Supplemental Material to "A dictionary between finite spectral triples and Kaluza-Klein data"

Draft, 3 November 2026

This is the supplement of the paper *A dictionary between finite spectral triples and Kaluza-Klein data*. Its parts keep the numbers of the full Supplemental Material, so that a citation of a part in the paper refers to the part of the same name here.

Parts contained. Lemma parts: 05, 16, 22, 23, 24, 25, 28, 29, 30, 31, 32, 40, 35, 36, 37, 38, 41, 42, 43, 46, 47, 48, 49, 50, 51, 52.

The full Supplemental Material, with all parts of the project, exists as `supplemental.pdf` and `supplemental_mobile.pdf` in the same folder as this file. A reference inside a part to a part not listed here points to the full version.

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Part 05

The geometric identification exists: the minimal grading-preserving spectral truncation of a suitable chiral interval is exactly Connes–Lott, with

$$|m| = \frac{\pi}{2l}, \qquad d_{\text{trunc}} = \frac{2l}{\pi}.$$

The truncated endpoint states are smeared states on the original interval. Moreover, this interval has no zero mode, which limits the physical interpretation of both extremes.

Part 1. Deconstructed chain

Statement. There is a natural family with opposite endpoint chiralities, an exact Connes–Lott member at $N = 2$, and endpoint distance converging to l. The answer depends on the representation of the site algebra; the familiar lattice-distance anomaly is not universal. (Parts 1 and 2 were refereed by Claude Opus 5.5 on 5 November 2026, report in historical/reviews/2026-11-05_opus_check_05_21_34.md: correct as written, every formula recomputed, with the scope of the three convergence statements made explicit at the end of the Statement of Part 2.)

Hypotheses. We consider complex even spectral triples, without imposing additional real-structure or KO-dimension axioms. Let $N \geq 2$, $n = N - 1$, and

$$A_N = \mathbb{C}^N, \qquad H_N = \text{span}\{L_1, \dots, L_{N-1}, R_2, \dots, R_N\}.$$

Thus interior sites carry both components, site 1 only L, and site N only R. Define

$$\pi_N(a) = \text{diag}(a_1, \dots, a_{N-1}, a_2, \dots, a_N), \qquad \Gamma_N = \text{diag}(I_n, -I_n).$$

If S_n is the lower shift, set

$$B_N = h_N(I_n - S_n), \qquad D_N = \begin{pmatrix} 0 & B_N \\ B_N^* & 0 \end{pmatrix}.$$

These are forward/backward differences:

$$(D_N\psi)_{L_i} = h_N(R_{i+1} - R_i), \qquad (D_N\psi)_{R_j} = h_N(L_{j-1} - L_j),$$

with the missing components interpreted as

$$R_1 = 0, \qquad L_N = 0.$$

There are nearest-neighbour couplings and onsite L-R terms at interior sites.

For compatibility with the spectral truncation in Part 3, choose

$$\theta_N = \frac{\pi}{4N-2}, \qquad h_N = \frac{\pi}{4l\sin\theta_N} \sim \frac{N}{l}. \tag{1}$$

This fixes the smallest positive eigenvalue to $\pi/(2l)$ at every N.

Proof.

The representation is faithful, $[\Gamma_N, \pi_N(a)] = 0$, and $\Gamma_N D_N = -D_N \Gamma_N$. At $N = 2$,

$$A_2 = \mathbb{C}^2, \quad H_2 = \mathbb{C}^2, \quad D_2 = \begin{pmatrix} 0 & h_2 \\ h_2 & 0 \end{pmatrix}, \quad h_2 = \frac{\pi}{2l}.$$

This is exactly the stated two-point triple; a diagonal unitary supplies any desired phase of m.

Ordering the basis as

$$L_1, R_2, L_2, R_3, \dots, L_{N-1}, R_N$$

makes D_N a path adjacency matrix with alternating signs. A diagonal unitary removes those signs. The usual sine eigenvectors of a path therefore give

$$\operatorname{spec}(D_N) = \left\{ 2h_N \cos\frac{j\pi}{2N-1} : 1 \le j \le 2N-2 \right\} = \{\pm\lambda_{N,r} : 1 \le r \le N-1\},$$

where

$$\boxed{\lambda_{N,r} = 2h_N \sin((2r-1)\theta_N).} \tag{2}$$

Consequently,

$$\boxed{K_N(t) = \operatorname{Tr} e^{-tD_N^2} = 2\sum_{r=1}^{N-1} e^{-4th_N^2 \sin^2((2r-1)\theta_N)}.} \tag{3}$$

For a real site function a, the onsite terms commute with $\pi_N(a)$, while

$$B_N \pi_R(a) - \pi_L(a) B_N = h_N \operatorname{diag}(a_2 - a_1, \dots, a_N - a_{N-1}).$$

Hence

$$\|[D_N, \pi_N(a)]\| = h_N \max_i |a_{i+1} - a_i|.$$

The triangle inequality bounds $|a_j - a_i|$ by $|j-i|/h_N$; the function $a_i = (i-1)/h_N$ attains this bound. Thus

$$\boxed{d_N(i,j) = \frac{|i-j|}{h_N}.} \tag{4}$$

In particular,

$$\boxed{d_N(1,N) = \frac{4l(N-1)}{\pi}\sin\frac{\pi}{4N-2} \longrightarrow l.} \tag{5}$$

One can instead choose the geometric normalization

$$h_N = \frac{N-1}{l}. \tag{6}$$

Then $d_N(1,N) = l$ exactly for every N, with the same continuum limit. However, its $N = 2$ mass is $1/l$, rather than the continuum lowest eigenvalue $\pi/(2l)$. Equations (1) and (6) distinguish spectral calibration from metric calibration at finite resolution.

Two other natural choices clarify the literature:

- **One-component staggered path.** Take $A = \mathbb{C}^M$, $H = \mathbb{C}^M$, alternating grading, and $D = h$ times path adjacency. Opposite endpoint chiralities require even M. Its spectrum is $2h\cos(j\pi/(M+1))$. For $k = |i-j|$,

$$\boxed{d(i,j) = \begin{cases} (k+1)/(2h), & k \text{ odd}, \\ \sqrt{k(k+2)}/(2h), & k \text{ even}. \end{cases}} \tag{7}$$

  In particular $d(1, M) = M/(2h)$. Thus $h = M/(2l)$ gives endpoint distance l, whereas $h = M/l$ gives $l/2$.

  Here is a direct proof of (7). Restrict the commutator to the connecting subpath and extend the function constantly outside it. After removing phases, its norm is that of weighted adjacency, with dimensionless edge weights $w_i = h|\Delta a_i|$. If its two bipartition sizes are p, q, the off-diagonal block C satisfies

$$\sum_i w_i = \mathbf{1}_p^T C \mathbf{1}_q \le \sqrt{pq}\,\|C\|.$$

  For $p = q$, alternating weights $1, 0, 1, \dots, 1$ attain the bound. For $p = q+1$, the weights

$$\frac{(q, 1, q-1, 2, \dots, 1, q)}{\sqrt{q(q+1)}}$$

  attain it: the normalized constant vectors on the two bipartitions are singular vectors with singular value 1, and positivity gives the norm 1. This proves (7).

  These are the nonadditive distances studied by Bimonte–Lizzi–Sparano, §5 and given explicitly by Atzmon, Eq. (30). Reduction to the connecting path also agrees with Iochum–Krajewski–Martinetti, Proposition 4.

- **Vertex-edge incidence Dirac.** Let $\partial : \mathbb{C}^N \to \mathbb{C}^{N-1}$ be the path difference matrix and use

$$D_{\text{inc}} = h \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix}.$$

  Its spectrum and heat trace are

$$\{0\} \cup \left\{ \pm 2h \sin \frac{k\pi}{2N} : 1 \le k < N \right\},$$

$$K_{\text{inc}}(t) = 1 + 2\sum_{k=1}^{N-1} e^{-4th^2 \sin^2(k\pi/(2N))}.$$

  These follow by substituting the cosine eigenvectors into the path Laplacian $\partial^*\partial$. Representing a site function on each edge by its value at the tail gives distance $|i-j|/h$. But H has dimension $2N-1$, has a zero mode, and both endpoint vertex components have the same grading. At $N = 2$, its spectrum is $0, \pm\sqrt{2}h$, so it is not Connes–Lott.

Thus the quoted integer deconstruction tower belongs to the incidence construction. Opposite chiral boundary conditions produce the shifted tower (2).

Status: PROVED

Part 2. Continuum limit

Statement. The chain above converges to an interval Dirac operator in three precise senses: ordered low eigenvalues, heat traces locally uniformly for $t > 0$, and pure-state metric spaces in Gromov–Hausdorff distance. It also has a controlled one-dimensional heat window. Here spectral convergence means that each eigenvalue lambda_{N,r} increases to (2r-1) m_1 at fixed r, so that the counting function on [-E, E] converges for every E outside spec(D_l); the Hilbert spaces differ and no resolvent convergence is asserted.

The heat traces converge uniformly on [eps, infinity) for every eps > 0. The Gromov-Hausdorff statement concerns the pure-state spaces with their Connes metrics, at distance at most l/(N-1) (referee, 5 November 2026).

Hypotheses. On $H_l = L^2([0,l],\mathbb{C}^2)$, define

$$D_l = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix}, \qquad \mathrm{Dom}\, D_l = \{(L,R) \in H^1 : R(0) = 0,\ L(l) = 0\}, \tag{8}$$

with scalar multiplication by $C^\infty([0,l])$ and grading $\Gamma = \mathrm{diag}(1,-1)$.

These are opposite local chiral conditions. Reflection parity is Γ at 0 and $-\Gamma$ at l; their composition gives antiperiodicity on the doubled interval.

Proof.

Integration by parts gives the boundary form

$$[\overline{L}\,\widetilde{R} - \overline{R}\,\widetilde{L}]_0^l.$$

The conditions in (8) form a maximal isotropic boundary subspace, so D_l is self-adjoint. Its resolvent is compact by the compact embedding $H^1 \hookrightarrow L^2$.

Solving $D_l\psi = \lambda\psi$ gives

$$\psi_\lambda(x) = c(\cos\lambda x, \sin\lambda x), \qquad \cos(\lambda l) = 0.$$

Therefore

$$\boxed{\mathrm{spec}(D_l) = \left\{ \pm\frac{(r-\frac{1}{2})\pi}{l} : r = 1,2,... \right\}.} \tag{9}$$

The kernel is zero. Squaring gives mixed Neumann–Dirichlet conditions on L, and Dirichlet–Neumann conditions on R.

For scalar f,

$$\|[D_l, f]\| = \|f'\|_\infty, \qquad d_l(x,y) = |x-y|. \tag{10}$$

Under normalization (1), with $m_1 = \pi/(2l)$,

$$\lambda_{N,r} = m_1 \frac{\sin((2r-1)\theta_N)}{\sin\theta_N} \longrightarrow (2r-1)m_1. \tag{11}$$

This proves low-eigenvalue convergence. Since

$$\sin y \geq \frac{2y}{\pi} \quad (0 \leq y \leq \pi/2),$$

we also have

$$\lambda_{N,r} \geq \frac{2r-1}{l}. \tag{12}$$

A summable Gaussian majorant now proves, uniformly on every $t \in [\varepsilon, T]$,

$$K_N(t) \longrightarrow K_l(t) = 2\sum_{r=1}^{\infty} e^{-t((r-\frac{1}{2})\pi/l)^2}. \tag{13}$$

Poisson summation gives

$$\boxed{K_l(t) = \frac{l}{\sqrt{\pi t}}\left(1 + 2\sum_{q=1}^{\infty}(-1)^q e^{-q^2l^2/t}\right).} \tag{14}$$

In particular, $K_l(t) \sim l/\sqrt{\pi t}$ as $t/l^2 \to 0$.

The finite-chain heat window has the following rigorous meaning. For every sequence $t_N > 0$ satisfying

$$\frac{t_N}{l^2} \to 0, \qquad \frac{N^2 t_N}{l^2} \to \infty, \tag{15}$$

one has

$$\boxed{K_N(t_N) \sim \frac{l}{\sqrt{\pi t_N}}, \qquad -2t_N \frac{K_N'(t_N)}{K_N(t_N)} \to 1.} \tag{16}$$

To prove this, put $u_r = \pi\sqrt{t_N}(r - \frac{1}{2})/l$. On bounded u_r-intervals, the sine argument in (11) tends uniformly to zero, so $\sqrt{t_N}\lambda_{N,r}/u_r \to 1$. The mesh tends to zero and its upper endpoint tends to infinity. Bound (12) controls the tails. Gaussian Riemann sums, and then the same sums weighted by $t_N\lambda_{N,r}^2$, prove both assertions.

Finally, identify site i with $x_i = (i-1)l/(N-1)$. Equation (4) gives

$$d_N(i,j) = \rho_N |x_i - x_j|, \qquad \rho_N = \frac{N-1}{l h_N} = 1 + O(N^{-1}).$$

The metric distortion and the grid mesh are both $O(l/N)$, proving Gromov–Hausdorff convergence to $[0, l]$.

The endpoint chirality here concerns allowed boundary components. It does not supply localized chiral zero modes.

Status: PROVED

Part 3. Minimal spectral truncation

Statement. The minimal nonzero grading-preserving spectral truncation of (8) is unitarily equivalent to Connes–Lott, with

$$\boxed{|m| = m_1 = \frac{\pi}{2l}, \qquad c = \frac{\pi}{2}.}$$

Its natural truncated endpoint states have distance $1/m_1 = 2l/\pi$. Literal endpoint evaluations do not descend through compression.

Hypotheses. Let

$$P = \mathbf{1}_{[0,E]}(|D_l|), \qquad m_1 \le E < 3m_1,$$

and give $S = P\,C([0,l])\,P$ its inherited operator-system structure. This is the compression construction described in Connes–van Suijlekom, §1.

Proof.

An orthonormal chiral basis of PH_l is

$$u_L(x) = \sqrt{\frac{2}{l}} \begin{pmatrix} \cos(\pi x/(2l)) \\ 0 \end{pmatrix}, \qquad u_R(x) = \sqrt{\frac{2}{l}} \begin{pmatrix} 0 \\ \sin(\pi x/(2l)) \end{pmatrix}.$$

In this basis,

$$PD_lP = \begin{pmatrix} 0 & m_1 \\ m_1 & 0 \end{pmatrix}, \qquad P\Gamma P = \mathrm{diag}(1,-1). \tag{17}$$

Compression of multiplication is

$$Q(f) = \begin{pmatrix} \frac{2}{l}\int_0^l f(x)\cos^2\frac{\pi x}{2l}\,dx & 0 \\ 0 & \frac{2}{l}\int_0^l f(x)\sin^2\frac{\pi x}{2l}\,dx \end{pmatrix}. \tag{18}$$

Since

$$Q(1) = I, \qquad Q(\cos(\pi x/l)) = \tfrac{1}{2}\,\mathrm{diag}(1,-1),$$

its range is exactly the full diagonal algebra:

$$\boxed{S = \mathbb{C} \oplus \mathbb{C}.} \tag{19}$$

Here the compressed operator system happens to be an algebra, although Q is not multiplicative.

Evaluation on the finite-dimensional space PH_l is bounded. At 0, only u_L has a nonzero boundary value; at l, only u_R does. The normalized evaluation vectors therefore define

$$\omega_0(\mathrm{diag}(a,b)) = a, \qquad \omega_l(\mathrm{diag}(a,b)) = b.$$

Using (17),

$$\|[PD_lP, \mathrm{diag}(a,b)]\| = m_1|a-b|,$$

and hence

$$\boxed{d_{\mathrm{trunc}}(\omega_0,\omega_l) = \frac{1}{m_1} = \frac{2l}{\pi} < l.} \tag{20}$$

Minimality follows because there is no zero mode and the grading exchanges the eigenspaces at $+m_1$ and $-m_1$. A nonzero grading-preserving spectral subspace must retain both.

There is an essential distinction between these states and literal evaluations. For

$$g(x) = \cos(3\pi x/l),$$

equation (18) gives $Q(g) = 0$, while $g(0) = 1$ and $g(l) = -1$. Thus neither endpoint evaluation factors through Q.

The pullbacks of the truncated endpoint states are instead the probability measures

$$d\mu_L = \frac{2}{l}\cos^2\frac{\pi x}{2l}\,dx, \qquad d\mu_R = \frac{2}{l}\sin^2\frac{\pi x}{2l}\,dx.$$

Their distance using the **original** continuum seminorm is

$$d_{D_l}(\mu_L,\mu_R) = \int_0^l \left|\int_0^x (d\mu_L - d\mu_R)\right| dx = \frac{4l}{\pi^2}. \tag{21}$$

The equality follows by integration by parts and is attained by $f = -x$. Equations (10), (20), and (21) concern three different state/metric comparisons.

Status: PROVED

Part 4. The two limits

Amendment (Terra 6.1, 3 Nov 2026; report). The statements of this part are correct for the stated bare chain, with one precision: the condition N/l -> infinity gives uniform convergence of the dispersion on bounded energy windows and of normalized heat traces and counting densities, and does not by itself give convergence of the unnormalized difference of heat traces (for N = floor(l^(5/4)) it diverges). Escapes for the empty light sector: a tuned bulk mass or a scaled boundary coupling.

Statement. At fixed finite rank, $m \to 0$ produces infinitely separated finite geometry, with heat dimension zero. A resolved interval requires a growing tower. As $m \to \infty$, the metric collapses, but the light Hilbert space is zero.

Hypotheses. Identify

$$m = m_1 = \frac{\pi}{2l},$$

use normalization (1), and hold physical energies or heat times fixed unless stated otherwise.

Proof.

For fixed N, $D_N = mT_N$ with T_N independent of m. Therefore, as $m \to 0$,

$$D_N \to 0, \qquad K_N(t) \to 2(N-1), \qquad d_N(i,j) \to \infty \quad (i \neq j).$$

For the minimal truncation,

$$K_2(t) = 2e^{-tm^2}.$$

Every finite N has ultraviolet heat dimension zero. Making its finitely many modes light does not change that conclusion.

For the full interval, a fixed energy E is in the resolved one-dimensional regime when

$$\frac{1}{l} \ll E.$$

For its finite-chain approximation, one additionally needs

$$\boxed{\frac{1}{l} \ll E \ll \frac{N}{l}.} \tag{22}$$

Thus, as $l \to \infty$, an exact continuum limit throughout a fixed positive energy window follows if $N/l \to \infty$. Equivalently, as $m \to 0$, one needs $Nm \to \infty$.

The continuum counting function and heat density make the emergence explicit:

$$\#\{|\lambda| \le E\} = 2\left\lfloor \frac{lE}{\pi} + \frac{1}{2} \right\rfloor, \qquad \frac{K_l(t)}{l} \longrightarrow \frac{1}{\sqrt{\pi t}} \quad (l \to \infty). \tag{23}$$

Based at an endpoint, the metric limit is a half-line; the opposite endpoint recedes to infinity.

The order of limits matters even for the normalized heat trace:

$$\lim_{l\to\infty}\lim_{N\to\infty} \frac{K_N(t)}{l} = \frac{1}{\sqrt{\pi t}}, \qquad \lim_{N\to\infty}\lim_{l\to\infty} \frac{K_N(t)}{l} = 0. \tag{24}$$

Likewise, taking $t \to 0$ at fixed N gives dimension zero, whereas taking $N \to \infty$ first gives the interval's dimension one.

As $m \to \infty$, all distances tend to zero. At fixed N, the algebra elements with uniformly bounded commutators reduce to constants. But every eigenvalue has magnitude at least m, so

$$\|(D_N - i)^{-1}\| \le \frac{1}{\sqrt{m^2+1}} \longrightarrow 0. \tag{25}$$

The same holds for D_l, after identifying the varying intervals by dilation. Its heat trace tends to zero at every fixed $t > 0$.

Thus the metric limit is a point, while the embedded resolvent limit has

$$H_{\text{light}} = \ker T_N = 0.$$

There is no finite operator limit on the original nonzero Hilbert space, and no surviving fermion pair at fixed energy.

Finally, the two-state spectral truncation exists for

$$m_1 \le E < 3m_1.$$

A genuine spectral cutoff $E \ll 1/l$ retains no modes. The ratio between the first two positive levels is always 3, so the minimal pair has no parametrically large separation from the next level.

Status: PROVED

Verdict on hypothesis H (theorem T1)

Supported for T1's geometric assertion. Connes–Lott is exactly the minimal nonzero grading-preserving spectral compression of the opposite-chirality interval above, with

$$l = \frac{\pi}{2|m|}.$$

The chain recovers its spectrum, heat window, and metric.

The precise replacement for the stronger interpretation of H is:

> The two sheets are truncated endpoint evaluation states, whose pullbacks are smeared over the interval. Refinement restores the interval. Decompactification produces a one-dimensional heat regime only when the number of retained modes grows sufficiently rapidly. Collapse identifies the metric points but removes all fermionic modes of this particular interval at fixed energy.

Accordingly, a nonempty two-state sector at $E \ll 1/l$, or surviving light fermions after collapse, is disproved by this model. The construction supplies no KO-6 or Standard Model identification.

Independent numerical check (Claude, 2 Nov 2026)

For l = 1 and N = 2, 3, 5, 9 we checked the following. The spectrum of D_N agrees with +/- 2 h_N sin((2r-1) theta_N) to 1e-14. The lowest positive eigenvalue equals pi/(2l) for every N. For random real site functions, ||[D_N, pi_N(a)]|| equals h_N max abs(a_{i+1} - a_i), which is the identity behind d_N(i,j) = abs(i-j)/h_N. The endpoint distance d_N(1,N) = 0.6366, 0.7869, 0.8844, 0.9398 tends to l. For the minimal truncation, Q(cos(3 pi x/l)) = 0, and the smeared-state distance 0.405290 matches 4l/pi^2. The truncated distance is 2l/pi = 0.6366, identical to d_2(1,2).

Literature (../LITERATURE.md, section A). The Kaluza-Klein reading of the two-point space is in Connes-Lott (sections I and III: "two copies of Z separated by a distance L"); the two points as points of a fifth dimension are in Viet-Wali 1996; the two-point model as an approximation of a manifold, to be related to dimensional reduction, is in Dimakis-Muller-Hoissen 1994 (section III, Example 4, and section VII). Connes-van Suijlekom (section 1) note that compressions P A P are operator systems, of which the chiral interval is an exception. The truncation theorem with |m| = pi/(2l) was not found in print.

Part 16

Noncommutative finite real triples from commutative data: truncating the coarse correspondence

Proved by Claude Opus 5.5 without Astra or Fable, in response to the challenge that the T0 note overread lemma 15 and the Farsi-Latrémolière limits. Signs and the level-two algebra were checked with numpy on random data (section "Numerical check"). The term "coarse correspondence" is that of Connes, *Noncommutative Geometry* (1994), chapter V, for the bimodule of Hilbert-Schmidt operators, and the form D = Delta + J Delta J^-1 is in Krajewski (arXiv:hep-th/9701081) and Paschke-Sitarz (arXiv:q-alg/9612029); all checked against the sources. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): the examples and Corollary 16.4 are correct; amendments to Proposition 16.1, Theorem 16.2, Corollary 16.3 and the caveats are incorporated below; Remark 16.5 and Example 16.6 come from the referee's assessment of what the lemma achieves. The two-sheet section was refereed separately (Terra 6.1, 3 Nov 2026; report), with one error in Corollary 16.8 and one in Example 16.9, corrected below.

Outcome. Compressing functions on a connected commutative space to a fixed spectral window produces a noncommutative finite algebra with no limit taken. With the standard reality J a J^-1 = a*, the inherited structure fails order zero exactly when that algebra is noncommutative (Proposition 16.1). With the parent Hilbert space doubled to H_0 (x) conj(H_0), order zero and order one hold for every such algebra (Theorem

16.2, Corollary 16.3); these two conditions are formal consequences of the tensor square, available for any finite even representation (Remark 16.5). The KO-6 signs and the count dim H_F = 2 k^2 are reached by a product with a Clifford point, which is an input and fails finite orientability (Corollary 16.4), and more faithfully by a parent with two sheets, which gives the form of the Chamseddine-Connes classification (Theorem 16.7, Corollary 16.8, Example 16.9). This selects no particular algebra and gives no dynamics.

Setting

Let (A_0, H_0, D_0, gamma_0) be an even spectral triple, A_0 commuting with gamma_0, with D_0 of compact resolvent. Let Sigma = -Sigma be a finite subset of R and P_0 = 1_Sigma(D_0), of rank k, so that P_0 commutes with gamma_0. Put V = P_0 H_0, graded by gamma_V, with d = D_0 restricted to V, and

```
Q(a) = P_0 a P_0 on V,     O_F = Q(A_0),     A_F = C*(O_F) in B(V).
```

O_F is an operator system, in general not an algebra, and A_F consists of even operators. Compression is standard: for functions it is the Toeplitz compression, and its use in truncating spectral triples is due to D'Andrea-Lizzi-Martinetti and Connes-van Suijlekom. The latter note that the C*-algebra generated by PAP is in many examples a noninformative full matrix algebra. Since Q(1) = 1, A_F is the bicommutant of O_F, so its structure is set by the operators commuting with all compressed functions, such as the grading in example (b) and j in Example 16.9. That is where the information lies, and the fact that A_F is noncommutative is modest.

Examples of A_F.

(a) *Chiral interval, lowest level.* For condition (c) with M = 0 and Sigma = {+/- pi/(2l)}, O_F = C + C = A_F (lemma 05).

(b) *Chiral interval, next level.* For Sigma = {+/- pi/(2l), +/- 3 pi/(2l)}, V has basis (c_1, 0), (c_2, 0), (0, s_1), (0, s_2), with c_r = (2/l)^{1/2} cos((r - 1/2) pi x/l) and s_r = (2/l)^{1/2} sin((r - 1/2) pi x/l). With C_n = (1/l) int f cos(n pi x/l),

Q(f) = (Q_L, Q_R), Q_L = [[C_0 + C_1, C_1 + C_2], [C_1 + C_2, C_0 + C_3]], Q_R = [[C_0 - C_1, C_1 - C_2], [C_1 - C_2, C_0 - C_3]].

The values C_0, C_1, C_2, C_3 are independent and determine Q(f), so O_F has dimension 4. Each of Q_L and Q_R alone ranges over all symmetric 2 x 2 matrices, which generate M_2(C), so A_F projects onto each summand. For f_1 = 1/2 + cos(pi x/l) + cos(2 pi x/l) + cos(3 pi x/l) and f_2 = 1/2 - cos(pi x/l) + cos(2 pi x/l) - cos(3 pi x/l),

```
Q(f_1) = ([[1, 1], [1, 1]], 0),     Q(f_2) = (0, [[1, -1], [-1, 1]]).
```

M_2(C) is simple, so multiplying Q(f_1) and Q(f_2) by elements of A_F gives M_2(C) + 0 and 0 + M_2(C). Hence A_F = M_2(C) + M_2(C), of dimension 8. The finite algebra of the interval itself is noncommutative at the second level.

(c) *Twistor space.* For the level-one Toeplitz compression on CP^3, Q(x_a) = Gamma_a/5 (Proposition 10.1), and these generate A_F = M_4(C) on V = C^4. Here V is the kernel of the Dolbeault-Dirac operator twisted by O(1), so d = 0 on V.

The standard correspondence

Proposition 16.1 (order zero fails with inherited reality). Let J_0 be a real structure on H_0 (antiunitary, J_0^2 = +/-1, with the grading sign), with J_0 a J_0^-1 = a* for all a in A_0 and J_0 D_0 = eps' D_0 J_0, and let J_F = J_0 restricted to V. Then J_F Q(a)* J_F^-1 = Q(a), and the data (A_F, V, d, J_F) satisfy order zero if and only if A_F is commutative. The same holds for a product with a fixed finite factor that itself satisfies order zero, with the compatible product real structure.

Proof. J_0 preserves V because Sigma = -Sigma. Then J_0 P_0 a* P_0 J_0^-1 = P_0 a P_0. Order zero would make Q(a) commute with Q(b)° = Q(b) for all a, b, so O_F and hence A_F would be commutative;

the converse is immediate, since J_F A_F J_F^-1 = A_F. For the product, take the unit in the finite factor: [1 (x) Q(a), 1 (x) Q(b)] = 1 (x) [Q(a), Q(b)]. ■

Together with Amendment 2 of lemma 07 (first order fails for C + C with inherited J_2), this is the correct form of the obstruction. It concerns the standard correspondence L^2(A_0) with the pointwise real structure.

The coarse correspondence

Let conj(H_0) be the conjugate Hilbert space, with canonical antiunitary xi -> xi^c, and with gamma^c, D^c the conjugate operators, gamma^c xi^c = (gamma_0 xi)^c and D^c xi^c = (D_0 xi)^c. On H = H_0 (x) conj(H_0), which is the coarse correspondence of A_0, equivalently the Hilbert-Schmidt operators on H_0, define for homogeneous xi, eta

```
pi(a) = a (x) 1,
DD = D_0 (x) 1 + gamma_0 (x) D^c,
Gamma = gamma_0 (x) gamma^c,
J (xi (x) eta^c) = (-1)^{|xi| |eta|} eta (x) xi^c.
```

Theorem 16.2. (A_0, H, DD, J, Gamma) is a real even spectral triple with signs J^2 = 1, J DD = DD J, J Gamma = Gamma J (KO-dimension 0). For even a, b with bounded commutators,

```
a° := J pi(a)\* J^-1 = 1 (x) (a\*)^c,     [pi(a), b°] = 0,     [[DD, pi(a)], b°] = 0,
```

and norm([DD, pi(a)]) = norm([D_0, a]). So if the original triple is metric, the new triple is metric with the same Connes distance on A_0. If the Weyl exponent p = lim log N_0(R)/log R of D_0 exists, that of DD is 2 p, since N_0(R/sqrt(2))^2 <= N_DD(R) <= N_0(R)^2.

Proof. J is antilinear, and J^2 = 1 because the sign is applied twice with the degrees exchanged. Gamma has eigenvalue (-1)^{|xi| + |eta|}, symmetric under the exchange. For the Dirac operator,

```
J DD (xi (x) eta^c) = (-1)^{(|xi| + 1) |eta|} eta (x) (D_0 xi)^c + (-1)^{|xi|} (-1)^{|xi| (|eta| +
↪  1)} D_0 eta (x) xi^c
                    = (-1)^{|xi| |eta|} [D_0 eta (x) xi^c + (-1)^{|eta|} eta (x) (D_0 xi)^c] = DD J
                     ↪  (xi (x) eta^c).
```

For even a, a* preserves degree, so J pi(a)* J^-1 (eta (x) xi^c) = eta (x) (a* xi)^c, which gives the formula for a° and order zero. Since a is even, [DD, pi(a)] = [D_0, a] (x) 1, which commutes with 1 (x) (b*)^c. Finally DD^2 = D_0^2 (x) 1 + 1 (x) (D^c)^2, because gamma_0 anticommutes with D_0. An eigenbasis of D_0 decomposes DD into finite-dimensional invariant blocks, so DD is essentially self-adjoint on the algebraic tensor product, with compact resolvent, and the spectrum is +/- (lambda_i^2 + lambda_j^2)^{1/2}. ■

Geometrically, when H_0 = L^2(S_X) on an even-dimensional spin manifold X, a charge conjugation commuting with D_X identifies conj(H_0) with H_0, H is L^2(S_{X x X}) and DD is the ordinary product Dirac operator. (The grading may acquire an overall sign under this identification.) So the data are a product-manifold Dirac operator on X x X with a restricted observable algebra C(X) (x) 1, and a J covering the flip of the two factors, which differs from the pointwise spin real structure. They count as commutative Kaluza-Klein data only with that restriction and that J: they are neither the ordinary spin triple of X x X with its full function algebra nor a finite-rank twist over X. The restriction is essential: on all of C(X x X), the compressions generate A_F (x) conj(A_F), and order zero generally fails again.

Corollary 16.3 (truncation). Let P = P_0 (x) conj(P_0). It commutes with DD, Gamma and J. It is a joint cutoff in the two factors, and need not be a spectral projection of DD: for absolute eigenvalues 1, 5, 7, sqrt(73) of D_0 with P_0 selecting the first three, the selected pair (5, 7) and the excluded pair (1, sqrt(73)) both give DD^2 = 74. The compression of pi(A_0) is O_F (x) 1; passing to the generated algebra gives the finite real even triple

```
T_F = (A_F, V (x) conj(V), D_F, J_F, Gamma_F),     D_F = d (x) 1 + gamma_V (x) d^c = Delta + J_F
↪  Delta J_F^-1,     Delta = d (x) 1,
```

with A_F acting on the left. It has KO-dimension 0, and satisfies order zero and order one for every A_F, noncommutative or not, with dim H_F = k^2. The family is exact at each truncation; no limit is taken.

Proof. The proof of Theorem 16.2 applies verbatim on V with arbitrary even a, b in B(V), since it uses only the grading and the tensor structure. For J_F Delta J_F^-1, apply J to (d (x) 1) J (xi (x) eta^c): the signs combine to (-1)^{|xi|}, giving gamma_V (x) d^c. ■

So H_F has the bimodule form of Connes' finite geometries, with A_F on the left and its opposite on the right, and D_F has the first-order form of Krajewski and Paschke-Sitarz.

Example (a) gives (C + C, C^2 (x) C^2, m (sigma_1 (x) 1 + sigma_3 (x) sigma_1)). There [[D_F, e], e°] = 0 for e = diag(1, 0), which repairs the first-order failure of Counterexample 08.4 at the price of doubling. Example (b) gives (M_2(C) + M_2(C), C^16, D_F).

Corollary 16.4 (KO-6 by a Clifford point). The product of T_F with (C, C^2, 0, sigma_1 K, sigma_3) is

```
(A_F, V (x) conj(V) (x) C^2, D_F (x) 1, J_F (x) sigma_1 K, Gamma_F (x) sigma_3),
```

with signs (+, +, -), order zero, order one and dim H = 2 k^2. It fails the usual zero-dimensional orientability axiom: every zero-cycle built from the left and opposite actions acts as T (x) 1_2, while the grading is Gamma_F (x) sigma_3.

Proof. (sigma_1 K)^2 = 1 and sigma_1 K sigma_3 = -sigma_3 sigma_1 K, while the D term is untouched. a° is a° of T_F tensored with 1. ■

Remark 16.5 (the order conditions are formal). For every finite even representation (A, V, d, gamma), the map V -> V (x) conj(V), d -> d (x) 1 + gamma (x) d^c gives a real bimodule with order zero and order one, using no prior finite real structure. So these two conditions follow from choosing the tensor square, and say nothing specific about A_F. The construction does not recover an arbitrary prescribed finite (H, D, J). What is specific to T0 is that A_F is generated by compressed functions, that the tensor square is itself the truncation of a commutative parent in the restricted sense above, and families such as the following.

Example 16.6 (a tower-decoupling family, from the referee). Let D_0 be the Dolbeault-Dirac operator on CP^3 twisted by O(1), with V = ker D_0 = C^4, and let DD_l = DD/l on the doubled parent. Its kernel is V (x) conj(V), and its gap grows like 1/l. Here P is a spectral projection. Compression followed by algebra generation gives the constant finite data A_F = M_4(C), dim H_F = 16, D_F = 0, and the Clifford point gives dim H = 32 with the KO-6 signs. Order one is automatic, since D_F = 0. The whole triple converges to (C, C^16, 0) by Theorem 15.3, consistently with lemma 15: the two statements concern the whole algebra and the truncation.

Two sheets: the form of the classification

Chamseddine and Connes (*Why the Standard Model*, arXiv:0706.3688, sections 2-3) classify irreducible pairs (A, J) with order zero and a separating vector: the centre of the complexified algebra is C, with A_C = M_k(C) on itself and J x = x*, or C + C, with A_C = M_k(C) + M_k(C) on itself and J(x, y) = (y*, x*). With centre C, J gamma = -gamma J is impossible. The pair of Corollary 16.4 is therefore reducible: the projection 1 (x) (1 + sigma_1)/2 commutes with A_F and J. The irreducible form comes from a parent with two sheets. At the finite level the bimodule (W (x) conj(V)) (+) (V (x) conj(W)) with its adjoint-flip J is already stated in Remark 2.9 of that paper; what we add is its realization from parents and compressed functions.

Theorem 16.7 (two-sheet coarse parent). Let (A_X, H_X, D_X, gamma_X) and (A_Y, H_Y, D_Y, gamma_Y) be even spectral triples. On H = (H_X (x) conj(H_Y)) (+) (H_Y (x) conj(H_X)), for even a in A_X and b in A_Y, put

```
pi(a, b) = (a (x) 1) (+) (b (x) 1),
DD = (D_X (x) 1 + gamma_X (x) D_Y^c) (+) (D_Y (x) 1 + gamma_Y (x) D_X^c),
Gamma = (gamma_X (x) gamma_Y^c) (+) (-gamma_Y (x) gamma_X^c),
J (xi (x) eta^c) = (-1)^{|xi| |eta|} eta (x) xi^c,
```

where J maps each summand to the other. Then J^2 = 1, J DD = DD J and J Gamma = -Gamma J, the KO-6 signs. Moreover

```
(a, b)° = (1 (x) (b\*)^c) (+) (1 (x) (a\*)^c),
```

so the first summand is an (A_X, A_Y)-bimodule and the second an (A_Y, A_X)-bimodule, and order zero and order one hold.

Proof. The computation of Theorem 16.2 applies with D_Y in place of the second factor: on xi (x) eta^c in the first summand, J DD and DD J both give (-1)^{|xi| |eta|} [D_Y eta (x) xi^c + (-1)^{|eta|} eta (x) (D_X xi)^c]. The sign in Gamma on the second summand gives J Gamma = -Gamma J. For a°, apply J, then pi(a, b)*, then J, as in Theorem 16.2: an element of the first summand goes to the second, where b* acts on the left, and returns with b* on the right factor. Since a and b are even, [DD, pi(a, b)] = ([D_X, a] (x) 1) (+) ([D_Y, b] (x) 1), which commutes with the opposite actions. ■

Corollary 16.8 (truncation; the form of the classification). For symmetric windows with V_X, V_Y of dimensions k_X, k_Y, and even subalgebras A_{F,X} in B(V_X), A_{F,Y} in B(V_Y), the compression

```
T_F = (A_{F,X} + A_{F,Y}, (V_X (x) conj(V_Y)) (+) (V_Y (x) conj(V_X)), D_F, J_F, Gamma_F)
```

has the KO-6 signs, order zero and order one, and dim H_F = 2 k_X k_Y. If gamma_{V_X} lies in A_{F,X} and gamma_{V_Y} in A_{F,Y}, the real orientability relation holds: with c = (gamma_{V_X}, gamma_{V_Y}) and z = (1, -1), Gamma_F = pi(z c) c°, a zero-cycle with coefficients in the algebra and its opposite. This is weaker than Gamma_F in pi(A_F). If V_X and V_Y are ungraded with k_X = k_Y = k and the generated algebras are full, then (A_F, H_F, J_F) is exactly the irreducible pair with centre C + C: identify both summands with M_k(C), so that J_F(x, y) = (y*, x*); an invertible matrix in each summand is a separating vector, and a projection commuting with both full matrix actions is scalar on each summand, with J equating the scalars. In all cases D_F commutes with the two sheet projections, so the sheets are not connected. It need not commute with the whole centre: for V_X = C^2, gamma_X = sigma_3, d_X = sigma_1, A_{F,X} = C + C acting diagonally and V_Y = C, the central element (diag(1, 0), 0) does not commute with D_F.

Proof. Theorem 16.7 applies on the truncation with arbitrary even operators. For orientability, pi(c) c° = (gamma_{V_X} (x) gamma_{V_Y}^c) (+) (gamma_{V_Y} (x) gamma_{V_X}^c), and z flips the sign of the second summand. The last statements are direct. ■

Example 16.9 (twistor level one, and M_2(H) + M_4(C)). Let D_0 be the Dolbeault-Dirac operator on CP^3 twisted by O(1), with kernel H^0(CP^3, O(1)) = C^4, and let j be the antiunitary with j^2 = -1 commuting with the Gamma_a of Proposition 10.1, which makes C^4 a quaternionic space H^2. Use this space twice, as W (with the functions of S^4 pulled back along the twistor projection) and as V (with the functions of CP^3).

1. A real function f on S^4 pulls back to a function invariant under w -> j w. So j Q(f) j^-1 = Q(f), and Q(f) is a Hermitian element of End_H(W). Since Q(x_a) = Gamma_a/5, the real algebra generated is End_H(W) = M_2(H), of real dimension 16.
2. For real functions on CP^3, Q(w* B w / w* w) = (B + tr(B) 1)/5 for Hermitian B, so Q is onto the Hermitian matrices, and the real algebra generated is End_C(V) = M_4(C), of real dimension 32.

Hence Corollary 16.8, with the parent gradings, which are trivial on the kernels, gives the real algebra A = M_2(H) + M_4(C) on H_F = Hom(V, W) (+) Hom(W, V) = E (-) E* of dimension 32, with J(x, y) = (y*, x*). These are the algebra, bimodule and real structure of Chamseddine and Connes. The generation is of real algebras from real functions: complex generation on the S^4 side would give M_4(C).

The nontrivial grading of W in that construction is not obtained from this parent. An even parent has [P gamma P, P a P] = 0 for every compressed function, while Gamma_5 = 5 Q(x_5) anticommutes with Q(x_1). So the grading has to be added after truncation, as abstract finite data, fixing an axis. The full algebra is then a graded prealgebra rather than the algebra of an even triple, with even part H + H + M_4(C), and Gamma_F(xi, eta) = (Gamma_5 xi, -eta Gamma_5), which is Proposition 3.2 of that paper; the real orientability relation holds on the even part. Similarly d = mu n^a Gamma_a with n orthogonal to the axis is the compression of a bounded real multiplication operator, but adding that operator to the

parent Dirac operator keeps neither the kernel as a spectral projection nor the parent grading as an odd perturbation, so d too is abstract data. It commutes with the two sheet projections and with the centre R + C of the full prealgebra, and not with the centre of the even algebra.

This bears on the step left open in canonical section 4, where M_2(H) in M_4(C) is a nested pair on one four-space, with "the independent E, F module roles" to be derived. The example closes that abstract finite-module problem after adding a second sheet: the nested actions on one four-space become independent summands acting on two Hom spaces. It does not turn four global sections into rank-four fibres over S^4, derive the two sheets from the fibration, or produce the Dirac operator connecting the two pieces that the selection of A_F = C + H + M_3(C) requires. The fibre-rank problem remains open.

The inputs are the second sheet, the restriction of one algebra to base functions, real algebra generation, the four-dimensional window, the grading axis and d. The content beyond the formal bimodule is the concrete compression calculation in items 1 and 2, already in the corpus, and its explicit match with the classified data.

Caveats

These results should be read as narrowly as the note they correct.

1. **Metric.** T_F is usually not a metric spectral triple. For A_F = B(V_+) + B(V_-) with V_+ of dimension at least 2 and d odd, the commutant of d in A_F contains the pairs (x, B^-1 x B) with x commuting with B B*, when B is the invertible block of d. In example (b), (diag(1, 0), diag(1, 0)) has L = 0, so d leaves a residual U(1) x U(1) of U(2) x U(2) unbroken. The Standard Model finite triple has the same feature, through M_3(C).
2. **Spectrum.** The eigenvalues of D_F are +/- (lambda_i^2 + lambda_j^2)^{1/2}. In example (a) they are +/- sqrt(2) m with Connes distance 1/m, so the doubled spectrum differs from the masses in d, and the mass relations of Theorem 9.6 and lemma 11 would have to be redone.
3. **Parent.** The doubled parent is a choice. Its algebra sees one factor of X x X, its J covers the flip, which is outside the spin real structure of X x X, and generating A_F from O_F is an additional operation. We have no dynamical reason for it. For an ordinary compact spin X of dimension n, its Weyl exponent is 2 n with metric dimension n, which bears on the counting of 9, 10 and 11 and has not been analyzed.
4. **KO-6.** The Clifford point is an input, as F_0 is in Theorem D, and it fails finite orientability. In Theorem 16.7 the sign comes from the block structure of two sheets, which is itself an input. The documents checked here establish neither a construction of two sheets from one connected parent nor a no-go theorem. The grading-reversing structure of lemma 08 is a candidate we have not tested.
5. **Count.** For k = 4, dim H = 32, the value in the classification of Chamseddine-Connes, and k = 4 occurs in examples (b) and (c). For k = 4 in examples (b) and (c) alone this is a count and nothing more: the classification's algebra is M_2(H) + M_4(C), acting differently, and those examples do not produce it. Example 16.9 does, as abstract data with the listed inputs.

Consequence for T0 and T0b

T0 asks for finite real triples from commutative Kaluza-Klein data under specified notions of convergence. For one specified truncation procedure (compression, then generation, on the doubled parent with restricted algebra and flip J), this lemma gives a noncommutative finite algebra with an explicit bimodule, and a family with the tower decoupled (Example 16.6). It does not establish the target under every meaning of ordinary Kaluza-Klein data. For T0b, it gives examples with the requested signs and first-order relations; with two sheets it reaches the form of the classification, and in Example 16.9 the real algebra M_2(H) + M_4(C), bimodule and real structure of Chamseddine and Connes, with the grading added as abstract data. It leaves open an operator connecting the two pieces and hence the selection of A_F, a parent that yields the grading and nonzero mass data, and the beyond-first-order and twisted frameworks. See the background note.

Numerical check

Python (numpy), run on 3 Nov 2026 in a temporary directory, not kept. On random odd d and even a, b for dim V_+ = dim V_- = 2 with J = M K: J^2 = 1, J DD = DD J, J Gamma = Gamma J, DD Gamma = -Gamma DD, order zero and order one all hold to machine precision, with norm([a, b]) = 7.67; the KO-6 signs of Corollary 16.4 hold. For Theorem 16.7 and Example 16.9, with k = 4, W graded by Gamma_5 and V ungraded: J^2 = 1, J Gamma = -Gamma J, J D = D J, D Gamma = -Gamma D, D = Delta + J Delta J^-1, order zero and order one for random a, b, and Gamma_F = pi(z c) c° all hold to machine precision; j^2 = -1 commutes with the Gamma_a, and x_a(j w) = x_a(w); the real algebras generated by Gamma_a, by i Gamma_a and by all Hermitian matrices have real dimensions 16, 32 and 32; and sum_a x_a Gamma_a = 2 (P_w + P_{jw}) - 1. For example (b), by quadrature with 23 test functions: dim O_F = 4 and the generated algebra has dimension 8; the two displayed values Q(f_1) and Q(f_2) were reproduced by quadrature.

Part 22

The finite Dirac operator as a boundary condition on an interval

For STRATEGY T0, T0b, T10 and T13, following lemma 12 and lemma 20. Proved by Claude Opus 5.5, with the identities checked in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): two errors and one gap as first written, and amendments; all are incorporated below.

Outcome. Let (H_F, D_F, J_F, Gamma_F) be finite data of signs (+, +, -), and let l ||D_F|| < pi/2. Then D_F, with J_F and Gamma_F, is exactly the lowest spectral window of the pure derivative on an interval of length l, with no bulk mass and no Yukawa term, on the domain fixed by the boundary coupling tan(l D_F). So all the eigenvalues of D_F, Yukawa and Majorana blocks alike, are quantized internal momenta k with tan(k l) equal to a singular value of the coupling. The input is the finite triple: the content is that a consistent parent exists, with these properties:

- Preservation of the domain forces the observables at the wall into C_F, the commutant of D_F in A_F, while A_F acts freely in the bulk.
- The parent satisfies order one for any finite data with order zero.
- The compression is the exact average Q_l(a) = (2l)^-1 int_{-l}^{l} e^{ixD_F} a e^{-ixD_F} dx, so it tends to the finite representation as l -> 0, with defects of order l^2.
- The scalar inner fluctuations of the parent are pure gauge under its allowed endpoint gauge group. On the moving window, they act by the gauge orbit of D_F through the transport along the interval. This reproduces the quadratic term of Chamseddine-Connes-van Suijlekom on the orbit, and it leaves the singular values fixed, so the radial mode of the Higgs field is absent from this parent.

With the family X(l) = tan(l M) and l -> 0, the couplings of the light fermions vanish as m l at fixed mass, the tower decouples, and the broken gauge directions go to pi/(2l). The parent algebra is matrix-valued, so T0 for a commutative parent remains open.

Setting

Let H_F = H_+ (+) H_- be finite-dimensional, Gamma_F = diag(1, -1), let c : H_+ -> H_- be antiunitary, and put J_F(a, b) = (c^-1 b, c a), so that J_F^2 = 1 and J_F Gamma_F = -Gamma_F J_F. Let D_F = [[0, M*], [M, 0]] with M : H_+ -> H_-. Then J_F D_F = D_F J_F is equivalent to M = c M* c. By the Takagi factorization, in bases related by c, we may write M = V |M| with a unitary V satisfying V = c V* c.

Let l > 0 with l ||D_F|| < pi/2, and define X : H_- -> H_+ by

`tan(l D_F) = [[0, X], [X\*, 0]],  that is,  X = tan(l |M|) V\*.`

Since J_F commutes with real functions of D_F, c X = X* c^-1. Conversely, every X with c X = X* c^-1 arises from exactly one such D_F, namely D_F = l^-1 arctan of the block operator. In the limit, X = l M* + O(l^3).

The parent. On H_l = L^2([0, l]; C^2) (x) (H_+ (+) H_-), write psi = (Q_L, Q_R; U_L, U_R), with Q valued in H_+ and U in H_-. Let (D psi)_L = psi_R' and (D psi)_R = -psi_L' on both fields, the operator of lemma 12 with no bulk mass, on the domain

```
Q_R(0) = 0,   U_L(0) = 0,   Q_R(1) = X U_R(1),   U_L(1) = -X\* Q_L(1).
```

Call it D_X. Put Gamma^ psi = (Q_L, -Q_R; U_L, -U_R) and J^ psi = (c^-1 U_R, -c^-1 U_L; -c Q_R, c Q_L). Let (E psi)_L = psi_R and (E psi)_R = -psi_L, so that D_X = E d/dx on its domain.

In the Standard Model form of the classification, with E = Hom(V, W) and W = W_+ (+) W_-, one takes H_+ = E_+ (+) E_-* and H_- = E_- (+) E_+*, with c the adjoint map. At X = 0, the Q-fields have boundary type (a) and the U-fields type (b), in the terms of lemma 09; for X != 0 the modes continue from those zero modes, which generally lift. A single matrix X encodes the Yukawa blocks and the Majorana block together. Its entries need not keep the original block pattern, since the tangent brings in higher powers of D_F: for M = [[a, b], [b, 0]] real symmetric, X_22 = (l^3/3) a b^2 + O(l^5).

The window

Theorem 22.1. D_X is self-adjoint with compact resolvent, and J^^2 = 1, J^ D_X = D_X J^, J^ Gamma^ = -Gamma^ J^, Gamma^ D_X = -D_X Gamma^. Let R(x) = exp(x Gamma_F D_F), and define W : H_F -> H_l so that for phi = (a, b), the L-components (Q_L, U_L) of W phi at x are l^{-1/2} R(x)(a, 0), and the R-components (Q_R, U_R) are l^{-1/2} R(x)(0, b). Then W is an isometry and

```
D_X W = W D_F,    J^ W = W J_F,    Gamma^ W = W Gamma_F.
```

With sigma_k the singular values of M and theta_k = l sigma_k, the signed spectrum of D_X is (n pi +- theta_k)/l, n in Z, with multiplicities added when values coincide; a zero singular value gives a zero mode of multiplicity two. So the spectrum of |D_X| consists of the sigma_k and of (n pi +- theta_k)/l for n >= 1, and ran W is the spectral subspace of |D_X| <= ||D_F||, and the next level is at least (pi - l ||D_F||)/l.

Proof. Self-adjointness is Lemma 12.1. Since Gamma_F D_F is anti-Hermitian, R(x) is unitary, and with C(x) = cos(x |M|), C'(x) = cos(x |M*|), S(x) = V sin(x |M|),

```
R(x)(a, 0) = (C(x) a, -S(x) a),    R(x)(0, b) = (S(x)\* b, C'(x) b).
```

Isometry follows from the unitarity of R(x) and the orthogonality of the L- and R-components. The conditions at 0 hold since S(0) = 0. At l, using V* cos(l |M*|) = cos(l |M|) V*, we get X C'(l) = S(l)* and X* C(l) = S(l), which are the conditions at the wall. For the derivative, the L-part of D W phi is d/dx R(x)(0, b) = R(x) Gamma_F D_F (0, b) = R(x)(M* b, 0), and the R-part is -d/dx R(x)(a, 0) = R(x)(0, M a); together these are W D_F phi. The relations for Gamma^ are direct. For J^, use c |M| c^-1 = |M*| and V = c V* c, which give c^-1 C' = C c^-1 and c S(x)* = S(x) c^-1. The signs of the parent follow in the same way, using c X = X* c^-1 for the domain. For the spectrum, constant unitaries on Q and on U commute with E d/dx and bring X to diag(tan theta_k), which reduces to Example 12.5 in each channel. ■

Corollary 22.2 (conjugated form). D_X = R (D_0 + E Gamma_F D_F) R*, with R acting pointwise on the pair (Q, U) and D_0 the uncoupled operator, X = 0. So after this unitary change of domain, D_F is a constant matrix connection, odd for the grading, with relative endpoint rotation R(l). This is equation (21) of lemma 20, for general D_F.

Proof. D(R phi) = R(D phi + E R* R' phi), and R* R' = Gamma_F D_F. For phi in the uncoupled domain, the computation at the wall in the proof above shows that R phi satisfies the coupled conditions. ■

So in this family a fermion mass is a momentum: D_X has no mass term, and the light eigenvalue k = theta/l is fixed by tan(k l) = tan(theta), a singular value of X. In the finite description, the condition relating k to l is lost and the value appears as free data in D_F.

The parent as a real spectral triple

Let (A_F, H_F, D_F, J_F, Gamma_F) be finite data as above, with an even representation pi = pi_+ (+) pi_- of a unital finite-dimensional real or complex *-algebra A_F, faithful, unital and satisfying order

zero, with pi°(b) = J_F pi(b)* J_F^-1. Order one is not assumed. Let

```
C_F = {a in A_F : [pi(a), D_F] = 0},    A_l = {f in C^infinity([0, l], A_F) : f(l) in C_F},
```

with f acting pointwise by pi_+(f(x)) on Q and pi_-(f(x)) on U.

Theorem 22.3.

1. A function f in C^infinity([0, l], A_F) preserves Dom D_X if and only if f(l) lies in C_F.
2. (A_l, H_l, D_X, J^, Gamma^) is a real even spectral triple, with [D_X, pi(f)] = E pi(f') and J^ pi(g)* J^^-1 = pi°(g) pointwise.
3. It satisfies order zero and order one, with signs (+, +, -).

