Supplemental Material to "The finite Dirac operator as a boundary condition on an interval"

Draft, 3 November 2026

This is the supplement of the paper *The finite Dirac operator as a boundary condition on an interval*. Its parts keep the numbers of the full Supplemental Material, so that a citation of a part in the paper refers to the part of the same name here.

Parts contained. Lemma parts: 18, 22, 23, 24, 25, 26.

The full Supplemental Material, with all parts of the project, exists as `supplemental.pdf` and `supplemental_mobile.pdf` in the same folder as this file. A reference inside a part to a part not listed here points to the full version.

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Part 18

T5b: vector masses in the spectral action at fixed cutoff

Prompt: prompts/18 (STRATEGY T5b). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5 with Python (numpy, scipy), by computations independent of the kernel derived here: a direct heat trace with constant a reproduces equation (2) for five choices of (m_1, m_2), to about 0.03%; a direct heat trace on Fourier modes with a_2(x_1) = a cos(q x_1), including the F_{12} and [T, V] vertices, reproduces the kernel (37) at three values of (m, q), to 0.1-0.3%, the size of the discretization error. The closed forms (16)-(19), the bound (28), the limits (27), the neutral matrix (30) recomputed from Standard Model charges, and the identities behind (31) also check. The largest value of the Taylor mass is M_W^2 = 0.275 Lambda^2, at m_t^2 = 2.02 Lambda^2. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): all ten numbered results are correct, with small amendments incorporated below. The referee added Propositions 11 to 14 and Remark 15, which widen the scope in both directions; the ratio in Proposition 14 was added by Claude.

Scope. Within the three-generation, zero-Majorana, top-only family at fixed cutoff, these results refute an infinitely heavy W, Z, t limit for the stated quadratic mass definitions. They do not rule out other finite data (Proposition 14), a different action or constraint, a joint limit with Lambda -> infinity, or a quantum effective theory. They do not show unbroken SU(2) x U(1) on the light space (Remark 15).

For the heat cutoff, on $\mathbb{R}^4$, we obtain the exact quadratic vector form without expanding in m/Λ. In the three-generation Standard Model finite representation with only the top Dirac mass nonzero, both the mass defined by the q^0/q^2 Taylor ratio and the unique positive timelike pole mass satisfy $M_W \sim \Lambda^2/m_t$ as $m_t/\Lambda \to \infty$ at fixed Λ. The broken vectors consequently do not decouple by becoming heavy, although the heavy fermion contribution disappears from the gauge coefficients. These statements concern the classical bosonic quadratic action

about a prescribed constant background; moreover, the positive Laplace class considered below does not itself select a finite nonzero stationary background when radial variations of that background are allowed.

1. Theorem (exact quadratic kernel).

Let

$$D_A = -i\gamma^\mu(\partial_\mu + iA_\mu) + \gamma_5 V, \qquad V = V_0 = V_0^\dagger,$$

on $\mathbb{R}^4$, with Hermitian Euclidean gamma matrices, $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$, $\gamma_5^2 = 1$, and four spin components. Let the finite trace be the literal trace on H_F, including every stated multiplicity, without a subsequent projection; this agrees with the bosonic trace of Chamseddine-Connes-Marcolli, which counts particles, antiparticles and four spin components. Take A smooth and compactly supported.

We use the vacuum-subtracted action, or equivalently its quadratic variation, since the constant-background trace on $\mathbb{R}^4$ has infinite volume. Write

$$A_\mu(q) = a_\mu(q)T, \qquad T = T^\dagger, \qquad q^\mu a_\mu(q) = 0,$$
$$\int_q = \int \frac{d^4q}{(2\pi)^4}.$$

Diagonalize V_0:

$$V_0 e_i = m_i e_i, \qquad m_i \in \mathbb{R}.$$

The signed eigenvalues are needed; their squares alone do not determine the answer.

For $S_t = \operatorname{Tr} e^{-tD_A^2}$, define

$$b(u) = u(1-u), \quad d_{ij} = m_i - m_j,$$

$$I_t(a, b; q^2) = \int_0^1 e^{-t[(1-u)a + ub + u(1-u)q^2]}\, du.$$

Then

$$S_t^{(2)} = \int_q |a_T(q)|^2\, C_{t,T}(q^2),$$

where

$$C_{t,T}(q^2) = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2 \left[\left(\frac{2}{t}+d_{ij}^2+q^2\right)I_t(m_i^2,m_j^2;q^2) - \frac{e^{-tm_i^2}+e^{-tm_j^2}}{t}\right]. \tag{1}$$

For several generators, replace $|T_{ij}|^2$ by $\mathrm{Re}((T_a)_{ij}(T_b)_{ji})$.

In particular, writing $C(q^2) = C_0 + q^2C_1 + O(q^4)$, we obtain

$$C_{0,t} = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2 \left[\left(\frac{2}{t}+d_{ij}^2\right)I_t(m_i^2,m_j^2;0) - \frac{e^{-tm_i^2}+e^{-tm_j^2}}{t}\right], \tag{2}$$

and

$$C_{1,t} = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2\int_0^1 e^{-t[(1-u)m_i^2+um_j^2]}\big[1-2b(u)-td_{ij}^2b(u)\big]\,du. \tag{3}$$

These formulas also apply, after integration over $t = s/\Lambda^2$, to

$$f(z) = \int_{(0,\infty)} e^{-sz}\,d\nu(s), \quad 0 < f(0) = \nu((0,\infty)) < \infty, \quad \int(s^{-2}+s^2)\,d\nu(s) < \infty, \tag{4}$$

where ν is positive. These hypotheses suffice for the Euclidean action density and the Taylor coefficients used below. They do not, by themselves, guarantee analytic continuation to every negative q^2.

Proof. With $\nabla_\mu = \partial_\mu + iA_\mu$,

$$D_A^2 = -\nabla^2 + V_0^2 - \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu} + \gamma^\mu\gamma_5[A_\mu, V_0].$$

Let $P_0 = -\partial^2 + V_0^2$ and write $D_A^2 = P_0 + P_1 + P_2$, according to degree in A. Duhamel expansion and cyclicity

give

$$S_t^{(2)} = -t\,\mathrm{Tr}(e^{-tP_0}P_2) + \frac{t^2}{2}\int_0^1 \mathrm{Tr}\,\left(e^{-(1-u)tP_0}P_1 e^{-utP_0}P_1\right) du. \tag{5}$$

The linear vertex from j, p to $i, p+q$ is

$$T_{ij}\left[(2p+q)\cdot a + \gamma^{\mu\nu}q_\mu a_\nu + (m_j - m_i)\gamma^\mu\gamma_5 a_\mu\right].$$

Completing the momentum square produces

$$(4\pi t)^{-2}e^{-t[(1-u)m_i^2 + um_j^2 + b(u)q^2]}.$$

For transverse a, the averaged spin trace of the vertex product is

$$4\left(\frac{2}{t} + d_{ij}^2 + q^2\right)|a|^2.$$

The P_2 term contributes

$$-\frac{1}{4\pi^2 t}\sum_{ij}|T_{ij}|^2 e^{-tm_i^2}|a|^2.$$

Symmetrizing in i, j proves (1). Differentiation proves (2)–(3).

The divided-difference structure is explicit:

$$I_t(a,b;0) = \frac{e^{-ta} - e^{-tb}}{t(b-a)}, \qquad I_t(a,a;0) = e^{-ta}, \tag{6}$$

and

$$\int_0^1 u(1-u)e^{-t[(1-u)a+ub]}\,du = \frac{1}{t^2}\partial_a\partial_b I_t(a,b;0). \tag{7}$$

Thus (2) uses a first divided difference, while (3) also uses its mixed derivative. The full momentum dependence follows from the convergent series

$$I_t(a,b;q^2) = \sum_{n=0}^{\infty}\frac{(-q^2/t)^n}{n!}\partial_a^n\partial_b^n I_t(a,b;0). \tag{8}$$

Repeated-node values are obtained continuously.

For completeness, the mass coefficient for (4) can be written entirely in dimensionless variables. Put

$$r_i = m_i/\Lambda, \quad x_i = r_i^2, \quad z(u) = (1-u)x_i + ux_j,$$
$$\Phi_1(z) = \int s^{-1}e^{-sz}\,d\nu(s).$$

The bracket replacing that in (2) is

$$\Lambda^2 \left[2 \int_0^1 \Phi_1(z(u))\, du + (r_i - r_j)^2 \int_0^1 f(z(u))\, du - \Phi_1(x_i) - \Phi_1(x_j) \right]. \tag{9}$$

The corresponding bracket in (3) is

$$\int_0^1 \left[(1-2b) f(z) + (r_i - r_j)^2 b f'(z) \right] du. \tag{10}$$

Each unweighted integral is a divided difference of an antiderivative. The moment assumptions justify the integrations and differentiations. □

On a finite flat torus, (5) remains exact, but the Gaussian momentum integral must be replaced by the spin-structure-dependent lattice sum. Explicitly, for a transverse polarization a, its two-vertex contribution per unit volume is

$$\frac{2t^2}{\mathrm{Vol}} \sum_{ij,p} |T_{ij}|^2 \int_0^1 e^{-t[(1-u)(p^2+m_i^2)+u((p+q)^2+m_j^2)]} \left(|(2p+q)\cdot a|^2 + (d_{ij}^2 + q^2)|a|^2 \right) du,$$

and the quadratic-vertex contribution is

$$-\frac{4t}{\mathrm{Vol}} \sum_{ij,p} |T_{ij}|^2 e^{-t(p^2+m_i^2)} |a|^2.$$

Consequently, the continuum formulas below are statements on $\mathbb{R}^4$, or in the infinite-volume limit. A finite torus can have additional holonomy dependence, including for constant commuting connections.

2. Lemma (gauge invariance and the Goldstone direction).

Let scalar fluctuations include the gauge orbit of V_0. Then the nonzero vector coefficient C_0 in (2) is compatible with gauge invariance.

For one generator, parameterize the orbit by

$$V(x) = e^{i\pi(x)T} V_0 e^{-i\pi(x)T}.$$

With our conventions,

$$a_\mu \longmapsto a_\mu - \partial_\mu \alpha, \qquad \pi \longmapsto \pi + \alpha.$$

The invariant combination is

$$B_\mu = a_\mu + \partial_\mu \pi. \tag{11}$$

Proof. Gauge covariance gives

$$D_{A^U,V^U} = U D_{A,V} U^{-1}.$$

Applying $U = e^{-i\pi T}$ transforms V to V_0 and a to B. Hence

$$S[a, e^{i\pi T} V_0 e^{-i\pi T}] = S[B, V_0]. \tag{12}$$

This establishes the Goldstone completion directly, without a derivative expansion.

For comparison, the longitudinal vector kernel is obtained from (1) by replacing the q^2 in its prefactor by $(1-2u)^2 q^2$ inside the integral. If $[T, V_0] = 0$, its integral vanishes by

$$\int_0^1 [2 + tq^2(1-2u)^2] e^{-tq^2 u(1-u)}\, du = 2.$$

For a broken generator the longitudinal kernel need not vanish; it belongs to the invariant combination (11).

If V_0 is not stationary, a Cartesian scalar Hessian alone need not annihilate the linear gauge-orbit tangent. The quadratic term in $e^{i\pi T} V_0 e^{-i\pi T}$, contracted with the scalar tadpole, must also be retained. Equation (12) includes that term. Transverse vectors do not mix quadratically with scalar fluctuations on this translation-invariant background. □

3. Corollary (the asymptotic normalization).

Let all finite eigenvalues be small relative to Λ, let f satisfy (4), and let Q below be positive definite on the chosen generator space (otherwise pass to the quotient by its null space). Define

$$Q_{ab} = \mathrm{Re}\,\mathrm{Tr}(T_a T_b), \qquad K_{ab} = \mathrm{Re}\,\mathrm{Tr}([T_a, V_0]^\dagger [T_b, V_0]).$$

At leading order,

$$C_0 = \frac{f(0)}{8\pi^2} K, \qquad C_1 = \frac{f(0)}{12\pi^2} Q,$$

and therefore

$$\boxed{\mathcal{M}^2_{\mathrm{Taylor}} = \frac{3}{2} Q^{-1/2} K Q^{-1/2}.} \tag{13}$$

This reproduces Proposition 13.5.

For the full three-generation Standard Model representation,

$$Q_{11} = 12, \qquad K_{11} = \sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2),$$

so

$$\boxed{8M_W^2 = \sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2).} \tag{14}$$

Proof. In (2), the orbital expression

$$\frac{2}{t} I_t(a, b; 0) - \frac{e^{-ta} + e^{-tb}}{t}$$

starts at order $t(a-b)^2$. The leading term is consequently d_{ij}^2. In (3), the leading integral is $\int_0^1 (1 - 2u(1-u))du = 2/3$, proving (13).

For $T_1 = \sigma_1/2$ on each left doublet, one generation of particles contributes 2 to Q_{11}. Charge conjugates and three generations give 12. A doublet with Dirac masses m_u, m_d contributes $\frac{1}{2}(m_u^2 + m_d^2)$ to the particle commutator trace, before color multiplicity. This proves (14).

Equivalently, the gauge coefficient is

$$\frac{f(0) g_2^2}{2\pi^2} = \frac{1}{4}.$$

For comparison with CCM's $\operatorname{Tr} F(D/\Lambda)$, take $F(v) = f(v^2)$; then $F(0) = f(0)$. The normalization and mass constraint are precisely equations (4.10) and (4.16), with the latter stated in Lemma 4.6 of the checked preprint. CCM describes the relation as "valid at unification scale." Chamseddine–Connes–Marcolli, pp. 37–39. □

The leading bound $M^2 \le 6\|V_0\|^2$ in Lemma 14 follows from (13) and $K \le 4\|V_0\|^2 Q$. Formula (13), rather than an exact finite-cutoff identity, is its applicable approximation.

4. Proposition (the exact charged-vector Taylor mass).

Let the finite data be the full three-generation Standard Model representation, including charge conjugates. Let

the Majorana matrix and every Dirac mass except one up-type color triplet be zero. Write

$$m = \mu\Lambda, \qquad x = \mu^2, \qquad r = e^{-x}.$$

Let $f(z) = e^{-z}$. Define

$$H(y) = \int_0^1 e^{-yu(1-u)}\,du,$$

$$g(y) = (2+y)H(y) - 2,$$

$$h_x(y) = (2+x+y)\int_0^1 e^{-xu-yu(1-u)}\,du - 1 - e^{-x}. \quad (15)$$

Then the full charged-vector kernel is

$$\boxed{C_W(q^2) = \frac{\Lambda^2}{4\pi^2}\left[\frac{9}{2}g(y) + \frac{3}{2}h_x(y)\right], \qquad y = q^2/\Lambda^2.} \quad (16)$$

Define

$$A(x) = h_x(0) = \frac{2[1-(1+x)e^{-x}]}{x},$$

$$w(x) = 3\int_0^1 u^2 e^{-xu}\,du = \frac{3[2-(x^2+2x+2)e^{-x}]}{x^3}. \quad (17)$$

Continuous values are understood at $x = 0$. The particle-trace kinetic coefficient is

$$K_W(x) = \frac{9}{2} + \frac{3}{2}w(x) > 0.$$

The Taylor mass, defined by $M^2_{W,\mathrm{Taylor}} = C_W(0)/C'_W(0)$, is

$$\boxed{\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{9A(x)}{4K_W(x)} = \frac{3[1-(1+x)e^{-x}]}{x[3+w(x)]}.} \quad (18)$$

Its limiting behavior is

$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{3}{8}x - \frac{23}{128}x^2 + O(x^3), \qquad \frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} \sim \frac{1}{x} \quad (x \to \infty). \quad (19)$$

Consequently,

$$\frac{m}{M_{W,\mathrm{Taylor}}} \longrightarrow \sqrt{\frac{8}{3}} \quad (\mu \to 0), \qquad \frac{m}{M_{W,\mathrm{Taylor}}} \sim \mu^2 \quad (\mu \to \infty). \quad (20)$$

Proof. On the particle space, $Q_{11} = 6$. The selected quark doublet contributes $3/2$, while the remaining doublets contribute $9/2$. Diagonalizing the massive left-right block produces eigenvalues $+m, -m$. The charged generator connects these to the massless down states; every such pair has $d_{ij}^2 = m^2$. Formula (1) therefore gives h_x for the selected doublet and g for the remaining doublets. Charge conjugation supplies the factor two in (16).

Integration by parts gives

$$h_x(y) = 2\int_0^1 u\,[x + 2y(1-u)]e^{-xu-yu(1-u)}\,du, \tag{21}$$

and

$$\boxed{h_x'(y) = 2\int_0^1 u^2[1 - y(1-u)]e^{-xu-yu(1-u)}\,du.} \tag{22}$$

The boundary term proving (22) is $\partial_u[u(1-u)e^{-xu-yu(1-u)}]$. Thus

$$h_x'(0) = \frac{2}{3}w(x), \qquad g'(0) = \frac{2}{3}.$$

This proves (18). Expanding the elementary expressions at zero and taking their large-x limits proves (19)–(20). □

The same Taylor result extends to every f in (4). Put

$$\ell = f(0), \qquad w_f(x) = \int w(sx)\,d\nu(s),$$
$$A_f(x) = \int \frac{A(sx)}{s}\,d\nu(s).$$

Then

$$\boxed{\frac{M_{W,\mathrm{Taylor}}^2}{\Lambda^2} = \frac{9A_f(x)}{4\left(\frac{9}{2}\ell + \frac{3}{2}w_f(x)\right)}.} \tag{23}$$

All denominators are positive, and dominated convergence gives

$$M_{W,\mathrm{Taylor}}^2 \sim \frac{3}{8}m^2 \quad (x \to 0),$$
$$\boxed{\frac{M_{W,\mathrm{Taylor}}^2}{\Lambda^2} \sim \frac{\int s^{-2}\,d\nu(s)}{\ell x} \quad (x \to \infty).} \tag{24}$$

Indeed, $A(z)/z \to 1$, $w(z) \to 1$ at zero, while $zA(z) \to 2$, $w(z) \to 0$ at infinity.

If the other Dirac masses are merely small and nonzero, the answer depends on those additional parameters and is given by (1). In particular, the vanishing limit in (19) assumes they are zero: masses of surviving light doublets can leave a residual vector mass.

5. Theorem (the charged-vector pole).

Under the heat-cutoff hypotheses of Proposition 4, define a timelike pole by analytic continuation of the quadratic inverse propagator:

$$C_W(-M^2_{W,\mathrm{pole}}) = 0. \tag{25}$$

For every $x > 0$, this equation has exactly one solution with $M^2_{W,\mathrm{pole}} > 0$. It is simple and has positive residue in the convention that the Euclidean propagator near the pole is a positive multiple of $(q^2 + M^2_{W,\mathrm{pole}})^{-1}$.

Let $z(x) = M^2_{W,\mathrm{pole}}/\Lambda^2$. Its exact specification is the unique positive root

$$\boxed{\frac{9}{2}g(-z) + \frac{3}{2}h_x(-z) = 0.} \tag{26}$$

Furthermore,

$$z(x) \sim \frac{3}{8}x \quad (x \to 0), \qquad z(x) \sim \frac{1}{x} \quad (x \to \infty), \tag{27}$$

and both mass definitions obey

$$\boxed{0 < M^2_{W,\mathrm{Taylor}},\ M^2_{W,\mathrm{pole}} \le \frac{\Lambda^2}{e}.} \tag{28}$$

Proof. Let

$$F_x(y) = \frac{9}{2}g(y) + \frac{3}{2}h_x(y).$$

Equation (22) implies $h_x'(y) > 0$ for $y \le 0$, and also $g'(y) > 0$, since $g = h_0$. Hence F_x is strictly increasing on that half-axis. We have

$$F_x(0) = \frac{3}{2}A(x) > 0.$$

As $y \to -\infty$, the prefactors in (15) become negative and both integrals grow exponentially, so $F_x(y) \to -\infty$. This

proves existence, uniqueness, simplicity, and the residue sign.

For $y \le 0$, equation (22) gives $g'(y) \ge 2/3$, and therefore

$$F_x'(y) \ge 3.$$

The mean-value theorem yields

$$z(x) \le \frac{1}{2} A(x).$$

The Taylor denominator gives the same bound for $M^2_{W,\mathrm{Taylor}}/\Lambda^2$. Finally,

$$A(x) = 2\int_0^1 xu\, e^{-xu}\, du \le \frac{2}{e},$$

proving (28).

As $x \to 0$, the bound implies $z(x) \to 0$, and $F_x'(y) \to 4$ near the root. As $x \to \infty$, $z(x) \to 0$, $h_x'(y) \to 0$ near the root, and $F_x'(y) \to 3$. Together with $F_x(0) \sim 3x/2$ and $F_x(0) \sim 3/x$, respectively, these prove (27). □

The pole mass governs the linearized dispersion relation of this analytically continued quadratic action. The Taylor mass governs its two-derivative truncation; they need not coincide near the cutoff. We assert no closed-form inversion of (26), no classification of complex poles, and no Lorentzian unitarity theorem.

The pole proof also extends to positive measures supported in a compact subset of $(0, \infty)$, using

$$F_{f,x}(y) = \int \frac{1}{s}\left[\frac{9}{2} g(sy) + \frac{3}{2} h_{sx}(sy)\right] d\nu(s).$$

It gives the analogues of (24) for the pole mass. For the larger class (4), only the Euclidean and Taylor conclusions have been established here; its moment assumptions alone do not justify that continuation.

6. Proposition (unbroken coefficients, neutral vectors, and weak angles).

Retain the heat-cutoff Standard Model family of Proposition 4. Normalize color generators by $\mathrm{tr}_{\mathbf{3}}\, t_a t_b = \frac{1}{2}\delta_{ab}$, and use $Q = T_3 + Y$. Define the full-trace coefficients by

$$S^{(2)}_{\mathrm{unbroken}} = \int \left(\kappa_{\mathrm{em}} F^2_{\mathrm{em}} + \kappa_s \sum_a G_a^2\right).$$

Then

$$\boxed{\begin{aligned}\kappa_{\rm em} &= \frac{1}{24\pi^2}\frac{16}{3}(5+r), \qquad \kappa_s = \frac{1}{24\pi^2}2(5+r),\\ \frac{\kappa_{\rm em}}{\kappa_s} &= \frac{8}{3}.\end{aligned}} \tag{29}$$

Both coefficients approach 5/6 of their all-massless values.

Let K denote the particle-trace neutral q^2 coefficient in coordinates (A_3, A_Y). Then

$$\boxed{\begin{aligned}K_{33} &= \frac{21}{4} + \left(\frac{3}{4} - \frac{3x}{8}\right) r,\\ K_{YY} &= \frac{103}{12} + \left(\frac{17}{12} - \frac{3x}{8}\right) r,\\ K_{3Y} &= -\frac{1}{4} + \left(\frac{1}{4} + \frac{3x}{8}\right) r.\end{aligned}} \tag{30}$$

The matrix K, the charged coefficient K_W, and the unbroken coefficients are positive for every $x \geq 0$.

Define the neutral-current angle from the current of the massive direction after diagonalizing the two-derivative neutral kinetic form:

$$J_Z \;\propto\; T_3 - \sin^2\theta_{\rm NC}\, Q.$$

Then

$$\boxed{\cos^2\theta_{\rm NC} = \frac{K_{YY} + K_{3Y}}{K_{33} + 2K_{3Y} + K_{YY}} = \frac{5}{8}.} \tag{31}$$

A different continuation, namely

$$\cos^2\theta_{\rm match} := \frac{K_{YY}}{K_W + K_{YY}},$$

satisfies

$$\boxed{\cos^2\theta_{\rm match}(0) = \frac{5}{8}, \qquad \cos^2\theta_{\rm match}(\infty) = \frac{103}{157}.} \tag{32}$$

Thus these definitions cease to describe one common electroweak angle.

Proof. Photon and color generators commute with V_0. Their coefficients therefore follow from (3) with $d_{ij} = 0$

on each invariant mass block. The charge traces give (29).

