent of Theorem 22.1 and a channel with singular value sigma of the Yukawa block, angle theta = l sigma and wall data X = tan(theta).

Statement.

1. The multiset {(n pi + theta)/l, (n pi - theta)/l : n in Z} determines theta modulo the group generated by theta -> -theta and theta -> theta + pi, and the orbit of theta meets [0, pi/2] in one point. In the interior the two progressions are disjoint.
2. At theta = 0 the spectrum is n pi/l with every value twice, including a zero mode of multiplicity two (periodic).
3. At theta = pi/2 the spectrum is (n + 1/2) pi/l with every value twice (antiperiodic), two copies of the spectrum of the chiral interval (31_interval_is_antiperiodic_circle.md). It is the limit X -> infinity, where the wall conditions tend to U_R(l) = Q_L(l) = 0 and the lowest eigenspace is degenerate.

So the restriction l norm(D_F) < pi/2 of lemma 22 is the choice of the fundamental domain.

Scope. Theorem 22.1 uses the strict interior, and theta = pi/2 is a boundary-condition limit with a degenerate window, so each end needs its own domain or limit. Equality of one channel spectrum does not identify complete physical backgrounds. The massless electroweak vectors at theta = pi/2 (M_W = M_Z = 0 at fixed l, with the fermion at pi/(2l)) are a statement of the two-grading model of lemma 23 (Theorem 23.6).

Proof. Lemma 40, Proposition 7.1, and the Referee proofs there: equality of two such multisets is equivalent to theta' = +-theta mod pi, by comparing the residues modulo pi/l, and n pi + theta = m pi - theta needs 2 theta in pi Z.

What it leaves open. The reading of the two ends as the two orientifold walls of type I', with the top angle as the position of the top's D8-brane, is a proposal of lemma 40; its proved part is 40_so32_wilson_line_endpoint.md. The multiplicities of k channels at a common angle are 35_wall_channel_multiplicity.md.

Part S173 (The window of the doubled torus with flux p carries the degenerate representation a on C^p and 0 on conj(C^p), and for p = 3 no unital M_3(C) representation on C^3 + conj(C^3) satisfies order zero with a unital opposite action)

The window of the doubled torus with flux p carries the degenerate representation a on C^p and 0 on conj(C^p), and for p = 3 no unital M_3(C) representation on C^3 + conj(C^3) satisfies order zero with a unital opposite action

Status. Counterexample, to the unital finite triple of the first version of Proposition 7.3, item 3. Constructed by Terra 6.1 in the referee report on the 11D line (historical/reviews/2026-11-04_terra_check_11d.md, section on Proposition 7.3), together with the unital repair, and written out in the Referee proofs of lemma 40; it is part of the ledger row `d11_torus_flux`, which records checks by hand by the author of lemma 40 (Claude Opus 5.5), by Terra 6.1 and by Claude Sonnet 5.5. The repaired datum M_p(C) + C is stated in the section on the algebra of paper/connes_kk_dictionary_note.tex.

Setting. The doubled torus of 40_doubled_torus_landau_window.md, with V = H^0(L (x) S_0) = C^p, V-bar its conjugate, and the representation generated by the compressed functions, rho(a) = a on V and 0 on V-bar.

Statement.

1. rho(1) is the identity of V and zero on V-bar, so the parent and the finite representations are degenerate on the second summand, and they lack the unital representation of the finite-data setting of lemma 22.
2. For p = 3 no complex-linear unital M_3(C) representation on C^3 + conj(C^3) satisfies order zero with a unital opposite action: the two commuting unital actions would make this six-dimensional space a module for M_3(C) (x) M_3(C)^op = M_9(C), whose nonzero modules have dimensions divisible by nine.
3. Unitizing the represented corner, A = M_p(C) + C with rho(a, lambda) = a on V and lambda on V-bar, gives a unital finite KO-6 datum with D = 0 and order zero, with one additional scalar summand.

Scope. Complex-linear representations of finite data on the doubled window.

Proof. Lemma 40, Referee proofs, Counterexample (referee) to the unital finite triple of Proposition 7.3: the dimension count for modules of M_9(C).

What it leaves open. The repair adds a scalar summand by hand. Deriving the full Standard Model algebra and its representation from such windows remains open.

Part S174 (On a flat torus with a line bundle of degree p, the doubled Dirac operator with J(psi, chi) = (C chi, -C psi) has the signs of KO-dimension six, a kernel C^p + conj(C^p) below Landau levels of order (p/A)^{1/2}, and compressions of the functions generating M_p(C), so the window carries the unital finite datum M_p(C) + C)

On a flat torus with a line bundle of degree p, the doubled Dirac operator with J(psi, chi) = (C chi, -C psi) has the signs of KO-dimension six, a kernel C^p + conj(C^p) below Landau levels of order (p/A)^{1/2}, and compressions of the functions generating M_p(C), so the window carries the unital finite datum M_p(C) + C

Status. Proposition (Proposition 7.3, with the referee amendment: the displayed action is nonunital on the doubled space, and a scalar summand gives a unital datum; the ledger records the row as conditional). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand, with Berezin-Toeplitz quantization of the torus recalled; checked by its author by hand, by Terra 6.1 by hand, whose independent proof of items 1 and 2 and of the generation step is written out in the Referee proofs of lemma 40, and by Claude Sonnet 5.5 by hand, who checked the compression in the theta-function basis for p = 2 to 5 (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_torus_flux`, with prior work recorded as partial (the fuzzy torus is known; the KO-6 doubling with restricted algebra follows lemma 16). The section on the algebra and the table row on M_p(C) of paper/connes_kk_dictionary_note.tex ("\supp{40}", "refereed with amendment").

Setting. T^2 a flat torus of area A with a spin structure, S_0 its flat spin line, L a line bundle of degree p >= 1 with a connection of constant curvature and L-bar = L^{-1}. H = L^2(S (x) L) + L^2(S (x) L-bar), D = D_L + D_{L-bar}, gamma = gamma_S on both summands, J(psi, chi) = (C chi, -C psi) with C the charge conjugation of the spin structure (antiunitary, C^2 = -1, C D = D C, C gamma_S = -gamma_S C), and C^infinity(T^2) acting by multiplication on the first summand.

Statement.

1. J^2 = 1, J D = D J and J gamma = -gamma J, the signs of KO-dimension six, and the algebra satisfies order zero.
2. ker D = H^0(L (x) S_0) + H^1(L^{-1} (x) S_0), isomorphic to C^p + conj(C^p), with gamma = +1 and -1 on the two summands; the nonzero levels are sqrt(2 abs(B) n), n >= 1, with B = 2 pi p/A, so they go to infinity as A -> 0 at fixed p.
3. The compressions of the characters to the lowest Landau level are nonzero Gaussian multiples of clock-and-shift operators U^m V^n with U V = e^{+-2 pi i/p} V U, and these span M_p(C).
4. With the algebra M_p(C) + C represented by rho(a, lambda) = a on C^p and lambda on conj(C^p), the opposite action J rho(b, mu)^* J^{-1} is mu on C^p and b^T on conj(C^p), which commutes with rho; this is a unital finite KO-6 datum with D = 0 and order zero.

Scope. At p = 3 the window has the form of the colour sector, M_3(C) on the quarks and its opposite on the antiquarks. The representation generated by the functions alone is degenerate on the second summand (40_counterexample_nonunital_torus_window.md), and the compression map is not multiplicative on the function algebra.

Proof. Lemma 40, Proposition 7.3, and the Referee proofs there: Riemann-Roch in genus one and Serre duality give the kernels; with the position split into guiding-centre and oscillator parts, P e^{ik.x} P = e^{-abs(k)^2/(4 abs(B))} e^{ik.R} P; tr(U^m V^n) = 0 unless m = n = 0 mod p, so the p^2 products are orthogonal.

What it leaves open. The quaternionic factor H from a torus with even flux and a real structure of

quaternionic type, and the full A_F from several tori or a rank-three bundle with flux. The reading for duality 1, with p counting IIB momentum on the nine-dimensional circle and SL(2, Z) acting by Morita equivalence, is a proposal, and whether p = 3 for colour has a reason there is open.

Part S175 (For the SO(32) Wilson line W_a = diag(1_30, R(2 pi a)), a = theta/pi in [0, 1/2], the centralizer is SO(32) at a = 0, SO(30) x SO(2) for 0 < a < 1/2 and S(O(30) x O(2)) at a = 1/2, where the (30)_{+-1} tower is antiperiodic, and the one-loop minimum of the flat family is a = 1/2)

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Status. Proposition (Proposition 7.2, with the referee amendment and the second-check derivation). Proved by Claude Opus 5.5 (branch 11D; ledger agent `claude`) by hand; checked by its author by hand, by Terra 6.1 by hand, whose independent proof of the full centralizer is written out in the Referee proofs of lemma 40, and by Claude Sonnet 5.5 by hand, whose derivation of the spin lift is written out in lemma 40 (reports: historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md). Ledger row `d11_far_wall` (depending on `d11_angle_interval`, `d11_wilson_top` and `l24_no_minimum`). No paper cites it separately.

Setting. By Proposition 6.1 of the 11D line (ledger row `d11_wilson_top`), the channel spectrum of 40_channel_angle_interval.md is that of the states (30)_{+-1} of SO(32) on a circle of radius R = l/pi with the Wilson line W_a, a = theta/pi.

Statement.

1. [0, 1/2] is the moduli interval of one Wilson-line eigenvalue modulo a -> -a and a -> a + 1.
2. The centralizer of W_a in SO(32) is SO(32) at a = 0, SO(30) x SO(2) for 0 < a < 1/2, and S(O(30) x O(2)) at a = 1/2, where W_{1/2} = diag(1_30, -1_2); its identity component is SO(30) x SO(2).
3. At a = 1/2 the (30)_{+-1} tower is (n + 1/2)/R, the antiperiodic tower of Proposition 1.1.
4. In Spin(32)/Z_2 the component of determinant -1 sends the lift of W_{1/2} to its negative, so the heterotic centralizer is connected; the SO(32) holonomy has period 1 in a, and its lift, in Spin(32) and in Spin(32)/Z_2, has period 2.
5. Under the hypotheses of Theorem 9 of lemma 24, with n_f > 0, the unique global minimum of the one-loop potential is t = sin^2 theta = 1, that is a = 1/2.

Scope. Item 5 assumes the flat massless determinant, its field content and wall conditions, and no further angle-dependent term; it derives no force on a supersymmetric type I' brane, whose D8 positions are flat directions.

Proof. Lemma 40, Proposition 7.2, its Referee proofs and its second-check derivation: a commuting orthogonal map preserves the eigenspaces of W_a and acts on the plane by a rotation when 0 < a < 1/2; at a = 1/2, with Clifford generators G_i, the lift of W_{1/2} is G_31 G_32, and G_1 G_31 conjugates it to its negative, which differs from 1 and from the volume element omega.

What it leaves open. The reading of W_a as fifteen D8-branes on one orientifold plane and the sixteenth at a position given by a, linear in a only at constant dilaton (37_angle_to_position_map.md), with a = 1/2 on the other plane and the one-loop vacuum carrying the top brane to the far wall, is a proposal; so is the reading of the same datum as an E8 holonomy on a Hořava-Witten wall in eleven dimensions.

Part S176 (The Pati-Salam finite data are orientable, with gamma = rho(p_L' - p_R') rho(1_4)^o - rho(1_4) rho(p_L' - p_R')^o, while the Standard Model data and the light block of colour plus electromagnetism admit no orientation cycle)

The Pati-Salam finite data are orientable, with gamma = rho(p_L' - p_R') rho(1_4)^o - rho(1_4) rho(p_L' - p_R')^o, while the Standard Model data and the light block of colour plus electromagnetism admit no orientation cycle

Status. Proposition (Proposition 41.4). Proved by Claude Fable 5.1 on 4 November 2026 by hand; confirmed by its author with numpy on the 32-state representation (the Pati-Salam cycle, script not kept). Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): correct with two amendments, the e_R obstruction and the hypothesis on D at the Pati-Salam step, both applied. Ledger row `l41_orient` (depending on `l17_no_higgs` and on the hypothesis `sheet_diagonal_ps`). The table row on orientability and Poincaré duality of paper/connes_kk_dictionary_note.tex.

Setting. The chain and the representation of 41_trace_forms_along_chain.md, with the Pati-Salam projections p_R', p_L' and p_4; an orientation cycle is c = sum rho(a_i) rho(b_i)^o with c = gamma. At the Pati-Salam step D is sheet-diagonal, of the form of Corollary 16.8 (a Pati-Salam-symmetric Yukawa with no Majorana block), since with the Majorana block order one fails there and Theorem 20.2 selects A_SM.

Statement.

1. The Pati-Salam data are orientable, with the cycle above.
2. The Standard Model data are not orientable: nu_R and J nu_R carry the same left character lambda and the same right character lambda and have opposite gradings, and so do e_R and J e_R (left character conj(lambda), right character lambda, in the real summand), so no sum of left-right products separates either pair; the failure does not depend on the presence of nu_R (Stephan, hep-th/0610097, for the KO-6 models with neutrinos).
3. The light block of the third step is not orientable: its kept mass blocks are nonzero and commute with B (Corollary 17.4), and a nonzero D commuting with both actions commutes with every orientation cycle while anticommuting with gamma; directly, d_L and d_R of one colour carry the same characters and opposite gradings. This proves the corpus statement rs_N9.

Scope. One generation with Dirac neutrinos in the stated representation, the strict projections of lemma 06 for the third step, and a sheet-diagonal D at the first. The same loss under a heavy centralizer in a four-summand toy model is orientability_under_heavy_centralizer.md; here it happens in the physical chain.

Proof. Lemma 41, Proposition 41.4: on the particles rho(1_4)^o is the identity and rho(1_4) vanishes, on the antiparticles the reverse, so the right side is +1 on p_L, -1 on p_R, -1 on p-bar_L and +1 on p-bar_R, which is gamma; a cycle takes the same value on two states with the same pair of characters, while gamma takes opposite values.

What it leaves open. A second independent reading. The result points to the topology of a Kaluza-Klein parent as the source of orientability, which the finite data along the chain do not keep (lemma 41, Scope).

Part S177 (Along the chain Pati-Salam, Standard Model, colour plus electromagnetism, the skew trace forms of the finite data have ranks 2, 2 and 0, so the trace-form Poincaré duality fails at every step, at the last because every isotypic range of the light block has graded dimension zero)

Along the chain Pati-Salam, Standard Model, colour plus electromagnetism, the skew trace forms of the finite data have ranks 2, 2 and 0, so the trace-form Poincaré duality fails at every step, at the last because every isotypic range of the light block has graded dimension zero

Status. Theorem (Lemma 41.1 and Theorem 41.2, with Remark 41.3). Proved by Claude Fable 5.1 on 4 November 2026 by hand; confirmed by its author with numpy on the 32-state representation (script not kept). Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): Lemma 41.1 correct, Theorem 41.2 correct with the three matrices recomputed, Remark 41.3 corrected (the vanishing of the form is the bimodule balance and not vectorlikeness). Ledger row `l41_forms` (depending on `l20_rank_one` and `l06_nuL`). The table row on orientability and Poincaré duality of paper/connes_kk_dictionary_note.tex.