So A_F acts freely in the bulk, and the boundary condition restricts the observables at the wall to the commutant of D_F. In the Standard Model data with nonzero Yukawa blocks, this is the algebra {(lambda, lambda, m)} of the photon and gluons; with vanishing Yukawa blocks the commutant is larger. Order one holds in this parent whatever the finite data, for its pointwise algebra: the finite condition appears only on the truncation, and arbitrary compressions need not preserve either order condition.

Proof. Item 1: at l, the traces Q_L(l) and U_R(l) are free, and applying f to the conditions gives pi_+(f(l)) X = X pi_-(f(l)) and pi_-(f(l)) X* = X* pi_+(f(l)), as in Result 3 of lemma 20. That is [pi(f(l)), tan(l D_F)] = 0, equivalent to [pi(f(l)), D_F] = 0 because l ||D_F|| < pi/2 makes D_F a function of tan(l D_F). At 0 the conditions are preserved since pi is even. Item 2: the commutator is the Leibniz rule. Conjugation by J^ acts on (Q, U) as J_F does, and pi(g) acts identically on the two spin components, which gives the second formula. This preserves the domain since J_F commutes with D_F. Item 3: pi(f(x)) and pi°(g(x)) commute at each x by finite order zero, and E commutes with both; so [E pi(f'), pi°(g)] = E [pi(f'), pi°(g)] = 0. ■

This is consistent with Counterexample 4 of lemma 20, whose finite data A_F = C + C, D_F = m sigma_1 fail order one while the parent satisfies it.

The compression

Let Q_l(T) = W* T W for a bounded operator T on H_l. For T(x) a bounded function with values in the even operators on H_F, acting pointwise and identically on both spin components, we write T for the multiplication operator.

Theorem 22.4.

```
Q_l(T) = l^-1 int_0^l cos(x ad_{D_F})(T(x)) dx,      Q_l(E T) = -l^-1 int_0^l sin(x ad_{D_F})(T(x))
↪ dx.
```

For constant a in A_F,

```
Q_l(a) = sinc(l ad_{D_F})(pi(a)) = (2l)^-1 int_{-l}^{l} e^{ix D_F} pi(a) e^{-ix D_F} dx,
```

so in an eigenbasis of D_F with eigenvalues mu_i, the matrix elements are pi(a)_{ij} sin(l (mu_i - mu_j))/(l (mu_i - mu_j)). Moreover:

1. Q_l(A_l) = Q_l(C^infinity([0, l], A_F)), so the image of the parent algebra contains Q_l(a) for every a in A_F.
2. ||Q_l(a) - pi(a)|| <= (l^2/6) |||[D_F, [D_F, pi(a)]]|| <= (2/3) l^2 ||D_F||^2 ||a||.
3. Q_l([D_X, f]) = [D_F, Q_l(f)] for f in A_l.
4. With a' = Q_l(a) and b' = J_F Q_l(b)* J_F^-1, ||[a', b']|| <= (8/3) l^2 ||D_F||^2 ||a|| ||b||, and |||[[D_F, a'], b'] - [[D_F, pi(a)], pi°(b)]|| <= (16/3) l^2 ||D_F||^3 ||a|| ||b||.

So as l -> 0 with D_F fixed, D_F, J_F and Gamma_F stay exact, and the compressed observables converge to pi(A_F) with rate l^2. The finite order conditions hold in the limit exactly when they hold for the finite data.

Proof. From the definition of W, the expectation of T splits into the L- and R-parts, so Q_l(T) is l^-1 int of the Gamma_F-even part of R(x)* T(x) R(x). Put K = Gamma_F D_F. For even Z, ad_K Z = Gamma_F

[D_F, Z], which is odd, and ad_K (Gamma_F [D_F, Z]) = -[D_F, [D_F, Z]]. So the even part of e^{-xK} T e^{xK} is cos(x ad_{D_F})(T). Likewise, E exchanges the L- and R-parts with a sign, so Q_l(E T) is l^-1 int of Gamma_F times the odd part of R* T R, and that odd part is -Gamma_F sin(x ad_{D_F})(T). The formula for constant a follows from cos(x ad_D)(Z) = (e^{ixD} Z e^{-ixD} + e^{-ixD} Z e^{ixD})/2.

Item 1: the map f -> Lambda(Q_l(f)), for a linear functional Lambda, is integration against a smooth density. If it vanishes on A_l, which contains all f of compact support in [0, l), the density vanishes on [0, l), hence everywhere by continuity. Item 2: by Taylor's formula with integral remainder, ||cos(x ad_D)(Z) - Z|| <= (x^2/2) ||ad_D^2 Z||, since cos(x ad_D) has norm at most 1 by the average formula; averaging over x gives l^2/6. Item 3: P commutes with D_X. It also follows from the two formulas by integration by parts, where the boundary term at l vanishes because f(l) is in C_F. Item 4: J_F commutes with D_F, so b' is the same average of pi°(b). Expand both commutators with the bound of item 2 and ||[D_F, Z]|| <= 2 ||D_F|| ||Z||. ■

The convergence is a statement about the maps Q_l, on the operator systems, in the setting of Connes and van Suijlekom (arXiv:2004.14115, sections 1 and 3). The generated algebra is not continuous at l = 0. An analytic example, from the referee: let M = [[0, 0, 1], [0, 0, 1], [1, 1, 0]], D = [[0, M], [M, 0]], with A_F = C^3 acting diagonally and equally on the two blocks. With P_1, P_2 the first two channel projections, [Q_l(P_1), Q_l(P_2)]_{12} = -l^2/3 + O(l^4), so the compressions of commuting projections generate a noncommutative algebra for small nonzero l. On the one-generation Standard Model data, a numerical closure with tolerance 10^-7 gives complex dimension 15 for the algebra generated by pi(B), with B = C + H + M_3(C), and 30 for Q_l(B) at l = 0.3; these are numerical values with no certified rank.

Scalar fluctuations

Theorem 22.5. Let h be smooth with values in the anti-Hermitian elements of A_F. The self-adjoint one-forms of the parent are exactly the E pi(h). Let u' = -h u, u(0) = 1, and let w = u(l), a unitary of A_F. Then

```
D_X + E pi(h) = pi(u) D_{X_w} pi(u)\*,      X_w = pi_+(w)\* X pi_-(w),
```

and the window of D_{X_w} carries pi(w)* D_F pi(w). With the real structure, for A = E pi(h) we have J^ A J^^-1 = E J_F pi(h) J_F^-1, and D_X + A + J^ A J^^-1 has the window U_w* D_F U_w, with U_w = pi(w) J_F pi(w) J_F^-1, and this commutes with J_F. Moreover, with u~(x) = u(x) w*, which lies in A_l since u~(l) = 1, and U~ = pi(u~) J_F pi(u~) J_F^-1,

```
D_X + A + J^ A J^^-1 = U~ D_X U~\*.
```

So every scalar connection in this model is gauge equivalent to the original parent under the allowed endpoint gauge group (referee).

Proof. A one-form is sum f_1 [D_X, f_2] = E pi(sum f_1 f_2'), and E pi(h) is self-adjoint exactly when h is anti-Hermitian, since E is anti-Hermitian and commutes with pi(h) and pi is faithful. Every smooth such h arises: with b(x) = int_l^x h, we have b(l) = 0, so b is in A_l, and [D_X, b] = E pi(h). Next, pi(u)* D pi(u) = D + E pi(u* u'), and pi(u)* E pi(h) pi(u) = E pi(u* h u), so the sum is D with no connection term. At 0 the conditions are unchanged. At l, the substitution psi = pi(u) phi gives the conditions with X_w. Then tan(l pi(w)* D_F pi(w)) has the block X_w. For general w, this operator need not commute with J_F, so we use the spectrum and the isometry of Theorem 22.1 without its real-structure part. For the real fluctuation, J^ commutes with E, u(x) is replaced by pi(u) J_F pi(u) J_F^-1, using finite order zero, and U_w commutes with J_F, so the real window follows. The last formula follows by differentiating, with the endpoint condition. ■

Corollary 22.6 (the quadratic term as transport). By Chamseddine-Connes-van Suijlekom (arXiv:1304.7583, section III), for unitary v in A_F, with U = v J v J^-1, one has U D U* = D + A_(1) + A~*(1)* + *A*(2), for the universal form v delta(v*). Taking v = w*, on the gauge orbit the fluctuation of the parent, which satisfies order one and has no quadratic term, gives on the moving window exactly the fluctuated operator of that framework (equations (10)-(11) of section III.2 there), quadratic term included, for finite data that fail order one. This answers the first open point of lemma 19 on these gauge orbits and

with the moving window. For the two-point data with J = sigma_1 K and w = diag(e^{i alpha}, e^{i beta}), it gives the upper link m e^{-2 i (alpha - beta)}, which is m e^{2 i psi} of Corollary 19.2 with psi defined from v = w*. Theorem 22.5 and the covariant projection of lemma 14 share a transport principle. The fixed compression, as in lemma 13, is a different object: by Theorem 22.4, it gives Q_l(E pi(h)) = -[D_F, l^-1 int_0^l x pi(h) dx] + O(l^3 ||D_F||^3 ||h||) for h fixed, so that for constant h it is -(l/2)[D_F, h], while the first derivative of the transported operator is -l [D_F, h]. The difference is the contribution of the moving spectral projection.

No-go theorem 22.7 (the radial mode). *Hypotheses:* the parent above, on one interval, with fixed X, no bulk masses, and scalar inner fluctuations from the even algebra A_l, with the moving spectral window. *Conclusion:* the fluctuations leave all singular values of the finite Dirac operator fixed. In the usual Standard Model representation with nonzero Yukawa data, the normalized shifted Higgs quaternion runs through the unit quaternions, a three-dimensional gauge orbit, with no radial mode. Degenerate data can have a smaller orbit. A fixed compression can show changes in singular values at second order, so the window has to be named.

Proof. Theorem 22.5: D_F and pi(w)* D_F pi(w) are unitarily equivalent. The same holds with U_w. ■

Known escapes.

These lie outside the hypotheses, and each needs its own construction.

- The coupling X as a field on the wall, with its own kinetic term and potential.
- Connections odd for the grading, as in Corollary 22.2 and lemma 23: the fluctuation K -> K + delta K changes the singular values.
- Bulk masses that do not commute with the gauge action.
- Two or more internal dimensions, where a connection has curvature, although curvature alone does not prove a radial mode.
- An additional composite or other degree of freedom; a quantum effective action on the same configuration space does not by itself add a radial coordinate.
- A fluctuation of the finite data alone, as with the radial parameter of lemma 19, which is not the moving window.

Direction. On the interval, the even connection is pure gauge up to its boundary effect, so the finite orbit is all it can reach. The physical Higgs modulus must then be sought at the wall, in odd directions, or in more dimensions. At the 10 -> 9 endpoint this agrees with T10, where the Higgs is frozen.

The gauge sector and the 10 -> 9 family

Proposition 22.8. Consider the product of M^4 with this parent, with algebra of functions f(y, x) valued in A_F and f(y, l) in C_F. Then the gauge one-forms sum f_1 d_mu f_2 take values in C_F at x = l. Assume, in addition, a flat product, a positive five-dimensional Yang-Mills kinetic form, the trivial background, unrestricted endpoint variation in the allowed directions, and no additional boundary kinetic or mass terms. Then the directions in the Lie algebra of C_F have masses n pi/l, n >= 0, and the orthogonal directions have masses (n - 1/2) pi/l, n >= 1. The physical gauge algebra comes from the skew-adjoint elements through the real gauge representation, with its kernel and any unimodularity condition.

Proof. The first statement is direct evaluation at x = l. The rest is the interval spectrum in the gauge A_5 = 0, with Neumann conditions at 0, and with Neumann or Dirichlet conditions at l. ■

In the Standard Model data with nonzero Yukawas, M_W = M_Z = pi/(2l). The Lie algebra of C_F has dimension 10 before unimodularity; with the unimodularity condition of Chamseddine-Connes-Marcolli (arXiv:hep-th/0610241, section 2.5), the massless modes are the photon and the gluons. These are the boundary conditions of Higgsless models (Csaki, Grojean, Murayama, Pilo and Terning, Phys. Rev. D 69 (2004) 055006), and the fermion formula tan(m L) = M_D L is equation (5.6) of Csaki, Grojean, Hubisz, Shirman and Terning (Phys. Rev. D 70 (2004) 015012); paper/LITERATURE.md, section C. Boundary spectral geometry for the Standard Model has been studied by Chamseddine and Connes (arXiv:1008.3980, section 5). What is added here, as far as our searches went, is the combination: the real structure, the finite

triple as an exact window, the wall algebra C_F, the compression formula and the fluctuations. A wider claim of novelty would need a fuller comparison.

Corollary 22.9 (the 10 -> 9 family). Fix a block M_0 for the light fermions and a block Theta/l on an orthogonal channel, compatible with J_F and the grading, with 0 < Theta < pi/2, for the top and if wanted for the Majorana block. Let D_F(l) have M(l) = M_0 (+) Theta/l, and let l -> 0. Then:

1. X(l) has the light block tan(l |M_0|) V_0* = l M_0* + O(l^3) and a heavy block with singular values tan(Theta): the light couplings vanish as m l at fixed mass m.
2. The heavy channel has mass Theta/l, the broken gauge directions have mass pi/(2l), and the other levels diverge.
3. The massless gauge sector is C_F. Let P_L be the projection reducing D_F(l) to the fixed light block D_L. For constant a, P_L Q_l(a) P_L = sinc(l ad_{D_L})(P_L pi(a) P_L), with ||P_L Q_l(a) P_L - P_L pi(a) P_L|| <= (2/3) l^2 ||D_L||^2 ||pi(a)||, also when pi(a) mixes light and heavy channels. The limiting light compression need not be a representation, since products can pass through the heavy subspace. The full estimate of Theorem 22.4 does not apply, because l ||D_F(l)|| tends to Theta. For example, with D_F(l) = sigma_1 (x) diag(m, Theta/l) and pi(a) = 1 (x) sigma_1, one has Q_l(a) = sinc(Theta - l m) pi(a), which tends to sinc(Theta) pi(a).
4. With l = pi/(2 M_W), we get X_q ~ (pi/2) m_q/M_W, which has the form of y_q = sqrt(2) m_q/v -> 0 at fixed mass, as the coupling vanishes while the mass stays. Identifying X_q with the four-dimensional Yukawa coupling is a hypothesis.
5. The ratio is m_t/M_W = 2 Theta/pi < 1, and M_W/M_Z = 1.

So the 10 -> 9 proposal is realized at this level as a limit l -> 0, with W, Z and top at the tower scale and the light fermions keeping explicit masses. Item 5 does not reach m_t/M_W = sqrt(8/3), which belongs to the interior. This is a statement in the stated family: lowest level, no bulk mass, strict wall algebra. Bulk masses for the top, and a changed gauge wall condition, are directions beyond it.

A numerical comparison for the interior (Python). A separate model, with a finite wall scalar and Neumann conditions at 0, lies outside Proposition 22.8 and No-go theorem 22.7. In it, the mass of a broken direction satisfies t tan t = s, with t = M l and s = g^2 v^2 l^2/4 (for the Z, s/cos^2 theta_0), which has the form of M_k tan(M_k pi R) = -V in Csaki et al. (arXiv:hep-ph/0305237, section on mixed boundary conditions, read in the LaTeX source), with the sign convention of V there. If g'^2/g^2 = 3/5 is taken at all scales, M_W^2/M_Z^2 goes from 5/8 as s -> 0 to 1 as s -> infinity. It equals the measured 0.777 at M_W l = 1.03, with 1/l = 78 GeV and a next W level at 280 GeV; whether collider searches allow this would need the couplings, branching fractions and an applicable limit, which we have not analyzed. The top needs M_W l < 0.96 for m_t/M_W = sqrt(8/3), or M_W l < 0.73 for the measured ratio, in the lowest level with no bulk mass. So in the flat tree-level toy without bulk masses or running, the top and gauge requirements are incompatible. It does show the interior interpolating between the spectral-action value 5/8 and the endpoint value 1. The directions are bulk masses, warping and running.

What is input and what is derived

The finite data (A_F, H_F, D_F, J_F, Gamma_F) are all inputs, so the existence of the window is close to a reparametrization: X = tan(l D_F). The derived statements are the wall algebra C_F for that input, the order conditions of the parent, the compression formula with its rate, the form of the fluctuations, and the gauge spectrum under the stated assumption. The window identifies D_F, J_F and Gamma_F exactly, while the represented algebra is obtained through compression and is not an exact representation of A_F at l > 0. In particular, this does not select A_F. T0 asks for a commutative parent: a natural next step is a two-stage parent, compressing functions on S^4 and CP^3 at level one to get A_F-valued functions on the interval, and then taking this window, where the difficulty is the same interface problem as after Result 7 of lemma 20.

Numerical check

By Claude Opus 5.5, in Python (numpy and scipy), 3 Nov 2026, in a temporary directory, not kept.

- Random complex symmetric M on C^5, l ||M|| = 1.3: D u_+(a) = u_-(M a) and D u_-(b) = u_+(M* b) by finite differences, to about 10^-9; boundary conditions and J^ to about 10^-15; norm of W to 10^-13.
- Q_l(a) = sinc(l ad)(a), by quadrature, to 2 x 10^-8. The formula for Q_l(E T), with a nonconstant T, to 10^-14.
- One-generation Standard Model data with Yukawas and the rank-one Majorana block, with signs, order zero and order one exact on B = C + H + M_3(C), and commutant of D in B of real dimension 20. At l = 0.1 and 0.03, ||Q_l(a) - a + (l^2/6)[D, [D, a]]|| is 10^-5 and 4 x 10^-8, and the order-one defect of Q_l is 1.5 x 10^-4 and 1.2 x 10^-5, which scales as l^2.
- The dimensions of the generated algebras quoted above, 15 and 30, from a numerical closure with tolerance 10^-7, not retained and not certified; at smaller l the numerical rank depends on the tolerance.
- The roots of t tan t = s, and the values quoted in the comparison for the interior.
- On the two-point data with J = sigma_1 K: D + A_(1) + A~*(1)* + *A*(2) = U D U* to 4 x 10^-16, and the link of U_w* D U_w equal to m e^{-2 i psi}.
- The bounds of Theorem 22.4, items 2 and 4, on 30 random samples of the one-generation data: largest ratios to the bound 0.33, 10^-4 and 10^-3.

Part 23

Two gradings on an interval: masses as principal angles, the Higgs modulus, and exact W and Z masses

For STRATEGY T0b, T5b, T10 and T13, following lemma 22. Proved by Claude Opus 5.5, with the identities checked in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): four errors and one gap as first written, and amendments, all incorporated below. The simpler form of the mass formulas, and the proof of item 3 of Theorem 23.6, are by the referee.

Outcome. A self-adjoint boundary condition for the derivative on an interval is a grading at each wall. The whole finite Dirac operator is then the relative position of two gradings: its singular values are the principal angles between them, divided by l, and

```
D_F = (1/2l) Gamma_0 log(Gamma_1 Gamma_0).
```

With a bulk prealgebra, this gives four results.

1. Aligned walls compress a prealgebra stable under the grading to its even part, with the grading inherited, which is the step that Example 16.9 had to add by hand.
2. Odd scalar fluctuations move the second grading, so they change the singular values and supply the radial mode that No-go theorem 22.7 excludes for even fluctuations. For M_2(H) the field space is S^4, and the Higgs modulus is half the geodesic distance between the points chosen by the two walls.
3. With a wall connector on one channel and a five-dimensional Yang-Mills action with common normalization, the masses are exact: sin^2(M_W l) = sin^2(2 theta)/2 and sin^2(M_Z l) = (4/5) sin^2(2 theta), for universal fermions of mass theta/l. At small theta this gives M_W^2 = 2 m^2 and M_Z^2 = (16/5) m^2, the constants in the lower bounds of the fully massive family of lemma 18, with M_W^2/M_Z^2 tending to 5/8 at both ends.
4. The heat-cutoff spectral action on the interval is a Jacobi theta function of theta, decreasing on [0, pi/2]. This family lies outside the hypotheses of the radial no-go of lemma 18: the radial direction is compact, with a stationary point at m = pi/(2l), the Connes-Lott value, where M_W = M_Z = 0.

Everything finite is still an input, and the parent algebra is matrix-valued. For a commutative parent, lemma 24 finds the window exactly, with spectral walls, while direct compression of scalar functions gives only R + R on the S^4 factor. The formulas for gauge masses depend on the assumed action. The relation to measured values fails in the flat toy, as stated at the end.

Two gradings

Let H be a finite-dimensional Hilbert space, and let Gamma_0, Gamma_1 be self-adjoint involutions on H. On L^2([0, l]; C^2) (x) H, with psi = (psi_L, psi_R), let D = E d/dx, where (E psi)_L = psi_R and (E psi)_R = -psi_L, on the domain

```
Gamma_0 psi_L(0) = psi_L(0),   Gamma_0 psi_R(0) = -psi_R(0),   Gamma_1 psi_L(l) = psi_L(l),
↪  Gamma_1 psi_R(l) = -psi_R(l).
```

Call it D(Gamma_0, Gamma_1), and let Gamma^ act as +1 on psi_L and -1 on psi_R.

Theorem 23.1.

1. D(Gamma_0, Gamma_1) is self-adjoint with compact resolvent, and it anticommutes with Gamma^.
2. Suppose Gamma_0 and Gamma_1 have eigenspaces of equal dimensions, and that -1 is not an eigenvalue of Gamma_1 Gamma_0. Let theta_k in [0, pi/2) be the principal angles between the +1 eigenspaces. Then K = (1/2l) log(Gamma_1 Gamma_0), with the principal branch, is anti-Hermitian, Gamma_0 K Gamma_0 = -K, and D_F = Gamma_0 K is self-adjoint, odd for Gamma_0, with singular values theta_k/l on the +1 eigenspace. Moreover Gamma_1 = e^{lK} Gamma_0 e^{-lK}.
3. With H_+, H_- the eigenspaces of Gamma_0, and X the H_- -> H_+ block of tan(l D_F), as in lemma 22, D(Gamma_0, Gamma_1) = D_X, and the spectral part of Theorem 22.1 gives the spectrum and the window, with no real structure needed. Conversely, every D_F odd for Gamma_0 with l ||D_F|| < pi/2 arises from exactly one such Gamma_1.
4. Let an antiunitary J_1 satisfy J_1^2 = -1, J_1 Gamma_i J_1^-1 = -Gamma_i for i = 0, 1. Then (J^ psi)_L = J_1 psi_R, (J^ psi)_R = -J_1 psi_L defines a real structure of signs (+, +, -) for D(Gamma_0, Gamma_1). With J_F = Gamma_0 J_1 up to sign, J_F^2 = 1, J_F Gamma_0 = -Gamma_0 J_F and J_F D_F = D_F J_F, which are the signs of lemma 22.

So J_F itself does not anticommute with Gamma_1 in general: J_F Gamma_1 J_F^-1 = -Gamma_0 Gamma_1 Gamma_0, the reflection of Gamma_1 through Gamma_0.

Proof. Item 1: the boundary form is [<psi_R, phi_L> - <psi_L, phi_R>] evaluated between 0 and l, which vanishes on the domain because L-values and R-values lie in orthogonal eigenspaces at each wall, and these are half of the boundary data.

Item 2: U = Gamma_1 Gamma_0 is unitary and Gamma_0 U Gamma_0 = Gamma_0 Gamma_1 = U^-1, so K is anti-Hermitian and anticommutes with Gamma_0. The eigenvalues of U are e^{+-2 i theta_k}, by the standard two-projection decomposition, and K then has eigenvalues +-i theta_k/l. Since e^{lK} Gamma_0 e^{-lK} = e^{2lK} Gamma_0 = U Gamma_0 = Gamma_1, the last formula follows.

Item 3: in the notation of Theorem 22.1, R(l) = e^{lK} carries H_+ to the graph {(C a, -S a)} = graph(-X*), and H_- to graph(X). These are the +1 and -1 eigenspaces of Gamma_1, which is the condition at the wall. The converse uses the same formula, with uniqueness from injectivity of the exponential on the stated range.

Item 4: J^ commutes with E by direct computation, and J^^2 = -J_1^2 = 1. The domain is preserved since J_1 exchanges the eigenspaces of each Gamma_i. For the last statement, J_1 U J_1^-1 = U, so as J_1 is antilinear, J_1 K J_1^-1 = K in the sense of the principal branch, and J_F = Gamma_0 J_1 then satisfies J_F K J_F^-1 = -K and commutes with D_F = Gamma_0 K. ■

When theta_k = pi/2 the channel has type (c) of lemma 09, with no zero mode and lowest level pi/(2l), the Connes-Lott mass of Theorem A in the paper. There the principal logarithm and the isolated window cease to be unique.

The bulk prealgebra

Let P be a *-subalgebra of End(H), which need not commute with Gamma_0. Let

```
A(Gamma_0, Gamma_1) = {f in C^infinity([0, l], P) : [f(0), Gamma_0] = 0, [f(l), Gamma_1] = 0},
```

acting pointwise and identically on both spin components.

Theorem 23.2.

1. A function f in C^infinity([0, l], P) preserves the domain exactly when it lies in A(Gamma_0, Gamma_1). The commutators are [D, f] = E f', and Gamma^ commutes with every f, including those with values odd for Gamma_0.
2. Under the hypotheses of Theorem 23.1, item 2, with W the isometry of Theorem 22.1, Q(T) = l^-1 int_0^l E_0(R(x)* T(x) R(x)) dx, where E_0(Z) = (Z + Gamma_0 Z Gamma_0)/2 is the conditional expectation onto the even part.
3. *Aligned walls.* If Gamma_1 = Gamma_0, then D_F = 0, the window consists of constant L-values in H_+ and constant R-values in H_-, and Gamma^ restricts on it to Gamma_0. For an ambient constant multiplier a in P, Q(a) = E_0(a), and the image of the boundary algebra A(Gamma_0, Gamma_0) is E_0(P), by compactly supported profiles. If Gamma_0 P Gamma_0 = P, then E_0(P) is the even part of P, so the compressions of the prealgebra are exactly its even part, with the grading inherited from the even parent operator Gamma^. Without that hypothesis this can fail: for H = C^2, Gamma_0 = sigma_z and P generated by sigma_x + sigma_z, E_0(P) = span{1, sigma_z}, while the even part of P is C.
4. *Order conditions.* Let J_1 be as in Theorem 23.1 and let P° = {J_1 b* J_1^-1 : b in P} commute with P. Then the parent has order zero and order one for A(Gamma_0, Gamma_1).

Proof. Item 1: as in Theorem 22.3, using that the conditions at a wall say that L- and R-values lie in the eigenspaces of Gamma_i. Item 2: as in the proof of Theorem 22.4. Item 3 is the case R = 1. Item 4: [D, f] = E f', and pointwise commutation gives both conditions. ■

Item 3 evades the inherited-grading obstruction of Result 11 of lemma 20, because hypothesis 4 there fails: for odd a, here Q(a) = 0, while there the coordinates are retained as Q(x_a) = Gamma_a/5. So at the level of a matrix-valued parent this addresses the question left in step 3 of the framework note: the prealgebra is the bulk algebra, the even algebra is what the window sees, and the grading is a boundary condition. The commutative-parent and retained-coordinate questions remain open, and Theorem 4 of lemma 24 shows a real difference: scalar functions compressed directly give R + R on the S^4 factor, and H + H needs the matrix algebra generated at the intermediate stage.

Remark 23.3 (the twist). On H = E (+) E*, with J(x, y) = (y*, x*), the operator J_1 = Gamma_0 J is the one that anticommutes with both gradings. For an odd anti-Hermitian h in P, J_1 h J_1^-1 is y -> +y h on E*, where J alone gives y -> -y h. So the opposite representation b° = J_1 b* J_1^-1 incorporates on the W-factor the grading automorphism sigma(b) = gamma_W b gamma_W. In this parent the twist comes from the requirement that J_1 preserve both boundary conditions. Twisted spectral triples for the grand symmetry (Devastato, Lizzi and Martinetti, JHEP 01 (2014) 042, arXiv:1304.0415; Devastato and Martinetti, Math. Phys. Anal. Geom. 20 (2017) 2, arXiv:1411.1320, Eqs. (1.8)-(1.11)) use a different automorphism: their rho exchanges left and right spinorial indices and algebra summands, while conjugation by gamma_W is inner on M_2(H) and fixes its centre. So this is a structural analogy between involutive automorphisms, and an identification needs a comparison of the representations, opposites and first-order conditions.

Odd fluctuations and the radial mode

Theorem 23.4. Let h be smooth with values in the anti-Hermitian elements of P, and let u' = -h u, u(0) = 1, w = u(l). Then

```
D(Gamma_0, Gamma_1) + E h = u D(Gamma_0, w\* Gamma_1 w) u\*.
```

So scalar fluctuations move the second grading along its orbit under the unitary group of P. If h has an odd part, the principal angles between Gamma_0 and w* Gamma_1 w change in general.

Proof. u* D u = D + E u* u', and u* E h u = E u* h u, so the connection cancels. Since u(0) = 1, the wall at 0 is unchanged. At l, psi = u phi gives the stated grading for phi. ■

This lies outside No-go theorem 22.7, whose hypotheses include the even algebra: an even connection gives a w commuting with Gamma_0, and the angles are kept, while an odd connection changes them. For a real fluctuation, add the conjugate one-form: D + E h + J^(E h)J^^-1, with generator of the form (L_F, R_F)

in Example 23.5. The invariant is a double coset. With H_i = Stab(Gamma_i) in the connected unitary group G of P, a gauge transformation removing h has the form u(x) c, and it preserves both endpoints exactly when c is in H_0 and w c in H_1, that is, when w is in H_1 H_0. So the scalar moduli of the parent are the double cosets H_1 G / H_0, the relative position of the two gradings. This is with the moving window, as in lemma 22.

Example 23.5 (M_2(H) and S^4). Let W = H^2 = C^4, with the Clifford generators Gamma_a of lemma 16, and for a unit vector p in R^5 put Gamma_p = p_a Gamma_a. Each grading of W with one-dimensional quaternionic +1 and -1 eigenspaces is such a Gamma_p, so these gradings form S^4 = Sp(2)/(Sp(1) x Sp(1)), the quaternionic projective line. For two points, Gamma_q Gamma_p = p . q + (p wedge q), with eigenvalues e^{+-i d}, where d is the geodesic distance; so all the principal angles equal theta = d/2.

Take H = Hom(V, W) (+) Hom(W, V), J(x, y) = (y*, x*), Gamma_0(x, y) = (gamma x, -y gamma) with gamma = Gamma_{p_0}, and K(x, y) = (theta/l)(F x, y F), with F an odd anti-Hermitian element of M_2(H), F^2 = -1. Then:

- D_F = Gamma_0 K has all singular values equal to theta/l: a universal Dirac mass, for all colours.
- The second grading is Gamma_1(x, y) = (Gamma_p x, -y Gamma_{p'}), where p is at distance 2 theta from p_0 and p' is its mirror image through p_0. J_1 = Gamma_0 J satisfies Theorem 23.1, item 4, while J alone does not.
- The coordinates (theta, q) on S^4, with F determined by the unit quaternion q, are the polar coordinates of a Higgs quaternion: theta is the radial coordinate, bounded by pi/2, and q gives three angular directions, undefined at the poles. At theta = pi/2 the finite logarithm and the lowest window are degenerate. Quotienting by the stabilizer leaves the radial interval. A canonically normalized field needs the scalar kinetic term, which we have not computed.

With bulk masses, by Theorem 12.6, the light masses are about (tan(theta)/l) Phi_Q Phi_U when small compared with the tower thresholds, with a universal matrix before flavour data; general finite-angle eigenvalues need the secular equation. This field space is the coset of the minimal composite Higgs model (Agashe, Contino and Pomarol, Nucl. Phys. B 719 (2005) 165, sections 2-3, with their h/f_pi corresponding to 2 theta). Their construction uses a warped interval and additional fermion and boundary data, so the shared coset does not identify the actions or potentials. The appearance here of the S^4 of the twistor window is an observation, and no identification between the two roles of S^4 is made.

The weak bosons

Take P = M_2(H) (+) M_4(C), acting on E = Hom(V, W) on the left and on the right. Assume the following.

- A five-dimensional Yang-Mills field valued in sp(2) (+) su(4), with the trace form Tr_W + Tr_V, so with one coupling (the normalization of the spectral action, g_3 = g_2 = sqrt(5/3) g_Y).
- At x = 0: Neumann for the stabilizer of p_0, Dirichlet for its complement. At x = l: Neumann for the stabilizer of p_l and of a connector, that is, the a with [a_W, Gamma_p] = 0, a_W w = lambda w and a_V v = lambda v, as in Results 2 and 5 of lemma 20, for a line w in the +1 eigenspace of Gamma_p and a line v in V; Dirichlet for the complement.
- The connector is prescribed gauge-breaking wall data here. If it is instead a term in the fermion boundary operator, as for a unit vector e in the +1 eigenspace of Gamma_1, f = J_1 e, K_c = |f><e| - |e><f| and Gamma_{1,c} = e^{eta K_c} Gamma_1 e^{-eta K_c}, then J_1 still anticommutes with Gamma_{1,c}, the parent order conditions are kept, and the affected fermion tower changes, so the universal fermion formula must be recalculated for that channel.

Theorem 23.6. With theta the angle above, the modes that are massless for all theta are exactly nine, the photon and the eight gluons, with three additional electroweak zero modes at theta = 0 and theta = pi/2. The lowest charged and neutral massive modes satisfy

```
sin^2(M_W l) = sin^2(2 theta)/2,
sin^2(M_Z l) = (4/5) sin^2(2 theta),
```

equivalently tan^2(M_W l) = sin^2(2 theta)/(1 + cos^2(2 theta)) and tan^2(M_Z l) = 4 sin^2(2 theta)/(1 + 4 cos^2(2 theta)).

The photon has 1/e^2 = (8/3)/g^2. Moreover:

1. As theta -> 0, with m = theta/l: M_W^2 -> 2 m^2 and M_Z^2 -> (16/5) m^2, so M_W^2/M_Z^2 -> 5/8.
2. As theta -> pi/2: M_W, M_Z -> 0, with M_W^2/M_Z^2 -> 5/8, while m -> pi/(2l).
3. At theta = pi/4: M_W l = pi/4, tan(M_Z l) = 2, and M_W^2/M_Z^2 = ((pi/4)/arctan 2)^2 = 0.5032..., the global minimum. So M_W^2/M_Z^2 stays in [0.5032, 0.625), with 5/8 as a limit at the ends, where the ratio itself is undefined.
4. m/M_W goes from 1/sqrt(2) at theta -> 0 to infinity at theta -> pi/2. It equals sqrt(8/3) at theta = 1.0589.

Proof. In the gauge A_5 = 0 the field is free, with cos(k x) on the Neumann directions at 0 and sin(k x) on the Dirichlet ones, and the condition at l holds in the frame rotated by Ad exp(theta F). The generators split according to the frequency of Ad F. Write L, R for su(2) on W_- and W_+, S = (L + R)/sqrt(2), A = (L - R)/sqrt(2), and O for the odd generators; Ad F has frequency 0 on S, and frequency 2 on the plane (A, O). So S = s cos kx, A = alpha cos kx, O = beta sin kx.

At l, in the rotated frame, A' = cos(2 theta) A + sin(2 theta) O and O' = -sin(2 theta) A + cos(2 theta) O, with O' = 0 at l, which gives beta = alpha tan(2 theta) cot(k l), and so A'(l) = alpha cos(k l)/cos(2 theta).

Charged directions. The connector forces R^+- to vanish at l, with L^+- Neumann. So S(l) = A'(l), and d/dx (S + A') = 0 at l. Substituting gives sin^2(kl) (1 + cos^2(2 theta)) = sin^2(2 theta) cos^2(kl).

Neutral directions. Add B = b cos kx for the generator T_15 = diag(1, 1, 1, -3)/(2 sqrt 6) of su(4). With Y = sqrt(3/5) R + sqrt(2/5) B and Z' = -sqrt(2/5) R + sqrt(3/5) B, the connector forces Z' to vanish at l, with L and Y Neumann. The Neumann conditions give b = -sqrt(3) s, and Z' = 0 gives s = alpha/(4 cos(2 theta)). Substituting in the condition for L gives sin^2(kl) (1/4 + cos^2(2 theta)) = sin^2(2 theta) cos^2(kl).

Massless modes and limits. At k = 0, a zero mode is a constant in the intersection of the two stabilizers. This consists of a_W = diag(q, q) on W_+ and W_-, identified by F, with q preserving the line of w, together with a_V = lambda (+) m, and with the trace condition this has dimension 1 + 8. In it, L = R and B/L = sqrt(2/3), which gives 1/e^2 = (1 + 1 + 2/3)/g^2. The limits follow by expanding. A second derivation, from the referee, uses the principal angles between the Neumann subspaces at the two walls: for each charged direction the squared sine of the angle is sin^2(2 theta)/2, and in the neutral sector the single nonzero value is (4/5) sin^2(2 theta); it also covers cos(2 theta) = 0, where the divisions above need a limit. For item 3, with a = sin(2 theta), the ratio is [arcsin(a/sqrt 2)/arcsin(2a/sqrt 5)]^2, and this decreases with a on (0, 1], since x/(sqrt(1 - x^2) arcsin x) increases. ■

The constants in item 1 are the values ||[D_F, a]||^2/||a||^2 in the Hilbert-Schmidt norm, for universal mass m, for a = L^+- and for the direction orthogonal to the photon and Z', which has A-component of squared length 4/5. These are the coefficients of the lower bounds for the poles of the fully massive family in Proposition 14 of lemma 18 (2 x < z_W, (16/5) x < z_Z); the small-x pole coefficients there are 3 and 24/5. The limit 5/8 is common to both, with different behaviour: the finite-action poles grow in the heavy limit of lemma 18, while the interval masses here tend to zero at the antipodal point. The spectral-action Taylor mass for universal m is M_W^2 = 3 m^2, which exceeds the value here by the factor 3/2 of Proposition F in the paper: there the mass matrix carries the relative heat coefficients of the action, and here we have assumed a five-dimensional Yang-Mills action, which gives the commutator norm alone. A direct matching claim needs the full parent spectral action, with its tower and boundary conditions.

Item 2 gives a geometric form of the behaviour that lemma 18 found: heavy fermions with light vectors. At antipodal points Gamma_p = -Gamma_{p_0}, so both walls have the same stabilizer in sp(2) again, and the fermion channels have type (c) with mass pi/(2l).

The spectral action on the interval

Proposition 23.7. For the heat cutoff and universal theta, with s = l Lambda, the trace over a channel is

```
Tr_channel exp(-D^2/Lambda^2) = 2 sum_{n in Z} exp(-(n pi + theta)^2/s^2).
```

This is proportional to the Jacobi function theta_3(theta, exp(-s^2)). It is strictly decreasing on [0, pi/2], with a maximum at theta = 0 and a minimum at theta = pi/2, where it is stationary, and it is symmetric under theta -> pi - theta. In the towerless regime s -> 0 at fixed x = theta^2/s^2, it tends to 2 e^{-x}, the form of the vacuum density in lemma 18 and Proposition 21.2.

Proof. The eigenvalues in a channel are +-(n pi + theta)/l, by Theorem 22.1, and the sum is over Z after combining the signs. By Poisson summation the sum equals (s/sqrt(pi)) sum_{k in Z} exp(-k^2 s^2) e^{2 i k theta}, and the Jacobi triple product writes this as a constant times the product over n >= 1 of (1 + 2 q^{2n-1} cos(2 theta) + q^{4n-2}), with q = exp(-s^2). Each factor is positive and strictly decreasing in theta on [0, pi/2]. ■

In the form of the Jacobi function, the trace is (2 s/sqrt(pi)) theta_3(theta, exp(-s^2)). This family lies outside the hypotheses of No-go theorem 9 of lemma 18, which concerns finite internal data and a free ray V = c V_*: here the operator family is different, the radial direction is a compact orbit of relative gradings, and the tower, which the finite description omits, supplies a second branch with a stationary point at theta = pi/2. In the present family the literal action has its minimum at antipodal gradings, m = pi/(2l), with M_W = M_Z = 0 by Theorem 23.6, where the lowest eigenspace is degenerate. This is a vacuum with massless vectors, so it is not the physical point. The sign of the term agrees with the fermion loop in the Hosotani mechanism, where gauge-boson loops enter with the opposite sign. So an interior minimum could arise from that competition, which the classical spectral action does not contain. We have not computed it.

The one-loop potential in the same family

From the exact spectra, the products over the towers are, relative to theta = 0 and in Euclidean momentum x = p l,

```
fermion channel:  1 + sin^2(theta)/sinh^2(x),
charged vectors:  1 + sin^2(2 theta)/(2 sinh^2(x)),
neutral vector:   1 + (4/5) sin^2(2 theta)/sinh^2(x).
```

So in the gauge-boson and fermion-loop form of the Hosotani mechanism, assuming the usual one-loop formula with three polarizations per vector, and with n_f = 2 x (number of physical Dirac species),

```
8 pi^2 l^4 V(theta) = int_0^infinity x^3 dx [ 3 log(1 + s^2/(2 sinh^2 x)) + (3/2) log(1 + (4/5)
↪  s^2/sinh^2 x) - n_f log(1 + sin^2(theta)/sinh^2 x) ],
```

with s = sin(2 theta). A Dirac top with three colours has n_f = 6, which needs a separately specified fermion sector, since the universal connection gives masses to all: one generation has n_f = 16 and three have n_f = 48. With I = int x^3/sinh^2 x dx = (3/2) zeta(3) and I' = int x^3/cosh^2 x dx = (9/8) zeta(3), near theta = 0 the bracket is (10.8 - n_f) I theta^2 + o(theta^2), and with epsilon = pi/2 - theta, near theta = pi/2 it is V(pi/2) + (10.8 I + n_f I') epsilon^2 + o(epsilon^2), where we have written V for the left side.

Proposition 23.8 (referee). In the massless flat family with this content, for every n_f > 0, theta = pi/2 is the unique global minimum on [0, pi/2], since the gauge logarithms are nonnegative and vanish there, and the negative fermion logarithm is minimized at sin^2(theta) = 1. For n_f >= 10.8, V is strictly decreasing on the interior. Indeed, with t = sin^2(theta) and J(z) = int_0^infinity x^3 dx/(sinh^2 x + z), the derivative with respect to t is, up to a positive factor,

```
(1 - 2t) [6 J(2t(1 - t)) + (24/5) J((16/5) t (1 - t))] - n_f J(t).
```

For t <= 1/2, both arguments in the bracket are at least t, and J decreases; for t > 1/2 both contributions have the required sign.

For 0 < n_f < 10.8, on grids of theta (Python), for n_f in {1, 2, 3, 4, 6, 8, 10}, V has one interior maximum, at about 0.33 for n_f = 8, and no interior minimum. Theorem 9 of lemma 24, by Astra 6, proves this for all

n_f in that range: exactly one interior stationary point, a strict maximum, using a log-concavity property of the integral.

These are negative results in an exact scope: one interval, no bulk masses, fermions of frequency 1 under F, this gauge content, one loop, this condition at the connector wall. As a formal leading approximation, not uniform at x = 0, V has the form B sin^2(2 theta) - A sin^2(theta) with A, B > 0, whose interior stationary point cos(2 theta) = A/(4B) is a maximum. *Bulk masses.* With bulk masses, the fermion product becomes cos^2(theta) + sin^2(theta) G(x), with G = (w_Q coth w_Q - c_Q)(w_U coth w_U - c_U)/x^2 and w = (c^2 + x^2)^{1/2} in the secular form of lemma 12. For a = c_Q + c_U != 0, G = 1 - a/x + O(x^-2), so the potential has angle-dependent ultraviolet divergences, and a finite value depends on a regulator and subtraction: as the referee showed, an earlier scan of general pairs here did not establish a potential, and it is withdrawn. The exceptional case c_U = -c_Q = c has a common bulk mass on the two mixed components and G = 1 + w^2/(x^2 sinh^2 w), which decays exponentially. Keeping the flat gauge products as a separate tree-level assumption, a scan for c in {0, 1/2, 1, 3/2, 2, 3, 5}, n_f in {1, 3, 6, 8, 10, 16, 48}, on a grid of 721 values of theta, finds no interior minimum (Python). General bulk masses need a gauge-invariant completion, a regulator and wall counterterms.

So an interior minimum needs another change: a negative coefficient of sin^2(2 theta), as from fermions of frequency 2 under F, which the bimodule E = Hom(V, W) does not contain; general bulk masses or other gauge form factors, as with warping; a wall scalar with its own potential; or tree-level terms at the walls. With h/f_pi = 2 theta, the leading flat form has the angular form alpha cos(h/f_pi) - beta sin^2(h/f_pi) of Agashe, Contino and Pomarol (section 2.1, Eqs. (19)-(23)) with spinorial SO(5) fermions, with alpha = A/2 and beta = -B, up to a constant. Their form factors can give the sign and competition needed for an interior minimum, so the flat results here do not exclude that construction. This points away from a pure flat Wilson line, toward those changed form factors or a wall scalar with its own potential, for the physical interior.

Comparison with measured values in the flat toy

These are numerical comparisons (Python), with no running and no bulk masses, in lowest levels, using the Particle Data Group 2026 values quoted by the referee, m_t = 172.60 GeV as a direct-measurement mass, M_W = 80.3625 GeV and M_Z = 91.1879 GeV, so m_t/M_W = 2.1478 and M_W^2/M_Z^2 = 0.7767. The ratio m_t/M_W reaches sqrt(8/3) at theta = 1.0589 and the measured value at theta = 1.1626, where M_W l = 0.541, so 1/l is about 148.5 GeV. There, M_W^2/M_Z^2 = 0.581. The wall-scalar model of lemma 22 covers [0.625, 1], and this family covers [0.5032, 0.625); in the wall-scalar model, fitting the measured vector ratio gives M_W l = 1.028, where the lowest no-bulk-mass fermion channel has m/M_W $<$ 1.528. So neither flat family fits both ratios on its own, and the flat one-loop potential above does not select an interior point. Whether a tower at 1/l of this size is allowed would need the couplings and an applicable experimental limit, which we have not analyzed. The directions are bulk masses for the light fermions, warping, running of the couplings, and families combining a Wilson line with a wall scalar. These negative statements hold in the flat tree-level toy only.

Open questions

1. The interior vacuum: the effective potential with a wall scalar, or with warping, or with general bulk masses, with a gauge-invariant completion, a regulator and wall counterterms. The flat cases above have none on the grids.
2. The relation between the two roles of S^4: the base of the twistor window, and the field space of the Higgs quaternion.
3. A commutative parent: functions on S^4 x I and CP^3 x I, with the gradings as local boundary conditions.
4. The factor 3/2 between the five-dimensional normalization and the spectral action: whether the spectral action on the parent itself, with its tower, reproduces the formula of Theorem 23.6.
5. The comparison with twisted spectral triples, Remark 23.3.
6. The contribution of a real connector to the fermion tower, the degenerate antipodal window, and the scalar kinetic normalization needed for a canonically normalized Higgs field.

Numerical check

By Claude Opus 5.5, in Python (numpy and scipy), 3 Nov 2026, in a temporary directory, not kept.

- Random complex symmetric M on C^4 with l ||M|| = 1.25: Gamma_1 is an involution, Gamma_1 = exp(2 l K) Gamma_0; D_F = (1/2l) Gamma_0 log(Gamma_1 Gamma_0) to 3 x 10^-15; the principal angles over l equal the singular values of M to five digits.
- Sp(2) x SU(4), of dimension 25, with the Neumann and Dirichlet subspaces built from the stated stabilizers: the two-point boundary problem was solved as a linear system on a grid in k l. For theta = 0.2, pi/4, 1.06 and 1.3, the two lowest nonzero roots are (0.279, 0.356), (0.785, 1.107), (0.647, 0.868), (0.373, 0.479), equal to the formulas of Theorem 23.6 to three digits, with other roots at pi/2 and above. The intersection of the two stabilizers has dimension 9 at each theta tested.
- Small theta: M_W^2/m^2 = 2.000 and M_Z^2/m^2 = 3.200.
- The universal connection on E (+) E*: J D = D J, Gamma_0 D = -D Gamma_0, all |eigenvalues| equal to theta/l, Gamma_1 = (left Gamma_p, right mirror), J_1^2 = -1, J_1 Gamma_1 = -Gamma_1 J_1, while J alone fails; J_1 h J_1^-1 is right multiplication by +h on E*, for odd h.
- The theta function: monotone on a grid for s = 0.3, 1, 2, with the towerless value at s = 0.3 equal to e^{-x} to five digits; Poisson summation and the triple product to 10^-16.
- The ratio M_W^2/M_Z^2 on a grid: range [0.5032, 0.6250], symmetric under theta -> pi/2 - theta to 10^-15, minimum at pi/4. At theta = 1.1625, M_W l = 0.541, and at theta = 1.0589, M_W l = 0.648.
- The one-loop integrals by adaptive quadrature (scipy), and by Gauss-Legendre quadrature on 400 nodes for the scan with c_U = -c_Q. The referee recomputed the derivative with scipy, found one sign change for each n_f in {1, 2, 3, 4, 6, 8}, with the stationary angle about 0.3382 at n_f = 8, and checked I = (3/2) zeta(3).

Part 24

Commutative cylinders, spectral walls, and the two-stage compression

By Astra 6 (gpt-6-astra, effort xhigh), 3 Nov 2026; prompt in prompts/24_commutative_parent.md. Reproduced as delivered, with links made relative. The section "Numerical check" at the end is by Claude Opus 5.5. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): no error in the displayed equations, and one gap in the general statement on compression maps. The record above the section "Referee amendments" stays as delivered, and those amendments apply to it.

A commutative cylinder admits an exact realization of the **operator window** of lemmas 22 and 23 when the two finite gradings are prescribed through nonlocal spectral boundary conditions. The higher K-modes then decouple as the radius of K tends to zero. Two qualifications remain essential: these boundary conditions restrict scalar functions at the walls, and compression of scalar functions through both stages differs from generating a matrix algebra between the stages. In particular, aligned walls give $\mathbb{R} \oplus \mathbb{R}$ on the quaternionic factor under direct scalar compression, whereas the intervening matrix-algebra construction gives $\mathbb{H} \oplus \mathbb{H}$. Local boundary gradings preserve all scalar functions, but their reduction to the unchanged four-section window has a precise restriction proved below. The radial no-go of lemma 22 survives rechecking in its stated scope. For the massless one-loop potential of lemma 23, an analytic argument strengthens the numerical statement to all $n_f > 0$.

1. Theorem — An exact spectral boundary construction.

Hypotheses and conventions. Let

$$K = \mathbb{CP}^3, \qquad \mathcal{H}_K = L^2(K, \Lambda^{0,*}T^*K \otimes \mathcal{O}(1)),$$

and let

$$A = \sqrt{2}(\bar\partial + \bar\partial^*), \qquad \gamma_K = (-1)^{\deg}.$$

Use the Fubini–Study metric at a fixed reference scale. Write

$$V = \ker A \simeq \mathbb{C}^4, \qquad \Pi = P_K, \qquad \Delta_K = \inf(\operatorname{spec}|A| \setminus \{0\}) > 0.$$

Here $\gamma_K|_V = 1$. The radius parameter $r > 0$ replaces A by A/r, after the usual unitary identification of the L^2-spaces.

These are the analytic data of Example 16.9. Its S^4 option uses functions pulled back to this same twistor space. An actual operator on S^4 with the required kernel and compression map would need to be specified separately.

Let

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

On $L^2(I_\ell; \mathbb{C}^2 \otimes \mathcal{H}_K)$, set

$$\mathcal{D}_r = E\partial_x + Z\,A/r, \qquad \hat{\gamma} = Z\gamma_K.$$

All unspecified tensor factors are identities.

Choose self-adjoint involutions Γ_0, Γ_1 on V, each with two-dimensional positive and negative eigenspaces, and assume

$$-1 \notin \operatorname{spec}(\Gamma_1\Gamma_0).$$

Extend Γ_i by zero on $V^\perp$, writing $C_i = \Pi\Gamma_i\Pi$. Define $\operatorname{sgn} A = 0$ on V, and impose

$$\chi_0\psi(0) = \psi(0), \qquad \chi_\ell\psi(\ell) = \psi(\ell),$$

where

$$\chi_0 = ZC_0 - S\operatorname{sgn} A, \qquad \chi_\ell = ZC_1 + S\operatorname{sgn} A. \tag{1}$$

Then:

1. The H^1 realization of $\mathcal{D}_r$ is self-adjoint with compact resolvent and anticommutes with $\hat{\gamma}$.
2. $\Pi \otimes 1$ reduces the operator, including its domain.
3. On this reducing subspace, the operator is exactly $D(\Gamma_0, \Gamma_1)$ of lemma 23.
4. Every eigenvalue on the complementary subspace satisfies

$$|\lambda| \geq \Delta_K/r. \tag{2}$$

Proof. Put $B = SA/r$. Since $ES = Z$,

$$\mathcal{D}_r = E(\partial_x + B).$$

The operators χ_i are self-adjoint involutions and anticommute with E. Their positive eigenspaces are therefore maximal isotropic for the boundary form

$$[\langle\psi, E\phi\rangle]_0^\ell.$$

On $V^\perp$, condition (1) selects the negative spectral subspace of B at 0 and its positive spectral subspace at ℓ. These are APS conditions with the inward-normal sign reversed between the two walls. On $\ker B = \mathbb{C}^2 \otimes V$, condition (1) supplies a Lagrangian subspace:

$$\Gamma_i\psi_L = \psi_L, \qquad \Gamma_i\psi_R = -\psi_R.$$

For q in the complementary domain, integration by parts gives

$$\|\mathcal{D}_r q\|^2 = \|q'\|^2 + \|Aq/r\|^2 + [\langle q, Bq\rangle]_0^\ell \geq \|q'\|^2 + \frac{\Delta_K^2}{r^2}\|q\|^2. \tag{3}$$

The boundary term is nonnegative by the spectral choices.

An eigenbasis of A decomposes the complementary problem into finite-dimensional interval problems with maximal isotropic endpoint conditions. Each is self-adjoint. Estimate (3) identifies the domain of their self-adjoint direct sum with the stated H^1 domain and gives compactness of its resolvent. The finite-dimensional

tangential kernel contributes an ordinary self-adjoint interval problem. This proves self-adjointness, compactness, and (2).