A Dirac species of mass m, with neutral charges q_L^a, q_R^a, contributes to K

$$e^{-x}\left[q_L^a q_L^b + q_R^a q_R^b - \frac{x}{2}(q_L^a - q_R^a)(q_L^b - q_R^b)\right]. \tag{33}$$

Applying (33) to the three top colors, and retaining all other states, proves (30).

Write

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad B_\infty = \begin{pmatrix} 21/4 & -1/4 \\ -1/4 & 103/12 \end{pmatrix},$$
$$U = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 17/12 \end{pmatrix}.$$

Then

$$K = B_\infty + rU - \frac{3xr}{8} vv^T.$$

We have $B_\infty \geq 5I$, $U \geq 0$, and $xr \leq 1/e$. Hence

$$K \geq \left(5 - \frac{3}{4e}\right) I > 0.$$

Positivity of K_W follows from (17).

The neutral mass direction is v, while the photon direction is $(1, 1)$. The kinetic-orthogonal massive field is proportional to $K^{-1}v$, giving the first expression in (31). Directly,

$$K_{33} + K_{3Y} = 5 + r, \qquad K_{YY} + K_{3Y} = \frac{5}{3}(5 + r),$$

which proves (31). Taking limits proves (32). $\square$

The neutral mass also tends to zero. More precisely,

$$\boxed{\frac{M_{Z,\mathrm{Taylor}}^2}{\Lambda^2} = \frac{9}{4} xr\, v^T K^{-1} v \sim \frac{2}{3} x e^{-x} \quad (x \to \infty).} \tag{34}$$

For $x > 0$ there is likewise one positive massive neutral pole, besides the photon pole. To prove this, put $B = B_\infty + rU$. The exact neutral kernel, apart from the common factor $\Lambda^2/(4\pi^2)$, is

$$B\, g(y) + \frac{3}{2} xr\, H(y) vv^T. \tag{35}$$

After congruence by $B^{-1/2}$, its massive eigenvalue is

$$g(y) + \beta H(y), \qquad \beta = \frac{3}{2} x r \, v^T B^{-1} v > 0.$$

Its timelike root obeys

$$z - 2 + \frac{2}{H(-z)} = \beta. \tag{36}$$

The left side increases from zero to infinity, because

$$\frac{d}{dz}\left(z - 2 + \frac{2}{H(-z)}\right) = 1 - \frac{2\,\mathbb{E}_z[b]}{H(-z)} \geq \frac{1}{2},$$

where $\mathbb{E}_z$ uses density proportional to $e^{zb(u)}$. The pole is therefore unique and has positive residue. Since $v^T B_\infty^{-1} v = 8/27$,

$$\frac{M_{Z,\mathrm{pole}}^2}{\Lambda^2} \sim \frac{2}{3} x e^{-x}.$$

In particular, M_W^2/M_Z^2 does not continue to equal $5/8$.

For every f in (4), equations (29)–(33) extend by replacing 1 by $\ell = f(0)$, r by $f(x)$, and xr by $-xf'(x)$. Positivity follows by averaging the positive heat-cutoff matrices. The ratio $8/3$, the neutral-current value $5/8$, and the matching limit $103/157$ persist. At the endpoint both neutral masses vanish, so the distinguished massive neutral direction—and hence its angle—requires continuation from finite x.

7. Counterexample (positive cutoff does not ensure positive broken kinetic coefficients).

Let $H_F = \mathbb{C}^2$,

$$V_0 = m\sigma_3, \qquad T = \sigma_1, \qquad x = m^2/\Lambda^2,$$

with the same spacetime and trace conventions. For the heat cutoff,

$$C_T(q^2) = \frac{\mathrm{Tr}(T^2)\Lambda^2}{8\pi^2} e^{-x} \left[(2 + 4x + y)H(y) - 2\right]. \tag{37}$$

Its q^2 Taylor coefficient is proportional to

$$e^{-x}(1 - x).$$

It vanishes at $x = 1$ and becomes negative for $x > 1$. Thus the two-derivative truncation has a wrong-sign vector kinetic term in this example.

Nevertheless, the full quadratic form has a unique positive real timelike pole with positive residue, satisfying

$$\boxed{4x < \frac{M^2_{\rm pole}}{\Lambda^2} < 4x + 2.} \tag{38}$$

Consequently $M_{\rm pole} \sim 2m$ diverges at fixed cutoff.

Proof. The generator connects the eigenvalues $m, -m$, so $d^2_{ij} = 4m^2$ while both squared eigenvalues equal m^2. Substitution in (1) proves (37). Its Taylor ratio is

$$M^2_{\rm Taylor} = \frac{6m^2}{1-x},$$

where this is a positive mass only for $x < 1$.

The pole equation is (36) with $\beta = 4x$. The monotonicity proof there gives uniqueness and the residue sign. Evaluating (37) at $y = -4x$ gives $2H(-4x) - 2 > 0$, whereas at $y = -4x - 2$ it gives -2; this proves (38).

For general f satisfying (4), the Taylor kinetic coefficient in this example is proportional to

$$f(x) + xf'(x) = \frac{d}{dx}[xf(x)]. \tag{39}$$

It must be negative somewhere: $xf(x) > 0$ for $x > 0$, but $xf(x) \to 0$ at infinity under (4). A nonnegative derivative everywhere would contradict these facts. □

This counterexample concerns arbitrary finite data; the top-only Standard Model family is outside it. It also distinguishes a wrong-sign **Taylor truncation** from a negative-residue pole of the **full** form factor. We have not proved absence of complex-pole or other nonlocal problems in either model.

8. Counterexample (heavy fermions do not dynamically select the centralizer).

Let the data be those of Proposition 4, and let

$$\mathfrak{g}_{V_0} = \{X \in \mathfrak{g}_{\rm SM} : [X, V_0] = 0\}.$$

For nonzero top mass this is the represented unbroken gauge algebra $\mathfrak{su}(3) \oplus \mathfrak{u}(1)_{\mathrm{em}}$. The limit $x \to \infty$ at fixed Λ does **not** produce a theory containing only these vector directions by quadratic mass decoupling.

Proof. For fixed y, equation (21) implies

$$h_x(y) \longrightarrow 0.$$

Consequently

$$\boxed{C_W(q^2) \longrightarrow \frac{\Lambda^2}{4\pi^2}\frac{9}{2}g(q^2/\Lambda^2).} \tag{40}$$

For positive y,

$$g(y) = 4y \int_0^1 u(1-u)e^{-yu(1-u)}\,du > 0.$$

Thus a charged generator outside $\mathfrak{g}_{V_0}$ retains a finite nonzero quadratic kinetic form and a nonvanishing propagator at every fixed positive momentum. Its mass approaches zero. Equation (35) similarly leaves the positive matrix $B_\infty g(y)$, including the additional neutral direction. These facts contradict decoupling of the broken vectors through diverging masses or vanishing propagators. □

A narrower decoupling statement is true. If we **restrict the connection from the outset** to the centralizer, the finite mass blocks are invariant and the heat trace factorizes. On the heavy block its contribution carries e^{-sm^2/Λ^2}, which tends to zero. The restricted action therefore converges to the light-block spectral action. This proves decoupling of a finite fermion block under that restriction; it does not prove dynamical elimination of the other gauge fields.

The amendment to Lemma 03 Part C is therefore:

- Its transverse kinetic weights, including $w_L(x,0) = w(x)$, and its two weak-angle formulas are correct under their stated conventions.
- They must be accompanied by the mass coefficient (2), including the quadratic-vertex subtraction.
- A centralizer-only endpoint does not follow from those weights. In the present exact quadratic family, the broken-vector masses vanish while their remaining kinetic coefficients are positive.

9. No-go theorem (finite radial vacuum for the positive Laplace class).

Let f satisfy (4), and suppose the allowed scalar configurations include

$$V = cV_*, \qquad c > 0, \qquad V_* \neq 0,$$

with $A = 0$. Let the action be the literal spectral action, without an added scalar potential or a constraint fixing c. Then no finite $c > 0$ is stationary along this radial direction.

Proof. On $\mathbb{R}^4$, the constant-background action density is

$$\mathcal{U}(c) = \frac{\Lambda^4}{4\pi^2} \int s^{-2} \operatorname{Tr}_{H_F} e^{-sc^2 V_*^2/\Lambda^2} \, d\nu(s).$$

Differentiating gives

$$\boxed{\mathcal{U}'(c) = -\frac{c\Lambda^2}{2\pi^2} \int s^{-1} \operatorname{Tr}_{H_F} \left(V_*^2 e^{-sc^2 V_*^2/\Lambda^2} \right) d\nu(s) < 0.} \tag{41}$$

The inequality is strict because $V_* \neq 0$ and ν is nonzero and positive. The same argument applies on a finite flat torus before performing its positive momentum sum. □

Thus the masses above are conditional quadratic-background quantities. In the top-only family, the ordinary Higgs radial variation supplies the direction used in this theorem. A finite symmetry-breaking minimum of the polynomial asymptotic scalar potential cannot automatically be regarded as a minimum of this full positive-Laplace spectral action.

10. Corollary (comparison of heavy limits).

Let the background family and mass definitions be those above.

At fixed Λ, with $\mu \to \infty$:

quantity, heat cutoff	behavior
m_t	$\mu\Lambda \to \infty$
$M_{W,\mathrm{Taylor}}$, $M_{W,\mathrm{pole}}$	$\sim \Lambda/\mu \to 0$
$M_{Z,\mathrm{Taylor}}$, $M_{Z,\mathrm{pole}}$	$\sim \sqrt{2/3}\,\Lambda\mu e^{-\mu^2/2} \to 0$
m_t/M_W	$\sim \mu^2 \to \infty$

(42)

At fixed $0 < \mu < \infty$, with $\Lambda \to \infty$, all four vector masses in (42) instead grow linearly with Λ, as does m_t. A controlled asymptotic family can also satisfy

$$\Lambda \to \infty, \qquad \mu \to 0, \qquad \mu\Lambda \to \infty;$$

then $M_W, m_t \to \infty$ while

$$m_t/M_W \to \sqrt{8/3}.$$

Proof. The first assertions follow from (19), (27), and (34)–(36). At fixed positive μ, their dimensionless mass functions are fixed and positive, so multiplication by Λ proves the joint-limit assertions. The final assertion follows from the small-μ limits. □

Accordingly, the obstruction has three distinct scopes. The CCM identity cannot be extrapolated to $\mu \to \infty$ at fixed cutoff because it belongs to the asymptotic truncation. The exact top-only quadratic action obstructs a fixed-cutoff heavy-W limit for the heat cutoff under both mass definitions, and for class (4) under the Taylor definition. A family of actual nonzero stationary vacua of the unmodified positive-Laplace action encounters the separate obstruction (41). Counterexample 7 excludes extending the first of these exact mass conclusions to arbitrary finite representations.

11. Proposition (the radial derivative for general f; referee). On flat $\mathbb{R}^4$, for a sufficiently integrable real cutoff f, with λ_j^2 the eigenvalues of V_*^2,

$$\mathcal{U}'(c) = -\frac{c\Lambda^2}{2\pi^2} \sum_j \lambda_j^2 \int_{c^2\lambda_j^2/\Lambda^2}^{\infty} f(z)\, dz. \tag{43}$$

For nonnegative f, a finite $c > 0$ is stationary exactly when every tail associated with a nonzero λ_j vanishes. So if f has nonzero support arbitrarily far out, there is no finite stationary point on this ray; if f has compact support ending at R, there is a stationary plateau once every nonzero $c^2\lambda_j^2/\Lambda^2 \geq R$. Sign-changing cutoffs can have cancellations. A nonconstant cutoff flat near zero lies outside (4), since a nonzero positive measure gives $f'(0) < 0$; (43) covers the nonnegative ones. A radial path with an additional fixed noncommuting mass matrix need not have the form cV_*. □

12. Proposition (the top-only Taylor mass for general f; referee). For nonnegative smooth compactly supported f with $f(0) > 0$, the formulas of Proposition 4 hold with

$$A_f(x) = \frac{2}{x}\int_0^x zf(z)\,dz, \qquad w_f(x) = \frac{3}{x^3}\int_0^x z^2 f(z)\,dz,$$

by approximating f with its derivatives by real combinations of e^{-nz}. Hence

$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} \sim \frac{\int_0^\infty zf(z)\,dz}{f(0)\,x}.$$

This includes cutoffs flat near zero. For a nonanalytic cutoff there need be no analytic continuation defining the pole. □

13. Proposition (several heavy doublets; referee). For the heat cutoff, let P be the total particle-trace weight of complete massless doublets, and let heavy doublet i, of weight $d_i > 0$, have one Dirac member of mass m_i and one massless member. Then

$$C_W(y) = \frac{\Lambda^2}{4\pi^2}\Big(Pg(y) + \sum_i d_i h_{x_i}(y)\Big),$$
$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{3}{2}\,\frac{\sum_i d_i A(x_i)}{P + \sum_i d_i w(x_i)}.$$

For finitely many doublets, if $P > 0$ and all $x_i \to \infty$, the Taylor mass tends to zero. If $P = 0$ and all $x_i = x$, then $M^2_{W,\mathrm{Taylor}} \sim m^4/(2\Lambda^2)$, the form factor has a unique positive pole with $M^2_{W,\mathrm{pole}} \sim m^2/2$, and the whole unnormalized kernel tends to zero at every fixed Euclidean momentum. The divergent Taylor and pole scales differ, and neither alone establishes a healthy limiting gauge theory. □

14. Proposition (the fully massive family; referee; the complete proof by Astra 6 follows the sketch). Give both members of every Standard Model doublet the same Dirac mass m, with no Majorana masses, and let z_W, z_Z be the charged and massive neutral poles in units of Λ^2. The charged form factor is proportional to $e^{-x}[g(y) + 2xH(y)]$, and

$$2x < z_W < 2x + 2, \qquad \tfrac{16}{5}x < z_Z < \tfrac{16}{5}x + 2.$$

So both pole masses diverge with m at fixed cutoff, and

$$\frac{M_{W,\mathrm{pole}}^2}{M_{Z,\mathrm{pole}}^2} \longrightarrow \frac{5}{8} \qquad (x \to \infty),$$

as also for $x \to 0$, where $z_W \sim 3x$ and $z_Z \sim \frac{24}{5}x$. In between the ratio differs, reaching about 0.66 near $x = 3$ (Python, solving the pole equations). The kernel tends to zero at every fixed Euclidean momentum, and the charged Taylor kinetic coefficient is proportional to $e^{-x}(1 - x/2)$, negative for $x > 2$.

Proof. The species with equal mass have eigenvalues $\pm m$. The charged generator connects them with $d^2 = 0$ and $d^2 = 4m^2$ in equal weights, which gives the charged form. For the neutral generators the diagonal charge has $d = 0$ and the off-diagonal charge $(q_L - q_R)/2 = T_3 v/2$ has $d^2 = 4m^2$; on the particle trace the kernel is $e^{-x}[\mathrm{diag}(6, 10)g + 12xHvv^T]$, so $\beta = 12x\, v^T\, \mathrm{diag}(6, 10)^{-1} v = \frac{16}{5}x$, and for the charged direction $\beta = 2x$. Equation (36) gives the bounds, since $H(-z) > 1$. □

Complete proof (Astra 6, 3 November 2026; written here from its validation report, item 11, so that the lemma is self-contained). Take the heat cutoff, three generations with right-handed neutrinos, no Majorana masses, and one common positive Dirac mass m for every species. Let $x = m^2/\Lambda^2$, $y = q^2/\Lambda^2$, $H(y) = \int_0^1 e^{-yu(1-u)}\, du$ and $g(y) = (2 + y)H(y) - 2$. Every mass eigenvalue is $+m$ or $-m$, so the exact quadratic kernel of Theorem 1 for a generator entry is proportional to $e^{-x}[g(y) + ((m_i - m_j)^2/\Lambda^2)H(y)]|T_{ij}|^2$. A charged generator connects equal and opposite mass signs with equal weights, the average of $(m_i - m_j)^2$ is $2m^2$, and the charged kernel is proportional to $e^{-x}[g(y) + 2xH(y)]$. In the neutral particle trace, with $v = (1, -1)^T$, the charge and mass-commutator traces give $Q = \mathrm{diag}(6, 10)$ and $C_{\mathrm{neutral}}(y) \propto e^{-x}[Qg(y) + 12xH(y)vv^T]$, whose determinant factors as $\det Q\, g(y)[g(y) + 12x\, v^T Q^{-1} v\, H(y)]$; since $v^T Q^{-1} v = 1/6 + 1/10 = 4/15$, the massive neutral factor has coefficient $16x/5$. Charge conjugation multiplies the common normalization and moves no root. Both massive pole equations therefore have the form $(2 + \beta - z)H(-z) = 2$ with $\beta_W = 2x$ and $\beta_Z = 16x/5$. Let $h(z) = H(-z)$ and $F(z) = z - 2 + 2/h(z)$. For $z \geq 0$, $h(z) \geq 1$ and $0 < h'(z) = \int_0^1 u(1-u)e^{zu(1-u)}\, du \leq h(z)/4$,

so $F'(z) = 1 - 2h'(z)/h(z)^2 \geq 1 - 1/(2h(z)) \geq 1/2$; with $F(0) = 0$ and $F(z) \to \infty$, each $\beta > 0$ has a unique positive solution, at which $h(z) > 1$, so $z = \beta + 2 - 2/h(z)$ gives $\beta < z < \beta + 2$: the displayed bounds. For small z, $h(z) = 1 + z/6 + O(z^2)$ and $F(z) = \frac{2}{3}z + O(z^2)$, so $z_W \sim 3x$, $z_Z \sim 24x/5$ and the ratio tends to 5/8; for $x \to \infty$ the bounds give $z_W = 2x + O(1)$, $z_Z = 16x/5 + O(1)$, again 5/8. At fixed Euclidean momentum every kernel entry is e^{-x} times a function growing at most linearly in x, hence tends to zero. Finally $H'(0) = -1/6$ and $g'(0) = 2/3$, so the charged Taylor kinetic coefficient is proportional to $e^{-x}(\frac{2}{3} - \frac{x}{3}) = \frac{2}{3}e^{-x}(1 - x/2)$, negative for $x > 2$. A quadrature and root calculation (Python 3, SciPy, not kept) gave $z_W = 7.4996$, $z_Z = 11.3746$, ratio 0.6593 at $x = 3$. □

This is a counterexample to any general no-go for diverging pole locations at fixed cutoff. It has no light fermions, and the physical quality of that limit is open.

15. Remark (the light space does not carry unbroken $SU(2) \times U(1)$; referee). In the top-only limit the broken masses vanish, but this does not restore the electroweak group on the whole light space. The light space retains b_L after removing t_L. If P_{light} projects off the top mass block, then on b_L we have $P_{\text{light}} T_1 P_{\text{light}} = P_{\text{light}} T_2 P_{\text{light}} = 0$ and $P_{\text{light}} T_3 P_{\text{light}} = -\frac{1}{2}$, which do not obey the $SU(2)$ relations. Correspondingly $K_W = 9/2$, $K_{33} = 21/4$, $K_{3Y} = -1/4$ in the limit, while a linearly realized unbroken action would have equal weak coefficients and no mixing. The remaining complete massless doublets carry the ordinary representation, and their contribution is gauge invariant. If both members of every removed doublet are heavy, the retained light space can be invariant under the full group; a full limiting theory still needs control of the remaining fields, normalizations, interactions and vacuum. "The heavy fermion leaves the loop" is shorthand for the suppression in this classical trace.

The light-space statement should be read together with the separate role of the top (lemma 21): among single charged species only an up-type quark can be removed while keeping cos^2 theta_NC = 5/8 and the ratio 8/3, and removing it leaves the five other quarks and the leptons, on which SU(3) x U(1)_em acts with b_L paired

with b_R.

16. Remark (the CCM relation as the small-mass form of a weighted sum rule; Claude). For the heat cutoff and doublets with one massive member, Proposition 13 reads

$$8\kappa M^2_{W,\mathrm{Taylor}} = \sum_f N_f m_f^2\, \rho(m_f^2/\Lambda^2),$$
$$\rho(x) = \frac{A(x)}{x} = \frac{2[1-(1+x)e^{-x}]}{x^2},$$

with $N_f = 2d_f$ and $\kappa = (P + \sum_i d_i w(x_i))/6$. As $x \to 0$, $\rho, \kappa \to 1$, which is the CCM relation (14). The relation is therefore the leading term of an expansion, and its range is $m \ll \Lambda$. For the top alone, $\kappa = (3+w)/4$; m/M_W departs from $\sqrt{8/3}$ by 5% at $m \approx 0.45\Lambda$, and M_W peaks at about 0.52Λ at $m \approx 1.42\Lambda$ (Python). The maximum is unique, at $x = 2.024$ with $M_W/\Lambda = 0.5242$ (referee, by a sign argument and Python). A massive diagonal contribution to the trace is exponentially suppressed above the cutoff, while the mixed loop with a massless partner has the algebraic $\rho \sim 2/x^2$ at large x; a vacuum with no other mass leaves the W mass with no source. Whether the classical action is meaningful for $m > \Lambda$ is a separate question, on which these formulas do not decide. In the top-only heat-cutoff family, the ratio differs from $\sqrt{8/3}$ for every $x > 0$, since $3 + w - 4\rho$ vanishes at zero and is increasing, so the limit with M_W and m_t diverging together and $m_t/M_W \to \sqrt{8/3}$ needs $\Lambda \to \infty$ faster than m_t (Corollary 10). This necessity is for that family: with one complete massless lepton doublet and one member of each other doublet of common mass m, the weighted sum equals the unweighted CCM sum at some $x_* > 0$ (referee).

As algebraic checks, we used **Python 3 with SymPy** to verify the integration-by-parts identities, the displayed small-x Taylor series, and $v^T B_\infty^{-1} v = 8/27$. The arguments above provide the proofs; these computations introduce no fitted constants.

Consequence for T5b. At fixed cutoff, the specified top-only Standard Model spectral action does not realize an infinitely heavy W, Z, t family: the top grows while both vector masses tend to zero, for the heat cutoff and the stated quadratic mass definitions. A joint limit with

$\Lambda \to \infty$ permits all three background mass scales to diverge, but neither centralizer-only decoupling nor a stationary Higgs vacuum follows from that scaling. These conclusions concern the classical bosonic action and contain no quantum corrections or claims about the physical Standard Model at those scales. Other finite data behave differently: with every doublet heavy and no light fermions, the pole masses diverge at fixed cutoff with M_W^2/M_Z^2 -> 5/8 (Proposition 14). With a massless complete doublet retained, the Taylor mass tends to zero (Proposition 13). So in the families treated here, a heavy W and Z at fixed cutoff is found only without a massless complete doublet. Light nonzero masses are not treated: they leave a residual vector mass that depends on them.

Part 22

The finite Dirac operator as a boundary condition on an interval

For STRATEGY T0, T0b, T10 and T13, following lemma 12 and lemma 20. Proved by Claude Opus 5.5, with the identities checked in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): two errors and one gap as first written, and amendments; all are incorporated below.

Outcome. Let (H_F, D_F, J_F, Gamma_F) be finite data of signs (+, +, -), and let l ||D_F|| < pi/2. Then D_F, with J_F and Gamma_F, is exactly the lowest spectral window of the pure derivative on an interval of length l, with no bulk mass and no Yukawa term, on the domain fixed by the boundary coupling tan(l D_F). So all the eigenvalues of D_F, Yukawa and Majorana blocks alike, are quantized internal momenta k with tan(k l) equal to a singular value of the coupling. The input is the finite triple: the content is that a consistent parent exists, with these properties:

- Preservation of the domain forces the observables at the wall into C_F, the commutant of D_F in A_F, while A_F acts freely in the bulk.
- The parent satisfies order one for any finite data with order zero.

- The compression is the exact average Q_l(a) = (2l)^-1 int_{-l}^{l} e^{ixD_F} a e^{-ixD_F} dx, so it tends to the finite representation as l -> 0, with defects of order l^2.
- The scalar inner fluctuations of the parent are pure gauge under its allowed endpoint gauge group. On the moving window, they act by the gauge orbit of D_F through the transport along the interval. This reproduces the quadratic term of Chamseddine-Connes-van Suijlekom on the orbit, and it leaves the singular values fixed, so the radial mode of the Higgs field is absent from this parent.

