Supplemental Material to "Explicit masses and the chiral limit in odd reductions of almost-commutative geometries"

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This is the supplement of the paper *Explicit masses and the chiral limit in odd reductions of almost-commutative geometries*. Its parts keep the numbers of the full Supplemental Material, so that a citation of a part in the paper refers to the part of the same name here.

Parts contained. Lemma parts: 17, 27.

The full Supplemental Material, with all parts of the project, exists as `supplemental.pdf` and `supplemental_mobile.pdf` in the same folder as this file. A reference inside a part to a part not listed here points to the full version.

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Part 17

Explicit masses in the reduced triple, the chiral limit and the set of reductions

Routine part of STRATEGY T10. Proved by Claude Opus 5.5 without Astra or Fable. The identities of Theorems 17.1 and 17.6 were checked with numpy on random finite models (section "Numerical check"). Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): of 10 items, 1 correct, 4 needed amendments, and 5 were wrong as first written (Corollary 17.4, Remark 17.7, Proposition 17.8, Proposition 17.9 and the consequence for T10, each by overreach). All corrections are incorporated below. Proposition 17.10 is the referee's.

Outcome. The fermion action of a product M x F, restricted to its physical subspace, is exactly the action of the odd triple M x T on an unconstrained spinor, with the Yukawa block becoming the constant matrix d in D_M + gamma_5 d (Theorem 17.1). A paired Dirac sector gives one Dirac action for either sign s, up to a change of Grassmann variables (Corollary 17.2); other sectors, such as a neutral one, have Majorana-type actions. On a subalgebra commuting with D_F there are no scalar fluctuations, so the mass is explicit (Proposition 17.3). At M = 0 the symmetry of the quark sector is U(N_f) x U(N_f) (Proposition 17.5). The admissible involutions of the quark sector are the Sigma in U(N_f) with Sigma* M Hermitian, which are exactly the stationary points of the mass term Re tr(M Sigma*); at M = 0 they form a homogeneous space of the chiral group (Theorem 17.6). All of this is kinematic. A pion kinetic term does not come from the fixed spectral action (Proposition 17.10). The condensate requires a measure on gauge fields, infinite volume and an order of limits (Proposition 17.8).

"D = 9" below is the formal KO-label 4 + 5 of Theorem D. The metric dimension of M x T is 4. The label exists only where an involution of sign s = -1 exists: charged paired sectors have one, and the strict light

spectrum with one or three unpaired neutrinos has none, since its H_0 has odd dimension (lemma 06, Result 7).

Setting

Let M be a compact Riemannian spin 4-manifold with (D_M, J_M, gamma_5), where J_M^2 = -1, J_M D_M = D_M J_M, J_M gamma_5 = gamma_5 J_M and gamma_5 D_M = -D_M gamma_5. Let F = (A_F, H_F, D_F, J_F, gamma_F) be a finite real triple with the KO-6 signs. On L^2(S) (x) H_F put

```
D = D_M (x) 1 + gamma_5 (x) D_F,     J = J_M (x) J_F,     Gamma = gamma_5 (x) gamma_F,
```

with J^2 = -1, J D = D J and J Gamma = -Gamma J. Let H^+ be the Gamma = +1 subspace. The fermion action is (1/2) <J xi, D xi> on Grassmann variables xi in H^+ (Chamseddine-Connes-Marcolli 2007).

Let U be an admissible involution in the sense of Theorem D: U^2 = 1, U unitary, U gamma_F = -gamma_F U, [U, a] = [U, a°] = [U, D_F] = 0 and U J_F = s J_F U. On H_0 = {gamma_F = +1} put d = U D_F and j = U J_F, restricted to H_0, so that j^2 = s and j d = s d j.

The reduced action

Theorem 17.1. Define R : L^2(S) (x) H_0 -> H^+ by R(phi (x) x) = phi_+ (x) x + phi_- (x) U x, with phi_+- the chiral parts of phi. Then R is unitary, and

```
<J R psi, D R psi'> = <jj psi, DD psi'>,     DD = D_M (x) 1 + gamma_5 (x) d,
```

where jj = J_M (x) j for s = +1 and jj = gamma_5 J_M (x) j for s = -1, with jj^2 = -s and jj DD = s DD jj. Hence the action is (1/2) <jj psi, DD psi> on an unconstrained Grassmann spinor psi. Moreover DD^2 = D_M^2 (x) 1 + 1 (x) d^2, and if f(D^2/Lambda^2) is trace class, Tr f(D^2/Lambda^2) = 2 Tr f(DD^2/Lambda^2).

Proof. R is unitary because U exchanges the two gamma_F-eigenspaces. Let UU = 1 (x) U, which commutes with D and is an involution, so that <J xi, D xi'> = <UU J xi, UU D xi'>.

Put r x = (x + U x)/sqrt(2). Then D_F r = r d, since D_F r x = (D_F x + U D_F x)/sqrt(2) with U D_F x = d x and D_F x = U d x. Also (1 + UU) R/sqrt(2) = 1 (x) r. Since UU commutes with D, this gives UU D R = R DD.