Every boundary projector commutes with Π, proving reduction. Finally, $\{\gamma_K, A\} = 0$ implies $\{\hat{\gamma}, \mathcal{D}_r\} = 0$, and $\hat{\gamma}$ preserves both boundary subspaces. ■

Define

$$K_F = \frac{1}{2\ell}\log(\Gamma_1\Gamma_0), \qquad D_F = \Gamma_0 K_F, \qquad R(x) = e^{xK_F},$$

with the principal logarithm. Let $P_\pm = (1 \pm \Gamma_0)/2$. The finite-window embedding is

$$W_\ell v(x) = \ell^{-1/2}\begin{pmatrix} R(x)P_+ v \\ R(x)P_- v \end{pmatrix}. \tag{4}$$

Direct differentiation gives

$$\mathcal{D}_r W_\ell = W_\ell D_F, \qquad \hat{\gamma} W_\ell = W_\ell \Gamma_0.$$

Thus the prescribed finite operator and grading are exact, provided $\Delta_K/r > \|D_F\|$. The next eigenvalue in absolute value is at least

$$\min\left\{\frac{\Delta_K}{r}, \frac{\pi - \ell\|D_F\|}{\ell}\right\}. \tag{5}$$

---

2. Proposition — Local walls and spectral walls have different scalar domain algebras.

Hypotheses. Use the product differential expression of Result 1.

For the local option, let $G_i(y)$ be smooth self-adjoint bundle involutions and impose

$$G_i\psi_L = \psi_L, \qquad G_i\psi_R = -\psi_R \quad \text{at } x = i.$$

Assume

$$[G_i, \gamma_K] = 0, \qquad [G_i, \sigma_A(y,\xi)] = 0 \quad (\xi \in T_y^*K), \tag{6}$$

where σ_A is the tangential principal symbol.

Then these local conditions give a self-adjoint elliptic realization with compact resolvent, and **every** smooth scalar function on $K \times I_\ell$ preserves its domain.

For the spectral option (1), the maximal smooth scalar $*$-algebra preserving the domain is

$$\mathcal{A}_{\mathrm{APS}} = \{f \in C^\infty(K \times I_\ell) : f(\,\cdot\,,0), f(\,\cdot\,,\ell) \text{ are constant on } K\}. \tag{7}$$

The same conclusion holds for the subalgebra pulled back from $S^4 \times I_\ell$.

Proof. For the local option, the boundary involution is ZG_i. It anticommutes with the normal matrix E. Condition (6) makes it anticommute with the principal symbol of the adapted tangential operator SA/r. Hence its positive eigenspace is an elliptic Lagrangian boundary condition. The corresponding local ellipticity and adjoint-domain criteria are described in Bär–Ballmann, Corollary 4.18 and Example 4.20. Primary reference

Scalar multiplication commutes pointwise with G_i, so it preserves the endpoint conditions and the H^1 domain. Moreover,

$$[\mathcal{D}_r, f] = E\,\partial_x f + Z[A, f]/r$$

is bounded for each $r > 0$.

For spectral walls, preservation by a scalar $*$-algebra is equivalent to

$$[M_{f_i}, \chi_i] = 0.$$

Independence of the spin matrices Z, S gives

$$[M_{f_i}, C_i] = 0, \qquad [M_{f_i}, \operatorname{sgn} A] = 0.$$

Since $C_i^2 = \Pi$, the first equation implies

$$[M_{f_i}, \Pi] = 0. \tag{8}$$

Consequently multiplication by f_i preserves $H^0(K, \mathcal{O}(1))$. If s is such a section, then

$$0 = \bar{\partial}(f_i s) = (\bar{\partial} f_i)s.$$

The sections of $\mathcal{O}(1)$ have no common zero, so $\bar{\partial} f_i = 0$ everywhere. A holomorphic function on compact connected $\mathbb{CP}^3$ is constant. Constants plainly preserve (1), proving (7). ■

The hypotheses in (6) are a sufficient local ellipticity condition. An arbitrary bundle grading supplies a pointwise isotropic condition, but self-adjointness on an H^1 domain and compact resolvent still require an ellipticity check. Likewise, prescribing a grading only on V leaves the boundary problem incomplete until a condition on $V^\perp$ is supplied.

---

3. Proposition — Compression of a local grading and the unchanged level-one window.

Hypotheses. Let G be a self-adjoint bundle involution acting by multiplication, and let

$$g = \Pi G \Pi|_V, \qquad Q_K(f) = \Pi M_f \Pi|_V.$$

Keep the level-one space and scalar representation of Example 16.9 unchanged.

Then

$$1_V - g^2 = \big((1-\Pi)G\Pi\big)^*\big((1-\Pi)G\Pi\big). \tag{9}$$

In particular, g is a grading exactly when $[G, \Pi] = 0$. In that event,

$$[g, Q_K(f)] = 0 \quad \text{for every scalar } f. \tag{10}$$

For either the full $\mathbb{CP}^3$ scalar algebra or the S^4 pullback algebra, this forces

$$g = 1_V \quad \text{or} \quad g = -1_V. \tag{11}$$

Proof. Expand $\Pi G^2 \Pi = \Pi$ using $1 = \Pi + (1 - \Pi)$; this gives (9). Vanishing of its positive right-hand side implies $(1-\Pi)G\Pi = 0$, and self-adjointness gives the other off-diagonal block.

Since scalar multiplication commutes with G, reduction then gives (10). The compressed $\mathbb{CP}^3$ functions generate $M_4(\mathbb{C})$. The compressed real S^4 functions generate $M_2(\mathbb{H})$, whose complex-linear commutant on $\mathbb{C}^4$ is also $\mathbb{C}1$. Thus a self-adjoint involution satisfying (10) is ± 1. ■

For **aligned local walls** with $[G, A] = 0$, the finite zero-mode window is explicitly

$$v \longmapsto \ell^{-1/2} \begin{pmatrix} P_+(g)v \\ P_-(g)v \end{pmatrix}.$$

The induced grading is g, and scalar compression is

$$P_+(g)Q_K(f)P_+(g) + P_-(g)Q_K(f)P_-(g) = Q_K(f).$$

By (11), the unchanged four-section window has trivial grading. Its generated algebra remains $M_2(\mathbb{H})$ or $M_4(\mathbb{C})$, according to the scalar algebra chosen. The globally constant choices $G = \pm 1$ give concrete self-adjoint examples with this window and a separated higher spectrum.

There is also an explicit local field illustrating failure of reduction. Add a trivial auxiliary $\mathbb{C}^4$ bundle and put

$$G(y) = \sum_{a=1}^{5} y_a \Gamma_a^{\text{aux}},$$

where $y \in S^4$ is the twistor projection and the Γ_a satisfy $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$. This is a local involution satisfying (6), but now the tangential kernel has dimension 16, and

$$g = \frac{1}{5}\sum_{a=1}^{5} \Gamma_a^{\text{section}} \otimes \Gamma_a^{\text{aux}}.$$

Using $\operatorname{tr}(\Gamma_a \Gamma_b) = 4\delta_{ab}$,

$$\operatorname{tr}(g^2) = \frac{16}{5} \neq 16.$$

Thus g is not a grading. Equation (9) even gives

$$\|(1-\Pi)G\Pi\| \geq \frac{2}{\sqrt{5}},$$

independently of r. This is a bound on boundary-trace mixing; it does not identify the actual low spectral window of the coupled local problem.

A local $G(y)$ specifies a **field of gradings**. It does not select a marked point p. In the tautological quaternionic line bundle,

$$\Pi_y = \frac{1 + y_a \Gamma_a}{2},$$

the natural field $2\Pi_y - 1$ acts as $+1$ on the fibre $\operatorname{ran}\Pi_y$. By contrast, the finite grading

$$\Gamma_p = p_a \Gamma_a = 5Q_K(p \cdot y)$$

uses a separately chosen point p. Its action on global section coefficients is nonlocal.

The restriction proved here concerns the unchanged level-one space, a local multiplication grading, and its compression as an actual involution. A different coupled window, additional fibres, or spectral boundary conditions remain available directions.

---

4. Theorem — The composed compression map and the aligned-wall algebra.

Hypotheses. Use the exact spectral construction of Result 1. For a scalar multiplier $f(y, x)$, let

$$T_f(x) = Q_K(f(\,\cdot\,, x)), \qquad \mathcal{E}_0(T) = \frac{1}{2}(T + \Gamma_0 T \Gamma_0).$$

Use real-valued functions and real algebra generation when discussing $M_2(\mathbb{H})$.

Then the compression directly from the scalar cylinder to the finite window is

$$\boxed{\Phi_\ell(f) = \frac{1}{\ell}\int_0^\ell \mathcal{E}_0\big(R(x)^* T_f(x) R(x)\big)\, dx.} \tag{12}$$

It is independent of r.

For aligned walls $\Gamma_1 = \Gamma_0 = \Gamma_p$, direct scalar compression on the S^4 factor generates

$$\mathbb{R} \oplus \mathbb{R}. \tag{13}$$

If instead one first generates $M_2(\mathbb{H})$ and then applies the interval compression, the result is

$$\mathbb{H} \oplus \mathbb{H}. \tag{14}$$

Proof. Insert (4) on both sides of the multiplication operator. Orthogonality of the two interval spin components gives exactly (12).

For real S^4 functions, Example 16.9 implies

$$Q_K(C^\infty(S^4,\mathbb{R})) = \mathrm{span}_{\mathbb{R}}\{1,\Gamma_1,\dots,\Gamma_5\}. \tag{15}$$

Indeed all compressed functions are quaternionic Hermitian matrices; that space has real dimension six, and the displayed six matrices belong to the image and are independent.

For aligned walls $R = 1$. Clifford multiplication gives

$$\mathcal{E}_p(\Gamma_a) = p_a\Gamma_p.$$

Therefore the image of direct scalar compression is

$$\mathrm{span}_{\mathbb{R}}\{1,\Gamma_p\}.$$

This space is already the algebra $\mathbb{R}\oplus\mathbb{R}$, proving (13).

On the other hand,

$$\mathcal{E}_p(M_2(\mathbb{H})) = \{a\in M_2(\mathbb{H}) : [a,\Gamma_p]=0\} = \mathbb{H}\oplus\mathbb{H},$$

proving (14). The noncommuting products needed to generate the quaternionic diagonal blocks have been inserted between the two compressions in this second procedure. ■

On the single $\mathbb{CP}^3$ factor, real scalar compression is onto the Hermitian 4×4 matrices. A chosen $2+2$ aligned grading therefore gives all Hermitian block-diagonal matrices and generates

$$M_2(\mathbb{C})\oplus M_2(\mathbb{C}).$$

For the **chosen two-sheet coarse correspondence** of Example 16.9, with

$$H_0 = \mathrm{Hom}(V,W)\oplus\mathrm{Hom}(W,V), \qquad \Gamma_0(\xi,\eta) = (\Gamma_p\xi, -\eta\Gamma_p),$$

the same calculation gives

$$\boxed{\mathrm{alg}_{\mathbb{R}}(\text{direct aligned scalar compression}) = \mathbb{R}\oplus\mathbb{R}\oplus M_4(\mathbb{C}).} \tag{16}$$

The V-side matrices act on the left of η and commute with its right grading, so that factor survives unchanged.

Endpoint restrictions do not change these finite-dimensional images. If a linear functional vanishes on compressions of functions supported in $0<x<\ell$, its smooth matrix-entry density vanishes in the interior and hence everywhere. Finite-dimensional separation then gives equality with the unrestricted image, exactly as in Result 7 of lemma 20.

At the intermediate stage, however, spectral walls still give scalar endpoint values. Even after algebra generation, those values remain scalar. Enlarging them to the full stabilizers of Γ_0,Γ_1, as in lemma 23, is another change of the parent algebra. Equality of finite compression images does not identify the parent gauge groups or one-forms.

---

5. Theorem — Radius collapse, interval collapse, and controlled mixing.

Hypotheses. First use the exact construction of Result 1. Identify the varying Hilbert spaces by metric rescaling. For resolvent formulas, fix units in which the nonreal spectral parameter is i.

Then, at fixed ℓ,

$$\left\|(\mathcal{D}_r - i)^{-1} - \iota_\Pi(D(\Gamma_0,\Gamma_1) - i)^{-1}\iota_\Pi^*\right\| \le \frac{r}{\Delta_K}. \tag{17}$$

Thus radius collapse gives the entire matrix interval operator in generalized norm-resolvent sense.

If D_F is fixed, $\ell\to0$, and

$$\Gamma_1(\ell) = e^{\ell\Gamma_0 D_F}\Gamma_0 e^{-\ell\Gamma_0 D_F},$$

then $r \to 0$ and $\ell \to 0$ decouple both towers and leave the finite operator D_F. No relation between the two rates is required in this exactly reducing construction.

Proof. The resolvent is block diagonal with respect to Π. Estimate (2) bounds its complementary block by r/Δ_K, proving (17). For the finite-window limit, (5) bounds the complementary resolvent by

$$\max\left\{\frac{r}{\Delta_K}, \frac{\ell}{\pi - \ell\|D_F\|}\right\},$$

which tends to zero. ■

A useful mixing statement requires additional hypotheses. Suppose a transformed problem has a common domain and the form

$$\widetilde{\mathcal{D}}_r = \mathcal{D}_r + V_r, \qquad V_r = V_r^*, \qquad \|V_r\| \le b_r.$$

Write

$$v_r = \|(1-\Pi)V_r\Pi\|.$$

For an eigenvector $\psi = p + q$, with $p = \Pi\psi$, $q = (1-\Pi)\psi$, and $|\lambda| \le L$, assume

$$g_r = \Delta_K/r - b_r - L > 0.$$

Then

$$\boxed{\|q\| \le \frac{v_r}{g_r}\|p\|, \qquad \|\text{Schur correction at } \lambda\| \le \frac{v_r^2}{g_r}.} \tag{18}$$

Proof. The complementary eigenvalue equation gives

$$q = -(\mathcal{D}_{r,Q} + V_{QQ} - \lambda)^{-1} V_{QP} p.$$

The inverse has norm at most g_r^{-1}. Substituting this formula into the p-equation gives the second bound. ■

For bounded b_r, v_r, both errors are $O(r/\Delta_K)$. If a common-domain construction instead produces terms of order $1/\ell$, sufficient conditions include

$$\frac{rv_r}{\Delta_K} \to 0, \qquad \frac{rv_r^2}{\Delta_K} \to 0,$$

together with convergence of the Π-diagonal perturbation.

An arbitrary local $G_i(y)$ generally fails $[G_i, \Pi] = 0$. Although the differential expression commutes with Π, its domain then does not. Such a boundary change is not automatically a bounded perturbation satisfying (18). The explicit local field in Result 3 has nonvanishing boundary mixing at every radius. No convergence to the prescribed two-gradings operator is established here for that nonreducing local problem.

At the level of observables, radius collapse leaves (12) unchanged. For an x-independent $T = Q_K(f)$,

$$\|\Phi_\ell(f) - \mathcal{E}_0(T)\| \le \ell\|D_F\|\,\|T\|. \tag{19}$$

If T is even for Γ_0, the stronger formula of Theorem 22.4 applies:

$$\Phi_\ell(f) = \operatorname{sinc}(\ell\ \mathrm{ad}_{D_F})(T),$$

$$\|\Phi_\ell(f) - T\| \le \frac{\ell^2}{6}\|[D_F,[D_F,T]]\|. \tag{20}$$

Equation (19) follows by integrating $\|R^*TR - T\| \le 2x\|D_F\|\|T\|$; equation (20) follows by taking the even part of the conjugation series.

Consequently there is convergence of the **compression maps on their scalar operator systems**. The limit is $\mathcal{E}_0 Q_K$. It does not turn that image into every element of $\mathcal{E}_0(P)$; Result 4 computes the difference explicitly. Algebra generation remains a separate operation.

6. Proposition — The second sheet, reality, even algebra, and connector remain prescribed data.

Hypotheses. Consider the scalar cylinder construction above and, separately, its extension using the two-sheet coarse correspondence of Example 16.9.

The resulting distinctions are:

Structure	What follows under the stated choices
Second sheet and the two Hom spaces	They are inputs in the coarse correspondence. A single four-section cylinder window does not supply them.
$J_1^2 = -1,\ J_1\Gamma_i = -\Gamma_i J_1$	These are additional compatibility conditions on the prescribed gradings and antiunitary.
$\mathbb{H} \oplus \mathbb{H} \oplus M_4(\mathbb{C})$	This is the derived even part after choosing the quaternionic real form, the grading, and intermediate algebra generation. Direct aligned scalar compression gives (16).
Connector and $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$	The connector's lines and coupling are chosen. Their intertwining algebra is then derived.

Proof. The one-factor construction starts with $V = \mathbb{C}^4$, and (4) has domain V. The two-sheet construction instead starts with

$$(W \otimes \overline{V}) \oplus (V \otimes \overline{W}).$$

Geometrically, its analytic realization uses product factors and a restricted scalar algebra, as Theorem 16.7 specifies. It is additional correspondence data over the scalar spaces.

For reality, suppose an antiunitary $\mathcal{J}_1$ on the tangential Hilbert space satisfies

$$\mathcal{J}_1^2 = -1, \quad \mathcal{J}_1 A = -A\mathcal{J}_1, \quad [\mathcal{J}_1, \gamma_K] = 0,$$

and its restriction J_1 to V anticommutes with both Γ_i. Then

$$\widehat{J} = E\mathcal{J}_1$$

preserves (1) and obeys

$$\widehat{J}^2 = 1, \qquad [\widehat{J}, \mathcal{D}_r] = 0, \qquad \widehat{J}\hat{\gamma} = -\hat{\gamma}\widehat{J}.$$

On the window,

$$J_F = \Gamma_0 J_1.$$

These identities follow directly from the stated commutation relations. They specify sufficient conditions; the natural $\mathcal{O}(1)$ cylinder has not supplied such an extension or its order-zero property here.

In particular, the quaternionic antiunitary j used to recognize $M_2(\mathbb{H})$ commutes with the Γ_a. It does not itself satisfy the required anticommutation with Γ_p.

The even-algebra assertion is the centralizer calculation in Result 4 after intermediate generation.

Finally, choose

$$w \in W_+, \quad v \in V, \quad \|w\| = \|v\| = 1, \quad \kappa \neq 0,$$

and prescribe

$$T_\kappa \xi = \kappa(w^* \xi v)\, v w^*.$$

Its intertwining conditions require the lines $\mathbb{C}w, \mathbb{C}v$ to be reducing, with the same scalar action. A quaternion preserving $\mathbb{C}w$ lies in its chosen copy of $\mathbb{C}$; $v^{\perp}$ carries arbitrary $M_3(\mathbb{C})$, and W_- carries arbitrary $\mathbb{H}$. Hence

$$\mathcal{A}(T_\kappa) = \{(\lambda \oplus q, \lambda \oplus m)\} \simeq \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}),$$

as in Result 2 of lemma 20. ■

A marked $p \in S^4$ fixes a quaternionic line, but choosing w also requires a complex line within it. The second line v, the coupling, and an interface realizing the connector remain necessary. Scalar agreement at marked points has unchanged compression image by Result 7 of lemma 20, so it supplies none of this selection by itself.

---

7. No-go theorem — Recheck of the radial statement in lemma 22.

Hypotheses. Let the parent be exactly the one-interval model of lemma 22: fixed wall coupling X, zero bulk mass, the even algebra $\mathcal{A}_\ell$, scalar inner fluctuations Eh with $h^* = -h$, and the **moving spectral window**. For real fluctuations, include the conjugate one-form and assume the stated order-zero representation.

Then every fluctuated finite operator has the same singular values as D_F. This is an exact finite-amplitude spectral statement; it uses no action or loop expansion.

Proof. Solve

$$u' = -hu, \qquad u(0) = 1, \qquad w = u(\ell),$$

and set $v(x) = u(x)w^*$. Then

$$v(\ell) = 1, \qquad v(0) = w^*.$$

Since h is even, w preserves the boundary splitting at 0; the identity at ℓ preserves the fixed coupled wall. Thus v preserves the original domain, and

$$vD_X v^* = D_X + Eh.$$

For the real fluctuation, the analogous implementing unitary is

$$vJ_F vJ_F^{-1};$$

order zero separates the two factors in its derivative.

The full operators are unitarily equivalent, so their moving spectral windows are unitarily equivalent. In the finite identifications of lemma 22, the operators are $w^* D_F w$, or $U_w^* D_F U_w$ in the real case. Their singular values coincide. ■

The moving-window hypothesis matters. For example, with $D_F = m\sigma_1$ and $h = i\sigma_3$, fixed compression gives

$$Q_\ell(Eh) = -\frac{1 - \cos(2m\ell)}{2m\ell}\sigma_2.$$

Adding a multiple of this operator to the fixed $m\sigma_1$ changes its singular values at second order.

The verdict agrees with the author's amended theorem and Terra's independent endpoint-preserving proof. In the usual nondegenerate Standard Model representation, the normalized shifted Higgs quaternion lies on the unit-quaternion orbit; its norm is fixed.

The known escapes are a variable wall coupling, odd connection directions, suitable bulk masses, additional internal directions with nontrivial connection data, or additional elementary or composite fields. A quantum effective action on the same configuration space does not add a radial coordinate. The proposed $K \times I_\ell$ parent has tangential directions and tangential one-forms, so this theorem does not decide its full fluctuation space.

---

8. Lemma — A logarithmic concavity property of the one-loop integral.

Hypotheses. Let

$$\mathcal{J}(z) = \int_0^\infty \frac{x^3}{\sinh^2 x + z}\, dx, \qquad z > 0,$$

and define

$$e(z) = -\frac{z\mathcal{J}'(z)}{\mathcal{J}(z)}.$$

Then $0 < e(z) < 1$, and e is nondecreasing.

Proof. Positivity and

$$0 < \frac{z}{\sinh^2 x + z} < 1$$

show that $e(z)$, expressed as a weighted average of this fraction, lies strictly between zero and one.

Put $x = e^v$, $z = e^u$. Then

$$\mathcal{J}(e^u) = \int_{\mathbb{R}} \exp\left(4v - \log(\sinh^2(e^v) + e^u)\right)\, dv. \tag{21}$$

The function

$$b(v) = 2\log\sinh(e^v)$$

is convex, because, with $x = e^v$,

$$b''(v) = 2x(\coth x - x\ \operatorname{csch}^2 x) > 0.$$

The last inequality is equivalent to $\sinh x \cosh x > x$.

The function $(u, v) \mapsto \log(e^{b(v)} + e^u)$ is therefore convex. The integrand in (21) is jointly log-concave. Marginal integration preserves log-concavity by Prékopa's theorem, so $u \mapsto \log \mathcal{J}(e^u)$ is concave. We use precisely Theorem 6, p. 342, of Prékopa's original paper.

Finally,

$$\frac{d}{du}\log \mathcal{J}(e^u) = -e(e^u).$$

Concavity makes this derivative nonincreasing, proving the assertion. ■

---

9. Theorem — The massless one-loop potential has no interior local minimum.

Hypotheses. Let the action be exactly the one-loop determinant displayed in lemma 23: a flat interval, zero bulk masses, frequency-one fermions, the stated charged and neutral gauge towers and multiplicities, and the prescribed connector wall condition. Assume that no fermion connector modifies the displayed fermion tower, and add no wall potential or other angle-dependent term.

Let $n_f > 0$, set $t = \sin^2\theta$, and write

$$\mathcal{V}(t) = 8\pi^2\ell^4 V(\theta).$$

Then:

1. $t = 1$, equivalently $\theta = \pi/2$, is the unique global minimum.
2. If $n_f \geq 54/5$, the potential is strictly decreasing on the interior.
3. If $0 < n_f < 54/5$, there is exactly one interior stationary point, and it is a strict maximum.

Proof. Define

$$F(z) = \int_0^\infty x^3 \log\left(1 + \frac{z}{\sinh^2 x}\right)\, dx, \qquad F'(z) = \mathcal{J}(z).$$

The exact potential is

$$\mathcal{V}(t) = 3F(2t(1-t)) + \frac{3}{2}F\left(\frac{16}{5}t(1-t)\right) - n_f F(t). \tag{22}$$

Its gauge terms are nonnegative and vanish at $t = 1$. Since F is strictly increasing, the fermion term is uniquely minimized there. This proves item 1.

Differentiate:

$$\mathcal{V}'(t) = (1-2t)\left[6\mathcal{J}(2t(1-t)) + \frac{24}{5}\mathcal{J}\left(\frac{16}{5}t(1-t)\right)\right] - n_f\mathcal{J}(t). \tag{23}$$

For $t \geq 1/2$, this is strictly negative.

For $0 < t < 1/2$ and $c \geq 2$, put

$$R_c(t) = (1-2t)\frac{\mathcal{J}(ct(1-t))}{\mathcal{J}(t)}.$$

Let $a = ct(1-t)$. Then $a \geq t$, and Result 8 gives $e(a) \geq e(t)$. Consequently,

$$\begin{aligned}\frac{d}{dt}\log R_c(t) &= -\frac{2}{1-2t} - \frac{1-2t}{t(1-t)}e(a) + \frac{e(t)}{t} \\ &\leq -\frac{2}{1-2t} + \frac{e(t)}{1-t} \\ &< -\frac{2}{1-2t} + \frac{1}{1-t} < 0.\end{aligned}$$

Thus every R_c is strictly decreasing. Since $\mathcal{J}(0)$ is finite and positive,

$$R_c(0+) = 1, \qquad R_c(1/2-) = 0.$$

Therefore

$$H(t) = 6R_2(t) + \frac{24}{5}R_{16/5}(t)$$

decreases strictly from $54/5$ to zero. Equation (23) becomes

$$\mathcal{V}'(t) = \mathcal{J}(t)\,[H(t) - n_f] \quad (0 < t < 1/2).$$

For $n_f \geq 54/5$, its sign is negative. For $0 < n_f < 54/5$, it changes sign exactly once, from positive to negative. Since $t = \sin^2\theta$ is strictly increasing on the interior, the same stationary-point classification holds in θ. ■

This proves the massless statement for the entire parameter range, including the previously numerical range. No numerical computation is used in this proof.

The recorded author and Terra referee calculations agree on the sampled massless cases. The logarithmic-concavity proof above is an additional single-agent derivation; it has not received a fresh independent model review in this turn. The recorded agreement therefore should not be described as a multi-model review of this new proof.

The escapes remain changes to the stated determinant: frequency-two fermions, modified form factors through warping or other geometry, suitable bulk-mass sectors, wall potentials, and wall kinetic or interaction terms. The full effective action of the proposed $K \times I_\ell$ parent, including its K-tower, has not been computed here.

---

10. Proposition — Scope of the bulk-mass scans.

Hypotheses. Retain the flat gauge terms of (22) as an independent assumption. Replace the fermion factor by the expression in lemma 23:

$$1 + t(G(x) - 1), \qquad G(x) = \frac{(w_Q \coth w_Q - c_Q)(w_U \coth w_U - c_U)}{x^2}, \quad w_i = \sqrt{x^2 + c_i^2}.$$

If $c_U = -c_Q = c$, then the potential is finite and $t = 1$ is its unique global minimum for every $n_f > 0$. The reported absence of additional interior **local** minima at nonzero c remains a numerical statement here.

If $a = c_Q + c_U \neq 0$, the displayed unrenormalized integral has angle-dependent ultraviolet divergences.

Proof. For opposite masses,

$$G(x) = 1 + \frac{w^2}{x^2 \sinh^2 w}, \qquad w = \sqrt{x^2 + c^2}.$$

The added fraction is positive and decays exponentially at infinity. The negative fermion logarithm is strictly decreasing in t. The gauge logarithms are nonnegative and vanish at $t = 1$, proving the global-minimum assertion.

For general masses,

$$G(x) = 1 - \frac{a}{x} + \frac{a^2}{2x^2} + O(x^{-3}),$$

and hence

$$\log(1 + t(G - 1)) = -\frac{ta}{x} + \frac{t(1-t)a^2}{2x^2} + O(x^{-3}).$$

Multiplication by the integration measure $x^3 dx$ produces angle-dependent divergences. A finite potential therefore requires a specified completion, regulator, and counterterms. ■

The opposite-mass scan in lemma 23 samples finitely many masses, multiplicities, and angles. It cannot exclude unsampled local minima. General bulk masses and altered gauge form factors remain directions for constructing an interior vacuum; these conclusions leave that question open.

Consequence for T0 and T0b

For T0, a commutative scalar representation with prescribed spectral walls gives an exact interval operator window and controlled tower decoupling, while local walls preserve the full scalar algebra under the ellipticity hypotheses above. The derivation of a nontrivial local grading and the desired algebra from one coupled scalar window remains open, and intermediate algebra generation must be stated explicitly. For T0b, the second sheet, compatible reality, quaternionic grading choice, and connector remain inputs; their algebraic consequences can be derived once those choices are supplied. Neither the radial theorem nor the specified one-loop potential closes the commutative-parent problem.

Referee amendments

From Terra 6.1. These apply to the text above.

- **Result 2.** Write G_0 at x = 0 and G_l at x = l. Equation (6) supplies the principal-symbol condition of Bar-Ballmann, Corollary 4.18 and Theorem 4.11; their Example 4.20 assumes more. On the unaugmented Dolbeault bundle, the Clifford representation is irreducible, so a fibre endomorphism commuting with every sigma_A is scalar, and a smooth involution of this form is +1 or -1. Additional fibres and other local elliptic projectors lie outside this sufficient criterion.
- **Result 3.** The statement on marked points concerns the natural field displayed there. A local field can encode a chosen point: on the auxiliary bundle, G_p(y) = p_a Gamma_a^aux is a local constant field whose coefficients contain p.
- **Result 4.** Equation (16) has the two Hom spaces and the two scalar algebras of Example 16.9 as inputs. The endpoint argument proves equality of the finite-dimensional linear images, and does not identify the parent algebras, norms, gauge groups or one-forms.
- **Result 5, gap in the final assertion.** Convergence of compression maps needs a common domain and a collapse map. With aligned walls, D_F = 0 and f_l(y, x) = 6 (x/l)(1 - x/l), both endpoint values vanish while Phi_l(f_l) = 1. A correct form: for F(y, s) fixed on K x [0, 1] and f_l(y, x) = F(y, x/l), Phi_l(f_l) tends to E_0 Q_K(int_0^1 F(., s) ds). For x-independent ambient multipliers the limit is E_0 Q_K, by (19). The rate discussion after (18) also needs control of the denominator: a sufficient addition is r (b_r + L) <= (1 - epsilon) Delta_K. The case b_r = O(1/l) follows from the stronger estimate |D_{r,Q}| >= (Delta_K^2/r^2 + pi^2/(4 l^2))^{1/2}, from the one-ended Poincare inequality on the positive and negative B-spectral components.

- **Result 6.** At the finite level, if the gradings are Gamma_p, Gamma_q, a unit n orthogonal to p and q gives J_1 = Gamma_n j, with J_1^2 = -1 and the required anticommutation. Extending this choice to the parent remains a separate task.
- **Lemma 8 and Theorem 9.** An independent check of the key monotonicity, without log-concavity or Prekopa's theorem: with rho(s) = (arsinh sqrt(s))^3/(2 sqrt(s (1 + s))), J(z) = int_0^infinity rho(s)/(s + z) ds; with L(v) = log rho(e^v), L'(v) = 3 tanh(x)/(2x) - (1 + tanh^2 x)/2 for v = 2 log sinh x, which is strictly decreasing. For z = e^u, with the probability density in y proportional to e^y rho(e^{u+y})/(1 + e^y), e(e^u) is the mean of 1/(1 + e^y), and its derivative in u is Cov(1/(1 + e^y), L'(u + y)) > 0, as both are strictly decreasing. Theorem 9 is for theta in [0, pi/2]. At theta = pi/2 the determinant has its continuous value, while the principal-logarithm window is degenerate. Its hypotheses are the flat massless one-loop determinant, its gauge multiplicities and connector wall condition, frequency-one fermions, no fermion-connector modification, and no other angle-dependent term.
- **Result 10.** The divergence is stated for nonzero fermion multiplicity, with no cancellation by additional sectors. The opposite-mass scan in lemma 23 is a numerical result on its sampled parameters.

Numerical check

By Claude Opus 5.5, in Python (numpy, scipy), 3 Nov 2026, in a temporary directory, not kept: E_p(Gamma_a) = p_a Gamma_p to 10^-15, which gives equation (13); tr(g^2) = 16/5 for the auxiliary field of Result 3; on grids, e(z) lies in (0, 1) and is nondecreasing for z in [10^-3, 50]; H(t) is strictly decreasing on (0, 1/2), with H near 54/5 at small t and near 0 at t = 1/2; for n_f in {1, 3, 6, 8, 10}, H - n_f has one sign change, with the stationary angle 0.34 at n_f = 8, which agrees with the referee of lemma 23. Results 1, 2, 5 and 6 were read and not separately computed.

Part 25

The weak-boson sum rule and the effective dimension of the interval

For STRATEGY T5b, T13 and T1, following lemma 18, lemma 22 and lemma 23. Proved by Claude Opus 5.5, with checks in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): two errors and two gaps as first written, and amendments, all incorporated below.

Outcome. Unfolded, the interval parent is a circle of length 2l, and the parent operator is -i d/dx (x) 1 + 1 (x) D_F: the tower is D_F shifted by the momenta k_n = n pi/l. So the exact quadratic kernel of lemma 18 applies termwise over the tower. For small masses, at leading order, for the heat cutoff, the Taylor mass of a vector with constant profile in the transported frame is

```
M^2 = ((3 - d_eff)/2) ||[D_F, T]||^2 / ||T||^2,      d_eff(s) = d log S / d log s,      S(s) =
↪  sum_{n in Z} exp(-n^2 pi^2/s^2),   s = l Lambda,
```

in Hilbert-Schmidt norms, where d_eff is the spectral dimension at the scale of the cutoff of the unfolded circle, equivalently of the paired Neumann and Dirichlet interval: 0 for l Lambda « 1 and 1 for l Lambda » 1. For the Standard Model representation, at the same order,

```
M_W^2 = ((3 - d_eff)/24) sum_f N_f m_f^2,       m_t/M_W = (8/(3 - d_eff))^{1/2} for the top alone,
```

with M_W^2/M_Z^2 = 5/8 for all s. So the relation 8 M_W^2 = sum N_f m_f^2 of the asymptotic spectral action is the zero-dimensional value, with m_t/M_W = sqrt(8/3). A resolved interval gives 12 M_W^2 = sum N_f m_f^2, with m_t = 2 M_W, as five-dimensional covariance requires for this operator on the circle. The endpoint m_t = 2 M_W is known in flat gauge-Higgs models with suitable representations (references below); in the inspected records the referee did not find the coefficient (3 - d_eff)/24 or its crossover, and priority remains open pending a wider search. The factor 3/2 in Proposition F of the paper, the finite-action coefficient relative to the Yang-Mills commutator quotient, is the ratio of these two endpoints. The crossover is at l Lambda about 2.

Unfolding

Let D_X = E d/dx be the parent of lemma 22, and let D_0 + E Gamma_F D_F be its form after the transport of Corollary 22.2, where D_0 is the operator with X = 0.

Proposition 25.1. On the circle x in [-l, l] with periodic conditions, for psi in the domain of D_0, put

```
Psi_Q(x) = Q_L(|x|) + i sgn(x) Q_R(|x|),      Psi_U(x) = U_R(|x|) - i sgn(x) U_L(|x|).
```

After multiplying by 1/sqrt(2), this is a unitary from the parent Hilbert space onto L^2(circle) (x) (H_+ (+) H_-), which maps the H^1 domain of D_0 onto the periodic H^1 domain, and it carries D_0 + E Gamma_F D_F to

```
-i d/dx (x) 1 + 1 (x) D_F.
```

So the signed spectrum is k_n + mu_i, with k_n = n pi/l for n in Z and mu_i the eigenvalues of D_F. An even operator T = T_Q (+) T_U on H_F, constant in x in this transported frame and acting identically on both spin components, is carried to 1 (x) T.

Proof. Since Q_R(0) = Q_R(l) = 0 and U_L(0) = U_L(l) = 0, the functions Psi are continuous and periodic, and every smooth function on the circle arises from its even and odd parts. For x > 0, -i d/dx Psi_Q = Q_R' - i Q_L' = (D psi)_L + i (D psi)_R on Q, and -i d/dx Psi_U = -i U_R' - U_L' = (D psi)_R - i (D psi)_L on U, which are the transforms of D_0 psi; x < 0 is the same with the odd parts. For the connection term, with M* the block of D_F, the term E Gamma_F D_F sends Psi_Q to M* Psi_U and Psi_U to M Psi_Q. ■

In this form D_F is a constant connection on the circle, with holonomy exp(2 i l D_F) up to the convention for parallel transport, and the two walls are the fixed points of the reflection that carries D_F to -D_F. With U = exp(-i pi Gamma_0/4), the identity of lemma 23 reads Gamma_1 Gamma_0 = exp(2 l Gamma_0 D_F) = U exp(2 i l D_F) U*. This form of the interval as a circle with a reflection is the usual orbifold description. It also gives a one-line proof of the spectrum in Theorem 22.1.

The frame. By Corollary 22.2, D_X = R (D_0 + E Gamma_F D_F) R*, with R(x) = exp(x Gamma_F D_F). A generator T constant after transport is R(x) T R(x)* in the original frame of D_X, which generally varies with x and has an odd bulk component. Conversely, a constant generator in the original frame need not preserve the coupled wall, if it is broken. So the field used here requires an admissible enlarged gauge algebra, as in the prealgebra P of lemma 23, and it is generally not a field of the even algebra of lemma 22.

The kernel over the tower

Let A_mu(y) = a_mu(y) T on M^4 = R^4, with T even and constant along x, and consider D = -i gamma^mu (d_mu + i A_mu) + gamma_5 V with V = -i d/dx (x) 1 + 1 (x) D_F. Theorem 1 of lemma 18 has this form, with V_0 = V, whose eigenvalues are k_n + mu_i, and T_{n i, n' j} = delta_{n n'} T_{ij}.

Theorem 25.2. With d_ij = mu_i - mu_j, for the heat cutoff t = 1/Lambda^2, the kernel is

```
C_t(q^2) = (1/(8 pi^2)) sum_{n in Z} sum_{ij} |T_ij|^2 [ (2/t + d_ij^2 + q^2) I_t((k_n + mu_i)^2,
↪  (k_n + mu_j)^2; q^2) - (e^{-t (k_n + mu_i)^2} + e^{-t (k_n + mu_j)^2})/t ],
```

with I_t as in lemma 18. The sum converges, and it holds without expansion in D_F. Expanding in D_F, with C_t = C_0 + q^2 C_1 + …, and with t = 1, so with k_n = n pi/(l Lambda) dimensionless,

```
C_0 = (1/(8 pi^2)) ||[D_F, T]||^2 sum_n e^{-k_n^2} (1 - (2/3) k_n^2) + O(||D_F||^4),
C_1 = (1/(8 pi^2)) (2/3) ||T||^2 sum_n e^{-k_n^2} + O(||D_F||^2).
```

So, with <k^2> = sum k_n^2 e^{-k_n^2} / sum e^{-k_n^2}, at fixed s and for fixed nonzero T,

```
M^2 = C_0/C_1 = (3/2 - <k^2>) ||[D_F, T]||^2/||T||^2 + O(||D_F||^4 / Lambda^2).
```

Proof. V and T split into finite-dimensional blocks indexed by n, so equation (1) of lemma 18 applies to symmetric finite sets of blocks, and the limit is justified by Gaussian bounds, uniform for D_F in a bounded neighborhood of zero. The same bounds justify the Taylor expansion under the sum. Terms odd in k cancel

between n and -n: exactly, the bracket at q = 0 is d_ij^2 int_0^1 E(u) [1 - u(1-u)(2k + mu_i + mu_j)^2] du, with E(u) = exp(-(1-u)(k + mu_i)^2 - u (k + mu_j)^2), whose next term is d_ij^2 e^{-k^2} ((2k^3 - 5k)/3)(mu_i + mu_j), and the bracket at order q^2 begins e^{-k^2} (2/3 - (2k/3)(mu_i + mu_j)). For the leading term of the expansion, put a_i = (k + mu_i)^2 = k^2 + p mu_i + mu_i^2 with p = 2k. To second order in mu,

```
I(a_i, a_j) = e^{-k^2} [1 - (p/2)(mu_i + mu_j) - (mu_i^2 + mu_j^2)/2 + (p^2/6)(mu_i^2 + mu_j^2 +
↪  mu_i mu_j)],
```

and e^{-a_i} = e^{-k^2} [1 - p mu_i - mu_i^2 + p^2 mu_i^2/2]. Therefore

```
2 I(a_i, a_j) - e^{-a_i} - e^{-a_j} = -(p^2/6) d_ij^2 e^{-k^2},
```

and adding d_ij^2 I gives d_ij^2 e^{-k^2} (1 - p^2/6) = d_ij^2 e^{-k^2} (1 - (2/3) k^2). Finally, sum_ij |T_ij|^2 d_ij^2 = ||[D_F, T]||^2. For C_1, at D_F = 0 the bracket in equation (3) of lemma 18 has d_ij = 0 and integrates to 2/3. ■

Differentiating directly, <k^2> = (1/2) d log S/d log s = (s/2) d log S/ds, with S(s) = sum_n exp(-n^2 pi^2/s^2). This S is the heat trace of the Dirac operator of the unshifted circle of length 2l at t = 1/Lambda^2, equivalently the paired Neumann and Dirichlet trace on the interval, with S_N = (S + 1)/2 and S_D = (S - 1)/2 separately. So d_eff = d log S/d log s = -2 d log Tr e^{-t D^2}/d log t is its spectral dimension at that scale. For nonzero D_F, the parent trace has e^{-t (k_n + mu_i)^2}, and S is the unperturbed trace used in the leading expansion. Hence, at leading order,

```
M^2 = ((3 - d_eff)/2) ||[D_F, T]||^2/||T||^2.
```

The two limits. We first take the coefficient at small D_F/Lambda at fixed s, and then the limit in s. Explicitly, d_eff ~ (4 pi^2/s^2) exp(-pi^2/s^2) as s -> 0, and d_eff = 1 - 4 s^2 exp(-s^2) + O(s^2 exp(-2 s^2)) as s -> infinity.

1. For s « 1, only n = 0 contributes. Then d_eff = 0, the factor is 3/2, and this is the towerless regime of lemma 18 and Proposition F of the paper.
2. For s » 1, the sum is a Gaussian integral, with <k^2> = 1/2. Then d_eff = 1, and the factor is 1. This is forced by covariance on the unfolded circle: V is the fifth component of a five-dimensional Dirac operator with constant A_5 = D_F, and the local five-dimensional heat coefficient contains F_{MN} F^{MN}, with F_{5 mu} = i [D_F, A_mu] up to convention and with the same coefficient as F_{mu nu}^2. So the mass is the Hilbert-Schmidt commutator norm alone, as at small angle for the Yang-Mills parent of Theorem 23.6. On a general space with boundary, the bulk coefficient alone does not exclude boundary contributions; here the operator with these transported fields is unitarily equivalent to the circle problem, which is the reason they are absent.

In table form, at small masses:

s = l Lambda	0.3	1	1.5	2	2.5	3	5
(3 - d_eff)/2	1.5000	1.4990	1.3935	1.1414	1.0240	1.0022	1.0000

Corollary 25.3 (the sum rule). For the three-generation Standard Model representation, with the literal particle and antiparticle trace, with Dirac masses small compared with Lambda and with 1/l, and with T a charged generator of SU(2)_L, at leading order,

```
M_W^2 = ((3 - d_eff)/24) sum_f N_f m_f^2,
```

which at d_eff = 0 is 8 M_W^2 = sum N_f m_f^2, and at d_eff = 1 is 12 M_W^2 = sum N_f m_f^2. For the top alone,

```
m_t/M_W = (8/(3 - d_eff))^{1/2},
```

from sqrt(8/3) = 1.633 at d_eff = 0 to 2 at d_eff = 1. The neutral direction scales with the same factor, so M_W^2/M_Z^2 = 5/8 at all s to this order, and the massless directions keep zero mass coefficient, while their kinetic coefficients can change. These are leading-order statements: already in the towerless top-only family, M_W^2 = (3/8) m_t^2 - (23/128) m_t^4/Lambda^2 + … (equation (19) of lemma 18).

Proof. The trace gives ||T||^2 proportional to Q_11 = 12 and |||[D_F, T]||^2 proportional to K_11 = sum N_f m_f^2, which gives the coefficient. At d_eff = 0 this is the CCM relation (hep-th/0610241, Eq. (4.16)), as recovered in Proposition 4 and Remark 16 of lemma 18 at small x. The factor is common to all generators in Theorem 25.2. ■

In terms of Yukawa couplings, with m_t = y_t v/sqrt(2) and M_W = g v/2, this reads y_t^2 = 4 g^2/(3 - d_eff), so y_t = 2 g/sqrt(3) at d_eff = 0, the top-dominance value of CCM (Eq. (5.27) there), and y_t = sqrt(2) g at d_eff = 1.

Remark (positive Laplace cutoffs). For f(z) = int e^{-tz} dnu(t) as in equation (4) of lemma 18, with t measured in units of 1/Lambda^2, the same computation gives at leading order

```
M^2 = ((3 - d_nu)/2) ||[D_F, T]||^2/||T||^2,     S_nu(s) = int S(s/sqrt(t)) dnu(t),     d_nu = d
↪  log S_nu/d log s = 2 <t k^2>_nu,
```

where the average has weights dnu(t) e^{-t k_n^2}. Under the moment hypotheses there, dominated convergence gives the same endpoints, and the crossover and its corrections depend on nu. A general smooth cutoff, such as one flat near zero, need not belong to this class.

Scope

- This is the Taylor quotient of a prescribed constant-profile vector, at leading order in D_F, about a prescribed background and not a demonstrated vacuum. Taylor and pole masses differ in general. Calling it the leading term of a lowest-mode mass also needs an admissible gauge field, a positive kinetic form and an isolated lowest branch; on the circle, translation invariance separates the zero Fourier sector, within which the generator matrix must still be diagonalized. For finite angles in the Yang-Mills parent, the exact masses are in Theorem 23.6, which expands as M_W^2 = 2 m^2 - (4/3) m^4 l^2 + … and M_Z^2 = (16/5) m^2 - (64/75) m^4 l^2 + …, in agreement with d_eff = 1 at leading order. That theorem is for its own Yang-Mills boundary problem, and it does not establish the full finite-cutoff gauge spectrum of the spectral action.
- T must be even and constant in the transported frame, so it preserves both walls there, and it needs the enlarged gauge algebra noted above. Vectors that are Dirichlet at a strict wall, as in Proposition 22.8, are a different boundary problem.
- All fields have flat profiles on one interval, with no bulk masses. In particular, for the top-only statement, the kinetic term counts all twelve doublets of three generations on the same interval, with the same S. With bulk masses to localize light fields, the spectra and overlaps change, and the kinetic coefficient must be recomputed from all eigenstates and overlaps.
- The spectral action is taken as a function of the fermion operator at a matching scale, with no running.
- For a boundary with reflection, the walls contribute heat-kernel boundary terms in general. Here the unfolding removes them for this operator and these generators.
- Other priority: the endpoint m_t = 2 M_W appears at tree level in Cacciapaglia, Csaki and Park, Fully radiative electroweak symmetry breaking, arXiv:hep-ph/0510366, section 5, for a suitable rank-four representation, and a limit approaching 2 M_W in Adachi and Maru, arXiv:1804.06012, Eq. (3.23). These are representation-dependent mechanisms, and they do not give the spectral-action crossover. Universal propagation alone does not impose it in ordinary universal extra dimensions (Cordero-Cid et al., arXiv:1108.2926, section 2).

Reading

This connects three items of the project.

1. *The emergence of a dimension.* The heat dimension of the interval, which distinguishes the labels 10 and 11, enters the mass relation directly. As the cutoff resolves the interval, the coefficient moves from

the finite-space value to the five-dimensional value.

2. *Proposition F and lemma 13.* The factor 3/2 is the finite-action coefficient relative to the Yang-Mills commutator quotient, and in this constant-profile calculation the two are the two ends of one function. Proposition F also contains a bounded-fermion against tower-scale obstruction and overlap conditions, which keep their separate hypotheses.
3. *The measured ratio.* With the Particle Data Group 2026 listings, m_t = 172.60 +- 0.27 GeV as a direct-measurement Monte Carlo mass, and M_W = 80.3625 +- 0.0077 GeV, the ratio is 2.148, which would formally correspond to d_eff = 1.27, outside the range. With a running mass of about 163 GeV (Yoon and Peskin, arXiv:1810.12352, section 3.3 and Eq. (23), for a pole mass of 173 GeV, with about 147 GeV at 2 TeV), it is 2.03, which corresponds to d_eff = 1.06. Dividing a running mass by an on-shell W mass is only a formal orientation and not a matching test at a common scale, and no running from the matching scale has been computed, so we draw no conclusion from this comparison. The known spectral-action value y_t = 2 g/sqrt(3) is the d_eff = 0 boundary condition. A tower near the weak scale, as in the separate unrefereed estimate of the steer note, item 9, would use the d_eff = 1 boundary condition y_t = sqrt(2) g, at a matching scale of that order, in a specified effective theory with thresholds and a renormalization prescription. In a one-loop orientation with universal bulk fields (the steer note, item 11, no referee), the running approaches an infrared quasi-fixed point near 211 to 232 GeV, nearly independently of the boundary value, so that family does not support the closeness of 2 M_W to the running top mass.

Numerical check

By Claude Opus 5.5, in Python (numpy), 3 Nov 2026, in a temporary directory, not kept. The kernel of lemma 18, with quadrature in u, was summed over the tower, with explicit 4 x 4 level blocks for the operator on the interval, independently of the unfolding.

- Random complex K_1 and Hermitian T_Q, T_U on C^3, with ||K|| = 10^-3 min(s, 1): the ratio of C_0/C_1 to the Hilbert-Schmidt quotient agrees with 3/2 - <k^2> to six digits, for s in {0.3, 1, 1.5, 2, 2.5, 3, 5, 10}.
- Three generations, top only, with T = sigma_x/2 on the U-fields: m_t/M_W agrees with (8/(3 - d_eff))^{1/2} to six digits on the same grid, from 1.63299 at s = 0.3 to 2.00000 at s = 5.
- The neutral direction, with weights for lepton and colour: M_Z^2/m^2 goes from 4.801 to 3.200 for universal mass, and M_W^2/M_Z^2 stays at 0.625 to four digits.
- Finite mass in the top-only family: M^2/m^2 at s = 5 is 0.2500, 0.2498, 0.2477, 0.2409, 0.2241, 0.1478 for m/Lambda = 0.02, 0.1, 0.3, 0.6, 1, 2, and at s = 0.5 it is 0.3749, 0.3732, 0.3591, 0.3145, 0.2277, 0.0555, the latter matching lemma 18.

Part 28

Two sheets as the two ends of one connected commutative parent

For STRATEGY T0 and T0b, following lemma 16 (Example 16.9), lemma 24 (Proposition 6) and lemma 09. Propositions 28.1 to 28.4 and 28.13 are by Claude Opus 5.5. Results 28.8 to 28.12 and 28.14 to 28.21 are by Astra 6, job 28 (report, with the full proofs; prompt), checked by Claude Opus 5.5 by hand and in Python. Proposition 28.13 and the numerics are refereed by Terra 6.1 (report), with its corrections applied. The window is a conjecture with numerical support; the numbers come from a radial solve in Python (numpy and scipy), described in the section Numerical check; the script is not kept.

Correction, 3 Nov 2026. The first version of Result 28.5 and of the numbers in Remark 28.7 (commits dd1b14c, 530fa06) used the wrong sign of the spin connection in the Python slice operator. That changed the singular values and the boundary fibre parts, so those numbers are withdrawn, including the next singular value 1.04, the value 1.35 on the representation 16, and the boundary fibre parts -0.011 to -0.853. The exponents at the origin and the multiplicities are unaffected. Astra 6 did not reproduce the old boundary values and its proofs do not use them. We found the error when that operator gave a core zero mode in the representation 20, against the exact count of Proposition 28.11. The singular values, fibre parts and

densities below are recomputed, and the script tests the operator against exact statements: {A_0, A_1} = 0, A_0^2 equal to the Casimir formula on S^4, D^2 on (0, 0)-forms with values in O(1) at the Fubini-Study radius equal to j(j + 4), and A_2 equal to its curvature contraction (the last test proposed by Terra 6.1).

Update, 4 Nov 2026. Lemma 29 proves Conjecture 28.6 in case (b) for the metric class of its Setting 29.0, and in case (a) at the Fubini-Study radius, refereed by Terra 6.1. Item 3 below and the Status of Proposition 28.13 describe the position before that lemma.

Outcome.

1. *The mechanism is proved at the level of compression maps.* A smooth function on the disc bundle X has independent values on the zero section S^4 and on the boundary CP^3, and at the zero section it is a function on S^4 alone (Proposition 28.8). For prescribed windows over this X, real functions generate exactly M_2(H) + M_4(C) at every localization scale (Theorem 28.21). In Example 16.9 the second sheet and the restriction to S^4 were inputs, so this is a connected source for them.
2. *The two ends of an actual local operator are understood.* There is a self-adjoint operator with a local wall (Theorem 28.9). The core carries exactly four states in the representation 4 (Proposition 28.11). At the wall, large mass keeps a tangential mixing (Lemma 28.12). For the pull-back connection the boundary mode then has the fibre part -w(rho) in closed form, with w(rho) = 2(rho^2+2)^2/(9+2(rho^2+2)^2), which reaches the level-one value of Example 16.9 only as rho -> infinity (Proposition 28.13). This is a statement on the finite wall model, and its convergence from the full operator is numerical. A connection preserving the line summands in a collar, with the Fubini-Study metric at the wall, gives the level-one value in that model.
3. *The isolated window of the whole operator stays a conjecture* (Conjecture 28.6): what is missing is the normal reductions with their errors, the matching, and a coercive estimate uniform in the mass and in all representations of Sp(2) (Lemma 28.14).
4. *The algebra M_2(H) + M_4(C) is the algebra of the limit map.* Generated algebras need not converge when compression maps do (Counterexample 28.17). At the sample points resolved numerically, the core mode has a nonzero density breaking the quaternionic structure, empirically of order lambda^2; if it is nonzero in the continuum, Proposition 28.16 gives M_4(C) for the core block at those masses. This is numerical evidence and does not establish the exact algebra at any finite mass.