With the family X(l) = tan(l M) and l -> 0, the couplings of the light fermions vanish as m l at fixed mass, the tower decouples, and the broken gauge directions go to pi/(2l). The parent algebra is matrix-valued, so T0 for a commutative parent remains open.

Setting

Let H_F = H_+ (+) H_- be finite-dimensional, Gamma_F = diag(1, -1), let c : H_+ -> H_- be antiunitary, and put J_F(a, b) = (c^-1 b, c a), so that J_F^2 = 1 and J_F Gamma_F = -Gamma_F J_F. Let D_F = [[0, M*], [M, 0]] with M : H_+ -> H_-. Then J_F D_F = D_F J_F is equivalent to M = c M* c. By the Takagi factorization, in bases related by c, we may write M = V |M| with a unitary V satisfying V = c V* c.

Let l > 0 with l ||D_F|| < pi/2, and define X : H_- -> H_+ by

```
tan(l D_F) = [[0, X], [X\*, 0]],   that is,   X =
↪  tan(l |M|) V\*.
```

Since J_F commutes with real functions of D_F, c X = X* c^-1. Conversely, every X with c X = X* c^-1 arises from exactly one such D_F, namely D_F = l^-1 arctan of the block operator. In the limit, X = l M* + O(l^3).

The parent. On H_l = L^2([0, l]; C^2) (x) (H_+ (+) H_-), write psi = (Q_L, Q_R; U_L, U_R), with Q valued in H_+ and U in H_-. Let (D psi)_L = psi_R' and (D psi)_R = -psi_L' on both fields, the operator of lemma 12 with no bulk mass, on the domain

```
Q_R(0) = 0,   U_L(0) = 0,   Q_R(l) = X U_R(l),
↪  U_L(l) = -X\* Q_L(l).
```

Call it D_X. Put Gamma^ psi = (Q_L, -Q_R; U_L, -U_R) and J^ psi = (c^-1 U_R, -c^-1 U_L; -c Q_R, c Q_L). Let (E psi)_L = psi_R and (E psi)_R = -psi_L, so that D_X = E d/dx on its domain.

In the Standard Model form of the classification, with E = Hom(V, W) and W = W_+ (+) W_-, one takes H_+ = E_+ (+) E_-* and H_- = E_- (+) E_+*, with c the adjoint map. At X = 0, the Q-fields have boundary type (a) and the U-fields type (b), in the terms of lemma 09; for X != 0 the modes continue from those zero modes, which generally lift. A single matrix X encodes the Yukawa blocks and the Majorana block together. Its entries need not keep the original block pattern, since the tangent brings in higher powers of D_F: for M = [[a, b], [b, 0]] real symmetric, X_22 = (l^3/3) a b^2 + O(l^5).

The window

Theorem 22.1. D_X is self-adjoint with compact resolvent, and J^^2 = 1, J^ D_X = D_X J^, J^ Gamma^ = -Gamma^ J^, Gamma^ D_X = -D_X Gamma^. Let R(x) = exp(x Gamma_F D_F), and define W : H_F -> H_l so that for phi = (a, b), the L-components (Q_L, U_L) of W phi at x are l^{-1/2} R(x)(a, 0), and the R-components (Q_R, U_R) are l^{-1/2} R(x)(0, b). Then W is an isometry and

```
D_X W = W D_F,    J^ W = W J_F,    Gamma^ W = W
↪ Gamma_F.
```

With sigma_k the singular values of M and theta_k = l sigma_k, the signed spectrum of D_X is (n pi +- theta_k)/l, n in Z, with multiplicities added when values coincide; a zero singular value gives a zero mode of multiplicity two. So the spectrum of |D_X| consists of the sigma_k and of (n pi +- theta_k)/l for n >= 1, and ran W is the spectral subspace of |D_X| <= ||D_F||, and the next level is at least (pi - l ||D_F||)/l.

Proof. Self-adjointness is Lemma 12.1. Since Gamma_F D_F is anti-Hermitian, R(x) is unitary, and with C(x) = cos(x |M|), C'(x) = cos(x |M*|), S(x) = V sin(x |M|),

```
R(x)(a, 0) = (C(x) a, -S(x) a),    R(x)(0, b) =
↪ (S(x)\* b, C'(x) b).
```

Isometry follows from the unitarity of R(x) and the or-

thogonality of the L- and R-components. The conditions at 0 hold since S(0) = 0. At l, using V* cos(l |M*|) = cos(l |M|) V*, we get X C'(l) = S(l)* and X* C(l) = S(l), which are the conditions at the wall. For the derivative, the L-part of D W phi is d/dx R(x)(0, b) = R(x) Gamma_F D_F (0, b) = R(x)(M* b, 0), and the R-part is -d/dx R(x)(a, 0) = R(x)(0, M a); together these are W D_F phi. The relations for Gamma^ are direct. For J^, use c |M| c^-1 = |M*| and V = c V* c, which give c^-1 C' = C c^-1 and c S(x)* = S(x) c^-1. The signs of the parent follow in the same way, using c X = X* c^-1 for the domain. For the spectrum, constant unitaries on Q and on U commute with E d/dx and bring X to diag(tan theta_k), which reduces to Example 12.5 in each channel. ■

Corollary 22.2 (conjugated form). D_X = R (D_0 + E Gamma_F D_F) R*, with R acting pointwise on the pair (Q, U) and D_0 the uncoupled operator, X = 0. So after this unitary change of domain, D_F is a constant matrix connection, odd for the grading, with relative endpoint rotation R(l). This is equation (21) of lemma 20, for general D_F.

Proof. D(R phi) = R(D phi + E R* R' phi), and R* R' = Gamma_F D_F. For phi in the uncoupled domain, the computation at the wall in the proof above shows that R phi satisfies the coupled conditions. ■

So in this family a fermion mass is a momentum: D_X has no mass term, and the light eigenvalue k = theta/l is fixed by tan(k l) = tan(theta), a singular value of X. In the finite description, the condition relating k to l is lost and the value appears as free data in D_F.

The parent as a real spectral triple

Let (A_F, H_F, D_F, J_F, Gamma_F) be finite data as above, with an even representation pi = pi_+ (+) pi_- of a unital finite-dimensional real or complex *-algebra A_F, faithful, unital and satisfying order zero, with pi°(b) = J_F pi(b)* J_F^-1. Order one is not assumed. Let

```
C_F = {a in A_F : [pi(a), D_F] = 0},    A_l = {f in
 ↪  C^infinity([0, l], A_F) : f(l) in C_F},
```

with f acting pointwise by pi_+(f(x)) on Q and pi_-(f(x)) on U.

Theorem 22.3.

1. A function f in C^infinity([0, l], A_F) preserves Dom D_X if and only if f(l) lies in C_F.
2. (A_l, H_l, D_X, J^, Gamma^) is a real even spectral triple, with [D_X, pi(f)] = E pi(f') and J^ pi(g)* J^^-1 = pi°(g) pointwise.
3. It satisfies order zero and order one, with signs (+, +, -).

So A_F acts freely in the bulk, and the boundary condition restricts the observables at the wall to the commutant of D_F. In the Standard Model data with nonzero Yukawa blocks, this is the algebra {(lambda, lambda, m)} of the photon and gluons; with vanishing Yukawa blocks the commutant is larger. Order one holds in this parent whatever the finite data, for its pointwise algebra: the finite condition appears only on the truncation, and arbitrary compressions need not preserve either order condition.

Proof. Item 1: at l, the traces Q_L(l) and U_R(l) are free, and applying f to the conditions gives pi_+(f(l)) X = X pi_-(f(l)) and pi_-(f(l)) X* = X* pi_+(f(l)), as in Result 3 of lemma 20. That is [pi(f(l)), tan(l D_F)] = 0, equivalent to [pi(f(l)), D_F] = 0 because l ||D_F|| < pi/2 makes D_F a function of tan(l D_F). At 0 the conditions are preserved since pi is even. Item 2: the commutator is the Leibniz rule. Conjugation by J^ acts on (Q, U) as J_F does, and pi(g) acts identically on the two spin components, which gives the second formula. This preserves the domain since J_F commutes with D_F. Item 3: pi(f(x)) and pi°(g(x)) commute at each x by finite order zero, and E commutes with both; so [E pi(f'), pi°(g)] = E [pi(f'), pi°(g)] = 0. ■

This is consistent with Counterexample 4 of lemma 20, whose finite data A_F = C + C, D_F = m sigma_1 fail order one while the parent satisfies it.

The compression

Let Q_l(T) = W* T W for a bounded operator T on H_l. For T(x) a bounded function with values in the even operators on H_F, acting pointwise and identically on

both spin components, we write T for the multiplication operator.

Theorem 22.4.

```
Q_l(T) = l^-1 int_0^l cos(x ad_{D_F})(T(x)) dx,
 ↪  Q_l(E T) = -l^-1 int_0^l sin(x ad_{D_F})(T(x))
 ↪  dx.
```

For constant a in A_F,

```
Q_l(a) = sinc(l ad_{D_F})(pi(a)) = (2l)^-1
 ↪  int_{-l}^{l} e^{ix D_F} pi(a) e^{-ix D_F} dx,
```

so in an eigenbasis of D_F with eigenvalues mu_i, the matrix elements are pi(a)_{ij} sin(l (mu_i - mu_j))/(l (mu_i - mu_j)). Moreover:

1. Q_l(A_l) = Q_l(C^infinity([0, l], A_F)), so the image of the parent algebra contains Q_l(a) for every a in A_F.
2. ||Q_l(a) - pi(a)|| <= (l^2/6) |||[D_F, [D_F, pi(a)]]|| <= (2/3) l^2 ||D_F||^2 ||a||.
3. Q_l([D_X, f]) = [D_F, Q_l(f)] for f in A_l.
4. With a' = Q_l(a) and b' = J_F Q_l(b)* J_F^-1, |||[a', b']|| <= (8/3) l^2 ||D_F||^2 ||a|| ||b||, and |||[[D_F, a'], b'] - [[D_F, pi(a)], pi°(b)]|| <= (16/3) l^2 ||D_F||^3 ||a|| ||b||.

So as l -> 0 with D_F fixed, D_F, J_F and Gamma_F stay exact, and the compressed observables converge to pi(A_F) with rate l^2. The finite order conditions hold in the limit exactly when they hold for the finite data.

Proof. From the definition of W, the expectation of T splits into the L- and R-parts, so Q_l(T) is l^-1 int of the Gamma_F-even part of R(x)* T(x) R(x). Put K = Gamma_F D_F. For even Z, ad_K Z = Gamma_F [D_F, Z], which is odd, and ad_K (Gamma_F [D_F, Z]) = -[D_F, [D_F, Z]]. So the even part of e^{-xK} T e^{xK} is cos(x ad_{D_F})(T). Likewise, E exchanges the L- and R-parts with a sign, so Q_l(E T) is l^-1 int of Gamma_F times the odd part of R* T R, and that odd part is -Gamma_F sin(x ad_{D_F})(T). The formula for constant a follows from cos(x ad_D)(Z) = (e^{ixD} Z e^{-ixD} + e^{-ixD} Z e^{ixD})/2.

Item 1: the map f -> Lambda(Q_l(f)), for a linear functional Lambda, is integration against a smooth density.

If it vanishes on A_l, which contains all f of compact support in [0, l), the density vanishes on [0, l), hence everywhere by continuity. Item 2: by Taylor's formula with integral remainder, ||cos(x ad_D)(Z) - Z|| <= (x^2/2) ||ad_D^2 Z||, since cos(x ad_D) has norm at most 1 by the average formula; averaging over x gives l^2/6. Item 3: P commutes with D_X. It also follows from the two formulas by integration by parts, where the boundary term at l vanishes because f(l) is in C_F. Item 4: J_F commutes with D_F, so b' is the same average of pi°(b). Expand both commutators with the bound of item 2 and ||[D_F, Z]|| <= 2 ||D_F|| ||Z||. ■

The convergence is a statement about the maps Q_l, on the operator systems, in the setting of Connes and van Suijlekom (arXiv:2004.14115, sections 1 and 3). The generated algebra is not continuous at l = 0. An analytic example, from the referee: let M = [[0, 0, 1], [0, 0, 1], [1, 1, 0]], D = [[0, M], [M, 0]], with A_F = C^3 acting diagonally and equally on the two blocks. With P_1, P_2 the first two channel projections, [Q_l(P_1), Q_l(P_2)]_{12} = -l^2/3 + O(l^4), so the compressions of commuting projections generate a noncommutative algebra for small nonzero l. On the one-generation Standard Model data, a numerical closure with tolerance 10^-7 gives complex dimension 15 for the algebra generated by pi(B), with B = C + H + M_3(C), and 30 for Q_l(B) at l = 0.3; these are numerical values with no certified rank.

Scalar fluctuations

Theorem 22.5. Let h be smooth with values in the anti-Hermitian elements of A_F. The self-adjoint one-forms of the parent are exactly the E pi(h). Let u' = -h u, u(0) = 1, and let w = u(l), a unitary of A_F. Then

```
D_X + E pi(h) = pi(u) D_{X_w} pi(u)\*,      X_w =
 ↪  pi_+(w)\* X pi_-(w),
```

and the window of D_{X_w} carries pi(w)* D_F pi(w). With the real structure, for A = E pi(h) we have J^ A J^^-1 = E J_F pi(h) J_F^-1, and D_X + A + J^ A J^^-1 has the window U_w* D_F U_w, with U_w = pi(w) J_F pi(w) J_F^-1, and this commutes with J_F. Moreover, with u~(x) = u(x) w*, which lies in A_l since u~(l) = 1, and U~ = pi(u~) J_F pi(u~) J_F^-1,

```
D_X + A + J^ A J^^-1 = U~ D_X U~\*.
```

So every scalar connection in this model is gauge equivalent to the original parent under the allowed endpoint gauge group (referee).

Proof. A one-form is sum f_1 [D_X, f_2] = E pi(sum f_1 f_2'), and E pi(h) is self-adjoint exactly when h is anti-Hermitian, since E is anti-Hermitian and commutes with pi(h) and pi is faithful. Every smooth such h arises: with b(x) = int_l^x h, we have b(l) = 0, so b is in A_l, and [D_X, b] = E pi(h). Next, pi(u)* D pi(u) = D + E pi(u* u'), and pi(u)* E pi(h) pi(u) = E pi(u* h u), so the sum is D with no connection term. At 0 the conditions are unchanged. At l, the substitution psi = pi(u) phi gives the conditions with X_w. Then tan(l pi(w)* D_F pi(w)) has the block X_w. For general w, this operator need not commute with J_F, so we use the spectrum and the isometry of Theorem 22.1 without its real-structure part. For the real fluctuation, J^ commutes with E, u(x) is replaced by pi(u) J_F pi(u) J_F^-1, using finite order zero, and U_w commutes with J_F, so the real window follows. The last formula follows by differentiating, with the endpoint condition. ■

Corollary 22.6 (the quadratic term as transport). By Chamseddine-Connes-van Suijlekom (arXiv:1304.7583, section III), for unitary v in A_F, with U = v J v J^-1, one has U D U* = D + A_(1) + A~*(1)* + *A*(2), for the universal form v delta(v*). Taking v = w*, on the gauge orbit the fluctuation of the parent, which satisfies order one and has no quadratic term, gives on the moving window exactly the fluctuated operator of that framework (equations (10)-(11) of section III.2 there), quadratic term included, for finite data that fail order one. This answers the first open point of lemma 19 on these gauge orbits and with the moving window. For the two-point data with J = sigma_1 K and w = diag(e^{i alpha}, e^{i beta}), it gives the upper link m e^{-2 i (alpha - beta)}, which is m e^{2 i psi} of Corollary 19.2 with psi defined from v = w*. Theorem 22.5 and the covariant projection of lemma 14 share a transport principle. The fixed compression, as in lemma 13, is a different object: by Theorem 22.4, it gives Q_l(E pi(h)) = -[D_F, l^-1 int_0^l x pi(h) dx] + O(l^3 ||D_F||^3 ||h||) for h fixed, so that for constant

h it is -(1/2)[D_F, h], while the first derivative of the transported operator is -1 [D_F, h]. The difference is the contribution of the moving spectral projection.

No-go theorem 22.7 (the radial mode). *Hypotheses:* the parent above, on one interval, with fixed X, no bulk masses, and scalar inner fluctuations from the even algebra A_1, with the moving spectral window. *Conclusion:* the fluctuations leave all singular values of the finite Dirac operator fixed. In the usual Standard Model representation with nonzero Yukawa data, the normalized shifted Higgs quaternion runs through the unit quaternions, a three-dimensional gauge orbit, with no radial mode. Degenerate data can have a smaller orbit. A fixed compression can show changes in singular values at second order, so the window has to be named.

Proof. Theorem 22.5: D_F and pi(w)* D_F pi(w) are unitarily equivalent. The same holds with U_w. ■

Known escapes.

These lie outside the hypotheses, and each needs its own construction.

- The coupling X as a field on the wall, with its own kinetic term and potential.
- Connections odd for the grading, as in Corollary 22.2 and lemma 23: the fluctuation K -> K + delta K changes the singular values.
- Bulk masses that do not commute with the gauge action.
- Two or more internal dimensions, where a connection has curvature, although curvature alone does not prove a radial mode.
- An additional composite or other degree of freedom; a quantum effective action on the same configuration space does not by itself add a radial coordinate.
- A fluctuation of the finite data alone, as with the radial parameter of lemma 19, which is not the moving window.

Direction. On the interval, the even connection is pure gauge up to its boundary effect, so the finite orbit is all it can reach. The physical Higgs modulus must then be sought at the wall, in odd directions, or in more dimensions. At the 10 -> 9 endpoint this agrees with T10, where the Higgs is frozen.

The gauge sector and the 10 -> 9 family

Proposition 22.8. Consider the product of M^4 with this parent, with algebra of functions f(y, x) valued in A_F and f(y, l) in C_F. Then the gauge one-forms sum f_1 d_mu f_2 take values in C_F at x = l. Assume, in addition, a flat product, a positive five-dimensional Yang-Mills kinetic form, the trivial background, unrestricted endpoint variation in the allowed directions, and no additional boundary kinetic or mass terms. Then the directions in the Lie algebra of C_F have masses n pi/l, n >= 0, and the orthogonal directions have masses (n - 1/2) pi/l, n >= 1. The physical gauge algebra comes from the skew-adjoint elements through the real gauge representation, with its kernel and any unimodularity condition.

Proof. The first statement is direct evaluation at x = l. The rest is the interval spectrum in the gauge A_5 = 0, with Neumann conditions at 0, and with Neumann or Dirichlet conditions at l. ■

In the Standard Model data with nonzero Yukawas, M_W = M_Z = pi/(2l). The Lie algebra of C_F has dimension 10 before unimodularity; with the unimodularity condition of Chamseddine-Connes-Marcolli (arXiv:hep-th/0610241, section 2.5), the massless modes are the photon and the gluons. These are the boundary conditions of Higgsless models (Csaki, Grojean, Murayama, Pilo and Terning, Phys. Rev. D 69 (2004) 055006), and the fermion formula tan(m L) = M_D L is equation (5.6) of Csaki, Grojean, Hubisz, Shirman and Terning (Phys. Rev. D 70 (2004) 015012); paper/LITERATURE.md, section C. Boundary spectral geometry for the Standard Model has been studied by Chamseddine and Connes (arXiv:1008.3980, section 5). What is added here, as far as our searches went, is the combination: the real structure, the finite triple as an exact window, the wall algebra C_F, the compression formula and the fluctuations. A wider claim of novelty would need a fuller comparison.

Corollary 22.9 (the 10 -> 9 family). Fix a block M_0 for the light fermions and a block Theta/l on an orthogonal channel, compatible with J_F and the grading, with 0 < Theta < pi/2, for the top and if wanted for the Majorana block. Let D_F(l) have M(l) = M_0 (+) Theta/l, and let l -> 0. Then:

1. X(l) has the light block tan(l |M_0|) V_0* = l M_0* + O(l^3) and a heavy block with singular values tan(Theta): the light couplings vanish as m l at fixed mass m.
2. The heavy channel has mass Theta/l, the broken gauge directions have mass pi/(2l), and the other levels diverge.
3. The massless gauge sector is C_F. Let P_L be the projection reducing D_F(l) to the fixed light block D_L. For constant a, P_L Q_l(a) P_L = sinc(l ad_{D_L})(P_L pi(a) P_L), with ||P_L Q_l(a) P_L - P_L pi(a) P_L|| <= (2/3) l^2 ||D_L||^2 ||pi(a)||, also when pi(a) mixes light and heavy channels. The limiting light compression need not be a representation, since products can pass through the heavy subspace. The full estimate of Theorem 22.4 does not apply, because l ||D_F(l)|| tends to Theta. For example, with D_F(l) = sigma_1 (x) diag(m, Theta/l) and pi(a) = 1 (x) sigma_1, one has Q_l(a) = sinc(Theta - l m) pi(a), which tends to sinc(Theta) pi(a).
4. With l = pi/(2 M_W), we get X_q ~ (pi/2) m_q/M_W, which has the form of y_q = sqrt(2) m_q/v -> 0 at fixed mass, as the coupling vanishes while the mass stays. Identifying X_q with the four-dimensional Yukawa coupling is a hypothesis.
5. The ratio is m_t/M_W = 2 Theta/pi < 1, and M_W/M_Z = 1.

So the 10 -> 9 proposal is realized at this level as a limit l -> 0, with W, Z and top at the tower scale and the light fermions keeping explicit masses. Item 5 does not reach m_t/M_W = sqrt(8/3), which belongs to the interior. This is a statement in the stated family: lowest level, no bulk mass, strict wall algebra. Bulk masses for the top, and a changed gauge wall condition, are directions beyond it.

A numerical comparison for the interior (Python). A separate model, with a finite wall scalar and Neumann conditions at 0, lies outside Proposition 22.8 and No-go theorem 22.7. In it, the mass of a broken direction satisfies t tan t = s, with t = M l and s = g^2 v^2 l^2/4 (for the Z, s/cos^2 theta_0), which has the form of M_k tan(M_k pi R) = -V in

Csaki et al. (arXiv:hep-ph/0305237, section on mixed boundary conditions, read in the LaTeX source), with the sign convention of V there. If g'^2/g^2 = 3/5 is taken at all scales, M_W^2/M_Z^2 goes from 5/8 as s -> 0 to 1 as s -> infinity. It equals the measured 0.777 at M_W l = 1.03, with 1/l = 78 GeV and a next W level at 280 GeV; whether collider searches allow this would need the couplings, branching fractions and an applicable limit, which we have not analyzed. The top needs M_W l < 0.96 for m_t/M_W = sqrt(8/3), or M_W l < 0.73 for the measured ratio, in the lowest level with no bulk mass. So in the flat tree-level toy without bulk masses or running, the top and gauge requirements are incompatible. It does show the interior interpolating between the spectral-action value 5/8 and the endpoint value 1. The directions are bulk masses, warping and running.

What is input and what is derived

The finite data (A_F, H_F, D_F, J_F, Gamma_F) are all inputs, so the existence of the window is close to a reparametrization: X = tan(l D_F). The derived statements are the wall algebra C_F for that input, the order conditions of the parent, the compression formula with its rate, the form of the fluctuations, and the gauge spectrum under the stated assumption. The window identifies D_F, J_F and Gamma_F exactly, while the represented algebra is obtained through compression and is not an exact representation of A_F at l > 0. In particular, this does not select A_F. T0 asks for a commutative parent: a natural next step is a two-stage parent, compressing functions on S^4 and CP^3 at level one to get A_F-valued functions on the interval, and then taking this window, where the difficulty is the same interface problem as after Result 7 of lemma 20.

Numerical check

By Claude Opus 5.5, in Python (numpy and scipy), 3 Nov 2026, in a temporary directory, not kept.