For the real structure, J_F x = U j x and U J_F U = s J_F. As J_M preserves chirality,

```
UU J R(phi (x) x) = J_M phi_+ (x) j x + s J_M phi_- (x) U j x = R((J_M phi_+ + s J_M phi_-) (x) j
↪  x) = R jj (phi (x) x).
```

Therefore <J R psi, D R psi'> = <UU J R psi, UU D R psi'> = <R jj psi, R DD psi'> = <jj psi, DD psi'>. The square follows from gamma_5 D_M = -D_M gamma_5. Finally, D is the direct sum of its restrictions to the eigenspaces of UU, on which it is equivalent to D_M + gamma_5 d and D_M - gamma_5 d, and conjugation by gamma_5 sends the first to minus the second, so their squares are unitarily equivalent. ■

So the doubling is removed by the chirality projection, and the physical space is exactly the Hilbert space of M x T.

Corollary 17.2 (a paired sector gives one Dirac fermion, for both signs). Let H_0 = span(e_L, conj(e_R)) with m > 0 and psi = (psi_1, psi_2). For (d_+, j_+) = (m I, sigma_1 K) and (d_-, j_-) = (diag(m, -m), [[0, 1], [-1, 0]] K) of lemma 06, the action is

```
S_+ = <J_M psi_2, (D_M + gamma_5 m) psi_1>     and     S_- = <gamma_5 J_M psi_2, (D_M + gamma_5 m)
↪  psi_1>
```

respectively, with S_-(psi_1, psi_2) = S_+(psi_1, gamma_5 psi_2). After the charge-conjugation matrix is absorbed, the left argument is an independent Euclidean variable psi-bar, in the Grassmann-bilinear sense of Chamseddine-Connes-Marcolli, section 4. With R_5 = exp(i pi gamma_5/4), R_5 (D_M + gamma_5 m) R_5 = D_M + i m. So the action is that of one Dirac fermion of explicit mass m, whichever sign s is used, up to the displayed invertible change of Grassmann variables.

Proof. The total form is antisymmetric, so on Grassmann variables the two cross terms are equal, which cancels the factor 1/2. The rotation uses R_5 D_M R_5 = D_M. ■

This agrees with Theorem D, item 5: at the level of the classical fermion action, s labels two coordinate descriptions of one bilinear form, and no anomaly or theta shift follows from changing s alone. The rotation R_5 is a formal chiral change of both independent variables; in a quantized theory its Jacobian must be included.

An admissible reduction need not have this paired form. For gamma_F = sigma_3, J_F = sigma_1 K, D_F = m sigma_1 and U = sigma_1 we have s = +1, H_0 is one-dimensional, and the action is the Majorana-type (1/2) <J_M psi, (D_M + gamma_5 m) psi>. The neutral sector of the strict light spectrum has this form with m = 0.

Proposition 17.3 (no Higgs on a commuting subalgebra). Let B in A_F be a unital subalgebra with [D_F, B] = 0, and let U commute with B and B°. The inner fluctuation of D by C^infinity(M) (x) B contains only Clifford one-forms, D_A commutes with UU, and D_A reduces to

```
DD_A = D_M(A) + gamma_5 (x) d,
```

with d constant on M. If [D_F, B] != 0, the same reduction holds, and the fluctuation permits a finite scalar field d_A(x) in place of d.

Proof. For a = f (x) alpha and b = g (x) beta, we have a [D, b] = f [D_M, g] (x) alpha beta + f g gamma_5 (x) alpha [D_F, beta], whose second term vanishes. The term J A J^-1 is a Clifford one-form whose finite coefficients lie in B°. Since U commutes with B and with B°, Theorem 17.1 applies pointwise with D_M(A) in place of D_M. In the other case, U commutes with [D_F, beta], so the fluctuation preserves the reduction. ■

Corollary 17.4 (Yukawa couplings to explicit masses). In the representation of lemma 06, Result 5, with B = {(lambda, Q_lambda, m)} = C + M_3(C), u_L and u_R carry the same left character lambda and the same opposite colour action, d_L and d_R carry the left character conj(lambda) and the same opposite colour action, and e_L and e_R pair as well. So the kept mass blocks commute with B, in the strict projection of lemma 06 and also when only a vectorlike top pair is removed. Hence DD_A = D_M(A) + gamma_5 d, with d the direct sum of the kept mass matrices, and no Higgs field.

Two qualifications apply. First, U(B) contains U(1) x U(3), so before unimodularity or an inherited restriction from the Standard Model gauge group, two independent abelian gauge fields act on the surviving charged fermions; the reading "photon and gluons" needs that restriction. Second, a canonically normalized radial field couples to a fermion of fixed mass m_f with strength m_f/v, which tends to 0 if v tends to infinity at fixed gauge coupling. Under the CCM relation (lemma 11), keeping all Dirac masses fixed while M_W tends to infinity is impossible: some Dirac mass must become unbounded. It need not be the top; top dominance is a further hypothesis. The relation holds in the asymptotic matching regime, and the regime near the cutoff is T5b.

The chiral limit

Consider N_f quark flavours, with q_L, q_R in C^{N_f} (x) C^3, and q-bar_L, q-bar_R = J q_L, J q_R. Let D_F act on (q_L, q_R) by [[0, M], [M*, 0]] (x) 1_3, and by the complex conjugate block on the antiparticles. Let M_3(C) act on the antiparticles on the left and on the particles through its opposite action.