Setting. One generation with Dirac neutrinos in the representation of lemma 06, Result 5, with dim H = 32, J(p, p-bar) = (K p-bar, K p) and gamma = +1 on p_L and p-bar_R. The chain A_PS = H_R + H_L + M_4(C) -> A_SM = C + H + M_3(C) -> B = C + M_3(C), with A_SM the centralizer under the heavy Majorana block (Theorem 20.2, rank one) and B acting on the 18-dimensional light block of d, e and nu_L with their conjugates; the minimal projection of C inside B is (1, Q_1, 0) = p_1 + p_2. One minimal projection per summand, and the trace form Q_ij = Tr(gamma rho(p_i) rho(p_j)^o), with b^o = J b^* J^{-1}.

Statement.

1. (Lemma 41.1) For a finite real triple with J^2 = +-1, J gamma = -gamma J, order zero and an even algebra, Q_ij = -Q_ji.
2. (Theorem 41.2) Q_PS = [[0, 0, -2], [0, 0, 2], [2, -2, 0]], Q_SM = [[0, -2, -2], [2, 0, 2], [2, -2, 0]] and Q_B = 0, of ranks 2, 2 and 0, with kernels spanned by p_R' + p_L' and by p_1 + p_2 - p_3. So the trace-form Poincaré duality fails at each step: at the first two because a skew form on a K_0 of odd rank is singular, at the third because the form vanishes.
3. (Remark 41.3) Q_ij is the graded dimension of the range of rho(p_i) rho(p_j)^o, and it vanishes identically exactly when every such range has equal numbers of states of the two gradings, the bimodule balance condition behind the doubling of lemma 06. This is weaker than physical vectorlikeness: the light block keeps the unpaired nu_L, with [V_L] - [V_R] = (1)_0 on SU(3) x U(1)_em, while Q_B = 0, because the form sees the full real space, where the neutral class cancels against its conjugate and lambda and conj(lambda) share one projection of the real summand C.

Scope. One generation with Dirac neutrinos; with three generations the matrices multiply by three and nothing else changes. The test is the trace form on minimal projections, and a real Kasparov duality is a different question. Lemma 41.1 is the observation of Ćaćić for KO-dimension 6, recorded in the corpus as rs_N7.

Proof. Lemma 41, Lemma 41.1 and Theorem 41.2: Tr(J A J^{-1}) = conj(Tr A) and order zero give the skew symmetry, and each entry counts, with the sign of gamma, the states on which p_i and p_j^o both act as the identity.

What it leaves open. A second independent reading (the ledger's consensus threshold). The direction lemma 41 names: along the chain the finite data keep the KO-6 signs, order zero and the first-order condition of Theorem 20.2, the data that lemma 22 derives from the interval, and the failed axioms are those a Kaluza-Klein parent would have to supply through its topology.

Part S178 (The left-action traces of the abelian generator of C and of the common colour phase are (4i, 12i) on the 32 states, (2i, 12i) with the Majorana pair removed, (-2i, 6i) with the up-type sector removed and (-4i, 6i) on the light block, and only the inherited unimodularity condition gives the photon its charges)

The left-action traces of the abelian generator of C and of the common colour phase are (4i, 12i) on the 32 states, (2i, 12i) with the Majorana pair removed, (-2i, 6i) with the up-type sector removed and (-4i, 6i) on the light block, and only the inherited unimodularity condition gives the photon its charges

Status. Proposition (Proposition 41.5). Proved by Claude Fable 5.1 on 4 November 2026 by hand. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): the first version used the element (i, 0, 0) on the spaces where only B acts and counted 26 states where there are 20; the referee corrected the generator, the counts and the traces, and added the charge computation; the proposition stands as corrected. Ledger row `l41_unimod`. The table row of paper/connes_kk_dictionary_note.tex for lemma 41.

Setting. The chain and the representation of 41_trace_forms_along_chain.md. X_lambda is the generator of the abelian direction of the summand C, taken in each algebra as its own element: (i, 0, 0) in A_SM, acting as diag(i, -i) (x) I_4 on p_R and as i on the lepton line of the antiparticles; (i, Q_i, 0) in B, acting in addition as diag(i, -i) (x) I_4 on p_L. X_c is the common colour phase, i on the three colours of the antiparticles. The unimodularity condition of Chamseddine, Connes and Marcolli is Tr A = 0 on the left action (the adjoint action u J u J^{-1} has determinant one for every unitary).

Statement.

1. Full space, 32 states, A_SM: Tr X_lambda = 4i, Tr X_c = 12i.
2. Majorana pair removed, 30 states, A_SM: 2i, 12i.
3. Up-type sector removed with nu_R kept, 20 states, B: -2i, 6i.
4. Light block of the third step, 18 states, B: -4i, 6i.

So the condition imposed on the parent gives theta_lambda + 3 theta_c = 0 (the infinitesimal form of lambda det m = 1, reproducing lambda^2 (det m)^4 at the 30-state step), and reimposed on the light block it gives -4 theta_lambda + 6 theta_c = 0. Under the adjoint action, normalized so that the electron has charge -1, the inherited condition gives (u, d, e, nu) the charges (2/3, -1/3, -1, 0) and the reimposed one gives (1/6, -5/6, -1, 0): only the inherited lift gives the photon.

Scope. One generation with Dirac neutrinos in the stated representation. The inherited lift is the one Proposition 22.8 uses, where the ten generators of u(C_F) reduce to the nine of the photon and the gluons.

Proof. Lemma 41, Proposition 41.5: the phases are counted state by state; Q_i is traceless on p_R and on p_L, so on the full space only the antilepton line contributes; removing u_L and u_R (phase +i each, three colours) and their conjugates (no abelian phase) gives -3i on each of p_L and p_R against the 4i of the antilepton line.

What it leaves open. A second independent reading. Whether the two abelian embeddings can be told apart in the window of the main paper, where lemma 22 inherits the lift from the parent, is a question for the parent, which fixes the lift.

Part S179 (The Sp(2)-orbits of the pair of marked lines (Cw, Cv) of the connector are classified by a = abs(h(w, v)) and b = abs(omega(w, v)), with a^2 + b^2 <= 1, and the orbits of the connector by a, b, abs(kappa) and the phase of kappa alpha^2 when alpha = h(w, v) is nonzero)

The Sp(2)-orbits of the pair of marked lines (Cw, Cv) of the connector are classified by a = abs(h(w, v)) and b = abs(omega(w, v)), with a^2 + b^2 <= 1, and the orbits of the connector by a, b, abs(kappa) and the phase of kappa alpha^2 when alpha = h(w, v) is nonzero

Status. Proposition (Proposition 42.3). Proved by Claude Fable 5.1 on 4 November 2026 by hand. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): the first version counted three moduli and missed the invariant kappa alpha^2; the referee supplied it, with T_kappa and T_(-kappa) as the counterexample, and the proposition stands as corrected. Ledger row `l42_moduli` (depending on `l20_rank_one`). No paper states it separately.

Setting. The connector of Theorem 20.2, the rank-one block T_kappa x = kappa K x K with K = v w^*: a complex line Cw in the core copy W, a complex line Cv in the wall copy V, and the coupling kappa, with G = Sp(2) acting on both copies of the 4 by the same representation (42_tangent_of_marked_line.md). h and omega are the G-invariant Hermitian and symplectic forms on C^4, with omega(x, y) = h(jx, y). The centralizer of T_kappa is C + H + M_3(C) (20_rank_one_block_classified_algebra.md).

Statement.

1. The G-orbits of the pair of lines (Cw, Cv) are classified by a = abs(h(w, v)) and b = abs(omega(w, v)), with a^2 + b^2 <= 1: a^2 + b^2 is the squared norm of the component of v in the quaternionic line L of w, a the overlap with l = Cw and b the overlap with jl.
2. The G-orbits of the connector T_kappa are classified by a, b, abs(kappa) and, when alpha = h(w, v) is nonzero, the phase of kappa alpha^2 = tr T_kappa(1), with 1 the identity in Hom(V, W), fixed by G. So the connector has four real moduli when alpha is nonzero and three when alpha = 0; a symmetry larger than the diagonal Sp(2), such as an independent phase on V, removes the fourth.

Scope. The proposition counts moduli; it does not identify them with a vacuum expectation value, fixes no coupling and computes no potential.

Proof. Lemma 42, Proposition 42.3: G acts transitively on the unit sphere of H^2 with stabilizer Sp(1)_a, which acts transitively on each sphere of L^perp; writing v = alpha w + beta jw + u with u in L^perp and beta = omega(w, v), rephasing w by e^{i phi} and v by e^{i theta} sends (alpha, beta) to (e^{i(theta - phi)} alpha, e^{i(theta + phi)} beta) and removes both phases of the pair of lines. The rephasings that fix T_kappa = kappa K x K multiply kappa by e^{-2i(theta - phi)} and alpha^2 by e^{2i(theta - phi)}, so kappa alpha^2 is invariant under them and under G; T_kappa and T_(-kappa) on the same pair of lines with alpha nonzero have opposite invariants and lie in different orbits.

What it leaves open. A parent in which the connector, and with it these moduli, is a field; Theorem 28.21 prescribes the windows, and the connector is not derived.

Part S180 (With one copy of the 4 of Sp(2) in each degree of the end grading, the off-diagonal part of the window operator is a scalar lambda_M I_4, invariant under Sp(2), so the equivariant window operator carries no Higgs representation; G-breaking off-diagonal blocks do)

With one copy of the 4 of Sp(2) in each degree of the end grading, the off-diagonal part of the window operator is a scalar lambda_M I_4, invariant under Sp(2), so the equivariant window operator carries no Higgs representation; G-breaking off-diagonal blocks do

Status. Proposition (Proposition 42.1). Proved by Claude Fable 5.1 on 4 November 2026 by hand, from Proposition 28.19 (Schur's lemma). Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): correct for the G-invariant block, with the escape through G-breaking blocks added. Ledger row `l42_no_higgs_in_window` (depending on `l28_tunnelling`). The table row on the Higgs doublet of paper/connes_kk_dictionary_note.tex.

Setting. The eight-state window W + V of lemma 29 on the disc bundle (29_window_case_b.md, 29_window_case_a.md), two copies of the 4 of G = Sp(2), one in each degree of the end grading: the core copy W, on which real functions of S^4 compress to M_2(H), and the wall copy V, on which real functions of CP^3 compress to M_4(C).

Statement.

1. After equivariant identifications the window operator is d_M = [[0, conj(lambda_M) I_4], [lambda_M I_4, 0]], and a phase makes lambda_M >= 0.
2. So its off-diagonal part is a multiple of the identity, G-invariant, and it transforms trivially under the isotropy group of any marked point.
3. The question of item (ii) of CORPUS_SALVAGE, whether that part transforms as T_p S^4, the Higgs representation of the first-stage note of the corpus, has the answer no for the G-invariant block.
4. (Escape, from the referee.) Hom(W, V) = 4 (x) 4 = 1 + 5 + 10 as a representation of G contains T_p S^4 = (2, 2) twice after restriction to the isotropy group of p, once in the 5 and once in the 10; a G-breaking off-diagonal block at p, such as d = mu n^a Gamma_a of Example 16.9 with n orthogonal to the axis, carries the representation of item (ii).

Scope. The Sp(2)-equivariant window of lemma 29, with one copy of the 4 in each degree of the end grading; the G-invariant block.

Proof. Lemma 42, Proposition 42.1, by Proposition 28.19: an equivariant map between two copies of the irreducible 4 is a scalar. The decomposition 4 (x) 4 = 1 + 5 + 10 under Sp(2) = Spin(5) and its restriction to Spin(4) give item 4.

What it leaves open. Where the Higgs representation sits: in the variation of the marked lines of the connector of lemma 20, once the connector is a field (42_tangent_of_marked_line.md, 42_connector_moduli.md), and in the G-breaking blocks, which the vector sector of the window (STRATEGY T14) would have to supply. The connector is an input of the window, since Theorem 28.21 prescribes the windows and the connector is not derived.

Part S181 (For a marked complex line l in the quaternionic sheet W = H^2 with quaternionic line L over p in S^4, T_L S^4 = (2, 2) = Hom_H(L, L^perp) and T_l CP^3 = (2)_{+-1} + (1)_{+-2} under Sp(1)_a x U(1)_b: the Higgs doublet of hypercharge 1/2 along the base and a charged singlet of hypercharge 1 along the twistor fibre)

For a marked complex line l in the quaternionic sheet W = H^2 with quaternionic line L over p in S^4, T_L S^4 = (2, 2) = Hom_H(L, L^perp) and T_l CP^3 = (2)_{+-1} + (1)_{+-2} under Sp(1)_a x U(1)_b: the Higgs doublet of hypercharge 1/2 along the base and a charged singlet of hypercharge 1 along the twistor fibre

Status. Proposition (Proposition 42.2). Proved by Claude Fable 5.1 on 4 November 2026 by hand; the representation theory is standard and is checked against the isotropy representation stated in lemma 28. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): correct, with the stabilizer dimensions and the ad-eigenvalues recomputed. Ledger row `142_tangent`. The table row on the Higgs doublet of paper/connes_kk_dictionary_note.tex.

Setting. G = Sp(2) acting on W = H^2 = C^4; K = Sp(1)_a x Sp(1)_b is the stabilizer of a quaternionic line L, with Sp(1)_b acting on L and Sp(1)_a on L^perp, and H = Sp(1)_a x U(1)_b the stabilizer of a complex line l in L, so G/K = S^4, G/H = CP^3 and K/H = S^2 is the twistor fibre (lemma 28). U(1)_b charges are normalized to +-1 on the doublet of Sp(1)_b, and Y = charge/2 as in the first-stage note of the corpus.

Statement.

1. The G-orbit of l = Cw is CP^3 = G/H and that of L = l + jl is S^4 = G/K.
2. T_L S^4 is the (2, 2) of Sp(1)_a x Sp(1)_b, identified with Hom_H(L, L^perp) with the action t -> q_a t q_b^{-1}.
3. T_l CP^3 = (2)_{+-1} + (1)_{+-2} under Sp(1)_a x U(1)_b, where (2)_{+-1} is the horizontal part, the restriction of T_L S^4, one complex doublet of the weak Sp(1)_a with hypercharge 1/2, and (1)_{+-2} is the tangent of the twistor fibre at l, one complex singlet of hypercharge 1 and electric charge +1.

Scope. The window of lemma 29 with its Sp(2)-equivariance, the connector of Theorem 20.2 as a marked pair of lines and a coupling, and the normalization Y = charge/2. Nothing here makes the marked lines dynamical or computes a potential.

Proof. Lemma 42, Proposition 42.2: sp(2) = sp(1)_a + sp(1)_b + m with m = (2, 2), the isotropy representation of S^4 (so(5) under so(4)); sp(1)_b = u(1)_b + (1)_{+-2} under its torus; and (2, 2) restricted to Sp(1)_a x U(1)_b is (2)_{+1} + (2)_{-1}.