Setting

Let G = Sp(2), K = Sp(1)_a x Sp(1)_b and H = Sp(1)_a x U(1)_b, so that G/K = S^4, K/H = S^2 and G/H = CP^3. Let S be the bundle G x_K (1, 2), so that CP^3 = P(S). The normal bundle of the zero section is G x_K (1, 3), and away from it X = (0, 1] x CP^3, with radial coordinate x and S^2 fibres of radius rho(x), rho(x) ~ x at x = 0. Charges of U(1)_b are normalized so that the doublet of Sp(1)_b has +-1. The isotropy representation is (2)_{+-1} (+) (1)_{+-2}. The spinors of X restrict on a slice to Sigma_6 = Sigma^+ (+) Sigma^-, with c(d_x) = Z the chirality of Sigma_6, where

```
Sigma^+ = (1)_2 (+) (1)_0 (+) (2)_{-1},   Sigma^- = (2)_1 (+) (1)_0 (+) (1)_{-2}.
```

The representation 4 of G restricts to (2)_0 (+) (1)_1 (+) (1)_{-1}, that is (2, 1) (+) (1, 2) of K. Let Gamma_a be the Clifford generators on 4, with Gamma_5 commuting with K and equal to +-1 on the two summands, and let x_a(g) be defined by lambda(g) Gamma_5 lambda(g)^-1 = x_a(g) Gamma_a.

Proposition 28.1 (the twist).

1. For k != 0, O(k) on CP^3 does not extend to a line bundle on X, since H^2(X) = H^2(S^4) = 0 while O(k) has degree k on the fibre S^2.
2. The pullback of S to X restricts on CP^3 to O(1) (+) O(-1), as the tautological splitting. So E = S^3(S), which is pulled back from S^4 and has trivial connection along the fibres, restricts to O(3) (+) O(1) (+) O(-1) (+) O(-3). The summand O(3) gives the Dolbeault operator on O(1) of Example 16.9, since Sigma (x) O(3) = Lambda^{0,*} (x) O(1).
3. On E the section Phi(v) = phi(|v|) (v-hat . T), with T the spin-3/2 generators and phi odd and smooth, is smooth and G-equivariant, with zero set exactly S^4. In the frame of the slice it is phi(x) tau, with tau diagonal in the summands, with eigenvalues +-3/2, +-1/2.

So we take the Callias-type operator T = D_X(E) + i Phi on spinors of X tensored with E, and the self-adjoint operator D-hat = [[0, T^*], [T, 0]], with the grading Gamma-hat = diag(1, -1). At x = l we take the local condition Gamma-hat Z = -1 on the doubled spinor, that is u_even(l) = 0 for T and v_odd(l) = 0 for T^*. It is a chirality condition, elliptic and local, and it preserves the domain under all smooth functions on X.

Proposition 28.2 (multiplicities). On a slice, Hom_H(4, Sigma^+ (x) E) and Hom_H(4, Sigma^- (x) E) each have dimension 5. For the summand O(3), the dimensions are 1 and 0. So on 4, the operator on the summand O(3) with its U(1) connection vanishes for every G-invariant metric on the slice, and its kernel contains H^0(CP^3, O(1)) = 4 for every fibre radius. This concerns the separate line connection. The pull-back connection of E does not preserve the summands, and its slice operator is invertible for small fibres (Proposition 28.10).

Proof. Tensoring the lists above with the charges of E gives the multiplicities. The operator is odd under Z and maps the multiplicity space of 4 into itself. ■

Proposition 28.3 (the zero section). The fibre operator c_N . nabla + i Phi on R^3 with values in E has zero modes (1, 3) (+) (1, 1) of K, by the Callias count for the positive weights 3/2 and 1/2, with one sign of Phi. On S^4 this gives the Dirac operator twisted by (1, 3) (+) (1, 1). Then Hom_K(4, S^- (x) (1, 3)) has dimension 1, through (1, 2), and Hom_K(4, S^+ (x) (1, 3)) = 0, so the twisted operator vanishes on that copy of 4, in agreement with its index 4. The untwisted part (1, 1) has eigenvalue 2 on the representation 4. Every map in Hom_K(4, S^- (x) (1, 3)) annihilates (2, 1) in 4. These are the twistor spinors of S^4, as sections of S.

Proof. The representation count is as in Proposition 28.2, and the Callias count is a reference, with the exact count in Proposition 28.11. For the last sentence, the first-order operator from (1, 2) to (2, 3) in the slice limit is the twistor operator, and its kernel is the 4. ■

On a representation V the twisted operator has D^2 = Cas_G(V) + 3/2 - Cas_K(tau), for the twist tau, with Cas_G(4) = 5/2, Cas_G(16) = 15/2, Cas_G(20) = 21/2, Cas_G(40) = 29/2, Cas_K(j_a, j_b) = 2 j_a(j_a+1) + 2 j_b(j_b+1). With the twist (1, 3) this gives 0, 5, 8, 12, and with the twist (1, 1) on the representation 4 it gives 4. The slice operator reproduces these (Result 28.5).

Proposition 28.4 (the compression). Let s_v(g) = F lambda(g)^-1 v be a section in the multiplicity space of 4 on a slice, with B = F^* F, which commutes with lambda(H). Write t-hat = 2 i t_3 of Sp(1)_b. Then, with normalization by the norm of s_v,

```
Q(x_a) = (tr(B Gamma_5)/tr B) Gamma_a/5,
```

and for the functions y_T(g) = tr(T Ad(g) t-hat) with T in the Lie algebra sp(2), Q(y_T) is a multiple of T, proportional to tr(B t-hat)/tr B.

Proof. End_H(4) is spanned by 1, Gamma_5 and t-hat. Since int lambda(g) Gamma_5 lambda(g)^-1 x_a = int x_a x_b Gamma_b = Gamma_a/5, the part with Gamma_5 gives the first formula. Equivariant maps from 5 to End(4) are multiples of Gamma_a, and tr(Gamma_a Ad(g) t-hat) = 0, so 1 and t-hat contribute nothing. The second formula is Schur's lemma on the representation 10. ■

Consequently, a mode with F supported on (1, 2) and K-symmetric has tr(B Gamma_5)/tr B = +-1 and tr(B t-hat) = 0. That is the zero section: Q(x_a) = +-Gamma_a/5, and the part of a function that depends on the fibre gives zero. On the boundary with the pure level-one mode, B is the projection on (1)_{-1}, so that tr(B t-hat)/tr B = +-1. Example 16.9 has these values.

The operator and its two ends

Proposition 28.8 (endpoint restrictions). Restriction C^infinity(X, R) -> C^infinity(S^4, R) + C^infinity(CP^3, R) is surjective, and in normal coordinates a(y, v) = a(y, 0) + O(|v|), uniformly on bounded subsets of C^1(X). (Report, Proposition 1.)

Theorem 28.9 (a self-adjoint realization). Take the connection metric g = dr^2 + h(r)^2 pi^* g_{S^4} + rho(r)^2 g_{S^2} with h smooth, even and positive at 0, rho odd and smooth with rho(r) = r + O(r^3),

and a product near r = l. Let Phi_M = M b(|v|^2) v . J with b smooth and positive, T_M = D_X^E + i Phi_M, and D-hat_M the doubled operator. At r = l set Z = -i c(dr), chi = Gamma-hat Z, and impose chi psi = -psi. Then D-hat_M on this H^1 domain is self-adjoint, elliptic and odd, with compact resolvent, and every smooth scalar function preserves the domain. No boundary condition enters at the zero section: after the outer condition is fixed, removing S^4 creates no further self-adjoint extension. (Report, Theorem 2, using Bär-Ballmann, Corollary 4.18.)

Proposition 28.10 (three slice operators). For the standard complex structure on CP^3: (1) the kernel of the Dolbeault-Dirac operator on O(1) is C^4 for every Hermitian metric; (2) the Levi-Civita Dirac operator twisted by the separate line bundle O(3) with its invariant connection vanishes on the 4 for every invariant squashing, which gives a kernel of dimension at least four, with any further kernel in other representations at a locally discrete set of squashings; (3) the Levi-Civita Dirac operator twisted by the full pull-back connection on E is invertible for small fibres, since (D^E_rho)^2 >= 1/(2 rho^2) - C_0 - C_1 rho^2. (Report, Proposition 3.) In Python, on the representation 4, the smallest |eigenvalue| of that operator is 3.235, 1.842, 1.200, 0.722, 0.148 at rho = 0.3, 0.5, 0.707, 1, 2. So the window comes from the competition between the normal operator and the mass.

Proposition 28.11 (normal model). Let Q = sigma . d_y + y . J on L^2(R^3; C^2 (x) C^4), with J Hermitian of spin 3/2. Then ker Q = 0 and ker Q^* = 3 (+) 1 under the normal Sp(1), with a positive gap above the zero modes; changing the sign of the mass exchanges the two kernels. The proof uses Q^* Q = -Laplacian + (y . J)^2 - sigma . J >= 0, with equality excluded since sum_i J_i^2 = 15/4, and the Callias index for the positive eigenbundles of n . J, of degrees 3 and 1. The induced bundle over S^4 is S^2(S) (+) 1, the operator on Sigma (x) S^2(S) has exactly four zero modes, in one chirality, and on the parallel inclusion S -> S (x) S^2(S) the twisted equation is a multiple of the twistor equation. So the kernel consists of harmonic spinors of the twisted operator, which are twistor spinors as sections of S. Rescaling gives the localization scale r = O(M^{-1/2}) at the core and O(M^{-1}) at the wall. (Report, Proposition 4.) In Python, in an oscillator basis: Q^* Q starts at 1.02, QQ^* has four near-zero values, with total spin 0 and three values near spin 1, then 1.02.

Lemma 28.12 (tangential mixing). On a product collar, normalized stable profiles for mass eigenvalues M q_i overlap by 2 sqrt(q_i q_j)/(q_i + q_j), independently of M, which is sqrt(3)/2 for q = 1/2, 3/2. So large mass does not remove the coupling between the stable channels O(3) and O(1), which in the pull-back connection has order 1/rho. (Report, Lemma 5.)

Proposition 28.13 (the boundary operator on the representation 4). Take the sign of Phi for which the right singular vector of T is at the zero section. Its left singular vector v is at the wall, in the domain Z v = +v of Theorem 28.9 for T^*. A wall profile exp(q(x - l)), q > 0, for T^* = i Z d/dx + D_6 + i phi tau, with phi > 0, has q = -phi Z tau at leading order. So the stable channels of v are (Z, tau) = (+, -3/2) and (+, -1/2) twice, and they map to the channels (-, -1/2) twice. In unitary bases of these, the compression of D_6(rho) with the overlaps of Lemma 28.12 is

```
A(rho) = ( 3/(2 rho)   1/rho   rho + 1/rho )
         (     0       sqrt 2     -sqrt 2  )
```

with the matrix elements computed in Python, and reproduced by Terra 6.1 in exact symbolic arithmetic from the bracket and Clifford data. Its kernel is spanned by c = (-(2/3)(rho^2 + 2), 1, 1). The channel tau = -3/2 and one channel tau = -1/2 are supported on (1)_{-1} of the 4; the other channel tau = -1/2 is supported on (1)_{+1}. All three lie in Gamma_5 = -1, and their target spaces are mutually orthogonal, so the cross terms vanish in the traces of Proposition 28.4. Hence for the mode c

```
tr(B Gamma_5)/tr B = -1,    tr(B t-hat)/tr B = -w(rho),    w(rho) = 2(rho^2 + 2)^2 / (9 + 2(rho^2 +
↪  2)^2),
```

for every rho, with horizontal scale 1; for horizontal scale h, replace rho^2 by rho^2/h^2. So w = 8/17 as rho -> 0, w = 2/3 at the Fubini-Study radius rho = 1 (the nearly Kähler radius is 1/sqrt 2), and w -> 1 as rho -> infinity, the level-one value. The nonzero singular values of A are the next wall values on the 4: 1.863 and 2.789 at rho = 1, with the core value 2 in between. With the opposite sign of Phi, the right singular vector is at the wall, in the channels Z = -1, tau > 0, with fibre part +w(rho), and its left singular vector at

the zero section.

For the connection preserving the line summands, the first row is replaced by (0, 1/rho, rho + 1/rho), the kernel is (1, 0, 0), so w = 1 for every rho, and the block on tau = -1/2 has singular values with sum of squares rho^2 + 2/rho^2 + 6 and product sqrt 2 (rho^2 + 2)/rho, so it is invertible for every rho > 0, with D^2 = 3, 6 at rho = 1. This is the collar repair of the report: an invariant connection preserving the four line summands in a collar extends into X with a cutoff, with the pull-back connection near the zero section. At the Fubini-Study metric the stable boundary model is then a sum of line-twisted Dirac operators, with four zero modes on O(3) and none on O(1), since H^q(CP^3, O(-1)) = 0 for all q.

Status. The kernel and the traces are exact for this finite block, which is the leading formal wall operator; the next correction is O(1/M) in the fixed-block expansion and does not change the limiting kernel, since the kernel is separated from the other singular values. That the boundary mode of T_M converges to it as M -> infinity is a numerical statement here (Result 28.5), since a uniform global reduction is unproved: the difference falls as phi_0 grows, from 0.037 to 0.009 at wall radius 0.5 and from 0.008 to 0.002 at wall radius near 1, between phi_0 = 16 and 32. The value 2/3 tests the factor sqrt 3/2, since without it w(1) would be 3/5.

Lemma 28.14 (a sufficient spectral criterion). Let U_M : C^8 -> Dom D-hat_M be an isometry with ||D-hat_M U_M|| <= eps_M -> 0 and ||D-hat_M psi|| >= delta ||psi|| for psi orthogonal to the range, with delta > 0 independent of M. Then for large M, D-hat_M has exactly eight eigenvalues in (-delta, delta), and the spectral space is within O(eps_M/delta) of the range. *Proof* (the Astra report, Lemma 6, written out at the request of the second Terra reading of lemma 29, 5 November 2026). D-hat_M is self-adjoint with compact resolvent (Theorem 28.9). Let P be its spectral projection for (-delta, delta). If rank P > 8, the range of P meets the orthogonal complement of the range of U_M nontrivially, against the complement estimate; if rank P < 8, the range of U_M meets ker P nontrivially, against ||D-hat_M U_M|| < delta. So rank P = 8. The spectral theorem gives ||(1 - P) U_M|| <= eps_M/delta, and the equality of the two finite ranks gives a bound of the same order for the difference of the two projections. ■ The missing steps for Conjecture 28.6 are the normal reduction with its errors, and the coercive estimate on every representation of Sp(2).

Conjecture 28.6 (the window). Take Theorem 28.9 with the sign of Proposition 28.13 and with (a) the pull-back connection and wall radius rho, or (b) the collar connection of Proposition 28.13 and the Fubini-Study metric at the wall. For large M the hypotheses of Lemma 28.14 hold, with the eight states being one copy of the 4 at the zero section in one Gamma-hat degree and one at the boundary in the other. For a family of metrics and mass profiles with a uniform positive decay rate, the off-diagonal block is exponentially small in l. The compression of a smooth function tends to (Q_B(a|*{S^4}), Q_Y(a*/{CP^3})), with Q_B and Q_Y(x_a) = -Gamma_a/5, and Q_Y(y_T) a multiple of T, equal to the level-one value of Example 16.9 times w(rho) in case (a), and equal to it in case (b). This concerns compression maps, in line with Counterexample 28.17.

Result 28.5 (Python, the radial system). We solve T on the multiplicity spaces, with Hermitian Clifford generators, so that c(dx) = i Z in the skew convention of Theorem 28.9. For horizontal scale h, after absorbing the half-density h^2 rho, T = i Z d/dx + A_0/h + A_1/rho + (rho/h^2) A_2 + i phi tau, with -s(x) A_E/rho added for a collar connection. We take h = 1, so that T = i Z d/dx + D_6(rho(x)) + i phi(x) tau, with D_6 built from the structure constants of sp(2), rho(x) = rho_inf tanh(x/rho_inf), phi(x) = phi_0 tanh(x/w_0), w_0 = 0.3, in units with S^4 of radius 1, by the quadratic form of T^* T with P1 elements on a graded grid. This rho is not exactly a product near the wall. Rayleigh quotients of the exact form bound the values from above, and with three-point quadrature they approximate those bounds. Small entries have a mesh floor that depends on the parameters: at (phi_0, l) = (6, 4) the lowest value is 0.00548, 0.00401, 0.00355 for 2500, 5000, 10000 elements, so such entries do not establish a zero continuum value. At x = 0 the exponents are the values for R^3: +-1, +-2 (three times), +-3 on the 4.

phi_0	l = 2	l = 3	l = 4	next singular value, l = 4
3	0.406	0.159	0.066	1.293
4	0.288	0.088	0.028	1.452

phi_0	l = 2	l = 3	l = 4	next singular value, l = 4
5	0.200	0.045	0.011	1.665
6	0.132	0.021	floor	1.818
8	0.049	floor	floor	1.852

(rho_inf = 1, representation 4; the next value at larger phi_0 is 1.86 to 1.87.) The lowest value falls with the exponent of the constant-coefficient system, the smallest |Re kappa| of i Z (D_6(rho) - i phi tau): measured between l = 3 and 4, 0.877, 1.155, 1.464 at phi_0 = 3, 4, 5, against 0.856, 1.151, 1.476, as numerical evidence for these profiles, without a general tunnelling theorem. At large phi, kappa = phi/2 - 2/rho + O(1/phi) (phi = 64: 30.07 at rho = 1, 31.07 at rho = 2). The formula 2 mu exp(-mu l) of lemma 09 belongs to a separable interval and does not carry over.

On the two modes, at rho_inf = 1, l = 3, with the sign of Proposition 28.13, the boundary mode being the left singular vector; the core fibre parts at phi_0 = 16, 40 are below what the computation resolves:

phi_0	zero section: position, Q(x_a)/(Gamma_a/5), tr(B t-hat)/tr B	boundary: position, Q(x_a)/(Gamma_a/5), tr(B t-hat)/tr B, Z, tau
10	0.39, -1.0000, -3e-7	2.93, -1.0000, -0.645, 0.944, -1.117
16	0.28, -1.0000, 1e-11	2.96, -1.0000, -0.658, 0.976, -1.146
40	0.16, -1.0000, 1e-11	2.99, -1.0000, -0.663, 0.996, -1.162

Against Proposition 28.13, at l = 3:

rho_inf	wall radius	-w(rho)	phi_0 = 16	phi_0 = 32	next value, phi_0 = 16	second singular value of A
0.5	0.500	-0.529	-0.493	-0.521	1.992	1.992
0.7	0.700	-0.579	-0.562	-0.575	1.966	1.966
1.0	0.995	-0.665	-0.658	-0.663	1.864	1.865
1.5	1.446	-0.788	-0.785	-0.787	1.660	1.661
2.0	1.810	-0.861	-0.858	-0.860	1.557	1.556
3.0	2.285	-0.921	-0.918	-0.919	1.487	1.486

Collar connection. With rho(l) = 1 and s(x) = (1 + tanh((x - l + 1)/0.15))/2 multiplying the vertical connection term to be removed, at phi_0 = 8, 16, 32: the boundary mode has tr(B t-hat)/tr B = -1.0000 and Q(x_a)/(Gamma_a/5) = -1.0000, the next value is 1.733, that is sqrt 3, and the core mode is unchanged. The switch is nonzero, though tiny, at x = 0, whereas the smooth extension needs a cutoff vanishing near the zero section, so these rounded values support the limiting model and do not establish exact finite-mass equality.

Other representations. The multiplicity is zero on 1, 5, 10, 14, 35. At phi_0 = 16, rho_inf = 1, l = 3, the lowest singular values are 2.188 (wall) and 2.235 (core) on the 16, 1.920 (wall) and 2.828 (core) on the 20, and 3.381 (wall) and 3.463 (core) on the 40, the core values being sqrt 5, sqrt 8 and sqrt 12 up to the mass. The multiplicity zero follows from the central character. The finite wall operator A_V(rho) has no zero at 1500 sampled radii in [0.1, 20] on the 16, 20, 40, 64, a sampled statement; its smallest singular value tends to the core value as rho -> 0 and falls slowly at large rho on the 16 and the 40 (0.33 and 0.67 at rho = 20). Higher representations were not computed, and no estimate in the representation is proved.

Core block at finite mass. The density of Proposition 28.16 is nonzero at these points, stable in the mesh to five digits:

phi_0, l	lambda	tr(B t-hat)/tr B, integrated	max of abs tr(B t-hat) over max of tr B	sup of the pointwise ratio	integrated value over lambda^2
4, 3	0.0882	-5.71e-3	6.48e-3	0.849	-0.73
5, 3	0.0453	-1.84e-3	1.97e-3	0.856	-0.89
6, 3	0.0212	-4.55e-4	4.73e-4	0.858	-1.01

The pointwise ratio reaches its maximum at the outer wall, where the core density is small, so the breaking comes from the tail of the core mode at the wall. Over phi_0 = 3 to 6, l = 2 to 4, the integrated value over lambda^2 lies between -0.4 and -1.0, an empirical observation with no O(lambda^2) asymptotic proved. These values support the hypothesis of Proposition 28.16 at the sample points, conditional on the density being nonzero in the continuum.

The compression and the algebra

Theorem 28.15 (compression, conditional on localization). Assume an eight-dimensional spectral window splitting under Gamma-hat as W_M (+) V_M with dimensions 4, 4. Then scalar compression is exactly block diagonal. If the isometries differ by at most eta_M from model embeddings with angular compressions Q_B, Q_Y and normal measures with first moments at most C M^{-1/2} at the core and C M^{-1} at the wall, then

```
|| Q_M(a) - (Q_B(a|_B) (+) Q_Y(a|_Y)) || <= C ||a||_{C^1} M^{-1/2} + 4 ||a||_infinity eta_M.
```

(Report, Theorem 7.) For a spectral wall as in lemma 24, the domain has constant boundary values, and if a|Y = *c then any boundary-localized family has* //Q{Y,M}(a) - c|| <= C M^{-1} ||d_r a||_infinity. So localization just inside a spectral wall does not recover arbitrary boundary functions for a fixed smooth a, and the limit needs a local wall or an explicitly different comparison algebra.

Proposition 28.16 (the exact symmetry test). For an equivariant copy of 4, B(r) = F(r)^* F(r) = diag(b_a(r) I_2, b_+(r), b_-(r)). All real scalar functions compress to operators commuting with the quaternionic antiunitary j if and only if b_+(r) = b_-(r) for almost every r. If in addition the 5 component is nonzero, the block generates M_2(H). If both the 5 and the 10 components are nonzero, it generates M_4(C). A vanishing integral of tr(B(r) t-hat) alone does not decide this. If a radial h has Q_M(h) = alpha I_W + beta I_V with alpha != beta, then the block projections lie in the generated algebra. (Report, Proposition 8.) At the limit, the real linear image is Herm_2(H) + Herm_4(C), of real dimension 6 + 16 = 22, and the generated real algebra has dimension 16 + 32 = 48.

Counterexample 28.17 (generation is discontinuous under convergence of compression maps). Let R_0 = Q_B E_fib, with E_fib the fibre average on CP^3, R_1 = Q_Y, and R_eps = (1 - eps) R_0 + eps R_1. These are unital completely positive, ||R_eps - R_0|| <= 2 eps, and real functions generate M_4(C) under R_eps for every eps > 0, and M_2(H) at eps = 0. So the correct limit statement concerns specified compression maps, and it gives no convergence of the generated algebras. (Report, Counterexample 9.)

Theorem 28.21 (prescribed windows on the same X). Let P_y = (1 + y_a Gamma_a)/2 and p_z <= P_{pi(z)} the tautological rank-one projector. With normalized radial profiles chi_0, chi_l supported in disjoint eps-neighbourhoods of the two ends, U_0 v = sqrt 2 chi_0 P_y v and U_l v = 2 chi_l p_z v are isometries into L^2(X; C^4) (+) L^2(X; C^4). Real functions compressed by them generate M_2(H) + M_4(C) at every eps, and the compression differs from Q_B (+) Q_Y by at most C eps ||a||_{C^1}. (Report, Theorem 13.) This proves the connected-space mechanism without Conjecture 28.6, with the window as input. Allowing nonlocal smoothing perturbations, (1 - P) D (1 - P) + P d P makes such a window reducing, which enlarges the operator class and keeps the window as input.

The finite data

Proposition 28.18 (charge conjugation and the wall). With Hermitian generators J_i, C = C_7 (x) j_E has C^2 = -1 and commutes with T_M, D-hat_M and Gamma-hat as a differential expression, and CZ = -ZC. So it exchanges the two local wall conditions and does not preserve the chosen domain. If a pointwise parent antiunitary preserves a window and commutes with real multiplication, every real compressed multiplier commutes with its restriction, which cannot hold on a generated full M_4(C). (Report, Proposition 10.) The complex boundary block therefore requires this loss of the inherited symmetry, or a different real correspondence.

Remark 28.7 (why one block is quaternionic and the other complex). Proposition 28.18 proves the signs argument of the first version, with one correction of wording: j_E commutes with the anti-Hermitian Lie algebra action and reverses the Hermitian generators, so that C commutes with i Phi. Numerically (Python, phi_0 = 16, l = 3, rho_inf = 1), exchanging the wall condition exchanges (Z, tau, tr(B t-hat)/tr B) = (0.976, -1.146, -0.658) and (-0.976, 1.146, 0.658) on the boundary mode, with the same singular values, and leaves the zero-section mode unchanged. The formal parent symmetry does not by itself prove that the finite-mass core space is preserved, since C changes the wall domain, and the breaking computed in Result 28.5 shows the size of the effect at the sample points.

Proposition 28.19 (tunnelling and the grading). If the window consists of one copy of the 4 in each Gamma-hat degree, then d_M = [[0, conj(lambda_M) I_4], [lambda_M I_4, 0]] after equivariant unitary identifications, by Schur's lemma, and lambda_M = (1/4) Tr(U_{Y,M}^* T_M U_{B,M}). The inherited grading is I_W (+) (-I_V), the auxiliary grading of the doubling, which labels the two ends under localization. The radial Clifford involution Z does not extend across the zero section, so it does not serve as an interval chirality. This grading leaves all four core states in one degree, so W = H (+) H, for the even algebra H + H + M_4(C), still needs a chosen quaternionic grading, such as a marked Gamma_p. (Report, Proposition 11.)

Proposition 28.20 (the doubled bimodule). With H = W (+) V, Theorem 16.2 gives K = H (x) conj(H), of dimension 64, with DD = d (x) 1 + Gamma-hat (x) d^c. Its odd part is exactly K^- = Hom(V, W) (+) Hom(W, V), with the desired bimodule actions, and order zero and order one hold exactly on the full K, for its actual even algebra. If lambda != 0, then DD maps K^- into K^+, P_- DD P_- = 0 and DD^2 = 2 lambda^2. So K^- is not a reducing subspace. On K^- the inherited tensor grading is constant; the sheet-sign grading of Theorem 16.7 gives J(x, y) = (y^*, x^*), J^2 = 1, J Gamma_6 = -Gamma_6 J, with zero Dirac operator. The tunnelling map W -> V does not provide the map Hom(V, W) -> Hom(W, V) of lemma 20. (Report, Proposition 12.)

Balance against Proposition 6 of lemma 24

(Report, Proposition 14, with the numerical statements updated.)

Structure	What follows
Connected scalar parent	Supplied by the chosen disc bundle X.
Independent endpoint functions	Derived, by restriction and extension.
Base-function restriction at the collapsed end	Derived geometrically; quantitative for localized states.
Four core states	Proved for the linear normal model and the resulting twisted operator on S^4.
Four boundary states	Proved for the separated collar model; for the pull-back connection, the finite block of Proposition 28.13, with the convergence numerical.
Isolated 4 + 4 window	Conjecture 28.6.
Limiting quaternionic core compression	Derived once convergence to the core model is established.
Exact finite-mass quaternionic block	Requires b_+ = b_-; numerically not satisfied at the sample points resolved.

Structure	What follows
Exact level-one boundary compression	Requires rho -> infinity or the separated collar connection.
End grading	Inherited from the chosen self-adjoint doubling.
Quaternionic grading axis	Input.
The two Hom spaces	The odd part of the coarse correspondence; the selection is additional.
KO-6 reality and order conditions	Exact after the specified correspondence and grading choices.
Nonzero connector of lemma 20	Input.
Mass value and dependence on l	Radial matching; equivariance gives only lambda I_4; the exponent is computed numerically in Result 28.5.

Choices that stay inputs: X, E = S^3(S), the hedgehog, the wall condition, the collar connection in case (b), and the doubling of Theorem 16.2. A double with two identical collapsed zero sections removes the outer boundary condition, but the same local core construction gives two quaternionic endpoint maps, so obtaining a complex block needs different structure at one end. T0 stays open, and this is a candidate for its first half.

Direction. The next analytic step is the coercive estimate of Lemma 28.14, in case (b) first, where both end models have exact kernels. Since Phi commutes with the Clifford symbol, D + iM Phi is a concentrating pair in the sense of Maridakis (arXiv:1510.07004, Definition 2.1 and Lemma 2.2), with [D, Phi] of order zero; the symbol condition goes back to Prokhorenkov and Richardson (arXiv:math/0307251). Two things are missing there. The spectral separation theorem of arXiv:2110.11654 (section 2, Assumption 3) needs a leading endomorphism independent of the normal direction, and this fails here, since (n . J)^2 depends on n for spin 3/2; so a separate core analysis or an anisotropic extension is needed. And the wall adds a boundary layer: on a half-line ||(d_t + M) f||^2 = ||f'||^2 + M^2 ||f||^2 - M |f(0)|^2, and exp(-Mt) is a zero mode away from the zero set of the mass. (Read by Terra 6.1 in arXiv HTML; the LaTeX sources were not available to it.)

Numerical check

By Claude Opus 5.5, in Python (numpy, scipy), 3 Nov 2026, in a script that ran the tests below as assertions and printed the tables of Result 28.5; by the rule of the project the script is not kept (removed on 4 Nov 2026; Terra 6.1 had reproduced its matrix of Proposition 28.13 in exact arithmetic). The slice operator on each multiplicity space is Hermitian, odd under Z, and closes on Hom_H. The structure constants reproduce [t_1, t_2] = t_3 in Sp(1)_a and Sp(1)_b. The decomposition D_6(rho) = A_0 + A_1/rho + A_2 rho holds. The form uses three-point Gauss quadrature on each element. The exact tests are {A_0, A_1} = 0, A_0^2 = 0, 4 on the 4, 5, 9 on the 16, 0, 8 on the 20, 12, 16 on the 40 (the value 0 on the 20 has the twist (1, 5) and gives no core zero mode), A_2 = -(i/8) sum C_ab^alpha c_a c_b c_alpha, and at rho = 1 with the line connection the values D^2 = 0, 5, 5, 12 on the summand O(3) of the 4, 16, 20, 40, against j(j + 4) for (0, 0)-forms in O(1); the whole summand also has the value 10 on the 20. The closed form of Proposition 28.13 is tested to ten digits, and the core mode density in the mesh.

Part 29

The eight-state window on the disc bundle

Written by Claude Opus 5.5 on 4 November 2026, for STRATEGY T0, without an Astra job. It continues lemma 28 and answers the prompt 29_window_coercive_estimate.md. Refereed by Terra 6.1 (report): case (b) stands with amendments, and the two errors it found in case (a), the definition of A_eff and the sign of the fibre part, are corrected below. The referee ran no computation. A fresh reading of the corrected case (a)

in the referee report on lemma 30 found one more sign and one missing step in Proposition 29.10; these are corrected. A second, fresh reading by Terra 6.1 (report, 5 November 2026, with the first reports withheld) found two errors, both repaired below: the range and quantifiers of delta (delta = 0 was a counterexample to the wording) and the sign of the Clifford commutator in the IMS step (the identity itself was right); it confirmed the chain of case (b) with amendments, the corrected case (a) with two added steps, and listed what a stand-alone paper needs (section "Second reading" below).

Outcome

1. **Conjecture 28.6 holds in case (b)** for the class of metrics and mass profiles of Setting 29.0: for each fixed 0 < delta < sqrt 3 there is M_delta such that, for M >= M_delta, D-hat_M has exactly eight eigenvalues in (-delta, delta) (Theorem 29.7).
2. **Conjecture 28.6 holds in case (a) at the Fubini-Study radius** rho = 1, with fibre part -2/3 (Theorem 29.11). At another wall radius, case (a) follows from one kernel condition K(rho), which reduces at each fixed radius to finitely many representations by ellipticity; an explicit bound identifying those representations remains to be written.
3. **The off-diagonal block is exponentially small**, |lambda_M| <= C exp(-(1 - eta)(M/2) int f dr), by an exact tunnelling identity and Agmon estimates (Proposition 29.8).
4. **The compression converges** to (Q_B(a|*{S^4}), Q_Y(a/*{CP^3})) with error O(M^{-1/2}) (Theorem 29.9). The statement concerns compression maps, in line with Counterexample 28.17.

The proof is short because Sp(2)-equivariance gives exact sums of squares at the core and at the wall. Each estimate holds on all of L^2 of a fixed manifold, with the boundary operators elliptic on CP^3 or S^4, so uniformity over the representations of Sp(2) needs no separate argument. The spectral separation hypothesis of Maridakis, which fails for this mass (historical/reviews/2026-11-03_terra_check_28.md, item 6), is never used. General forms of two identities are in the literature: the square with its zero-order cross term in the bulk (Maridakis, arXiv:1510.07004, Lemma 2.2), and a horizontal and vertical factorization with a curvature contraction (Kaad and van Suijlekom, arXiv:1710.03182, Proposition 18, under other hypotheses).

Scope. These are spectral statements for a chosen metric family, with flat fibres near the zero section. They give the finite window from the operator, for the sign of Proposition 28.13. They leave open: other metrics in the class of Theorem 28.9, including product-collar modifications of Bryant-Salamon-type metrics, and the complete Bryant-Salamon geometry (Theorem 28.9 assumes a product metric near a finite wall); the connector of Proposition 28.20; the quaternionic grading of Proposition 28.19; any dynamics selecting M, l or rho. Finite-mass generated algebras stay outside the statement (Counterexample 28.17).

Setting

Setting 29.0. Let X be the disc bundle of lemma 28, G = Sp(2), K = Sp(1)_a x Sp(1)_b, H = Sp(1)_a x U(1)_b, E = S^3(S), Phi_M = eps M f(r) n . J, and T_M = D_X^E + i Phi_M, with D-hat_M the doubled operator and the wall condition of Theorem 28.9: Z u = -u for T_M, and Z v = +v for T_M^*. Assume:

- (H1) g = dr^2 + pi^* g_{S^4} + rho(r)^2 g_{S^2}, with S^4 of radius 1, rho(r) = r on [0, 2e], and rho constant, equal to rho_l, on the collar [l - 2a, l]. Case (b) and Theorem 29.11 take rho_l = 1, the Fubini-Study metric.
- (H2) f(r) = r on [0, 2e], f = f_0 > 0 on the collar, and f > 0 on (0, l]. This is M b(|v|^2) v . J with b positive.
- (H3) The connection on E is the pull-back connection on r <= 2e. In case (b) it preserves the four line summands on the collar, with any invariant interpolation in between. In case (a) it is the pull-back connection throughout. We use radial parallel gauge: nabla_{d/dr} = d/dr. An invariant dr-component a(r) q could otherwise be added to a line-preserving connection; it is removed by radial parallel transport, which changes the displayed formulas by a phase on each line.
- (H4) eps = -gamma_b, where gamma_b is the chirality of Sigma_4 whose K-type is (1, 2). This is the sign of Proposition 28.13: its right singular vector is at the zero section. The opposite sign exchanges

T_M and T_M^* throughout. In the convention of the radial computation of lemma 28, gamma_b = +1.

On the collar put q = eps n . J, with eigenvalues +-3/2, +-1/2 on the line summands, and m = M f_0. In general q = -gamma_b n . J. In the notation of Proposition 28.13 and of the radial computation of lemma 28, q = -tau, and T_M = i Z d/dx + A + i m q on the collar, with x increasing towards the wall. The line q = +3/2 is the summand O(3), and q > 0 are the stable channels of Proposition 28.13.

Let C_R = ||A_2||, the norm on Sigma_6 (x) E of the curvature contraction A_2 = -(i/8) sum C_ab^alpha c_a c_b c_alpha; it does not depend on V, and C_R = 1 in Python (a finite C_R is all the proofs use). Let mu_1 > 0 be the smallest positive eigenvalue of Q^* Q for Q = sigma . d_y + y . J of Proposition 28.11. It is positive since the resolvent is compact, because (y . J)^2 >= |y|^2 / 4, and ker Q = 0: with M = 1, Q^* Q = (-Delta + |y|^2/4 - 3/2) + ((y . J)^2 - |y|^2/4) + (3/2 - sigma . J), three nonnegative terms; equality in the first forces a Gaussian times a constant vector, and equality in the second would then force J_i^2 v = v/4 for all i, against sum_i J_i^2 = 15/4 (referee). Python gives 1.02.

The zero section

Proposition 29.1 (exact splitting). Identify sections over the tube r <= 2e with K-equivariant maps s : G -> F, F = Sigma_4 (x) L^2(R^3; C^2 (x) C^4), with c(e_a) = gamma_a (x) 1 and c(f_alpha) = gamma_5 (x) sigma_alpha. Let D_H = i sum gamma_a e_a, with e_a acting as left-invariant derivatives on G, and Q^ = gamma_5 sigma . d_y + eps M y . J. Then on this tube T_M = T_0 + R, with T_0 = D_H + i Q^ and R Hermitian of order zero with |R| <= C_R |y|, and

```
T_0^* T_0 = D_H^2 + Q^^* Q^,    T_0 T_0^* = D_H^2 + Q^ Q^^*,
```

with D_H^2 commuting with Q^^* Q^ and Q^ Q^^*.

Proof. The fibres are flat and totally geodesic by (H1), so the Levi-Civita connection of X differs from the connection associated with the principal K-bundle G -> S^4 only by the O'Neill tensor. Its component is the curvature of N evaluated at v. This gives the zero-order term R, which equals r A_2 in the radial form D_6 = A_0 + A_1/rho + A_2 rho, of norm at most C_R |y|. For the pull-back connection, the remaining operator is D_H + i gamma_5 sigma . d_y + i Phi_M. The operator Q^ acts on the values of s, and it is K-equivariant, since sigma . d_y and y . J are invariant under the simultaneous rotation of y, of C^2 and of C^4. On the ambient space of fibre-valued functions on G, Q^ commutes with each e_a, because it does not depend on the point of G; the individual e_a need not preserve the K-equivariant subspace, but both Q^ and the summed operator D_H do. Since gamma_a anticommutes with gamma_5 and commutes with y . J, D_H Q^ = Q^^* D_H, where Q^^* = -gamma_5 sigma . d_y + eps M y . J. Therefore

```
T_0^* T_0 = (D_H - i Q^^*)(D_H + i Q^) = D_H^2 + Q^^* Q^ + i (D_H Q^ - Q^^* D_H) = D_H^2 + Q^^* Q^,
```

and likewise for T_0 T_0^*. Finally D_H Q^^* Q^ = Q^ Q^^* D_H, so D_H^2 commutes with both. ■

In radial form. T_0 = i Z d/dx + A_0 + A_1/rho + i phi q, and the cross terms vanish by {A_0, Z} = 0, [A_0, tau] = 0 and {A_0, A_1} = 0. these three identities were asserted on the representations 4, 16, 20, 40 (Python, script not kept).

Proposition 29.2 (the model spectrum). On the total space of N, with f = r and T_0 as above, let Pi be the projection onto ker T_0. Then:

1. ker T_0 is one copy of the 4 of G; ker T_0^* = 0.
2. ||T_0 u||^2 >= min(4, mu_1 M) ||u||^2 for u orthogonal to ker T_0, and ||T_0^* v||^2 >= min(4, mu_1 M) ||v||^2 for all v in the domain.

Proof. Write Q^_gamma for Q^ on chirality gamma. By Proposition 28.11, Q^_gamma has kernel exactly when gamma = -eps; this is the rank-four K-representation W_0 = (1, 3) + (1, 1), and on the orthogonal complement Q^^* Q^ >= mu_1 M, after the rescaling y -> M^{-1/2} y. The same bound holds on the other chirality, and for Q^ Q^^* with the chiralities exchanged, since Q^* Q and Q Q^* have the same nonzero spectrum. The fibrewise projection P_W onto W_0, on both chiralities of Sigma_4, commutes with D_H, by Proposition 29.1. (The projection onto the fibre kernel of Q^ also selects gamma = -eps, so

it does not commute with D_H, which is odd; it commutes with D_H^2.) On chirality -eps, Q^ vanishes on the range of P_W and is bounded below by sqrt(mu_1 M) on its complement; on chirality +eps, it is bounded below by sqrt(mu_1 M) throughout. On Sigma^gamma (x) W_0, D_H is the Dirac operator of S^4 twisted by the homogeneous bundle of W_0, so by Parthasarathy's formula D_H^2 = Cas_G(V) + 3/2 - Cas_K(tau), with Cas_K = 2 j (j + 1) on Sp(1)_b, that is 0 and 4 on W_0. Only V odd under -1 occur, with Cas_G(V) >= 5/2 and Cas_G(V) >= 15/2 for V not the 4. So D_H^2 >= 4 except for D_H^2 = 0 at V = 4, twist j = 1. This kernel needs Hom_K(4, Sigma^gamma (x) (1, 3)) != 0, which holds exactly for Sigma^gamma = (1, 2), where (1, 2) (x) (1, 3) contains (1, 2) once. By (H4), gamma_b = -eps, so T_0 sees this chirality and T_0^* the other. Decomposing u by chirality and by P_W, and using the sum of squares, gives both bounds. The connection metric on the total space N is complete, and the doubled model is a symmetric Dirac-type operator plus a smooth Hermitian zero-order potential, hence essentially self-adjoint on C_c^infinity(N); the identities and estimates extend from this core to its closure, and no boundary condition enters at infinity. The S^4 spectrum has a direct check in the literature: Dolan's homogeneous-instanton calculation (hep-th/0304037, section 2(ii), equations (40)-(46)) gives four zero modes for isospin 1 and the next squared value 5, with the untwisted squared spectrum starting at 4 (referee). The equivariant Callias calculation that identifies ker Q^* as 3 + 1, and not only its dimension, is stated here with the conventions of Proposition 28.11 and is one of the items a stand-alone paper must display. ■

Lemma 29.3 (core estimate). Let u be in H^1 with support in r < 2e. Then

```
|| |y| u || <= (2/M) ||T_0 u|| + sqrt(10/M) ||u||,
```

and the same holds with T_0^*. So ||T_M u|| >= (1 - 2 C_R/M) ||T_0 u|| - C_R sqrt(10/M) ||u||, and likewise for T_M^* with T_0^*. In particular, for such u with Pi u = 0, and for all such v,

```
||T_M u|| >= (2 - o(1)) ||u||,    ||T_M^* v|| >= (2 - o(1)) ||v||,    M -> infinity.
```

Proof. Q^^* Q^ = -Laplacian_y + M^2 (y . J)^2 - eps M gamma_5 sigma . J. The eigenvalues of sigma . J are 3/2 and -5/2, and (y . J)^2 >= |y|^2/4. Hence M^2 || |y| u ||^2 / 4 <= ||Q^ u||^2 + (5/2) M ||u||^2 <= ||T_0 u||^2 + (5/2) M ||u||^2, by Proposition 29.1. The rest is the triangle inequality with |R| <= C_R |y|, and Proposition 29.2. ■

The bulk

Lemma 29.4 (bulk estimate). Let w be in H^1 with support in a compact subset of 0 < r < l, and let f_1 be the minimum of f on that support. Then

```
||T_M w||^2, ||T_M^* w||^2 >= (M^2 f_1^2 / 4 - C_Phi M) ||w||^2,
```

with C_Phi the supremum of the operator norm of the Clifford contraction sum_i c(e_i) nabla_{e_i} (f n . J). With 0 < r_1 < r_2 < l and a radial cutoff psi equal to 1 on [r_1, r_2], with compact support in 0 < r < l, we have for u in Dom T_M,

```
||u||^2_{L^2(r_1 <= r <= r_2)} <= C M^{-2} (||T_M u||^2 + ||u||^2).
```

Proof. ||(D + i Phi) w||^2 = ||D w||^2 + ||Phi w||^2 + <w, i [D, Phi] w>, with [D, Phi] of order zero and pointwise bounded independently of V, and Phi^2 >= M^2 f^2 / 4. For the second statement, choose 0 <= psi <= 1 equal to one on the compact set under consideration; the first estimate applied to psi u with T_M (psi u) = psi T_M u + [D, psi] u gives ||psi u||^2 <= 2 (||T_M u||^2 + ||d psi||_infinity^2 ||u||^2) / (M^2 f_psi^2/4 - C_Phi M), with f_psi the minimum of f on the support of psi. Applied to compact sets containing the supports of the d chi_i and of chi_b, this gives the O(M^{-2}) bounds used in the transition regions below. ■

The interpolation of the connection in (H3) takes place in this region, where it enters only through C_Phi.

The wall

Lemma 29.5 (line-twisted operators at the Fubini-Study metric). Let D_k be the Dirac operator on CP^3 twisted by O(k) with its invariant connection, k odd, at rho = 1. Then D_k^2 = C_2(L) + 3/2 - (3/8) k^2 on the SU(4)-type L, with C_2 normalized by C_2(4) = 15/8. Consequently:

1. On O(3): ker D_3 is one copy of the 4 of Sp(2), in Z = +1; and D_3^2 >= 5 on the V-isotypic part for V not the 4.
2. On O(1): ker D_1 = 0; D_1^2 >= 3 on the 4 and D_1^2 >= 6 on the V-isotypic part for V not the 4.

So the smallest nonzero |eigenvalue| on O(3) + O(1), over all V, is kappa_w = sqrt 3, attained on the 4; the lowest wall value on the 16 is sqrt 5.

Proof. The form of D_k^2 is Parthasarathy's formula for SU(4)/U(3), since the twist is a character of U(3). The Sp(2)-invariant connection on a line summand is unique, because the isotropy representation has no trivial summand, so it is the Chern connection. The normalization: j(j + m + 3) is the eigenvalue of 2 box on (0,0)-forms in O(m) on the type [m + j, 0, j], with C_2([m + j, 0, j]) - C_2([m, 0, 0]) = j (j + m + 3), which fixes the coefficient of C_2; scalar curvature 12 gives the constant 3/2; and the term in k^2 is fixed by the kernel of D_3 on the 4. Two checks: D_0^2 = 4 on the type 6, the Kirchberg limit R/3 at R = 12 (Ann. Global Anal. Geom. 4 (1986) 291-325); and the values in the Python table below. The kernel statements are H^q(CP^3, O(1)) = C^4 at q = 0 only, and H^q(CP^3, O(-1)) = 0, with K^{1/2} = O(-2). The types of odd 4-ality with C_2 < 55/8 are the 4, its conjugate, the 20 and its conjugate (enumerated in Python). The 4 occurs on O(3) once, by the multiplicity on the slice, and this copy is the kernel; since each 20 restricts to Sp(2) as 4 + 16, neither 20 occurs on O(3). So V != 4 has C_2 >= 55/8 on O(3), giving D_3^2 >= 5. On O(1), the 4 and its conjugate restrict only to the 4 of Sp(2), so V != 4 has C_2 >= 39/8, giving D_1^2 >= 6; and on the 4 Frobenius reciprocity gives the relevant SU(4) types 4 and 20, each in both chiralities, with Casimirs 15/8 and 39/8, hence the two squared values 3 and 6 on the slice block. The chirality of ker D_3 is that of (0,0)-forms, and the slice block fixes it as Z = +1 for q = +3/2 (Python). The line-twisted spectrum on odd complex projective spaces is in print: Ben Halima, Comment. Math. Univ. Carolin. 49 (2008), Theorem 1, with the tautological line O(-1) and scalar curvature 3; taking its exponent m = -k and multiplying its squared eigenvalues by 4 for scalar curvature 12 reproduces the zero and the gap 5 for k = 3 and the gap 3 for k = 1 (referee); the Sp(2) branching and the Z-sign are recorded here in the present conventions. ■

Lemma 29.6 (wall estimate in case (b)). Let w be in H^1 with support in the collar, vanishing near its inner end, and p(x) = sqrt(3 m) exp((3/2) m (x - l)), so that the kernel of T_M^* on the half-cylinder is p (x) ker D_3. Then:

1. If Z w = -w at r = l, then ||T_M w||^2 >= min(m^2/4, 3) ||w||^2.
2. If Z w = +w at r = l, and w is orthogonal to p (x) ker D_3, then ||T_M^* w||^2 >= min(m^2/4, 3) ||w||^2.

Proof. On the collar, A commutes with q and with d/dx, and anticommutes with Z. Writing w = w_+ + w_- by Z, integration by parts gives, for T_M,

```
||T_M w||^2 = ||(d + m q) w_+||^2 + ||A w_+||^2 + ||(d - m q) w_-||^2 + ||A w_-||^2,
```

where the sum of the real parts of the mixed terms is the boundary term obtained by integrating d/dx <w_+, A w_->, which vanishes because w_+(l) = 0 (the individual mixed terms need not be purely imaginary). For T_M^* the same holds with q replaced by -q and with w_-(l) = 0. There are three one-dimensional cases on (-infinity, l]. (i) If g(l) = 0, then ||(d +- m q) g||^2 = ||g'||^2 + m^2 q^2 ||g||^2. (ii) With g(l) free, ||(d + m |q|) g||^2 = ||g'||^2 + m^2 q^2 ||g||^2 + m |q| |g(l)|^2. (iii) With g(l) free, ||(d - m |q|) g||^2 is the form of -d^2 + m^2 q^2 with a Robin condition at l, whose spectrum consists of 0, on the profile sqrt(2 m |q|) exp(m |q| (x - l)), and of [m^2 q^2, infinity). Cases (i) and (ii) give the lower bound m^2/4. Case (iii) arises exactly in the channels (Z, q) = (-, q > 0) for T_M and (+, q > 0) for T_M^*. On those, expand in eigenvectors of A^2: the term ||A w||^2 is at least 3 ||w||^2, except on ker D_3 in Z = +1, which by Lemma 29.5 only arises for T_M^* at q = 3/2, where the Robin gap (9/4) m^2 applies. ■

The window

Theorem 29.7 (the window, case (b)). Under Setting 29.0 with rho_l = 1 and connection (b), for each fixed 0 < delta < sqrt 3 there is M_delta such that, for M >= M_delta, D-hat_M has exactly eight eigenvalues in (-delta, delta). The spectral subspace splits under Gamma-hat as W_M (+) V_M, each a

copy of the 4 of Sp(2), with W_M within O(M^{-1/2}) of the cut-off core modes in the domain of T_M, and V_M within O(M^{-1/2}) of the cut-off wall modes p (x) ker D_3 in the domain of T_M^*.

Proof. We verify Lemma 28.14. Take radial functions chi_c, chi_b, chi_w with chi_c^2 + chi_b^2 + chi_w^2 = 1, chi_c = 1 on r <= e with support in r < 2e, chi_w = 1 on [l - a, l] with support in the collar. Let U_M span (chi_c u_i, 0) and (0, chi_w v_k), where u_i is a basis of ker T_0 and v_k of p (x) ker D_3, with Gram-Schmidt. Radial functions preserve the wall condition.

Trial states. T_M (chi_c u_i) = [D, chi_c] u_i + R chi_c u_i, of norm O(M^{-1/2}), since u_i has Gaussian decay at the scale M^{-1/2}; and T_M^* (chi_w v_k) = [D, chi_w] v_k, which is O(exp(-(3/2) m a)). So eps_M = O(M^{-1/2}).

Coercivity. Since [D, chi] = c(d chi) is skew-adjoint with c(d chi)^* c(d chi) = |d chi|^2 (its square is -|d chi|^2: for instance [D, x] = i Z on the collar), while sum_i chi_i d chi_i = 0,

```
||T_M u||^2 = sum_i ||T_M chi_i u||^2 - sum_i || |d chi_i| u ||^2,
```

and likewise for T_M^*. The functions d chi_i vanish outside the bulk region e <= r <= l - a. So by Lemma 29.4,

```
(1 + C M^{-2}) ||T_M u||^2 >= ||T_M chi_c u||^2 + ||T_M chi_w u||^2 - C M^{-2} ||u||^2,    ||chi_b
↪  u||^2 <= C M^{-2} (||T_M u||^2 + ||u||^2).
```

If u is orthogonal to every chi_c u_i, then Pi (chi_c u) = 0 exactly, since <u_i, chi_c u> = <chi_c u_i, u>, so Lemmas 29.3 and 29.6 (1) give ||T_M u||^2 >= (3 - o(1)) ||u||^2. If v is orthogonal to every chi_w v_k, then chi_w v is orthogonal to p (x) ker D_3, so Lemmas 29.3 and 29.6 (2) give the same for T_M^*. Since ||D-hat (u, v)||^2 = ||T_M u||^2 + ||T_M^* v||^2, Lemma 28.14 gives the count and the distance to the range. The window is G-invariant and close to the range of U_M, so as representations W_M and V_M are copies of the 4. ■

Remark. The gap is set by the wall at sqrt 3, with the core next at 2. On the 16, 20 and 40 the lowest values are sqrt 5, sqrt 5, sqrt 12.

Proposition 29.8 (tunnelling). Let T_M u = lambda v and T_M^* v = lambda u be a singular pair of the window, with unit vectors. By Proposition 28.19, the four singular values coincide. For a radial chi equal to 1 near r = 0 and 0 near r = l,

```
lambda (<u, chi u> - <v, chi v>) = <v, c(d chi) u>.
```

For every eta in (0, 1) there are M_eta and C_eta with

```
|lambda_M| <= C_eta exp(-(1 - eta)(M/2) int_e^{l - a} f dr),    M >= M_eta.
```

Proof. The function chi u lies in Dom T_M, since chi vanishes near the wall. So <T_M^* v, chi u> = <v, T_M chi u>, and T_M chi u = chi lambda v + c(d chi) u. This gives the identity.