Proposition 17.5 (the symmetry restored at M = 0). Let G(M) be the group of unitaries on this sector commuting with both actions of M_3(C), gamma_F, J_F and D_F. Then

```
G(M) = {(V_L, V_R) in U(N_f) x U(N_f) : V_L M = M V_R},
```

acting by V_L, V_R on q_L, q_R and by the complex conjugates on q-bar_L, q-bar_R. So G(0) = U(N_f) x U(N_f), G(m I) = U(N_f)_V for m != 0, and G(M) = U(1)^{N_f}_V for M positive diagonal with distinct entries. With m_u = m_d = 0 and m_s != 0 in one block, G contains U(2) x U(2). If V is

also required to commute with the electric charge, through lambda in B, then each V_L, V_R is diagonal between u and d, and G(0) = U(1)^4 for two flavours.

Proof. The colour identity separates particles from antiparticles, since its left action is zero on particles and the identity on antiparticles. The commutant of the colour matrix algebra is scalar on colour, so V = V' (x) 1_3 on each. The grading separates L and R, and J_F fixes the antiparticle part. Commutation with D_F gives V_L M = M V_R, whose adjoint with unitarity gives the other block equation. ■

The nonabelian flavour transformations are not colour gauge transformations, and at M = 0 the axial part is an ungauged symmetry of the reduced triple. If the central U(1) of U(3) is retained in the gauge group, it contains a common vector phase, which an imposed unimodularity condition can remove. The Lagrangian symmetry U(1)_A is anomalous in the quantum theory, which the classical data do not show.

Theorem 17.6 (the set of reductions). Let B contain only the colour algebra. The admissible involutions of sign s on this sector are exactly the operators, tensored with 1_3,

```
U_{Sigma,s} = [[0, Sigma], [Sigma\*, 0]] on (q_L, q_R),     s [[0, conj(Sigma)], [Sigma^T, 0]] on
↪  (q-bar_L, q-bar_R),
```

with Sigma in U(N_f) and Sigma* M Hermitian. This condition holds if and only if Sigma is a stationary point of V_M(Sigma) = -Re tr(M Sigma*) on U(N_f). The reduced mass operator on H_0 = (q_L, q-bar_R) is

```
d_{Sigma,s} = (Sigma M\* (x) 1_3) (+) (s conj(Sigma\* M) (x) 1_3).
```

The group G(0) acts by Sigma -> V_L Sigma V_R*. Hence:

1. At M = 0 the set is all of U(N_f), for each s, and the action is transitive with stabilizer of Sigma = 1 the diagonal U(N_f): the reductions form the homogeneous space (U(N_f) x U(N_f))/U(N_f)_V.
2. At M = m I with m > 0 it is {Sigma : Sigma^2 = 1, Sigma* = Sigma}, the disjoint union of the Grassmannians of C^{N_f}.
3. For M positive diagonal with distinct entries, it is the 2^{N_f} matrices diag(+-1, …, +-1).
4. For M invertible, the unique Sigma with Sigma* M positive is the polar part of M. It is the unique global minimum of V_M, and it is the unique Sigma for which the first block of d_{Sigma,s} is positive.

With the electric charge in B, the admissible set is the intersection of this set with the commutant of the charge. For two differently charged flavours this keeps diagonal phases at M = 0, and only signs +-1 for positive nonzero diagonal masses.

Proof. An admissible U commutes with both colour actions, so it acts on each of q, q-bar as U' (x) 1_3; it reverses the grading, so it is off-diagonal between L and R; U^2 = 1 with U unitary gives the form [[0, Sigma], [Sigma*, 0]] with Sigma unitary. Then U J_F = s J_F U fixes the antiparticle block. Commutation with D_F is Sigma M* = M Sigma* and Sigma* M = M* Sigma. If H = Sigma* M is Hermitian, then Sigma M* = Sigma H Sigma* is Hermitian, so the two are equivalent. Along Sigma exp(i t X), X Hermitian, the derivative of Re tr(M Sigma*) at t = 0 is Im tr(H X), which vanishes for all Hermitian X exactly when H is Hermitian. The formula for d is direct, using q-bar_R = J q_R. Item 1 is immediate. For item 2 the condition is Sigma* = Sigma. For item 3, H^2 = M* M forces H to be diagonal with entries +- m_i. Item 4 is von Neumann's trace inequality, Re tr(M Sigma*) <= tr |M|, with equality exactly at the polar part, and the first block is Sigma H Sigma*. ■

Remark 17.7 (scope). Item 1 is an equivariant statement: the reductions form a homogeneous space of the chiral group, with the same stabilizer as in the symmetry breaking of QCD. The mass condition has an elementary origin, since it says that H = Sigma* M is Hermitian, and for invertible M the positive choice is the polar decomposition. Its equality with the stationarity equation of -Re tr(M Sigma*) is exact, and it captures all stationary points, including saddles and maxima. Vacuum alignment (Dashen) asks which stationary point minimizes a physical energy, and the data here contain no such energy; the stationarity calculation is our derivation, and Dashen is cited for the vacuum-selection idea alone.