- Random complex symmetric M on C^5, l ||M|| = 1.3: D u_+(a) = u_-(M a) and D u_-(b) = u_+(M* b) by finite differences, to about 10^-9; boundary conditions and J^ to about 10^-15; norm

of W to 10^-13.
- Q_l(a) = sinc(l ad)(a), by quadrature, to 2 x 10^-8. The formula for Q_l(E T), with a nonconstant T, to 10^-14.
- One-generation Standard Model data with Yukawas and the rank-one Majorana block, with signs, order zero and order one exact on B = C + H + M_3(C), and commutant of D in B of real dimension 20. At l = 0.1 and 0.03, ||Q_l(a) - a + (l^2/6)[D, [D, a]]|| is 10^-5 and 4 x 10^-8, and the order-one defect of Q_l is 1.5 x 10^-4 and 1.2 x 10^-5, which scales as l^2.
- The dimensions of the generated algebras quoted above, 15 and 30, from a numerical closure with tolerance 10^-7, not retained and not certified; at smaller l the numerical rank depends on the tolerance.
- The roots of t tan t = s, and the values quoted in the comparison for the interior.
- On the two-point data with J = sigma_1 K: D + A_(1) + A~*(1)* + *A*(2) = U D U* to 4 x 10^-16, and the link of U_w* D U_w equal to m e^{-2 i psi}.
- The bounds of Theorem 22.4, items 2 and 4, on 30 random samples of the one-generation data: largest ratios to the bound 0.33, 10^-4 and 10^-3.

Part 23

Two gradings on an interval: masses as principal angles, the Higgs modulus, and exact W and Z masses

For STRATEGY T0b, T5b, T10 and T13, following lemma 22. Proved by Claude Opus 5.5, with the identities checked in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): four errors and one gap as first written, and amendments, all incorporated below. The simpler form of the mass formulas, and the proof of item 3 of Theorem 23.6, are by the referee.

Outcome. A self-adjoint boundary condition for the derivative on an interval is a grading at each wall. The whole finite Dirac operator is then the relative position of

two gradings: its singular values are the principal angles between them, divided by l, and

```
D_F = (1/2l) Gamma_0 log(Gamma_1 Gamma_0).
```

With a bulk prealgebra, this gives four results.

1. Aligned walls compress a prealgebra stable under the grading to its even part, with the grading inherited, which is the step that Example 16.9 had to add by hand.
2. Odd scalar fluctuations move the second grading, so they change the singular values and supply the radial mode that No-go theorem 22.7 excludes for even fluctuations. For M_2(H) the field space is S^4, and the Higgs modulus is half the geodesic distance between the points chosen by the two walls.
3. With a wall connector on one channel and a five-dimensional Yang-Mills action with common normalization, the masses are exact: sin^2(M_W l) = sin^2(2 theta)/2 and sin^2(M_Z l) = (4/5) sin^2(2 theta), for universal fermions of mass theta/l. At small theta this gives M_W^2 = 2 m^2 and M_Z^2 = (16/5) m^2, the constants in the lower bounds of the fully massive family of lemma 18, with M_W^2/M_Z^2 tending to 5/8 at both ends.
4. The heat-cutoff spectral action on the interval is a Jacobi theta function of theta, decreasing on [0, pi/2]. This family lies outside the hypotheses of the radial no-go of lemma 18: the radial direction is compact, with a stationary point at m = pi/(2l), the Connes-Lott value, where M_W = M_Z = 0.

Everything finite is still an input, and the parent algebra is matrix-valued. For a commutative parent, lemma 24 finds the window exactly, with spectral walls, while direct compression of scalar functions gives only R + R on the S^4 factor. The formulas for gauge masses depend on the assumed action. The relation to measured values fails in the flat toy, as stated at the end.

Two gradings

Let H be a finite-dimensional Hilbert space, and let Gamma_0, Gamma_1 be self-adjoint involutions on H. On L^2([0, l]; C^2) (x) H, with psi = (psi_L, psi_R), let D = E d/dx, where (E psi)_L = psi_R and (E psi)_R

= -psi_L, on the domain

```
Gamma_0 psi_L(0) = psi_L(0),   Gamma_0 psi_R(0) =
 ↪  -psi_R(0),   Gamma_1 psi_L(l) = psi_L(l),
 ↪  Gamma_1 psi_R(l) = -psi_R(l).
```

Call it D(Gamma_0, Gamma_1), and let Gamma^ act as +1 on psi_L and -1 on psi_R.

Theorem 23.1.

1. D(Gamma_0, Gamma_1) is self-adjoint with compact resolvent, and it anticommutes with Gamma^.
2. Suppose Gamma_0 and Gamma_1 have eigenspaces of equal dimensions, and that -1 is not an eigenvalue of Gamma_1 Gamma_0. Let theta_k in [0, pi/2) be the principal angles between the +1 eigenspaces. Then K = (1/2l) log(Gamma_1 Gamma_0), with the principal branch, is anti-Hermitian, Gamma_0 K Gamma_0 = -K, and D_F = Gamma_0 K is self-adjoint, odd for Gamma_0, with singular values theta_k/l on the +1 eigenspace. Moreover Gamma_1 = e^{lK} Gamma_0 e^{-lK}.
3. With H_+, H_- the eigenspaces of Gamma_0, and X the H_- -> H_+ block of tan(l D_F), as in lemma 22, D(Gamma_0, Gamma_1) = D_X, and the spectral part of Theorem 22.1 gives the spectrum and the window, with no real structure needed. Conversely, every D_F odd for Gamma_0 with l ||D_F|| < pi/2 arises from exactly one such Gamma_1.
4. Let an antiunitary J_1 satisfy J_1^2 = -1, J_1 Gamma_i J_1^-1 = -Gamma_i for i = 0, 1. Then (J^ psi)_L = J_1 psi_R, (J^ psi)_R = -J_1 psi_L defines a real structure of signs (+, +, -) for D(Gamma_0, Gamma_1). With J_F = Gamma_0 J_1 up to sign, J_F^2 = 1, J_F Gamma_0 = -Gamma_0 J_F and J_F D_F = D_F J_F, which are the signs of lemma 22.

So J_F itself does not anticommute with Gamma_1 in general: J_F Gamma_1 J_F^-1 = -Gamma_0 Gamma_1 Gamma_0, the reflection of Gamma_1 through Gamma_0.

Proof. Item 1: the boundary form is [<psi_R, phi_L> - <psi_L, phi_R>] evaluated between 0 and l, which

vanishes on the domain because L-values and R-values lie in orthogonal eigenspaces at each wall, and these are half of the boundary data.

Item 2: U = Gamma_1 Gamma_0 is unitary and Gamma_0 U Gamma_0 = Gamma_0 Gamma_1 = U^-1, so K is anti-Hermitian and anticommutes with Gamma_0. The eigenvalues of U are e^{+-2 i theta_k}, by the standard two-projection decomposition, and K then has eigenvalues +-i theta_k/l. Since e^{lK} Gamma_0 e^{-lK} = e^{2lK} Gamma_0 = U Gamma_0 = Gamma_1, the last formula follows.

Item 3: in the notation of Theorem 22.1, R(l) = e^{lK} carries H_+ to the graph {(C a, -S a)} = graph(-X*), and H_- to graph(X). These are the +1 and -1 eigenspaces of Gamma_1, which is the condition at the wall. The converse uses the same formula, with uniqueness from injectivity of the exponential on the stated range.

Item 4: J^ commutes with E by direct computation, and J^^2 = -J_1^2 = 1. The domain is preserved since J_1 exchanges the eigenspaces of each Gamma_i. For the last statement, J_1 U J_1^-1 = U, so as J_1 is antilinear, J_1 K J_1^-1 = K in the sense of the principal branch, and J_F = Gamma_0 J_1 then satisfies J_F K J_F^-1 = -K and commutes with D_F = Gamma_0 K. ∎

When theta_k = pi/2 the channel has type (c) of lemma 09, with no zero mode and lowest level pi/(2l), the Connes-Lott mass of Theorem A in the paper. There the principal logarithm and the isolated window cease to be unique.

The bulk prealgebra

Let P be a *-subalgebra of End(H), which need not commute with Gamma_0. Let

```
A(Gamma_0, Gamma_1) = {f in C^infinity([0, 1], P) :
 ↪  [f(0), Gamma_0] = 0, [f(1), Gamma_1] = 0},
```

acting pointwise and identically on both spin components.

Theorem 23.2.

1. A function f in C^infinity([0, 1], P) preserves the domain exactly when it lies in A(Gamma_0,

Gamma_1). The commutators are [D, f] = E f', and Gamma^ commutes with every f, including those with values odd for Gamma_0.

2. Under the hypotheses of Theorem 23.1, item 2, with W the isometry of Theorem 22.1, Q(T) = l^-1 int_0^l E_0(R(x)* T(x) R(x)) dx, where E_0(Z) = (Z + Gamma_0 Z Gamma_0)/2 is the conditional expectation onto the even part.
3. *Aligned walls.* If Gamma_1 = Gamma_0, then D_F = 0, the window consists of constant L-values in H_+ and constant R-values in H_-, and Gamma^ restricts on it to Gamma_0. For an ambient constant multiplier a in P, Q(a) = E_0(a), and the image of the boundary algebra A(Gamma_0, Gamma_0) is E_0(P), by compactly supported profiles. If Gamma_0 P Gamma_0 = P, then E_0(P) is the even part of P, so the compressions of the prealgebra are exactly its even part, with the grading inherited from the even parent operator Gamma^. Without that hypothesis this can fail: for H = C^2, Gamma_0 = sigma_z and P generated by sigma_x + sigma_z, E_0(P) = span{1, sigma_z}, while the even part of P is C.
4. *Order conditions.* Let J_1 be as in Theorem 23.1 and let P° = {J_1 b* J_1^-1 : b in P} commute with P. Then the parent has order zero and order one for A(Gamma_0, Gamma_1).

Proof. Item 1: as in Theorem 22.3, using that the conditions at a wall say that L- and R-values lie in the eigenspaces of Gamma_i. Item 2: as in the proof of Theorem 22.4. Item 3 is the case R = 1. Item 4: [D, f] = E f', and pointwise commutation gives both conditions. ■

Item 3 evades the inherited-grading obstruction of Result 11 of lemma 20, because hypothesis 4 there fails: for odd a, here Q(a) = 0, while there the coordinates are retained as Q(x_a) = Gamma_a/5. So at the level of a matrix-valued parent this addresses the question left in step 3 of the framework note: the prealgebra is the bulk algebra, the even algebra is what the window sees, and the grading is a boundary condition. The commutative-parent and retained-coordinate questions remain open, and Theorem 4 of lemma 24 shows a real difference: scalar functions compressed directly give R + R on the S^4 factor,

and H + H needs the matrix algebra generated at the intermediate stage.

Remark 23.3 (the twist). On H = E (+) E*, with J(x, y) = (y*, x*), the operator J_1 = Gamma_0 J is the one that anticommutes with both gradings. For an odd anti-Hermitian h in P, J_1 h J_1^-1 is y -> +y h on E*, where J alone gives y -> -y h. So the opposite representation b° = J_1 b* J_1^-1 incorporates on the W-factor the grading automorphism sigma(b) = gamma_W b gamma_W. In this parent the twist comes from the requirement that J_1 preserve both boundary conditions. Twisted spectral triples for the grand symmetry (Devastato, Lizzi and Martinetti, JHEP 01 (2014) 042, arXiv:1304.0415; Devastato and Martinetti, Math. Phys. Anal. Geom. 20 (2017) 2, arXiv:1411.1320, Eqs. (1.8)-(1.11)) use a different automorphism: their rho exchanges left and right spinorial indices and algebra summands, while conjugation by gamma_W is inner on M_2(H) and fixes its centre. So this is a structural analogy between involutive automorphisms, and an identification needs a comparison of the representations, opposites and first-order conditions.

Odd fluctuations and the radial mode

Theorem 23.4. Let h be smooth with values in the anti-Hermitian elements of P, and let u' = -h u, u(0) = 1, w = u(l). Then

```
D(Gamma_0, Gamma_1) + E h = u D(Gamma_0, w\* Gamma_1
↪  w) u\*.
```

So scalar fluctuations move the second grading along its orbit under the unitary group of P. If h has an odd part, the principal angles between Gamma_0 and w* Gamma_1 w change in general.

Proof. u* D u = D + E u* u', and u* E h u = E u* h u, so the connection cancels. Since u(0) = 1, the wall at 0 is unchanged. At l, psi = u phi gives the stated grading for phi. ■

This lies outside No-go theorem 22.7, whose hypotheses include the even algebra: an even connection gives a w commuting with Gamma_0, and the angles are kept, while an odd connection changes them. For a real fluctuation, add the conjugate one-form: D + E h + J^(E

h)J^^-1, with generator of the form (L_F, R_F) in Example 23.5. The invariant is a double coset. With H_i = Stab(Gamma_i) in the connected unitary group G of P, a gauge transformation removing h has the form u(x) c, and it preserves both endpoints exactly when c is in H_0 and w c in H_1, that is, when w is in H_1 H_0. So the scalar moduli of the parent are the double cosets H_1 G / H_0, the relative position of the two gradings. This is with the moving window, as in lemma 22.

Example 23.5 (M_2(H) and S^4). Let W = H^2 = C^4, with the Clifford generators Gamma_a of lemma 16, and for a unit vector p in R^5 put Gamma_p = p_a Gamma_a. Each grading of W with one-dimensional quaternionic +1 and -1 eigenspaces is such a Gamma_p, so these gradings form S^4 = Sp(2)/(Sp(1) x Sp(1)), the quaternionic projective line. For two points, Gamma_q Gamma_p = p . q + (p wedge q), with eigenvalues e^{+- i d}, where d is the geodesic distance; so all the principal angles equal theta = d/2.

Take H = Hom(V, W) (+) Hom(W, V), J(x, y) = (y*, x*), Gamma_0(x, y) = (gamma x, -y gamma) with gamma = Gamma_{p_0}, and K(x, y) = (theta/l)(F x, y F), with F an odd anti-Hermitian element of M_2(H), F^2 = -1. Then:

- D_F = Gamma_0 K has all singular values equal to theta/l: a universal Dirac mass, for all colours.
- The second grading is Gamma_1(x, y) = (Gamma_p x, -y Gamma_{p'}), where p is at distance 2 theta from p_0 and p' is its mirror image through p_0. J_1 = Gamma_0 J satisfies Theorem 23.1, item 4, while J alone does not.
- The coordinates (theta, q) on S^4, with F determined by the unit quaternion q, are the polar coordinates of a Higgs quaternion: theta is the radial coordinate, bounded by pi/2, and q gives three angular directions, undefined at the poles. At theta = pi/2 the finite logarithm and the lowest window are degenerate. Quotienting by the stabilizer leaves the radial interval. A canonically normalized field needs the scalar kinetic term, which we have not computed.

With bulk masses, by Theorem 12.6, the light masses are about (tan(theta)/l) Phi_Q Phi_U when small com-

pared with the tower thresholds, with a universal matrix before flavour data; general finite-angle eigenvalues need the secular equation. This field space is the coset of the minimal composite Higgs model (Agashe, Contino and Pomarol, Nucl. Phys. B 719 (2005) 165, sections 2-3, with their h/f_pi corresponding to 2 theta). Their construction uses a warped interval and additional fermion and boundary data, so the shared coset does not identify the actions or potentials. The appearance here of the S^4 of the twistor window is an observation, and no identification between the two roles of S^4 is made.

The weak bosons

Take P = M_2(H) (+) M_4(C), acting on E = Hom(V, W) on the left and on the right. Assume the following.

- A five-dimensional Yang-Mills field valued in sp(2) (+) su(4), with the trace form Tr_W + Tr_V, so with one coupling (the normalization of the spectral action, g_3 = g_2 = sqrt(5/3) g_Y).
- At x = 0: Neumann for the stabilizer of p_0, Dirichlet for its complement. At x = l: Neumann for the stabilizer of p_l and of a connector, that is, the a with [a_W, Gamma_p] = 0, a_W w = lambda w and a_V v = lambda v, as in Results 2 and 5 of lemma 20, for a line w in the +1 eigenspace of Gamma_p and a line v in V; Dirichlet for the complement.
- The connector is prescribed gauge-breaking wall data here. If it is instead a term in the fermion boundary operator, as for a unit vector e in the +1 eigenspace of Gamma_1, f = J_1 e, K_c = |f><e| - |e><f| and Gamma_{1,c} = e^{eta K_c} Gamma_1 e^{-eta K_c}, then J_1 still anticommutes with Gamma_{1,c}, the parent order conditions are kept, and the affected fermion tower changes, so the universal fermion formula must be recalculated for that channel.

Theorem 23.6. With theta the angle above, the modes that are massless for all theta are exactly nine, the photon and the eight gluons, with three additional electroweak zero modes at theta = 0 and theta = pi/2. The lowest charged and neutral massive modes satisfy

```
sin^2(M_W l) = sin^2(2 theta)/2,
```

```
sin^2(M_Z l) = (4/5) sin^2(2 theta),
```

equivalently tan^2(M_W l) = sin^2(2 theta)/(1 + cos^2(2 theta)) and tan^2(M_Z l) = 4 sin^2(2 theta)/(1 + 4 cos^2(2 theta)).

The photon has 1/e^2 = (8/3)/g^2. Moreover:

1. As theta -> 0, with m = theta/l: M_W^2 -> 2 m^2 and M_Z^2 -> (16/5) m^2, so M_W^2/M_Z^2 -> 5/8.
2. As theta -> pi/2: M_W, M_Z -> 0, with M_W^2/M_Z^2 -> 5/8, while m -> pi/(2l).
3. At theta = pi/4: M_W l = pi/4, tan(M_Z l) = 2, and M_W^2/M_Z^2 = ((pi/4)/arctan 2)^2 = 0.5032..., the global minimum. So M_W^2/M_Z^2 stays in [0.5032, 0.625), with 5/8 as a limit at the ends, where the ratio itself is undefined.
4. m/M_W goes from 1/sqrt(2) at theta -> 0 to infinity at theta -> pi/2. It equals sqrt(8/3) at theta = 1.0589.

Proof. In the gauge A_5 = 0 the field is free, with cos(k x) on the Neumann directions at 0 and sin(k x) on the Dirichlet ones, and the condition at l holds in the frame rotated by Ad exp(theta F). The generators split according to the frequency of Ad F. Write L, R for su(2) on W_- and W_+, S = (L + R)/sqrt(2), A = (L - R)/sqrt(2), and O for the odd generators; Ad F has frequency 0 on S, and frequency 2 on the plane (A, O). So S = s cos kx, A = alpha cos kx, O = beta sin kx.

At l, in the rotated frame, A' = cos(2 theta) A + sin(2 theta) O and O' = -sin(2 theta) A + cos(2 theta) O, with O' = 0 at l, which gives beta = alpha tan(2 theta) cot(k l), and so A'(l) = alpha cos(k l)/cos(2 theta).

Charged directions. The connector forces R^+- to vanish at l, with L^+- Neumann. So S(l) = A'(l), and d/dx (S + A') = 0 at l. Substituting gives sin^2(kl) (1 + cos^2(2 theta)) = sin^2(2 theta) cos^2(kl).

Neutral directions. Add B = b cos kx for the generator T_15 = diag(1, 1, 1, -3)/(2 sqrt 6) of su(4). With Y = sqrt(3/5) R + sqrt(2/5) B and Z' = -sqrt(2/5) R + sqrt(3/5) B, the connector forces Z' to vanish at l, with L and Y Neumann. The Neumann conditions give b = -sqrt(3) s, and Z' = 0 gives s = alpha/(4 cos(2 theta)).

Substituting in the condition for L gives sin^2(kl) (1/4 + cos^2(2 theta)) = sin^2(2 theta) cos^2(kl).

Massless modes and limits. At k = 0, a zero mode is a constant in the intersection of the two stabilizers. This consists of a_W = diag(q, q) on W_+ and W_-, identified by F, with q preserving the line of w, together with a_V = lambda (+) m, and with the trace condition this has dimension 1 + 8. In it, L = R and B/L = sqrt(2/3), which gives 1/e^2 = (1 + 1 + 2/3)/g^2. The limits follow by expanding. A second derivation, from the referee, uses the principal angles between the Neumann subspaces at the two walls: for each charged direction the squared sine of the angle is sin^2(2 theta)/2, and in the neutral sector the single nonzero value is (4/5) sin^2(2 theta); it also covers cos(2 theta) = 0, where the divisions above need a limit. For item 3, with a = sin(2 theta), the ratio is [arcsin(a/sqrt 2)/arcsin(2a/sqrt 5)]^2, and this decreases with a on (0, 1], since x/(sqrt(1 - x^2) arcsin x) increases. ■

The constants in item 1 are the values ||[D_F, a]||^2/||a||^2 in the Hilbert-Schmidt norm, for universal mass m, for a = L^+- and for the direction orthogonal to the photon and Z', which has A-component of squared length 4/5. These are the coefficients of the lower bounds for the poles of the fully massive family in Proposition 14 of lemma 18 (2 x < z_W, (16/5) x < z_Z); the small-x pole coefficients there are 3 and 24/5. The limit 5/8 is common to both, with different behaviour: the finite-action poles grow in the heavy limit of lemma 18, while the interval masses here tend to zero at the antipodal point. The spectral-action Taylor mass for universal m is M_W^2 = 3 m^2, which exceeds the value here by the factor 3/2 of Proposition F in the paper: there the mass matrix carries the relative heat coefficients of the action, and here we have assumed a five-dimensional Yang-Mills action, which gives the commutator norm alone. A direct matching claim needs the full parent spectral action, with its tower and boundary conditions.

Item 2 gives a geometric form of the behaviour that lemma 18 found: heavy fermions with light vectors. At antipodal points Gamma_p = -Gamma_{p_0}, so both walls have the same stabilizer in sp(2) again, and the fermion channels have type (c) with mass pi/(2l).

The spectral action on the interval

Proposition 23.7. For the heat cutoff and universal theta, with s = l Lambda, the trace over a channel is

```
Tr_channel exp(-D^2/Lambda^2) = 2 sum_{n in Z}
 ↪  exp(-(n pi + theta)^2/s^2).
```

This is proportional to the Jacobi function theta_3(theta, exp(-s^2)). It is strictly decreasing on [0, pi/2], with a maximum at theta = 0 and a minimum at theta = pi/2, where it is stationary, and it is symmetric under theta -> pi - theta. In the towerless regime s -> 0 at fixed x = theta^2/s^2, it tends to 2 e^{-x}, the form of the vacuum density in lemma 18 and Proposition 21.2.

Proof. The eigenvalues in a channel are +-(n pi + theta)/l, by Theorem 22.1, and the sum is over Z after combining the signs. By Poisson summation the sum equals (s/sqrt(pi)) sum_{k in Z} exp(-k^2 s^2) e^{2 i k theta}, and the Jacobi triple product writes this as a constant times the product over n >= 1 of (1 + 2 q^{2n-1} cos(2 theta) + q^{4n-2}), with q = exp(-s^2). Each factor is positive and strictly decreasing in theta on [0, pi/2]. ■

In the form of the Jacobi function, the trace is (2 s/sqrt(pi)) theta_3(theta, exp(-s^2)). This family lies outside the hypotheses of No-go theorem 9 of lemma 18, which concerns finite internal data and a free ray V = c V_*: here the operator family is different, the radial direction is a compact orbit of relative gradings, and the tower, which the finite description omits, supplies a second branch with a stationary point at theta = pi/2. In the present family the literal action has its minimum at antipodal gradings, m = pi/(2l), with M_W = M_Z = 0 by Theorem 23.6, where the lowest eigenspace is degenerate. This is a vacuum with massless vectors, so it is not the physical point. The sign of the term agrees with the fermion loop in the Hosotani mechanism, where gauge-boson loops enter with the opposite sign. So an interior minimum could arise from that competition, which the classical spectral action does not contain. We have not computed it.