What it leaves open. Whether the fibre direction (1)_{+-2} is a dynamical scalar, a gauge direction of U(1)_b, or frozen by the collar connection of lemma 29, which preserves the line summands; this is the question of the grading axis of Proposition 28.19, which marks L. A parent in which the lines of the connector vary along S^4, a Higgs field as a section over the base; lemma 43 covers horizontal changes of the connection and leaves the kernel-moving collar connection open.

Part S182 (Changing the coefficient connection by a horizontal alpha changes the Berry connection on a constant-rank kernel bundle by the fibrewise compression P alpha P, exactly, and every connection on the kernel bundle is a Berry connection of such a change exactly when the compression Gamma(F_b, u(E)) -> u(K_b) is surjective at every point, that is, when the products s_i s_j^* of a kernel frame are linearly independent)

Changing the coefficient connection by a horizontal alpha changes the Berry connection on a constant-rank kernel bundle by the fibrewise compression P alpha P, exactly, and every connection on the kernel bundle is a Berry connection of such a change exactly when the compression Gamma(F_b, u(E)) -> u(K_b) is surjective at every point, that is, when the products s_i s_j^* of a kernel frame are linearly independent

Status. Theorem (Proposition 43.1 and Theorem 43.2). Proved by Claude Fable 5.1 on 4 November 2026 by hand. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): correct with three amendments, applied: the mean-curvature term of the unitary Berry connection, the transposed adjoint, and the exclusion of vertical components of alpha, which change the fibre operators and move the kernels. Ledger row `l43_berry`. The table row on the connection on the window of paper/connes_kk_dictionary_note.tex.

Setting. pi: P -> B a fibre bundle with compact fibres F_b and a horizontal distribution; E -> P a Hermitian vector bundle with a unitary connection nabla; D = (D_b) a smooth family of fibrewise elliptic self-adjoint operators whose kernels K_b have constant dimension k, with projections P_b. The Berry connection is nabla^K_X s = P_b nabla_{X^h} s, with X^h the horizontal lift; it is unitary when horizontal transport preserves the fibre measure, and in general the unitary connection adds (1/2) k(X) with k the mean-curvature form of the fibres (Bismut; Berline-Getzler-Vergne, chapter 9), a term common to all the connections compared. C_b(X) = P_b X P_b on Gamma(F_b, End(E)) has the adjoint C_b^*(Y) = sum_{ij} Y_{ij} s_i s_j^* in a unitary frame s_1, ..., s_k of K_b.

Statement.

1. (Proposition 43.1) For nabla' = nabla + alpha with alpha in Omega^1(P, u(E)) vanishing on vertical vectors, the Berry connection is nabla^K + beta with beta_b(X) = C_b(alpha(X^h) restricted to F_b), exactly; a fibrewise twisted Dirac operator uses only the vertical part of the coefficient connection, so such an alpha leaves the family and its kernels unchanged. A vertical component changes D_b, moves the kernels and changes the Berry connection at first order through the second fundamental form; such components are excluded.
2. (Theorem 43.2) The following are equivalent: every unitary connection on K -> B is the Berry connection of nabla + alpha for some horizontal alpha; C_b : Gamma(F_b, u(E)) -> u(K_b) is surjective for every b; for every b and a unitary frame of K_b, the k^2 sections s_i s_j^* of End(E) over F_b are linearly independent.
3. When these hold, alpha(X^h) = C_b^*(C_b C_b^*)^{-1} beta_b(X), extended by zero on vertical vectors, is a smooth preimage.

Scope. Constant-rank kernels, compact fibres, unitary connections with volume-preserving horizontal transport or the mean-curvature term; changes of the connection with no vertical component. The theorem is linear algebra fibre by fibre with a smoothness remark.

Proof. Lemma 43, Proposition 43.1 and Theorem 43.2: with the kernels unchanged, P_b applied to alpha(X^h) s is C_b(alpha(X^h)) s; C_b is surjective on u(K_b) exactly when C_b^* is injective, its kernel being closed under the adjoint; and C_b C_b^* is positive, invertible and smooth in b.

What it leaves open. A second independent reading. The engineered connection of the corpus on the Berger family of S^5 with flux (1, 1, 2), with its integrality statement for eta^2, stays a record that the theorem does not redo (Remark 43.4). The collar connection of lemma 29 is a vertical change at the wall and moves the kernel, so it is outside the theorem; a statement that covers it must include the motion of the kernels (STRATEGY T15, open). The application to the two sheets of lemma 29 for scalar changes is 43_scalar_changes_cp3_s4.md.

Part S183 (Scalar changes of the coefficient connection attain every connection on the kernel bundle exactly when the functions <s_i, s_j> are independent on the fibre, which holds for H^0(CP^3, O(1)) and fails for functions pulled back from S^4, whose compressed image has real dimension six)

Scalar changes of the coefficient connection attain every connection on the kernel bundle exactly when the functions <s_i, s_j> are independent on the fibre, which holds for H^0(CP^3, O(1)) and fails for functions pulled back from S^4, whose compressed image has real dimension six

Status. Corollary (Corollary 43.3). Proved by Claude Fable 5.1 on 4 November 2026 by hand. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): both computations confirmed; the last clause of the first version, which read the collar connection of lemma 29 as an instance of the matrix-valued case, was an error and is withdrawn (that connection acts at the wall, on the CP^3 copy, and moves the kernel). Ledger row `143_scalar` (depending on `143_berry` and `110_toeplitz`). The table row on the connection on the window of paper/connes_kk_dictionary_note.tex.

Setting. The family of 43_berry_connection_criterion.md with alpha restricted to scalar horizontal forms, alpha = i a (x) 1_E with a real; the kernel H^0(CP^3, O(1)) of the Dolbeault-Dirac operator twisted by O(1), with unitary frame s_i = z_i/abs(z); and the twistor fibration CP^3 -> S^4.

Statement.

1. Scalar changes attain every connection on K exactly when the k^2 functions <s_i, s_j> on F_b are linearly independent.
2. On CP^3, <s_i, s_j> = z-bar_i z_j/abs(z)^2 are sixteen independent functions, so every connection on the rank-four kernel bundle is the Berry connection of an abelian change of the coefficient connection; in matrix form Q(w^* B w/w^* w) = (B + tr B)/5 (Example 16.9, item 2, from the fourth-moment formula of Proposition 10.1), a bijection of Hermitian matrices with eigenvalues 1/5 and 1.
3. For functions pulled back from S^4 the compressed image is the real span of 1 and Gamma_1, ..., Gamma_5, of dimension six (a real function from S^4 compresses to a Hermitian element commuting with j, Example 16.9, item 1; the span is attained by Proposition 10.1), so scalar changes along the base attain only the connections whose difference from nabla^K lies in that span, six of sixteen directions.

Scope. For scalar changes this separates the two sheets of lemma 29: the wall copy, functions on CP^3, attains every connection; the core copy, functions on S^4, six directions of sixteen. Whether matrix-valued horizontal changes of a restricted class reach the other ten is a further application of criterion 3 that has not been made.

Proof. Lemma 43, Corollary 43.3: Theorem 43.2 with the scalar subalgebra in place of Gamma(u(E)); the second-moment identity maps Herm(4) onto itself, and Example 16.9 with Proposition 10.1 gives the statement for S^4.

What it leaves open. A second independent reading. The collar connection of lemma 29 is outside the statement, so the question of STRATEGY T15, whether that connection is a choice or a construction, stays

open; the compression of the twistor coordinates used in item 3 is 10_toeplitz_twistor_coordinates.md.

Part S184 (On the neutral block of the seesaw family the light mass is 2 t^{2 alpha} abs(y)^2 / (t + sqrt(t^2 + 4 t^{2 alpha} abs(y)^2)), zero for alpha < 1/2, abs(y)^2 at alpha = 1/2 and divergent above)

On the neutral block of the seesaw family the light mass is 2 t^{2 alpha} abs(y)^2 / (t + sqrt(t^2 + 4 t^{2 alpha} abs(y)^2)), zero for alpha < 1/2, abs(y)^2 at alpha = 1/2 and divergent above

Status. Proposition (Proposition 44.2). Proved by Claude Fable 5.1 on 4 November 2026, by hand (a 2 x 2 characteristic polynomial). Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): confirmed, with the eigenvalues at t = 10^4 for alpha = 1/4, 1/2, 3/4 checked against the asymptotics (Python); amendments on abs(y)^2, on the spectrum as plus and minus the roots, and on the bounded blocks, applied. Ledger row `l44_light_mass`. The cancellation-free form of the root is the one recorded in the corpus (historical/corpus/research_checks_knowledge.md, section 2.6) with m = t^alpha y. Addresses STRATEGY T2c.

Setting. The neutral block spanned by nu_L and nu_R with their conjugates, with the Majorana entry t on (nu_R, J nu_R) and the Dirac entry s = t^alpha y between nu_L and nu_R, as in 44_seesaw_scaling_retained_algebra.md: the symmetric mass matrix [[0, s], [s, t]] of lemma 06, Proposition 6.

Statement. The spectrum of t T_R + t^alpha V_nu on the four states is plus and minus the roots mu_+- of mu^2 - t mu - t^{2 alpha} abs(y)^2 = 0 (characteristic polynomial lambda^4 - (t^2 + 2 abs(s)^2) lambda^2 + abs(s)^4), and the light mass is abs(mu_-) with

```
mu_-(t) = - 2 t^{2 alpha} abs(y)^2 / (t + sqrt(t^2 + 4 t^{2 alpha} abs(y)^2)) ~ - abs(y)^2 t^{2
↪  alpha - 1}.
```

It tends to zero for alpha < 1/2, to -abs(y)^2 for alpha = 1/2 (the Schur complement -V_nu T_R^{-1} V_nu, the seesaw mass), and diverges for alpha > 1/2. In one generation the other Yukawa blocks act on other species and leave the neutral block alone.

Hypotheses. One generation; the block structure of lemma 06, Result 5.

Proof. Lemma 44, Proposition 44.2: the 2 x 2 block [[0, s], [s, t]] has the displayed characteristic polynomial, and the small root is written without cancellation.

What it leaves open. Nothing in the statement; its use is the comparison with the retained algebra (44_seesaw_scaling_retained_algebra.md), which shows that, under the bounded-commutator criterion, the regime alpha = 1/2 with a finite light mass is reached in the algebra C + M_3(C).

Part S185 (Along the seesaw scaling the algebra retained by bounded commutators is C + M_3(C) for any growing Dirac block, so under that criterion a finite seesaw mass and the Standard Model algebra belong to two different limits)

Along the seesaw scaling the algebra retained by bounded commutators is C + M_3(C) for any growing Dirac block, so under that criterion a finite seesaw mass and the Standard Model algebra belong to two different limits

Status. Proposition (Proposition 44.1), with Corollary 44.3. Proved by Claude Fable 5.1 on 4 November 2026, by hand. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): the commutants recomputed and confirmed; amendments applied on the

criterion (the bounded-commutator criterion agrees with the order-one criterion of Theorem 20.2 for T_R and differs from it for V_nu), on the conjugate sector, on y nonzero and on the bounded blocks. Ledger row `144_algebra` (depending on `120_rank_one`, Astra 6, checked by Terra 6.1). Addresses STRATEGY T2c.

Setting. One generation in the representation of lemma 06, Result 5. T_R is the Majorana block on (nu_R, J nu_R), V_nu the neutrino Dirac block y(|nu_L><nu_R| + h.c.) with its conjugate, y nonzero, S the other Yukawa blocks; the family is D_t = t T_R + t^alpha V_nu + S, t -> infinity, 0 <= alpha < 1. The retained algebra is the set of a in A_PS = H_R + H_L + M_4(C) with sup_t |||[D_t, rho(a)]|| finite.

Statement.

1. For alpha = 0 the retained algebra is A_PS cap {T_R}' = C + H + M_3(C) (Theorem 20.2).
2. For 0 < alpha < 1 it is A_PS cap {T_R}' cap {V_nu}' = {(lambda, Q_lambda, m)} = C + M_3(C), the centralizer of the top block of lemma 06 and the commutant C_F of lemma 22 for the Standard Model data; the conjugate states give no condition and m is unconstrained.
3. (Corollary 44.3) Under this criterion a finite light neutrino mass at the heavy Majorana limit needs alpha = 1/2 (44_light_mass_along_scaling.md), along which the retained algebra is C + M_3(C); the Standard Model algebra needs alpha = 0, along which the light mass vanishes as 1/t. The two coincide only at finite t, where the commutators with A_SM are bounded by t^{1/2} abs(y) up to a constant and |||[S, rho(a)]||.
4. Under the order-one criterion of Theorem 20.2 the conclusion differs: V_nu is part of the Standard Model Dirac operator and satisfies order one on A_SM, so A_SM stays admissible at every alpha. The two criteria agree for T_R and differ for V_nu. Read physically (lemma 20, Result 9), an unbounded commutator is a gauge direction whose mass grows: a Dirac neutrino mass growing without bound breaks SU(2).

Hypotheses. The strict limit t -> infinity with the bounded-commutator criterion of the corpus (the criterion behind the heavy projection of lemma 06); one generation (any nonzero entry of Y_nu gives the same conclusion for three).

Proof. Lemma 44, Proposition 44.1: the coefficients of t and t^alpha must commute with rho(a), and rho(a) V_nu nu_R = V_nu rho(a) nu_R forces q_11 = lambda and q_21 = 0, so q = Q_lambda.

What it leaves open. Which criterion governs the retained algebra along a scaling where a block of the Standard Model Dirac operator itself grows; the two criteria part company exactly here. Which algebra the physical seesaw selects is a question about a finite t, where the Dirac neutrino mass is of the electroweak scale. A seesaw keeping the Standard Model algebra under the bounded-commutator criterion needs a bounded Dirac coupling with the light mass from a lower scale, or a second neutral state pairing with nu_L (lemma 06). In the interval parent every block of D_F is wall data, and in nine dimensions the Majorana block could not be a bulk mass (Remark 44.4).

Part S186 (The 18-state orientable KO-6 bimodule of the corpus is the two-sheet construction of lemma 16 over the two orbit types of the Weyl crossed product)

The 18-state orientable KO-6 bimodule of the corpus is the two-sheet construction of lemma 16 over the two orbit types of the Weyl crossed product

Status. Theorem (Theorem 45.2), with Proposition 45.1 on the fibres of the crossed product. Proved by Claude Fable 5.1 on 4 November 2026, by hand, from the finite content of Corollary 16.8 (row `116_two_sheets`, refereed by Terra 6.1) and standard crossed-product facts. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): both results correct with amendments, applied (the complex algebra is not the complexification of A_F; the Z_2 characters enter the selection; the intersection form over the complex algebra; the formal use of Corollary 16.8; the grading as a chosen central element); two errors in Remark 45.3 corrected (six irreducible repre-

sentations of C*(W), not eight; the corpus obtains H at the orbit type (a, a, a), not at the fixed point). Ledger rows `l45_fibres`, `l45_two_sheets`. Addresses STRATEGY T16; it proves structurally the corpus row rs_N18 (the finite quotient) and identifies rs_N6 (the 18-state bimodule).