So the theorem shows the shape of the set of reductions. Identifying a reduction with a condensate direction would need an explicit map to a quantum order parameter, which we do not have. In QCD, the anomalous

singlet axial direction also means that the full classical U(N_f) cannot be identified without qualification with the pion manifold.

For invertible M, every stationary point other than the polar part has at least one negative eigenvalue of Sigma* M; for singular M there may be a continuum of minima, as for M = diag(1, 0) and Sigma = diag(1, exp(i theta)). A negative entry in the reduced mass block comes from choosing another reduction of one fixed even triple, and is not a physical mass-sign change at fixed fermion measure. A physical sign change is different: for one Dirac flavour in a background with n_0 zero modes, the determinants at m and -m differ by the factor (-1)^{n_0}, the factor associated with a shift of theta by pi.

Spacetime-dependent Sigma

Proposition 17.10 (referee; no kinetic term from the fixed action). Let U(x) be a pointwise admissible involution on M, with c^mu = [D_M, x^mu] and B_mu = (U d_mu U) restricted to H_0. With R defined pointwise,

```
R^-1 UU D R = D_M + c^mu P_- B_mu + gamma_5 d(x),
```

where P_- = (1 - gamma_5)/2. This operator is generally not self-adjoint. So:

1. If Sigma(x) only changes coordinates for the fixed operator D, the spectral action of D does not depend on Sigma, and conjugating D by a smooth unitary preserves its spectrum. Neither produces a two-derivative term.

2. A kinetic term needs an additional choice of operator. For D_new = D_M + gamma_5 d_Sigma(x) on flat M, with M = m I and Sigma* = Sigma, Sigma^2 = 1, the heat expansion of Tr f(D_new^2/Lambda^2) contains

   ```
   (N_c f(0) m^2/(4 pi^2)) int tr(d_mu Sigma\* d^mu Sigma),
   ```

   tracing over both reduced blocks. The sign is positive when f(0) > 0. At M = 0 this operator has no dependence on Sigma.

Proof. The first display is direct differentiation in Theorem 17.1. Since B_mu is anti-Hermitian and P_- c^mu = c^mu P_+, the derivative term has adjoint c^mu P_+ B_mu. For the coefficient, D_new^2 = -(nabla^2 + E) with E containing -i gamma^mu gamma_5 d_mu d, and tr over spinors of the square of that term is 4 tr(d_mu d d^mu d); then the a_4 coefficient (1/16 pi^2) tr(E^2/2) gives (1/(8 pi^2)) tr_{H_0}(d_mu d d^mu d), and the two blocks of d give 2 N_c m^2 tr(d_mu Sigma d^mu Sigma). ■

So the idea of a pion as a spacetime-dependent reduction, in the form we first proposed, does not produce a kinetic term: the kinetic term needs a nonzero m and a new operator, or a quantum effective action. On the even triple, D_M + gamma_5 S(x) with S built from Phi = m Sigma(x) is self-adjoint for every unitary Sigma. Its kinetic term is the Higgs-type term of an inner fluctuation, available when A_F contains flavour algebras acting on L and R, and then flavour gauge fields accompany it. The coefficient is proportional to N_c m^2, as in constituent-quark estimates of f_pi^2 (quoted from memory); here it is a classical spectral-action coefficient, and m is not derived.

What the spectral data say about the condensate

Proposition 17.8. Let D_M(A) have eigenvalues lambda_n, with n_0 zero modes, on M of volume V, and for m > 0 put

```
c_V(m; A) = (1/V) sum_n m/(lambda_n^2 + m^2) = (1/V) d/dm log |det (D_M(A) + gamma_5 m)|,
```

which is formal on a compact four-manifold and needs ultraviolet regularization, with subtraction of divergent terms including m Lambda^2, m^3 log Lambda and possibly curvature terms.

1. With a subtraction compatible with the pairing +-lambda whose finite remainder is O(m), for example odd and analytic in m, at fixed A on compact M, c_V(m; A) = n_0/(V m) + O(m) as m -> 0. A subtraction merely regular at m = 0 may leave a constant. So at finite volume and fixed background, no nonzero limit exists beyond the zero-mode term.

2. Let rho_m(lambda) = lim_{V -> infinity} <(1/V) sum_n delta(lambda - lambda_n)>*m for a measure* <.>m on gauge fields. Assume that the family is equicontinuous at 0 with rho_m(0) -> rho_0(0), that the tails and the ultraviolet subtraction are controlled uniformly in m, and that the infinite-volume limit may be taken inside the integral. Then lim*{m -> 0}* *lim*{V -> infinity} = pi rho_0(0). This has the form of the Banks-Casher relation as stated by Giusti and Lüscher (the original, Nucl. Phys. B 169, eqs. (49), (50), has neither the factor pi nor an order of limits with the volume); the sign and flavour normalization of a condensate depend on convention, and the hypotheses here are ours.

Equicontinuity alone is insufficient: rho_m(lambda) = (1 + (1/2) sin(1/m)) exp(-lambda^2) gives no limit.