The one-loop potential in the same family

From the exact spectra, the products over the towers are, relative to theta = 0 and in Euclidean momentum x = p l,

```
fermion channel:  1 + sin^2(theta)/sinh^2(x),
charged vectors:  1 + sin^2(2 theta)/(2 sinh^2(x)),
neutral vector:   1 + (4/5) sin^2(2
 ↪  theta)/sinh^2(x).
```

So in the gauge-boson and fermion-loop form of the Hosotani mechanism, assuming the usual one-loop formula with three polarizations per vector, and with n_f = 2 x (number of physical Dirac species),

```
8 pi^2 l^4 V(theta) = int_0^infinity x^3 dx [ 3
 ↪  log(1 + s^2/(2 sinh^2 x)) + (3/2) log(1 + (4/5)
 ↪  s^2/sinh^2 x) - n_f log(1 + sin^2(theta)/sinh^2
 ↪  x) ],
```

with s = sin(2 theta). A Dirac top with three colours has n_f = 6, which needs a separately specified fermion sector, since the universal connection gives masses to all: one generation has n_f = 16 and three have n_f = 48. With I = int x^3/sinh^2 x dx = (3/2) zeta(3) and I' = int x^3/cosh^2 x dx = (9/8) zeta(3), near theta = 0 the bracket is (10.8 - n_f) I theta^2 + o(theta^2), and with epsilon = pi/2 - theta, near theta = pi/2 it is V(pi/2) + (10.8 I + n_f I') epsilon^2 + o(epsilon^2), where we have written V for the left side.

Proposition 23.8 (referee). In the massless flat family with this content, for every n_f > 0, theta = pi/2 is the unique global minimum on [0, pi/2], since the gauge logarithms are nonnegative and vanish there, and the negative fermion logarithm is minimized at sin^2(theta) = 1. For n_f >= 10.8, V is strictly decreasing on the interior. Indeed, with t = sin^2(theta) and J(z) = int_0^infinity x^3 dx/(sinh^2 x + z), the derivative with respect to t is, up to a positive factor,

```
(1 - 2t) [6 J(2t(1 - t)) + (24/5) J((16/5) t (1 -
 ↪  t))] - n_f J(t).
```

For t <= 1/2, both arguments in the bracket are at least t, and J decreases; for t > 1/2 both contributions have the required sign.

For 0 < n_f < 10.8, on grids of theta (Python), for n_f

in {1, 2, 3, 4, 6, 8, 10}, V has one interior maximum, at about 0.33 for n_f = 8, and no interior minimum. Theorem 9 of lemma 24, by Astra 6, proves this for all n_f in that range: exactly one interior stationary point, a strict maximum, using a log-concavity property of the integral.

These are negative results in an exact scope: one interval, no bulk masses, fermions of frequency 1 under F, this gauge content, one loop, this condition at the connector wall. As a formal leading approximation, not uniform at x = 0, V has the form B sin^2(2 theta) - A sin^2(theta) with A, B > 0, whose interior stationary point cos(2 theta) = A/(4B) is a maximum. *Bulk masses.* With bulk masses, the fermion product becomes cos^2(theta) + sin^2(theta) G(x), with G = (w_Q coth w_Q - c_Q)(w_U coth w_U - c_U)/x^2 and w = (c^2 + x^2)^{1/2} in the secular form of lemma 12. For a = c_Q + c_U != 0, G = 1 - a/x + O(x^-2), so the potential has angle-dependent ultraviolet divergences, and a finite value depends on a regulator and subtraction: as the referee showed, an earlier scan of general pairs here did not establish a potential, and it is withdrawn. The exceptional case c_U = -c_Q = c has a common bulk mass on the two mixed components and G = 1 + w^2/(x^2 sinh^2 w), which decays exponentially. Keeping the flat gauge products as a separate tree-level assumption, a scan for c in {0, 1/2, 1, 3/2, 2, 3, 5}, n_f in {1, 3, 6, 8, 10, 16, 48}, on a grid of 721 values of theta, finds no interior minimum (Python). General bulk masses need a gauge-invariant completion, a regulator and wall counterterms.

So an interior minimum needs another change: a negative coefficient of sin^2(2 theta), as from fermions of frequency 2 under F, which the bimodule E = Hom(V, W) does not contain; general bulk masses or other gauge form factors, as with warping; a wall scalar with its own potential; or tree-level terms at the walls. With h/f_pi = 2 theta, the leading flat form has the angular form alpha cos(h/f_pi) - beta sin^2(h/f_pi) of Agashe, Contino and Pomarol (section 2.1, Eqs. (19)-(23)) with spinorial SO(5) fermions, with alpha = A/2 and beta = -B, up to a constant. Their form factors can give the sign and competition needed for an interior minimum, so the flat results here do not exclude that construction. This points away from a pure flat Wilson line, toward those changed

form factors or a wall scalar with its own potential, for the physical interior.

Comparison with measured values in the flat toy

These are numerical comparisons (Python), with no running and no bulk masses, in lowest levels, using the Particle Data Group 2026 values quoted by the referee, m_t = 172.60 GeV as a direct-measurement mass, M_W = 80.3625 GeV and M_Z = 91.1879 GeV, so m_t/M_W = 2.1478 and M_W^2/M_Z^2 = 0.7767. The ratio m_t/M_W reaches sqrt(8/3) at theta = 1.0589 and the measured value at theta = 1.1626, where M_W l = 0.541, so 1/l is about 148.5 GeV. There, M_W^2/M_Z^2 = 0.581. The wall-scalar model of lemma 22 covers [0.625, 1], and this family covers [0.5032, 0.625); in the wall-scalar model, fitting the measured vector ratio gives M_W l = 1.028, where the lowest no-bulk-mass fermion channel has m/M_W < 1.528. So neither flat family fits both ratios on its own, and the flat one-loop potential above does not select an interior point. Whether a tower at 1/l of this size is allowed would need the couplings and an applicable experimental limit, which we have not analyzed. The directions are bulk masses for the light fermions, warping, running of the couplings, and families combining a Wilson line with a wall scalar. These negative statements hold in the flat tree-level toy only.

Open questions

1. The interior vacuum: the effective potential with a wall scalar, or with warping, or with general bulk masses, with a gauge-invariant completion, a regulator and wall counterterms. The flat cases above have none on the grids.
2. The relation between the two roles of S^4: the base of the twistor window, and the field space of the Higgs quaternion.
3. A commutative parent: functions on S^4 x I and CP^3 x I, with the gradings as local boundary conditions.
4. The factor 3/2 between the five-dimensional normalization and the spectral action: whether the spectral action on the parent itself, with its tower, reproduces the formula of Theorem 23.6.

5. The comparison with twisted spectral triples, Remark 23.3.
6. The contribution of a real connector to the fermion tower, the degenerate antipodal window, and the scalar kinetic normalization needed for a canonically normalized Higgs field.

Numerical check

By Claude Opus 5.5, in Python (numpy and scipy), 3 Nov 2026, in a temporary directory, not kept.

- Random complex symmetric M on C^4 with l ||M|| = 1.25: Gamma_1 is an involution, Gamma_1 = exp(2 l K) Gamma_0; D_F = (1/2l) Gamma_0 log(Gamma_1 Gamma_0) to 3 x 10^-15; the principal angles over l equal the singular values of M to five digits.
- Sp(2) x SU(4), of dimension 25, with the Neumann and Dirichlet subspaces built from the stated stabilizers: the two-point boundary problem was solved as a linear system on a grid in k l. For theta = 0.2, pi/4, 1.06 and 1.3, the two lowest nonzero roots are (0.279, 0.356), (0.785, 1.107), (0.647, 0.868), (0.373, 0.479), equal to the formulas of Theorem 23.6 to three digits, with other roots at pi/2 and above. The intersection of the two stabilizers has dimension 9 at each theta tested.
- Small theta: M_W^2/m^2 = 2.000 and M_Z^2/m^2 = 3.200.
- The universal connection on E (+) E*: J D = D J, Gamma_0 D = -D Gamma_0, all |eigenvalues| equal to theta/l, Gamma_1 = (left Gamma_p, right mirror), J_1^2 = -1, J_1 Gamma_1 = -Gamma_1 J_1, while J alone fails; J_1 h J_1^-1 is right multiplication by +h on E*, for odd h.
- The theta function: monotone on a grid for s = 0.3, 1, 2, with the towerless value at s = 0.3 equal to e^{-x} to five digits; Poisson summation and the triple product to 10^-16.
- The ratio M_W^2/M_Z^2 on a grid: range [0.5032, 0.6250], symmetric under theta -> pi/2 - theta to 10^-15, minimum at pi/4. At theta = 1.1625, M_W l = 0.541, and at theta = 1.0589, M_W l = 0.648.
- The one-loop integrals by adaptive quadrature

(scipy), and by Gauss-Legendre quadrature on 400 nodes for the scan with c_U = -c_Q. The referee recomputed the derivative with scipy, found one sign change for each n_f in {1, 2, 3, 4, 6, 8}, with the stationary angle about 0.3382 at n_f = 8, and checked I = (3/2) zeta(3).

Part 24

Commutative cylinders, spectral walls, and the two-stage compression

By Astra 6 (gpt-6-astra, effort xhigh), 3 Nov 2026; prompt in prompts/24_commutative_parent.md. Reproduced as delivered, with links made relative. The section "Numerical check" at the end is by Claude Opus 5.5. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): no error in the displayed equations, and one gap in the general statement on compression maps. The record above the section "Referee amendments" stays as delivered, and those amendments apply to it.

A commutative cylinder admits an exact realization of the **operator window** of lemmas 22 and 23 when the two finite gradings are prescribed through nonlocal spectral boundary conditions. The higher K-modes then decouple as the radius of K tends to zero. Two qualifications remain essential: these boundary conditions restrict scalar functions at the walls, and compression of scalar functions through both stages differs from generating a matrix algebra between the stages. In particular, aligned walls give $\mathbb{R} \oplus \mathbb{R}$ on the quaternionic factor under direct scalar compression, whereas the intervening matrix-algebra construction gives $\mathbb{H} \oplus \mathbb{H}$. Local boundary gradings preserve all scalar functions, but their reduction to the unchanged four-section window has a precise restriction proved below. The radial no-go of lemma 22 survives rechecking in its stated scope. For the massless one-loop potential of lemma 23, an analytic argument strengthens the numerical statement to all $n_f > 0$.

1. Theorem — An exact spectral boundary construction.

Hypotheses and conventions. Let

$$K = \mathbb{CP}^3, \qquad \mathcal{H}_K = L^2\big(K, \Lambda^{0,*}T^*K \otimes \mathcal{O}(1)\big)\,,$$

and let

$$A = \sqrt{2}(\bar\partial + \bar\partial^*), \qquad \gamma_K = (-1)^{\deg}.$$

Use the Fubini–Study metric at a fixed reference scale. Write

$$V = \ker A \simeq \mathbb{C}^4, \qquad \Pi = P_K,$$
$$\Delta_K = \inf(\operatorname{spec}|A| \setminus \{0\}) > 0.$$

Here $\gamma_K|_V = 1$. The radius parameter $r > 0$ replaces A by A/r, after the usual unitary identification of the L^2-spaces.

These are the analytic data of Example 16.9. Its S^4 option uses functions pulled back to this same twistor space. An actual operator on S^4 with the required kernel and compression map would need to be specified separately.

Let

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

On $L^2(I_\ell; \mathbb{C}^2 \otimes \mathcal{H}_K)$, set

$$\mathcal{D}_r = E\partial_x + Z\,A/r, \qquad \hat\gamma = Z\gamma_K.$$

All unspecified tensor factors are identities.

Choose self-adjoint involutions Γ_0, Γ_1 on V, each with two-dimensional positive and negative eigenspaces, and assume

$$-1 \notin \operatorname{spec}(\Gamma_1\Gamma_0).$$

Extend Γ_i by zero on $V^\perp$, writing $C_i = \Pi\Gamma_i\Pi$. Define $\operatorname{sgn} A = 0$ on V, and impose

$$\chi_0\psi(0) = \psi(0), \qquad \chi_\ell\psi(\ell) = \psi(\ell),$$

where

$$\chi_0 = ZC_0 - S\operatorname{sgn} A, \qquad \chi_\ell = ZC_1 + S\operatorname{sgn} A. \quad (1)$$

Then:

1. The H^1 realization of $\mathcal{D}_r$ is self-adjoint with compact resolvent and anticommutes with $\hat{\gamma}$.
2. $\Pi \otimes 1$ reduces the operator, including its domain.
3. On this reducing subspace, the operator is exactly $D(\Gamma_0, \Gamma_1)$ of lemma 23.
4. Every eigenvalue on the complementary subspace satisfies

$$|\lambda| \geq \Delta_K / r. \tag{2}$$

Proof. Put $B = SA/r$. Since $ES = Z$,

$$\mathcal{D}_r = E(\partial_x + B).$$

The operators χ_i are self-adjoint involutions and anticommute with E. Their positive eigenspaces are therefore maximal isotropic for the boundary form

$$[\langle \psi, E\phi \rangle]_0^\ell.$$

On $V^\perp$, condition (1) selects the negative spectral subspace of B at 0 and its positive spectral subspace at ℓ. These are APS conditions with the inward-normal sign reversed between the two walls. On $\ker B = \mathbb{C}^2 \otimes V$, condition (1) supplies a Lagrangian subspace:

$$\Gamma_i \psi_L = \psi_L, \qquad \Gamma_i \psi_R = -\psi_R.$$

For q in the complementary domain, integration by parts gives

$$\|\mathcal{D}_r q\|^2 = \|q'\|^2 + \|Aq/r\|^2 + [\langle q, Bq \rangle]_0^\ell \geq \|q'\|^2 + \frac{\Delta_K^2}{r^2} \|q\|^2. \tag{3}$$

The boundary term is nonnegative by the spectral choices.

An eigenbasis of A decomposes the complementary problem into finite-dimensional interval problems with maximal isotropic endpoint conditions. Each is self-adjoint. Estimate (3) identifies the domain of their self-adjoint direct sum with the stated H^1 domain and gives compactness of its resolvent. The finite-dimensional tangential kernel contributes an ordinary self-adjoint interval problem. This proves self-adjointness, compactness, and (2).

Every boundary projector commutes with Π, proving reduction. Finally, $\{\gamma_K, A\} = 0$ implies $\{\hat{\gamma}, \mathcal{D}_r\} = 0$, and $\hat{\gamma}$ preserves both boundary subspaces. ■

Define

$$K_F = \frac{1}{2\ell}\log(\Gamma_1\Gamma_0), \qquad D_F = \Gamma_0 K_F, \qquad R(x) = e^{xK_F},$$

with the principal logarithm. Let $P_{\pm} = (1 \pm \Gamma_0)/2$. The finite-window embedding is

$$W_\ell v(x) = \ell^{-1/2}\begin{pmatrix} R(x)P_+ v \\ R(x)P_- v \end{pmatrix}. \tag{4}$$

Direct differentiation gives

$$\mathcal{D}_r W_\ell = W_\ell D_F, \qquad \hat{\gamma} W_\ell = W_\ell \Gamma_0.$$

Thus the prescribed finite operator and grading are exact, provided $\Delta_K/r > \|D_F\|$. The next eigenvalue in absolute value is at least

$$\min\left\{\frac{\Delta_K}{r}, \frac{\pi - \ell\|D_F\|}{\ell}\right\}. \tag{5}$$

---

2. Proposition — Local walls and spectral walls have different scalar domain algebras.

Hypotheses. Use the product differential expression of Result 1.

For the local option, let $G_i(y)$ be smooth self-adjoint bundle involutions and impose

$$G_i\psi_L = \psi_L, \qquad G_i\psi_R = -\psi_R \quad \text{at } x = i.$$

Assume

$$[G_i, \gamma_K] = 0, \qquad [G_i, \sigma_A(y,\xi)] = 0 \quad (\xi \in T_y^*K), \tag{6}$$

where σ_A is the tangential principal symbol.

Then these local conditions give a self-adjoint elliptic realization with compact resolvent, and **every** smooth scalar function on $K \times I_\ell$ preserves its domain.

For the spectral option (1), the maximal smooth scalar $*$-algebra preserving the domain is

$$\mathcal{A}_{\mathrm{APS}} = \{f \in C^\infty(K \times I_\ell) : f(\,\cdot\,,0), f(\,\cdot\,,\ell) \text{ are constant on } K\}. \tag{7}$$

The same conclusion holds for the subalgebra pulled back from $S^4 \times I_\ell$.

Proof. For the local option, the boundary involution is ZG_i. It anticommutes with the normal matrix E. Condition (6) makes it anticommute with the principal symbol of the adapted tangential operator SA/r. Hence its positive eigenspace is an elliptic Lagrangian boundary condition. The corresponding local ellipticity and adjoint-domain criteria are described in Bär–Ballmann, Corollary 4.18 and Example 4.20. Primary reference

Scalar multiplication commutes pointwise with G_i, so it preserves the endpoint conditions and the H^1 domain. Moreover,

$$[\mathcal{D}_r, f] = E\,\partial_x f + Z[A, f]/r$$

is bounded for each $r > 0$.

For spectral walls, preservation by a scalar $*$-algebra is equivalent to

$$[M_{f_i}, \chi_i] = 0.$$

Independence of the spin matrices Z, S gives

$$[M_{f_i}, C_i] = 0, \qquad [M_{f_i}, \operatorname{sgn} A] = 0.$$

Since $C_i^2 = \Pi$, the first equation implies

$$[M_{f_i}, \Pi] = 0. \tag{8}$$

Consequently multiplication by f_i preserves $H^0(K, \mathcal{O}(1))$. If s is such a section, then

$$0 = \bar\partial(f_i s) = (\bar\partial f_i)s.$$

The sections of $\mathcal{O}(1)$ have no common zero, so $\bar\partial f_i = 0$ everywhere. A holomorphic function on compact connected $\mathbb{CP}^3$ is constant. Constants plainly preserve (1), proving (7). ■

The hypotheses in (6) are a sufficient local ellipticity condition. An arbitrary bundle grading supplies a pointwise isotropic condition, but self-adjointness on an H^1 domain and compact resolvent still require an ellipticity check. Likewise, prescribing a grading only on V leaves the boundary problem incomplete until a condition on $V^\perp$ is supplied.

3. Proposition — Compression of a local grading and the unchanged level-one window.

Hypotheses. Let G be a self-adjoint bundle involution acting by multiplication, and let

$$g = \Pi G \Pi|_V, \qquad Q_K(f) = \Pi M_f \Pi|_V.$$

Keep the level-one space and scalar representation of Example 16.9 unchanged.

Then

$$1_V - g^2 = \big((1-\Pi)G\Pi\big)^*\big((1-\Pi)G\Pi\big). \tag{9}$$

In particular, g is a grading exactly when $[G, \Pi] = 0$. In that event,

$$[g, Q_K(f)] = 0 \quad \text{for every scalar } f. \tag{10}$$

For either the full $\mathbb{CP}^3$ scalar algebra or the S^4 pullback algebra, this forces

$$g = 1_V \quad \text{or} \quad g = -1_V. \tag{11}$$

Proof. Expand $\Pi G^2 \Pi = \Pi$ using $1 = \Pi + (1-\Pi)$; this gives (9). Vanishing of its positive right-hand side implies $(1-\Pi)G\Pi = 0$, and self-adjointness gives the other off-diagonal block.

Since scalar multiplication commutes with G, reduction then gives (10). The compressed $\mathbb{CP}^3$ functions generate $M_4(\mathbb{C})$. The compressed real S^4 functions generate $M_2(\mathbb{H})$, whose complex-linear commutant on $\mathbb{C}^4$ is also $\mathbb{C}1$. Thus a self-adjoint involution satisfying (10) is ± 1. ■

For **aligned local walls** with $[G, A] = 0$, the finite zero-mode window is explicitly

$$v \longmapsto \ell^{-1/2} \begin{pmatrix} P_+(g)v \\ P_-(g)v \end{pmatrix}.$$

The induced grading is g, and scalar compression is

$$P_+(g)Q_K(f)P_+(g) + P_-(g)Q_K(f)P_-(g) = Q_K(f).$$

By (11), the unchanged four-section window has trivial grading. Its generated algebra remains $M_2(\mathbb{H})$ or $M_4(\mathbb{C})$,

according to the scalar algebra chosen. The globally constant choices $G = \pm 1$ give concrete self-adjoint examples with this window and a separated higher spectrum.

There is also an explicit local field illustrating failure of reduction. Add a trivial auxiliary $\mathbb{C}^4$ bundle and put

$$G(y) = \sum_{a=1}^{5} y_a \Gamma_a^{\mathrm{aux}},$$

where $y \in S^4$ is the twistor projection and the Γ_a satisfy $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$. This is a local involution satisfying (6), but now the tangential kernel has dimension 16, and

$$g = \frac{1}{5} \sum_{a=1}^{5} \Gamma_a^{\mathrm{section}} \otimes \Gamma_a^{\mathrm{aux}}.$$

Using $\mathrm{tr}(\Gamma_a \Gamma_b) = 4\delta_{ab}$,

$$\mathrm{tr}(g^2) = \frac{16}{5} \neq 16.$$

Thus g is not a grading. Equation (9) even gives

$$\|(1 - \Pi) G \Pi\| \geq \frac{2}{\sqrt{5}},$$

independently of r. This is a bound on boundary-trace mixing; it does not identify the actual low spectral window of the coupled local problem.

A local $G(y)$ specifies a **field of gradings**. It does not select a marked point p. In the tautological quaternionic line bundle,

$$\Pi_y = \frac{1 + y_a \Gamma_a}{2},$$

the natural field $2\Pi_y - 1$ acts as $+1$ on the fibre $\mathrm{ran}\, \Pi_y$. By contrast, the finite grading

$$\Gamma_p = p_a \Gamma_a = 5 Q_K (p \cdot y)$$

uses a separately chosen point p. Its action on global section coefficients is nonlocal.

The restriction proved here concerns the unchanged level-one space, a local multiplication grading, and its compression as an actual involution. A different coupled window, additional fibres, or spectral boundary conditions remain available directions.

4. Theorem — The composed compression map and the aligned-wall algebra.

Hypotheses. Use the exact spectral construction of Result 1. For a scalar multiplier $f(y, x)$, let

$$T_f(x) = Q_K(f(\cdot\,, x)), \qquad \mathcal{E}_0(T) = \frac{1}{2}(T + \Gamma_0 T \Gamma_0).$$

Use real-valued functions and real algebra generation when discussing $M_2(\mathbb{H})$.

Then the compression directly from the scalar cylinder to the finite window is

$$\boxed{\Phi_\ell(f) = \frac{1}{\ell}\int_0^\ell \mathcal{E}_0\big(R(x)^* T_f(x) R(x)\big)\, dx.} \tag{12}$$

It is independent of r.

For aligned walls $\Gamma_1 = \Gamma_0 = \Gamma_p$, direct scalar compression on the S^4 factor generates

$$\mathbb{R} \oplus \mathbb{R}. \tag{13}$$

If instead one first generates $M_2(\mathbb{H})$ and then applies the interval compression, the result is

$$\mathbb{H} \oplus \mathbb{H}. \tag{14}$$

Proof. Insert (4) on both sides of the multiplication operator. Orthogonality of the two interval spin components gives exactly (12).

For real S^4 functions, Example 16.9 implies

$$Q_K(C^\infty(S^4, \mathbb{R})) = \mathrm{span}_{\mathbb{R}}\{1, \Gamma_1, \dots, \Gamma_5\}. \tag{15}$$

Indeed all compressed functions are quaternionic Hermitian matrices; that space has real dimension six, and the displayed six matrices belong to the image and are independent.

For aligned walls $R = 1$. Clifford multiplication gives

$$\mathcal{E}_p(\Gamma_a) = p_a \Gamma_p.$$

Therefore the image of direct scalar compression is

$$\mathrm{span}_{\mathbb{R}}\{1, \Gamma_p\}.$$

This space is already the algebra $\mathbb{R} \oplus \mathbb{R}$, proving (13).