Setting. X = T^6 = A x E with the Weyl group W = S_3 x Z_2 of SU(3) x SU(2) acting, X/W = CP^2 x CP^1; B = C(X) x W. The fixed point x_0 and a three-point orbit O with stabilizer S_2 x Z_2 are the two orbit types used; the selected representations are quotients of the crossed product, never spectral windows of multiplication operators (the corpus, section 6.13, shows no finite-rank projection compresses C(T^6) multiplicatively).

Statement.

1. (Proposition 45.1) Evaluation at x_0 maps B onto C*(W) = C^4 + M_2(C) + M_2(C), which has six irreducible representations; restriction to O maps B onto C(O) x W = M_3(C*(S_2 x Z_2)) = M_3(C)^4; selecting the trivial character and the standard representation of S_3, each with a character of Z_2, at x_0, and one character of the stabilizer on O, gives a surjection B -> C + M_2(C) + M_3(C), the complex algebra with M_2(C) in place of H.
2. (Theorem 45.2) The finite content of Corollary 16.8 with the sheets V_X = C + C^2 (algebra C + M_2(C), grading G = diag(1, -1, -1) as a chosen central element, d_X = D_v) and V_Y = C^3 (algebra M_3(C), d_Y = 0) gives H_F = Hom(V_Y, V_X) + Hom(V_X, V_Y), of dimension 18, with J_F(x, y) = (y^dagger, x^dagger), Gamma_F(x, y) = (G x, -y G) and D_F(x, y) = (D_v x, y D_v): the KO-6 signs, order zero, order one, nonzero one-forms and the real orientability relation Gamma_F = pi(z c) c° with c = (G, 1), z = (1, -1). This is, in Hilbert space, D, J and grading, the bimodule of the corpus, section 6.5. Over the complex algebra its intersection form on minimal projections is [[0, 0, 1], [0, 0, -1], [-1, 1, 0]], skew of rank 2 on a K_0 of rank 3 with kernel (1, 1, 0); the corpus form [[0, 0, 1], [0, 0, -2], [-1, 2, 0]] is the one over the real algebra C + H + M_3(C), where the minimal projection of H has rank two.

Hypotheses. Finite groups and finite orbits; the Dirac operator D_v and the grading are inputs; the two representations are selected by hand among the six of C*(W) and the four characters of the stabilizer.

Proof. Lemma 45, Proposition 45.1 and Theorem 45.2; the proofs are there: evaluation at a fixed point respects the covariance relation, Green's imprimitivity gives the orbit fibre, and the data satisfy the hypotheses of Theorem 16.7, whose grading uses the sheet sign and the central element G of the fixed-point fibre, so no spinor bundle on T^6 and no spin lift of the Weyl action enters.

What it leaves open. (a) The quaternionic real form H inside M_2(C): at the fixed point the lift of the inversion of E is central and the spin-sector algebra complexifies, and the corpus obtains C + H at the orbit type (a, a, a), 3a = 0, with a point of E that is not 2-torsion (canonical section 10), so the two halves of T16 as framed use different first sheets. (b) A Dirac operator from the geometry in place of D_v, the coupled-window problem of lemma 20 with a discrete group, where the spin lift and its obstruction rs_N20 return. (c) A criterion for the selection of the two representations. Poincaré duality fails in the trace form at rank 2, as at every step of lemma 41.

Part S187 (A Yukawa block X = q + i q' on a doublet, with q and q' quaternions, gives the two members the masses m_+-^2 = |q|^2 + |q'|^2 +- 2 |Im(conj(q) q')|: equal exactly when X is quaternionic up to a complex factor, as every block compressed from a function on S^4 is; one massless member exactly when X has rank one with the two halves of equal norm; and when split, the masses fix a maximal torus of SU(2)_R, the hypercharge circle of the connector exactly when its line is an eigenline)

A Yukawa block X = q + i q' on a doublet, with q and q' quaternions, gives the two members the masses m_+-^2 = |q|^2 + |q'|^2 +- 2 |Im(conj(q) q')|: equal exactly when X is quaternionic up to a complex factor, as every block compressed from a function on S^4 is; one massless member exactly when X has rank one with the two halves of equal norm; and when split, the masses fix a maximal torus of SU(2)_R, the hypercharge circle of the connector exactly when its line is an eigenline

Status. Proposition (Proposition 46.2), with Corollary 46.4. Proved by Claude Fable 5.1 on 5 November 2026 by hand; the mass formula and the rank-one limit were checked with numpy on random quaternions (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_46.md): the mass formula confirmed on 2000 random pairs, the convention of L_q corrected, the case q = 0 added, the identification of the rank-one block with the top-only family of lemma 21 withdrawn (the block is shared with the lepton doublet), the identification of the circle with lemma 42's U(1)_b made conditional on the eigenline condition, the stabilizers added; the formula is a restatement of the bidoublet mass matrices of the left-right and spectral Pati-Salam literature in the variables (q, q'); applied. Ledger rows `l46_doublet_masses` (depending on `l46_odd_blocks`) and `l46_hypercharge_direction` (depending on `l46_doublet_masses`, `l42_tangent` and `l20_rank_one`). Not yet in a paper.

Setting. The sheet W = H^2 with axis p of 46_odd_blocks_bidoublet.md; L_p and L_p^perp identified with H = C^2 through j, so that X in Hom_C(L_p, L_p^perp) is L_q + i L_{q'} with q, q' quaternions acting by left multiplication, the j-linear and the j-antilinear halves. The Yukawa block of a sheet-diagonal operator of Corollary 16.8 is [[0, X^*], [X, 0]] (x) 1_V, the same for the lepton and the three colours, and the masses of the two members of a doublet are the singular values of X.

Statement.

1. m_+-^2 = |q|^2 + |q'|^2 +- 2 |Im(conj(q) q')|.
2. m_+ = m_- exactly when conj(q) q' is real, that is X = z L_q with z complex; the odd part of the compression of any real function on S^4 is j-linear and gives equal masses.
3. m_- = 0 < m_+ exactly when q' = q u with u a unit imaginary quaternion, that is X = L_q (1 + i L_u) = 2 L_q P_u of rank one, with the two halves of equal norm; tensored with 1_V this gives the up-type quark and the Dirac neutrino the mass m_+ and the down-type quark and the charged lepton zero, so the top-only family of lemma 21 needs a lepton block different from the quark block. Equivalently det X = |q|^2 - |q'|^2 + 2 i Re(conj(q) q'), and rank one is the null-vector condition.
4. With |q'| = t |q| and Re(conj(q) q') = |q| |q'| cos phi, m_+-^2 = |q|^2 (1 + t^2 +- 2 t sin phi).
5. (Corollary 46.4) If m_+ != m_-, the commutant of X^* X in Sp(1)_b is a circle with weight lines the two mass eigenlines in L_p, a maximal torus of the kind lemma 42 marked by hand, equal to lemma 42's U(1)_b (the stabilizer of the connector's line) exactly when that line is one of its weight lines; under it T_p S^4 restricts to one complex doublet of hypercharge 1/2. With the axis p under the quaternionic line of the connector's line w, the Majorana state is a Dirac-mass eigenstate exactly when w is an eigenline of X^* X. The stabilizer of X in K_p is the diagonal Sp(1) (custodial) for equal

masses and a circle for split ones, and X and the connector's line leave a common circle unbroken exactly under the eigenline condition. For a j-linear block neither the torus nor the position of w is determined.

Scope. One generation; three generations replace q and q' by 3 x 3 quaternionic matrices with the same structure in the singular values. The block is an input; nothing here derives it or a potential.

Proof. Lemma 46, Proposition 46.2 and Corollary 46.4: X^* X = |q|^2 + |q'|^2 + 2 i L_w with w = Im(conj(q) q'), and left multiplication by an imaginary quaternion has eigenvalues +- i |w|; the rank-one case is the equality case of |w| <= |q| |q'| <= (|q|^2 + |q'|^2)/2; the commutant of a Hermitian 2 x 2 matrix with distinct eigenvalues in SU(2) is its maximal torus.

What it leaves open. An independent reading. A parent in which the block is the compression of a function on the wall sheet, so that the up-down splitting is the fibre dependence of that function, and a potential driving the third-generation block to rank one (lemma 46, Scope).

Part S188 (On the quaternionic sheet W = H^2 with grading axis p, the Hermitian blocks odd for the grading form the complex bidoublet Hom_C(L_p, L_p^perp) = (2, 2) (x) C of Sp(1)_a x Sp(1)_b; the quaternionic half is T_p S^4, the odd part of the compressed coordinates of S^4, and the anti-quaternionic half lies outside M_2(H), inside the compression of the functions on CP^3)

On the quaternionic sheet W = H^2 with grading axis p, the Hermitian blocks odd for the grading form the complex bidoublet Hom_C(L_p, L_p^perp) = (2, 2) (x) C of Sp(1)_a x Sp(1)_b; the quaternionic half is T_p S^4, the odd part of the compressed coordinates of S^4, and the anti-quaternionic half lies outside M_2(H), inside the compression of the functions on CP^3

Status. Proposition (Proposition 46.1). Proved by Claude Fable 5.1 on 5 November 2026 by hand; the Clifford relations, the quaternionic structure, the dimensions 16, 8 and 16 and the K_p-invariance of the two four-dimensional spans were checked with numpy (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_46.md): correct with amendments, the existence and phase of j stated and item 3 sharpened to the fibre-odd functions on CP^3; applied. Ledger row `l46_odd_blocks` (depending on `l10_toeplitz` and `l16_two_sheets`). Not yet in a paper.

Setting. W = C^4 with the Hermitian Clifford generators Gamma_1, ..., Gamma_5 and the antiunitary j, j^2 = -1, commuting with them (Example 16.9: the kernel H^0(CP^3, O(1)) with the quaternionic structure of the twistor space; Proposition 10.1: Q(x_a) = Gamma_a/5). End_H(W) = M_2(H) is the real span of 1, Gamma_a, Gamma_a Gamma_b, with unitary group G = Sp(2). For p in S^4, Gamma_p = p^a Gamma_a has the quaternionic eigenlines L_p and L_p^perp, stabilizer K_p = Sp(1)_a x Sp(1)_b (Sp(1)_b on L_p, Sp(1)_a on L_p^perp), and Odd_p is the set of Hermitian Y with Gamma_p Y = -Y Gamma_p.

Statement.

1. Odd_p = {[[0, X^*], [X, 0]] : X in Hom_C(L_p, L_p^perp)}, real dimension 8, the complex bidoublet (2, 2) (x)_R C of K_p.
2. Its j-linear part Odd_p cap M_2(H) is the real span of the Gamma_a with a orthogonal to p: the maps with X in Hom_H(L_p, L_p^perp) = T_p S^4, which by Proposition 10.1 are the odd parts of the compressed coordinates of S^4.
3. Its j-antilinear part is the real span of the i Gamma_a Gamma_p, a orthogonal to p, which is i times the (2, 2) inside sp(2); it lies outside M_2(H), hence outside the compression of the real functions of

S^4, and it is the compression of the real functions on CP^3 odd under the fibre involution [w] -> [jw] (Q(f o j) = j Q(f) j^{-1}); complex functions on the base with the projections of the grading also reach it.

4. End_C(W) = 1 + 5 + 10 under G restricts to K_p as (1, 1); (1, 1) + (2, 2); (3, 1) + (1, 3) + (2, 2), with the spans Gamma_p, the Gamma_a (a orthogonal to p), the Gamma_a Gamma_b (a, b orthogonal to p) and the Gamma_a Gamma_p; the even part is End(L_p) + End(L_p^perp).

Scope. Finite linear algebra on the quaternionic sheet; the axis p as an input (Proposition 28.19); the identification of the two halves with the two sheets is by representation type, not a derivation of the block from a function.

Proof. Lemma 46, Proposition 46.1: a Hermitian Y anticommuting with Gamma_p exchanges its eigenlines; Hom_H between two quaternionic lines is a copy of H, spanned by the four Gamma_a with a orthogonal to p; Y -> j Y j^{-1} is an involution of Odd_p whose -1 part is spanned by the i Gamma_a Gamma_p; the decomposition of 4 (x) 4 under Spin(5) and Spin(4).

What it leaves open. An independent reading. The role of the two halves in the masses is 46_doublet_mass_formula.md; the cross blocks of the window operator are 46_window_cross_blocks.md.

Part S189 (In the eight-state window of lemma 29 every cross block between the core and the wall states of a bounded multiplication operator is bounded by the exponential that bounds the tunnelling mass, the cross block of a coordinate of S^4 is c_M Gamma_a, and a window operator with block lambda + mu n . Gamma has the eigenvalues +- (lambda^2 + |mu|^2 +- 2 lambda Re mu)^{1/2}, each of multiplicity two)

In the eight-state window of lemma 29 every cross block between the core and the wall states of a bounded multiplication operator is bounded by the exponential that bounds the tunnelling mass, the cross block of a coordinate of S^4 is c_M Gamma_a, and a window operator with block lambda + mu n . Gamma has the eigenvalues +- (lambda^2 + |mu|^2 +- 2 lambda Re mu)^{1/2}, each of multiplicity two

Status. Proposition (Proposition 46.3), conditional on the weighted estimates in the proof of Proposition 29.8 (lemma 29, as amended by its second reading). Proved by Claude Fable 5.1 on 5 November 2026 by hand; the spectrum was checked with numpy (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_46.md): the hypothesis corrected from a bounded operator to a bounded multiplication operator (counterexample B = v u^*), the weights of Proposition 29.8 extended to the whole bulk, the statement read as an upper bound and as the first-order block of T_M + B, the phase of c_M as an invariant relative to lambda_M and c_M != 0 left open; applied. Ledger row `l46_cross_blocks` (depending on `l29_tunnelling` and `l28_tunnelling`). Not yet in a paper.

Setting. The window W_M + V_M of Theorem 29.7, u a unit core state and v a unit wall state, B the multiplication by a bounded measurable endomorphism field of E; the identification W_M = V_M = 4 of Proposition 28.19; a window operator with off-diagonal block d = lambda + mu n^a Gamma_a, lambda >= 0, mu complex, n a unit vector in R^5.

Statement.

1. For every eta in (0, 1), |<v, B u>| <= C_eta ||B|| exp(-(1 - eta)(M/2) int_e^{l - a} f dr) for M >= M_eta: every off-diagonal block of the compression of a bounded multiplication operator to the eight

states is bounded by the exponential that bounds lambda_M; no lower bound is proved, and the block is the first-order term of the window block of T_M + B.

2. For B the multiplication by a coordinate x_a of S^4 pulled back to X, P_V x_a P_W = c_M Gamma_a, with |c_M| bounded as in item 1, arg(c_M conj(lambda_M)) an invariant and c_M != 0 not shown: the window operator can carry G-breaking blocks in the 5, hence the (2, 2) of item (ii) of CORPUS_SALVAGE, at that order.
3. The eight eigenvalues of the window operator with block d are +-|lambda + mu| and +-|lambda - mu|, each of multiplicity two, on the quaternionic lines L_n and L_n^perp of the two copies of the 4.

Scope. The window of Theorem 29.7 in case (b); bounded local operators; the Agmon weights of Proposition 29.8 for both the core and the wall states.