Proof. Item 1: for a cutoff preserving the pairs +-lambda, the determinant has absolute value m^{n_0} times the product of (lambda^2 + m^2) over lambda > 0, and the nonzero eigenvalues are bounded away from 0. Item 2: m/(lambda^2 + m^2) is pi times the Poisson kernel at scale m. ■

So the spectral triple supplies the operator D_M + gamma_5 m, the density rho_A of each fixed operator, and the continuity of m -> 0 at fixed background, since DD^2 = D_M^2 + m^2. The gauge-averaged density, the regulated determinant or Pfaffian measure, the sequence of volumes and the order of limits need further input. A spectral action may contribute to a proposed gauge weight, but a single compact spectral triple does not specify the functional integral or the thermodynamic limit.

A proof of existence of QCD with fermions would supply a measure. It would not by itself give rho_0(0) > 0. The points to check when it appears are those in STRATEGY T10: whether m_q = 0 is included, and whether infinite volume is reached. With the anti-self-adjoint operator slashed-D = -i D_M and a compatible regulator, det(slashed-D + m) = m^{n_0} times the product over lambda > 0 of (lambda^2 + m^2), which is positive for m > 0; this positivity is part of the method of Vafa and Witten.

The chiral limit on the interval

Proposition 17.9. In the scalar-algebra realization F_0 x I_l of Theorem 08.7, DD = D_M (x) 1 + gamma_5 (x) (D_0 + m Gamma_0), so the interval mass is an explicit mass term with the internal grading. The limit m -> 0 has two distinct routes:

1. *Localization (lemma 09).* With bulk mass -mu and mu l > 1, m = lambda_0 = mu sech(kappa_0 l), of order 2 mu exp(-mu l), which is positive at finite parameters. At fixed mu > 0, l -> infinity gives m -> 0 as an exponentially small residue. The relevant control is the dimensionless suppression: an absolute limit needs a scaling hypothesis, since mu = 2/l gives m -> 0 at fixed mu l, and mu l -> infinity does not ensure m -> 0 if mu grows fast enough. The localization and overlap mechanism is that of domain-wall fermions (Kaplan; Shamir), though this is neither lattice operator.
2. *Boundary Yukawa (lemma 12).* At X = 0 the zero modes are exact (Theorem 12.2), with boundary types (a) and (b). For flavour-degenerate bulk masses, D_0 commutes with U(N_Q) x U(N_U) on the whole tower, and at any l the boundary coupling X preserves exactly the stabilizer {(V_Q, V_U) : V_Q X = X V_U}. So X acts as a spurion located on the wall x = l, and the symmetry is exact at finite l when X = 0. In the towerless limit X = l Y_hat, M_F is proportional to X, and for N_Q = N_U the stabilizer is G(M_F) of Proposition 17.5. At finite l, M_F is not proportional to X: for one flavour with zero bulk masses, the light eigenvalue is arctan(abs(X))/l.

The boundary-coupled operator of route 2 is a different construction from the tensor-product truncation of Theorem 08.7, and an identification of the two at finite l would need a proof.

Proof. Route 1 is Theorem 9.4 and Lemma 9.5, and the scaling examples are direct. Route 2 follows from the invariance of the domain of D_X (lemma 12), Theorem 12.7 and Example 12.5. ■

Caveats

1. **Classical data.** Everything here is at the level of the classical fermion operator. Anomalies, the running of masses and confinement do not appear.
2. **QED.** With the photon present, the axial group reduces as stated in Proposition 17.5, and the set of reductions reduces as stated in Theorem 17.6.

3. **Regime.** Corollary 17.4 inherits the regime of the CCM relation.
4. **Label.** A reduction is a rewriting of one even triple. It gives no parent nine-dimensional geometry, in agreement with Theorem D, and the KO-5 label exists only where an s = -1 involution exists.

Consequence for T10

The reduced action is (1/2) <jj psi, (D_M(A) + gamma_5 d) psi> on an unconstrained spinor, whenever the chosen admissible involution exists. A paired charged sector gives one Dirac action of explicit mass for either sign, up to a change of variables; a neutral unpaired sector gives a Majorana-type action and has only the KO-7 reduction, so the formal label 9 is available only for sectors with the required s = -1 involution. The symmetry restored at m_u = m_d = 0 is U(2) x U(2) on the colour-only algebra, and the reductions there form the homogeneous space U(2), with the mass condition picking out the stationary points of the mass term. The spectral data state the operator, the symmetry, this homogeneous space and the density of each fixed operator. The condensate and the pions as dynamical objects need the QCD measure and a quantum effective action.

Numerical check

Python (numpy, scipy), run on 3 Nov 2026 in a temporary directory, not kept. On a random finite model with KO-4 signs for M (dimension 8) and a quark sector with N_f = 3 and random complex M, for three sign patterns of Sigma and both s: U^2 = 1, U gamma_F = -gamma_F U, [U, D_F] = 0 and U J_F = s J_F U hold to machine precision; R is unitary; the form <J R psi, D R psi'> equals <jj psi, DD psi'> to about 10^-14 and is antisymmetric; j^2 = s and j d = s d j; the formula for d_{Sigma,s} holds, after a first version of it was found wrong by this check; finite differences confirm that V_M is stationary at these Sigma; a generic unitary Sigma gives norm([U, D_F]) of about 10. R_5 (D_M + gamma_5 m) R_5 = D_M + i m holds to machine precision. The spin trace in Proposition 17.10 was evaluated with explicit Dirac matrices (tr of the square equals 4 |a|^2).