For the decay of u, let xi = psi exp(g) be radial, with psi = 0 on r <= e/2 and psi = 1 on r >= e, with g = 0 on r <= e, g' = (1 - eta)(M/2) f on [e, r_*], and g constant on r >= r_*. Since xi^2 u lies in Dom T_M,

```
||T_M (xi u)||^2 = lambda^2 ||xi u||^2 + || |d xi| u ||^2.
```

Choose the partition so that the derivative of the Agmon weight is supported where the bulk cut-off is one. The IMS lower bound, with no core piece and with Lemma 29.6 (1), or Proposition 29.10 in case (a), on the collar, reads int (M^2 f^2/4 - C M) chi_b^2 |xi u|^2 + kappa_w^2 ||chi_w xi u||^2 - sum_i || |d chi_i| xi u ||^2 <= lambda^2 ||xi u||^2 + || |d xi| u ||^2, and the transition terms are controlled by the bulk estimate of Lemma 29.4. On the bulk the excess over (g')^2 is at least (eta/4)(M^2 f^2 / 4) for M large, with f >= f_1 > 0, so ||exp(g) u||{L^2(r >= e)} <= C_eta. *For v we use the same argument from the wall inward, with g-tilde = 0 on [l - a, l], with -g-tilde' = (1 - eta)(M/2) f, and with Lemma 29.3 replacing Lemma 29.6 on the core, since T_M^* has no low states there. Take chi with |d chi| <= 2/s, supported on [r_, r_ + s] in the bulk. By Theorem 29.7, <u, chi u> -> 1 and <v, chi v> -> 0, so the identity gives |lambda| <= (4/s) ||u||{[r_, r_ + s]} ||v||{[r_, r_ + s]}. Run the weights with eta' = eta/2, and take s so small that (1 - eta')(I - int{r_}^{r_ + s} f) >= (1 - eta) I, with I = int_e^{l - a} f. This gives the stated exponent.*

The absorption uses lambda -> 0, supplied by the trial states. The factor 1/2 holds on every channel, since (n . J)^2 >= 1/4; for the adjoint, the cut-off vanishes near the wall and the core estimate replaces the wall estimate. For a family with l -> infinity the collar widths, the cut-off derivatives, the connection bounds and the positive lower bound of f on the bulk have to be uniform, and the excluded end lengths are absorbed in the spare eta (referee). ■

For a family with l -> infinity, M >= M_eta, with uniform bounds on the end geometry, the cutoffs, the connection and the positive bulk mass, the constants are uniform, so lambda is exponentially small in l at the rate (1 - eta)(M/2) inf f. This is the uniform decay rate required by Conjecture 28.6, as amended by the referee.

Theorem 29.9 (compression). Under Theorem 29.7 or Theorem 29.11, for a in C^1(X),

```
|| Q_M(a) - (Q_B(a|_{S^4}) (+) Q_Y(a|_{CP^3})) || <= C ||a||_{C^1} M^{-1/2},
```

where Q_B(a) = int a(x) 2 P_x d mu(x), with P_x = (1 - x . Gamma)/2, and Q_Y(a) = int a(z) B(z) d mu(z). In case (b), B(z) = 4 p_z, the level-one map of Example 16.9. In case (a) at radius rho, B(z) = 2((1 + w) p_z + (1 - w) p_{jz}), with p_{jz} the quaternionic partner of p_z and w = w(rho); so Q_Y(x_a) = -Gamma_a/5, Q_Y(y_T) is w(rho) times the level-one value, and the signed fibre part is -w(rho). Restriction to the two ends is surjective (Proposition 28.8), so real functions on X generate M_2(H) + M_4(C) under the limit map, by lemma 10 and Example 16.9, for every w(rho) > 0.

Proof. Theorem 28.15, with eta_M = O(M^{-1/2}) from Theorems 29.7 and 29.11, after choosing equivariant unitary identifications of the window spaces with the model copies of the 4. The normal first moments are O(M^{-1/2}) at the core, by the Gaussian decay of the model, and O(M^{-1}) at the wall, by the profiles. At the core, F in Hom_K(4, (1, 2) (x) (1, 3)) kills (2, 1), so B = 2 P_x in the normalization Q_B(1) = I. The sign in P_x is the value tr(B Gamma_5)/tr B = -1 of Proposition 28.13. At the wall, the pointwise inner product of different line summands vanishes, so the density is a sum over channels with weights from Proposition 28.13. ■

Case (a)

On the collar in case (a), A = A_d + A_o, where A_d = D_line preserves the lines and A_o = A_E/rho has order zero, with [A_o, q] != 0. In Python, the block of A_E between q = 3/2 and q = 1/2 has norm sqrt 3 and ||A_E|| = 2, on the full fibre, hence for all V. Let P be the projection onto {sum_{q > 0} p_q g_q : Z g_q = -g_q}, with p_q = sqrt(2 m q) exp(m q (x - l)), and Y the same with Z = +1, with Pi_Y the projection onto the second space. Let F = Pi_Y (A_d + A_o) P, from the range of P to Y, and let A_eff be the operator on the sum of both ranges with blocks F and F^*. Identified with sections over CP^3 of the lines q > 0, it is, by Lemma 28.12,

```
A_eff = D_3(rho) (+) D_1(rho) + (sqrt 3 / 2) B(rho),
```

with B(rho) the block of A_o between the stable channels, of norm sqrt 3 / rho. This is self-adjoint and elliptic on the fixed manifold CP^3, so its nonzero spectrum has a gap kappa_a(rho) > 0, uniformly in V. On the 4, F^* is the matrix A(rho) of Proposition 28.13, from Y to the range of P.

Condition K(rho). The kernel of A_eff(rho) is exactly one copy of the 4, in Z = +1, so F is injective and ker F^* is that copy.

Proposition 29.10 (wall estimate in case (a)). Assume K(rho). For w as in Lemma 29.6, with the kernel p (x) ker D_3 replaced by the kernel of the half-cylinder operator T_M^*, the two statements of Lemma 29.6 hold with the constant 3 replaced by kappa_a(rho)^2 - O(1/m). The half-cylinder kernel is exactly one copy of the 4 for m large, with fibre part tending to -w(rho) at rate O(1/m).

Proof. T_d = i Z d/dx + A_d + i m q preserves the block structure: T_d P = Pi_Y T_d P, since T_d (p_q g) = p_q A_d g. Also T_d maps the complement of P to the complement of Y, because for y in Y the function T_d^* y = A_d y lies in P. On the complement of P, the sum of squares of Lemma 29.6, with the channels with Robin kernel removed, gives the bound m/2 both for T_d : P^perp -> Y^perp and for its adjoint, so this block is bijective; for m > 2 ||A_o|| its bounded perturbation B_22 = (1 - Pi_Y)(T_d

+ A_o)(1 - P) is bijective with ||B_22^{-1}|| <= (m/2 - ||A_o||)^{-1}, which is the surjectivity the Schur complement F^* - B_21^* (B_22^*)^{-1} B_12^* needs, with B_12 = Pi_Y A_o (1 - P) and B_21 = (1 - Pi_Y) A_o P (referee). So with w = w_1 + w_2 and w_1 = P w,

```
||T_M w|| >= ||(1 - Pi_Y) T_M w|| >= (m/2) ||w_2|| - ||A_o|| ||w||,
||T_M w|| >= ||Pi_Y T_M w|| >= ||F w_1|| - ||A_o|| ||w_2||.
```

Eliminating ||w_2||, one gets (1 + 2 ||A_o|| / m) ||T_M w|| >= ||F w_1|| - (2 ||A_o||^2 / m) ||w||. Under K(rho), F is bounded below by kappa_a(rho), and F^* is bounded below on the complement of its kernel. The same elimination holds for T_M^*, with P and Y and F and F^* exchanged. On the half-cylinder T_M is thus injective with closed range, so T_M^* is surjective; its kernel is the kernel of the Schur complement of the block T_d + (1 - Pi_Y) A_o (1 - P), whose inverse has norm at most (m/2 - ||A_o||)^-1. So this Schur complement differs from F^* by at most ||A_o||^2 / (m/2 - ||A_o||), and the graph of the Schur solution gives ||P_{ker T_M^*} - P_{ker F^*}|| = O(1/m). This projection estimate turns orthogonality to the exact half-cylinder kernel into orthogonality to ker F^* up to O(1/m), which gives the adjoint bound. Off the 4, this is invertible. On the 4 the block is a 2 x 3 matrix of rank 2, so its kernel has dimension exactly 1, within O(1/m) of (-(2/3)(rho^2 + 2), 1, 1). ■

Theorem 29.11 (the window in case (a)). Under Setting 29.0 with the pull-back connection, K(1) holds, and for each fixed 0 < delta < min(2, kappa_a(1)) there is M_delta such that for M >= M_delta the conclusion of Theorem 29.7 holds at the Fubini-Study radius, with the wall modes replaced by the kernel of Proposition 29.10, and with fibre part tending to -w(1) = -2/3. Here kappa_a(1) >= sqrt 5 - 3/2 off the 4, and it equals 1.863 on the 4, the squared values there being (45 +- 3 sqrt 33)/8. At another wall radius the same conclusion holds under K(rho).

Proof. By Lemma 29.5, on V != 4 we have |D_3 (+) D_1| >= sqrt 5, while ||(sqrt 3/2) B(1)|| = 3/2. So A_eff is invertible there, with lower bound sqrt 5 - 3/2 > 0. On the 4, F^* is A(1), with kernel of dimension one in Z = +1, the vector (-2, 1, 1), and nonzero singular values 1.863 and 2.789, by Proposition 28.13. So K(1) holds. The proof of Theorem 29.7 then applies with Proposition 29.10 in place of Lemma 29.6. The trial wall states are cut-offs of the exact half-cylinder kernel: on the 4-isotypic component the adjoint kernel equation is v' = (m Z q + i Z A_4) v, with A_4 a fixed finite Hermitian matrix anticommuting with Z, so the coefficient is Hermitian, every L^2 solution on the negative half-cylinder lies for large m in its positive spectral subspace, whose eigenvalues are at least m/2 - ||A_4||, and its mass beyond distance a is O(exp(-(m/2 - ||A_4||) a)); the cut-off defect is exponentially small. The finite matrix A(rho) of Proposition 28.13 is used here as given; its derivation from the geometric data, with bases and signs, is not in the live text and is an obligation for a stand-alone paper (referee). ■

Other radii. For K(rho), only finitely many V can carry a kernel, those with |D_3(rho) (+) D_1(rho)| <= 3/(2 rho), by ellipticity. In Python, on the 16, 20, 40 and for rho in [0.2, 4], the compressed block has smallest singular value at least 1.30. A check of K(rho) on all relevant V at a given radius needs a lower bound for D_line(rho) on all V, which we have not written down away from rho = 1.

Remarks

Remark 29.12 (the index). The summands of E on CP^3 have indices 4, 0, 0, -4, of total zero, as equivariant cobordism invariance requires: E extends over X (Witten 1983, preprint pp. 43-44). The window has one copy of the 4 in each Gamma-hat degree, so its Sp(2)-index is 4 - 4 = 0. So index theory allows lambda != 0, and the eight states come from the estimates above. The sign in (H4) is consistent with this: with the opposite sign, the end-model modes at the core belong to T_M^* and those on O(-3) at the wall to T_M (Proposition 28.13), so the two ends never lie in the same Gamma-hat degree. At finite mass the full operator can have nonzero lambda, so these states need not lie in exact kernels. The relative sign of the two ends enters through the chirality of ker D_3. It follows from the slice weights, which contain the 4 once in Z = +1 and not in Z = -1, and the sign of the core is the chirality statement behind (H4). If the O(3) zero mode lay in Z = -1 with q > 0, it would be a wall zero mode of T_M, in conflict with Lemma 29.6 (1). As a check in Python, the mean position of the modes is 0.51 at the core for T_M and 2.96 at the wall for the adjoint problem, with l = 3.

Remark 29.13 (physical units). All statements are in shape units. Under the dilation g -> R^2 g, with D -> D/R and the potential divided by R, the operator is unitarily equivalent to D-hat_M / R. So the light mass is lambda_M / R, the gap delta / R, and their ratio is unchanged, while the compression maps stay the same; this observation is from the audit of 3 November 2026 by Astra 6, section 5.2, conditional there on the estimates proved here. A family with R -> infinity, with lambda_M / delta small, closes the physical gap and keeps the eight states separated. Whether a dynamics selects such a family is open. The limiting geometry of such a family is a separate question: nothing here identifies it with a complete Bryant-Salamon manifold.

Remark 29.14 (other metrics). In the class of Theorem 28.9, T_M - T_0 also contains first-order terms with coefficients O(r^2), and M (b(r^2) - b(0)) y . J. One route shrinks the core piece to r < M^{-1/2 + s}, where Lemma 29.4 controls the complement since M^2 r^2 dominates. We have not carried this out, and the horizontal first-order terms need a bound in terms of D_H that takes account of the unbounded curvature terms.

Second reading (GPT-6.1 Terra, 5 November 2026)

The second reading (report) found no counterexample to the corrected Theorem 29.7 with 0 < delta < sqrt 3, confirmed the core-wall channel assignment and the relative sign of Remark 29.12, and confirmed the corrected case (a) (the overlap factor sqrt 3/2, the Schur complement, the fibre sign -w(rho), the kernel (-2, 1, 1) with squared singular values (45 +- 3 sqrt 33)/8) against the finite matrix of Proposition 28.13. Its verdict on a stand-alone mathematics paper on case (b): not ready in the present form. What such a paper must display, in the referee's order: the spinor and curvature conventions of Proposition 29.1; a self-contained equivariant Callias kernel calculation identifying ker Q^* as 3 + 1; the domains and essential self-adjointness on N; the SU(4)-to-Sp(2) eligibility and chirality calculation of Lemma 29.5; the repaired IMS and weighted estimates with their cut-offs, and the uniformity in l for the family of lemma 30; the derivation of the finite matrix A(rho) of Proposition 28.13 from the geometry; and a literature comparison crediting the published end-model spectra and localization methods.

Prior work identified by the referee, with the passages it checked (the identifiers are in paper/LITERATURE.md, section L): the line-twisted CP^3 spectra of Lemma 29.5 are in Ben Halima (2008), Theorem 1; the S^4 instanton spectrum of Proposition 29.2 is in Dolan, hep-th/0304037; the zero-order cross-term identity of Lemma 29.4 is Maridakis, arXiv:1510.07004, Lemma 2.2, and arXiv:2110.11654 develops normal-model localization under a direction-independence assumption that (n . J)^2 violates here; Kaad and van Suijlekom, arXiv:1710.03182, Proposition 18, give a horizontal-vertical factorization with a curvature contraction under other fibration hypotheses; Bismut and Lebeau (1991), sections VIII-IX, equations (9.155)-(9.156), give model-space projection and spectral-counting arguments of the same kind; Le Peutrec (Witten Laplacian with Neumann condition), Parker, arXiv:2307.00694, Theorem 1.1, and Schulz-Baldes and Stoiber, arXiv:2203.15014, sections 3 and 5, are the methodological ancestors of the boundary localization and the IMS argument; Kottke, arXiv:0909.5661, section 5, is the families Callias framework behind Proposition 28.11. The passages checked do not state the combined eight-state theorem with this S^4 core, CP^3 wall, anisotropic spin-3/2 mass and local boundary condition; this is a finding about the checked records, within a search that did not exhaust forward citations or review databases.

Numerical check

Python (numpy, scipy), scripts not kept by the rule of the project. The exact inputs were asserted: [A, q] = 0 and {A, Z} = 0 at rho = 1 with the line-preserving connection, the values of Lemma 29.5 against the formula, the norms used above, and the identities of Proposition 29.1. The direct solve is a test of the prediction: the limit values below were fixed by Proposition 29.2 and Lemma 29.5 before the run, and a failure would have refuted the argument. Setting: rho(x) = x for x <= 0.4, rho(x) = 1 for x >= 1.2, phi = M rho, l = 3, with the connection switched on [0.6, 1.2], and P1 elements at 2500 cells. Lowest singular values of T_M:

V	M = 8	M = 16	M = 32	predicted limit
4	small, 1.7321, 1.9861, 2.4495	small, 1.7321, 1.9963, 2.4495	small, 1.7323, 1.9991, 2.4497	0, sqrt 3, 2, sqrt 6
16	2.2245, 2.2361, 2.4495	2.2324, 2.2367, 2.4495	2.2351, 2.2409, 2.4497	sqrt 5 (core), sqrt 5 (wall), sqrt 6
20	2.2361, 2.8148, 2.8284	2.2367, 2.8248, 2.8284	2.2409, 2.8275, 2.8286	sqrt 5 (wall), sqrt 8 (core), sqrt 8 (wall)
40	3.4461, 3.4641, 3.4641	3.4585, 3.4641, 3.4645	3.4626, 3.4642, 3.4672	sqrt 12 three times

The wall values have a mesh error growing with M, of size 0.005 at M = 32 on the 16 and 20. The small value on the 4 is uninformative in this table: it halves when the mesh doubles, so it is a mesh floor, and Proposition 29.8 is proved without it. The Python wall values of the adjoint problem round to -1.00000 for the fibre part and the Gamma_5 part at M = 8, 16, 32, consistent with the limiting values of Theorem 29.9 in case (b); the theorem establishes the convergence, and the printed decimals do not establish exact equality at finite M. The scripts behind the table are not kept, by the rule of the project, so the table has not been reproduced by a referee.

Part 30

The massless limit as a joint limit: a finite window and an opening tower

Written by Claude Opus 5.5 on 4 November 2026, for STRATEGY T14. Theorem 30.1, Corollary 30.2, Proposition 30.3, Corollary 30.4 and Proposition 30.6 were proposed in an audit of commit b2ddf98 by Astra 6 (assessment in the review); the proofs below were redone against the source lemmas. Theorem 30.5 and Proposition 30.7 are new here and rest on lemma 29, whose referee is queued. Refereed by Terra 6.1, together with the note (report); its corrections are applied and have not been read again, since the Codex quota is spent.

Outcome

The claim that W and Z become massless as a dimension opens needs a family. Along a general path to M_W = M_Z = 0 nothing forces a tower to open: lemma 23 has theta -> 0 at fixed l, with the tower spacing fixed. The families below have the hierarchy

```
m_weak << E_eps << Delta_KK -> 0,
```

so a moving low-energy window keeps a fixed finite number of states. The hierarchy alone does not make the count at fixed energy diverge: the signed spectrum 0, +-eps^alpha, +-1, +-2, … has the hierarchy with a count bounded at fixed energy below 1. So the divergence at fixed positive energy is a separate property, proved below for each family.

1. On the exact interval this holds with an internal dimension of one (Theorem 30.1), also with localized end modes (Proposition 30.3).
2. On the cylinder of lemma 24 and on the disc bundle of lemma 29 it holds with seven internal dimensions. On the disc bundle the window comes from the operator, with one copy of the 4 at each end, and the count at fixed energy has seven-dimensional Weyl growth (Theorem 30.5).
3. On the disc bundle the core block of the window is quaternionic up to a defect that is O(M^{-1/2}) in general, and exponentially small when the connection preserves the quaternionic structure of E, for all bounded real observables (Propositions 30.6, 30.7).

Scope. These are statements about operators, compression maps and spectra along families given by hand. No dynamics selects them (Remark 30.8). The vector masses of Corollary 30.2 belong to the five-dimensional Yang-Mills toy of lemma 23. On the disc bundle only the fermion window is treated, so the relation between lambda and M_W is unproved there.

The interval

Theorem 30.1 (window and tower on the interval). Let (H_F, D_, *J_F, Gamma_F) satisfy the hypotheses of Theorem 22.1, with the algebra A_F represented by pi as in Theorem 22.4, r = dim H_F, and for 0 < alpha < 1, l_ > 0* and small eps > 0 let D_F(eps) = eps D_, *l_eps = l_* eps^{-alpha}, with D_eps the interval operator D_X of Theorem 22.1 for these data. Let Delta_eps be the smallest element of spec |D_eps| outside spec |eps D_*|. Then for eps small:

1. W_eps^* D_eps W_eps = eps D_*, with J_F and Gamma_F exact.
2. Delta_eps = (pi / l_) *eps^alpha - eps //D_||*.
3. ||Q_eps(a) - pi(a)|| <= (2/3) l_^2 *//D_||^2* eps^{2(1 - alpha)} ||a||.
4. For alpha < beta < 1, E_* > 0 and E_eps = E_* eps^beta, the count N_eps(E_eps) = #{|D_eps| <= E_eps} equals r.
5. For fixed E > 0, N_eps(E) ~ (2 r l_* E / pi) eps^{-alpha}.

Proof. The condition l_eps ||D_F(eps)|| = l_* ||D_// *eps^{1 - alpha} < pi/2 holds for small eps. Theorem 22.1 gives item 1, and the next level is (pi - theta_max)/l = pi/l - //D_F||, which is item 2. Theorem 22.4 (2) gives item 3. For item 4, eps //D_||* « eps^beta « eps^alpha. For item 5, the signed spectrum is (n pi +- theta_k)/l_eps, which is r arithmetic progressions of spacing pi / l_eps, counted with multiplicity also when singular values coincide or vanish, each with (2 E l_eps / pi)(1 + o(1)) points in [-E, E]. ■

Theorem 22.1 starts from supplied finite data and a matrix-valued parent algebra, and the wall data change with eps by hand. A fixed Majorana block does not scale as eps D_*, so applying the theorem to an electroweak Dirac block needs a reducing sector or a treatment of the heavy neutral sector.

Corollary 30.2 (vector poles in the toy of lemma 23). For the universal channel of lemma 23, with m_eps = eps m_* and theta_eps = l_eps m_eps -> 0,

```
M_W = arcsin(sin(2 theta_eps)/sqrt 2) / l_eps,    M_Z = arcsin(2 sin(2 theta_eps)/sqrt 5) / l_eps,
```

so M_W^2 = 2 m_eps^2 (1 + O(theta_eps^2)) and M_Z^2 = (16/5) m_eps^2 (1 + O(theta_eps^2)). The vector poles follow the hierarchy. The constants 2 and 16/5 belong to this five-dimensional action. The small-mass pole coefficients of the fully massive finite spectral action are 3 and 24/5 (lemma 18, Proposition 14, and lemma 23), for the family with both members of every doublet at the same Dirac mass and no Majorana mass. Lemma 25 connects the two pairs at the level of the leading coefficient: its M^2 = ((3 - d_eff(s))/2) |||[D_F, T]||^2 / ||T||^2, with s = l Lambda, is the leading Taylor quotient of the heat-cutoff action for a generator constant in the transported frame, so the coefficient is 3 and 24/5 at d_eff = 0, and 2 and 16/5 at d_eff = 1. This does not identify the spectral-action pole problem with the Yang-Mills toy, and a joint-limit use needs a remainder uniform in s, since lemma 25 expands at fixed s. Along Lambda = E_eps we have s = l_* E_* eps^{beta - alpha} -> 0, the finite value, and at fixed Lambda > 0 we have s -> infinity, the interval value. Varying the cutoff of the action is an extra prescription: an observational cutoff alone does not change the action.

Proposition 30.3 (a localized family). Take M = -mu and the mixed conditions of lemma 9, with mu_* > 0, 0 < alpha < 1, mu_eps = mu_* eps^alpha and the length l_eps^loc = c_eps / mu_eps, where c_eps = (1 - alpha) log(1/eps) + c_0. This length differs from the l_eps of Theorem 30.1. Let s = kappa l be the root of s = c tanh s, and m = mu sech s. Then

```
m_eps ~ 2 mu_* exp(-c_0) eps,    Delta_eps ~ mu_* eps^alpha,    (1/kappa) / l_eps = 1/s -> 0.
```

With the left profile L(x) proportional to exp(-kappa x) - r exp(kappa x), r = exp(-2 kappa l), the following identities are exact:

```
r = (m / (mu + kappa))^2,    mu - kappa = m^2 / (mu + kappa).
```

Proof. Lemma 9.5 gives m = 2 mu exp(-c)(1 + o(1)) and kappa/mu -> 1. The first oscillatory root has pi < k_1 l < 3 pi/2 and lambda_1^2 = mu^2 + k_1^2, so Delta/mu -> 1. From mu = kappa coth s and m = kappa / sinh s we get mu + kappa = kappa exp(s) / sinh s, hence m / (mu + kappa) = exp(-s). The second identity is m^2 = mu^2 - kappa^2. ■

The localization width and the length both grow, with a shrinking ratio. The finite two-point distance 1/m and the length l are both divergent, with (1/m)/l ~ exp(c)/(2c), so a metric convergence statement would need its own comparison maps.

Corollary 30.4 (the cylinder of lemma 24). Use the operator D_r = E d_x + Z A / r on the interval times CP^3, with A as in lemma 24, finite data on V = ker A = C^4 with gradings of type (2, 2) as in lemma 24, so that r = 4, the scaling of Theorem 30.1, and r_eps = r_* eps^{-gamma}, 0 < gamma < 1. Other values of r need an enlarged tangential bundle, which we do not construct. Then for max(alpha, gamma) < beta < 1, the cutoff E_* eps^beta isolates the finite window, since the next level is at least min((Delta_K / r_) *eps^gamma, (pi / l_)* eps^alpha - eps ||D_*||), by lemma 24, (5). At fixed E > 0 the count is proportional to l_eps r_eps^6 E^7, the seven-dimensional Weyl growth, by Dirichlet and Neumann bracketing of the squared product operator as l_eps and r_eps grow. For gamma = alpha the geometry expands homothetically in all seven directions.

The window here is prescribed by the spectral walls of lemma 24.

The disc bundle

Theorem 30.5 (joint limit on the disc bundle). Fix a shape as in Setting 29.0, with connection (b), or with connection (a) at rho = 1. Let delta > 0 be as in Theorem 29.7 or 29.11, G_0 = int_e^{l - a} f dr, 0 < alpha < beta < 1, E_* > 0, eta in (0, 1), and

```
M_eps = (2 (1 - alpha) / ((1 - eta) G_0)) log(1/eps),    R_eps = eps^{-alpha},    D_eps =
↪  D-hat_{M_eps} / R_eps.
```

By the dilation of Remark 29.13, which is exact under the canonical spinor identification and the L^2 unitary factor R^{-7/2}, D_eps is the operator of the metric R_eps^2 g with potential (M_eps / R_eps) f(s / R_eps) n . J in the physical radial coordinate s = R_eps r, of size M_eps / R_eps -> 0. For eps small:

1. D_eps has exactly eight eigenvalues in (-delta eps^alpha, delta eps^alpha), and they are +-m_eps with multiplicity 4 (or zero with multiplicity 8), where m_eps <= C eps. The next level is at most C log(1/eps) eps^alpha, so it tends to zero.
2. For E_eps = E_* eps^beta the count N_eps(E_eps) = #{|D_eps| <= E_eps} equals 8.
3. The compression converges to the endpoint maps, with error C ||a||_{C^1} (log 1/eps)^{-1/2} in shape coordinates.
4. For fixed E > 0, N_eps(E) ~ C_X E^7 eps^{-7 alpha}, with C_X = 64 omega_7 Vol_g(X) / (2 pi)^7 the Weyl constant of the elliptic operator D-hat_0 on X, where omega_7 is the volume of the unit ball and 64 = 8 x 4 x 2 counts the spinors, E and the doubling.

Proof. Item 1: Theorem 29.7 or 29.11 applies since M_eps -> infinity, Proposition 28.19 gives the multiplicity, and Proposition 29.8 gives lambda <= C exp(-(1 - eta)(M_eps/2) G_0) = C eps^{1 - alpha}, hence m_eps = lambda / R_eps <= C eps. The bound on the next level: the shape-unit value is at most s_9(D-hat_0) + c M_eps, with the comparison below. Item 2: eps « eps^beta « eps^alpha. Item 3: Theorem 29.9, with M_eps^{-1/2} of the stated order. Item 4: D-hat_M = D-hat_0 + M V on the same domain, with V bounded by c = (3/2) sup f, so by min-max N(D-hat_0, Lambda - c M) <= N(D-hat_M, Lambda) <= N(D-hat_0, Lambda + c M). Take Lambda = E R_eps, so that M_eps / Lambda -> 0, and use the Weyl law for the fixed self-adjoint elliptic operator D-hat_0. ■

So the eight finite states stay separated at the moving scale, and at fixed physical energy the spectrum has the growth of a seven-dimensional internal space, for eleven dimensions in M^4 x X. Item 1 is an upper bound on m_eps, since a lower bound on lambda is unproved. A family with l_eps growing at fixed M has the same form, with the rate of Proposition 29.8. The constants M_eta, C_eta of Proposition 29.8 are fixed once the shape and eta are fixed, so no uniformity in a varying shape is used. The limit at fixed energy resolves a neighbourhood whose shape is fixed and whose size grows, so the pointed limit is flat at a fixed interior point and a half-space at a fixed boundary point. Nothing here identifies it with a complete Bryant-Salamon manifold, or with a solution of field equations.

The quaternionic defect

For the 4-copy of the window at the core, let B(r) = F(r)^* F(r) = diag(b_a I_2, b_+, b_-) as in Proposition 28.16, let j be the quaternionic antiunitary commuting with Sp(2), and P_H(A) = (A + j A j^-1)/2. Put

```
eta_t = int |b_+(r) - b_-(r)| dr / int tr B(r) dr.
```

Proposition 30.6 (all bounded real observables). For the normalized compression Q of this mode, ||Q(f) - P_H Q(f)|| <= 2 eta_t for every real f with ||f||_infinity <= 1. If X_1, …, X_k are such compressions, then ||X_1 … X_k - P_H(X_1 … X_k)|| <= 2 k eta_t.

Proof. With angular Haar measure of total mass 1, Schur's lemma gives the angular average (tr B / 4) I, so Q(f) = (4 / int tr B) int int f(r, g) lambda(g)^-1 B(r) lambda(g) dg dr. Since j commutes with lambda(g), Q(f) - P_H Q(f) is the same integral with B - P_H B. Here j exchanges b_+ and b_- and fixes the block b_a, so ||B - P_H B|| = |b_+ - b_-|/2. The triangle inequality gives the first bound. For the second, write X-tilde = j X j^-1, so that X - P_H X = (X - X-tilde)/2 and ||X_i - X-tilde_i|| <= 4 eta_t. Telescoping, X_1 … X_k - X-tilde_1 … X-tilde_k is a sum of k terms, each a product of contractions with one factor X_i - X-tilde_i, so its norm is at most 4 k eta_t; dividing by 2 gives 2 k eta_t. ■

So bounded expressions become quaternionic when eta_t -> 0. An unrestricted generated algebra may use coefficients of size 1/eta_t and retain the other directions, as in Counterexample 28.17.

Proposition 30.7 (the size of the defect). Under Theorem 29.7 or 29.11, eta_t = O(M^{-1/2}). Assume moreover that the quaternionic structure j_E of E is parallel on all of X. This holds for the pull-back connection and for the interpolation (1 - s(r)) nabla^pull + s(r) nabla^line with real s, and it can fail for other invariant interpolations allowed by (H3). Then for every eta in (0, 1) there are M_eta and C_eta with

```
eta_t <= C_eta exp(-(1 - eta) M int_{2e}^{l - a} f dr),    M >= M_eta.
```

This is the exponent of the bound on lambda in Proposition 29.8 doubled, with the integral starting at 2e. A bound in terms of lambda itself would need a lower bound on lambda, which is open.

Proof. For the first bound, fix a normalized equivariant model F_0 at the core with C F_0 = F_0 for the local real structure below, so that j B_0 j^-1 = B_0. Theorem 29.7 gives ||u - chi_c F_0|| = O(M^{-1/2}) in L^2, and |b_+ - b_-| <= (|F| + |C F|) |F - C F| gives the bound, by the Cauchy-Schwarz inequality.

For the second, j_E is parallel and j_E (n . J) j_E^-1 = -n . J, so C = C_7 (x) j_E commutes with D_X^E and with i Phi_M, hence with T_M as a differential expression, with C^2 = -1, and it exchanges the wall conditions (Proposition 28.18). For the line-preserving connection, j_E P_q j_E^-1 = P_{-q} gives j_E nabla^line j_E^-1 = nabla^line, and real affine interpolation keeps this. On Hom_G(4, -) put (C F) = C F j^-1, which is antilinear with square +1. It commutes with D_H, with the full fibre term i Q^ and with R separately, since each part of the Clifford structure transforms in the same way, so it acts on the one-dimensional space Hom_G(4, ker T_0) as a real structure. Fix F_0 there with C F_0 = F_0, so that B_0 is j-invariant. Let u = F be the mode with T_M u = lambda v, T_M^* v = lambda u, with the phase for which <chi_c F_0, u> is real, with chi_c as in Theorem 29.7. Put zeta = u - C u and nu = v - C v. In the interior,

```
T_M zeta = lambda nu,    Pi (chi_c zeta) = 0,
```

where the second holds since <chi_c F_0, C u> is the conjugate of <chi_c F_0, u>. Take the weights of the proof of Proposition 29.8 starting at r = 2e and meeting at r_* = l - a: g = 0 on r <= 2e, g' = (1 - eta)(M/2) f on [2e, l - a], g constant beyond; and g-tilde = G - g on r <= l - a, g-tilde = 0 beyond, with G = g(l - a). So g-tilde is constant on r <= 2e. That proof gives ||exp(g) u|| <= C_eta on X, and ||exp(g-tilde) v|| <= C_eta on r <= l - a/2, and lambda <= C_eta exp(-G). The same bounds hold for C u and C v, since C is pointwise isometric.

Let xi = psi-tilde exp(g-tilde), with psi-tilde = 1 on r <= l - a and 0 on r >= l - a/2. Then ||T_M (xi zeta)|| <= lambda ||xi nu|| + || |d xi| zeta ||. Since xi is constant on the support of chi_c, Pi (chi_c xi zeta) = 0, so Lemmas 29.3 and 29.4 and the partition of Theorem 29.7 bound ||T_M (xi zeta)|| from below by

the bulk term with (M/2) f and by (2 - o(1)) on the core. On the bulk |d xi| = (1 - eta)(M/2) f xi, which is absorbed with the factor eta to spare, leaving

```
||xi zeta|| <= C (lambda + ||u||_{L^2(1 - a <= r <= 1 - a/2)}) <= C exp(-G).
```

Finally B - j B j^-1 = F^* F - (C F)^* (C F), so |b_+ - b_-| <= (|F| + |C F|) |zeta| pointwise in r. On r <= l - a, the Cauchy-Schwarz inequality with the weight exp(g) gives

```
int (|F| + |C F|) |zeta| <= 2 ||exp(g) u|| ||exp(-g) zeta|| = 2 ||exp(g) u|| exp(-G) ||xi zeta|| <=
↪  C exp(-2 G),
```

since exp(-g) = exp(g-tilde - G). On r >= l - a, both |F|^2 and |C F|^2 integrate to at most C exp(-2 G). ■

The samples in section 6.1 of the audit (Python, in the pull-back connection, rows reproduced by Claude Sonnet 5.5) have eta_t / lambda^2 between 0.5 and 0.9. They vary the length at fixed mass, with profiles outside Setting 29.0, so they are a qualitative comparison and do not test this statement as M -> infinity. Under the added hypothesis the defect comes from the wall, which breaks C: for an operator with the same interior and no wall, on a complete manifold, Hom_G(4, ker T) carries a real structure, and if it is one-dimensional the core block is exactly quaternionic.

Remark 30.8 (what is missing, and the direction). The next steps are, in order:

1. The vector sector on the family of Theorem 30.5: the kinetic matrix and mass Hessian of the vector fluctuations, H_eps u = M^2 K_eps u, and then the full momentum-dependent kernel. The aim is M_W, M_Z ~ c lambda / R with finite nonzero couplings. The tunnelling eigenvalue alone does not establish this.
2. The gauge normalization. For a bulk Yang-Mills field with fixed microscopic coupling and constant profile, g_4^-2 = Vol(X) / g_11^2, so plain dilation gives g_4 proportional to R^{-7/2}, which must be separated from restoration at finite coupling. Gauge fields from isometries have Killing overlap integrals instead.
3. The action in the radius and the electroweak order parameter. A toy V = a r^-p + b eps^2 r^q with a, b, p, q > 0 has its unique minimum at r proportional to eps^{-2/(p+q)}, so alpha = 2/(p+q) lies in (0, 1) if p + q > 2. The repository has not derived such a potential, and adding it by hand would derive nothing.
4. The connector of Proposition 28.20, and the origin of the quaternionic grading of Proposition 28.19.

Numerical check

No statement here needs a computation. The audit tabulates the localized family in Python (mu_* = 1, alpha = 1/2, c_0 = 2), and Claude Sonnet 5.5 recomputed the rows; for eps = 10^-2, c = 4.3026 and m = 2.710 x 10^-3, against the asymptotic 2.707 x 10^-3.

Part 31

The interval of H as an eleventh dimension

Lemma 31. Written as file 01 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README). Propositions 1.1, 1.4 and 1.5 are proved here; Proposal 1.3 is a dictionary to be tested. The joint-limit family of the section of that name is from the Astra 6 audit of 3 Nov 2026 (historical/reviews/2026-11-03_astra_audit_d11.md, §3 and §4), restated here in circle language and checked. Attempted by Claude (Anthropic), 3 Nov 2026; checks in Python, checks_11d.py, sections F, F2 and H. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

Referee note (GPT-6.1 Terra, 4 Nov 2026). Proposition 1.1 and the joint family's Weyl constant are independently checked; the unfolded circle is scalar, and even extensions of arbitrary smooth interval functions need not be smooth at the fixed points. See the referee report. Proposition 1.5 is amended at its statement below.

Referee note on Proposition 1.4 (GPT-6.1 Terra). The endpoint value, minimum and small-angle coefficient are independently checked under the stated one-loop hypotheses; see the report.

Correction (3 Nov 2026). There was an error in the first version of Proposition 1.4: it read the potential of lemma 24 as the whole energy of the walls, but that potential is taken relative to theta = 0 and omits the theta-independent Casimir energy. The corrected statement is below; the reading "the walls attract" is withdrawn. The first version of Proposal 1.3 also placed the W at the bottom of the tower, M_W ~ 1/(2R), along every path to v = 0; after the audit (its §2.2), that holds only on the scale-free path of Proposition 1.5.

Setting

Hypothesis H of STRATEGY.md reads the two sheets of the finite geometry as the minimal truncation of an interval I_l with opposite chiral boundary conditions, with l ~ 1/M_W. Its three regimes carry the labels 9, 10 and 11. These are KO labels. At the 11 endpoint (v -> 0, l -> infinity) the heat dimension is 4 + 1 = 5, since the finite factor F keeps heat dimension 0 for every v (canonical §2.1). To read 11 as a metric dimension, the KO-6 factor must itself be resolved into a six-dimensional space, which gives the count

```
11 = 4 (spacetime) + 1 (the interval) + 6 (a space resolving F).
```

The files of this folder take the six-dimensional space to be B_6 = CP^2 x S^2, the base of Witten's spaces (02), and study two ways of completing 4 + 6 to eleven dimensions (03). This file treats the "1", and the path along which v -> 0 opens it.

Proposition 1.1 (the chiral interval is the antiperiodic circle)

Statement. Let D_l act on H_l = L^2([0, l], C^2) as in lemma 05, eq. (8): D_l(L, R) = (R', -L'), with R(0) = 0 and L(l) = 0, grading Gamma = diag(1, -1) and the scalar functions C^infinity([0, l]). Let S_R be the circle of length 2l, that is of radius R = l/pi, with coordinate x in [-l, l] and antiperiodic functions u(x + 2l) = -u(x). The map

```
(W(L, R))(x) = (L(|x|) + i sgn(x) R(|x|)) / sqrt 2
```

is a unitary H_l -> L^2_antiperiodic(S_R) with

1. W D_l W^* = -i d/dx on its antiperiodic domain;
2. W Gamma W^* = P, the reflection (Pu)(x) = u(-x);
3. W f W^* = multiplication by f(|x|), so the algebra goes to the reflection-even functions;
4. the walls x = 0 and x = l go to the two fixed points x = 0 and x = l ~ -l of the reflection.

Hence spec D_l = {(k + 1/2)/R : k in Z}, each eigenvalue simple, which is (9) of lemma 05, and the Connes-Lott mass of the minimal truncation is m = pi/(2l) = 1/(2R).

Proof. The inverse map is L(x) = (u(x) + u(-x))/sqrt 2, R(x) = (u(x) - u(-x))/(i sqrt 2) for x in [0, l], so W is a bijection, and |u(x)|^2 + |u(-x)|^2 = |L(x)|^2 + |R(x)|^2 gives unitarity. For H^1 data, u is continuous at 0 exactly when R(0) = 0, and u(l) + u(-l) = sqrt 2 L(l), so the antiperiodic matching u(-l) = -u(l) is the condition L(l) = 0. On x > 0, -i (L + iR)' = R' + i(-L'), the image of (R', -L') = D_l(L, R). On x < 0, u(x) = (L(-x) - i R(-x))/sqrt 2 and -i u'(x) = (R'(-x) - i(-L'(-x)))/sqrt 2, which is again the image of (R', -L'). Gamma(L, R) = (L, -R) maps to L(|x|) - i sgn(x) R(|x|) = u(-x). Items 3 and 4 are immediate. The antiperiodic spectrum of -i d/dx on a circle of length 2l is pi(2k + 1)/(2l) = (k + 1/2)/R. ■

So the two sheets of H are the two fixed points of a reflection on a circle whose spinors are antiperiodic, and the grading of the finite geometry is that reflection. With finite data D_F supplied as in lemma 22, the parent unfolds to -i d/dx + D_F on the periodic circle of length 2l (lemma 25, Proposition 25.1), and the angle theta interpolates between that circle (theta = 0) and the antiperiodic circle of Proposition 1.1 (theta

= pi/2): D_F is a constant connection, a Wilson line, and the wall angles theta_k = l mu_k (mu_k the eigenvalues of D_F) are its holonomy phases.

The joint family: weak masses below a descending tower

At fixed l, the vector masses of lemma 23 vanish as theta -> 0 and also as theta -> pi/2, while the tower spacing pi/l stays fixed; vanishing masses alone open no dimension (audit §2.2). The audit gives a family in which they do.

Proposition (Astra 6, audit §3; restated). Let D_F(epsilon) = epsilon D_* and l_epsilon = l_* epsilon^{-alpha}, 0 < alpha < 1, with wall data X = tan(l D_F) as in lemma 22. Then the exact window is W^* D W = epsilon D_, *the rest of the spectrum lies at |lambda| >= pi/l_epsilon - epsilon ||D_||*, the compression error is at most (2/3) l_^2 ||D_||^2 epsilon^{2(1-alpha)} ||a|| (Theorem 22.4, item 2), and for alpha < beta < 1 a cutoff E_epsilon = E_* epsilon^beta contains exactly the r = dim H_F finite states, while at every fixed E > 0 the number of states N_epsilon(E) grows like l_epsilon E, which is proportional to epsilon^{-alpha}.

In the circle language of lemma 25 (the periodic circle, on which D_F is a Wilson line; the window is its n = 0 mode) this reads: the radius R_epsilon = l_epsilon/pi goes to infinity, and the Wilson-line angles theta_epsilon = l_epsilon epsilon mu_k go to zero like epsilon^{1-alpha}. The hierarchy is

```
m_weak ~ epsilon  <<  E_epsilon ~ epsilon^beta  <<  1/R_epsilon ~ epsilon^alpha  ->  0.
```

In the five-dimensional Yang-Mills toy of lemma 23 the vector poles along this family are M_W^2 = 2 m^2 (1 + O(theta^2)) and M_Z^2 = (16/5) m^2 (1 + O(theta^2)), so M_W^2/M_Z^2 -> 5/8 (audit §3.1, from Theorem 23.6).

Check (Claude). The parent space of lemma 22 is L^2([0, l]; C^2) ⊗ H_F, of fibre dimension 2r. By Theorem 22.1 its signed spectrum is (n pi +- theta_k)/l, n in Z, two progressions of spacing pi/l for each of the r/2 singular values, so N(E) = 2 r l E/pi + O(r), the constant of the audit (the scalar circle of Proposition 1.1 gives 2 E l/pi; with fibre H_F the count is 2 r l E/pi). The window, gap and compression statements follow from Theorems 22.1 and 22.4 as stated. (Python, section H: exact spectra for random singular values.)

The audit also gives a localized family (its §4): a bulk mass mu_epsilon = mu_* epsilon^alpha and mu l = (1 - alpha) log(1/epsilon) + c_0, with light mass m ~ 2 mu_* e^{-c_0} epsilon, gap ~ mu_* epsilon^alpha, and the exact tail identity e^{-2 kappa l} = (m/(mu + kappa))^2. Its table of roots is reproduced in section H (Python, bisection on tanh s = s/c and tan y = y/c): the four light masses and next masses agree with the audit to the digits printed there.

Referee note (GPT-6.1 Terra, 4 Nov 2026). The localized asymptotics and tail identity are independently checked analytically; the length includes a logarithmic factor. The numerical table was not recomputed. See the referee report.

Proposal 1.3 (the circle of H as the M-theory circle)

Take the circle of Proposition 1.1 to be the eleventh direction of M-theory over a ten-dimensional type IIA background, so R = g_s l_s. Then, along the joint family:

H	circle of Proposition 1.1	M-theory / string side (recalled, to check)
v -> 0 along the joint family, label 11	R -> infinity, Wilson-line angles -> 0, m_weak R -> 0	strong coupling: D0-brane bound states of mass n/(g_s l_s) become light and are the momentum modes of the eleventh dimension (Witten 1995); the weak masses lie below this tower

H	circle of Proposition 1.1	M-theory / string side (recalled, to check)
v finite, label 10, two sheets	R finite, fixed points of a reflection	M-theory on an interval with two walls (Hořava-Witten); here with walls of opposite chirality
v -> infinity, label 9, sheets merge	R -> 0, antiperiodic spinors	a circle with antiperiodic fermions shrinking: bulk fermions get masses >= 1/(2R); compare type 0 strings, which have no bulk spacetime fermions (Bergman-Gaberdiel 1999), and the walls of opposite chirality of Fabinger-Hořava 2000

Each row matches a statement already in the corpus: the 11 endpoint needs the tower (canonical §2.1, lemma 05 Part 4), and at the 9 endpoint the bare interval keeps no light fermion, so light fermions must come from another sector (STRATEGY, T1 and T3). On the string side the same happens: the light fermions of a type 0 theory come from open strings on branes. The proposal therefore places the light fermions of the 9 endpoint on branes or walls, and 03 develops a brane origin for them.

Tests.

1. Along the joint family m_weak/M_KK ~ epsilon^{1 - alpha} -> 0, so the weak scale is far below 1/(g_s l_s); the scale-free path of Proposition 1.5, with M_W R fixed, is the other possibility and keeps no finite window.
2. Gauge couplings. A five-dimensional gauge field with constant profile on the interval has g_4^{-2} = l/g_5^2 = pi R/g_5^2 (audit §7.2 for the general volume formula), so at fixed g_5 the four-dimensional couplings vanish as R -> infinity, and M_W = g v/2 could vanish through g. The limit sought keeps g finite, which needs g_5^2 ~ R, and this must be shown for the vectors in question before the masses are read as restoration of the symmetry.
3. The path must be selected by an action: Proposition 1.5 shows what the flat interval alone selects.

Proposition 1.4 (the angle-dependent one-loop term at the vacuum angle)

Statement. Under the hypotheses of Theorem 9 of lemma 24 (flat interval, zero bulk masses, frequency-one fermions, the stated gauge towers and multiplicities, no wall potential), the angle-dependent part of the one-loop potential, taken relative to theta = 0 as in lemma 23, has its minimum at theta = pi/2 with value

```
V(pi/2) - V(0) = - 93 zeta(5) n_f / (512 pi^2 l^4),
```

and near theta = 0 it is (3 zeta(3)/(16 pi^2)) (54/5 - n_f) theta^2/l^4 + o(theta^2). At theta = pi/2 the exact vector poles of lemma 23 give M_W = M_Z = 0, with the fermion at the tower scale m = pi/(2l).

Proof. In the notation of lemma 24, V = $\mathcal{V}$/(8 pi^2 l^4), with F(0) = 0, and $\mathcal{V}$(1) = -n_f F(1) since the gauge terms vanish at t = 1. With 1 + 1/sinh^2 x = coth^2 x and log coth x = 2 sum over odd k of e^{-2kx}/k,

```
F(1) = 2 int_0^infinity x^3 log coth x dx = 4 sum_{k odd} (1/k) 3!/(2k)^4 = (3/2) sum_{k odd}
↪  k^{-5} = (3/2)(31/32) zeta(5) = 93 zeta(5)/64.
```

Near t = 0, $\mathcal{V}$'(0) = (6 + 24/5 - n_f) J(0) with J(0) = int x^3/sinh^2 x dx = (3/2) zeta(3), the coefficient already given in lemma 23. Python: F(1) = 1.506786, J(0) = 1.803085, series against quadrature to 10^-6. ■

Reading. The θ-independent Casimir energy of the towers, a_0/l^4, is not part of this potential and is not computed in lemmas 23 or 24; the direction in which l moves depends on the sign of a_0 - 93 zeta(5)

n_f/(512 pi^2). The minimum over the angle is the point with massless W and Z at fixed l, which opens no dimension (lemma 23 already notes that it is not the physical point).

Proposition 1.5 (a scale-free potential with an isolated stationary angle fixes M_W l)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The derivative identity is correct, and an isolated stationary angle gives $l \propto 1/m$. The universal conclusion has a flat-direction counterexample $G(\theta) = \theta^4$; for the displayed one-loop determinant the small-angle obstruction is recovered when $n_f \neq 54/5$. See the referee report.

Statement. Let V(l, theta) = l^{-4} G(theta) with G of class C^1, the form of every one-loop potential of a flat interval whose only scales are l and the wall angles (lemmas 23 and 24, with a_0 included in G).

1. If the angle is free, V is monotone in l at every fixed theta with G(theta) != 0. At the minimizing angle theta_min the length runs away: to l -> infinity if G(theta_min) > 0, to l -> 0 if G(theta_min) < 0. Stationary points exist only on the set G = G' = 0.
2. If the angle is tied to a fixed mass, theta = l m (as for X = tan(l M) in lemma 22), the stationary lengths satisfy theta G'(theta) = 4 G(theta) at theta = l m. So l = theta_*/m for a root theta_* that does not depend on m, and M_W l, M_Z l (functions of theta alone by Theorem 23.6) are the same at every m.

Hence, when the roots of theta G' - 4 G are isolated, a stationary branch has l = theta_*/m: the path l ~ 1/M_W of H's original statement, alpha = 1 in the scaling of the joint family (Remark 10.7 of 10 for a small fixed ratio). A root interval, as for G = theta^4 near 0, makes every alpha < 1 stationary and selects none (referee amendment). The joint family with alpha < 1 needs a second scale in the action: a bulk mass (the localized family, G(l m, l mu)), the cutoff of the spectral action (lemma 25, G(theta, l Lambda)), curvature, flux or a wall term.

Proof. d/dl [l^{-4} G(l m)] = l^{-5} (theta G'(theta) - 4 G(theta)) at theta = l m. For item 1, d/dl V = -4 l^{-5} G(theta) and d/dtheta V = l^{-4} G'(theta). ■

Amendment (4 Nov 2026). The window can separate on the scale-free path, with a fixed small ratio, if the light mass is exponentially small in a stabilized shape parameter, and a_0 is now computed. See 10, Remark 10.7.

Consequence for the modulus target of the audit (§7.1). The audit proposes V = V_*(a r^{-p} + b epsilon^2 r^q) with a, b, p, q > 0, whose minimum r ~ epsilon^{-2/(p+q)} gives alpha = 2/(p + q) < 1 when p + q > 2. In the flat one-loop family the epsilon-dependent term near theta = 0 is (3 zeta(3)/(16 pi^2)) (54/5 - n_f) m^2/l^2, so q = -2, and with the Casimir term p = 4 the stationary point has l ~ 1/m, alpha = 1, in agreement with item 2. A term growing with l at fixed v, as the ansatz requires, is absent from the flat family and must come from one of the second scales listed above.

What remains open here

- Whether the identification of Proposal 1.3 survives the gauge sector: lemmas 13, 14 and 25 tie M_W to the tower with model-dependent coefficients, which here become a relation between M_W, g_s and l_s, and test 2 above requires g_5^2 ~ R.
- The theta-independent Casimir coefficient a_0 of the flat family, which decides the runaway direction of Proposition 1.5, item 1.
- A potential with a second scale that selects alpha < 1: the localized family of the audit is the first candidate, with G(l m, l mu) computable from the exact secular equations of lemma 09.
- A fermion with both a fibre momentum and a base twist: on a nontrivial circle bundle (Witten spaces, file 02) the antiperiodic shift corresponds to a spin structure whose fibre circle does not lift the circle action. Which Witten spaces carry such a structure is an open item for the next job.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 1.4. *From the referee report on the 11D line, section 'Proposition 1.4: the one-loop endpoint and small-angle coefficient', transferred verbatim in substance on 4 Nov 2026.*

Hypotheses: those of Theorem 9 of lemma 24, including n_f > 0 and its normalization. The result is relative to theta = 0 and leaves the theta-independent Casimir term open.

Let

```
F(t) = int_0^infinity x^3 log(1 + t/sinh^2 x) dx,   t = sin^2 theta,
```

so that, by equation (22) of lemma 24, $\mathcal{V}$(t) = 8 pi^2 l^4 V(theta) = 3 F(2t(1 - t)) + (3/2) F((16/5) t(1 - t)) - n_f F(t). For every x > 0 the integrand increases strictly with t, so F is strictly increasing, and F(0) = 0. The two gauge terms are nonnegative and vanish at t = 1, where their arguments vanish; the fermion term -n_f F(t) is strictly decreasing for n_f > 0. Hence t = 1 is the unique global minimum.

Endpoint value. At t = 1, 1 + 1/sinh^2 x = coth^2 x. With q = e^{-2x}, coth x = (1 + q)/(1 - q), so

```
log coth x = log(1 + q) - log(1 - q) = 2 sum_{k >= 1, k odd} e^{-2kx}/k,     int_0^infinity x^3
↪  e^{-2kx} dx = 3!/(2k)^4.
```

All terms are positive, so the series may be integrated term by term, and

```
F(1) = 2 int_0^infinity x^3 log coth x dx = 4 sum_{k odd} (1/k) 3!/(2k)^4 = (3/2) sum_{k odd}
↪  k^{-5} = (3/2)(1 - 2^{-5}) zeta(5) = 93 zeta(5)/64.
```

Since $\mathcal{V}$(0) = 0 and $\mathcal{V}$(1) = -n_f F(1), division by 8 pi^2 l^4 gives the stated endpoint value V(pi/2) - V(0) = -93 zeta(5) n_f/(512 pi^2 l^4).

Small-angle coefficient. Differentiation under the integral at t = 0 is allowed because x^3/sinh^2 x is integrable on (0, infinity). With 1/sinh^2 x = 4 e^{-2x}/(1 - e^{-2x})^2 = 4 sum_{n >= 1} n e^{-2nx},

```
F'(0) = int_0^infinity x^3/sinh^2 x dx = 4 sum_{n >= 1} n 3!/(2n)^4 = (3/2) sum_{n >= 1} n^{-3} =
↪  (3/2) zeta(3),
```

which is J(0) in the proof of Proposition 1.4. The two gauge arguments 2t(1 - t) and (16/5) t(1 - t) have derivatives 2 and 16/5 at t = 0, with prefactors 3 and 3/2, so

```
V'(0) = (3 * 2 + (3/2)(16/5) - n_f) F'(0) = (54/5 - n_f)(3/2) zeta(3).
```

With t = sin^2 theta = theta^2 + o(theta^2), V(theta) - V(0) = $\mathcal{V}$'(0) theta^2/(8 pi^2 l^4) + o(theta^2) = (3 zeta(3)/(16 pi^2))(54/5 - n_f) theta^2/l^4 + o(theta^2), the claimed coefficient. The exact vector formulas of lemma 23 vanish at theta = pi/2, where the lowest fermion mass is pi/(2l).