Proof. Lemma 46, Proposition 46.3: the weights g and g-tilde of Proposition 29.8 sum to the constant (1 - eta)(M/2) int f on the bulk and bound the end regions, so int |u| |v| is exponentially small; Schur's lemma for the 5 inside End_C(4) = 1 + 5 + 10; d^* d = lambda^2 + |mu|^2 + 2 lambda Re(mu) n . Gamma.

What it leaves open. An independent reading. Whether the cross blocks of operators in the 10 (two-forms of S^4, or functions on CP^3 whose harmonics contain the 10) are nonzero at that order; the Yukawa block of the bimodule is not a cross block of the window (Proposition 28.20), and lemma 46 places it in End(W) instead.

Part S190 (On the Berger five-sphere the Killing-form coupling ratio of the central circle with charge c on the coordinates to the SU(3) is e^2/g_s^2 = (3 + eta^2)/(24 eta^2 c^2); with the spinor charge unit c = 2/3 it is 3(eta^2 + 3)/(32 eta^2), equal to Connes' matching value 3/8 at the round point)

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Status. Proposition (Proposition 47.1) with Corollary 47.2. Proved by Claude Fable 5.1 on 5 November 2026 by hand; the sphere moments, the formula at three values of eta and the charges of the coordinates in Lambda^2(4) were checked with numpy (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_47_48.md): Proposition 47.1 correct (moments and the form at four values of eta reproduced), Corollary 47.2 correct with amendments (the coupling formula is Weinberg's, in the form of Barnes, Gorbatov, Intriligator and Wright; the eta-curve belongs to the Einstein-term reduction, the round-point value being source-independent by Schur's lemma; the enhancement at the round point proved through the Ricci eigenvalues; lemma 03 reread); the round-point value is implicit in print in holographic form, with c = 2/3 the superconformal R-charge. Ledger rows `l47_killing_form` and `l47_coupling_ratio`. The lead came from GitHub issue 1 of the repository (round point, by Schur's lemma). Not yet in a paper.

Setting. S^5 in C^3 with the Hopf fibre scaled by eta (Berger metric g_eta, volume V_eta = eta V_1); U(3) isometric for every eta, SO(6) at eta = 1. Killing vectors K_T(z) = i T z for Hermitian traceless T on C^3 (tr T^2 = 1/2 for the colour generators) and K_c(z) = i c z for the central circle with charge c on the coordinates. The tree-level Kaluza-Klein kinetic form is <K_a, K_b> = int g(K_a, K_b) dvol (Weinberg 1983, after Witten's ansatz of 1981), with g_a^{-2} proportional to <K_a, K_a>; the isometry group is U(3) x| Z_2 for eta != 1 and O(6) at eta = 1, and g_eta is Einstein only at eta = 1.

Statement.

1. (Proposition 47.1) <K_T, K_T'>_eta = V_eta (3 + eta^2) tr(T T')/12, <K_c, K_c>_eta = V_eta

eta^2 c^2, <K_T, K_c>_eta = 0.

2. (Corollary 47.2) e^2/g_s^2 = (3 + eta^2)/(24 eta^2 c^2). The two formulas of the parked job 04 are c = 1/3 and c = 2/3. At the round point the Killing spinors form the 4 of SU(4), which restricts to U(3) as 3_{-1/3} + 1_{1} with the singlet at unit charge, and the coordinates carry 2/3 in Lambda^2(4) = 6; with this unit e^2/g_s^2 = 3(eta^2 + 3)/(32 eta^2), which is 3/8 at eta = 1, also directly by Schur's lemma on the trace form of su(4) (tr Q^2 = 4/3 against tr T^2 = 1/2). In the Einstein-term reduction the function decreases from infinity to 3/32 as eta grows, taking 3/8 at the round point only; a source supporting the non-Einstein shape changes the curve off the round point and leaves 3/8 at eta = 1 unchanged, by Schur's lemma.

Scope. Tree-level reduction of the Einstein-Hilbert term; the Berger family; the charge unit as an input fixed by the spinor at the round point; the reading of SU(3) as colour and of the central circle as electromagnetism is that of the 9 end of T18. The running and the physical value of the ratio are outside the statement.

Proof. Lemma 47, Proposition 47.1 and Corollary 47.2: the vertical component of K_T at z is <z, T z>; the moments int <T z, T' z> = tr(T T')/3 and int <z, T z> <z, T' z> = tr(T T')/12 on the unit sphere of C^3; the branching of the 4 and of its exterior square under U(3).

What it leaves open. A second independent reading. The general rule (referee): at an enhanced point where su(3) + u(1) sits in a simple algebra, the ratio is tr T^2 / tr Q^2 in any representation, so the matching value appears exactly when the embedding gives Q the SU(5) index; this reduces the question for the other geometric families of T18 to an index computation.

Part S191 (The lift of the Weyl group S_3 x Z_2 to the spinors of T^6 is (Dic_3 x Z_4)/Z_2; the spinorial part of R[Dic_3] at the orbit type (a, a, a) is C + H, complexified at the fixed point by the central lift of the inversion; the two-sheet bimodule of lemma 45 with its first sheet there carries the real Standard Model algebra C + H + M_3(C))

The lift of the Weyl group S_3 x Z_2 to the spinors of T^6 is (Dic_3 x Z_4)/Z_2; the spinorial part of R[Dic_3] at the orbit type (a, a, a) is C + H, complexified at the fixed point by the central lift of the inversion; the two-sheet bimodule of lemma 45 with its first sheet there carries the real Standard Model algebra C + H + M_3(C)

Status. Theorem (Theorem 48.3), with Propositions 48.1 and 48.2. Proved by Claude Fable 5.1 on 5 November 2026 by hand, from lemma 45 and the representation theory of the binary dihedral group, without consulting the corpus construction that reached the same algebra at the same orbit type (as recorded by the referee of lemma 45). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_47_48.md), with an explicit Cl(6), the lifted groups, the spinor decompositions, the fibres and the bimodule axioms checked numerically: Propositions 48.1 and 48.2 correct with amendments (the lift group is also S_3 x Z_4; a matrix corrected), Theorem 48.3 correct with amendments (the corner at O_1 against the quotient at O_2 is a choice; the selection statement restated as the forcing of the orbit types by the target); the misdescription of lemma 45's complex algebra corrected. Ledger rows `l48_spin_lift`, `l48_spinorial_algebras`, `l48_real_two_sheets` (depending on `l45_fibres`, `l45_two_sheets`, `l16_two_sheets`). Not yet in a paper.

Setting. X = A x E = T^6 with the flat metric and the periodic spin structure, W = S_3 x Z_2 acting by permutations of the z_i and by the inversion iota of E; the lift group W-tilde, the preimage of dW in Spin(6); the real crossed product C(X, R) x W-tilde and its spinorial quotient B_spin (u_{-1} = -1). The orbit types of lemma 45: the fixed point, the orbits of stabilizer S_3 (points (a, a, a), 3a = 0, with a point

of E that is not 2-torsion; size two), and the three-point orbit with stabilizer S_2 x Z_2.

Statement.

1. (Proposition 48.1) Transpositions act on T A = a (x) R^2 as rotations by pi in a two-plane and lift to elements of order four (rs_N20: no homomorphic lift of S_3, nor of any subgroup containing a transposition without the inversion); three-cycles lift to order three; the lift of S_3 is Dic_3; iota lifts to a central element of square -1; W-tilde = (Dic_3 x Z_4)/Z_2, of order 24, isomorphic to S_3 x Z_4 since the twisted copy of S_3 pairing odd permutations with the inversion lifts homomorphically. The periodic spin structure is the only S_3-invariant one on A.
2. (Proposition 48.2) R[Dic_3] = R + R + M_2(R) + C + H, and its spinorial part is C + H: the conjugate pair of characters in which the lifted transposition acts as +-i, and the faithful representation, which is quaternionic. At the fixed point the spinorial part of R[W-tilde] is C + C + M_2(C), every spinorial representation being of complex type because the central lift of iota acts as +-i, so there is no quaternionic summand there even over R (lemma 45 used the untwisted complex group algebra at that point). At the three-point orbit it is C + C. Each spinorial irreducible representation of these three groups occurs in the spinor fibre.
3. (Theorem 48.3) The fibre of B_spin at an orbit of type (a, a, a) is M_2(C + H), with corner C + H; at the three-point orbit M_3(C + C), one pair giving M_3(C). The data of Theorem 45.2 with the first sheet at such an orbit (the corner there, the full quotient at the three-point orbit: a choice) give the 18-state bimodule with the KO-6 signs, order zero, order one, nonzero one-forms and the real orientability relation over the real algebra C + H + M_3(C), with the corpus intersection form [[0, 0, 1], [0, 0, -2], [-1, 2, 0]]. The spinorial condition fixes the algebra on each of the six orbit types, and the orbit type (a, a, a) is the only one with a quaternionic summand, the three-point orbit the only one with matrix size three, so the target forces the two orbit types; V_X keeps one of the two conjugate characters, a choice of submodule of the spinor fibre chi_1 + chi_3 + V_f, on which C acts as (lambda, conj(lambda)) and no choice is made.

Scope. Flat T^6, periodic spin structure, finite orbits; D_v and the grading as inputs; the real forms decided by the Frobenius-Schur types of the spinorial representations of the lifted stabilizers.

Proof. Lemma 48, Propositions 48.1, 48.2 and Theorem 48.3: rho(tau) (x) 1 = 1 - 2P with P a projection of rank two; Dic_3 from the orders 3 and 4 and the relation x a x^{-1} = a^{-1}; the character table of Dic_3 with its Frobenius-Schur types; a central element acting as +-i forces complex type; Green's imprimitivity with real coefficients and the twisted action; Theorem 16.7 is formal for the real form.

What it leaves open. A second independent reading. Item (b) of Remark 45.3, a Dirac operator from the geometry: the torus Dirac operator with the W-tilde action is now in place and its compression to the two sheets is the next computation of T16; a real structure from the geometry in place of the adjoint flip of Corollary 16.8; the choices (corner against quotient, one character or the whole fibre, the pair at the three-point orbit); the physical reading of the orbit of type (a, a, a) against the fixed point. Leads from the referee: the whole spinor fibre at that orbit carries, through Corollary 16.8, a 24-state bimodule over C + H + M_3(C), the bimodule form of one quark generation, with the grading as the A-chirality restricted to the stabilizer algebra.

Part S192 (The spinor fibre of the four-torus with its Clifford action and chirality is the quaternionic sheet of Example 16.9 with the chirality as the grading axis; the lifted Weyl stabilizer generates C + H with C the Cartan of H_R spanned by the lifted transposition, which is the hypercharge circle; the hypercharge-invariant Yukawa blocks are the diagonal ones, and the whole fibre carries one quark generation)

The spinor fibre of the four-torus with its Clifford action and chirality is the quaternionic sheet of Example 16.9 with the chirality as the grading axis; the lifted Weyl stabilizer generates C + H with C the Cartan of H_R spanned by the lifted transposition, which is the hypercharge circle; the hypercharge-invariant Yukawa blocks are the diagonal ones, and the whole fibre carries one quark generation

Status. Propositions 49.1 to 49.5. Proved by Claude Fable 5.1 on 5 November 2026 by hand, from lemmas 46 and 48; Proposition 49.5 is the lead of the referee of lemma 48, checked numerically there, written with its proof. The Clifford structure, the lifts, their action on the two chiralities, the real span of dimension 6 and the invariant Yukawa blocks (complex dimension 2 under the lifted transposition, 0 under Dic_3) were checked with numpy (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_49.md), who rebuilt every object: correct with amendments (orientation making Sigma_R the line of H_R; order-six lifts; the eigenline condition for lemma 42's torus; the unitary group {+-1}; nonzero block and the restriction condition in 49.5) and one phrase in error in 49.3 (the invariance group is the circle of x on both chiralities, the C_F of Chamseddine, Connes and Marcolli); the identifications with their Proposition 2.11, Definition 2.20 and Proposition 2.30 are the referee's. Ledger rows `149_sheet_is_spinor_fibre`, `149_hypercharge_is_transposition`, `149_invariant_yukawa`, `149_orbit_types_groups`, `149_quark_generation`. Not yet in a paper.

Setting. The flat four-torus A = a (x) R^2 of lemma 48 with the periodic spin structure; Sigma_4 = Sigma_+ + Sigma_-, Spin(4) = Sp(1)_+ x Sp(1)_-, the quaternionic structure j, the Clifford multiplication c(v) and the chirality Gamma; Gamma_a = c(e_a), Gamma_5 = Gamma; the lifted stabilizer Dic_3 = <a, x> of the orbit type (a, a, a).

Statement.

1. (49.1) (Sigma_4, Gamma_1, …, Gamma_5, j) is the sheet W = H^2 of Example 16.9 with axis p = e_5: Gamma_p is the chirality, L_p and L_p^perp are the two chiralities, K_p = Spin(4), and the bidoublet Hom_H(L_p, L_p^perp) = T_p S^4 is the Clifford action of the tangent space of A, a Spin(4)-equivariant isomorphism R^4 -> Hom_H(Sigma_+, Sigma_-). The even algebra is End_H(Sigma_+) + End_H(Sigma_-) = H_R + H_L.
2. (49.2) The three-cycles act trivially on one chirality Sigma_R and by order three on the other Sigma_L; x acts with eigenvalues +-i on both. The real algebra generated by Dic_3 is C + H with C = span_R(1, x|_{Sigma_R}) acting as (lambda, conj(lambda)) on Sigma_R, the Cartan of H_R, and H = End_H(Sigma_L): the Standard Model subalgebra of the Pati-Salam pair. The circle of the lifted transposition is the T_3R torus of Sp(1)_R, the same for the three transpositions; it is lemma 42's U(1)_b exactly when the connector's line is one of its weight lines, and Corollary 46.4's torus exactly when the mass eigenlines are its weight lines.
3. (49.3) With x acting as L_i on both lines, the odd blocks commuting with x (the circle on both chiralities, whose span is the C_F of Chamseddine, Connes and Marcolli, their photon condition on D_F) are X = diag(y_1, y_2), complex dimension two, the Y_u, Y_d of the spectral Standard Model; none is invariant under the circle on Sigma_R alone; X = L_a + i L_c with a = (y_1 + conj(y_2))/2, c = (y_1 - conj(y_2))/(2i); the quaternionic ones have y_2 = conj(y_1). No nonzero odd block

commutes with all of Dic_3.

4. (49.4) The stabilizers in S_3 of the points of A are S_3 (nine points), (mirror planes) and trivial; the lifted stabilizer algebras on Sigma_4 are C + H, C (the circle of x on both chiralities, the C_F restricted to the first sheet and the unbroken circle of Corollary 46.4) and R, with unitary groups U(1) x SU(2), U(1) and {+-1}. The Clifford element of a displacement along a mirror is a quaternionic block commuting with x.
5. (49.5) Corollary 16.8 with V_X = Sigma_4, A_{F,X} = C + H, gamma_X = Gamma (for the orientation of the Setting), a nonzero d_X, V_Y = C^3, M_3(C) gives a 24-state bimodule over C + H + M_3(C), the quark summand H_F^(3) of Chamseddine, Connes and Marcolli for one generation with the grading the chirality of A, with the KO-6 signs, order zero, order one, nonzero one-forms and the orientability relation, and the intersection form [[0, 0, 2], [0, 0, -2], [-2, 2, 0]] they print; the 18-state Hilbert space of Theorem 48.3 is the sub-bimodule on one weight line, preserved by D_F exactly when X vanishes on the other.