Part 27

The axial anomaly on the reduced action, and the two-flavour discrete term

For STRATEGY T10 and T13, following lemma 17 and open question 6 of the steer note. Proved by Claude Opus 5.5, using the index theorem and Witten 1982 as references. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): two errors of interpretation, in Propositions 27.4 and 27.5, and amendments, all incorporated below.

Outcome.

1. On the reduced operator D_M + gamma_5 d, with masses commuting with the gauge field, the regulated axial trace has an exact form at every cutoff: each flavour block contributes its index, weighted by f(m^2/Lambda^2) (Theorem 27.1). The regulated mass term equals it identically (Proposition 27.3). As Lambda -> infinity this is the anomaly with the coefficient 2 N_c tr(T^3 Q^2) = 1, on the action of Theorem 17.1 (Corollary 27.2).
2. The operator of Proposition 17.10 at M = 0, in independent quark and antiquark variables, is the massless Dirac operator with a pure-gauge connection on one chirality. So for two flavours and a reduction Sigma : S^4 -> SU(2) in the nontrivial class, the spectral action is unchanged, and the Weyl Pfaffian transported along connections from 0 to that pure gauge has the relative sign (-1)^{N_c} (Proposition 27.5). The same sign appears in Witten's anomaly, in the discrete term of the pion sigma model, and in the mod-two index on S^5 of the research state. The sigma field represents a class in the reduced spin bordism of SU(2) in dimension four, and the anomaly a class in Omega_5^Spin(BSU(2)); these agree through the anomaly comparison and are distinct groups. The corpus S^5 is an internal space, and the S^5 here is the compactified S^4 x R of an auxiliary flow.

Setting

Let M be a closed Riemannian spin 4-manifold. Following Corollary 17.2, a paired sector gives one Dirac fermion: we take the operator

```
DD = D_M(A) + gamma_5 d,   d = diag(m_f) on E = C^{N_c} (x) C^{N_f},   m_f >= 0,
```

on L^2(S) (x) E, with action psi-bar DD psi, and with A the gauge connection of the commuting subalgebra: colour and electromagnetism, so that A is diagonal in the flavour index, with charge Q_f on flavour f. For topologically nontrivial backgrounds, E is replaced by the colour and electromagnetic bundles E_f, with charges normalized so that the flavour representations are globally defined. Let T be a Hermitian matrix on C^{N_f}, diagonal in the same basis, with eigenvalues tau_f. Then d and T commute with A, and Theorem 17.1 gives DD^2 = D_M(A)^2 + d^2.

Let f be a cutoff with f(DD^2/Lambda^2) trace class, for example in the positive Laplace class of lemma 18. Choose gamma_5 = +1 on S^+, and let ind_f = n_{f,+} - n_{f,-} be the index of D_f^+ : S^+ (x) E_f -> S^- (x) E_f.

The rotation psi -> exp(i alpha gamma_5 T) psi, psi-bar -> psi-bar exp(i alpha gamma_5 T) sends DD to D_M + gamma_5 d exp(2 i alpha gamma_5 T), since gamma_5 anticommutes with D_M. So the classical action is invariant at d = 0. Its Jacobian, regulated by the spectrum of DD, is exp(-2 i alpha Tr(gamma_5 T f(DD^2/Lambda^2))), up to the orientation convention.

The axial trace

Theorem 27.1. For every Lambda > 0,

```
Tr(gamma_5 T f(DD^2/Lambda^2)) = sum_f tau_f f(m_f^2/Lambda^2) ind_f.
```

Proof. On block f, DD^2 = D_f^2 + m_f^2, with D_f = D_M(A) on that block. The operator D_f anticommutes with gamma_5, so it maps the eigenspace of D_f^2 with eigenvalue mu > 0 isomorphically between the two chiralities. So these eigenspaces contribute zero to the trace with gamma_5, and only ker D_f remains, with weight f(m_f^2/Lambda^2) and signed dimension ind_f. ■

No expansion in m/Lambda enters. By the index theorem,

```
ind_f = (1/8 pi^2) int tr_f(F wedge F) - (N_c/24) p_1(M),
```

with F Hermitian, and with F = Q_f e F_em + g G on block f, tr_f(F wedge F) = N_c Q_f^2 e^2 F_em wedge F_em + g^2 tr_c(G wedge G), since G is traceless. Reversing the chirality convention reverses the index.