Counterexample (referee) to the universal conclusion of Proposition 1.5. *From the referee report on the 11D line, section 'Proposition 1.5: scale-free potentials and the joint path', transferred verbatim in substance on 4 Nov 2026.*

Verdict of the referee: an error in the universal conclusion, and correct with an amendment for isolated stationary angles. The derivative identities in the proof hold. The universal conclusion, that every stationary family has constant angle and that a second scale is necessary for the existence of a joint family, fails by the counterexample below.

At fixed m > 0 put theta = l m. Then

```
d/dl V(l, l m) = l^{-5} (theta G'(theta) - 4 G(theta)).
```

If a continuous stationary branch chooses an isolated positive root theta_, *then theta_* is constant along the branch and l = theta_*/m. A root interval gives a flat direction instead.

Counterexample. Let G(theta) = theta^4 on a neighbourhood 0 <= theta < delta, with a C^1 extension to the angle domain. There

```
V(l, l m) = l^{-4} (l m)^4 = m^4
```

is independent of l. For any 0 < alpha < 1, the family m = epsilon, l = epsilon^{-alpha} has theta = l m = epsilon^{1 - alpha} < delta for all sufficiently small epsilon, so it is stationary, and l m -> 0. The vector mass formula of Theorem 23.6 gives M_W l ~ sqrt 2 l m -> 0. The potential leaves this path undetermined and does not dynamically select it. A smooth extension can also be chosen with G(theta) >= theta^4 on the rest of the fundamental interval; then V(l, l m) >= m^4, with equality on l m < delta, and the example is a flat family of minima.

The specified flat one-loop determinant. By Proposition 1.4, with a_0 the theta-independent Casimir coefficient,

```
G(theta) = a_0 + c theta^2 + o(theta^2),    theta G'(theta) - 4 G(theta) = -4 a_0 - 2 c theta^2 +
↪  o(theta^2),
```

where c = (3 zeta(3)/(16 pi^2))(54/5 - n_f) is proportional to 54/5 - n_f (the expansion of G' follows from that of $\mathcal{V}$', since $\mathcal{V}$ is C^1 at t = 0). If c != 0, there is no sequence of positive stationary angles tending to zero, whether or not a_0 = 0: for a_0 != 0 the right side tends to -4 a_0, and for a_0 = 0 it equals -2 c theta^2 (1 + o(1)). In particular this covers every integer n_f under the displayed determinant. The tuned case c = 0 requires its higher-order term to be checked. The comparison with the audit's ansatz with q > 0, in the paragraph on the modulus target, is correct: the displayed quadratic contribution is l^{-4} c (l m)^2 = c m^2/l^2.

Free angle. For a free angle, d/dl V = -4 l^{-5} G(theta); a nonzero minimized G produces the stated runaway, and a zero minimum leaves the length flat. At an endpoint of a constrained angle interval, the stationarity condition on G' must be understood one-sided.

Scope and direction. The corrected fixed-angle result concerns isolated stationary roots of a scale-free C^1 potential. A flat modulus, a tuned higher-order potential, or additional dimensionful terms leave other families possible. A mechanism that selects a joint path with finite gauge couplings remains to be constructed.

Independent proof of the asymptotics of the localized family (audit, its §4). *From the referee report on the 11D line, section 'Audit section 4 and file 01: localized joint family', transferred verbatim in substance on 4 Nov 2026.*

The asymptotic light mass, the gap, the localization ratio and the exact tail identity follow analytically from the interval equations. The localized length includes a logarithmic factor, and so differs from the pure-power length of the boundary-angle family above. The numerical table of section H was not recomputed by the referee.

Let c = mu l, s = kappa l, and s = c tanh s, the light-mode condition kappa = mu tanh(kappa l). For c > 1 there is a positive solution, and as c -> infinity it has s -> infinity and s/c = tanh s -> 1. Moreover

```
0 < c - s = c (1 - tanh s) = 2c e^{-2s}/(1 + e^{-2s}) <= 2c e^{-2s} -> 0.
```

Therefore

```
m = mu sech s = 2 mu e^{-s}/(1 + e^{-2s}) ~ 2 mu e^{-s} ~ 2 mu e^{-c}.
```

With mu = mu_* epsilon^alpha and c = (1 - alpha) log(1/epsilon) + c_0, e^{-c} = e^{-c_0} epsilon^{1 - alpha}, which gives m ~ 2 mu_* e^{-c_0} epsilon and l = c/(mu_* epsilon^alpha). The localization width is 1/kappa, so its ratio to l is 1/s ~ 1/c.

The first oscillatory root y_1 = k_1 l of tan y = y/c satisfies pi < y_1 < 3 pi/2, since y/c > 0 is small for large c. The gap above the light pair is therefore

```
Delta = sqrt(mu^2 + k_1^2) = mu sqrt(1 + (k_1 l/c)^2) ~ mu = mu_* epsilon^alpha.
```

Finally, s = c tanh s reads mu = kappa coth s, and with it m = mu sech s = kappa/sinh s. Then

```
m/(mu + kappa) = 1/(sinh s (coth s + 1)) = 1/(cosh s + sinh s) = e^{-s},    m^2 = kappa^2/sinh^2 s
↪  = kappa^2 (coth^2 s - 1) = (mu - kappa)(mu + kappa),
```

and the first identity squared is e^{-2 kappa l} = (m/(mu + kappa))^2. This proves both displayed tail identities.

Part 32

Witten spaces as the hypercharge quotient of the five-dimensional representation

Lemma 32. Written as file 02 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README). Theorem 2.1 and Propositions 2.2 to 2.4 and 2.6 are proved here, Theorem 2.1 and Proposition 2.6 conditionally as the referee notes say; Proposition 2.5 is a negative result in a stated scope, refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5. Attempted by Claude (Anthropic), 3 Nov 2026, with a remark from the steer of the same day on the mixing of u and d. Checks: checks_11d.py, sections A to D, Python with exact fractions. Referee pending.

Referee note (GPT-6.1 Terra, 4 Nov 2026). The exclusion in Proposition 2.5 has an independent analytic proof. The required-bidegrees lemma and the general assertion that u^c, d^c must have the same flux image require a common ten-dimensional chirality; the exclusion itself also holds without that extra assumption. See the referee report; other reviewed items are noted separately.

Referee note on Theorem 2.1 and Propositions 2.2, 2.4 (GPT-6.1 Terra). The quotient and tensor-centre calculations are independently checked. The cited M^{111} identification is conditional on the recalled fibration; the original quotient metric needs rescaling to be Sasaki-Einstein, and the action on holomorphic functions is dual to the action on U. See the report.

Referee note on Proposition 2.3 (GPT-6.1 Terra). The Gysin, Smith normal form and spin calculations are independently checked; the cited Kreck-Stolz paper was unavailable. See the report.

Setting

Witten's spaces M^{pqr} = G/H, with G = SU(3) x SU(2) x U(1) and H = SU(2) x U(1) x U(1), are the seven-dimensional spaces of least dimension with the isometry group of the Standard Model, so 4 + 7 = 11 (Witten 1981; paper/LITERATURE.md §F). Castellani, D'Auria and Fré gave them Einstein metrics; Killing spinors exist only for p = q, where the AdS4 vacuum has N = 2 and the space is M^{111} (Fabbri, Fré, Gualtieri, Termonia, hep-th/9903036). Fabbri et al. (hep-th/9907219, §4) write the cone over M^{111} as the Kähler quotient of C^5 by a circle acting with weights 2 on a triplet U and -3 on a doublet V, with D-term 2|U|^2 = 3|V|^2, and M^{111} as a circle bundle over P^2 x P^1. In later notation it is M^{3,2}.

These weights are six times the hypercharges of the 5-bar of SU(5). The rest of the file develops that observation.

Theorem 2.1 (M^{111} is the hypercharge quotient of the 5-bar)

Statement. Let 5bar = C^3 + C^2 carry the Standard Model action on the left-handed fields d^c in (3bar, 1)_{1/3} and L in (1, 2)_{-1/2}; write U for d^c and V for L, so 6Y = 2 on U and -3 on V. Let mu_Y = (2|U|^2 - 3|V|^2)/6 be the moment map of hypercharge, and

```
Y_{3,2} = (mu_Y^{-1}(0) ∩ S^9) / U(1)_Y.
```

1. Y_{3,2} is a smooth, simply connected seven-manifold, the circle bundle of O(3, 2) over CP^2 x CP^1 (up to orientation), and its cone is the Kähler quotient C^5 // U(1)_Y of Fabbri et al.
2. The cone is Calabi-Yau (Gorenstein, sum of weights 0) because tr Y = 0 on the 5-bar, that is, because U(1)_Y lies in SU(5).

3. The group acting effectively on Y_{3,2} through the quotient is (U(3) x U(2))/U(1)_Y. Its subgroup S(U(3) x U(2))/U(1)_Y is PSU(3) x SO(3), and the overall phase of U(5) supplies one further circle, the Reeb (R-symmetry) direction.
4. The functions on Y_{3,2} that are polynomial in U, V and their conjugates have U(1)_Y-charge zero; the holomorphic ones are spanned by the monomials of degree (3k, 2k) in (U, V), whose span as tensors is Sym^{3k}(3bar) x Sym^{2k}(2) and as functions its dual Sym^{3k}(3) x Sym^{2k}(2): for k = 1 the functions form the (10, 3) of Fabbri et al.

Proof. On mu_Y^{-1}(0) ∩ S^9 we have |U|^2 = 3/5 and |V|^2 = 2/5, a product of round spheres S^5 x S^3. The circle acts by (lambda^2 U, lambda^{-3} V); a fixed point would need lambda^2 = lambda^3 = 1, so lambda = 1 and the action is free. The quotient of S^5 x S^3 by the full torus T^2 of phases of U and V is CP^2 x CP^1, and Y_{3,2} -> CP^2 x CP^1 is a principal bundle for T^2/U(1)_Y. A character lambda_U^m lambda_V^n of T^2 is trivial on U(1)_Y exactly when 2m - 3n = 0, so the characters of T^2/U(1)_Y are generated by (m, n) = (3, 2), whose line bundle is O(3) ⊠ O(2) up to the sign convention; this gives item 1 and item 4. Simple connectivity follows from the exact homotopy sequence of U(1) -> S^5 x S^3 -> Y_{3,2}. Item 2: the sum of weights is 3 * 2 + 2 * (-3) = 0. Item 3: an element (g, h) of U(3) x U(2) acts trivially on the quotient exactly when (g, h) = (lambda^2, lambda^{-3}), because mu_Y^{-1}(0) spans C^5. The intersection of SU(3) x SU(2) with U(1)_Y consists of lambda^6 = 1, generated by (e^{2 pi i/3} I_3, -I_2), the centre Z_3 x Z_2. ■

So the isometry group of the Witten space with Killing spinors, SU(3) x SU(2) x U(1) (Fabbri et al.), is the centralizer of hypercharge in U(5), divided by hypercharge. The SU(3) x SU(2) part is the Standard Model one; the remaining circle is the R-symmetry, and hypercharge itself has been quotiented. Witten assigned the U(1) of G to hypercharge; in this realization hypercharge is a different U(1) (03, Proposal 3.1).

Proposition 2.2 (uniqueness of the count 7)

Statement. (a) Among the quotients Y_{a,b} = (S^5 x S^3)/U(1), with weights b on C^3 and -a on C^2 and gcd(a, b) = 1, a, b > 0, the cone is Calabi-Yau only for (a, b) = (3, 2). (b) For a complex representation of dimension n and a free circle action, (S^{2n-1} ∩ mu^{-1}(0))/U(1) has dimension 2n - 3; this is 7 only for n = 5. Among representations of SU(5), with hypercharge as the circle, only the 5 and the 5-bar give a seven-dimensional quotient.

Proof. (a) The Calabi-Yau condition is 3b - 2a = 0. (b) Dimension count: 2n - 1 for the sphere, minus one for the moment map, minus one for the circle. The irreducible representations of SU(5) of dimension 5 are the 5 and the 5-bar. ■

Witten's count reads: 11 = 4 + (2 * 5 - 3). The seven internal dimensions come from the smallest representation of SU(5), with hypercharge as the circle. For the 10 the quotient has dimension 17, and for the 5 + 10 it has 27.

Proposition 2.3 (topology of Y_{a,b})

Statement. Let Y_{a,b} -> B = CP^2 x CP^1 be the circle bundle with Euler class a h_1 + b h_2, a != 0. Then H^2(Y) = Z + Z/gcd(a, b), b_3(Y) = 0, H^4(Y) is a group of order a^2, cyclic when gcd(a, b) = 1, and Y is spin exactly when a is odd and b is even. For (a, b) = (3, 2): H^2 = Z, H^4 = Z/9, spin. The pullback H^2(B) = Z^2 -> H^2(Y) = Z is (x, y) -> 2x - 3y up to sign: the two divisor classes h_1 = [{l(U) = 0}] and h_2 = [{m(V) = 0}] go to 2 and -3, the values of 6Y on d^c and L.

Proof. The Gysin sequence H^{k-2}(B) -> H^k(B) -> H^k(Y) -> H^{k-1}(B) -> H^{k+1}(B), with the first map the cup product with c = a h_1 + b h_2 and H^*(B) = Z[h_1, h_2]/(h_1^3, h_2^2). In degree 2 the cokernel of 1 -> (a, b) is Z^2/(a, b). In degree 4 the map H^2(B) -> H^4(B) has matrix [[a, 0], [b, a]] in the bases (h_1, h_2), (h_1^2, h_1 h_2), with determinant a^2 and Smith invariants (gcd(a, b), a^2/gcd(a, b)); since H^3(B) = 0 this gives H^4(Y), and the same matrix is injective, so H^3(Y) = 0. For spin: w_2(B) = h_1 mod 2, and the kernel of H^2(B; Z/2) -> H^2(Y; Z/2) is spanned by c mod 2; so w_2(Y) = 0 exactly when (1, 0) = (a, b) mod 2. For (3, 2), Z^2/(3, 2) -> Z by (x, y) -> 2x - 3y is an isomorphism. ■ (Python: Smith invariants and the spin condition, section C.)

The product member CP^2 x S^2 x S^1 of Witten's family (a = b = 0) is not spin.

Proposition 2.4 (the Z_6 of the Standard Model)

Statement. (a) Every monomial in U and V, and every state of the exterior algebra Λ^*(5bar), satisfies 2t + 3d + 6Y = 0 mod 6, with t the triality (1 for 3, -1 for 3bar) and d = 2j mod 2. (b) The U(1)_Y-neutral monomials have t = 0 and d = 0. (c) Every tensor field on Y_{3,2} carries a representation of (U(3) x U(2))/U(1)_Y, on which SU(3) x SU(2) acts through PSU(3) x SO(3). So no tensor Kaluza-Klein mode on Y_{3,2}, on a background invariant under the displayed action, is a colour triplet or a weak doublet; spinors are open.

Proof. (a) holds for U (t = -1, d = 0, 6Y = 2) and V (t = 0, d = 1, 6Y = -3) and is additive. (b) U^alpha V^beta is neutral when 2 alpha = 3 beta, so 3 | alpha and 2 | beta. (c) The effective group of Theorem 2.1 acts on the frame bundle and on every tensor bundle, and its intersection with SU(3) x SU(2) is the centre Z_6 (Theorem 2.1, item 3). ■

(a) is the statement that the Standard Model group is (SU(3) x SU(2) x U(1))/Z_6 = S(U(3) x U(2)). In the quotient realization the Z_6 is the kernel of SU(3) x SU(2) -> Isom(Y_{3,2}). States with fractional hypercharge are those charged under the quotient circle, and they cannot be functions or tensors on Y_{3,2}. For spinors the isometry group may lift to a double cover; whether fermionic Kaluza-Klein modes on Y_{3,2} can carry half-integer weak isospin or triality is open (next job, item 2 of the README).

Chiral zero modes on the base with abelian fluxes

Witten's argument makes Kaluza-Klein fermions on an odd-dimensional space vectorlike, for gauge fields acting by isometries. The even base B_6 = CP^2 x CP^1 escapes it: abelian fluxes on B_6 give chiral four-dimensional fermions through the index of the twisted Dirac operator. This section asks which fluxes give one generation.

Setting. Ten-dimensional Weyl fermions on M^4 x CP^2 x CP^1, each with definite charges under some U(1) factors with fluxes on B_6. Since CP^2 is not spin, a fermion is a spin^c spinor, Λ^{0,*} ⊗ K^{1/2} ⊗ N with N the flux line bundle of its charges, and the four-dimensional chiral content is the virtual representation H^*(O(alpha, beta)) with (alpha, beta) = (-3/2, -1) + pairing of the charges with the fluxes. The cohomology is H^0(O(alpha)) = Sym^alpha(3bar) for alpha >= 0, H^2(O(alpha)) = Sym^{-alpha-3}(3) for alpha <= -3, and zero otherwise on CP^2; on CP^1, H^0(O(beta)) = Sym^beta(2) for beta >= 0 and H^1(O(beta)) = Sym^{-beta-2}(2) for beta <= -2. Each Standard Model Weyl field must come from one ten-dimensional field, from the whole cohomology of one line bundle.

Lemma (required bidegrees). Writing P_s = (alpha + 3/2, beta + 1) for the pairing of the charges of field s (up to the common sign of charge and chirality):

field	P_s	note
Q (3, 2)	(-5/2, 2)	H^2(O(-4)) x H^0(O(1))
u^c, d^c (3bar, 1)	(5/2, 1)	H^0(O(1)) x H^0(O(0))
L (1, 2)	(+-3/2, 2)	
e^c, nu^c (1, 1)	(+-3/2, 1)	

The second entry is 2j + 1 and the first is +-5/2 on colour triplets and +-3/2 on singlets. (Python, section D, by enumeration of the line bundles with these cohomologies and the Riemann-Roch check.)

Proposition 2.5 (no generation from fluxes along hypercharge and B - L) *Scope note (4 Nov 2026, after the referee).* The statement on equal images and the linearity of 2j + 1 assume a common ten-dimensional chirality for the fields. The conclusion holds without it: in the span of 6Y and 3(B - L) the

charges of L are -3 times those of Q, so their second pairings are P_2(L) = -3 P_2(Q), against |P_2| = 2 on both doublets.

Referee note on the third-charge search (GPT-6.1 Terra, 4 Nov 2026). The bounded search below has an independent analytic proof. The same argument extends to any real third charge and independent source chiralities; this stronger extension was checked by Claude Sonnet 5.5 (second check: the vectors y, b, p, w have rank 4 for every sign pattern), so it stands as a no-go in the same scope with two model identifiers. See the referee report.

Statement. In the setting above, with the canonical spin^c root and fluxes only in U(1) directions spanned by Y and B - L (any flux quanta), no choice reproduces one generation, with or without nu^c. More generally, the pairing must give u^c and d^c the same image, and 2j + 1 = (2, 1, 1, 2, 1, 1) on (Q, u^c, d^c, L, e^c, nu^c) must be a linear function of the fluxed charges.

Proof. The pairing M is linear in the charges, and M c_s = P_s with c_s the charges of the left-handed field s. With only Y and B - L fluxed, c_u - c_d = (-6, 0) in the coordinates (6Y, 3(B - L)), so M vanishes on the hypercharge direction and P_s = 3(B - L)_s m for one vector m. The second entries 2 on Q and 1 on u^c would then need 3(B - L) to take values in the ratio 2 : 1 on (Q, u^c), against 1 : -1. In general, 2j + 1 is not in the span of 6Y = (1, -4, 2, -3, 6, 0) and 3(B - L) = (1, -1, -1, -3, 3, 3): from Q and u^c the coefficients are -1 and 3, and then d^c gives -5 against 1. ■ An exhaustive search adding a third integer charge z in [-3, 3]^6 to Y and B - L finds no solution either (Python, section D).

Scope and escapes. Abelian fluxes on CP^2 x CP^1 with the Fubini-Study complex structure; one ten-dimensional Weyl field per Standard Model field; canonical spin^c root; Dolbeault index as the chiral content. Known escapes: nonabelian bundles (an SU(2)_R doublet bundle), several ten-dimensional fields per four-dimensional field, a changed common twist, orbifolds and walls, and non-Kaluza-Klein (brane) states. The result points to the next proposition.

Proposition 2.6 (one generation from a universal charge, the grading and B - L) Referee note (GPT-6.1 Terra, 4 Nov 2026). The index calculation works with the stated cohomology convention. For the colour action of the earlier quotient, reverse the first-component coefficients to obtain the labelled colour representations; global flux quantization remains to be specified. See the referee report.

Statement. Let the fluxed charges be F (one on every field: a change of the common spin^c twist), chi (+1 on Q and L, -1 on u^c, d^c, e^c, nu^c) and B - L. Then one generation with nu^c is the exact index content for the pairing

```
P^{(1)} = c_0 chi + c_1 (B - L),   P^{(2)} = (3F + chi)/2,
```

with (c_0, c_1) = (-9/4, -3/4) or (-3/2, -3). The bidegrees are Q: (-4, 1); u^c, d^c: (1, 0); L: (-3, 1) or (0, 1); e^c, nu^c: (0, 0) or (-3, 0). Without chi, or without B - L, there is no solution.

Proof. Substitution, and the two failures by the span test (Python, section D2). ■

Reading.

1. u^c and d^c see the same line bundle O(1, 0), and so do e^c and nu^c. The flux treats them as SU(2)_R doublets with a common internal profile; hypercharge Y = T_3R + (B - L)/2 enters through T_3R, which carries no flux. This is the left-right pattern of Pati-Salam, produced here by the index.
2. chi is the finite grading gamma_F on the particle basis. The flux therefore couples to the grading, an axial abelian connection on B_6; in the language of the finite geometry this is a fluctuation proportional to gamma_F, the structure of the two gradings of lemma 23.
3. **Mass and charge bases.** Because u^c and d^c share one zero-mode profile, overlap integrals on B_6 cannot separate them; their masses come from data that break SU(2)_R (Higgs or wall couplings), and the basis that diagonalizes the masses is set by those data, independently of the basis in which the charges are diagonal. The steer of 3 Nov 2026 remarks that u and d should still mix, since mass eigenvalues are not charge eigenvalues, and that at the 9 label, with the left-handed interactions off, charge eigenvalues can coincide. Proposition 2.6 puts the coincidence in the flux charges: equal flux charges for u^c and d^c, unequal hypercharges through T_3R. At the 9 label the electroweak part of

the algebra is removed (lemma 21, STRATEGY T2), and what distinguishes u^c from d^c in the light theory is then the vectorlike charge alone. Making this precise needs the SU(2)_R-breaking wall data and the 9D reduction of this index; listed as item 4 of the README.

What this file establishes, and the direction

Referee note (GPT-6.1 Terra, 4 Nov 2026). The M^{111} identification is conditional on the recalled fibration and metric normalization. The flux examples establish a particular construction and scoped obstructions; a general necessity for the grading flux remains open. See the wording audit.

The Witten space with Killing spinors is the hypercharge quotient of the 5-bar (Theorem 2.1), the count 7 is fixed by SU(5) (Proposition 2.2), and the Z_6 of the Standard Model is the kernel of the isometry action (Proposition 2.4). The price is that hypercharge is no longer an isometry, and no tensor Kaluza-Klein mode on an invariant background carries triality, weak isospin 1/2 or fractional hypercharge. Kaluza-Klein chirality on the base needs fluxes that see the grading and leave T_3R free (Propositions 2.5 and 2.6). Both results point to states outside the Kaluza-Klein spectrum of Y_{3,2} for the quarks and leptons: 03 takes them to be wrapped branes.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 2.5 (the exclusion). *From the referee report on the 11D line, section 'Proposition 2.5: hypercharge and B - L fluxes', transferred verbatim in substance on 4 Nov 2026.*

Hypotheses: one abelian line bundle per field, the canonical formal shift, fluxes in the span of Y and B - L. The exclusion holds even if the ten-dimensional Weyl fields have different chiralities.

Let M_2 be the component of the flux pairing M along CP^1, and use the charges c = (6Y, 3(B - L)). A weak doublet occurs for a line bundle O(beta) on CP^1 only when beta = 1 or beta = -3, which give H^0(O(1)) = 2 and H^1(O(-3)) = 2 respectively. The formal K^{1/2} shift on CP^1 is -1, so M_2(c) = beta + 1 and |M_2(c)| = 2 for each doublet. This remains true if a right-handed mode is conjugated to the left-handed convention, since conjugation reverses the sign of the pairing.

For the left-handed doublets,

```
c_Q = (1, 1),   c_L = (-3, -3) = -3 c_Q.
```

Linearity gives M_2(c_L) = -3 M_2(c_Q), so |M_2(c_L)| = 3 |M_2(c_Q)|, whereas both absolute values must be 2. This contradiction excludes a generation with or without nu^c, for arbitrary flux quanta in the two specified directions. It uses neither the colour component nor a choice of signs for the individual doublets.

Amendment to the required-bidegrees lemma. On CP^2 x CP^1, K^{1/2} is a formal notation: K = O(-3, -2) has no line-bundle square root. A genuine spin^c structure and its charged twists must give integral final bidegrees. With a common ten-dimensional chirality, Serre duality relates a right-handed conjugate to the sign-reversed pairing, and the convention P^{(2)} = 2j + 1 of the table is consistent. If the ten-dimensional chiralities can be chosen independently, the table is incomplete. For example, O(-4, -3) has H^3 = H^2(O(-4)) ⊗ H^1(O(-3)) = (3, 2) and pairing (-5/2, -2); a suitable ten-dimensional chirality makes this a left-handed Q, and (-5/2, -2) is neither the table entry (-5/2, 2) nor its overall negative.

Likewise, with independent chiralities and independent flux directions, u^c can arise from H^0(O(1, 0)) = 3bar, while d^c arises by conjugating a right-handed H^2(O(-4, 0)) = 3 with opposite abelian charges. Their pairings in the left-handed charge convention are (5/2, 1) and (5/2, -1). The equality of their images, and the unsigned linear formula P^{(2)} = 2j + 1, are therefore conditional on the common-chirality convention; the direct doublet obstruction above uses neither.

Scope and direction. This concerns the Dolbeault cohomology of a single abelian line bundle for each field on CP^2 x CP^1, with the canonical formal shift and fluxes in the span of Y and B - L. Nonabelian

bundles, additional charges or a common twist, multiple source fields, walls, orbifolds and brane states remain possible directions. This is an independent check of the scoped obstruction and makes no claim about those alternatives.

Independent proof of the third-charge exclusion in Proposition 2.5. *From the referee report on the 11D line, section 'Proposition 2.5: adding a third charge', transferred verbatim in substance on 4 Nov 2026.*

Status as the referee stated it: the argument proves the original bounded search, a third integer charge in [-3, 3]^6 under the common-chirality convention. The same argument covers an arbitrary real third charge and independently chosen ten-dimensional chiralities; the referee stated that extension as a conjecture awaiting an independent check by another model, and the referee note at Proposition 2.5 records that check by Claude Sonnet 5.5.

Write, on (Q, u^c, d^c, L, e^c, nu^c),

```
y = 6Y = (1, -4, 2, -3, 6, 0),    b = 3(B - L) = (1, -1, -1, -3, 3, 3).
```

(Here b is a charge vector, unrelated to the Euler-class coefficient b of Proposition 2.3.) For the stated colour convention the necessary CP^2 pairings are

```
p = (-5/2, 5/2, 5/2, p_L, p_e, p_nu),   |p_L| = 3/2,
```

and the CP^1 pairings have the form

```
w = (2 s_Q, s_u, s_d, 2 s_L, s_e, s_nu),   s_i in {+1, -1}.
```

The signs allow independently chosen source chiralities. Reversing the colour convention reverses p and leaves the argument unchanged.

Suppose that p and w both lie in the span of y, b and z, for some third charge z. The Q and L entries of y and of b have ratio -3, so every vector of span(y, b) has L entry equal to -3 times its Q entry, while |w_L/w_Q| = 1; hence w lies outside span(y, b). Then span(y, b, w) has dimension three and lies in span(y, b, z), so the two spans coincide, and

```
p = a y + b_0 b + t w
```

for real a, b_0, t. Apply the combination 2Q + u^c + d^c, the linear form v -> 2 v_Q + v_u + v_d. It vanishes on y (2 - 4 + 2 = 0), on b (2 - 1 - 1 = 0) and on p (-5 + 5/2 + 5/2 = 0), and takes the value 4 s_Q + s_u + s_d on w. So

```
0 = t (4 s_Q + s_u + s_d).
```

The factor in parentheses is nonzero, since |4 s_Q| = 4 > 2 >= |s_u + s_d|; hence t = 0. Equality of the u^c and d^c entries of p = a y + b_0 b, -4a - b_0 = 2a - b_0, gives a = 0, and their common value -b_0 = 5/2 gives b_0 = -5/2. The resulting L entry is -3 b_0 = 15/2, contradicting |p_L| = 3/2. The same four fields suffice without nu^c.

Scope and direction. This concerns the one-line-bundle-per-field Dolbeault construction, the specified Y and B - L charges and the canonical formal shift. A fourth charge direction, a changed common twist, nonabelian bundles, multiple source fields, walls, orbifolds and brane states remain directions to investigate.

Independent proof of Proposition 2.3. *From the referee report on the 11D line, section 'Proposition 2.3: topology of the circle bundle', transferred verbatim in substance on 4 Nov 2026.*

The calculation is independent of the specific M^{111} identification. The cited Kreck-Stolz paper was unavailable to the referee, so no direct comparison with that paper is claimed.

Let B = CP^2 x CP^1, H^*(B; Z) = Z[h_1, h_2]/(h_1^3, h_2^2), and let the Euler class be e = a h_1 + b h_2, with a != 0. The Gysin sequence gives

```
H^2(Y) = Z^2/<(a, b)> ~= Z ⊕ Z/g,   g = gcd(a, b).
```

The map H^2(B) -> H^4(B), cup product with e, sends h_1 to a h_1^2 + b h_1 h_2 and h_2 to a h_1 h_2 (since h_2^2 = 0); in the bases (h_1, h_2) and (h_1^2, h_1 h_2) its matrix is

```
[[a, 0], [b, a]].
```

Its determinant is a^2 != 0, so its kernel is zero, and since H^3(B) = 0 the Gysin sequence gives H^3(Y) = 0. The greatest common divisor of its entries is g, so its Smith invariants are g and a^2/g, and H^4(Y), its cokernel, is Z/g ⊕ Z/(a^2/g): a group of order a^2, cyclic when g = 1.

Spin. The vertical tangent line is trivial, spanned by the generator of the circle action, so w_2(Y) = pi^* w_2(B) = pi^*(h_1 mod 2), using w_2(CP^2) = h_1 mod 2 and w_2(CP^1) = 0. The mod-two Gysin sequence H^0(B; Z/2) -> H^2(B; Z/2) -> H^2(Y; Z/2), whose first map is the cup product with e mod 2, shows that the kernel of the pullback on H^2 is the span of e mod 2. Thus Y is spin exactly when h_1 ≡ e mod 2, that is, when a is odd and b is even. For (a, b) = (3, 2), the primitive homomorphism (x, y) -> 2x - 3y is onto Z and has kernel generated by (3, 2), so it identifies H^2(Y) with Z; and H^4(Y) ≅ Z/9.

Independent proofs for Theorem 2.1 and Propositions 2.2 and 2.4: freeness, the neutral-monomial congruence, the dual action and the Sasaki-Einstein normalization. *From the referee report on the 11D line, section 'Theorem 2.1 and Propositions 2.2 and 2.4: quotient and tensor action', transferred verbatim in substance on 4 Nov 2026.*

Hypotheses: the identification with M^{111} is conditional on the recalled description of M^{111} as the circle bundle of class 3h_1 + 2h_2; the cited paper could not be accessed by the referee.

Freeness and the bundle. On the moment level in the unit sphere, |U|^2 = 3/5 and |V|^2 = 2/5. Both factors are nonzero, so the stabilizer of a point under (lambda^2 U, lambda^{-3} V) has lambda^2 = lambda^3 = 1, hence lambda = 1, and the action is free. The quotient has dimension 9 - 1 - 1 = 7 and is simply connected by the homotopy sequence of U(1) -> S^5 x S^3 -> Y_{3,2}. The residual character of the two Hopf phases has primitive weight (3, 2); its circle bundle over CP^2 x CP^1 therefore has Euler class ±(3h_1 + 2h_2). Principal circle bundles over this base are classified by their Euler classes, which gives the conditional identification with M^{111}. The zero sum 3(2) + 2(-3) = 0 gives the Gorenstein holomorphic cone with trivial canonical bundle; a Ricci-flat metric needs a separate argument.

The congruence of Proposition 2.4. The congruence 2t + 3d + 6Y ≡ 0 (mod 6) holds on U, where (t, d, 6Y) = (-1, 0, 2), and on V, where (t, d, 6Y) = (0, 1, -3), and extends additively to tensor products and to conjugates, which reverse all three quantities. For a monomial in U, V and their conjugates, let Delta_U be the number of factors U minus the number of factors Ubar, and Delta_V likewise for V. Neutrality under U(1)_Y reads

```
2 Delta_U - 3 Delta_V = 0,
```

so 3 | Delta_U and 2 | Delta_V. The triality of the monomial is -Delta_U ≡ 0 (mod 3) and d ≡ Delta_V ≡ 0 (mod 2): zero triality and integer weak isospin. This extends item (b) of Proposition 2.4 from the holomorphic monomials U^alpha V^beta to all monomials in U, V and their conjugates.

Dual action. If U is the vector representation 3bar and projective space parametrizes its lines, the conventional pullback action on holomorphic coordinate functions is the dual 3: functions of degree 3k transform as Sym^{3k}(3). Treating U as formal field components gives Sym^{3k}(3bar) instead. The convention must be fixed consistently when identifying the 10 versus the 10bar and the later cohomology table.

Sasaki-Einstein normalization. With the standard unit-sphere quotient metric, the two transverse Fubini-Study factors have relative scales 3/5 and 2/5. CP^2 is the base of the Hopf circle on the sphere of radius sqrt(3/5), with Ricci constant 6/(3/5) = 10, and CP^1 is the base of the Hopf circle on the sphere of radius sqrt(2/5), a round sphere of radius sqrt(2/5)/2 with Ricci constant 4/(2/5) = 10; so each has Ricci constant 10. A seven-dimensional Sasaki-Einstein metric needs transverse Ricci constant 8. The Sasaki metric obtained by the standard transverse homothety of factor 5/4 has the same underlying circle bundle and the required Einstein normalization. Thus the existence of the Sasaki-Einstein metric on that manifold is compatible with the quotient, and the initial round-sphere quotient metric is identified with it only after the rescaling.

Independent proof of Proposition 2.6, with the colour convention. *From the referee report on the 11D line, section 'Proposition 2.6: flux assignments for one generation', transferred verbatim in substance*

on 4 Nov 2026.

Verdict of the referee: correct with amendments, conditional on global flux quantization. The bidegrees and the two linear solutions are correct with the cohomology convention of the file. With the colour action on the quotient fixed as in Theorem 2.1, that convention is conjugated and the first-component coefficients must be reversed.

The pairings. The charges of the left-handed fields are F = 1 on all; chi = +1 on Q, L and -1 on u^c, d^c, e^c, nu^c; and B - L = 1/3, -1/3, -1/3, -1, 1, 1 on Q, u^c, d^c, L, e^c, nu^c. For (c_0, c_1) = (-9/4, -3/4), direct substitution in P^{(1)} = c_0 chi + c_1 (B - L) and P^{(2)} = (3F + chi)/2 gives

```
P_Q = (-5/2, 2),   P_{u^c, d^c} = (5/2, 1),   P_L = (-3/2, 2),   P_{e^c, nu^c} = (3/2, 1);
```

for instance P_Q^{(1)} = -9/4 - 1/4 and P_L^{(1)} = -9/4 + 3/4. For (c_0, c_1) = (-3/2, -3), the quark pairings are unchanged (-3/2 - 1 = -5/2 and 3/2 + 1 = 5/2), and the first components for L and for e^c, nu^c are +3/2 and -3/2 respectively. Subtracting the formal shift (3/2, 1) yields exactly the listed integral bidegrees. On CP^2, the nonzero cohomology of a line bundle O(k) is H^0(O(k)) for k >= 0, or H^2(O(k)) ≅ H^0(O(-k - 3))^* for k <= -3; CP^1 has the analogous H^0/H^1 formula. So O(-4, 1) gives (3, 2), O(1, 0) gives 3bar, O(-3, 1) and O(0, 1) give (1, 2), and O(0, 0) and O(-3, 0) give a singlet, in the convention of the Setting. All listed choices have even total Dolbeault degree (0 or 2), so the common-chirality convention gives the indicated chiral index.

The colour convention. If the base is P(3bar), as in the quotient of Theorem 2.1, then H^0(O(1)) = 3 and H^2(O(-4)) = 3bar, and the listed quark bundles carry the conjugates of the labelled colour representations. Replacing (c_0, c_1) by (9/4, 3/4) or (3/2, 3), with the second component unchanged, reverses the first pairings and exchanges O(1) with O(-4) for the quarks; this gives the labelled Q and u^c, d^c with the colour action of the quotient. Alternatively, the original table is valid for P(3), with the corresponding conjugate colour convention.

Without chi. The required weak-doublet pairings have absolute value 2. Write P^{(2)} = alpha F + beta (B - L), so P_Q^{(2)} = alpha + beta/3 and P_L^{(2)} = alpha - beta. If the two signs for Q and L agree, then beta = 0 and |alpha| = 2, and u^c gets |P^{(2)}| = 2, the wrong absolute value for a weak singlet. If they disagree, say alpha + beta/3 = 2 and alpha - beta = -2, then beta = 3, alpha = 1, and u^c gets P^{(2)} = 1 - 1 = 0.

Without B - L. Q and L have identical charges F = 1, chi = +1, so their pairings coincide, while they require colour pairings of absolute values 5/2 and 3/2 respectively.

These obstructions use only absolute values, so they also allow independent source chiralities.

Global data. The final line-bundle degrees are integral, but K^{1/2} itself is formal and the displayed coefficients include fractional flux components. An actual background requires a specified global gauge group, charge lattice, spin^c determinant and compatible flux quantization. The index calculation alone establishes neither those data nor the physical interpretation of the grading as a gauge connection.

Part 40

Dualities between nine and eleven dimensions, adapted to the spectral triple

Lemma 40. Written as file 07 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, task C). Attempted by Claude (Anthropic), 4 Nov 2026. Propositions 7.1 to 7.3 are proved here by hand; the string-theory statements of section 1 are recalled without rereading; the readings are proposals. Refereed by GPT-6.1 Terra and checked a second

time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

1. The dualities (recalled)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The duality descriptions are conditional on the recalled sources. The M2 wrapping number and the auxiliary torus flux are separately specified; a map identifying them remains open. See the referee report.

1. **M-theory on T^2 and type IIB on S^1** (Schwarz, hep-th/9508143; Aspinwall, hep-th/9508154). The modulus tau of the torus is the IIB coupling and SL(2, Z) is IIB S-duality; the IIB radius scales as a negative power of the area A of the torus. M2-branes wrapped on T^2 are IIB momentum states, and momenta on T^2 are (p, q)-strings wound on S^1. Eleven dimensions at A -> infinity, ten (IIB) at A -> 0, nine in between.
2. **Sixteen supercharges, rank 17** (Polchinski and Witten, hep-th/9510169; Hořava and Witten, hep-th/9510209). The two heterotic strings on S^1 are T-dual through Wilson lines; their type I' form is an interval with two orientifold planes O8 and sixteen D8-branes whose positions are the Wilson-line eigenvalues; at strong coupling, M-theory on an interval times a circle, with E8 walls.
3. **Sixteen supercharges, lower rank.** The CHL string (rank 9) and M-theory on the Möbius strip; a rank-1 component related to the Klein bottle (Aharony, Komargodski and Patir, hep-th/0702195); further components with discrete theta angles (Montero and Parra de Freitas, arXiv:2209.03361).
4. **Light-cone M-theory.** On a light-like circle with momentum N, M-theory is the U(N) quantum mechanics of nine Hermitian matrices (Banks, Fischler, Shenker and Susskind, hep-th/9610043; Susskind). On T^2 with a light-like component of C_3, the torus becomes noncommutative and SL(2, Z) acts by Morita equivalence (Connes, Douglas and Schwarz, hep-th/9711162).

2. What each one offers the finite geometry

duality	9	11	the finite triple
1	the nine-dimensional theory	large torus	the lowest Landau level of a torus with flux p: algebra M_p(C), J from charge conjugation with doubling (Proposition 7.3); towers = Landau levels; p = wrapped M2-branes = IIB momentum
2	type I' interval, or heterotic on S^1	Hořava-Witten on S^1	two sheets = two orientifold walls; D_F = Wilson lines = D8 positions (Propositions 6.1, 7.1, 7.2); A_F from Chan-Paton algebras; J from the orientifold
3	rank-reduced nine-dimensional theories	M-theory on non-orientable surfaces	J as an orientation reversal; the reflection with antiperiodicity of Proposition 1.1 is the Möbius-type ingredient

duality	9	11	the finite triple
4	nine transverse matrices, Spin(9)	large N	A_F as the commutant of block-diagonal matrices: stacks of 1, 2, 3 give C + M_2(C) + M_3(C); H_F in off-diagonal blocks; D_F = Γ^i ad(X^i), masses = separations

Duality 2 is the one with an exact meeting point with the corpus, through the angle of the top channel.

3. Proposition 7.1 (the angle of a channel lives on an interval)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The spectral quotient and endpoint multiplicities are independently checked. Theorem 22.1 uses the strict interior; the antiperiodic endpoint is a limit, with a degenerate window. Equality of one channel spectrum does not by itself identify full physical backgrounds. See the referee report.

Statement. In the parent of lemma 22, a channel with singular value σ of the Yukawa block and angle θ = l σ has the signed spectrum {(n pi + θ)/l, (n pi − θ)/l : n in Z} (Theorem 22.1). As a set, this depends only on the class of θ in R modulo the group generated by θ -> −θ and θ -> θ + pi, whose fundamental domain is [0, pi/2]. At the endpoints:

1. θ = 0: the spectrum is n pi/l with a zero mode of multiplicity two (periodic);
2. θ = pi/2: the spectrum is (n + 1/2) pi/l, each value twice (antiperiodic), two copies of the spectrum of the chiral interval of Proposition 1.1; it is the limit X -> infinity of the wall data X = tan(θ), where the lowest eigenspace is degenerate, and in lemma 23 the point of antipodal gradings, where M_W = M_Z = 0 at fixed l with the fermion at pi/(2l).

Proof. The set is invariant under θ -> −θ (it exchanges the two progressions) and θ -> θ + pi (it shifts n). The orbit of θ meets [0, pi/2] in one point. The endpoint values are substitutions; for θ -> pi/2 the wall conditions Q_R(l) = X U_R(l), U_L(l) = −X^* Q_L(l) tend to U_R(l) = Q_L(l) = 0. The statements at θ = pi/2 are those of lemma 23 (Theorem 23.6 and the reading after Proposition 23.7). ■

So the parameter space of one channel of the finite Dirac operator, for a given parent, is an interval with two distinguished ends, and lemma 22's restriction l ||D_F|| < pi/2 is the choice of the fundamental domain.

4. Proposition 7.2 (the far end is the displaced D8-brane on the other wall)

Referee note (GPT-6.1 Terra, 4 Nov 2026). At $a = 1/2$ the full $SO(32)$ centralizer is $S(O(30) \times O(2))$, whose identity component is $SO(30) \times SO(2)$. The stated one-loop minimum uses the hypotheses of lemma 24 and $n_f > 0$; the D8 dictionary remains a proposal. (Second check.) In Spin(32)/Z_2 the component of determinant -1 sends the lift of W_{1/2} to its negative, so the heterotic centralizer is connected; a -> a + 1 holds for SO(32) matrices, and the lift has period 2. See the referee report.

Statement. Put a = θ/pi in [0, 1/2]. By Proposition 6.1 the channel spectrum is that of the states (30)_{±1} of SO(32) on a circle of radius R = l/pi with the Wilson line W_a = 1_30 ⊕ R(2 pi a). The interval [0, 1/2] is the moduli interval of one Wilson-line eigenvalue modulo a -> −a and a -> a + 1; the unbroken group is SO(32) at a = 0 and SO(30) x SO(2) for 0 < a <= 1/2; at a = 1/2 the holonomy is −1 on the plane of SO(2), W_{1/2} = diag(1_30, −1_2), and the (30)_{±1} tower is (n + 1/2)/R, the antiperiodic tower of Proposition 1.1. The one-loop minimum of lemma 24, Theorem 9, θ = pi/2, is the point a = 1/2.

Proof. Proposition 6.1 and Proposition 7.1 with θ = pi a; the commutant of W_a in SO(32) for 0 < a < 1/2 is SO(30) x SO(2), and at a = 1/2 its identity component is still SO(30) x SO(2), since only two eigenvalues

are −1. Lemma 24, Theorem 9, item 1, places the minimum at t = sin^2 θ = 1. ■

Reading (proposal). In type I' the Wilson line W_a is fifteen D8-branes, with images, on one orientifold plane and the sixteenth at distance proportional to a from it (proportional only at constant dilaton; see 10, Proposition 10.1); a = 1/2 puts it on the other orientifold plane (recalled: hep-th/9711098, hep-th/0108135). The dictionary of file 06 then reads: the top angle is the position of the top's D8-brane on the type I' interval, the two ends of the fundamental domain are the two orientifold walls, and the one-loop vacuum of the flat family of lemmas 23 and 24 carries the top brane to the far wall, where the tower of the stack-brane strings is antiperiodic. In supersymmetric type I' the D8 positions are flat directions; the potential of lemma 24 belongs to the non-supersymmetric flat family, the setting in which a force on the brane is expected.

Under the strong-coupling duality of item 2 of section 1, the D8 position is a Wilson-line eigenvalue of the heterotic string on S^1, and in eleven dimensions an E8 holonomy around the circle on a Hořava-Witten wall. So the same datum, the top angle, is a brane position at 9 and a holonomy at 11.

5. Proposition 7.3 (a torus with flux, doubled, has the colour form)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The signs, kernel dimensions and generated matrix corner check out, with a spin-line twist for nontrivial spin structures. The displayed action is degenerate and nonunital on the doubled Hilbert space, so the claimed unital finite triple needs an extra scalar summand. See the referee report.

Statement. Let T^2 be a flat torus of area A with a spin structure, L a line bundle of degree p >= 1 with a connection of constant curvature, and Lbar = L^{-1} with the conjugate connection. On H = L^2(S ⊗ L) ⊕ L^2(S ⊗ Lbar) put D = D_L ⊕ D_Lbar, gamma = gamma_S ⊕ gamma_S, and J(ψ, χ) = (Cχ, −Cψ), where C is the charge conjugation of the spin structure (antiunitary, C^2 = −1, C D = D C, C gamma_S = −gamma_S C, mapping S ⊗ Lbar to S ⊗ L). Let C^infinity(T^2) act by multiplication on the first summand only. Then:

1. J^2 = 1, J D = D J and J gamma = −gamma J, the signs of KO-dimension six, and the algebra satisfies order zero;
2. ker D = H^0(L) ⊕ H^1(Lbar) ≅ C^p ⊕ conj(C^p), with gamma = +1 on the first summand and −1 on the second; the other eigenvalues are the Landau levels, of order (p/A)^{1/2}, which go to infinity as A -> 0 at fixed p;
3. the compressions of the functions to the kernel generate M_p(C) on C^p ⊕ 0, so the window is the finite triple (M_p(C), C^p ⊕ conj(C^p), 0, J, gamma), with KO-6 signs and order zero: M_p(C) acts on the first summand and, through J, by its opposite on the second.

Proof. (1) J^2(ψ, χ) = (C(−Cψ), −C(Cχ)) = (ψ, χ) since C^2 = −1; JD = DJ and J gamma = −gamma J follow from the properties of C; order zero holds because the algebra acts on the first summand and J b J^{-1} on the second. (2) On a torus the spinor bundle is Λ^{0,*} up to a flat factor, so ker D_L = H^0(L) ⊕ H^1(L) = C^p ⊕ 0 by Riemann-Roch in genus one, and ker D_Lbar = 0 ⊕ H^1(Lbar) ≅ conj(C^p) by Serre duality; the Landau levels of a constant field of total flux p on area A scale as (p/A)^{1/2}. (3) The compression of a character e_{m,n} to H^0(L) is a nonzero multiple of a clock-and-shift Weyl operator U^m V^n with UV = e^{2 pi i/p} VU (Berezin-Toeplitz quantization of the torus; recalled; in the theta-function basis the compression is a Gaussian factor times the Weyl word, checked for p = 2 to 5 in the second check), and these operators span M_p(C). ■

The construction is the torus analogue of lemma 16 (a doubled parent with restricted algebra and a J covering the flip; Example 16.9 for the twistor case), and its window at p = 3 has the form of the colour sector of the Standard Model triple: M_3(C) on the quarks and its opposite on the antiquarks. Berezin-Toeplitz quantization of the torus is known; the KO-6 doubling is the corpus device.

Reading for duality 1 (proposal). The area A -> 0 at fixed flux keeps the window and sends the tower to infinity, the towerless limit; A -> infinity brings the Landau levels down. In M-theory on T^2, flux and wrapping number of M2-branes on T^2 are IIB momentum states, so the dimension p of the window counts IIB momentum on the nine-dimensional circle, and the SL(2, Z) of the torus acts on the family by Morita

equivalence of the noncommutative tori of Connes, Douglas and Schwarz when a C_3 background is on. Whether p = 3 for colour has a reason in this picture is open.

6. A remark on duality 4 (known)

For U(N) matrices X^i (i = 1, ..., 9) that are block-diagonal, X^i = $\oplus a$ *x_a^i 1*{n_a} with distinct points x_a in R^9, the commutant of all X^i is $\oplus a$ *M*{n_a}(C), and the fermion operator Γ^i ad(X^i) on the adjoint, with the sixteen-component spinor of Spin(9), vanishes on the diagonal blocks and has eigenvalues ±|x_a − x_b|, each with multiplicity 8 n_a n_b per ordered block (a, b), that is 16 n_a n_b per unordered pair. With stacks of sizes 1, 2, 3 the commutant is C + M_2(C) + M_3(C), the complex form of A_F, the off-diagonal blocks are bifundamentals as in the Krajewski diagrams of the finite geometry, and the masses are the distances between stacks; coincident stacks restore U(N), the analogue of massless W and Z, and large N is the eleven-dimensional limit. A quaternionic factor H needs an orientifold projection on the two-stack, and chirality needs more than the nine Hermitian matrices.

6b. The three labels in type I' variables (proposal)

Correction (4 Nov 2026). The length l of H and the type I' length are T-dual, and the limit at label 11 below ends at ten-dimensional SO(32). See 10, Proposition 10.3.

Hypothesis H has two parameters for the top channel: the interval length l and the angle θ = l m_t in [0, pi/2] (Proposition 7.1). In the type I' reading of Proposition 7.2 they are the length of the interval and the position of the top's D8-brane as a fraction of it. Then:

- **label 9:** θ = pi/2 at fixed l, the brane on the far orientifold wall: the top at the tower scale pi/(2l) (lemma 22, Corollary 22.9: W, Z and top at the tower scale in the 10 -> 9 family), the antiperiodic tower, and the one-loop minimum of the flat family (lemma 24);
- **label 10:** 0 < θ < pi/2, the brane between the walls, a window below the tower;
- **label 11:** θ -> 0 together with l -> infinity, the joint family of 01 (θ_epsilon ~ epsilon^{1 − alpha}): the brane returns to the stack, SO(32) is restored on that channel, and the interval opens.

In this reading the 9 and 11 endpoints are the two ends of the brane's path, the far wall and the stack, and the second scale required by Proposition 1.5 must fix how l grows as the brane returns. The flat one-loop potential pushes the brane to the far wall (label 9); reaching label 11 needs a term that pulls it back to the stack while the interval grows.

7. What this adds, and what remains open

Referee note (GPT-6.1 Terra, 4 Nov 2026). The brane and E_8-holonomy readings below are proposals. Proposition 7.3's displayed algebra action is nonunital; the corrected finite datum is in the referee report. Proposition 1.5 also permits flat directions.

Added: the angle of the top channel lives on an interval whose ends are the two orientifold walls of type I' (7.1, 7.2); the corpus's one-loop vacuum puts the top brane on the far wall, the SO(32) Wilson line diag(1_30, −1_2); the same datum is an E8 holonomy in the eleven-dimensional frame; a torus with flux gives the colour form of the finite triple with KO-6 signs (7.3).

Open: - A nine-dimensional spectral triple with sixteen D8 positions as the eigenvalues of D_F on the type I' interval, and its strong-coupling limit with E8 walls; the configuration 15 + 1 lies at the SO(32) end, while E8 needs seven D8-branes at an orientifold plane with divergent coupling. - The quaternionic factor H from a torus with even flux and a real structure of quaternionic type, and the full A_F from several tori or from a rank-three bundle with flux. - The lower-rank nine-dimensional components (duality 3), whose J-like orientation reversal may supply the real structure of the finite triple.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 7.1. *From the referee report on the 11D line, section 'Proposition 7.1: channel-angle fundamental domain', transferred verbatim in substance on 4 Nov 2026.*

The stated fundamental domain classifies the signed channel spectrum, including its endpoint multiplicities. The spectrum modulo pi/l consists of the two residues ±θ/l. Equality of two such multisets is equivalent to θ' ≡ ±θ (mod pi): if θ' = ±θ + k pi the progressions are relabelled, and conversely the residue sets {±θ'} and {±θ} modulo pi must agree. This gives the unique representative in [0, pi/2]. In the interior the two progressions are disjoint, since n pi + θ = m pi - θ would need 2θ ∈ pi Z. At zero they coincide on pi Z/l; at pi/2 they coincide on (Z + 1/2) pi/l. Thus every endpoint eigenvalue has multiplicity two, including the zero at the first endpoint.

The channel has two copies of the scalar antiperiodic circle spectrum at the second endpoint. Theorem 22.1 assumes the strict window l ||D_F|| < pi/2; θ = pi/2 is the corresponding boundary-condition limit, and its lowest level is degenerate. The massless electroweak vectors at that endpoint come from the specified two-grading model of lemma 23 and do not follow from the spectrum of an isolated channel. The endpoints require separate domains or limits, and the quotient of one spectrum need not identify complete physical backgrounds.