Scope. The flat A with the periodic spin structure; the convention on which chirality is Sigma_R depends on the orientation and the lift; the E factor and the inversion are outside; the motion of the point along A is a reading, the proved content being the stabilizer algebras.

Proof. Lemma 49, Propositions 49.1 to 49.5: Cl(4) with its chirality is Cl(5) on C^4; isoclinic rotations lie in one Sp(1) factor and d tau is a rotation by pi in one plane, so x has two imaginary components; the image of R[Dic_3] is its spinorial part C + H (lemma 48); a map commuting with L_i preserves its eigenlines; the three-cycles have no fixed vectors on Sigma_L; the stabilizers of points of A by the coordinates; Theorem 16.7 formally for the real subalgebra.

What it leaves open. A second independent reading. The lepton generation, which needs a connecting operator identifying the two copies of C (the order-one condition of Chamseddine, Connes and Marcolli, Proposition 2.11; the connector of lemma 20); the Dirac operator of the torus at a point of a mirror with a constant three-form term, the candidate that supplies both halves of an x-invariant Yukawa block (the anti-quaternionic half is i times the Clifford multiplication by a three-form, which no displacement supplies); the E factor, which doubles the fibre.

Part S193 (The compression of 5 w'^* B w'/w'^* w' on CP^3 is B + tr(B): the grading and the projectors on the marked lines are compressions of height functions, and the top block is the anticommutator of a compressed base coordinate and a compressed height function of the wall sheet)

The compression of 5 w'^* B w'/w'^* w' on CP^3 is B + tr(B): the grading and the projectors on the marked lines are compressions of height functions, and the top block is the anticommutator of a compressed base coordinate and a compressed height function of the wall sheet

Status. Proposition 50.2. Proved by Claude Fable 5.1 on 5 November 2026 by Schur's lemma and the two normalizations Q(1) = 1 and Q(5 x_a) = Gamma_a (Proposition 10.1), with a check by the sphere moments; Monte Carlo on CP^3 with 4 x 10^5 points (numpy, not kept) for Q(x_1), Q(5 |<., w>|^2) and Q(5 <., B .>). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_50.md): correct with amendments (the identity is Example 16.9, item 2, extended complex-linearly; the sign of delta' is fixed by the definition of J_w; the Clifford identity carries a factor 2). Ledger row `l50_height_functions`. Not yet in a paper.

Setting. The compression Q of lemmas 10 and 16 to the level-one kernel H^0(CP^3, O(1)) = C^4, in the convention of lemma 10 (Q(f) = 4 int f P_{w'} dmu, extended complex-linearly), used in lemma 46, item 3: Q(1) = 1 and Q(5 x_a) = Gamma_a. The Fubini-Study distance d, cos d([w'], [w]) = |<w', w>|/(|w'| |w|).

Statement. For every B in End_C(C^4), Q(5 w'^* B w'/w'^* w') = B + tr(B) 1. Hence

1. 5 Q(p . x) = Gamma_p = 2 Pi_p - 1: the grading is the compression of the height function 5 cos d(., p) of S^4 at the axis.
2. Q(5 cos^2 d(., [w]) - 1) = P_w: the projector on the connector's line is the compression of the height function of CP^3 at the marked point; likewise P_v = Q_V(h_v) on the second sheet.
3. The block of Proposition 50.1 is Y = m {5 Q(n . x), Q(h_w)}; the Majorana block is supported on Q(h_w) Hom(V, W) Q_V(h_v); the j-antilinear half of Y is i c(delta') Gamma_p with delta' = -J_w h/2 (sign fixed by the definition of J_w; -J_mu h/2 for any rank-one block c(h) P_mu), the three-form term of lemma 49 whose dual is the quarter turn of the Higgs vector in the complex structure of the Majorana point, and lemma 49's formulas give m_+ = m, m_- = 0 for delta = h/2, delta' orthogonal of the same length.

Scope. The convention of lemma 10, Q(5 x_a) = Gamma_a; the identity is Example 16.9, item 2, extended complex-linearly, the level-one case of Berezin's covariant and contravariant symbols on CP^3.

Proof. Lemma 50, Proposition 50.2: B -> Q(f_B) is U(4)-equivariant, End(C^4) = C + sl(4) under conjugation, and the two normalizations fix the two constants; the moments E|z_1|^4 = 1/10, E|z_1 z_2|^2 = 1/20 on the unit sphere of C^4 give the check 2 P_w + (1 - P_w); the three-form by Proposition 46.1 (3) and the comparison of the two j-antilinear blocks on w.

What it leaves open. The same identities for the compressions of lemma 29's window at finite mass, where the core compression of a function has a j-antilinear half of order M^{-1/2} (lemma 46, lead 3).

Part S194 (The rank-one Yukawa block with the eigenline condition is the Clifford action of a Higgs vector times the projector on the connector's line: one point of the twistor fibre carries the Majorana block and the top's right-handed states)

The rank-one Yukawa block with the eigenline condition is the Clifford action of a Higgs vector times the projector on the connector's line: one point of the twistor fibre carries the Majorana block and the top's right-handed states

Status. Proposition 50.1. Proved by Claude Fable 5.1 on 5 November 2026 by hand, from lemmas 20, 42 and 46; the isometry n -> c(n) w, the spectrum and the two halves of the block, the complex structure J_w and the action of the circle were checked with numpy (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_50.md): one error corrected (the up-type assignment is an input: with the universal block both cases keep 5/8 and 8/3) and amendments (notation; the hypercharge -1/2 of the doublet in J_w; the stabilizer proof; prior work in Chamseddine-Connes-Marcolli, Proposition 3.5). Ledger row `l50_top_block`. Not yet in a paper.

Setting. The quaternionic sheet W = H^2 with axis p, the lines L_p and L_p^perp, K_p = Sp(1)_a x Sp(1)_b, the Clifford action c(n) = n^a Gamma_a for n in T_p S^4; the connector of Theorem 20.2 on the lines Cw in L_p and Cv in V, with centralizer A_SM = {(lambda, q, m)}, lambda acting on w and conj(lambda) on jw; the dictionary of Chamseddine, Connes and Marcolli, lambda on nu_R and u_R.

Statement. Let X be a Yukawa block of a doublet with masses m_- = 0 < m_+ = m, X = m ell mu^* (Proposition 46.2 (2)).

1. n -> c(n) w is a real isometry of T_p S^4 onto L_p^perp; so X = c(h) P_mu for a unique h in T_p S^4 with |h| = m, the Hermitian block is Y = {c(h), P_mu}, with j-linear half c(h)/2 and j-antilinear half {c(h), P_mu - P_{j mu}}/2, both of operator norm m/2.
2. The eigenline condition of Corollary 46.4 (Cw an eigenline of X^* X) holds exactly when C mu = Cw or C mu = C jw, exactly when the block and the connector leave a common circle unbroken.

3. C mu = Cw gives the mass to the up-type right-handed states nu_R and u_R and C mu = C jw to e_R and d_R; with the universal block both cases keep 5/8 and 8/3 (Proposition 21.1, last statement); for a quark-only block only the first does, and the first case is taken as an input. Then X = c(h) P_w: the point [w] of the twistor fibre over p carries the Majorana block (on P_w Hom(V, W) P_v) and the massive right-handed states (on P_w Hom(V, W)), and the massive left-handed line is C c(h) w.
4. The joint stabilizer is the circle U(1)_h with b_theta = e^{i theta} on w and a_theta = e^{i theta} on c(h) w; c(n) commutes with it; the complex structure J_w on T_p S^4, c(J_w n') w = i c(n') w, is orthogonal, and U(1)_b acts on (T_p S^4, J_w) by scalars: it is the complex Higgs doublet (2)_{+-1} of Proposition 42.2 (hypercharge -1/2 in J_w, the conjugate doublet through which u_R couples; +1/2 in -J_w = J_{jw}), with h a vector of it. The blocks commuting with U(1)_h on both chiralities are c(n)(y_1 P_w + y_2 P_{jw}), the diagonal blocks of Proposition 49.3; with three generations, c(n)(Y_u (x) P_w + Y_d (x) P_{jw}), and the top-only family is Y_u = m_t P_g, Y_d = 0.

Scope. Finite linear algebra; the axis, the connector and the block are inputs; one generation except in item 4; the up-type assignment is an input, supported by Proposition 21.1 for a quark-only block.

Proof. Lemma 50, Proposition 50.1: the anticommutator of c(n) and c(n') is 2 n . n' and their commutator is anti-Hermitian, which gives the isometry; X^* X = m^2 P_mu has eigenlines C mu and C j mu; the stabilizer computation of Corollary 46.4 for a rank-one block; the action t -> t b^{-1} of Proposition 42.2 on Hom_H(L_p, L_p^perp).

What it leaves open. A parent in which h, [w] and v are fields; the three-generation order-one defect of the split block is the same computation and is stated in lemma 50 without a separate proof.

Part S195 (The centralizer in A_SM of the top-only Dirac operator is C + M_3(C) with the summand C embedded in H along the top's left-handed line: the algebra of the label 9, the retained algebra of lemma 44 and the corpus row rs_F1)

The centralizer in A_SM of the top-only Dirac operator is C + M_3(C) with the summand C embedded in H along the top's left-handed line: the algebra of the label 9, the retained algebra of lemma 44 and the corpus row rs_F1

Status. Proposition 50.4. Proved by Claude Fable 5.1 on 5 November 2026 by hand; the real dimension 20 of the centralizer and the forced embedding were checked with numpy (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_50.md): correct with amendments (the gauge group up to a finite group; the transfer of Proposition 41.5 by an Sp(1)_a conjugation; a lepton block on another left line drops the dimension to 19). Ledger row `l50_top_centralizer`. Not yet in a paper.

Setting. D_F = Y_q (x) (1 - P_v) + S_{T_kappa} on H_F with Y_q = {c(h), P_w}, h != 0, the lepton block zero (the top-only family, one generation), the connector on w and v, and A_SM = A(T_kappa) = {(lambda, q, m)} of Theorem 20.2.

Statement. {a in A_SM : [D_F, pi(a)] = 0} = {(lambda, lambda_h, m)} = C + M_3(C), of real dimension 20, lambda_h the embedding of lambda in H_L with weight line C c(h) w, q c(h) w = lambda c(h) w. Its unitary group is U(1) x U(3); with the unimodularity inherited from A_SM (Proposition 41.5) it acts, up to a finite group, as U(1)_em x SU(3) with the charges (2/3, -1/3, -1, 0): the gauge group of the label 9. On it there is no scalar fluctuation and the top mass is explicit (Proposition 17.3). The algebra coincides with lemma 44's retained algebra along the seesaw scaling, with the same embedding when the block is universal, and with the corpus row rs_F1, whose embedding is now fixed by h.

Scope. Algebraic centralizer by the criterion [D_F, a] = 0; Counterexample 18.8 shows that mass decoupling in the spectral action does not select it dynamically; one generation; the connector commutes with all of

A_SM, so kappa plays no role.

Proof. Lemma 50, Proposition 50.4: [Y_q, a_W] on w, jw and L_p^perp; the condition q c(h) w = lambda c(h) w for the normal operator q; j-linearity of q fixes it on j c(h) w; Proposition 41.5 for the charges.

What it leaves open. A rank-one lepton block on the same J_w-line gives the same centralizer (dimension 20, referee); for a rank-one lepton block {c(h_l), P_w} whose left line C c(h_l) w differs from C c(h) w, the centralizer drops to R + M_3(C), real dimension 19 (referee): q must have two non-proportional lambda-eigenvectors, which forces lambda to be real, so the alignment of the Higgs vectors of the blocks is what keeps the electromagnetic circle; a dynamical selection, outside the spectral action at fixed cutoff.

Part S196 (A sheet-diagonal Yukawa block split along projections of the second sheet keeps order one exactly when the projections commute with the second-sheet algebra: universal on the Pati-Salam M_4(C), with the lepton Yukawas equal to the quark ones, and separable exactly on the connector's algebra C + M_3(C), with the lepton line the connector's line)

A sheet-diagonal Yukawa block split along projections of the second sheet keeps order one exactly when the projections commute with the second-sheet algebra: universal on the Pati-Salam M_4(C), with the lepton Yukawas equal to the quark ones, and separable exactly on the connector's algebra C + M_3(C), with the lepton line the connector's line

Status. Proposition 50.3. Proved by Claude Fable 5.1 on 5 November 2026 by hand; the order-one defects of the split block on M_4(C) (nonzero) and on C + M_3(C) (zero to 10^-14), and of the universal block on M_4(C) (zero), were checked with numpy on random data (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_50.md): one error corrected in the closing sentence (the top-only block alone keeps order one on H_R + H_L + C + M_3(C), real dimension 28; the connector reduces the sum to A_SM) and amendments (Krajewski's first-order rule as the general criterion; the split block lies outside the compressed form d_X (x) 1). Ledger row `l50_split_after_connector`. Not yet in a paper.

Setting. The data of Corollary 16.8 with V_X = W, A_X an even algebra containing the grading Gamma_p, V_Y = V = C^4 with A_Y in M_4(C); projections P_1, ..., P_r on V summing to 1 and odd Hermitian blocks Y_i on W; D x = sum Y_i x P_i, D y = sum P_i y Y_i.

Statement. D is self-adjoint, odd, J_F-invariant and of order zero, with order-one defect sum_i [Y_i, a_X] x [b_Y, P_i]. (1) Equal Y_i: order one for every A_Y. (2) P_i commuting with A_Y: order one for every Y_i. (3) Linearly independent Y_i: order one exactly when every P_i commutes with A_Y; for Y_l (x) P_v + Y_q (x) (1 - P_v), exactly when Y_l = Y_q or [A_Y, P_v] = 0. (4) On A_Y = M_4(C) the block is universal: the Dirac neutrino and the up quark, the charged lepton and the down quark, share their masses in each generation (the unified Yukawas k^nu = k^u, k^e = k^d of the spectral Pati-Salam model); the commutant of P_v in M_4(C) is C + M_3(C), the second-sheet algebra of a connector on v (Theorem 20.2), on which the split block keeps order one. (5) If a split block on a line v' and a connector on v both keep order one with colour End(v^perp) retained, then Cv' = Cv. In general, with the grading in A_X, order one holds exactly when D commutes with the right action of A_Y (Krajewski's rule). So the top-only family of lemma 21 (Y_l = 0, Y_q = {c(h), P_w} (x) P_g) is a sheet-diagonal operator on the bimodule of Corollary 16.8, outside its compressed form d_X (x) 1, with order one exactly when the second-sheet algebra lies in the commutant C + M_3(C) of P_v, all of H_R + H_L being allowed on the first sheet; with the connector on the same line added, order one of the sum holds exactly on A(T_kappa) = C + H + M_3(C).