Corollary 27.2 (the anomaly on the reduced action). Let f be continuous at 0, and let Lambda -> infinity at fixed masses. If tr T = 0, with the flavours in the same colour bundle, then

```
2 Tr(gamma_5 T f(DD^2/Lambda^2)) -> f(0) 2 N_c tr(T Q^2) (e^2/8 pi^2) int F_em wedge F_em,
```

with no gluonic and no gravitational term. For two flavours with T = T^3 and Q = diag(2/3, -1/3), the coefficient is 2 N_c tr(T^3 Q^2) = 1, so that the right side is f(0) (e^2/16 pi^2) int F_em,mu nu F-tilde_em^mu nu, with F-tilde^mu nu = (1/2) epsilon^mu nu rho sigma F_rho sigma. A singlet T = 1 on N_f flavours gives in addition the gluonic term f(0) 2 N_f (g^2/8 pi^2) int tr_c(G wedge G) and the gravitational term -f(0) (N_c N_f/12) p_1(M). ■

The normalization f(0) = 1 is the usual convention for a regulator, and it is the value that makes the Jacobian the anomaly. For a smooth cutoff class in which the heat expansion can be used, with E and Omega defined by the Laplace-type square, the local form, with a smooth function chi on M inserted, follows at Lambda -> infinity from the a_4 coefficient, (1/16 pi^2) tr(gamma_5 T (E^2/2 + Omega^2/12)), which gives the integrands above. The other a_4 terms and the lower coefficients have vanishing gamma_5 spin trace in this problem, and the factor exp(-t d^2) commutes with the heat kernel and changes only higher orders in t. So the algebraic coefficient quoted in the steer note, item 6, now has the regulated trace and measure that it lacked.

Proposition 27.3 (the mass term equals it at every cutoff). Let m_f > 0 for all f. Then

```
Tr(f(DD^2/Lambda^2) d T DD^-1) = Tr(gamma_5 T f(DD^2/Lambda^2)) = sum_f tau_f f(m_f^2/Lambda^2)
↪ ind_f.
```

Proof. DD^-1 = (D_M + gamma_5 d)(D_M^2 + d^2)^-1. The term with D_M has zero trace, by conjugation with gamma_5, since all other factors commute with gamma_5. The remaining term is Tr(f gamma_5 T d^2 (D_M^2 + d^2)^-1). On the nonzero spectrum of D_f it vanishes by the pairing of Theorem 27.1, and on ker D_f the factor m_f^2/m_f^2 is 1. ■

Since {gamma_5 T, D_M + gamma_5 d} = 2 d T, the insertion is psi-bar d T psi in this mass convention, pseudoscalar in a conventional scalar-mass basis. So if the Jacobian and the composite mass insertion are given the same specified regularization, this is the integrated Ward identity in a fixed background at every Lambda, with a common factor f(m_f^2/Lambda^2) on both sides. The trace identity alone constructs neither an independently regulated measure nor a local current, and the local identity has a current-divergence term as well. The surviving zero modes contribute the signed count n_{f,+} - n_{f,-}, whereas the scalar density of Proposition 17.8 sees the unsigned count n_0 = n_{f,+} + n_{f,-}.

Proposition 27.4 (a heavy flavour at fixed cutoff). With the weights of Theorem 27.1, for T^3 on a doublet (t, b) with m_b/Lambda -> 0, the electromagnetic coefficient is

```
2 N_c (1/2) (Q_t^2 f(x) - Q_b^2 f(0)) / f(0),   x = m_t^2/Lambda^2.
```

It equals 1 at x = 0. For the heat cutoff it is (4 exp(-x) - 1)/3, which tends to -N_c Q_b^2 = -1/3 as x -> infinity, with gluonic and gravitational regulated traces proportional to f(x) - f(0). ■

This is a trace in the fixed-cutoff family of lemma 18, where the top is heavy at fixed Lambda. An explicitly massive flavour breaks the axial symmetry, so this calculation alone does not establish a Wess-Zumino term in the low-energy theory. Such a term can be required once the retained symmetry, its background fields or spurions, and the anomaly-matching prescription are specified, as in D'Hoker and Farhi (Nucl. Phys. B 248 (1984) 59 and 77, named in the steer note, and not read here). The weights depend on the chosen regulator, and Proposition 27.3 carries them on both sides of the Ward identity. Making f part of a finite-scale spectral action, as in lemmas 18 and 25, does not by itself identify the regulator of the fermion measure with that action or make these weights observable.

The two-flavour discrete term

Take N_f = 2 and M = 0, with electromagnetism off, so that SU(2)_L x SU(2)_R is a symmetry of the reduced sector (Proposition 17.5). By Theorem 17.6, at M = 0 the admissible involutions are U_Sigma, with Sigma in U(N_f), exchanging the two eigenspaces of gamma_F by Sigma. Let Sigma(x) be a smooth map, and take the pointwise reduction of Proposition 17.10. Its operator at d = 0 is

```
D_M + c^mu P_- B_mu,   B_mu = Sigma d_mu Sigma^*,        (1)
```

up to the convention for which side carries Sigma, in independent quark and antiquark variables. On the full H_0 = (q_L, q-bar_R) the two blocks of (U_Sigma d U_Sigma) are Sigma d Sigma^* and Sigma^T d conj(Sigma). On the chirality selected by P_- this is the massless Dirac operator with the pure-gauge connection B of SU(2), and on the other chirality it has no flavour connection. The operator is generally not self-adjoint. The colour connection is unaffected.

Let Sigma : S^4 -> SU(2) generate pi_4(SU(2)) = Z_2, on the round S^4, with colour background zero, so that D_M has no kernel. Such a Sigma cannot be joined to 1 through maps to SU(2). So let Pf_-(A_t) be the square root of Witten 1982 for one Weyl doublet, transported along the path of connections A_t = t B, with the endpoint fibres compared by the gauge transformation.