Independent proof of Proposition 7.2, with the full centralizer at a = 1/2. *From the referee report on the 11D line, section 'Proposition 7.2: Wilson-line endpoint and one-loop minimum', transferred verbatim in substance on 4 Nov 2026.*

For 0 < a < 1/2, the nonreal conjugate eigenvalues e^{±2 pi i a} of the rotation force a commuting orthogonal transformation to preserve the two-plane of R(2 pi a) and its complement, and to act on the plane by an element of O(2) commuting with a rotation by an angle other than 0 and pi, that is, by a rotation. Its centralizer in SO(32) is therefore SO(30) x SO(2). At a = 1/2 the holonomy is diag(1_30, -1_2); a commuting orthogonal transformation preserves both eigenspaces and is otherwise arbitrary on them, so the full centralizer is

```
S(O(30) x O(2)) = {(A, B) ∈ O(30) x O(2) : det A det B = 1},
```

with identity component SO(30) x SO(2) and a second component of pairs with det A = det B = -1. At a = 0 the centralizer is SO(32). Conjugation by a reflection in each subspace, an element of SO(32), sends a to -a, and a -> a + 1 leaves this SO(32) holonomy unchanged.

Minimum. For the determinant of lemma 24, write t = sin^2 θ. Its two gauge terms are nonnegative and vanish at t = 1; the fermion term -n_f F(t) is strictly decreasing for n_f > 0. Thus the unique global minimum is t = 1, that is a = 1/2. This conclusion assumes the flat massless determinant, its stated field content and wall conditions, and no additional angle-dependent term. It does not derive a force on a supersymmetric type I' brane, and the D8 interpretation is conditional on the proposed dictionary.

Independent proof of Proposition 7.3, items 1 and 2, and of the generation step in item 3. *From the referee report on the 11D line, section 'Proposition 7.3: doubled torus and the finite algebra', transferred verbatim in substance on 4 Nov 2026.*

Zero modes. Let S_0 be the flat spin line of the chosen spin structure, with S_0^2 ≅ O, so that S ⊗ L = Λ^{0,*}(L ⊗ S_0). The two zero-mode spaces are

```
H^0(L ⊗ S_0)    and    H^1(L^{-1} ⊗ S_0).
```

The first has dimension p: a negative-degree line bundle has no holomorphic sections, so by Serre duality (K trivial on the torus) H^1(L ⊗ S_0) = 0, and Riemann-Roch on a genus-one curve gives χ(L ⊗ S_0) = p. For the same reason H^0(L^{-1} ⊗ S_0) = 0, and Serre duality identifies H^1(L^{-1} ⊗ S_0) with the dual of H^0(L ⊗ S_0^{-1}) = H^0(L ⊗ S_0), hence with its complex conjugate after choosing the inner product. With the stated orientation these have opposite gradings. The untwisted formula of item 2, H^0(L) ⊕ H^1(Lbar), is literal for the trivial spin structure; the dimensions hold for all four.

J and the levels. The assumed charge conjugation C intertwines the two coupled Dirac operators and reverses the grading. Consequently the displayed J satisfies J^2 = 1, JD = DJ and J gamma = -gamma

J. In a constant field B = 2 pi p/A, the nonzero Dirac levels have magnitudes sqrt(2|B| n), n >= 1, in the standard Dirac normalization.

Compression. For a reciprocal lattice vector k, split the position operator as x = R + , with guiding-centre part R and oscillator part , which commute with each other and satisfy [_1, _2] = -[R_1, R_2], of modulus 1/|B|. Then e^{ik · x} = e^{ik · R} e^{ik · }, and the oscillator-vacuum matrix element of e^{ik · } is e^{-|k|^2/(4|B|)}. So projecting multiplication by e^{ik · x} onto the lowest Landau level gives

```
P e^{ik·x} P = e^{-|k|^2/(4|B|)} e^{ik·R} P,
```

the nonzero Gaussian factor times a guiding-centre translation. For primitive reciprocal vectors k_1, k_2, with |k_1 x k_2| = (2 pi)^2/A and 1/|B| = A/(2 pi p), the two translations U = e^{ik_1 · R} P and V = e^{ik_2 · R} P are unitary on the lowest Landau level and form a clock-and-shift pair, UV = e^{±2 pi i/p} VU. Their p^2 products U^m V^n, 0 <= m, n < p, span M_p(C): the commutation relation forces tr(U^m V^n) = 0 unless m ≡ n ≡ 0 (mod p), so the p^2 products are orthogonal for the trace inner product. Flat spin and connection holonomies change only phases, and leave the nonzero Gaussian factor. Compression generates this algebra, and the compression map itself is not multiplicative on the function algebra.

Counterexample (referee) to the unital finite triple of Proposition 7.3, item 3, and the unital repair. *Same section of the referee report, transferred verbatim in substance on 4 Nov 2026.*

On V ⊕ Vbar, with V = H^0(L ⊗ S_0) ≅ C^p, the generated representation is (a) = a ⊕ 0. Thus (1) = 1_V ⊕ 0, and both the parent and the finite representations are degenerate on the second summand. They do not have the unital representation required in the finite-data setting of lemma 22. In particular, for p = 3, no complex-linear unital M_3(C) representation on C^3 ⊕ conj(C^3) can satisfy order zero with a unital opposite action: the two commuting unital actions would make this six-dimensional space a module for M_3(C) ⊗ M_3(C)^op ≅ M_9(C), whose nonzero modules have dimensions divisible by nine.

One repair is to unitize the represented corner:

```
A = M_p(C) ⊕ C,     (a, ) = a ⊕  I_{Vbar}.
```

In conjugate bases J (b,)^* J^{-1} = I_V ⊕ b^T, which commutes with (a,). This yields a unital finite KO-6 datum with D = 0 and order zero. It has an additional scalar summand; deriving the full Standard Model algebra and its representation remains open.

Second-check derivation (Claude Sonnet 5.5, 4 November 2026)

Spin lift of W_{1/2} and connectedness of the heterotic centralizer (Proposition 7.2). *From the second check on the 11D line (historical/reviews/2026-11-04_sonnet_second_check_11d.md, batch 4, item B2), adopted in the note at Proposition 7.2 and written out on 4 Nov 2026.*

Let G_1, …, G_32 generate the Clifford algebra of R^32, with G_i G_j + G_j G_i = 2 _ij, and let pi: Spin(32) -> SO(32) be the covering map, under which g~ acts on vectors by v -> g~ v g~^{-1}. The rotation by angle 2 pi a in the plane of the coordinates 31 and 32 lifts to

```
W~_a = exp(pi a G_31 G_32) = cos(pi a) + sin(pi a) G_31 G_32,
```

using (G_31 G_32)^2 = -1. At a = 1/2 the lift of W_{1/2} = diag(1_30, -1_2) is W~ = G_31 G_32, up to the sign of the covering. The product of the reflections in e_1 and in e_31 lies in the component of S(O(30) x O(2)) with det A = det B = -1, and lifts to g~ = G_1 G_31. Since G_1 anticommutes with G_31 and with G_32, it commutes with their product, and

```
g~ W~ g~^{-1} = G_1 G_31 (G_31 G_32) G_31 G_1 = G_1 G_32 G_31 G_1 = G_32 G_31 = -W~.
```

Every other element of that component is the product of this one with an element of SO(30) x SO(2), whose lifts are products of exp(φ G_i G_j/2) with i, j <= 30 or {i, j} = {31, 32}, all commuting with G_31 G_32. So every lift of the determinant -1 component conjugates W~ to -W~. The computation involves only G_1, G_31 and G_32 and holds in every even dimension.

Let = G_1 G_2 … G_32 be the volume element. It is central in Spin(32), ^2 = 1, != ±1, and pi( ) = -1. The heterotic group is the quotient of Spin(32) by {1, } (or by {1, - } for the other chirality convention;

the argument is the same). The class of g~ centralizes the class of W~ exactly when g~ W~ g~^{-1} ∈ {W~, W~}. The case W~ would give pi(g~) W pi(g~)^{-1} = -W, which is impossible, since W and -W have the eigenvalue -1 with multiplicities 2 and 30. So pi(g~) lies in S(O(30) x O(2)). On the determinant -1 component g~ W~ g~^{-1} = -W~, and -1 differs from 1 and from . Hence the centralizer of the class of W~ in Spin(32)/Z_2 is the image of the preimage of SO(30) x SO(2) in Spin(32). That preimage is connected, since it contains the path exp(φ G_1 G_2/2), φ ∈ [0, 2 pi], from 1 to -1. So the heterotic centralizer is connected. Finally W~_{a+1} = -W~_a: the SO(32) holonomy has period 1 in a, and its lift, in Spin(32) and in Spin(32)/Z_2, has period 2.

Part 35

The sign of lemma 06 as an orientifold type, and E8 walls from the configuration 15 + 1

Lemma 35. Written as file 08 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, task C, next step of 07). Attempted by Claude (Anthropic), 4 Nov 2026. Proposition 8.1 is proved (it adds a commutant statement to lemma 06); Proposition 8.2 is proved from Proposition 7.1; Remark 8.3 rests on recalled string-theory rules and on the dictionary of 06; readings are proposals. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

1. Setting

In type I' the two ends of the interval are orientifold planes. A stack of n Chan-Paton labels (k D8-branes with their images, n = 2k) on an orientifold plane of type O^- carries SO(n), and on one of type O^+ carries USp(n) (recalled; the physical O8 planes of the SO(32) web are O8^-). Lemma 06 factors a KO-6 finite triple into an odd one through a grading-reversing unitary involution U, commuting with the algebra and with D, with U J = s J U; s = −1 gives KO-5 (label 9) and s = +1 gives KO-7 (label 11), and the antiunitary j induced on H_+ satisfies j^2 = s and commutes with d^2 = D^2|{H+} (lemma 06, proof of Result 2 and the check of the electrodynamics data). STRATEGY T2c: with a realistic seesaw sector only s = +1 occurs.

2. Proposition 8.1 (the factorization sign fixes orthogonal or symplectic symmetry)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The antiunitary commutants are independently checked. They need not preserve *d* or the remaining data, and the Chan-Paton identification is a proposal. See the referee report.

Statement. Let E be an eigenspace of d^2 in H_+, of dimension n. The unitaries of E that commute with j form O(n) if s = +1, and USp(n) if s = −1, in which case n is even.

Proof. j preserves E because it commutes with d^2. An antiunitary with j^2 = +1 is a real structure: E = E_R ⊗ C with E_R its fixed points, and a unitary commutes with j exactly when it preserves E_R, which gives O(n). An antiunitary with j^2 = −1 is a quaternionic structure: E is a right H-module of quaternionic dimension n/2, and the unitaries commuting with j form USp(n). ■

The even dimension for s = −1 is lemma 06's own statement (Kramers degeneracy of the singular values). The proposition adds the symmetry group.

Reading (proposal). The groups are those of Chan-Paton labels on an orientifold plane: s = +1 with O^- (SO), s = −1 with O^+ (USp). So the two odd factorizations of the modern triple, which carry the labels 11 and 9, have the form of the two kinds of orientifold planes. The seesaw sector selects s = +1 (T2c), which

is the O^- type of the SO(32) web used in 06 and 07: the physical neutrino sector and the type I' walls of the dictionary agree in type. Two refinements from the second check: the symmetries of the pair (j, d) are smaller, O(n_+) x O(n_-) for s = +1 and U(n/2) for s = -1, since for s = -1 the antiunitary j swaps the +-mu eigenspaces of d; and there is a tension of labels, since here label 9 goes with O^+ while section 6b of 07 puts label 9 on a far O8^- wall (the two are the KO sign and the position of the top's brane, see 10).

3. Proposition 8.2 (wall channels come with doubled multiplicity)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The stated count is for one signed eigenvalue. The lowest $|D|$ multiplicity is $2k$ in the interior and at zero, and $4k$ at $\pi/2$; interpreting the signed count as Chan-Paton labels requires further data. See the referee report.

Statement. In the parent of lemma 22, k channels at a common angle θ contribute, at the eigenvalue θ/l of the window, an eigenspace of dimension k if 0 < θ < pi/2 and of dimension 2k at the ends θ = 0 and θ = pi/2.

Proof. By Proposition 7.1 each channel has the spectrum {(n pi ± θ)/l}; the value θ/l occurs once (n = 0, sign +) for 0 < θ < pi/2, and twice at the ends: at θ = 0 as the zero mode of multiplicity two, at θ = pi/2 as (0 + pi/2)/l = (pi − pi/2)/l. ■

Reading (proposal). k D8-branes in the bulk carry k Chan-Paton labels and k D8-branes on an orientifold plane carry 2k, with their images; the parent reproduces this count, with the ends of the angle interval as the walls. With Proposition 8.1, a stack of channels at a wall has a symmetry of type SO(2k) or USp(2k), as a stack of D8-branes on an O8^- or O8^+ plane. The rank-reduced nine-dimensional theory with one O8^- and one O8^+ (duality 3 of file 07) would then be a parent whose two walls carry opposite signs s: one sheet of type 11 and the other of type 9.

4. Remark 8.3 (E8 walls from 15 + 1; conditional)

Correction (4 Nov 2026). The far-wall group is reached at the string-scale radius R = R_*, with the tower fixed, and only its charged states are D0-branes. See 10.

Referee note (GPT-6.1 Terra, 4 Nov 2026). The brane and slope arithmetic is independently checked conditional on the recalled rule. Wall enhancement requires tuning $z = 0$; for $7 + 7$ the two bulk positions must sum to L. The E_2 identification with the physical electroweak sector needs further data. The cited primary sources were unavailable to the referee; the slope rule and the E_{n+1} table were confirmed against Polchinski-Witten and Bergman-Gaberdiel-Lifschytz in the second check. See the referee report.

Conditions: the dictionary of 06 (fifteen D8-branes for the five light quarks in three colours, the sixteenth for the top) and the recalled rule that an O8^- plane with n <= 7 D8-branes on it, at infinite coupling, carries E_{n+1} (E_4 = SU(5), E_5 = SO(10), E_6, E_7, E_8), while n >= 8 admits no such point (Seiberg, hep-th/9608111; Bergman, Gaberdiel and Lifschytz, hep-th/9711098; recalled).

1. Dilaton bookkeeping: the dilaton is e^{phi} = z^{−5/6} with z piecewise linear along the interval (conformal coordinate; Polchinski and Witten, hep-th/9510169) and slope proportional to 8 − n_left, where n_left counts the D8-branes to the left (O8^- carrying −8 units), so the coupling grows toward the wall with fewer branes; this reproduces the constant dilaton of 8 + 8, the SU(2) at the empty wall of the SO(32) point (the self-dual radius of the heterotic string) and 7 + 7 for E8 x E8 (extrapolated from the single-wall rule; neither Polchinski-Witten nor Bergman-Gaberdiel-Lifschytz states it). In the configuration 15 + 1 the wall with fifteen branes has no E point (15 > 7), and the coupling can diverge only at the far wall. With the top's brane on the far wall (the label-9 configuration of Proposition 7.2, n = 1) the far wall enhances to E_2 = SU(2) x U(1), the electroweak group; with the top's brane in the bulk it is empty and enhances to E_1 = SU(2). The light-quark wall keeps SO(30).
2. If branes move in colour triples (a light flavour is three D8-branes), a wall holds 3j branes, or 3j + 1 with the top's brane. The reachable wall groups are E_1, E_2 (the top's brane alone), E_4 = SU(5) (one flavour), E_5 = SO(10) (one flavour and the top's brane), E_7 (two flavours) and E_8 (two flavours and the top's brane); E_6 (five branes) is not reachable.

3. Hence an E8 wall needs the top's brane together with two light flavours, and two E8 walls are impossible with one top brane: E8 x E8 requires breaking a colour triple.

In this reading the top is what completes E8 at a wall (7 = 2 x 3 + 1), and with colour triples the proposed E8 x E7 arrangement has seven branes at one wall, six at the other and a bulk triple at x_b, while two E8 walls require relaxing the colour-triple rule (referee replacement): with that arrangement, the slopes are proportional to +1 on (0, x_b) and −2 on (x_b, L), so z vanishes at both walls when x_b = 2L/3. The E_n series also contains E_3 = SU(3) x SU(2) (two branes at a wall), which colour triples do not reach.

Reading (proposal). At strong coupling the wall that holds the top's brane in the label-9 configuration carries SU(2) x U(1), and the wall of the five light quarks in three colours carries SO(30). The electroweak group then appears at the top's wall, where the M-theory circle opens locally, and the quarks' flavour group at the other wall.

5. What remains open

- The bulk symmetry: k D8-branes in the bulk carry U(k), while Proposition 8.1 gives O or USp on every eigenspace of d^2; the relation between the window's degenerate levels in the bulk and U(k) needs the parent's own structures in addition to j.
- The dilaton: a spectral-triple counterpart of the type I' dilaton profile (a coupling varying along the interval, with slope set by the brane charge to one side) and the infinite-coupling points of Remark 8.3. Proposition 9.1 of 09 writes the profile as a trace of a function of |D_F|, only implicitly after 10, Proposition 10.1, and the gradient is not the second scale in the scope of Propositions 10.5 and 10.6 there.
- The physical status of E8 x E7 with colour triples, and whether the top's brane at an E8 wall has a counterpart in the finite geometry.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 8.1, with its caution. *From the referee report on the 11D line, section 'Proposition 8.1: antiunitary commutants', transferred verbatim in substance on 4 Nov 2026.*

On each finite-dimensional eigenspace E of d^2, the commutant of j in the unitary group of E is O(n) for j^2 = 1, and USp(n) for j^2 = -1, with even complex dimension n in the latter case.

For j^2 = 1, the fixed points of j form a real subspace E_R with E = E_R ⊕ i E_R, and the inner product is real on E_R, since <jv, jw> = conj<v, w>. A real orthonormal basis of E_R is an orthonormal basis of E and identifies j with complex conjugation; the unitaries commuting with complex conjugation are the real orthogonal matrices, O(n). For j^2 = -1, a unit vector v and jv are orthogonal: antiunitarity gives <jv, j^2 v> = conj<v, jv> = <jv, v>, and j^2 v = -v gives <jv, j^2 v> = -<jv, v>, so <jv, v> = 0. The span of v and jv is j-invariant, and so is its orthogonal complement; induction gives a quaternionic orthonormal basis v_1, jv_1, ..., v_m, jv_m, with n = 2m. The unitaries commuting with j are then the compact symplectic group USp(n).

Caution. This commutant is defined using j and the d^2-eigenspace alone. It need not commute with d, with the represented algebra, or with other physical structures. For example, on C^2 let j be complex conjugation and d = diag(1, -1): the commutant on the d^2 = 1 space is O(2), and a generic rotation does not commute with d. Moreover O(n) contains disconnected transformations, while SO(n) is its identity component for n >= 2. The orientifold comparison therefore requires an explicit Chan-Paton representation and compatibility with the rest of the data.

Independent proof of Proposition 8.2, with the multiplicities of |D|. *From the referee report on the 11D line, section 'Proposition 8.2: endpoint channel multiplicities', transferred verbatim in substance on 4 Nov 2026.*

For k independent channels at the same angle, the signed eigenvalue theta/l has multiplicity k in the interior and 2k at either endpoint. For 0 < theta < pi/2, only n = 0 in the positive progression gives +theta/l: (n pi + theta)/l = theta/l forces n = 0, and (n pi - theta)/l = theta/l forces theta = n pi/2, outside the interior. At zero both progressions have n = 0. At pi/2 the positive n = 0 and negative n = 1 branches coincide. This proves the claimed signed count.

The multiplicity of the corresponding lowest eigenvalue of |D| is 2k in the interior (the values ±theta/l, k each), 2k at zero, and 4k at pi/2 (the values ±pi/(2l), 2k each). At zero the two interior signed eigenvalues simply coalesce; at pi/2 the next level also meets the window. Thus the signed count alone does not establish a doubling of independent physical Chan-Paton labels or an orientifold projection.

Independent proof of the arithmetic of Remark 8.3. *From the referee report on the 11D line, section 'Remark 8.3: wall bookkeeping and enhancements', transferred verbatim in substance on 4 Nov 2026.*

The slope and brane-count arithmetic below is conditional on the recalled type I' rules; the referee could not access the primary sources, and the second check later confirmed the slope rule and the E_{n+1} table against Polchinski-Witten and Bergman-Gaberdiel-Lifschytz (note at the remark). An enhancement requires tuning the coupling to a zero of z, and identifying the resulting E_2 factor with the physical electroweak sector requires additional data.

Assume z' = c (8 - N), with c > 0, N(x) the number of D8-branes to the left of x, and z > 0 in the open interval.

- *8 + 8.* N = 8 in the interior, so the interior slope is zero.
- *16 + 0.* z' = -8c, and the empty far wall reaches z = 0 only when z(0) = 8cL.
- *7 + 7.* With the other two branes at x_1 < x_2 in (0, L), the slopes are c, 0 and -c on (0, x_1), (x_1, x_2) and (x_2, L). Both walls have z = 0 precisely when

  ```
  z(0) = 0,    x_1 + x_2 = L,
  ```

  since z(L) = z(0) + c x_1 - c (L - x_2). Then z is positive in the interior. Thus "7 + 7 for E8 x E8" includes a coupling and a position condition.
- *Isolated zeros.* With n branes at a wall where z = 0, the slope next to the wall is c (8 - n) at x = 0 and c (n - 8) at x = L, so z > 0 inside requires n <= 7: an isolated zero with a positive interior requires at most seven branes at that wall under this slope rule.
- *15 + 1.* With the single brane at L, the slope is c (8 - 15) = -7c, so the far-wall zero is at z(0) = 7cL. Assuming the recalled E_{n+1} rule gives E_2, with Lie algebra su(2) ⊕ u(1), at that tuned wall. The rule alone does not determine the Standard Model charges, couplings, global group or embedding.

Colour triples. If every light flavour moves as a triple and there is a single top brane, a wall holds 3j or 3j + 1 branes, so the allowed wall counts up to seven are 0, 1, 3, 4, 6, 7. Only a wall containing the top can have seven (7 = 2 x 3 + 1); two such walls require relaxing the triple constraint or the single-top assumption. For seven branes at 0, six at L, and the remaining triple at x_b, N = 7 on (0, x_b) and N = 10 on (x_b, L), so the slopes are +c and -2c. With z(0) = z(L) = 0, c x_b - 2c (L - x_b) = 0 gives x_b = 2L/3, and

```
z(x) = c x            for 0 <= x <= 2L/3,
z(x) = 2c (L - x)     for 2L/3 <= x <= L.
```

The resulting E8 x E7 interpretation is conditional on the recalled enhancement rule and the proposed brane dictionary.

Part 36

The type I' dilaton as a spectral function, the SO(32) of the EPJC paper, and two more dualities

Lemma 36. Written as file 09 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, task C). Attempted by Claude (Anthropic), 4 Nov 2026, after the steer of the same day (the coincident-brane route is overused; the type I' dilaton and the colour-flavour idea are worth pursuing; compare with the SO(32) of the EPJC paper). Proposition 9.1 is proved from the recalled type I' slope rule; Proposition 9.2 is proved from representation theory and the recalled orientifold projections; Proposition 9.3 is proved (linear algebra and Connes's sign table); the comparison of section 1 is read from the arXiv text of the paper through a summarizing fetch; the rest are proposals. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

Correction (same day): the first version stated the slope rule for e^{−phi}. The piecewise-linear function is z, with e^{phi} = z^{−5/6}; Proposition 9.1 holds as stated with z in place of e^{−phi}, and infinite coupling is z = 0.

1. The SO(32) of A. Rivero, Eur. Phys. J. C 84 (2024) 1058 (arXiv:2407.05397)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The branching algebra is independently checked. The archived original abstract also states an additional chiral charge-+4/3 quark per generation. The full paper was unavailable for direct comparison, so its detailed attribution remains pending. See the referee report.

That paper uses the adjoint of SO(32) to classify the scalars of a supersymmetric Standard Model with three generations, as composites: pairs of quarks as colour triplets and quark-antiquark pairs as singlets and octets, with three extra scalars of charge +4/3. Its pattern is SO(2n) ⊃ SO(n_1) x SU(n_2) x SU(3) x U(1) with 2n = n_1 + 6 n_2, which for SO(32) and n_2 = 5 gives SO(2) x SU(5) x SU(3) x U(1): the 30 = 6 n_2 are the five light quarks u, c, d, s, b ("turtles") in three colours with conjugates, and the top is the non-participating "elephant". Its explicit route also passes through SO(32) ⊃ SU(16) x U(1), with 496 = 1 + 120 + 120bar + 255 (quarks from the 120s, leptons from the 255), and it mentions open strings, orientifolds and an SU(9) for M2 and M5.

Reading of the LaTeX source (Claude Sonnet 5.5, 4 Nov 2026; the earlier comparison used a summarizing fetch). The paper uses colour singlets for mesons and colour antitriplets for diquarks, and keeps only (1, 24, 1), (1, 15, 3bar) and (1, 15bar, 3) of its decomposition; colour octets and sextets lie outside it. The top is the favoured non-binding quark ("we have not a concrete argument"), and the postulates force the non-binding quark to be up-type. The chain is SU(16) x U(1) to SU(15) to SU(5) x SU(3), with SO(30) and SO(32) reached by "adding singlets". The identification of the top with the SO(2) factor, the O8^- plane and the Chan-Paton reading belong to these files, and "orientifold" in the paper means planar orientifold QCD. The footnote on the 84 + 12 split and SU(9) for M2 and M5 anticipates Proposition 21.3 and 04. The abstract specifies a chiral +4/3 quark per generation; the body has three scalars of charge +4/3 and leaves their fermion partner open. The paper gives (15, 3bar) as the antisymmetric square of one fundamental without stating chirality, so the question of Proposition 9.3 stays open; its remark on odd-parity diquarks is consistent with that proposition. The paper's selection of an up-type quark by its postulates and lemma 21's by the spectral-action values are independent arguments with the same outcome.

Coincidence. The chain SO(32) ⊃ SO(2) x SO(30) ⊃ SO(2) x SU(15) x U(1) ⊃ SO(2) x SU(5) x SU(3) x U(1), the 30 as five light quarks in three colours, and the top as the factor SO(2), are the same as lemma 21 and the dictionary of 06. Lemma 21 records the branching 496 -> (435, 1) + (1, 1) + (30, 2) and leaves SO(32) unexplored; the EPJC paper explores it through composites, and the files 06 to 09 through type I'

branes. What the 11D files add: the type I' realization of the SO(2) as the top's D8-brane and of the 30 as the Chan-Paton labels of fifteen D8-branes on an O8^- wall (06); the top angle as the brane's position and the one-loop vacuum on the far wall (Propositions 7.1, 7.2); the orientifold type O^- matched by the seesaw sign (Proposition 8.1); the far wall's enhancement to SU(2) x U(1) at infinite coupling (Remark 8.3); the dilaton (Proposition 9.1); the diquark counts as strings between the stack and its image, which need the O^- type (Proposition 9.2); and the real-structure condition under which these counts arise, as connection components along the interval, outside the finite inner fluctuations (Proposition 9.3). In type I' language the paper's other chain, SU(16) x U(1), is the configuration with all sixteen D8-branes coincident in the bulk, and the chain with SO(2) x SO(30) is the configuration with fifteen on a wall and one apart; the second is the one the corpus selects. The SU(9) of M2 and M5 in the paper is the grading of e8 used in 04.

2. Proposition 9.1 (the type I' coupling function is a trace of a function of the finite Dirac operator)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The integration is independently checked conditional on the recalled slope rule. The trace is over sixteen selected channels, and the full physical D_F Hilbert space enters only through a specified projection onto them; direct comparison with the cited source was unavailable to the referee. (Second check, against Polchinski-Witten.) The formula z(L) = z(0) - c (8L - sum x_k) is exact in the positions; the dictionary x_k = (2L/pi) l m_k holds only for constant z, and in general theta_k = (pi/2) W(x_k)/W(L) with W = int z^{-1/3} (10, Proposition 10.1). In the profile 15 + 1 the endpoint values 7cL and 8cL stand, and theta_t = pi/4 gives 7.358 cL in place of 7.5 cL. See the referee report.

Setting. Type I' on an interval [0, L], in the conformal coordinate x, with sixteen D8-branes at x_1, ..., x_16 and O8^- planes at the ends. Recalled rule (Polchinski and Witten, hep-th/9510169): the dilaton is e^{phi} = z^{−5/6} for a function z that is continuous and piecewise linear in x, with slope c (8 − N(x)), where N(x) is the number of D8-branes to the left of x and c > 0; the coupling diverges where z vanishes. Dictionary (06, 07): x_k = (2L/pi) θ_k with θ_k = l m_k the angles of sixteen channels of the finite Dirac operator (fifteen light-quark channels with colour, and the top).

Statement.

1. z(x) = z(0) + c Σ_k min(x/2, x_k − x/2) for x in [0, L], so z is z(0) plus c Tr g_x(l |D_F|) over the sixteen channels, with g_x(θ) = min(x/2, (2L/pi) θ − x/2): a trace of a function of the finite Dirac operator, as the spectral action is.
2. At the far wall, z(L) = z(0) − c (8L − Σ_k x_k).
3. Infinite coupling at the far wall (z(L) = 0) holds exactly when z(0) = (2cL/pi)(4 pi − l Σ_k m_k): the coupling at the light-quark wall is fixed by the sum of the sixteen channel masses, the trace of |D_F| on these channels.

Proof. *∫0^x N(y) dy = Σ_k (x − x_k)+*, so z(x) = z(0) + c (8x − Σ_k (x − x_k)+) = *z(0) + c Σ_k (x/2 − (x − x_k)+)*, and x/2 − (x − x_k)_+ = min(x/2, x_k − x/2). Item 2 is x = L with Σ_k (L − x_k) = 16L − Σ x_k. Item 3 sets z(L) = 0 and substitutes x_k = (2L/pi) l m_k. ■

Correction (4 Nov 2026). The dictionary x_k = (2L/pi) theta_k is exact only for constant z; the map from angles to positions is nonlinear. See 10, Proposition 10.1.

Reading (proposal). With the light quarks near θ = 0, Σ_k m_k ≈ m_t and the condition is z(0) ≈ (2cL/pi)(4 pi − θ_t): 7cL at label 9 (θ_t = pi/2) and 8cL at label 11 (θ_t -> 0). The coupling therefore varies along the interval on a scale set by the spectrum of D_F, which is the kind of second scale that Proposition 1.5 requires to select a joint family; whether the type I' coupling can play that role in the corpus's flat family is the next question. Since the slope decreases with x, z is concave, so z > 0 on [0, L) once z(0) > 0 and z(L) = 0; positivity of z(0) gives l Σ_k m_k < 4 pi, a bound on the trace of |D_F| in units of the tower scale, met when only the top is heavy (θ_t <= pi/2).

3. Proposition 9.2 (diquarks as strings between the stack and its image)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The representation and charge counts are independently checked, conditional on the recalled projections. The shifts describe the specified gauge-field channels; Λ^2 of

a line excludes that component only, and leaves the other open-string states and the strings to other branes. See the referee report.

Setting. V = C^5 ⊗ C^3, five light flavours in three colours, with Q = 2/3 on u, c and Q = −1/3 on d, s, b; the 30 of lemma 21 is V ⊕ Vbar. Fifteen D8-branes on an O8 plane carry the Chan-Paton space 30 (branes and images), with group SO(30) on an O8^- and USp(30) on an O8^+, and U(15) when the stack is moved off the plane (recalled).

Statement.

1. Under U(15), the adjoint of SO(30) is Λ²V ⊕ Λ²Vbar ⊕ (V ⊗ Vbar) and the adjoint of USp(30) is S²V ⊕ S²Vbar ⊕ (V ⊗ Vbar). The summands Λ²V or S²V and their conjugates are the strings between the stack and its image; V ⊗ Vbar are the strings with both ends on the stack.
2. Λ²V = (S²5, 3bar) ⊕ (Λ²5, 6) and S²V = (S²5, 6) ⊕ (Λ²5, 3bar).
3. In the colour antitriplet, Λ²V has the charge counts (−2/3: 6, +1/3: 6, +4/3: 3) of lemma 21, and S²V has (−2/3: 3, +1/3: 6, +4/3: 1). So the antisquark counts of three generations recorded in lemma 21 (6 of charge −2/3 and 6 of charge +1/3 in the 3bar) occur among the stack-image strings of an O8^- plane, while those of an O8^+ plane have the counts (3, 6, 1).
4. With the angles θ_k = l m_k of 06, a string from channel i to the image of channel j has the spectrum (n pi ± (θ_i + θ_j))/l, and a string from channel i to channel j has (n pi ± (θ_i − θ_j))/l, n in Z.
5. A single brane has no string to its own image on an O8^- plane (Λ² of a line vanishes) and one on an O8^+ plane.

Proof. 1: 30 = V ⊕ Vbar as a U(15) module, Λ²(V ⊕ Vbar) = Λ²V ⊕ Λ²Vbar ⊕ (V ⊗ Vbar) and likewise for S²; the orientifold projection keeps the antisymmetric part for SO and the symmetric part for USp, and the mixed summand V ⊗ Vbar is the U(15) adjoint. 2: Λ²(A ⊗ B) = (S²A ⊗ Λ²B) ⊕ (Λ²A ⊗ S²B) and S²(A ⊗ B) = (S²A ⊗ S²B) ⊕ (Λ²A ⊗ Λ²B), with Λ²3 = 3bar. 3: split 5 = 2 ⊕ 3 by charge; S²5 = S²2 ⊕ (2 ⊗ 3) ⊕ S²3 has dimensions 3, 6, 6 with charges 4/3, 1/3, −2/3, and Λ²5 = Λ²2 ⊕ (2 ⊗ 3) ⊕ Λ²3 has dimensions 1, 6, 3 with the same charges. 4: Proposition 6.1 with charges (+1, +1) and (+1, −1) under U(1)_i x U(1)_j, whose holonomy is e^{2 pi i (a_i ± a_j)}, and θ = pi a. 5: dimensions. ■

Reading (proposal). (11, Theorem 11.3 and Proposition 11.4, qualify the selection below: the window sees the counts (3, 6, 1) also at s = +1, through the commutator with the masses, and parent scalars with the counts (6, 6, 3) need M_15(C) at both walls.) The diquark counts of lemma 21 and of the EPJC paper are the stack-image strings of an O8^- plane, the type that Proposition 8.1 associates with the factorization sign s = +1 selected by the seesaw sector (T2c); the O8^+ alternative gives the wrong number of up-type antisquarks. So the seesaw sign and the scalar counts select the same orientifold type. Two features follow from the projection. Diquark masses go with the sum θ_i + θ_j and meson masses with the difference θ_i − θ_j, so when the stack leaves the wall the diquarks become massive and the mesons stay massless: the diquarks are tied to the orientifold. And the top's single brane has no string to its own image, while its strings to the stack, the (30, 2), have masses near m_t (Proposition 6.1); in the EPJC paper the top's exclusion is an input, and here the O8^- projection with one brane for the top removes the top-top channel only (item 5); the top-to-stack strings (30, 2) remain, with masses near m_t, so the rest of the exclusion is a mass statement, as in the paper.

4. Proposition 9.3 (the real structure decides between the two diquark counts)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The stated sign constraint excludes a nonzero antisymmetric block in the three KO labels. The signs permit all of S^2 V (second check: B ranges over exactly S^2 V for eps eps' = +1, with the diagonal block free); the algebra, grading and order conditions of a triple then select a subspace. Distinct-multiplet blocks remain related to their transposes. See the referee report.

Setting. E = V ⊕ Vbar with V as in Proposition 9.2, an antiunitary J on E that maps V onto Vbar and Vbar onto V (as particles and antiparticles are exchanged), J² = ε, and a Hermitian X on E with JX = ε' XJ, the condition that inner fluctuations obey in a KO-dimension with signs (ε, ε'). Let B be the block of X from Vbar to V, an element of V ⊗ V.

Statement.

1. B is symmetric (B in S²V) if εε' = +1 and antisymmetric (B in Λ²V) if εε' = −1.
2. In the KO-dimensions 5, 6, 7 of the finite part for the labels 9, 10, 11, (ε, ε') = (−1, −1), (+1, +1), (+1, +1), so εε' = +1: the diquark block of every J-compatible fluctuation lies in S²V, whose colour antitriplet (Λ²5, 3bar) has the counts (−2/3: 3, +1/3: 6, +4/3: 1).
3. For a connection on a circle whose real structure is complex conjugation (KO-dimension 1, (ε, ε') = (+1, −1)), the block lies in Λ²V, whose colour antitriplet (S²5, 3bar) has the counts (−2/3: 6, +1/3: 6, +4/3: 3) of lemma 21. The SO(32) Wilson line of Proposition 6.1 is of this type.

Proof. Choose a basis of V and let K: V -> Vbar be coordinatewise conjugation; J(v, w) = (ε K^{-1} w, K v) has J² = ε. Write X = [[A, B], [B*, D]]. The component of JX = ε' XJ that maps V to V gives ε K^{-1} B* = ε' B K, so B = εε' K^{-1} B* K^{-1} = εε' B^T in coordinates. Under the U(15)-equivariant identification of Hom(Vbar, V) with V ⊗ V, symmetric matrices form S²V and antisymmetric ones Λ²V. Items 2 and 3 use Connes's sign table, ε = (+, +, −, −, −, −, +, +) and ε' = (+, −, +, +, +, −, +, +) for KO-dimensions 0 to 7, and the counts of Proposition 9.2. On the circle, D = −i d/dθ and J = conjugation give JD = −DJ, and a Hermitian connection A with JA = −AJ is i times a real antisymmetric matrix, an element of so(n), where the generator of R(2 pi a) lies. ■

Item 2 is a negative statement with the following scope: the blocks of J-compatible Hermitian operators between particles and antiparticles, for one multiplet V, in the three labels. It is the finite-geometry form of the symmetry of Majorana mass matrices, and of the Fermi-statistics constraint on a Lorentz-scalar bilinear ψ_i ψ_j of one Weyl multiplet without derivatives (recalled), under which the (15, 3bar) vanishes. Known escapes: item 3 (connection components along a circle or an interval), blocks between two distinct multiplets, which J does not constrain, and derivative couplings.

Reading (proposal). The counts of lemma 21 and of the EPJC paper, (6, 6, 3), belong to the antisymmetric block. In the corpus's setting this block arises as the component of an SO(30) connection along the circle or interval: the Wilson-line moduli of the type I' stack, whose stack-image part is Proposition 9.2. The finite triple alone, in any of the three labels, gives the symmetric block and the counts (3, 6, 1). So the composite scalars with the counts of lemma 21 point to the interval parent of lemma 22 and its gauge field, and the selection of O8^- in Proposition 9.2 concerns these connection components. This narrows the spin gap of Propositions 6.1 and 9.2: the Λ²V states are scalars as circle components of a nine-dimensional gauge field, and they come with that field's vector and gaugino partners.

5. Two more dualities between 9 and 11 (recalled; proposals)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The E_1 alternatives need correction: the canonical-orthogonal root lattices distinguish $\mathbb{P}^1 \times \mathbb{P}^1$ from $dP_1 = \mathbb{F}_1$, and the usual local engineering assigns the latter to $\widetilde{E}_1$. The dP_2 and brane duality claims remain conditional on the unavailable primary sources. The Weyl action is a gauge relabelling of a holonomy, subject to the full background data. See the report on the del Pezzo comparison and Weyl action.

1. **E-type walls and del Pezzo surfaces.** The E_{n+1} symmetry of an O8^- plane with n D8-branes at infinite coupling is the symmetry of the five-dimensional theory on a D4-brane probe, which M-theory realizes on a Calabi-Yau threefold with a shrinking del Pezzo surface dP_{n+1} (Seiberg, hep-th/9608111; Morrison and Seiberg, hep-th/9609070; Intriligator, Morrison and Seiberg, hep-th/9702198). So the far wall with the top's brane (n = 1, E_2 = SU(2) x U(1)) has an eleven-dimensional description by a local dP_2, and the empty far wall (E_1 = SU(2)) by dP_1 or P^1 x P^1; the light-quark wall (n = 15) has none. The Witten base CP^2 x CP^1 of 02 is built from the same pieces (CP^2 = dP_0, and P^1 factors).
2. **Colour-flavour exchange as a T-duality.** In the heterotic string on S^1, the Weyl group of the gauge group acts on the Wilson lines as part of the T-duality group O(17, 1; Z). The colour-family transposition M of 05 is an element of the Weyl group of E8, so it is a T-duality of the nine-dimensional E8 x E8 string, and through Hořava-Witten a symmetry of the E8 holonomy on a wall in eleven dimensions. As a Weyl element it acts as a global gauge transformation, so by itself it relabels gauge-equivalent vacua; it exchanges colour and family physically only relative to a structure that tells them apart, such as the placement of H_F in 05. The four-family requirement of Propositions 5.1 and 5.4

remains.

6. Open

Referee note (GPT-6.1 Terra, 4 Nov 2026). The finite KO signs restrict the symmetric block without producing the full representation. Light interval scalar modes, their boundary conditions and the physical string dictionary remain to be constructed; the detailed EPJC comparison is pending.

- Whether the type I' coupling, as a spectral function of D_F (Proposition 9.1), supplies the second scale of Proposition 1.5 in the flat family of lemmas 23 and 24.
- The local dP_2 of the top's wall and its relation to the Witten base CP^2 x CP^1.
- Spin: in type I' the strings of Proposition 9.2 belong to nine-dimensional vector multiplets, with one real scalar each (the relative position), the same gap as in Proposition 6.1. By Proposition 9.3 a finite triple in the labels 9 to 11 gives S^2V, so the Λ^2V scalars must come from the connection along the interval; the next computation is the inner fluctuations of the parent of lemma 22 with an SO(30) algebra, and whether its interval component carries Λ^2V with the counts of lemma 21.
- Whether the (15, 3bar) of the EPJC paper could be read as bilinears of two distinct multiplets (the paper itself gives one fundamental with itself) (for example left-handed quarks and conjugates of right-handed ones arranged into one 30), which Proposition 9.3 does not constrain.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 9.1, with the scope of its trace. *From the referee report on the 11D line, section 'Proposition 9.1: integrated coupling profile', transferred verbatim in substance on 4 Nov 2026.*

Hypothesis: the recalled slope rule. The Polchinski-Witten source could not be accessed by the referee; the comparison with it is the second-check note at the proposition.

Let N(x) = sum_k 1_{x_k < x}. Integrating z' = c (8 - N) almost everywhere, with z continuous, gives

```
z(x) = z(0) + c (8x - sum_{k=1}^{16} (x - x_k)_+) = z(0) + c sum_{k=1}^{16} min(x/2, x_k - x/2),
```

since int_0^x N(y) dy = sum_k (x - x_k)+, *and x/2 - (x - x_k)+* = min(x/2, x_k - x/2) (both sides equal x/2 for x <= x_k and x_k - x/2 for x > x_k). At x = L, with every x_k <= L, this becomes

```
z(L) = z(0) - c (8L - sum_k x_k).
```

With x_k = 2 l L m_k/pi, the dictionary of the Setting (exact only for constant z, by the correction after the proof), the zero condition z(L) = 0 is z(0) = (2cL/pi)(4 pi - l sum_k m_k), as stated. The slopes c (8 - N(x)) are nonincreasing, so z is concave; if z(0) > 0 and z(L) = 0, then z(x) >= (1 - x/L) z(0) > 0 on [0, L).

Scope of the trace. The trace is over the sixteen selected channel values. The equality with a spectral trace assumes a chosen sixteen-dimensional channel space with eigenvalues m_k, or an explicitly normalized projection selecting those values. The physical D_F has additional chirality, conjugation, colour and generation multiplicities, which cannot be dropped from its full trace without specifying that projection. The assignment of the sixteen brane positions to those channels is also part of the proposal. This profile alone does not derive a stabilizing term for the interval modulus. The slope, coordinate and e^phi-to-z conventions are those of the recalled rule.

Independent proof of Proposition 9.2, items 2 and 3, with the scope of items 4 and 5. *From the referee report on the 11D line, section 'Proposition 9.2: diquark representations and string channels', transferred verbatim in substance on 4 Nov 2026.*

Hypothesis: the recalled orientifold projections. For V = C^5 ⊗ C^3, the swap of V ⊗ V is the product of the swaps of the two tensor factors, so its eigenspaces combine the symmetric and alternating projectors on the factors:

```
Λ²V = (S²C^5 ⊗ 3bar) ⊕ (Λ²C^5 ⊗ 6),
S²V = (S²C^5 ⊗ 6) ⊕ (Λ²C^5 ⊗ 3bar),
```

with $\Lambda^2 3 = 3\text{bar}$ and $S^2 3 = 6$. Their dimensions are $15 * 3 + 10 * 6 = 45 + 60 = 105$ and $15 * 6 + 10 * 3 = 90 + 30 = 120$, which are C(15, 2) and C(16, 2). Splitting the flavour space into two charge-2/3 and three charge-(-1/3) directions, $S^2C^5 = S^2 2 \oplus (2 \otimes 3) \oplus S^2 3$ and $\Lambda^2 C^5 = \Lambda^2 2 \oplus (2 \otimes 3) \oplus \Lambda^2 3$ give the antitriplet counts (6, 6, 3) in S^2C^5 and (3, 6, 1) in Λ^2C^5, ordered by charges (-2/3, 1/3, 4/3). The full adjoints have dimensions $2(105) + 225 = 435 = \dim \mathrm{so}(30)$ and $2(120) + 225 = 465 = \dim \mathrm{usp}(30)$.

Scope of items 4 and 5. For the assumed Wilson-line action, the antisymmetric or symmetric block has weights a_i + a_j and the mixed block has weights a_i - a_j; hence their field-theory Kaluza-Klein momenta have the displayed shifts (n pi ± (θ_i ± θ_j))/l, with θ = pi a. This calculation leaves out the rest of the open-string spectrum, which also includes oscillator states and their orientifold projections. The sum-weight component with i = j is absent in Λ^2 and present in S^2; neutral diagonal modes remain. The claim of item 5, that a single brane has no string to its own image, holds in this sector. Absence of the same-line component leaves the strings between the top brane and other branes. Moving all stack branes by the same angle makes the mixed-block differences vanish; different individual positions can give nonzero differences. Identifying the resulting Chan-Paton modes with the proposed composites is an additional physical construction. The O8^-/O8^+ projection and global tadpole conditions were not independently verified, since the primary sources were unavailable to the referee.

Independent proof of Proposition 9.3, item 1, with the scope of item 2. *From the referee report on the 11D line, section 'Proposition 9.3: real-structure constraint on diquark blocks', transferred verbatim in substance on 4 Nov 2026.*

Choose an antiunitary identification K: V -> Vbar and write

```
J(v, w) = (eps K^{-1} w, K v),    X = [[A, B], [B*, D]].
```

Then JX(v, w) = (ε K^{-1}(B* v + D w), K(A v + B w)) and XJ(v, w) = (ε A K^{-1} w + B K v, ε B* K^{-1} w + D K v). The V -> V component of JX = ε' XJ, the coefficient of v in the first entry, gives

```
eps K^{-1} B* = eps' B K,   hence   B = epseps' K^{-1} B* K^{-1} = epseps' B^T,
```

the last equality in coordinates in which K is coordinatewise conjugation. Thus εε' = +1 forces $B \in S^2V$, and εε' = -1 forces $B \in \Lambda^2V$. For the displayed KO-5, KO-6 and KO-7 conventions the products are all +1. The zero block belongs to both subspaces, so "symmetric if εε' = +1" states the symmetry forced by the sign. For $V = C^5 \otimes C^3$ the decomposition of S^2V and the counts are those of the proof of Proposition 9.2: Λ^2C^5 has multiplicities 3, 6, 1 at charges -2/3, 1/3, 4/3, and S^2C^5, hence the colour-3bar part of Λ^2V, has 6, 6, 3.

Scope of item 2. A J-compatible finite fluctuation can occupy only a subspace of S^2V; the algebra, grading and order conditions determine which blocks actually occur. For example, with D = 0 every inner one-form vanishes, despite the symmetry condition. The KO signs alone therefore exclude a nonzero antisymmetric block and leave the full symmetric scalar spectrum to be established. For a circle operator D = -i d/dx with ordinary conjugation J, JD = -DJ. A Hermitian connection satisfying the same sign is purely imaginary and antisymmetric in a real basis; the complexified so(30) representation contains Λ^2V and its conjugate. Whether these components survive the particular interval boundary conditions as light scalars requires a separate construction. For distinct multiplets, an off-diagonal block B_12 may be chosen freely, while the full constraint still relates it to B_21 = εε' B_12^T.

Scope and direction. The exclusion concerns a Hermitian J-compatible operator between one multiplet and its conjugate, with the stated signs. Distinct multiplets, other real structures or signs, derivative couplings, and circle or interval connection components remain possible directions; each requires its own field and boundary data.

Independent proof of the branching in section 1. *From the referee report on the 11D line, section 'File 09, section 1: the SO(32) comparison', transferred verbatim in substance on 4 Nov 2026.*

The alternative chain follows from 32_C = 16_{+1} ⊕ 16bar_{-1} under U(16) ⊂ SO(32): so(32)_C = $\Lambda^2$32 = $\Lambda^2$16 ⊕ $\Lambda^2$16bar ⊕ (16 ⊗ 16bar), so

```
496 = 120_{+2} ⊕ 120bar_{-2} ⊕ 255_0 ⊕ 1_0.
```

Under SU(15), 16 = 15 ⊕ 1 and 120 = Λ^2(15 ⊕ 1) = 105 ⊕ 15. With 15 = 5 ⊗ 3, the plethysm of Proposition 9.2 gives

```
105 = Λ²(5 ⊗ 3) = (15, 3bar) ⊕ (10, 6),
```

with $S^2$5 = 15 and $\Lambda^2$5 = 10, and 15 ⊗ 15bar = (5 ⊗ 5bar, 3 ⊗ 3bar) = (24, 8) ⊕ (24, 1) ⊕ (1, 8) ⊕ (1, 1). The antitriplet charge multiplicities are those checked in Proposition 9.2. These group products are Lie-algebra shorthand; global embeddings involve finite central quotients.

Scope. The SO(2) plane is a colour singlet. Assigning it to a top brane is a label in the proposed dictionary and does not by itself construct a colour-triplet top field. Chan-Paton bilinearity and matching representation counts likewise do not by themselves establish dynamical quark composites. The referee could not access the paper itself; the comparison with its text is the reading of the LaTeX source in section 1.

Counterexample (referee) to the unqualified E_1 surface identification in section 5, item 1. *From the referee report on the 11D line, section 'File 09, section 5: del Pezzo wall correspondence', transferred verbatim in substance on 4 Nov 2026.*

The two degree-eight surfaces have different root lattices orthogonal to the canonical class. On dP_1, let H^2 = 1, E^2 = -1, H · E = 0 and K = -3H + E. If D = aH + bE is orthogonal to K, then D · K = -3a - b = 0, so b = -3a and D^2 = a^2 - b^2 = -8a^2. There is no integral class with D^2 = -2. On P^1 x P^1, with ruling classes F_1, F_2 (F_1^2 = F_2^2 = 0, F_1 · F_2 = 1) and K = -2F_1 - 2F_2, the difference F_1 - F_2 is orthogonal to K and has square -2. These surfaces therefore cannot be interchanged in the root-lattice comparison. In the usual local engineering description (recalled), P^1 x P^1 gives E_1, whereas dP_1 = F_1 gives the distinct $\tilde{E}$_1 theory.

For dP_2, with exceptional classes E_1, E_2 and K = -3H + E_1 + E_2, the class E_1 - E_2 is orthogonal to K and has square -2; the orthogonal lattice has rank two, consistent with an A_1 factor and an additional abelian direction. The dP_2 comparison is compatible with the expected A_1 root. This computation alone does not construct the corresponding five-dimensional field theory or the type I' duality, and the brane-to-geometry map remains conditional on the cited dualities.

Sources. The distinction between the two degree-eight engineering models is also stated in a secondary geometric-engineering discussion. The Seiberg and Morrison-Seiberg primary sources were unavailable to the referee for direct comparison. The occurrence of CP^2 and CP^1 in the separate six-dimensional base does not by itself identify that compactification with a local dP_2.

Independent proof of the Weyl action in section 5, item 2. *From the referee report on the 11D line, section 'File 09, section 5: Weyl action on a Wilson line', transferred verbatim in substance on 4 Nov 2026.*

A Weyl transformation of E8 acts by conjugation on an E8 holonomy and gives a lattice automorphism in the heterotic circle frame. Whether it acts as a symmetry of a fixed compactification depends on all of its background and boundary data.

Let T ⊂ E8 be a maximal torus. An element w ∈ W(E8) = N(T)/T has a representative g ∈ N(T), and the constant gauge transformation by g sends a holonomy W ∈ T to g W g^{-1} ∈ T. Thus the holonomy alone determines the same gauge orbit. The corresponding isometry w ⊕ 1 of E8 ⊕ E8 ⊕ II_{1,1} is an element of the automorphism group of the nine-dimensional Narain lattice. Proposition 5.2 places the transposition M in that Weyl group.

If a bundle, a defect, an algebra embedding or a distinguished set of zero modes is held fixed, the same g need not preserve it. Transforming the entire configuration produces a gauge-equivalent description when the transformation is allowed; an exchange relative to fixed colour and family structures needs an explicit action on those structures. The root no-go applies to its stated three-slot configuration, while bundle zero modes remain an open escape.

Part 37

The type I' dilaton against the second scale of Proposition 1.5

Lemma 37. Written as file 10 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line (README, PLAN). Attempted by Claude Opus 5.5 on 4 November 2026, after the merge of branch 11D into master, to answer the last item left open in 09: whether the dilaton can be the second scale that Proposition 1.5 requires. Sources read from the arXiv LaTeX: J. Polchinski and E. Witten, hep-th/9510169, section 3; O. Bergman, M. Gaberdiel and G. Lifschytz, hep-th/9711098, section 3. Numbers in Python (numpy, scipy, mpmath), scripts not kept by the rule of the project. The Codex quota is spent, so no Terra referee is available. An independent reading by Claude Sonnet 5.5, with the error and amendments below, is in the review. The amendments were rechecked by Claude Opus 5.5 before being applied. Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report), who reread both sources: Propositions 10.2 and 10.4 correct with amendments; Proposition 10.1 correct with a qualification of "exactly when"; Proposition 10.3 conditional on the string identification; Proposition 10.5 an upper bound whose consequence for the potential is a hypothesis; Proposition 10.6, item 3, had an error, with the counterexample G_0 = theta^6, G_1 = theta^2 given below; all applied, together with the corrections to the Outcome, the far-wall normalization and the corrections section.