On the other hand,

$$\mathcal{E}_p(M_2(\mathbb{H})) = \{a \in M_2(\mathbb{H}) : [a, \Gamma_p] = 0\} = \mathbb{H} \oplus \mathbb{H},$$

proving (14). The noncommuting products needed to generate the quaternionic diagonal blocks have been inserted between the two compressions in this second procedure. ■

On the single $\mathbb{CP}^3$ factor, real scalar compression is onto the Hermitian 4×4 matrices. A chosen $2+2$ aligned grading therefore gives all Hermitian block-diagonal matrices and generates

$$M_2(\mathbb{C}) \oplus M_2(\mathbb{C}).$$

For the **chosen two-sheet coarse correspondence** of Example 16.9, with

$$H_0 = \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W, V), \qquad \Gamma_0(\xi, \eta) = (\Gamma_p \xi, -\eta \Gamma_p),$$

the same calculation gives

$$\boxed{\mathrm{alg}_{\mathbb{R}}(\text{direct aligned scalar compression}) = \mathbb{R} \oplus \mathbb{R} \oplus M_4(\mathbb{C}).} \tag{16}$$

The V-side matrices act on the left of η and commute with its right grading, so that factor survives unchanged.

Endpoint restrictions do not change these finite-dimensional images. If a linear functional vanishes on compressions of functions supported in $0 < x < \ell$, its smooth matrix-entry density vanishes in the interior and hence everywhere. Finite-dimensional separation then gives equality with the unrestricted image, exactly as in Result 7 of lemma 20.

At the intermediate stage, however, spectral walls still give scalar endpoint values. Even after algebra generation, those values remain scalar. Enlarging them to the full stabilizers of Γ_0, Γ_1, as in lemma 23, is another change of the parent algebra. Equality of finite compression images does not identify the parent gauge groups or one-forms.

5. Theorem — Radius collapse, interval collapse, and controlled mixing.

Hypotheses. First use the exact construction of Result 1. Identify the varying Hilbert spaces by metric rescaling. For resolvent formulas, fix units in which the nonreal spectral parameter is i.

Then, at fixed ℓ,

$$\left\|(\mathcal{D}_r - i)^{-1} - \iota_\Pi (D(\Gamma_0, \Gamma_1) - i)^{-1} \iota_\Pi^*\right\| \le \frac{r}{\Delta_K}. \qquad (17)$$

Thus radius collapse gives the entire matrix interval operator in generalized norm-resolvent sense.

If D_F is fixed, $\ell \to 0$, and

$$\Gamma_1(\ell) = e^{\ell \Gamma_0 D_F} \Gamma_0 e^{-\ell \Gamma_0 D_F},$$

then $r \to 0$ and $\ell \to 0$ decouple both towers and leave the finite operator D_F. No relation between the two rates is required in this exactly reducing construction.

Proof. The resolvent is block diagonal with respect to Π. Estimate (2) bounds its complementary block by r/Δ_K, proving (17). For the finite-window limit, (5) bounds the complementary resolvent by

$$\max\left\{\frac{r}{\Delta_K}, \frac{\ell}{\pi - \ell\|D_F\|}\right\},$$

which tends to zero. ■

A useful mixing statement requires additional hypotheses. Suppose a transformed problem has a common domain and the form

$$\widetilde{\mathcal{D}}_r = \mathcal{D}_r + V_r, \qquad V_r = V_r^*, \qquad \|V_r\| \le b_r.$$

Write

$$v_r = \|(1 - \Pi) V_r \Pi\|.$$

For an eigenvector $\psi = p + q$, with $p = \Pi\psi$, $q = (1 - \Pi)\psi$, and $|\lambda| \le L$, assume

$$g_r = \Delta_K / r - b_r - L > 0.$$

Then

$$\boxed{\|q\| \le \frac{v_r}{g_r}\|p\|, \qquad \|\text{Schur correction at } \lambda\| \le \frac{v_r^2}{g_r}.} \tag{18}$$

Proof. The complementary eigenvalue equation gives

$$q = -(\mathcal{D}_{r,Q} + V_{QQ} - \lambda)^{-1} V_{QP} p.$$

The inverse has norm at most g_r^{-1}. Substituting this formula into the p-equation gives the second bound. ■

For bounded b_r, v_r, both errors are $O(r/\Delta_K)$. If a common-domain construction instead produces terms of order $1/\ell$, sufficient conditions include

$$\frac{r v_r}{\Delta_K} \to 0, \qquad \frac{r v_r^2}{\Delta_K} \to 0,$$

together with convergence of the Π-diagonal perturbation.

An arbitrary local $G_i(y)$ generally fails $[G_i, \Pi] = 0$. Although the differential expression commutes with Π, its domain then does not. Such a boundary change is not automatically a bounded perturbation satisfying (18). The explicit local field in Result 3 has nonvanishing boundary mixing at every radius. No convergence to the prescribed two-gradings operator is established here for that nonreducing local problem.

At the level of observables, radius collapse leaves (12) unchanged. For an x-independent $T = Q_K(f)$,

$$\|\Phi_\ell(f) - \mathcal{E}_0(T)\| \le \ell \|D_F\| \, \|T\|. \tag{19}$$

If T is even for Γ_0, the stronger formula of Theorem 22.4 applies:

$$\Phi_\ell(f) = \operatorname{sinc}(\ell \operatorname{ad}_{D_F})(T),$$

$$\|\Phi_\ell(f) - T\| \le \frac{\ell^2}{6} \|[D_F, [D_F, T]]\|. \tag{20}$$

Equation (19) follows by integrating $\|R^* T R - T\| \le 2x\|D_F\|\|T\|$; equation (20) follows by taking the even part of the conjugation series.

Consequently there is convergence of the **compression maps on their scalar operator systems**. The limit

is $\mathcal{E}_0 Q_K$. It does not turn that image into every element of $\mathcal{E}_0(P)$; Result 4 computes the difference explicitly. Algebra generation remains a separate operation.

---

6. Proposition — The second sheet, reality, even algebra, and connector remain prescribed data.

Hypotheses. Consider the scalar cylinder construction above and, separately, its extension using the two-sheet coarse correspondence of Example 16.9.

The resulting distinctions are:

Structure	What follows under the stated choices
Second sheet and the two Hom spaces	They are inputs in the coarse correspondence. A single four-section cylinder window does not supply them.
$J_1^2 = -1,\ J_1\Gamma_i = -\Gamma_i J_1$	These are additional compatibility conditions on the prescribed gradings and antiunitary.
$\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$	This is the derived even part after choosing the quaternionic real form, the grading, and intermediate algebra generation. Direct aligned scalar compression gives (16).
Connector and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$	The connector's lines and coupling are chosen. Their intertwining algebra is then derived.

Proof. The one-factor construction starts with $V = \mathbb{C}^4$, and (4) has domain V. The two-sheet construction instead starts with

$$(W \otimes \overline{V}) \oplus (V \otimes \overline{W}).$$

Geometrically, its analytic realization uses product factors and a restricted scalar algebra, as Theorem 16.7 spec-

ifies. It is additional correspondence data over the scalar spaces.

For reality, suppose an antiunitary $\mathcal{J}_1$ on the tangential Hilbert space satisfies

$$\mathcal{J}_1^2 = -1, \quad \mathcal{J}_1 A = -A\mathcal{J}_1, \quad [\mathcal{J}_1, \gamma_K] = 0,$$

and its restriction J_1 to V anticommutes with both Γ_i. Then

$$\widehat{J} = E\mathcal{J}_1$$

preserves (1) and obeys

$$\widehat{J}^2 = 1, \qquad [\widehat{J}, \mathcal{D}_r] = 0, \qquad \widehat{J}\widehat{\gamma} = -\widehat{\gamma}\widehat{J}.$$

On the window,

$$J_F = \Gamma_0 J_1.$$

These identities follow directly from the stated commutation relations. They specify sufficient conditions; the natural $\mathcal{O}(1)$ cylinder has not supplied such an extension or its order-zero property here.

In particular, the quaternionic antiunitary j used to recognize $M_2(\mathbb{H})$ commutes with the Γ_a. It does not itself satisfy the required anticommutation with Γ_p.

The even-algebra assertion is the centralizer calculation in Result 4 after intermediate generation.

Finally, choose

$$w \in W_+, \quad v \in V, \quad \|w\| = \|v\| = 1, \quad \kappa \neq 0,$$

and prescribe

$$T_\kappa \xi = \kappa(w^* \xi v)\, v w^*.$$

Its intertwining conditions require the lines $\mathbb{C}w, \mathbb{C}v$ to be reducing, with the same scalar action. A quaternion preserving $\mathbb{C}w$ lies in its chosen copy of $\mathbb{C}$; $v^\perp$ carries arbitrary $M_3(\mathbb{C})$, and W_- carries arbitrary $\mathbb{H}$. Hence

$$\mathcal{A}(T_\kappa) = \{(\lambda \oplus q, \lambda \oplus m)\} \simeq \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),$$

as in Result 2 of lemma 20. ■

A marked $p \in S^4$ fixes a quaternionic line, but choosing w also requires a complex line within it. The second line v, the coupling, and an interface realizing the connector remain necessary. Scalar agreement at marked points has unchanged compression image by Result 7 of lemma 20, so it supplies none of this selection by itself.

7. No-go theorem — Recheck of the radial statement in lemma 22.

Hypotheses. Let the parent be exactly the one-interval model of lemma 22: fixed wall coupling X, zero bulk mass, the even algebra $\mathcal{A}_\ell$, scalar inner fluctuations Eh with $h^* = -h$, and the **moving spectral window**. For real fluctuations, include the conjugate one-form and assume the stated order-zero representation.

Then every fluctuated finite operator has the same singular values as D_F. This is an exact finite-amplitude spectral statement; it uses no action or loop expansion.

Proof. Solve

$$u' = -hu, \qquad u(0) = 1, \qquad w = u(\ell),$$

and set $v(x) = u(x)w^*$. Then

$$v(\ell) = 1, \qquad v(0) = w^*.$$

Since h is even, w preserves the boundary splitting at 0; the identity at ℓ preserves the fixed coupled wall. Thus v preserves the original domain, and

$$vD_X v^* = D_X + Eh.$$

For the real fluctuation, the analogous implementing unitary is

$$vJ_F vJ_F^{-1};$$

order zero separates the two factors in its derivative.

The full operators are unitarily equivalent, so their moving spectral windows are unitarily equivalent. In the finite identifications of lemma 22, the operators are $w^* D_F w$, or $U_w^* D_F U_w$ in the real case. Their singular values coincide. ■

The moving-window hypothesis matters. For example, with $D_F = m\sigma_1$ and $h = i\sigma_3$, fixed compression gives

$$Q_\ell(Eh) = -\frac{1 - \cos(2m\ell)}{2m\ell}\sigma_2.$$

Adding a multiple of this operator to the fixed $m\sigma_1$ changes its singular values at second order.

The verdict agrees with the author's amended theorem and Terra's independent endpoint-preserving proof. In the usual nondegenerate Standard Model representation, the normalized shifted Higgs quaternion lies on the unit-quaternion orbit; its norm is fixed.

The known escapes are a variable wall coupling, odd connection directions, suitable bulk masses, additional internal directions with nontrivial connection data, or additional elementary or composite fields. A quantum effective action on the same configuration space does not add a radial coordinate. The proposed $K \times I_\ell$ parent has tangential directions and tangential one-forms, so this theorem does not decide its full fluctuation space.

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8. Lemma — A logarithmic concavity property of the one-loop integral.

Hypotheses. Let

$$\mathcal{J}(z) = \int_0^\infty \frac{x^3}{\sinh^2 x + z}\, dx, \qquad z > 0,$$

and define

$$e(z) = -\frac{z\mathcal{J}'(z)}{\mathcal{J}(z)}.$$

Then $0 < e(z) < 1$, and e is nondecreasing.

Proof. Positivity and

$$0 < \frac{z}{\sinh^2 x + z} < 1$$

show that $e(z)$, expressed as a weighted average of this fraction, lies strictly between zero and one.

Put $x = e^v$, $z = e^u$. Then

$$\mathcal{J}(e^u) = \int_{\mathbb{R}} \exp\left(4v - \log(\sinh^2(e^v) + e^u)\right)\, dv. \tag{21}$$

The function

$$b(v) = 2\log\sinh(e^v)$$

is convex, because, with $x = e^v$,

$$b''(v) = 2x(\coth x - x\ \mathrm{csch}^2 x) > 0.$$

The last inequality is equivalent to $\sinh x \cosh x > x$.

The function $(u, v) \mapsto \log(e^{b(v)} + e^u)$ is therefore convex. The integrand in (21) is jointly log-concave. Marginal integration preserves log-concavity by Prékopa's theorem, so $u \mapsto \log \mathcal{J}(e^u)$ is concave. We use precisely Theorem 6, p. 342, of Prékopa's original paper.

Finally,

$$\frac{d}{du} \log \mathcal{J}(e^u) = -e(e^u).$$

Concavity makes this derivative nonincreasing, proving the assertion. ■

---

9. Theorem — The massless one-loop potential has no interior local minimum.

Hypotheses. Let the action be exactly the one-loop determinant displayed in lemma 23: a flat interval, zero bulk masses, frequency-one fermions, the stated charged and neutral gauge towers and multiplicities, and the prescribed connector wall condition. Assume that no fermion connector modifies the displayed fermion tower, and add no wall potential or other angle-dependent term.

Let $n_f > 0$, set $t = \sin^2 \theta$, and write

$$\mathcal{V}(t) = 8\pi^2 \ell^4 V(\theta).$$

Then:

1. $t = 1$, equivalently $\theta = \pi/2$, is the unique global minimum.
2. If $n_f \geq 54/5$, the potential is strictly decreasing on the interior.
3. If $0 < n_f < 54/5$, there is exactly one interior stationary point, and it is a strict maximum.

Proof. Define

$$F(z) = \int_0^\infty x^3 \log\left(1 + \frac{z}{\sinh^2 x}\right) dx, \qquad F'(z) = \mathcal{J}(z).$$

The exact potential is

$$\mathcal{V}(t) = 3F(2t(1-t)) + \frac{3}{2} F\left(\frac{16}{5} t(1-t)\right) - n_f F(t). \quad (22)$$

Its gauge terms are nonnegative and vanish at $t = 1$. Since F is strictly increasing, the fermion term is uniquely minimized there. This proves item 1.

Differentiate:

$$\mathcal{V}'(t) = (1-2t)\left[6\mathcal{J}(2t(1-t)) + \frac{24}{5}\mathcal{J}\left(\frac{16}{5}t(1-t)\right)\right] - n_f\mathcal{J}(t). \tag{23}$$

For $t \geq 1/2$, this is strictly negative.

For $0 < t < 1/2$ and $c \geq 2$, put

$$R_c(t) = (1-2t)\frac{\mathcal{J}(ct(1-t))}{\mathcal{J}(t)}.$$

Let $a = ct(1-t)$. Then $a \geq t$, and Result 8 gives $e(a) \geq e(t)$. Consequently,

$$\begin{aligned}\frac{d}{dt}\log R_c(t) &= -\frac{2}{1-2t} - \frac{1-2t}{t(1-t)}e(a) + \frac{e(t)}{t} \\ &\leq -\frac{2}{1-2t} + \frac{e(t)}{1-t} \\ &< -\frac{2}{1-2t} + \frac{1}{1-t} < 0.\end{aligned}$$

Thus every R_c is strictly decreasing. Since $\mathcal{J}(0)$ is finite and positive,

$$R_c(0+) = 1, \qquad R_c(1/2-) = 0.$$

Therefore

$$H(t) = 6R_2(t) + \frac{24}{5}R_{16/5}(t)$$

decreases strictly from $54/5$ to zero. Equation (23) becomes

$$\mathcal{V}'(t) = \mathcal{J}(t)\,[H(t) - n_f] \quad (0 < t < 1/2).$$

For $n_f \geq 54/5$, its sign is negative. For $0 < n_f < 54/5$, it changes sign exactly once, from positive to negative. Since $t = \sin^2\theta$ is strictly increasing on the interior, the same stationary-point classification holds in θ. ■

This proves the massless statement for the entire parameter range, including the previously numerical range. No numerical computation is used in this proof.

The recorded author and Terra referee calculations agree on the sampled massless cases. The logarithmic-concavity proof above is an additional single-agent derivation; it has not received a fresh independent model review

in this turn. The recorded agreement therefore should not be described as a multi-model review of this new proof.

The escapes remain changes to the stated determinant: frequency-two fermions, modified form factors through warping or other geometry, suitable bulk-mass sectors, wall potentials, and wall kinetic or interaction terms. The full effective action of the proposed $K \times I_\ell$ parent, including its K-tower, has not been computed here.

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10. Proposition — Scope of the bulk-mass scans.

Hypotheses. Retain the flat gauge terms of (22) as an independent assumption. Replace the fermion factor by the expression in lemma 23:

$$1 + t(G(x) - 1),$$
$$G(x) = \frac{(w_Q \coth w_Q - c_Q)(w_U \coth w_U - c_U)}{x^2},$$
$$w_i = \sqrt{x^2 + c_i^2}.$$

If $c_U = -c_Q = c$, then the potential is finite and $t = 1$ is its unique global minimum for every $n_f > 0$. The reported absence of additional interior **local** minima at nonzero c remains a numerical statement here.

If $a = c_Q + c_U \neq 0$, the displayed unrenormalized integral has angle-dependent ultraviolet divergences.

Proof. For opposite masses,

$$G(x) = 1 + \frac{w^2}{x^2 \sinh^2 w}, \qquad w = \sqrt{x^2 + c^2}.$$

The added fraction is positive and decays exponentially at infinity. The negative fermion logarithm is strictly decreasing in t. The gauge logarithms are nonnegative and vanish at $t = 1$, proving the global-minimum assertion.

For general masses,

$$G(x) = 1 - \frac{a}{x} + \frac{a^2}{2x^2} + O(x^{-3}),$$

and hence

$$\log(1 + t(G - 1)) = -\frac{ta}{x} + \frac{t(1-t)a^2}{2x^2} + O(x^{-3}).$$

Multiplication by the integration measure x^3dx produces angle-dependent divergences. A finite potential therefore requires a specified completion, regulator, and counterterms. ■

The opposite-mass scan in lemma 23 samples finitely many masses, multiplicities, and angles. It cannot exclude unsampled local minima. General bulk masses and altered gauge form factors remain directions for constructing an interior vacuum; these conclusions leave that question open.

Consequence for T0 and T0b

For T0, a commutative scalar representation with prescribed spectral walls gives an exact interval operator window and controlled tower decoupling, while local walls preserve the full scalar algebra under the ellipticity hypotheses above. The derivation of a nontrivial local grading and the desired algebra from one coupled scalar window remains open, and intermediate algebra generation must be stated explicitly. For T0b, the second sheet, compatible reality, quaternionic grading choice, and connector remain inputs; their algebraic consequences can be derived once those choices are supplied. Neither the radial theorem nor the specified one-loop potential closes the commutative-parent problem.

Referee amendments

From Terra 6.1. These apply to the text above.

- **Result 2.** Write G_0 at x = 0 and G_l at x = l. Equation (6) supplies the principal-symbol condition of Bar-Ballmann, Corollary 4.18 and Theorem 4.11; their Example 4.20 assumes more. On the unaugmented Dolbeault bundle, the Clifford representation is irreducible, so a fibre endomorphism commuting with every sigma_A is scalar, and a smooth involution of this form is +1 or -1. Additional fibres and other local elliptic projectors lie outside this sufficient criterion.
- **Result 3.** The statement on marked points concerns the natural field displayed there. A local field can encode a chosen point: on the auxiliary bundle, G_p(y) = p_a Gamma_a^aux is a local constant

field whose coefficients contain p.

- **Result 4.** Equation (16) has the two Hom spaces and the two scalar algebras of Example 16.9 as inputs. The endpoint argument proves equality of the finite-dimensional linear images, and does not identify the parent algebras, norms, gauge groups or one-forms.
- **Result 5, gap in the final assertion.** Convergence of compression maps needs a common domain and a collapse map. With aligned walls, D_F = 0 and f_l(y, x) = 6 (x/l)(1 - x/l), both endpoint values vanish while Phi_l(f_l) = 1. A correct form: for F(y, s) fixed on K x [0, 1] and f_l(y, x) = F(y, x/l), Phi_l(f_l) tends to E_0 Q_K(int_0^1 F(., s) ds). For x-independent ambient multipliers the limit is E_0 Q_K, by (19). The rate discussion after (18) also needs control of the denominator: a sufficient addition is r (b_r + L) <= (1 - epsilon) Delta_K. The case b_r = O(1/l) follows from the stronger estimate |D_{r,Q}| >= (Delta_K^2/r^2 + pi^2/(4 l^2))^{1/2}, from the one-ended Poincare inequality on the positive and negative B-spectral components.
- **Result 6.** At the finite level, if the gradings are Gamma_p, Gamma_q, a unit n orthogonal to p and q gives J_1 = Gamma_n j, with J_1^2 = -1 and the required anticommutation. Extending this choice to the parent remains a separate task.
- **Lemma 8 and Theorem 9.** An independent check of the key monotonicity, without log-concavity or Prekopa's theorem: with rho(s) = (arsinh sqrt(s))^3/(2 sqrt(s (1 + s))), J(z) = int_0^infinity rho(s)/(s + z) ds; with L(v) = log rho(e^v), L'(v) = 3 tanh(x)/(2x) - (1 + tanh^2 x)/2 for v = 2 log sinh x, which is strictly decreasing. For z = e^u, with the probability density in y proportional to e^y rho(e^{u+y})/(1 + e^y), e(e^u) is the mean of 1/(1 + e^y), and its derivative in u is Cov(1/(1 + e^y), L'(u + y)) > 0, as both are strictly decreasing. Theorem 9 is for theta in [0, pi/2]. At theta = pi/2 the determinant has its continuous value, while the principal-logarithm window is degenerate. Its hypotheses are the flat massless one-loop determinant, its gauge multiplicities and connector wall condition, frequency-one

fermions, no fermion-connector modification, and no other angle-dependent term.

- **Result 10.** The divergence is stated for nonzero fermion multiplicity, with no cancellation by additional sectors. The opposite-mass scan in lemma 23 is a numerical result on its sampled parameters.

Numerical check

By Claude Opus 5.5, in Python (numpy, scipy), 3 Nov 2026, in a temporary directory, not kept: E_p(Gamma_a) = p_a Gamma_p to 10^-15, which gives equation (13); tr(g^2) = 16/5 for the auxiliary field of Result 3; on grids, e(z) lies in (0, 1) and is nondecreasing for z in [10^-3, 50]; H(t) is strictly decreasing on (0, 1/2), with H near 54/5 at small t and near 0 at t = 1/2; for n_f in {1, 3, 6, 8, 10}, H - n_f has one sign change, with the stationary angle 0.34 at n_f = 8, which agrees with the referee of lemma 23. Results 1, 2, 5 and 6 were read and not separately computed.

Part 25

The weak-boson sum rule and the effective dimension of the interval

For STRATEGY T5b, T13 and T1, following lemma 18, lemma 22 and lemma 23. Proved by Claude Opus 5.5, with checks in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): two errors and two gaps as first written, and amendments, all incorporated below.

Outcome. Unfolded, the interval parent is a circle of length 2l, and the parent operator is -i d/dx (x) 1 + 1 (x) D_F: the tower is D_F shifted by the momenta k_n = n pi/l. So the exact quadratic kernel of lemma 18 applies termwise over the tower. For small masses, at leading order, for the heat cutoff, the Taylor mass of a vector with constant profile in the transported frame is

```
M^2 = ((3 - d_eff)/2) ||[D_F, T]||^2 / ||T||^2,
 ↪  d_eff(s) = d log S / d log s,      S(s) = sum_{n
 ↪  in Z} exp(-n^2 pi^2/s^2),   s = l Lambda,
```

in Hilbert-Schmidt norms, where d_eff is the spectral di-

mension at the scale of the cutoff of the unfolded circle, equivalently of the paired Neumann and Dirichlet interval: 0 for l Lambda « 1 and 1 for l Lambda » 1. For the Standard Model representation, at the same order,

```
M_W^2 = ((3 - d_eff)/24) sum_f N_f m_f^2,
 ↪  m_t/M_W = (8/(3 - d_eff))^{1/2} for the top
 ↪  alone,
```

with M_W^2/M_Z^2 = 5/8 for all s. So the relation 8 M_W^2 = sum N_f m_f^2 of the asymptotic spectral action is the zero-dimensional value, with m_t/M_W = sqrt(8/3). A resolved interval gives 12 M_W^2 = sum N_f m_f^2, with m_t = 2 M_W, as five-dimensional covariance requires for this operator on the circle. The endpoint m_t = 2 M_W is known in flat gauge-Higgs models with suitable representations (references below); in the inspected records the referee did not find the coefficient (3 - d_eff)/24 or its crossover, and priority remains open pending a wider search. The factor 3/2 in Proposition F of the paper, the finite-action coefficient relative to the Yang-Mills commutator quotient, is the ratio of these two endpoints. The crossover is at l Lambda about 2.