Scope. Sheet-diagonal operators with projections on the second sheet; the three-generation form is the same computation.

Proof. Lemma 50, Proposition 50.3: block computation of [[D, a], b°]; with a_X = Gamma_p the first factors -2 Gamma_p Y_i are independent; the commutant of a rank-one projection in M_4(C); a rank-one projection commuting with End(v^perp) vanishes on v^perp.

Prior work. Krajewski (arXiv:hep-th/9701081, sections 3.4, 4.1 and 4.2): the first-order rule of which the criterion is a case; Chamseddine-Connes-Marcolli, Theorem 2.21; Chamseddine and Connes (arXiv:0706.3688, section 4); Chamseddine, Connes and van Suijlekom (arXiv:1304.8050, sections 4, 5 and 8): the unified case k^nu = k^u, k^e = k^d, the decoupling of Sigma, and the Standard Model vacuum aligning Delta, Sigma and phi on the lines of Propositions 50.1 and 50.3; the forcing of the shared lepton line (item 5) was not found (paper/LITERATURE.md, section P).

What it leaves open. A parent whose wall data produce the projections, that is a lepton line as a field.

Part S197 (The torus Dirac operator with a Wilson line and a torsion commutes with the lift of a transposition exactly when both data lie in its mirror plane; then the window block commutes with the lifted transposition on both chiralities, the condition [D_F, C_F] = 0 of Chamseddine, Connes and Marcolli, and these are all the invariant blocks; nonzero data along a mirror break S_3 to that transposition, and no nonzero data are invariant under S_3)

The torus Dirac operator with a Wilson line and a torsion commutes with the lift of a transposition exactly when both data lie in its mirror plane; then the window block commutes with the lifted transposition on both chiralities, the condition [D_F, C_F] = 0 of Chamseddine, Connes and Marcolli, and these are all the invariant blocks; nonzero data along a mirror break S_3 to that transposition, and no nonzero data are invariant under S_3

Status. Proposition 51.2. Proved by Claude Fable 5.1 on 5 November 2026 by hand; the covariance x c(v) x^{-1} = c(d tau v), the commutation exactly for data in the mirror plane and the fixed space of a three-cycle were checked with numpy (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_51_52.md): correct with amendments (the Wilson line as a gauge class; the wording on the symmetric points). Ledger row `l51_photon_condition`. Not yet in a paper.

Setting. As in Proposition 51.1; tau a transposition of S_3 acting on A = a (x) R^2 by rho(tau) (x) 1, with mirror plane P_tau = n_tau^perp (x) R^2 (the tangent plane of the mirror z_i = z_j, a complex line of A) and lift x of order four (lemma 48), tau-tilde psi(z) = x psi(tau^{-1} z).

Statement. tau-tilde D_{A,H} tau-tilde^{-1} = D + c(d tau A) + i c(d tau eta) Gamma, so D_{A,H} commutes with tau-tilde exactly when A and eta lie in P_tau (since A is a class modulo 2 pi Lambda^*, the gauge-covariant condition d tau A - A in 2 pi Lambda^* reduces to d tau A = A for |A| < pi/l). Then Y = c(A) + i c(eta) Gamma commutes with x on both chiralities, that is with the algebra C = span_R(1, x) of the mirror stratum (Proposition 49.4), the restriction of C_F to the first sheet, so [D_F, C_F] = 0 (Proposition 49.3) holds; the blocks commuting with x are exactly these, a real four-dimensional family. A nonzero A or eta in P_tau is fixed by tau only (P_tau meets P_tau' in 0; a three-cycle fixes no vector), so D_{A,H} is S_3-invariant exactly when A = eta = 0, and a first sheet at a symmetric point carries no Yukawa block from these data. For v in P_tau^perp, x c(v) x^{-1} = -c(v): the charged Higgs directions.

Scope. The lift of the transposition acting on the whole torus with the periodic spin structure; the identification with C_F is that of Proposition 49.3.

Proof. Lemma 51, Proposition 51.2: D commutes with lifted isometries; x in Spin(4) commutes with Gamma; the two halves are independent; the dimension count against Proposition 49.3; the planes P_tau meet in 0 since the normals n_tau are independent in the two-dimensional a.

What it leaves open. A derivation of the choice of mirror; the three mirrors as three photon circles (lemma 49).

Part S198 (For a Wilson line and a torsion in a mirror plane the doublet has equal masses exactly when the torsion is parallel to the Wilson line, masses |A| +- |eta| exactly when perpendicular, and a massless member exactly when eta = +- J A, the quarter turn in the complex structure of the elliptic curve; the up and down Yukawas are A + J_w eta and A - J_w eta in the complex structure J_w = +-J of the Majorana line; the finite data at the mirror stratum are the quark sector of the label 9 with explicit masses)

For a Wilson line and a torsion in a mirror plane the doublet has equal masses exactly when the torsion is parallel to the Wilson line, masses |A| +- |eta| exactly when perpendicular, and a massless member exactly when eta = +- J A, the quarter turn in the complex structure of the elliptic curve; the up and down Yukawas are A + J_w eta and A - J_w eta in the complex structure J_w = +-J of the Majorana line; the finite data at the mirror stratum are the quark sector of the label 9 with explicit masses

Status. Proposition 51.3. Proved by Claude Fable 5.1 on 5 November 2026 by hand; the rank-one case eta = +- J A, the perpendicular case and the fixed space of a three-cycle were checked with numpy (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_51_52.md): one error corrected in item 2 (the heavier member is fixed by the sign of the oriented area <J_w A, eta>, which the orientation of A does not fix) and amendments (the intrinsic form |y_mu| = |A + J_mu eta|; the correlated sign in item 3; the orientability relation fails at the mirror). Ledger row `l51_rank_one_pattern`. Not yet in a paper.

Setting. A, eta in the mirror plane P_tau with the complex structure J = 1 (x) J_E of A; phi the angle between them; the finite data of Corollary 16.8 with V_X = Sigma_4, A_{F,X} = span_R(1, x), d_X = Y, V_Y = C^3, M_3(C).

Statement. m_+-^2 = |A|^2 + |eta|^2 +- 2 |A| |eta| sin phi. (1) Equal masses exactly when eta is parallel to A (the custodial case). (2) m_+ = |A| + |eta|, m_- = ||A| - |eta|| exactly when eta is perpendicular to A, eta in R J A; then for |eta| <= |A| the Wilson line is the mean mass of the doublet and the torsion half its splitting, m_up = |A| + |eta| and m_down = |A| - |eta| when eta is a positive multiple of -J_w A, exchanged for the opposite sign (the up-type member on the Majorana line w, an input). (3) m_- = 0 exactly when eta = +- J A; then m_+ = 2 |A| and Y = {c(2A), P_mu} with J_mu = -+J for eta = +-J A, the block c(h) P_w of lemma 50 with h = 2A when the Majorana line is mu. (4) On each weight line mu of x the coupling has modulus |A + J_mu eta|; with the up-type member on w, y_u = A + J_w eta and y_d = A - J_w eta in (P_tau, J_w), so |y_u|^2 = |A|^2 + |eta|^2 - 2 <J_w A, eta>. The finite data at the mirror stratum are a 24-state bimodule over C + M_3(C) with the KO-6 signs, order zero and order one, with every one-form zero, explicit masses (Proposition 17.3) and the algebra equal to the centralizer of

the block (Proposition 50.4), the label 9; the orientability relation fails, since the sums of products a b° lie in span_R(1, x) (x) M_3(C) while Gamma_F takes opposite signs on the two +i weight lines.

Scope. One generation, one Wilson line and one three-form; no potential selects eta = +- J A.

Proof. Lemma 51, Proposition 51.3: the formula in a two-plane; the perpendicular of A in P_tau is R J A; m_- = 0 forces |A| = |eta| and sin phi = 1; comparison with Proposition 50.2 (3); Proposition 49.4 for c(A) = L_a and Proposition 49.3 for the couplings; Theorem 16.7 formally; [Y, x] = 0 kills the one-forms.

What it leaves open. Three generations; a relation between H and A from an action or a quantization condition.

Part S199 (On the flat four-torus with a constant connection and a constant three-form the Dirac operator is c(k + A) + i c(eta) Gamma on each momentum sector; the constant spinors are its lowest window for |A| + |eta| < pi/l, and there it is the Yukawa block c(A) + i c(eta) Gamma, with the Wilson line as the custodial half and the three-form as the splitting half, of masses m_+-^2 = |A|^2 + |eta|^2 +- 2 |A wedge eta|)

On the flat four-torus with a constant connection and a constant three-form the Dirac operator is c(k + A) + i c(eta) Gamma on each momentum sector; the constant spinors are its lowest window for |A| + |eta| < pi/l, and there it is the Yukawa block c(A) + i c(eta) Gamma, with the Wilson line as the custodial half and the three-form as the splitting half, of masses m_+-^2 = |A|^2 + |eta|^2 +- 2 |A wedge eta|

Status. Proposition 51.1. Proved by Claude Fable 5.1 on 5 November 2026 by hand; the three-form identity, the mass formula on 200 random pairs and the isolation of the window on 400 random pairs with the momenta enumerated were checked with numpy (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_51_52.md): correct with amendments (Agricola's normalization verified; the quaternion identity for |A wedge eta|). Ledger row `l51_window`. Not yet in a paper.

Setting. The flat four-torus A = R^4/Lambda of lemma 48 with the periodic spin structure, Sigma_4 the sheet of Proposition 49.1; D_{A,H} = D + c(A) + s i c(H) with A a constant one-form (a flat U(1) connection) and H a constant three-form, s i c(H) = i c(eta) Gamma with eta = +- s *H, s != 0 (s = 3/4 for the connection with torsion H, 1/4 for torsion H/3; Agricola, verified by the referee); 2 pi/l the length of the shortest nonzero dual vector.

Statement. D_{A,H} commutes with the translations and equals Y_k = c(k + A) + i c(eta) Gamma on the sector e^{i k . z} Sigma_4, an odd Hermitian block with j-linear half c(k + A) and j-antilinear half i c(eta) Gamma, of eigenvalues +-m_+(k), +-m_-(k), m_+-(k)^2 = |k + A|^2 + |eta|^2 +- 2 |(k + A) wedge eta|. For |A| + |eta| < pi/l the constant spinors are the lowest window, with every other eigenvalue of absolute value at least 2 pi/l - |A| - |eta|, and the compression to it is exactly Y = c(A) + i c(eta) Gamma, the Yukawa block of lemma 46 with q <-> A and q' <-> eta.

Scope. Constant data; the constant s and the sign of eta are conventions; the compression is exact since the window is reducing.

Proof. Lemma 51, Proposition 51.1: constant coefficients; X^* X = |v|^2 + |eta|^2 + 2 i c(v wedge eta) on L_p with c of a simple two-form of eigenvalues +-i times its norm; the triangle inequality for the window

and the lower bound m_-(k) >= |k + A| - |eta|; |Im(conj(q) q')| = |A wedge eta| from Re(conj(q) q') = q . q' and |conj(q) q'| = |q| |q'|.

What it leaves open. The three-generation form (flavour-dependent Wilson lines and three-forms); the E factor.

Part S200 (The lepton sub-bimodule of the two sheets has eight states and contains the connector; the connector's order-one condition reduces C + H + C' + M_3(C) to the Standard Model algebra, identifying the two copies of C, exactly when the Majorana line is a weight line of the lifted transposition; the trace-form pairing of the lepton generation over C + H is nondegenerate)

The lepton sub-bimodule of the two sheets has eight states and contains the connector; the connector's order-one condition reduces C + H + C' + M_3(C) to the Standard Model algebra, identifying the two copies of C, exactly when the Majorana line is a weight line of the lifted transposition; the trace-form pairing of the lepton generation over C + H is nondegenerate

Status. Proposition 52.1. Proved by Claude Fable 5.1 on 5 November 2026 by hand, from Theorems 16.7, 20.1 and 20.2 and Propositions 49.2, 49.5 and 50.3; the lepton pairing checked with numpy (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_51_52.md): correct with amendments (the dictionary needs the eigenline condition; the rank-one defect formula of the connector; the pairing nondegenerate over Q, the matrices of Chamseddine-Connes-Marcolli, Proposition 2.30 (3)). Ledger row `l52_lepton_line`. Not yet in a paper.

Setting. V_X = Sigma_4 with A_{F,X} = C + H (Proposition 49.2), V_Y = Cv + C^3 with A_{F,Y} = C' + M_3(C); H_F = Hom(V, Sigma_4) + Hom(Sigma_4, V), 32 states; D_F = D_Y + S_{T_kappa} with D_Y the split block of Proposition 50.3 and T_kappa the connector on the lines w in Sigma_R and v.

Statement. (i) H_F = H_l + H_q with H_l = Sigma_4 (x) conj(Cv) + Cv (x) conj(Sigma_4) (8 states) and H_q the quark generation of Proposition 49.5 (24 states), both invariant under the algebra, J_F, Gamma_F, D_Y and the connector, whose vectors e, f lie in H_l; H_l carries nu_R, e_R (the weight lines of x on the lepton line, w the Majorana line, under the eigenline condition), the left doublet and their conjugates. (ii) The largest subalgebra of C + H + C' + M_3(C) with order one for D_F (kappa != 0) is A_SM = {(lambda, q, lambda, m)} exactly when Cw is a weight line of x, the two copies of C identified, and R + H + M_3(C) otherwise; on A_SM, H_l is the summand H_F^(1) of Chamseddine, Connes and Marcolli with the Majorana block on nu_R. (iii) The trace-form pairing on H_l for p_1 = (1_C, 0, 1_{C'}, 0), p_2 = (0, 1_H, 0, 0), p_3 = (0, 0, 0, e_11) is [[0, 2, 0], [-2, 0, 0], [0, 0, 0]]: nondegenerate over Q over C + H (determinant 4), the matrices of Chamseddine-Connes-Marcolli, Proposition 2.30 (3), whose Corollary 2.31 obtains nondegeneracy over the three summands from the two classes; on H_q it is [[0, 0, 2], [0, 0, -2], [-2, 2, 0]] and on H_F the sum, skew of rank two (Theorem 41.2 up to sign).

Scope. One generation; the axis, connector and blocks are inputs; the sign convention of lemma 49.

Proof. Lemma 52, Proposition 52.1: P_v commutes with A_{F,Y}; the rank-one defect formula of S_{T_kappa}, which gives A(T_kappa) as the largest *-subalgebra, intersected with the four-summand algebra, a matrix diagonal in two different bases being scalar; the traces tr(Gamma a_p), tr(b_p) on the lepton line.

What it leaves open. Three generations; the orientability failure with nu_R (Proposition 41.4) is unchanged.