Proposition 27.5. For the operator (1), with N_c doublets of the chirality selected by P_-, the transport along A_t gives

```
Pf_-(B)^{N_c} = (-1)^{N_c} Pf_-(0)^{N_c}.
```

The spectral action of the fixed operator D does not depend on Sigma. The fixed classical form is unchanged as a form, while its expression in the fixed reduced variables acquires the connection in (1).

Proof. The display counts N_c Weyl doublets, on which we use the main result of Witten 1982: (det i D-slash)^(1/2) of one doublet has the opposite sign after a gauge transformation in the nontrivial class of pi_4(SU(2)) (Eq. (6), p. 325 there, read on 3 Nov 2026). The last sentences are item 1 of Proposition 17.10. ■

This is a relative sign for the stated Pfaffian transport and endpoint identification. The pointwise R_Sigma of Theorem 17.1 changes fermion coordinates, and pulling the original measure through that change preserves the value of the same integral, because the Jacobian of the measure accompanies the transformed action. So the sign is the comparison between that pulled-back measure and the Pfaffian continued along connections, and for N_c odd they differ between the two classes of reductions. Matching this to a pion field needs the sigma model and its anomaly-matching assumptions. The coupling that results has the sign of the discrete term in the N_f = 2 sigma model: Lee, Ohmori and Tachikawa (arXiv:2009.00033, section 2.5 and Eq. (2.29), read in the LaTeX source) define the coupling as (-1)^{N_c [sigma]}, with [sigma] the class in the reduced spin bordism group of SU(2) in dimension four, which is Z_2, on spin four-manifolds. The original discussion is Witten 1983 (*Global aspects of current algebra*), which we have not read. Proposition 17.10 constructs neither a pion action nor its measure: a kinetic term needs a new operator or a quantum effective action, and Proposition 27.5 supplies only the sign.

Remark 27.6 (the relation with the S^5 index). By Witten 1982, for the flow from A = 0 to A = B, the sign equals (-1) to the number of zero modes of the five-dimensional Dirac operator on S^4 x R. The two ends compactify, with the appropriate gauge trivializations, to the nontrivial SU(2) bundle on S^5, and footnote 2 there describes the invariant zero mode for a special connection. The general protected statement is the parity of the kernel. For isospin j the parity is that of 2 tr T^2 = 2 j (j + 1)(2j + 1)/3, which is odd exactly at j = 2k + 1/2 (LITERATURE, section G). So the discrete term, with N_c colours, has the parity of N_c times the mod-two index of the doublet on the nontrivial SU(2) bundle on S^5. After choices of identification, that bundle is the one in section 6.12 of the research state, and the operator D_{S^5,V} there has the same mod-two index, on the generator of Omega_5^Spin(BSU(2)) = Z_2.

The scope of this identification is the bundle and the index. In the research state the S^5 is an internal space, the base of the chain 7 -> 5. In Proposition 27.5 the S^5 is the compactification of spacetime and the auxiliary flow parameter, with the flavour group of the quarks as structure group. We have not proved a relation between those two spaces, or between their SU(2) groups, and open question 6 of the steer note stays open for that part.

Caveats

- The background is fixed in Theorem 27.1 and Proposition 27.3. A measure on gauge fields and the order of limits are separate, as in Proposition 17.8.
- T, d and A commute by hypothesis. So T^3 and the W are excluded: this is the commuting subalgebra of Proposition 17.3. Off-diagonal d is outside Theorem 27.1, where DD^2 has a term gamma_5 [d, D_M].
- Proposition 27.5 has colour background zero and M = 0. For a nonzero colour background the integral is complex in general, and the statement on the sign is on the class of Sigma through the flow, which we have not developed.
- The sign in Proposition 27.5 is a statement on the fermion measure at one configuration. A sum over Sigma needs a sigma model that is not constructed here.
- The regulated Jacobian by the spectrum of DD is one choice. It is gauge invariant under the commuting group, which is why the anomaly falls entirely on the axial current.

Consequence for T10

The chiral limit now has, on the reduced action: the anomaly with its coefficient, the Ward identity at every cutoff, the top against the cutoff, and the discrete term. Still to be supplied are a measure on gauge fields,

the map to a quantum order parameter, and the kinetic term of a pion field, which need a quantum effective action.

Numerical check

By Claude Opus 5.5, in Python (fractions, numpy), 3 Nov 2026, in a temporary directory, not kept. These check the arithmetic and the pairing, and the proofs stand without them.

- 2 N_c tr(T^3 Q^2) = 1 and tr T^3 = 0, in exact fractions. The third-generation coefficient for the heat cutoff at x = 0, 0.1, 1, 5, 50 is 1.0000, 0.8731, 0.1572, -0.3243, -0.3333.
- On T^2 x T^2 with fluxes n_1, n_2 and Landau spectra, truncated to 2000 levels: Tr(gamma_5 exp(-t(D^2 + m^2))) = exp(-t m^2) n_1 n_2 to 15 digits, at (n_1, n_2, t, m) = (1, 1, 0.3, 0.7), (2, 3, 0.05, 1.2), (4, 1, 1, 0).
- tr T^2 half-integer at j = 2k + 1/2 and integer at j = 2k + 3/2 for k up to 20, in exact fractions.