Unfolding

Let D_X = E d/dx be the parent of lemma 22, and let D_0 + E Gamma_F D_F be its form after the transport of Corollary 22.2, where D_0 is the operator with X = 0.

Proposition 25.1. On the circle x in [-l, l] with periodic conditions, for psi in the domain of D_0, put

```
Psi_Q(x) = Q_L(|x|) + i sgn(x) Q_R(|x|),
 ↪  Psi_U(x) = U_R(|x|) - i sgn(x) U_L(|x|).
```

After multiplying by 1/sqrt(2), this is a unitary from the parent Hilbert space onto L^2(circle) (x) (H_+ (+) H_-), which maps the H^1 domain of D_0 onto the periodic H^1 domain, and it carries D_0 + E Gamma_F D_F to

```
-i d/dx (x) 1 + 1 (x) D_F.
```

So the signed spectrum is k_n + mu_i, with k_n = n pi/l for n in Z and mu_i the eigenvalues of D_F. An even operator T = T_Q (+) T_U on H_F, constant in

x in this transported frame and acting identically on both spin components, is carried to 1 (x) T.

Proof. Since Q_R(0) = Q_R(l) = 0 and U_L(0) = U_L(l) = 0, the functions Psi are continuous and periodic, and every smooth function on the circle arises from its even and odd parts. For x > 0, -i d/dx Psi_Q = Q_R' - i Q_L' = (D psi)_L + i (D psi)_R on Q, and -i d/dx Psi_U = -i U_R' - U_L' = (D psi)_R - i (D psi)_L on U, which are the transforms of D_0 psi; x < 0 is the same with the odd parts. For the connection term, with M* the block of D_F, the term E Gamma_F D_F sends Psi_Q to M* Psi_U and Psi_U to M Psi_Q. ■

In this form D_F is a constant connection on the circle, with holonomy exp(2 i l D_F) up to the convention for parallel transport, and the two walls are the fixed points of the reflection that carries D_F to -D_F. With U = exp(-i pi Gamma_0/4), the identity of lemma 23 reads Gamma_1 Gamma_0 = exp(2 l Gamma_0 D_F) = U exp(2 i l D_F) U*. This form of the interval as a circle with a reflection is the usual orbifold description. It also gives a one-line proof of the spectrum in Theorem 22.1.

The frame. By Corollary 22.2, D_X = R (D_0 + E Gamma_F D_F) R*, with R(x) = exp(x Gamma_F D_F). A generator T constant after transport is R(x) T R(x)* in the original frame of D_X, which generally varies with x and has an odd bulk component. Conversely, a constant generator in the original frame need not preserve the coupled wall, if it is broken. So the field used here requires an admissible enlarged gauge algebra, as in the prealgebra P of lemma 23, and it is generally not a field of the even algebra of lemma 22.

The kernel over the tower

Let A_mu(y) = a_mu(y) T on M^4 = R^4, with T even and constant along x, and consider D = -i gamma^mu (d_mu + i A_mu) + gamma_5 V with V = -i d/dx (x) 1 + 1 (x) D_F. Theorem 1 of lemma 18 has this form, with V_0 = V, whose eigenvalues are k_n + mu_i, and T_{n i, n' j} = delta_{n n'} T_{ij}.

Theorem 25.2. With d_ij = mu_i - mu_j, for the heat cutoff t = 1/Lambda^2, the kernel is

```
C_t(q^2) = (1/(8 pi^2)) sum_{n in Z} sum_{ij}
 ↪  |T_ij|^2 [ (2/t + d_ij^2 + q^2) I_t((k_n +
 ↪  mu_i)^2, (k_n + mu_j)^2; q^2) - (e^{-t (k_n +
 ↪  mu_i)^2} + e^{-t (k_n + mu_j)^2})/t ],
```

with I_t as in lemma 18. The sum converges, and it holds without expansion in D_F. Expanding in D_F, with C_t = C_0 + q^2 C_1 + ..., and with t = 1, so with k_n = n pi/(l Lambda) dimensionless,

```
C_0 = (1/(8 pi^2)) ||[D_F, T]||^2 sum_n e^{-k_n^2}
 ↪  (1 - (2/3) k_n^2) + O(||D_F||^4),
C_1 = (1/(8 pi^2)) (2/3) ||T||^2 sum_n e^{-k_n^2} +
 ↪  O(||D_F||^2).
```

So, with <k^2> = sum k_n^2 e^{-k_n^2} / sum e^{-k_n^2}, at fixed s and for fixed nonzero T,

```
M^2 = C_0/C_1 = (3/2 - <k^2>) ||[D_F, T]||^2/||T||^2
 ↪  + O(||D_F||^4 / Lambda^2).
```

Proof. V and T split into finite-dimensional blocks indexed by n, so equation (1) of lemma 18 applies to symmetric finite sets of blocks, and the limit is justified by Gaussian bounds, uniform for D_F in a bounded neighborhood of zero. The same bounds justify the Taylor expansion under the sum. Terms odd in k cancel between n and -n: exactly, the bracket at q = 0 is d_ij^2 int_0^1 E(u) [1 - u(1-u)(2k + mu_i + mu_j)^2] du, with E(u) = exp(-(1-u)(k + mu_i)^2 - u (k + mu_j)^2), whose next term is d_ij^2 e^{-k^2} ((2k^3 - 5k)/3)(mu_i + mu_j), and the bracket at order q^2 begins e^{-k^2} (2/3 - (2k/3)(mu_i + mu_j)). For the leading term of the expansion, put a_i = (k + mu_i)^2 = k^2 + p mu_i + mu_i^2 with p = 2k. To second order in mu,

```
I(a_i, a_j) = e^{-k^2} [1 - (p/2)(mu_i + mu_j) -
 ↪  (mu_i^2 + mu_j^2)/2 + (p^2/6)(mu_i^2 + mu_j^2 +
 ↪  mu_i mu_j)],
```

and e^{-a_i} = e^{-k^2} [1 - p mu_i - mu_i^2 + p^2 mu_i^2/2]. Therefore

```
2 I(a_i, a_j) - e^{-a_i} - e^{-a_j} = -(p^2/6)
 ↪  d_ij^2 e^{-k^2},
```

and adding d_ij^2 I gives d_ij^2 e^{-k^2} (1 - p^2/6) = d_ij^2 e^{-k^2} (1 - (2/3) k^2). Finally, sum_ij |T_ij|^2 d_ij^2 = |||[D_F, T]|||^2. For C_1, at D_F = 0 the bracket in equation (3) of lemma 18 has d_ij = 0 and integrates to 2/3. ■

Differentiating directly, <k^2> = (1/2) d log S/d log s = (s/2) d log S/ds, with S(s) = sum_n exp(-n^2 pi^2/s^2). This S is the heat trace of the Dirac operator of the unshifted circle of length 2l at t = 1/Lambda^2, equivalently the paired Neumann and Dirichlet trace on the interval, with S_N = (S + 1)/2 and S_D = (S - 1)/2 separately. So d_eff = d log S/d log s = -2 d log Tr e^{-t D^2}/d log t is its spectral dimension at that scale. For nonzero D_F, the parent trace has e^{-t (k_n + mu_i)^2}, and S is the unperturbed trace used in the leading expansion. Hence, at leading order,

```
M^2 = ((3 - d_eff)/2) ||[D_F, T]||^2/||T||^2.
```

The two limits. We first take the coefficient at small D_F/Lambda at fixed s, and then the limit in s. Explicitly, d_eff ~ (4 pi^2/s^2) exp(-pi^2/s^2) as s -> 0, and d_eff = 1 - 4 s^2 exp(-s^2) + O(s^2 exp(-2 s^2)) as s -> infinity.

1. For s « 1, only n = 0 contributes. Then d_eff = 0, the factor is 3/2, and this is the towerless regime of lemma 18 and Proposition F of the paper.
2. For s » 1, the sum is a Gaussian integral, with <k^2> = 1/2. Then d_eff = 1, and the factor is 1. This is forced by covariance on the unfolded circle: V is the fifth component of a five-dimensional Dirac operator with constant A_5 = D_F, and the local five-dimensional heat coefficient contains F_{MN} F^{MN}, with F_{5 mu} = i [D_F, A_mu] up to convention and with the same coefficient as F_{mu nu}^2. So the mass is the Hilbert-Schmidt commutator norm alone, as at small angle for the Yang-Mills parent of Theorem 23.6. On a general space with boundary, the bulk coefficient alone does not exclude boundary contributions; here the operator with these transported fields is unitarily equivalent to the circle problem, which is the reason they are absent.

In table form, at small masses:

s = l Lambda	0.3	1	1.5	2	2.5	3	5
(3 - d_eff)/2	1.5000	1.4990	1.3935	1.1414	1.0240	1.0022	1.0000

Corollary 25.3 (the sum rule). For the three-generation Standard Model representation, with the literal particle and antiparticle trace, with Dirac masses small compared with Lambda and with 1/l, and with T a charged generator of SU(2)_L, at leading order,

```
M_W^2 = ((3 - d_eff)/24) sum_f N_f m_f^2,
```

which at d_eff = 0 is 8 M_W^2 = sum N_f m_f^2, and at d_eff = 1 is 12 M_W^2 = sum N_f m_f^2. For the top alone,

```
m_t/M_W = (8/(3 - d_eff))^{1/2},
```

from sqrt(8/3) = 1.633 at d_eff = 0 to 2 at d_eff = 1. The neutral direction scales with the same factor, so M_W^2/M_Z^2 = 5/8 at all s to this order, and the massless directions keep zero mass coefficient, while their kinetic coefficients can change. These are leading-order statements: already in the towerless top-only family, M_W^2 = (3/8) m_t^2 - (23/128) m_t^4/Lambda^2 + … (equation (19) of lemma 18).

Proof. The trace gives ||T||^2 proportional to Q_11 = 12 and ||[D_F, T]||^2 proportional to K_11 = sum N_f m_f^2, which gives the coefficient. At d_eff = 0 this is the CCM relation (hep-th/0610241, Eq. (4.16)), as recovered in Proposition 4 and Remark 16 of lemma 18 at small x. The factor is common to all generators in Theorem 25.2. ■

In terms of Yukawa couplings, with m_t = y_t v/sqrt(2) and M_W = g v/2, this reads y_t^2 = 4 g^2/(3 - d_eff), so y_t = 2 g/sqrt(3) at d_eff = 0, the top-dominance value of CCM (Eq. (5.27) there), and y_t = sqrt(2) g at d_eff = 1.

Remark (positive Laplace cutoffs). For f(z) = int e^{-tz} dnu(t) as in equation (4) of lemma 18, with t measured in units of 1/Lambda^2, the same computation gives at leading order

```
M^2 = ((3 - d_nu)/2) ||[D_F, T]||^2/||T||^2,
 ↪  S_nu(s) = int S(s/sqrt(t)) dnu(t),     d_nu = d
 ↪  log S_nu/d log s = 2 <t k^2>_nu,
```

where the average has weights dnu(t) e^{-t k_n^2}. Under the moment hypotheses there, dominated convergence gives the same endpoints, and the crossover and

its corrections depend on nu. A general smooth cutoff, such as one flat near zero, need not belong to this class.

Scope

- This is the Taylor quotient of a prescribed constant-profile vector, at leading order in D_F, about a prescribed background and not a demonstrated vacuum. Taylor and pole masses differ in general. Calling it the leading term of a lowest-mode mass also needs an admissible gauge field, a positive kinetic form and an isolated lowest branch; on the circle, translation invariance separates the zero Fourier sector, within which the generator matrix must still be diagonalized. For finite angles in the Yang-Mills parent, the exact masses are in Theorem 23.6, which expands as M_W^2 = 2 m^2 - (4/3) m^4 l^2 + … and M_Z^2 = (16/5) m^2 - (64/75) m^4 l^2 + …, in agreement with d_eff = 1 at leading order. That theorem is for its own Yang-Mills boundary problem, and it does not establish the full finite-cutoff gauge spectrum of the spectral action.
- T must be even and constant in the transported frame, so it preserves both walls there, and it needs the enlarged gauge algebra noted above. Vectors that are Dirichlet at a strict wall, as in Proposition 22.8, are a different boundary problem.
- All fields have flat profiles on one interval, with no bulk masses. In particular, for the top-only statement, the kinetic term counts all twelve doublets of three generations on the same interval, with the same S. With bulk masses to localize light fields, the spectra and overlaps change, and the kinetic coefficient must be recomputed from all eigenstates and overlaps.
- The spectral action is taken as a function of the fermion operator at a matching scale, with no running.
- For a boundary with reflection, the walls contribute heat-kernel boundary terms in general. Here the unfolding removes them for this operator and these generators.
- Other priority: the endpoint m_t = 2 M_W appears at tree level in Cacciapaglia, Csaki and Park, Fully radiative electroweak symmetry break-

ing, arXiv:hep-ph/0510366, section 5, for a suitable rank-four representation, and a limit approaching 2 M_W in Adachi and Maru, arXiv:1804.06012, Eq. (3.23). These are representation-dependent mechanisms, and they do not give the spectral-action crossover. Universal propagation alone does not impose it in ordinary universal extra dimensions (Cordero-Cid et al., arXiv:1108.2926, section 2).

Reading

This connects three items of the project.

1. *The emergence of a dimension.* The heat dimension of the interval, which distinguishes the labels 10 and 11, enters the mass relation directly. As the cutoff resolves the interval, the coefficient moves from the finite-space value to the five-dimensional value.
2. *Proposition F and lemma 13.* The factor 3/2 is the finite-action coefficient relative to the Yang-Mills commutator quotient, and in this constant-profile calculation the two are the two ends of one function. Proposition F also contains a bounded-fermion against tower-scale obstruction and overlap conditions, which keep their separate hypotheses.
3. *The measured ratio.* With the Particle Data Group 2026 listings, m_t = 172.60 +- 0.27 GeV as a direct-measurement Monte Carlo mass, and M_W = 80.3625 +- 0.0077 GeV, the ratio is 2.148, which would formally correspond to d_eff = 1.27, outside the range. With a running mass of about 163 GeV (Yoon and Peskin, arXiv:1810.12352, section 3.3 and Eq. (23), for a pole mass of 173 GeV, with about 147 GeV at 2 TeV), it is 2.03, which corresponds to d_eff = 1.06. Dividing a running mass by an on-shell W mass is only a formal orientation and not a matching test at a common scale, and no running from the matching scale has been computed, so we draw no conclusion from this comparison. The known spectral-action value y_t = 2 g/sqrt(3) is the d_eff = 0 boundary condition. A tower near the weak scale, as in the separate unrefereed estimate of the steer note, item 9, would use the d_eff = 1 boundary condition y_t = sqrt(2) g, at a matching scale of that order, in a specified

effective theory with thresholds and a renormalization prescription. In a one-loop orientation with universal bulk fields (the steer note, item 11, no referee), the running approaches an infrared quasi-fixed point near 211 to 232 GeV, nearly independently of the boundary value, so that family does not support the closeness of 2 M_W to the running top mass.

Numerical check

By Claude Opus 5.5, in Python (numpy), 3 Nov 2026, in a temporary directory, not kept. The kernel of lemma 18, with quadrature in u, was summed over the tower, with explicit 4 x 4 level blocks for the operator on the interval, independently of the unfolding.

- Random complex K_1 and Hermitian T_Q, T_U on C^3, with ||K|| = 10^-3 min(s, 1): the ratio of C_0/C_1 to the Hilbert-Schmidt quotient agrees with 3/2 - <k^2> to six digits, for s in {0.3, 1, 1.5, 2, 2.5, 3, 5, 10}.
- Three generations, top only, with T = sigma_x/2 on the U-fields: m_t/M_W agrees with (8/(3 - d_eff))^{1/2} to six digits on the same grid, from 1.63299 at s = 0.3 to 2.00000 at s = 5.
- The neutral direction, with weights for lepton and colour: M_Z^2/m^2 goes from 4.801 to 3.200 for universal mass, and M_W^2/M_Z^2 stays at 0.625 to four digits.
- Finite mass in the top-only family: M^2/m^2 at s = 5 is 0.2500, 0.2498, 0.2477, 0.2409, 0.2241, 0.1478 for m/Lambda = 0.02, 0.1, 0.3, 0.6, 1, 2, and at s = 0.5 it is 0.3749, 0.3732, 0.3591, 0.3145, 0.2277, 0.0555, the latter matching lemma 18.

Part 26

The interior vacuum and the matching scale in the flat parent

For STRATEGY T13, T5b and T12, following lemma 23, lemma 24, lemma 25 and the steer note, items 9 and 11. Proved by Claude Opus 5.5; the numerical statements are in Python and each is labelled. Refereed by Terra 6.1

(effort max, 3 Nov 2026; first report): three errors and two gaps as first written, and amendments. Proposition 26.3c and Result 26.8 were added after that report and answer its gap on the wall mass, and a second report corrected one error in the summary. The textual corrections of both are incorporated below, and the calculations they call for are listed under "Outstanding".

Outcome.

1. Pure frequency-2 fermion channels give the flat family an interior global minimum, and we classify it for n_2 >= n_c (Theorem 26.2). With frequency-2 channels alone it sits exactly at theta = pi/4, where M_KK = 4 M_W.
2. In the displayed channels of the multiplets containing Q_L and U_R, a pure channel carries an exact massless coloured exotic, and a wall mass of any finite positive size leaves a coloured state of charge 2/3 below M_W (Proposition 26.3c). So a top at 2 M_W in such a channel has a lighter partner, whose couplings are not analyzed. In E there is no such partner: m_1 = 2 M_W at theta = 1.1377, with M_KK = 444 GeV, and the vacuum then needs a separate sector.
3. In the step-threshold model of the parent, without frequency-2 multiplets, the left-right difference of the couplings runs only logarithmically (Proposition 26.5). The couplings then meet at mu_* = 2.2 x 10^14 GeV, with M_KK = 3.1 x 10^13 GeV (Result 26.6, one loop, conditional on the spectrum).
4. At a low tower, with the hypercharge coupling fitted by a wall kinetic term, the finite interval shifts M_W^2/M_Z^2 below its four-dimensional value by 0.05 at the top-compatible point, and by less than 0.006 only above about 1.2 TeV (Result 26.8, tree level).
5. These give two regimes of one family, stated at the end with hypotheses and escapes.

Setting

We keep the flat interval of length l and the gauge content, couplings and connector wall of Theorem 23.6. A fermion channel is *pure of frequency q* when its two wall gradings differ by a rotation through q theta in the chan-

nel. Its spectrum is +-(n pi + q theta)/l, n in Z, so that its product over the tower, relative to theta = 0, is 1 + sin^2(q theta)/sinh^2(x), with x = p l.

The bimodule E = Hom(V, W) is pure with q = 1 (Proposition 23.7). Let n_q >= 0 be twice the number of physical Dirac species in such channels, each colour counted once. With t = sin^2(theta), u = t (1 - t), and F, J as in Results 8 and 9 of lemma 24, the one-loop potential of the form displayed in lemma 23 is

```
8 pi^2 l^4 V = 3 F(2u) + (3/2) F((16/5) u) - n_1
 ↪  F(t) - n_2 F(4u),        (1)

dV/dt = (1 - 2t) G(u) - n_1 J(t),   G(u) = 6 J(2u) +
 ↪  (24/5) J((16/5) u) - 4 n_2 J(4u),        (2)
```

up to the factor 8 pi^2 l^4. Equation (1) assumes precisely these towers and multiplicities; a full representation can contain others. We use J(0) = (3/2) zeta(3) and J(1) = (9/8) zeta(3), so J(0)/J(1) = 4/3.

Lemma 26.1. Let R(u) = [6 J(2u) + (24/5) J((16/5) u)] / (4 J(4u)) on (0, 1/4]. Then R is strictly increasing, with R(0+) = 27/10 and R(1/4) = n_c, where

```
n_c = [6 J(1/2) + (24/5) J(4/5)] / (4 J(1)) =
 ↪  2.8855033... (numerical quadrature).
```

Proof. For c < 4, d/du log[J(c u)/J(4u)] = [e(4u) - e(c u)]/u, with e(z) = -z J'(z)/J(z). By Result 8 of lemma 24 and the covariance argument in the referee amendment there, which is strictly positive, e is strictly increasing. So both ratios J(c u)/J(4u), for c = 2 and c = 16/5, strictly increase. ■

The classification

Theorem 26.2. Let n_1, n_2 >= 0 in the family (1), with theta in [0, pi/2].

1. *n_1 = 0.* The potential is a function of u.
   - If n_2 <= 27/10, it increases with u: theta = 0 and theta = pi/2 are the two global minima, degenerate at n_2 = 27/10, where the leading term is (99/25) theta^4 log(1/theta^2), and theta = pi/4 is the maximum.
   - If 27/10 < n_2 < n_c, there are exactly two global minima, at theta_* and pi/2 - theta_,

*with theta_* in (0, pi/4), and theta = pi/4 is a local maximum.
    - If n_2 >= n_c, theta = pi/4 is the unique global minimum, with a quartic leading displacement at n_2 = n_c.
2. *n_1 > 0 and n_2 >= n_c.* Let n_* = (4/3)(4 n_2 - 54/5).
    - If n_1 < n_*, there is exactly one stationary point in the open interval. It is the strict global minimum, and it lies in (pi/4, pi/2).
    - If n_1 >= n_, *V decreases strictly on (0, pi/2), with unique global minimum at theta = pi/2, degenerate at n_1 = n_.*

The endpoints are stationary in theta in every case, since dt/d(theta) vanishes there. The range n_1 > 0, n_2 < n_c is not classified here.

Proof. Item 1: dV/du = 4 J(4u) (R(u) - n_2), so Lemma 26.1 gives the three cases. Item 2: since n_2 >= n_c >= R(u), G(u) = -4 J(4u) (n_2 - R(u)) <= 0 on (0, 1/4], with equality only at u = 1/4 and n_2 = n_c. For t <= 1/2, (2) gives dV/dt <= -n_1 J(t) < 0. For t > 1/2 write dV/dt = J(t) [Psi(t) - n_1], with Psi(t) = (2t - 1) |G(u)|/J(t). Here 2t - 1 increases, 1/J(t) increases, and |G(u)| increases with t, since u decreases with t and both factors J(4u) and n_2 - R(u) decrease in u. So Psi increases strictly from Psi(1/2) = 0 to Psi(1) = (4 n_2 - 54/5) J(0)/J(1) = n_*. ■

No numerical computation enters the proof, apart from the value of n_c. A Python check on a grid of 11600 pairs, n_2 from 3 to 60 and n_1 from 0 to 199, with theta on 20001 points, finds the stated stationary points in each case.

For the remaining integer counts: n_2 = 0 is Theorem 9 of lemma 24. For n_1 = 0 and n_2 = 1, 2, Theorem 26.2 gives both endpoints as global minima. For n_1 > 0 and n_2 = 1, 2, a scan of n_1 up to 200 (Python) finds no interior local minimum, which we have not proved. So at integer counts every interior minimum found has theta >= pi/4. The window 27/10 < n_2 < n_c, with its minima at small theta, contains no integer.

Proposition 26.2b (referee; no global minimum at small angle). For n_1 > 0 and 0 < t < 1/2, V(1 - t)

- V(t) = -n_1 [F(1 - t) - F(t)] < 0, so no point of (0, pi/4) is a global minimum. With item 1 of Theorem 26.2 for n_1 = 0, this covers all integer counts. It does not exclude metastable minima at small angle in the cases left to scans. ■

Corollary 26.3 (the spectrum at the minimum). Write g(y) = arcsin(sin(y)/sqrt(2)), so that M_W l = g(2 theta) by Theorem 23.6, for theta in (0, pi/2), where M_W > 0. On [pi/4, pi/2), m_1 l = theta and m_2 l = pi - 2 theta for pure channels.