Part S201 (A Majorana spinor of nine-dimensional Minkowski space is, under Spin(1, 3) x Spin(5), the complex space (2, 1) (x) 4: four left-handed Weyl spinors in the spinor fibre of the five-dimensional internal space, the sheet W = H^2, which under K_p is the left doublet and the right-handed pair, one lepton generation with nu_R; on its self-conjugate sector the masses are the torsion |eta| for both members, the sectors of nonzero momentum pairing into a generation and a mirror, and a Majorana mass for nu_R alone is wall data in the scope of constant bulk data)

A Majorana spinor of nine-dimensional Minkowski space is, under Spin(1, 3) x Spin(5), the complex space (2, 1) (x) 4: four left-handed Weyl spinors in the spinor fibre of the five-dimensional internal space, the sheet W = H^2, which under K_p is the left doublet and the right-handed pair, one lepton generation with nu_R; on its self-conjugate sector the masses are the torsion |eta| for both members, the sectors of nonzero momentum pairing into a generation and a mirror, and a Majorana mass for nu_R alone is wall data in the scope of constant bulk data

Status. Proposition 52.2. Proved by Claude Fable 5.1 on 5 November 2026 by hand (standard Clifford representation theory), with the Majorana conjugation of Spin(1, 8), its commutation with the spin generators, the exchange of the chiral eigenspaces and the commutant dimensions 4, 16 and 2 checked with numpy on an explicit representation (script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_51_52.md): one error in (iii) (the masses: the self-conjugate sector alone carries one generation, with the masses |eta|) and one in its proof (anticommuting components), corrected; amendments (two irreducible modules; e_R^c on w, nu_R^c on jw; the scope of the Majorana mass; the factorizations; commutant dimensions 32, 16, 2). Ledger row `l52_majorana_nine`. Not yet in a paper.

Setting. Spin(1, 8) with the Dirac spinor C^16; M^4 x K_5 with K_5 a spin five-manifold, structure group Spin(1, 3) x Spin(5) = Spin(1, 3) x Sp(2); for K_5 = A x S^1 the fibre is the sheet of Proposition 49.1 with c(e_5) = +-Gamma.

Statement. (i) C^16 = (2, 1) (x) 4 + (1, 2) (x) 4 under Spin(1, 3) x Spin(5); a Majorana conjugation C (antilinear, C^2 = 1, commuting with the Clifford action) exists and exchanges the summands, so the Majorana spinor R^16 is the complex space (2, 1) (x) 4. (ii) The 4 is the spinor fibre of K_5, the sheet W = H^2; under K_p = Sp(1)_a x Sp(1)_b it is (2, 1) + (1, 2), under Sp(1)_a x U(1)_b it is (2)_0 + (1)_{+1} + (1)_{-1}: one lepton generation with nu_R, with weak isospin Sp(1)_a and the T_3R circle; the quark generation is the same fibre tensored with C^3. (iii) The nine-dimensional mass bilinear vanishes (rs_F3), the masses are the internal Dirac operator on the fibre; a Majorana field has no U(1) charge and no Wilson line, so on A x S^1 its momenta lie in the dual lattice shifted by the spin structure; it admits the torsion of lemma 51 (C Gamma^{(3)} antisymmetric), and the mass operator on the sector (k, k_5) is c(k) + i c(eta) Gamma + k_5 Gamma with squared masses m_+-(k)^2 + k_5^2; the self-conjugate sector (k, k_5) = 0 carries exactly one generation, with the purely anti-quaternionic block i c(eta) Gamma and the equal masses |eta|, and a pair of sectors +-(k, k_5) != 0 carries a generation and a mirror, four Dirac fermions; a Majorana mass for nu_R alone is absent from Lorentz-invariant and constant bulk data (constant fluxes with a leg on S^1 give nu_R^c and e_R^c entries of equal modulus), wall data in that scope (Remark 44.4), and selects the KO-7 factorization for one generation (lemma 06, Lemma 4), with Dirac masses only both factorizations existing (Proposition 7 with R = 0).

Scope. Fibres at a point; no vacuum, gauge group or action constructed; the Wilson-line statement is the

absence of a U(1) charge on a Majorana field.

Proof. Lemma 52, Proposition 52.2: Cl(1, 8) as the graded tensor product of Cl(1, 3) and Cl(0, 5); conj(4) = 4 and conj(2, 1) = (1, 2); B conj(B) = 1 in signature (1, 8), C symmetric, C Gamma^{(3)} antisymmetric, with the antilinear Majorana conjugation exchanging the gamma_5-eigenspaces; the real form projects isomorphically onto the first summand; Proposition 49.1 for the fibre; Proposition 51.1 for the spectrum.

What it leaves open. A nine-dimensional parent with this field whose window is the lepton generation; the comparison of the quantized momenta with the charged-lepton masses.

Part S202 (Three lepton generations with nu_R (24 states) and the complexified adjoint of the flavour SU(5) of the five light quarks are isomorphic as representations of U(1)_em, with multiplicities 12, 6, 6; the adjoint is (3, 1)_0 + (1, 8)_0 + (1, 1)_0 + (2, 3-bar)_{+1} + (2-bar, 3)_{-1} under SU(2)_u x SU(3)_d x U(1)_em, and the charged states count (up flavours) x (down flavours) as the charged leptons count (chiralities) x (generations))

Three lepton generations with nu_R (24 states) and the complexified adjoint of the flavour SU(5) of the five light quarks are isomorphic as representations of U(1)_em, with multiplicities 12, 6, 6; the adjoint is (3, 1)_0 + (1, 8)_0 + (1, 1)_0 + (2, 3-bar)_{+1} + (2-bar, 3)_{-1} under SU(2)_u x SU(3)_d x U(1)_em, and the charged states count (up flavours) x (down flavours) as the charged leptons count (chiralities) x (generations)

Status. Proposition 52.3. Proved by Claude Fable 5.1 on 5 November 2026 by hand (charges and dimensions). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_51_52.md): correct with one amendment (the 3 of SU(3)_d for the states of charge -1, the 3-bar for their conjugates). Ledger row `l52_flavour_adjoint`. Not yet in a paper.

Setting. The five light quarks u, c, d, s, b of lemma 21 as the 5 of a flavour SU(5) with Q = diag(2/3, 2/3, -1/3, -1/3, -1/3); the lepton module H_l of Proposition 52.1 tensored with a generation space C^3.

Statement. 24 = (3, 1)_0 + (1, 8)_0 + (1, 1)_0 + (2, 3-bar)_{+1} + (2-bar, 3)_{-1}: twelve neutral real components spanning su(2)_u + su(3)_d + u(1), six complex components of charge +1 and their conjugates. H_l^{(3)} has the U(1)_em weights 0 (12), +1 (6), -1 (6), so H_l^{(3)} and sl(5, C) are isomorphic U(1)_em-modules; as left-handed Weyl fields the three generations are six neutral and six charged fields, pairing with the six neutral and six charged complex scalars of the adjoint in chiral multiplets of equal charge (the slepton counts 6, 6, 12 of lemma 21). The charged states number 2 x 3 on both sides; an isomorphism can map the generation index to the 3 of SU(3)_d for e_L, e_R (charge -1) and the 3-bar for their conjugates, and the pair (e_L, e_R) to the 2 of SU(2)_u (a reading). The top is absent from the 5.

Scope. Charges and dimensions; no supersymmetric theory and no field construction; the finer matching is a choice among isomorphisms.

Proof. Lemma 52, Proposition 52.3: block decomposition of sl(5) under S(U(2) x U(3)); the dictionary of the lepton states; the chiral-multiplet pairing by charge.

What it leaves open. A three-generation finite geometry whose generation space carries a representation of the flavour group.

Part S203 (Orientability is lost under a heavy centralizer while first order survives)

Orientability is lost under a heavy centralizer while first order survives

Status. Theorem, finite and explicit. Construction by the corpus (historical/corpus/connes_kk_research_state.md, section 6.6, ledger row rs_N8, 2 November 2026); proof by hand by Terra 6.1 on 3 November 2026 (historical/reviews/2026-11-03_terra_check_N8.md), with one clarification that the text adopted: the bridge exchange must be the normalized exchange that preserves the common colour factor. The proof below is Terra's, written out; nothing in it rests on the finite-matrix check the corpus reported. It is the basis of STRATEGY T17 (which axioms the finite triple keeps along the chain of Theorem 20.2).

Setting. Let A_4 = C_1 + H_2 + M_3(C)_3 + C_4 with elements (lambda, q, m, mu), V_1 = V_4 = C, V_2 = C^2, V_3 = C^3, and H_ij = Hom(V_j, V_i) with the Hilbert-Schmidt inner product. Let

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H_18 = H_13 + H_23 + H_31 + H_32,
H_32 = H_18 + (H'_13 + H'_31 + H_43 + H_34) + (H_14 + H_41),
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the primes marking copies distinct from those in H_18, of total complex dimension 18 + 12 + 2 = 32. The algebra acts by left multiplication, rho(a) X = a_i X on H_ij, and the opposite action is rho(b)^o X = J rho(b)^* J^{-1} X = X b_j, with J X = X^dagger. The grading gamma is +1 on H_13, H_32, H'_13, H_34, H_14 and -1 on H_23, H_31, H'_31, H_43, H_41 (the four bridge chiralities +, -, -, + in the displayed order and the spectator pair +, -). The Dirac operator is D_t = D_0 + t T, where D_0 is the operator of the 18-state bimodule, D_0(X, Y) = (D_v X, Y D_v) with D_v = [[0, v^dagger], [v, 0]] for a nonzero v: C -> C^2, extended by zero on the other fourteen states, and T is the normalized exchange H'_13 <-> H_43, H'_31 <-> H_34 that sends each row matrix to the same row matrix and each column matrix to the same column matrix (with V_1 = V_4 = C identified), and vanishes elsewhere. T is self-adjoint, T^2 = 1 on the bridge, and T commutes with J.

Theorem. For every real t:

1. (KO-6 signs) D_t is self-adjoint, J^2 = 1, J D_t = D_t J, J gamma = -gamma J, D_t gamma = -gamma D_t.
2. (Order zero) [rho(a), rho(b)^o] = 0 on H_32.
3. (First order) [[D_t, rho(a)], rho(b)^o] = 0 for all a, b in A_4.
4. (Parent orientability) With E_i = rho(e_i) and E_i^o = J E_i J^{-1}, the operator (E_1 - E_2 - E_4) E_3^o - E_3 (E_1^o - E_2^o - E_4^o) + E_1 E_4^o - E_4 E_1^o equals gamma on every block of H_32, independently of t.
5. (Heavy centralizer) ||[T, rho(a)]|| = |lambda - mu| on the row bridge and 0 elsewhere, so A_T = A_4 cap {T}' = {mu = lambda} = C + H + M_3(C), and ||[D_t, rho(a)]|| stays bounded as |t| -> infinity exactly for a in A_T.
6. (Spectral projection) With r = ||v|| > 0, the spectrum of D_t is 0 (multiplicity 8), +-r (6 each) and +-|t| (6 each). For r < Lambda < |t|, the spectral cutoff P_Lambda(D_t) is the projection onto ker T, with range H_18 + H_14 + H_41; it commutes with rho(A_4), J and gamma, and P D_t P = D_0 + 0.
7. (Faithful reduction) On the 20-dimensional light space, A_T acts faithfully: lambda on H_13, q on H_23, m on H_31.
8. (Orientability lost) On the spectator pair H_14 + H_41, every a in A_T acts as lambda_a I_2 and every opposite element as lambda_b I_2, while gamma = diag(1, -1) there. So no finite sum of left-right products equals gamma on the light space, and the operator-norm distance from gamma to the space of orientation operators is exactly 1 (the spectator pair gives the lower bound; the orientation operator of H_18 attains it).
9. (Nonzero one-forms) For a = e_1 + e_4 in A_T, [D_0, rho(a)] acts on the X-part of H_18 by left multiplication by [[0, -v^dagger], [v, 0]], which is nonzero.
10. (Distances) Before projection, d_{D_t}(omega_1, omega_4) = 1/|t| between the two scalar evaluation

states, attained by e_4/|t|; after projection alone, with the full A_4 retained, P D_t P commutes with e_4 and the distance is infinite. The intersection matrix on minimal projections is Q_4 = [[0,0,2,1],[0,0,-2,0],[-2,2,0,1],[-1,0,-1,0]], with Pfaffian -2 and determinant 4.

Proof. (1) D_v is self-adjoint and exchanges the +-1 eigenspaces of G = diag(1, -I_2); the bridge swap exchanges opposite chiralities; J X = X^dagger is antiunitary of square one for the Hilbert-Schmidt product and commutes with D_v-multiplication and with T. (2) Left and right matrix multiplications commute. (3) On H_18, D_0, rho(a) = ([D_v, L(a)] X, 0) with L(a) = diag(lambda, q), a left multiplication, which commutes with the right M_3(C) action. On H'_13 + H_43 the commutator [T, rho(a)] exchanges the two rows with the scalar coefficient (lambda - mu), and the opposite action is right multiplication by the same m_b on both rows; on H'_31 + H_34, T commutes with the common left m_a action. Hence the double commutator vanishes for every a, b. The normalization matters: with the exchange X -> X W, W = diag(1, -1, 1), and a = e_1, b_3 = E_12, X = (1, 0, 0), the double commutator is X(E_12 W - W E_12) = -2 X E_12, nonzero. (4) Each summand is a left-right product; evaluating on each of the ten blocks gives the sign of gamma there (for instance E_1 E_3^o is the identity on H_13 and H'_13, -E_3 E_1^o on H_31 and H'_31, E_1 E_4^o on H_14, -E_4 E_1^o on H_41), and no summand involves t. (5) The row bridge computation of [T, rho(a)]; the triangle inequality for the boundedness statement. (6) D_0 has eigenvalues 0 (on H_14 + H_41 and on the kernel of the D_v-multiplications, total 8 - 2 = 6 inside H_18 plus 2 spectators; count 8 in all), +-r with multiplicity 6 (three colours, two Hom directions) and T contributes +-|t| with multiplicity 6 (three row entries and three column entries); the three ranges are orthogonal, so P_Lambda(D_t) for r < Lambda < |t| is the projection off the bridge. It is a sum of block projections, hence commutes with rho(A_4), J and gamma. (7) Direct. (8) Centralization sets mu = lambda; on H_14 = Hom(V_4, V_1) the left action is lambda and the opposite action is mu = lambda, likewise on H_41; so the represented left-right products are scalar on the pair, and gamma is not. (9) Direct. (10) The upper bound uses the bridge matrix element |t| |mu - lambda|; equality takes lambda = q = m = 0, mu = 1/|t|. After projection D_0 has no entry involving e_4, so [P D_t P, e_4] = 0 and the distance is infinite. The matrix Q_4 is the trace pairing Tr(gamma rho(p_i) J rho(p_j) J^{-1}) on minimal projections. ■

What it says and what it does not. The example shows one mechanism by which opposite chiralities become indistinguishable to the reduced left-right algebra: the bounded-commutator selection (A_T) and the spectral projection together keep KO-6 signs, order zero, first order, faithfulness and nonzero one-forms, and lose orientability on a gauge-neutral spectator pair. Projection alone (with A_4 retained) keeps orientability and sends the two scalar states to infinite distance; centralization carries the metric identification d = 1/|t| -> 0 into the light theory. It is an engineered finite model: nothing derives T, the gap or the selection from a Kaluza-Klein geometry, and it does not fix the Weinberg angle. The same conclusion for the real chain Pati-Salam -> Standard Model -> colour + electromagnetism of lemmas 20 and 21 is STRATEGY T17.