1. At theta = pi/4, the minimum for n_1 = 0 and n_2 >= n_c: M_KK := pi/l = 4 M_W, m_1 = M_W, m_2 = 2 M_W, tan(M_Z l) = 2 and M_W^2/M_Z^2 = 0.5032…
2. As theta increases from pi/4 to pi/2, m_1/M_W increases from 1 to infinity and m_2/M_W decreases strictly from 2 to sqrt(2).
3. M_KK >= 4 M_W for all theta in (0, pi/2), and M_KK/M_W grows without bound as theta -> pi/2 and as theta -> 0.

Proof. Item 1 follows from Theorem 23.6. For item 2, g'(y) = cos(y)/(1 + cos^2 y)^(1/2) is positive and decreasing on (0, pi/2), so g is concave with g(0) = 0, and y/g(y) increases. With y = pi - 2 theta, this gives m_2/M_W = y/g(y). Item 3 follows from sin^2(M_W l) <= 1/2. ■

Evaluating the minimum (Python, on the root given by the proof, recomputed by the referee), with M_W = 80.36 GeV:

n_1	n_2	theta_*	M_KK (GeV)	m_1/ M_W	m_2/ M_W	M_W^2/ M_Z^2
0	>= 3	0.7854	321	1	2	0.503
6	8	0.941	342	1.27	1.71	0.525
14	8	1.138	444	2.00	1.52	0.576
16	8	1.186	491	2.31	1.50	0.586
16	6	1.446	1437	8.23	1.42	0.621
48	12	1.502	2597	15.5	1.42	0.624

A frequency-1 top with m_1/M_W = 2 needs theta = 1.13774, on the line n_1 = 2.71387 n_2 - 7.71725, with M_KK = 444 GeV. A larger tower needs n_1 close to n_* from below. With even counts this reaches 29 TeV at

(n_1, n_2) = (604, 116), and the frequency-1 fermions then sit near M_KK/2, as in item 2 of Theorem 23.6.

Remark (the quadratic coefficient). The coefficient of theta^2 in (1) is (54/5 - n_1 - 4 n_2) (3/2) zeta(3), which does not vanish for integer n_1, n_2. So in (1) the quadratic coefficient has no exact cancellation. This concerns that coefficient only: F(z) = J(0) z + (1/4) z^2 log z + O(z^2), so a theta^4 log(theta^2) term remains at a tuned small angle. For added bosons the statement needs a stated normalization: a real scalar determinant carries 1/2 and a real vector 3/2, and mixed boundary projections can give other prefactors.

Where frequency 2 lives

With multiplicities for the eigenvalues of F: the 4 has +-1, each twice; the 5 has 0 three times and +-2 once each; the 10 has 0 four times and +-2 three times each. So E = (4, 4bar) has no frequency-2 channel. For a top with Q_L in (2, 2) and U_R in the same multiplet, the charges Y = R_3 + (B - L)/2 require colour triplets with B - L = 4/3 for the 5, smallest in the 15, so (5, 15), of 75 components; or B - L = -2/3 for the 10, smallest in the 6, so (10, 6), of 60 components, with Q_L at R_3 = +1/2 and U_R at R_3 = +1. Also Lambda^2 E = (10, 6) (+) ((1 (+) 5), 10bar). The (5, 15) with light fermions localized is the pattern of Contino, Da Rold and Pomarol.

Proposition 26.3b (the frequency-2 plane does not separate the top). Take the (10, 6), with F the generator J_45 of so(5). The frequency-2 planes are (A^a, O^a), with A^a = (L^a - R^a)/sqrt(2) and O^a in (2, 2). The component O^+ is the left-handed top, with (T_3L, R_3) = (+1/2, +1/2), and its partner A^+ contains the right-handed top R^+ with weight 1/sqrt(2), together with the triplet component L^+. Let phi = 2 theta, A' = cos(phi) A + sin(phi) O, O' = -sin(phi) A + cos(phi) O, and S = (L + R)/sqrt(2). Consider the self-adjoint problem, by the referee: at x = 0, psi_R is Neumann on S, A and Dirichlet on O, and psi_L has the complementary conditions.

1. If at x = l, psi_R is Neumann on S and A' and Dirichlet on O', then (O^+, A^+) is a pure frequency-2 channel, and S_R is an exact massless coloured mode of charge 2/3.

2. If at x = l, psi_R is Dirichlet on O' and on S + A', and Neumann on S - A', the determinant is proportional to cos(a) (2 sin^2 a - sin^2 phi), with a = m l. So the angle-dependent tower obeys sin^2(m l) = sin^2(2 theta)/2, the equation of the charged vectors: m_t = M_W at every theta. There is also an angle-independent half-integer tower.
3. In the (5, 15) the same holds with the chiralities exchanged: the frequency-2 plane contains U_R and (t_L + X)/sqrt(2), with X the exotic of Y = 7/6.

Proof. These are the commutation relations of so(5) and the boundary problem, solved below as the limits of Proposition 26.3c. ■

These statements concern the displayed channel. A gauge-invariant completion needs conditions on complete wall multiplets, and the other modes of the multiplet also enter the potential.

Proposition 26.3c (a wall mass leaves a light state). In the problem above, normalize u = (S + A')/sqrt(2), v = (S - A')/sqrt(2), and let a wall Weyl field chi couple to g_u at x = l with parameter w: the self-adjoint operator is H(f, g, chi) = (-g', f', w g_u(l)), with domain f_u(l) = w chi, g_{O'}(l) = 0, f_v(l) = 0 and the stated conditions at x = 0 (referee). Let omega = w^2 l and a = m l. Eliminating chi for m != 0 gives f_u(l) = (omega/a) g_u(l). With s = sin(2 theta), the nonzero eigenvalues solve

```
2 a Delta sin(a) = omega cos(a) (Delta + sin^2 a),
 ↪  Delta = sin^2 a - s^2,        (3)
```

where omega = 0 gives Delta sin(a) = 0, item 1 above, with the decoupled wall Weyl zero mode counted separately, and omega -> infinity gives cos(a) (2 sin^2 a - s^2) = 0, item 2. For every theta in (0, pi/2) and every omega in (0, infinity), there is an eigenvalue with 0 < m < M_W.

At theta = pi/4 equation (3) factorizes as cos(a) (omega cos(2a) - a sin(2a)) = 0: the state m = pi/(2l) = 2 M_W does not depend on omega, and the other equation has a root in (0, pi/4) for every omega > 0, moving from 0 to M_W l = pi/4.

Proof. On x in [0, l], with f the left and g the right components and the massless bulk operator, f' = m g and g' = -m f. The conditions at 0 give f_O = alpha cos(m x), g_O = -alpha sin(m x), f_A = beta sin(m x), g_A = beta cos(m x), f_S = gamma sin(m x), g_S = gamma cos(m x). These enter three linear conditions at x = l, g_{O'}(l) = 0, f_v(l) = 0 and f_u(l) = (omega/a) g_u(l), and (3) is the vanishing of their determinant, which we use before any division, so that it covers cos(2 theta) = 0 as well. For the root, let u_W = M_W l = arcsin(s/sqrt(2)) <= pi/4. On 0 < a < u_W,

```
omega = 2 a tan(a) (s^2 - sin^2 a)/(s^2 - 2 sin^2 a)
```

is continuous and strictly increasing from 0 to +infinity (referee), and it takes every value in (0, infinity). ■

So in the displayed charge-2/3 channel a state lies below M_W for every finite positive wall mass, and in the infinite-mass limit the lowest root of this isolated channel is M_W. This does not exclude identifying a higher eigenstate with the top: equation (3) has the eigenvalue m = 2 M_W when omega = 4 u_W sin(2 u_W)/(1 + 2 cos(2 u_W)). For example, at theta = 1 this is omega = 2.04, with another state at 0.85 M_W (Python, referee). That higher state has nonzero O_L and R_R components; its physical identification and the couplings of the lighter state need analysis. This compares with the top-mass limits for low representations in flat gauge-Higgs models (Cacciapaglia, Csaki and Park; Adachi and Maru; massless exotics and their wall masses in Hosotani, Oda, Ohnuma and Sakamura). A complete gauge-invariant wall multiplet, other boundary mixings, bulk masses and the other towers need separate analysis, and they can add still lighter states.

The vacuum exists in case 2 as well. Let n_d be twice the number of Dirac species with the product 1 + sin^2(2 theta)/(2 sinh^2 x), in place of n_2. Then G(u) acquires -2 n_d J(2u), and for n_d >= 27/5 we have G <= 0 on (0, 1/4], since J(16u/5) <= J(2u). So with n_1 = 0, theta = pi/4 is the unique global minimum, by the argument of item 1 of Theorem 26.2. One Dirac colour triplet has n_d = 6. A scan (Python) gives minima at 0.8585 and 1.0011 for (n_d, n_1) = (12, 2), (12, 6), and no interior stationary point for n_d = 6 and n_1 = 2, 6. For E with n_1 and such a sector, m_1/M_W = 2 lies on the

line n_d = 5.292 + 0.6857 n_1; for example (n_1, n_d) = (4, 8) gives theta = 1.1423, with m_1/M_W = 2.026. Such a sector has its own state at M_W in that isolated channel.

The tower of the parent in the step-threshold model

Take the conditions of Theorem 23.6 at theta = 0, and let S = mu/M_KK with M_KK = pi/l. A massive vector, with its fifth component and ghosts, contributes -(7/2) tr Q^2 per level, the trace over real components.

generators	conditions (x = 0, x = l)	tower	SU(3)	SU(2)	U(1), GUT normalization
gluons	N, N	n	-21/2		
W_L	N, N	n		-7	
O = (2, 2), Y = +-1/2	D, D	n		-7/2	-21/10
R^+-, Y = +-1	N, D	n - 1/2			-21/5
X = 3 + 3bar, Y = +-2/3	N, D	n - 1/2	-7/2		-28/5
hypercharge combination	N, N	n			0
orthogonal neutral combination	N, D	n - 1/2			0

The totals -21/2 for SU(2) and -14 for SU(3) are -(7/2) C_2 of Sp(2) and SU(4). The Higgs is the zero mode of A_5 along O, counted in the Standard Model coefficients, and the nonzero modes of A_5 are the gauge partners counted in the vector towers. Three complete massless bulk generations of E add F = 8 to each coupling on the integer tower, in the limit theta -> 0; at nonzero angle, frequency-one masses are shifted to (n pi +- theta)/l. With the fermions on the walls, F = 0.

So above M_KK, with b = (41/10, -19/6, -7), we take as a model

```
alpha_i^-1(mu) = alpha_i^-1(M_Z) - (b_i/2 pi)
 ↪  log(mu/M_Z) - (1/2 pi) [ (B_i + F) I_n(S) + C_i
 ↪  I_h(S) ],        (4)

B = (-21/10, -21/2, -21/2),   C = (-49/5, 0, -7/2),
```

where I_n(S) is the sum of log(S/n) over n <= S, and I_h(S) the sum of log(S/(n - 1/2)) over n - 1/2 <= S. For large S, I_n ~ S - (1/2) log(2 pi S) and I_h ~ S - (1/2) log 2, so I_h - I_n ~ (1/2) log S + (1/2) log pi.

The model (4) omits the frequency-2 multiplets used for the vacuum above. A complete five-dimensional Dirac (10, 6) has level contributions (296/15, 24, 40/3), before its threshold splitting and zero-mode subtractions, and these are not a universal F. Boundary kinetic counterterms are also generated by loops and need a renormalization prescription (Carena, Tait and Wagner). So (4) is the running of this spectrum, and of the parent only as far as this spectrum is the parent's.

Proposition 26.4. In (4), the differences of the alpha_i^-1 are independent of F. ■

Proposition 26.5 (the left-right difference in Sp(2) is logarithmic). Define alpha_R^-1 = (5/3) alpha_1^-1 - (2/3) alpha_3^-1, the coupling of SU(2)_R in 1/g_Y^2 = 1/g_R^2 + (2/3)/g_4^2. Then, from (4),

```
alpha_R^-1(mu) - alpha_2^-1(mu) = D_0 - (44/3)/(2
 ↪  pi) log(mu/M_Z) + (7/pi) [I_h(S) - I_n(S)],
 ↪  (5)
```

with D_0 = alpha_R^-1(M_Z) - alpha_2^-1(M_Z), and it has no power of S. With the inputs (59.0, 29.6, 8.47), D_0 = 63.09.

Proof. The zero modes give (5/3)(41/10) + (2/3)(7) + 19/6 = 44/3. The integer tower gives (5/3)(-21/10 + F) - (2/3)(-21/2 + F) - (-21/2 + F) = 14, and the half-integer tower gives (5/3)(-49/5) - (2/3)(-7/2) = -14. ■

The cancellation concerns the difference within Sp(2); differences involving colour keep a power of S. This is the differential running of orbifold models (Nomura, Smith and Weiner). Without the tower the difference vanishes at 5 x 10^13 GeV. With a tower, for S » 1, it vanishes at

```
log(mu_LR/M_Z) = (3/23) [ 2 pi D_0 + 7 log(pi) - 7
 ↪  log(M_KK/M_Z) ].        (6)
```

So at M_KK = 321 GeV, the difference is 61.69 at S = 1 in (5), where I_h = log 2, against 60.15 from the Standard Model extrapolation alone, and by (6) it vanishes near 2 x 10^24 GeV. It lies above the Planck mass, 1.2 x 10^19 GeV, for M_KK below about 2 x 10^8 GeV.

Result 26.6 (the meeting point in (4); Python, one loop). The two conditions alpha_1 = alpha_2 = alpha_3 in (4) have the solution

```
M_KK = 3.15 x 10^13 GeV,   mu_* = 2.21 x 10^14 GeV,
 ↪  S = 7.04,
```

with alpha^-1 = 46.0 for F = 8 and 52.6 for F = 0. It is the only solution with S >= 1 (referee): eliminating log(mu/M_Z) between the left-right equation and the equation for alpha_2 = alpha_3 leaves a continuous function of S whose derivative between thresholds is (315 N_h - 161 N_n)/(44 S) > 0, with N_n, N_h the numbers of levels below S. Shifts of 0.1 in alpha_1^-1 or alpha_2^-1 and of 0.15 in alpha_3^-1, one at a time, move M_KK within 2.9 to 3.4 x 10^13 GeV, and all three together within 2.66 to 3.70 x 10^13 GeV. Larger effects are possible: one additional light colour-triplet Weyl fermion changes alpha_3^-1 by about -1.4 up to that scale, the Standard Model gauge part of the two-loop terms by about (-0.20, -0.27, +0.39) (referee), and finite wall terms and split thresholds are not included. So in this model the relation g_3^2 = g_2^2 = (5/3) g_Y^2 holds at one scale with no added thresholds, and that scale fixes the tower. An interval Pati-Salam model with a compactification threshold of order 10^13 GeV, with a different intermediate group and matter, is in Abe, Goto and Kawamura.

The top against the matching scale

In a flat parent with the Higgs as A_5, the tree-level relation at small angle, for a pure channel of frequency q, is y_q = (q/2) g, since m_q = q theta/l and M_W = sqrt(2) theta/l. The same covariant derivative fixes the local bulk vertices. Physical four-dimensional gauge and Yukawa couplings still receive different finite corrections and thresholds (Lim and Maru). So the absence of separate running holds for the local bulk coupling, in a stated matching scheme. This differs from item 11 of the steer note, where the Higgs is an independent bulk field, y_t runs with its own power law, and the matching sits at the crossings. Item 11 stays correct for that content, and the two calculations have different actions.

At finite angle a mass ratio does not give the Higgs coupling. With v_ang = 2 sqrt(2) theta/(g l), the coupling is

sqrt(2) dm/dv_ang. At theta = pi/4, the pure frequency-2 branches have slopes corresponding to |y| = g and are degenerate, so the coupling is a matrix on that subspace, while the tower with m = M_W has dm/d(theta) = 0 there (referee).

Result 26.7 (illustration; Python, one loop, Standard Model below M_KK). The running mass m_t solving y(m_t) v/sqrt(2) = m_t, from the boundary value y(M_KK) = r g_2(M_KK):

M_KK (GeV)	r = 1/2	r = 1/ sqrt(2)	r = 1	r = 2/ sqrt(3)	r = sqrt(2)
321	63	87	119	136	164
10^4	74	101	136	153	179
10^9	99	129	163	176	195
3.1 x 10^13	114	143	172	183	196

The measured value in the same convention is 163 GeV, and from it y_t/g_2 equals sqrt(2) at 265 GeV, 2/sqrt(3) at 2.8 x 10^5 GeV and 1 at 1.3 x 10^9 GeV. These are boundary-value calculations, and they are predictions of no spectrum: at low M_KK a real parent has other thresholds below M_KK, so Standard Model running there is an illustration. Reading the entry 164 GeV as the pure channel at theta = pi/4 fails for three reasons: the Higgs coupling at that angle is not sqrt(2) g, as above; the channel has its massless exotic and a second fermion at M_KK/2; and Proposition 26.3c gives a lighter partner for any wall mass.

Result 26.8 (the neutral problem with a wall term; Python, tree level). Let the bulk couplings of Sp(2) and SU(4) be separate, and let a wall kinetic term r at x = l act on the hypercharge combination Y-hat = cos(phi) R + sin(phi) B, with cos^2(phi) = alpha_2^-1/(alpha_2^-1 + (2/3) alpha_3^-1). We fix these, as an estimate, by Standard Model running to M_KK, with alpha_Y^-1 = (1 + r/l)(alpha_2^-1 + (2/3) alpha_3^-1). The charged sector is unchanged. At x = 0, S, A, B are Neumann and O is Dirichlet. At x = l, with L_l = (S + A')/sqrt(2), R_l = (S - A')/sqrt(2), p = cos(phi), q = sin(phi): O' = 0, -q R_l + p B = 0, d_x L_l = 0 and

d_x(p R_l + q B) = r m^2 (p R_l + q B). The lowest neutral root a = M_Z l solves, with t = r/l (referee),

```
(1 + t a cot a) sin^2 a = (sin^2(2 theta)/2) (1 +
↪  p^2 + t a cot a),
```

whose r = 0, one-coupling limit is Theorem 23.6, and which our 4 x 4 determinant reproduces.

theta	M_KK (GeV)	m_1/ M_W	M_W^2/ M_Z^2	four-dimensional value at M_KK	r/l
0.7854	321	1.00	0.629	0.762	1.63
1.1377	444	2.00	0.709	0.761	1.60
0.30	615	0.73	0.735	0.759	1.56
1.30	677	3.48	0.739	0.759	1.55
0.15	1199	0.71	0.750	0.756	1.48
0.08	2236	0.71	0.751	0.752	1.42
1.53	4378	26.5	0.749	0.749	1.35

So within this matching prescription the finite interval shifts the ratio by -0.13 at theta = pi/4 and by -0.05 at the theta where m_1/M_W = 2. In the rows of the table the shift falls below 0.006 from about 1.2 TeV, where m_1/M_W is either near 0.71 or above 8, as a numerical statement for this prescription. The four-dimensional number is a reference within this prescription. A physical precision comparison needs the coupling definitions, wavefunction overlaps, threshold matching and radiative corrections of the model, which can enter at the same order as a small shift. We have not done an electroweak fit, and the measured on-shell value, 0.7767, is not comparable with these columns without that.

Two regimes

We state the outcome in the exact scope of (1) and (4): one interval, flat profiles, this gauge content, one loop, no bulk masses. This is one independent review of a statement about one family, and broader consensus on its negative parts remains open. It leaves T13 open.

Low tower. With frequency-2 channels of sufficient count, the one-loop potential has a natural interior minimum, at theta >= pi/4 and M_KK from 321 GeV upward (Theorem 26.2). In E, m_1 = 2 M_W at theta = 1.1377, with M_KK = 444 GeV, and the frequency-2 sector that stabilizes it has its own light states in the displayed channels (Propositions 26.3b, 26.3c). A state at

2 M_W in a frequency-2 channel has a lighter partner, whose couplings are not analyzed. In (5), the left-right difference at the tower is about 60, so if only a hypercharge wall term and separate bulk couplings are admitted, that wall term is of that size in alpha^-1, comparable with twice the bulk value, as an estimate; general non-universal wall kinetic terms are another way to fit the couplings. The weak angle is then an input, and the finite interval shifts M_W^2/M_Z^2 by the amounts of Result 26.8. Whether a tower at this scale is allowed needs its couplings and an applicable experimental limit, which we have not analyzed.

High tower. The relation of one coupling holds in the model (4) at mu_* = 2.2 x 10^14 GeV, with M_KK = 3.15 x 10^13 GeV, and the weak angle runs down from it. With the boundary values of Result 26.7, r = 1 gives a running mass near 172 GeV and r = 1/2 gives 114 GeV. The small-angle relation M_W l = sqrt(2) theta then requires theta = 5.7 x 10^-12, and Proposition 26.2b shows that no such angle is a global minimum of (1) at integer counts. Metastable minima, in the cases left to scans, are not excluded by it.

Escapes. All of these are model-dependent: wall kinetic terms and separate bulk couplings, which give up the relation as a prediction; a wall scalar with its own potential; warping, whose spectra, counterterms and matching the flat sums do not determine; bulk masses; antiperiodic conditions; larger representations with brane mixing; boundary interactions.

Direction. Warping and larger representations are the standard directions for items 2 and 4 of the outcome. In this project the next computations are the same family in a warped interval, with the spectral action of lemma 25 at the level of its heat trace, and the items below.

Outstanding. The low-tower potential (1), the wall-mass spectrum (3), the wall kinetic term of Result 26.8 and the step-threshold running (4) use different simplified spectra, so their conclusions have not been combined into one model. A combined model needs: complete gauge-invariant wall multiplets, with their determinants in (1) and their thresholds in (4); the coupling matrix at finite angle; the couplings of the light partner in (3); and the matching of the neutral sector at one loop. These

calculations are open.

Priority

The sources are listed in LITERATURE, section J. The referee's search covered orbifold differential running, Pati-Salam interval scales, flat large-representation tops, SO(5) vector and adjoint embeddings, exotics and Wilson-line minima, and found no prior occurrence of the classification with n_c or of the pair (M_KK, mu_*) from (4), with forward citation chains incomplete. So priority for those exact statements is open. Hosotani et al. find theta_H = pi/2, corresponding to 2 theta = pi/2, in a warped model with other matter.

Numerical check

By Claude Opus 5.5, in Python (numpy, scipy), 3 Nov 2026, in a temporary directory, not kept; the referee recomputed the items marked in the report.

- J(0) and J(1) by quadrature against the closed forms, to 15 digits. The function e(z) is nondecreasing and R(u) is increasing on grids, with R(0+) = 2.7001 at u = 10^-5 and R(1/4) = 2.88550.
- Theorem 26.2 on the grid described after its proof, with no failure. For n_1 = 0: n_2 = 2.75 has minima at 0.1527 and 1.4180 and a maximum at pi/4, and n_2 = 2.90 has the minimum at pi/4.
- Equation (3) against a direct 3 x 3 determinant of the boundary conditions, for theta = 0.3, pi/4, 1.0, 1.1377 and omega from 10^-3 to 10^4: the lowest roots agree to four digits, and the lowest root lies below M_W l in every case.
- Equations (4), (5): the difference computed directly from (4) agrees with (5) to 13 digits at S = 1000. For S above 10^8 the closed sums lose precision in floating point, so (6) is used there.
- Result 26.6 by nested root finding, over a scan of 1401 values of M_KK. The inputs are those of the steer note, with no thresholds at the top or W.
- Result 26.7: one-loop top Yukawa equation of the steer note with no tower, below M_KK. The asymptotics of I_n, I_h and their difference, against the closed sums at S = 10.3, 100.3, 1000.3, to about 10^-3 or better.

- Result 26.8: the 4 x 4 determinant, with root finding on 6401 points. Its limit r = 0, one coupling against Theorem 23.6 at theta = 0.2, pi/4, 1.06, 1.1377, 1.3.