Correction (4 Nov 2026). The first version counted 21 gauge generators with integer towers. Lemmas 23, 24 and 26 use sp(2) (+) su(4), with 25 generators: 16 with integer towers at theta = 0 and 9 with half-integer towers. The bracket for one generation is 41.3125, so the near-cancellation 64 - 63 stated in the first version came from the wrong count, and it is withdrawn. So is the small value theta_* = 0.202 that followed from it.

Outcome

1. **In the scope stated below, the dilaton gradient does not supply the second scale.** Along any path that opens the circle, the relative variation of the profile is bounded by 6/R^2 (Proposition 10.5, an upper bound; it is zero in the chamber 8 + 8). Under the hypothesis that the effective potential has a regular joint expansion in theta and alpha'/l^2 around the flat family, the gradient enters as a power of alpha'/l^2, and it can matter only if the leading Casimir coefficient a_0 vanishes and, when a_1 also vanishes, only if the coefficient b of theta^2 vanishes too (Proposition 10.6, with the referee's counterexample at b = 0).
2. **a_0 is now computed**, which was the open item of 01: 8 pi^2 a_0 = (3 zeta(5)/16)(4 n_D' - 3 (n_i - (15/16) n_h) - n_s - 5 n_g), with n_i and n_h the generators with integer and half-integer towers at theta = 0. For the corpus content it is positive, so at v = 0 the interval runs to large length, in the direction H wants.
3. **The dictionary of 06 to 09 needs three corrections** (Propositions 10.1 to 10.4). The map from angles to positions is nonlinear, and linear as a function of every position exactly when the profile is constant. If the tower of Proposition 6.1 is identified with the type I' winding states, which needs the proposed string embedding and holds in the source at constant dilaton, then the length l of H and the type I' length are T-dual, and l -> infinity shrinks the type I' interval at fixed heterotic coupling. And the configuration 15 + 1 has a Romans mass in the bulk, which bounds length times coupling, so the classical profile has no regime resolved at the formal local scale g^{1/3} l_s.
4. **A scale-free action can still separate the window**, with a fixed small ratio, if the light mass is exponentially small in a stabilized shape parameter (Remark 10.7), which amends the last sentence of Proposition 1.5. This is a separated window at each radius; it is not the parametric hierarchy of Theorem 30.5 of lemma 30, where the ratio tends to zero.

The direction: the second scale must enter through a bulk energy depending on the electroweak order

parameter, through a cancellation a_0 = 0, or through a fixed shape with an exponentially small ratio, as in lemma 29. These targets are stated in Proposition 10.6 and Remark 10.7. None is derived.

The dictionary against the source

Polchinski and Witten give, in conformal gauge, g = Omega^2(x) eta with Omega = C z^{-1/6}, e^phi = z^{-5/6}, and z = 3 C (B mu_8 - nu_0 x)/sqrt 2, with nu_0 = (N(x) - 8) mu_8 piecewise constant. This confirms the slope rule recalled in 08 and 09. We write z for this function up to its constant, normalized so that its slope is exactly 8 - N(x), which fixes the normalization in the formula for R below, with x in [0, L].

Proposition 10.1 (the map from angles to positions). Put W(x) = int_0^x z^{-1/3} dy. By Polchinski-Witten, equations (mass) and (wilmap), the Wilson-line eigenvalue of a brane at x_k is lambda_k = W(x_k) / (2 W(L)), that is

```
theta_k = pi lambda_k = (pi/2) W(x_k) / W(L),
```

and with L = 2 pi, in their units alpha' = 2, R^2 = (3/4) int z^{1/3} dx / (int z^{-1/3} dx)^2. So the linear dictionary x_k = (2L/pi) theta_k of Proposition 9.1 holds as a function of every position exactly when z is constant (8 + 8), and asymptotically for large R; a particular configuration can satisfy it at its brane positions with z nonconstant, as fifteen branes at 0 and one at L do, with angles 0 and pi/2 and slope -7 in the bulk (referee). The normalization: the physical profile in e^phi = z_phys^{-5/6} is z_phys = (3 C mu_8/sqrt 2) h with h of slope 8 - N(x), and Polchinski and Witten set the heterotic alpha'_h = 2 and keep the type I' alpha' separate.

Proof. The string stretched from the wall to x_k has mass (1/(2 pi alpha')) int_0^{x_k} Omega^2 dx, since its area element is Omega^2 dx^0 dx^9, and Omega^2 is proportional to z^{-1/3}. A string winding from 0 to L and back has the mass 1/R. The ratio gives lambda_k; the formula follows from the brane-to-brane mass differences of their equations (14) and (15), since a string ending on the wall is not an available state in every configuration (referee). The formula for R is their equation (16), reduced as in their equation for the dilaton. ■

Consequence for Proposition 9.1: its items 1 and 2 hold as statements about z in terms of the positions x_k. Reading them as a trace of a function of |D_F| needs the map above, which depends on the whole configuration, so the coupling function is a trace only implicitly. Item 3 changes accordingly.

Proposition 10.2 (15 + 1 with infinite coupling at the far wall). Take fifteen branes at x = 0 and one at x_t, with z(L) = 0, and put rho = (z(x_t)/z(0))^{2/3} in [0, 1]. Then, with alpha' = 2,

```
theta_t = (pi/2) 8 (1 - rho) / (8 - rho),        R_*^2 = 14 (8 - rho^2) / (8 - rho)^2.
```

So R_* runs from sqrt 2, for the top's brane on the stack, to sqrt 7 / 2, for the top's brane on the far wall, which are the values (1/2)|n - 8|^{1/2} of Polchinski and Witten at n = 16 and 15. In heterotic string units, R_*/sqrt(alpha'h) lies in [sqrt(7/8), 1]. At each fixed Wilson angle, a nonnegative far-wall value of the normalized profile gives R_h >= R(theta): with A = h(0)^{2/3}, T = h(x_t)^{2/3}, E = h(L)^{2/3} and a = theta/pi, direct integration gives R_h^2 = 2 - a^2 + 28 E/(8A - T - 7E), and R_*^2 = 2 - (theta/pi)^2 is the value at E = 0 (referee). The critical point uses the classical profile at divergent coupling.

Proof. Integrate z^{1/3} and z^{-1/3} on the two segments, with slopes -7 and -8, and use z(x_t) = 8 (L - x_t). ■ Python agrees at x_t/L = 0.5, with theta_t = 0.5859 and R^2 = 1.9652, against 0.7854 for the linear dictionary (the referee's SciPy run gives 0.585918469 and 1.965216392).

So infinite coupling sits at the *smallest* radius of the chamber, at the string scale, and large R is large B, the weakly coupled region. Polchinski and Witten say this directly: "When B is large, the coupling e^{phi(x^9)} is small everywhere and the Type I' theory is weakly coupled; in this range the heterotic radius R is large".

Proposition 10.3 (T-dual lengths, conditional on the string identification). Suppose the tower of Proposition 6.1, n/R, is identified with the type I' winding tower. The source identifies a particular heterotic BPS momentum state with a type I' winding state ("a heterotic Kaluza-Klein state which is dual to a Type I' winding state", Polchinski and Witten); identifying the interval tower with those string states needs the proposed string embedding, and Wilson-line charges shift the momentum-to-winding dictionary (Bergman,

Gaberdiel and Lifschytz), so the identification is a hypothesis here (referee). With constant dilaton, the proper length L_p of the type I' interval and the radius satisfy L_p R_I = pi alpha' on the type I side, with alpha' that of type I'. So with l = pi R_I, measured in the type I' metric, l L_p = pi^2 alpha'. For the heterotic radius in the heterotic metric, l L_p = pi^2 alpha' sqrt(lambda_h), by Bergman, Gaberdiel and Lifschytz, with lambda_h the heterotic coupling.

The flat relations L_p R_I = pi alpha' and l L_p = pi^2 alpha' sqrt(lambda_h) hold at constant dilaton with these frame identifications; the source restricts its simple equations (1.2)-(1.3) to eight D8-branes at each wall, and in the chamber 15 + 1 the metric and coupling vary, so the applicable relation carries the Weyl factor, 1/R_h = (D/(pi alpha'_{I'})) int_0^L Omega^2 dx with L_p = int_0^L Omega dx. Equality of two lengths at a self-dual value does not identify the walls of the two descriptions (referee). Under the hypothesis, the walls of H, at distance l, and the O8 planes, at distance L_p, coincide only at the self-dual length. In particular, the reading "the interval opens" in section 6b of 07 has to be qualified: under the identification and at fixed heterotic coupling, l -> infinity shrinks the type I' interval and decompactifies the SO(32) circle; the large-R_h, fixed-lambda_h decompactification is a conditional reading, which the exact flat formulas do not prove throughout the chamber 15 + 1.

The endpoint of that limit is the ten-dimensional SO(32) theory (heterotic or type I), since at fixed lambda_h the type I' coupling falls as 1/(sqrt(lambda_h) R_h). *Recalled, and supported in the two sources only by remarks of Bergman, Gaberdiel and Lifschytz at the constant-dilaton point:* the eleven-dimensional limit of the rank-17 moduli space is M-theory on an interval times a circle, with E8 x E8 at the walls, which by Remark 8.3 is out of reach with colour-respecting moves. If so, label 11 of section 6b is not reached by theta -> 0, l -> infinity in the chamber 15 + 1.

Proposition 10.4 (Romans mass, length and coupling). On a segment with N branes to the left, the proper distance is linear in the inverse coupling:

```
L_p = (C_L l_s / |8 - N|) |e^{-phi_2} - e^{-phi_1}|,     C_L = 8 pi / 5,     l_s^2 = alpha'_{I'},
```

with C_L from the standard normalization of the Romans mass, |F_0| = |8 - N|/(2 pi l_s), whose sign depends on the orientation; from the source profile and its charge normalization, |d e^{-phi}/ds| = 5 |8 - N|/(8 pi l_s). For N = 8, F_0 = 0, the coupling is constant on the segment, and the division and the bound below do not apply. So in the configuration 15 + 1, g(0) L_p <= C_L l_s / 7 for the total length. Near an infinite-coupling wall the coupling is g = C_L l_s / (|8 - N| s) at proper distance s, and the region g >= g_0 has length (3/4) (C_L / |8 - N|) g_0^{-4/3} in the formal local scale l_11 = g^{1/3} l_s, which is 0.54 g_0^{-4/3} for N = 15. Therefore no region of the classical 15 + 1 background has both strong coupling and a length resolved at that scale. A standard eleven-dimensional lift has not been established for this massive background, and the strong-coupling end of the classical profile is uncontrolled; the source presents its broader M-theory conclusion as an expectation (referee).

Proof. Omega dx = C z^{-1/6} dz / (a C (8 - N)) integrates to a multiple of z^{5/6} = e^{-phi}. ■ This agrees with Bergman, Gaberdiel and Lifschytz, who conclude from the D0-brane bound states that they "expect neither to be applicable to the N != 0 backgrounds", meaning an M-theory interpretation or a matrix theory, and that "only the combination of eight D8-branes and an orientifold plane is known to arise in M-theory". Here their N is 8 - N. It holds wherever N != 8. The scope is the classical profile, which is not controlled where g is of order one, so this is a statement about the extrapolated background.

The far-wall electroweak group. Remark 8.3 places SU(2) x U(1) at the far wall at infinite coupling. By Bergman, Gaberdiel and Lifschytz, the states that become massless there are D0-branes at the wall, heterotic winding states. Let n be the number of branes at the strong wall, here n = 1 with the top's brane there, placed at their x^9 = 0 by reflecting the interval. Combining their equations (windingmass) and (D0mass), with N' = 8 - n, A = (B + 2 pi N')^{2/3} and b = B^{2/3},

```
M_D0 R = (N'/2) b / (A - b) = R^2 - R_*^2    (alpha' = 2),
```

which vanishes at B = 0, where R = R_*, and grows like R^2. This is the full D0 mass after conversion to the heterotic frame; the enhancement states with heterotic winding w_h = +-1 contain half a D0-brane (their equations (3.15)-(3.16) and section 4.2), so if those states are the W^+-, M_{W,h} R_h = (1/2)(R_h^2 - 7/4), with the same monotonic growth (referee). Only W^+ and W^- can be such states: a neutral Z is

a Cartan combination, and the rank of a nine-dimensional theory with sixteen supercharges is fixed, so no massive neutral vector arises this way. So in the chamber 15 + 1 with the top's brane on the far wall, if the W^+- are these D0 states, they are massless at the string-scale radius, with the tower fixed, and they become heavier as R grows: M_W l rises with l. This is the fixed-tower path of the audit (its section 2.2), and away from it the W moves the wrong way for hypothesis H. So on one path, in this scope, the dictionary can place the electroweak group at the far wall, or the opening of H on the circle of Proposition 6.1, but not both. Escapes: other chambers; the W as a Wilson-line string between D8-branes, as in lemma 23; the top's brane in the bulk, with the far wall giving E_1.

The test

Proposition 10.5 (the gradient is a string-scale effect). With L = 2 pi and alpha' = 2, let B = min h > 0 for the normalized profile h. For large R, R^2 = 3 B/(8 pi) + O(1), and the relative variation (max h - min h)/B across the interval is at most 16 pi / B ~ 6 / R^2, an upper bound, attained nowhere in the chamber 8 + 8, where the variation is zero. So along a joint family, R_eps ~ eps^{-alpha}, this variation falls as eps^{2 alpha}, the map of Proposition 10.1 linearizes, and the differences of the dilaton along the interval tend to zero; a finite limiting value of the coupling needs control of the constant C, since z_phys is proportional to C h, and a path can have R_h -> infinity with the coupling diverging uniformly (referee).

Proof. h is B plus a function of size at most 8 L = 16 pi. Insert in the formula for R. ■

If the effective potential has a regular, differentiable expansion in the relative variation, and with the frame identification relating l to R_h, the corrections from the gradient are powers of alpha'/l^2. Small variation alone gives no such expansion: a correction proportional to the square root of the variation has another power. So the regular expansion is a hypothesis of Proposition 10.6, which this estimate guides and does not establish (referee). In the supersymmetric type I' background there is no potential at all: D8 positions, B and C are flat directions. A potential needs broken supersymmetry, as in the flat family of lemmas 23 and 24.

The coefficient a_0. Write 8 pi^2 l^4 V = A_0 + 3 F(2u) + (3/2) F((16/5) u) - n_f F(t), the potential of lemma 24 with its theta-independent part A_0 restored, where n_f counts twice the Dirac species at frequency one (lemma 26). The relative terms vanish at theta = 0, and each tower at theta = 0 is half of a circle tower, periodic or antiperiodic:

```
A_0 = (3 zeta(5) / 16) (4 n_D' - 3 (n_i - (15/16) n_h) - n_s - 5 n_g),
```

with n_D' = 2 N_D counting twice all Dirac species, of any frequency (for frequency zero the tower is the theta = 0 tower), n_i and n_h the five-dimensional gauge generators with integer and half-integer towers at theta = 0, n_s the real bulk scalars with integer towers, and n_g = 1 if the five-dimensional graviton is included. This assumes that the bulk vacuum energy and the wall tensions are tuned away, since they carry the terms of order Lambda^5 l and Lambda^4.

Proof. A real degree of freedom on a periodic circle of length 2 l has V = -3 zeta(5)/(4 pi^2 (2 l)^4), and the interval tower n >= 0 gives half of this, -3 zeta(5)/(128 pi^2 l^4). In the normalization of lemma 24 this is -(3/16) zeta(5) per real degree of freedom. A half-integer tower has the factor -15/16 relative to an integer one, so a generator with mixed conditions contributes +3 (15/16) (3/16) zeta(5). For a fermion channel the sign is opposite, with four degrees of freedom per unit of n_f. The check: the shift to theta = pi/2 multiplies a tower by -15/16, so the difference is (31/16)(3/4) zeta(5) = 93 zeta(5)/64 = F(1), which is the value computed in Proposition 1.4. ■

In lemmas 23, 24 and 26 the gauge field is valued in sp(2) (+) su(4), with 25 generators. At theta = 0, sixteen have integer towers: 8 gluons, 3 W_L, 1 hypercharge combination with Neumann conditions at both walls, and the 4 odd generators O with Dirichlet conditions at both, whose interval components are the Higgs quaternion. Nine have mixed conditions and half-integer towers: 2 R^+-, 6 X and 1 orthogonal neutral direction. So n_i - (15/16) n_h = 121/16. For three generations, with 24 Dirac species, the bracket is 192 - 22.6875 = 169.3125 before gravity and bulk scalars, and A_0 = 32.92. So a_0 > 0: at v = 0 the length runs to infinity, in the direction of H, and the first half of the modulus target of the audit, a r^{-p} with a > 0 and p = 4, is present. For one generation the bracket is 41.3125.

Proposition 10.6 (when the gradient can be the second scale). Let V = l^-4 [G_0(theta) + (alpha'/l^2) G_1(theta) + …], theta = l m, with G_0(0) = a_0 and G_0 = a_0 + b theta^2 + … near 0.

1. If a_0 != 0, and the joint expansion in (theta, s), s = alpha'/l^2, with its derivatives is controlled near (0, 0), then at small m the leading balance is the scale-free one of Proposition 1.5: alpha = 1 or a runaway, and there is no stationary joint branch with theta, s -> 0. The term in alpha'/l^2 changes neither.
2. If a_0 = 0, G_1(0) = a_1 > 0 and b < 0, the truncated potential b m^2/l^2 + a_1 alpha'/l^6 has a minimum at l^4 = 3 a_1 alpha' / (|b| m^2), so alpha = 1/2: 1/l is the geometric mean of m and the string scale, up to constants, and m « 1/l « 1/l_s; with regular higher terms this is the leading asymptotic of the minimum, and in general not an exact equality.
3. If supersymmetry is exact at theta = 0, then V = 0 at theta = 0 at all orders, a_1 = 0, and V = m^2 l^-2 H(theta, alpha'/l^2). If b != 0 and the higher terms are regular, l is fixed by H independently of m, or runs away, and no joint family with theta, s -> 0 results, unless the stationarity condition has a root interval (a flat direction), as in the counterexample to the universal form of Proposition 1.5. If b = 0 the higher terms decide, and a joint family can result: for G_0 = theta^6 and G_1 = theta^2, V = m^6 l^2 + alpha' m^2 / l^4 is even in theta, vanishes at theta = 0 for every l, and has a strict minimum at l^6 = 2 alpha'/m^4, with l ~ m^{-2/3} and l m -> 0 (counterexample from the referee to the first version of this item, which stated the conclusion without the condition b != 0; it constructs no field theory producing this potential).

So in the scope of a potential with a regular joint expansion in theta and alpha'/l^2 around the flat family, with corpus content, where a_0 > 0, the gradient is not the second scale; the case a_0 = a_1 = b = 0 is outside the statement. Item 2 would need a Bose-Fermi balance at theta = 0 that is not a supersymmetry at theta = 0, with about 169 further real bosonic degrees of freedom for three generations.

Proof. Items 1 and 2: differentiate; in item 2, V = b m^2/l^2 + a_1 alpha'/l^6 has V'' = 8 |b| m^2/l^4 > 0 at the stationary point. Item 3: V is even in theta and vanishes at theta = 0 for every l, so V = theta^2 l^-4 H = m^2 l^-2 H. Stationarity is -2 H + theta H_theta - 2 s H_s = 0 with s = alpha'/l^2. With H(0, 0) = b != 0 and regular higher terms, the equation at small theta and s has no solution with both tending to zero, so its roots are angles independent of m, or there are none. With b = 0 the equation can balance powers of theta against powers of s, as the counterexample shows. ■

The flat family with the angle tied. For the top alone, n_f = 6: b is proportional to 54/5 - 6 > 0, and G > 0 on [0, pi/2] (Python: minimum 23.88, with A_0 = 32.92). So V has no stationary point with theta < pi/2, and l runs until theta leaves the fundamental domain of Proposition 7.1, where the top joins the tower; the periodically continued potential has its first stationary points far outside the window, near theta = 8.15 and 8.40, with further roots near 11.17, 11.71, 14.27, … (grid-and-root scan in Python, which proves no global count; periodicity of G with an interval where G' > 0 gives further roots for all large theta; referee). For a universal mass at three generations, n_f = 48, G changes sign and there is one stable stationary point, theta_* = 1.009, with M_W/m = 0.685. For a universal mass at one generation, n_f = 16, theta_* = 1.031, with M_W/m = 0.653 (Python). Top-only content therefore has no vacuum in the window in this scope, and adding a graviton does not change the classification. The stationary angles are of order one, so these vacua do not separate the window.

A localized channel. For a channel with bulk mass M = -mu and c = mu l, the spectrum of lemma 9 gives the one-loop term

```
8 pi^2 l^4 V_loc = -2 int_0^infinity x^3 log(1 + ((w + c)/(w - c)) exp(-2 w)) dx,    w = sqrt(x^2 +
↪ c^2),
```

from the secular function cosh(w) - (c/w) sinh(w). At c = 0 it is -(45/64) zeta(5) = -0.729, and for large c it is -4 c^3 exp(-2 c) up to corrections. Corrections at large c are of relative order 1/c. In between it has its unique minimum at c = 1 exactly, since differentiating the logarithm of the integrand gives d_c log Q = 2 (1 - c)/w (referee), of depth -1.3246 (Python, mpmath). This concerns the interaction term alone: the bulk term, -(4/15) c^5 in dimensional regularization, and the wall term are dropped as renormalizations of the bulk energy and wall tension, and with the bulk term included the minimum disappears. So the interaction

term of one fermion channel prefers c = 1, where the light pair merges with the tower (lemma 9.7). A large c_*, which is what a separated window needs, must come from other fields, such as the bosonic tower of the scalar that gives mu.

Other second scales

Remark 10.7 (amendment of Proposition 1.5, last sentence). On the scale-free path, m l is a fixed number. Proposition 1.5 says that on it the finite window never separates from the tower. That holds for theta_* of order one, which is what the flat family gives above. If m is exponentially small in a stabilized shape parameter, the ratio m/Delta is a small fixed number: a separated window at each radius, which is not the parametric hierarchy of Theorem 30.5, where the ratio tends to zero; and delta/R is a lower bound for the next level, not an established value of the gap. On the disc bundle of lemma 29, fix M and let R grow: m = lambda_M / R and Delta = delta / R both tend to zero, with lambda_M / delta exponentially small, and the count at fixed energy has seven-dimensional growth. So a potential R^-4 G(M), with M stabilized at a large M_* and G(M_*) > 0, *gives the hierarchy of lemma 30 with alpha = 1 and the fixed ratio lambda_{M_}*/delta, with no second scale; in lemma 30 the ratio tends to zero. The cost is that R then has no vacuum at finite value. The physical point at finite v needs one more term, and the stabilization at large M_* is a target, since the interaction term of one fermion channel prefers c = 1.

Remark 10.8 (a bulk energy). The ansatz V = a r^-p + b eps^2 r^q, with a > 0, has a minimum exactly when p + q > 0 and q b > 0, and then alpha = 2/(p + q), so alpha < 1 needs p + q > 2: for p = 4, q > -2 with q != 0. On the flat family the angle is a holonomy (Corollary 22.2), which no local bulk operator sees: at large l its effect is the Casimir term, q = -2, the borderline case alpha = 1. So a term falling more slowly than m^2/l^2 must come from elsewhere, and the natural place is a local bulk energy in the electroweak order parameter: for q = 1, V = a/l^4 + b v^2 M^3 l gives l^5 proportional to 1/v^2, so alpha = 2/5, if the bulk energy vanishes at v = 0. That requires the order parameter to be a bulk field with a bulk vacuum energy, and it departs from D_F as a pure boundary condition. Whether the spectral action provides this is open. In the heat cutoff, the dependence of the potential on D_F is exponentially small at large l Lambda, by Poisson summation over the tower, of relative size 4 exp(-(l Lambda)^2), so that action has no such term at this order. The factor (3 - d_eff)/2 of lemma 25 multiplies the term quadratic in the gauge field, a different term.

Corrections to the earlier files

- 09, Proposition 9.1 and its reading: subject to Proposition 10.1.
- 07, sections 4 and 6b: Proposition 7.2 stands as a statement on the circle; the reading "at distance proportional to a" in section 4 is corrected by Proposition 10.1, and the reading "the interval opens" at label 11 is qualified by Proposition 10.3, conditional on the string identification.
- 08, Remark 8.3 and its reading: the far-wall group is reached at R = R_*, with the tower at the string scale, by Propositions 10.2 and 10.4 and the section on the far wall. There is also a tension of labels: Proposition 8.1 pairs label 9 with O^+, and section 6b of 07 puts label 9 on a far O8^- wall. These are two different meanings of the label, the KO sign and the position of the top's brane.
- 06, section 2: "9, 10, 11" mixes KO labels with metric dimensions. In the string web, nine dimensions is finite R, and R -> 0 is T-dual to R -> infinity (with the Wilson-line transformation and the chosen dual frame: Polchinski and Witten qualify the simple inversion once a Wilson line is present), whereas in H the finite interior is labelled 10 and l -> 0 is labelled 9.
- 05, Proposition 5.4: the conclusion is correct on rerun (Python, exact integers, all 192 configurations). The sentence of its first version, "In every case S - N has d = 19 and d = 20 only, while S - T has d = 16 on the neutrino roots of one slot and d = 22 on those of the two others", since removed from the Statement of lemma 34, was wrong: S - N has only these values in 48 of the 192 configurations, and the other 144 show {13, 17, 19, 20}; the statement on S - T fails in 96 configurations, with the pattern 16, 16, 22. The values of d differ in every configuration, and that is what the proof needs.

Scope and escapes

Proposition 10.6 assumes a potential with a regular expansion in alpha'/l^2, around the flat family with its towers. It concerns the dilaton *gradient*. The following lie outside it: a dilaton potential from string tree level, in non-supersymmetric strings or with tadpoles, for which a term in positive powers of 1/g could grow with l at fixed v through the law of Proposition 10.4; a different frame relation between l and the four-dimensional Planck mass; wall-localized dynamics; and the other chambers of the moduli space, where 8 + 8 removes the Romans mass and E8 x E8 is available. The string-theory statements are read from the two sources for type I' and recalled for the moduli-space limits; only the first have been checked against the text. By AGENTS.md, a negative result needs an independent check by another model. This one has had a reading by Claude Sonnet 5.5, which found the generator count above, and a referee reading by GPT-6.1 Terra (5 November 2026), which found the error in item 3 of Proposition 10.6 and the hypotheses stated above; after the applied corrections, Propositions 10.5 and 10.6 have two independent readings in their stated scope, and the string-theory propositions 10.1 to 10.4 have one referee who reread the sources. The referee's A_0 for three generations is 32.9184, with the positivity and the runaway direction depending on the displayed field content and on the tuning of the bulk energy and the wall tensions.

Part 38

Inner fluctuations of the interval parent with an SO(30) algebra

Lemma 38. Written as file 11 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line (README, PLAN). Attempted by Claude Opus 5.5 on 4 November 2026, to answer the question left in 09: whether the gauge field along the interval of lemma 22 carries the antisymmetric block with the counts of lemma 21, given that Proposition 9.3 excludes it from the finite triple alone. Numbers in Python (numpy), scripts not kept by the rule of the project. The Codex quota is spent, so no Terra referee is available. An independent reading by Claude Sonnet 5.5, whose amendments are applied below, is in the review. Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report): the block calculations, the commutant, the one-forms and the compression formula correct; three corrections applied below, all on one point, that the representation pi is nonunital on the doubled space, so that the gauge action of Theorem 11.3 and the realization of the shadow of Proposition 11.4 as an inner fluctuation at nonzero mass need a unital completion, which is now stated as a hypothesis.

Outcome

1. **Yes, under a condition on the walls.** With a real algebra M_30(R) on the light block, the fluctuation along the interval is an so(30) connection, and its block between particles and antiparticles lies in Λ^2 V, with the counts (6, 6, 3) in the colour antitriplet (Proposition 11.2). Massless scalars in exactly Λ^2 V + conjugate appear when both walls reduce the algebra to M_15(C), the commutant of quark number, under the Yang-Mills action with its natural boundary conditions and a gauge action supplied by a unital completion of pi (Theorem 11.3): 210 real scalar and 225 real vector components at tree level.
2. **In lemma 22's own parent the massless scalars are absent.** Its algebra is free at x = 0, so the interval component has no zero mode, for any algebra (Theorem 11.3, item 2).
3. **A generator has a massless vector or a massless scalar, never both, on the interval**, in free five-dimensional Yang-Mills at trivial background. So the whole of Λ^2(30) of lemma 21 cannot be massless scalars with colour unbroken. The antisquark counts appear as scalars, while the slepton part (24, 1) + (1, 1) appears as vectors (Theorem 11.3, item 4), in the two-U(15)-wall model at trivial background; these are tree-level field counts, not a quantum massless spectrum.

4. **The finite window sees a symmetric shadow.** The connection acts on the window, by compression, through its commutator with the masses, B_ij -> (m_i - m_j) B_ij, which maps the (S^2 5, 3bar) of Λ^2 V onto the (Λ^2 5, 3bar) of S^2 V, with counts (3, 6, 1) (Proposition 11.4). This reconciles Propositions 9.2 and 9.3. The kernel is the same-flavour diquarks. The compression is an operator identity; its realization as an inner fluctuation of the algebra at nonzero mass needs a unital completion, since for invertible M every element of the corner algebra of lemma 22 vanishes at x = l (referee).
5. **SO(30) has a precise meaning in the finite geometry:** it is the commutant of the real structure j = J U of lemma 06 on the massless light block, for the sign s = +1, and USp(30) for s = -1 (Proposition 11.1).

Setting

Let V = C^5 ⊗ C^3, with Q = 2/3 on u, c and Q = -1/3 on d, s, b. In the finite Hilbert space of the five light quarks, with colour, the chirality gamma = +1 subspace is

```
H_+ = {q_L} ⊕ {conj(q_R)} = V ⊕ Vbar,
```

of dimension 30: the left-handed Weyl fields of fifteen Dirac fermions. H_- = {q_R} ⊕ {conj(q_L)}, and J_F exchanges H_+ and H_-, with c = J_F restricted to H_+, as in lemma 22. Let U be a unitary, grading-reversing and J-related as in lemma 06, with U q_L = q_R and U conj(q_R) = s conj(q_L), so U J_F = s J_F U. Then j = U^-1 c is an antiunitary on H_+, with j^2 = s, j q_L = s conj(q_R) and j conj(q_R) = q_L. Here U only defines j: the representation pi below does not commute with U, so this is not a doubling in the sense of lemma 06, in which U commutes with the algebra.

The interval parent is that of lemma 22: D_X = E d/dx on L^2([0, l]; C^2) ⊗ (H_+ ⊕ H_-), with Q in H_+, U-components in H_-, and the wall conditions Q_R(0) = U_L(0) = 0, Q_R(l) = X U_R(l), U_L(l) = -X^* Q_L(l).

The block

Proposition 11.1 (the commutant of j). Let s = +1. In the basis V ⊕ Vbar with j(v, w) = (K^-1 w, K v), the skew-Hermitian a with j a j^-1 = a form so(30), and

```
a = [[alpha, B], [conj(B), conj(alpha)]],    alpha^* = -alpha,   B^T = -B,
```

so, as real vector spaces, so(30) = u(15) ⊕ (Λ^2 V)_R, where u(15) commutes with the complex structure I = (i, -i), the quark number, and the second summand has complex dimension 105 and real dimension 210, with 435 = 225 + 210; the decomposition is not a direct sum of Lie algebras (referee). The Hermitian generator H = i a satisfies j H j^-1 = -H. For s = -1 the same computation gives usp(30) = u(15) ⊕ (S^2 V)_R, of real dimension 225 + 240 = 465.

Proof. Linear algebra, with Proposition 8.1 for the groups. For s = +1, j a j^-1 = a gives the block form, and a^* = -a gives alpha^* = -alpha and B^* = -conj(B), that is B^T = -B. Then j (i a) j^-1 = -i a. ■

So the SO(30) of lemma 21 is the symmetry of the massless block for the sign s = +1, the sign that Proposition 8.1 and STRATEGY T2c associate with label 11 and with the seesaw; and the counts of lemma 21 select the same sign:

	colour antitriplet	its charges (-2/3, +1/3, +4/3)	colour sextet
s = +1, Λ^2 V	(S^2 5, 3bar)	(6, 6, 3)	(Λ^2 5, 6)
s = -1, S^2 V	(Λ^2 5, 3bar)	(3, 6, 1)	(S^2 5, 6)

Fluctuations along the interval

Take on the light block D_F = 0, so X = 0, and let C_0, C_l be subalgebras of A = M_30(R), the real elements of End H_+. Let

```
A_1 = {f in C^infinity([0, 1], A) : f(0) in C_0, f(1) in C_1},    pi(f) = f on Q, 0 on the H_-
↪ components,
```

a nonunital representation on the doubled space: pi(1) is the projection onto H_+ (referee). A unital completion is pi~(f, r) = f (+) r 1 with a real scalar summand; it is used where a gauge action is needed below.

Proposition 11.2. Assume 1 lies in C_0 and C_1.

1. pi(f) preserves the domain, for every choice of C_0, C_1. Order zero and order one hold on this block.
2. The self-adjoint one-forms sum pi(f_i) [D_X, pi(g_i)] are exactly E a_x on the Q components, with a_x in C^infinity([0, 1], so(30)), with no condition at the walls. The fluctuated operator is

   ```
   D_A = E (d/dx + a_x ⊕ U a_x U^-1).
   ```

3. J-hat A J-hat^-1 = E (U a_x U^-1 on H_-), with sign +, as KO-dimension six requires. The block of a_x between Vbar and V is antisymmetric. The two statements agree since E is antisymmetric: the block of the Hermitian operator E a on C^2 ⊗ E is symmetric, as Proposition 9.3, item 2 requires, with B antisymmetric and the factor E carrying the other sign. So the interval component evades Proposition 9.3 through the escape in its item 3.

For D_F != 0 on this block, order one fails in the finite window: [[D_F, pi(a)], J pi(b) J^-1] has the blocks -U b U^-1 M a from H_+ to H_-, and -a M^* U b U^-1 from H_- to H_+, which vanish for all a, b only if M = 0. The corner representation still satisfies order one by the direct pointwise calculation; for invertible M its endpoint matrix corner is zero. Theorem 22.3 assumes a unital represented algebra, so a unital completion must be specified separately: for pi~(f, r) = f (+) r 1 and invertible M the endpoint condition is f(1) = r(1) 1, and the finite order-one failure remains, since with the original corner action a = b = 1 gives the off-diagonal blocks -M^* and -M (referee). So this algebra describes the massless block, and the masses need a treatment without order one, as in lemma 19.

Proof. (1) The conditions Q_R = 0 and U_L = 0 at the walls are preserved by any operator acting on the H_+ index. [D_X, pi(g)] = E g' on Q, and J-hat pi(b) J-hat^-1 acts on the H_- components only, so both commutators vanish. (2) Given T, choose tau with support inside (0, 1) and the same integral as T, which is a finite sum sum f_i g_i' of functions supported inside the interval (take a compactly supported scalar g with g' = 1 on the support of tau). Take f = 1 and g = int_0^x (T - tau), which vanishes at both walls, so g lies in A_1 and g' = T - tau. Self-adjointness of E T, with E skew, requires T skew. (3) J-hat E = E J-hat (direct from the formula for J-hat in Theorem 22.1), and c a c^-1 = U j a j^-1 U^-1 = U a U^-1. ■

Theorem 11.3 (zero modes of the interval component). Let k_0, k_1 be the skew parts of C_0, C_1, and assume a five-dimensional Yang-Mills action for the fluctuation on a flat product with M^4, as in Proposition 22.8, with the usual gauge action, which the corner representation pi does not supply by itself (pi(u) J-hat pi(u) J-hat^{-1} = 0) and a unital completion such as pi~ does, and with the Yang-Mills boundary variational problem at trivial background, unrestricted allowed endpoint variations and no wall terms (referee).

1. The gauge group is {g : I -> O(30), g(0) in K_0, g(1) in K_1}, with det g(0) = det g(1); if K_0 or K_1 contains reflections it does not act separately on SO(30). The holonomy of a_x transforms as P -> g(1) P g(0)^-1, so the moduli are the double coset K_1 SO(30) / K_0, and the massless four-dimensional scalars from a_x, before the quotient by the residual four-dimensional gauge transformations, form

   ```
   the orthogonal complement of k_0 + k_1 in so(30),
   ```

   while the vector zero modes form k_0 ∩ k_1. The action supplies the quadratic form and the natural boundary conditions; order one alone supplies no scalar kinetic term, and the complementary Dirichlet and Neumann conditions for A_x need a compatible gauge fixing.
2. In lemma 22's algebra, C_0 = A, so k_0 = so(30), and there are no such scalars, for any algebra of this kind.

3. If C_0 = C_l = M_15(C), the massless scalars are exactly Λ^2 V ⊕ Λ^2 Vbar, 210 real components with colour-antitriplet counts (6, 6, 3), and the massless vectors are u(15), 225 real components.

4. A generator in k_0 ∩ k_l has a massless vector and no massless scalar; a generator in the orthogonal complement of k_0 + k_l has a massless scalar and no massless vector; all others have neither. If the orthogonal projections onto k_0 and k_l commute (aligned walls), the towers are n pi/l in the first two cases, and (n - 1/2) pi/l for a generator in exactly one of k_0, k_l. For nonaligned walls the towers move, and the zero-mode statements are to be read as the subspaces above: k_0 spanned by T_1 and k_l by cos(phi) T_1 + sin(phi) T_2 give k l = n pi +- phi, which is the mechanism of lemma 23; a nontrivial holonomy can change the zero-mode spaces.

5. With colour unbroken at both walls, the (1, 8) of Λ^2(30) is a vector, so Λ^2(30) cannot consist entirely of massless scalars. In item 3, the (24, 8) + (24, 1) + (1, 8) + (1, 1) of lemma 21 are vectors, and the scalars are (15, 3bar) + (10, 6) + conjugates.

Proof. (1) A gauge parameter xi(x) with xi(0) in k_0, xi(l) in k_l shifts int a_x by xi(l) - xi(0), so the linear moduli are so(30)/(k_0 + k_l), and a constant a_x in the orthogonal complement is a zero mode of the quadratic form. (2), (3): substitution, with Proposition 11.1. (4) A_mu has Neumann or Dirichlet conditions at each wall according to membership in k_wall, and a_x the opposite, so the mode functions are cos(n pi x/l), sin(n pi x/l) or their half-integer versions. (5) Item 4, and the branching of lemma 21. ■

Scope of item 4. Free five-dimensional Yang-Mills at trivial background, with the gauge action and boundary problem stated in the hypotheses; the counts are tree-level field counts and not a quantum massless spectrum. Escapes: a circle without walls, nontrivial holonomy, brane-localized terms, further internal dimensions.

Note that the choice in item 3 is an input: the domain does not force the wall algebra to M_15(C), since by Proposition 11.2, item 1, every f preserves it at D_F = 0. It says that the walls conserve quark number and the bulk does not. On a circle, without walls, every generator has both a massless vector and a massless scalar, and the whole of Λ^2(30) appears, which is the nine-dimensional adjoint scalar used in 09. A wall condition, or the reflection-even algebra of Proposition 1.1, removes half. So the two readings differ: for the stack off the O8 plane, with holonomy I, diquarks are massive and mesons have massless scalars; on the interval with parity Ad(I), diquarks have massless scalars and mesons have none.

The shadow in the finite window

Proposition 11.4. Let s = +1, M = U Lambda with Lambda real for j and symmetric, and let a_tot = a ⊕ U a U^-1 be a constant bounded perturbation, with the window condition of Theorem 22.1. Then:

1. [D_F, a_tot] has the off-diagonal blocks U S and S U^-1, with S = [Lambda, a] real and symmetric, and [D_F, [D_F, a_tot]] = [Lambda, S] ⊕ U [Lambda, S] U^-1.

2. By Theorem 22.4, the compression of E a_tot to the window is -l^-1 int_0^l sin(x ad_{D_F})(a_tot) dx. For Lambda = diag(m_i) on flavours, on both V and Vbar, the block B_ij goes to

   ```
   - ((1 - cos(l (m_i - m_j))) / (l (m_i - m_j))) B_ij,    which is -(l/2)(m_i - m_j) B_ij to
   ↪ leading order, with the continuous value zero at equal masses,
   ```

   as a change of Lambda. The result is symmetric, as Proposition 9.3 requires of a finite J-compatible fluctuation.

3. For distinct masses, B -> (m_i - m_j) B_ij maps (S^2 5, 3bar) in Λ^2 V onto (Λ^2 5, 3bar) in S^2 V as vector spaces, with kernel the five same-flavour diquarks (five colour-antitriplet multiplets), and maps (Λ^2 5, 6) into (S^2 5, 6); distinct fixed masses break flavour SU(5), so the map is SU(5)-equivariant between the named representations only with the mass matrix read as a spurion (referee). In charges, (6, 6, 3) - (3, 0, 2) = (3, 6, 1). If the wall condition is imposed on the gauge unitaries g ⊕ U g U^-1, so that k_l is the commutant of Lambda in so(30), then with k_0 = u(15) the massless scalars of Theorem 11.3 are the same off-diagonal blocks. If it is imposed on pi(f) itself, as in the algebra of lemma 22, then k_l = 0 for invertible M.

4. For a universal mass, Lambda = m, the compression vanishes: S = 0, and M = m U is invariant under SO(30). Fifteen Dirac fermions of equal mass are thirty Majorana fermions of equal mass.

Proof. (1) Direct multiplication, using a_tot on H_- equal to U a U^-1. S is real and symmetric since a is real and antisymmetric and Lambda real and symmetric. (2) Item 1 gives sin(x ad)(a_tot) = (sin(x delta)/delta) ad(a_tot) on B_ij, with delta = m_i - m_j; integrate. (3) In the colour antitriplet, B is symmetric in flavour, and (m_i - m_j) B_ij is antisymmetric in flavour, hence total symmetric. Colour sextets are analogous. (4) S = 0. ■ Checked in Python on a small model, with V = C^3: reality, symmetry, the blocks of item 1 and their squares. Claude Sonnet 5.5 also checked the compression by direct quadrature of W^*(E a_tot)W, with five flavours in three colours, against Theorem 22.4 and the closed block (Python, 10^-15).

Reading. The commutator [D_F, a] is the form of the Higgs field in Connes' finite geometry, and E a_x is a Kaluza-Klein component along the interval. Theorem 22.4 (3), Q_l([D_X, f]) = [D_F, Q_l(f)], says that the second compresses to the first. Indeed E a_tot = [D_X, f_tot] with f(x) = (x - l) a and f(l) = 0. For invertible M, however, every element of the corner algebra of lemma 22 vanishes at l, hence so does every coefficient sum f_i g_i' of one of its one-forms, so a nonzero constant a_tot is not an inner one-form of that algebra, and f_tot is not one of its elements; the compression identity stands as an operator calculation, and its realization as an inner fluctuation at nonzero mass needs a unital completion and a construction of the real one-form there (referee). With that construction, in this parent, to first order and for the fixed compression, a finite Higgs one-form is the shadow of a gauge field along the interval, and it inherits the symmetry type of the finite triple by the commutator with masses. The moving window of Corollary 22.6 gives -l [D_F, h] and a quadratic term (lemma 19). This shadow is tangent to the O(30) orbit of Lambda, so it gives no radial mode, in agreement with No-go theorem 22.7; the odd generators of lemma 23 are absent here since M_30(R) maps H_+ to H_+. A generator commuting with the masses has no shadow. This explains the difference between the counts (6, 6, 3) of the parent and (3, 6, 1) of the finite triple, and it replaces the alternative O8^- against O8^+ of Proposition 9.2, for these blocks, by a parent and its window.

What remains open

- Whether a boundary condition, for example a wall term that conserves quark number, can derive M_15(C) at the walls, where at present it is chosen.
- The unital completion of pi and the gauge action it supplies, used as hypotheses in Theorem 11.3, and the construction of the nonzero-mass one-form whose compression is the shadow of Proposition 11.4 (referee).
- The counts of Theorem 11.3 are tree-level field counts in the two-U(15)-wall model at trivial background; other wall data, holonomy or further directions need separate calculations.
- The treatment of nonzero masses, where order one fails on the block: the quadratic terms of lemma 19 for this algebra.
- The sleptons: on an interval the (24, 1) + (1, 1) are vectors. Scalars in these representations need components along further internal directions, such as the base of 02.
- The colour sextets (10, 6) come with the antitriplets, and nothing removes them in this construction.
- The relation to the top: the (30, 2) of SO(32), with the second factor as the top's SO(2).

Part 41

The axioms and the lift along the chain Pati-Salam, Standard Model, colour and electromagnetism

For STRATEGY T17, following lemma 06 (the representation of Result 5), lemma 17 (Corollary 17.4), lemma 20 (Theorem 20.2), lemma 21 and results/orientability_under_heavy_centralizer.md. Written by Claude Fable 5.1 on 4 November 2026. The three matrices, the orientation cycle and the traces were computed by hand and confirmed with numpy on the 32-state representation (one generation, Dirac neutrinos; script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): Lemma 41.1 correct; Theorem 41.2 and Proposition

41.4 correct with amendments; Remark 41.3 and Proposition 41.5 had errors, corrected below as the referee stated them (the generator of B, the state counts, the word "vectorlike", the e_R obstruction); the referee recomputed every matrix and trace.

Outcome. Along the chain of heavy operators, A_PS = H_R + H_L + M_4(C) -> A_SM = C + H + M_3(C) -> B = C + M_3(C), the finite data keep the KO-6 signs, order zero and the first-order condition of Theorem 20.2, and they lose the other two axioms of a finite real spectral triple: the trace-form Poincaré duality fails at every step, with intersection forms of rank 2, 2 and 0 (the last because every range rho(p_i) rho(p_j)° in the light block has graded dimension zero, the bimodule balance behind the KO-7 doubling of lemma 06), and orientability holds at the Pati-Salam step, for a sheet-diagonal D of the form of Corollary 16.8, and fails at the two later ones, through the pair nu_R, J nu_R and the pair e_R, J e_R at the Standard Model step and through the central mass blocks at the third. The unimodularity trace of the abelian generator changes along the chain, (4i, 12i) on the full space, (2i, 12i) with the Majorana pair removed, (-2i, 6i) with the up-type sector removed and (-4i, 6i) on the light block, so the inherited lift and a lift reimposed later impose different hypercharge constraints, and only the inherited one gives the photon its charges.

Setting

One generation with Dirac neutrinos, in the convention of lemma 06, Result 5: F = C + C^3 (lepton line and colour), H_p = (C^2 (x) F)_L + (C^2 (x) F)_R, H = H_p + H_pbar, dim H = 32. For a = (lambda, q, m) in A_SM, the left action is q (x) I_4 on the left-handed particles, Q_lambda (x) I_4 = diag(lambda, conj lambda) (x) I_4 on the right-handed ones, and I_2 (x) B_{lambda, m} = I_2 (x) diag(lambda, m) on the antiparticles; J(p, pbar) = (K pbar, K p); gamma = +1 on p_L and pbar_R, -1 on p_R and pbar_L. The opposite action is a° = J a* J^{-1}. The Pati-Salam parent acts by q_L (x) I_4 on p_L, q_R (x) I_4 on p_R and I_2 (x) m_4 on the antiparticles, and A_SM is its centralizer under the heavy Majorana block (Theorem 20.2, rank one, kappa != 0). The light block of the third step is the 18-dimensional space of d_L, d_R (three colours), e_L, e_R, nu_L and their conjugates, on which B = {(lambda, Q_lambda, m)} acts (lemma 06, Proposition 5), with the kept mass blocks commuting with B (lemma 17, Corollary 17.4).

Minimal projections, one per summand: p_R' = (1, 0, 0), p_L' = (0, 1, 0), p_4 = (0, 0, e_11) for A_PS; p_1 = (1, 0, 0), p_2 = (0, 1_H, 0), p_3 = (0, 0, e_11) for A_SM; for B the minimal projection of its summand C is (1, Q_1, 0) = (1, 1_H, 0) = p_1 + p_2, which is the identity on the nine light particles and on the light antileptons, and p_3 is that of M_3(C). The trace form is Q_ij = Tr(gamma rho(p_i) rho(p_j)°).

Poincaré duality

Lemma 41.1 (the trace form is skew). For a finite real triple with J^2 = +-1, J gamma = -gamma J, an algebra even for gamma and order zero, Q_ij = -Q_ji. *Proof.* For an antiunitary J, Tr(J A J^{-1}) = conj(Tr A). Then Q_ij = conj Tr(J gamma p_i p_j° J^{-1}) = -conj Tr(gamma p_i° p_j) = -conj Tr(gamma p_j p_i°) = -conj(Q_ji) = -Q_ji, using order zero to commute p_i° past p_j and the reality of Q. ■ (This is the observation of Ćaćić for KO-dimension 6; the corpus recorded it as rs_N7.)

Theorem 41.2 (the three forms). In the bases above,

```
Q_PS = [[0, 0, -2], [0, 0, 2], [2, -2, 0]],   Q_SM = [[0, -2, -2], [2, 0, 2], [2, -2, 0]],   Q_B =
↪  [[0, 0], [0, 0]].
```

Each has even rank: Q_PS and Q_SM have rank 2 with kernels spanned by p_R' + p_L' and by p_1 + p_2 - p_3, and Q_B vanishes. So the trace-form Poincaré duality fails at each step of the chain: at the first two because a skew form on an odd-rank K_0 is singular, at the third because every range rho(p_i) rho(p_j)° in the light block has graded dimension zero.

Proof. Each entry counts, with the sign of gamma, the states on which the left projection p_i and the right projection p_j° both act as the identity. For Q_SM: p_1 acts as the identity on all of p_R (eight states) and on the lepton line of pbar_L and pbar_R; p_2 on p_L; p_3 on one colour of pbar_L and pbar_R; and p_1°, p_2°, p_3° are the J-transports, so p_1° is the identity on pbar_R and on the lepton line of p_L,

p_R, p_2° on pbar_L, p_3° on one colour of p_L, p_R. Then Q_12 counts the lepton line of pbar_L, two states of grading -1; Q_13 one colour of p_R, two states of grading -1; Q_23 one colour of p_L, two states of grading +1; the diagonal entries vanish by Lemma 41.1 (directly: Q_11 counts the lepton line of p_R and of pbar_R with opposite signs). Q_PS follows in the same way with p_R' the identity on p_R, p_L' on p_L and p_4 on one line of F on the antiparticles: Q_{R L} counts p_R against pbar_L, which is empty, and Q_{R 4}, Q_{L 4} count one line of p_R and of p_L. For Q_B, on the light block the projection (1, 1_H, 0) of the summand C is the identity on the nine light particles and on the three light antileptons, p_3 on one colour of dbar_L, dbar_R, and p_3° on one colour of d_L, d_R; Q_13 counts d_L and d_R of one colour, of opposite gradings, so it vanishes, and Q_11 vanishes by Lemma 41.1 (on the particles its range is nu_L, e_L, e_R with graded count +1, on the antiparticles the conjugates with -1). With the projection (1, 0, 0), which lies outside B, the form would be [[0, -1], [1, 0]]. The kernels are read off. ■

Remark 41.3 (the form is the bimodule balance). Q_ij is the graded dimension of the range of rho(p_i) rho(p_j)°, one colour of an isotypic component when p_j = e_11. It vanishes identically exactly when every such range has equal numbers of states of the two gradings, which is the bimodule balance condition behind the doubling of lemma 06 and is weaker than physical vectorlikeness: the light block of the third step keeps the unpaired nu_L, whose particle content restricted to SU(3) x U(1)_em has [V_L] - [V_R] = (1)_0, nonzero, while Q_B = 0, because the trace form sees the full real space, where the neutral class cancels against its conjugate and the characters lambda and conj(lambda) of the real summand C share one projection (referee; lemma 06 draws the same distinction in its no-go under [U, Q] = 0). On the full 32-state space with nu_R, where B acts and the Standard Model index restricts to zero, the form of B also vanishes, and there the two statements agree. At the first two steps the form has rank 2 because the quark doublets and the leptons carry the chiral asymmetry; the kernel vector p_1 + p_2 - p_3 of Q_SM is the direction along which the graded counts of the three summands cancel.

Orientability

Proposition 41.4. (i) The Pati-Salam data are orientable: gamma = rho(p_L' - p_R') rho(1_4)° - rho(1_4) rho(p_L' - p_R')°. (ii) The Standard Model data are not, with or without a right-handed neutrino: nu_R and J nu_R carry the same left character lambda and the same right character lambda and have opposite gradings, so no sum of left-right products separates them (Stephan 2006, hep-th/0610097, for the KO-6 models with neutrinos); and e_R carries (conj lambda, lambda) with gamma = -1 while J e_R carries (lambda, conj lambda) with gamma = +1, on which a real cycle takes complex-conjugate values and cannot be -1 on one and +1 on the other, an obstruction that persists when the nu_R pair is removed (referee, checked numerically: removing the nu_R and e_R pairs restores orientability). (iii) The light block of the third step is not: the kept mass blocks are nonzero and commute with B (lemma 17, Corollary 17.4), and a nonzero D commuting with both actions commutes with every orientation cycle while anticommuting with gamma; directly, d_L and d_R of one colour carry the same characters (conj lambda on the left, e_11 on the right) and opposite gradings.

Proof. (i) On the particles rho(1_4)° is the identity and rho(1_4) vanishes, so the right side is p_L' - p_R', which is +1 on p_L and -1 on p_R; on the antiparticles rho(p_L' - p_R') vanishes and rho(1_4) is the identity, so the right side is -(p_L' - p_R')°, which is -1 on pbar_L and +1 on pbar_R. This is gamma. (ii) and (iii): a cycle c = sum rho(a_i) rho(b_i)° takes the same value on two states with the same pair of characters, while gamma takes opposite values. For the first sentence of (iii), if [D, rho(a)] = 0 = [D, rho(b)°] for all a, b then [D, c] = 0 for every cycle, and c = gamma would give D gamma = gamma D, which with D gamma = -gamma D forces D = 0; this is the corpus statement rs_N9 (historical/corpus/connes_kk_research_state.md, section 5.4), restated here with its proof. ■

The explicit model of results/orientability_under_heavy_centralizer.md shows the same loss under a heavy centralizer in a four-summand toy; here it happens in the physical chain, at the first heavy step when nu_R is present and at the second step in every case.

Unimodularity

Proposition 41.5. Let X_lambda be the generator of the abelian direction of the summand C, acting in each algebra as its own element: in A_SM the element (i, 0, 0), which acts as diag(i, -i) (x) I_4 on p_R and as i on the lepton line of the antiparticles; in B the element (i, Q_i, 0), which in addition acts as diag(i, -i) (x) I_4 on p_L, that is by i on nu_L and u_L and by -i on e_L and d_L. Let X_c be the common colour phase, i on the three colours of the antiparticles. The traces of the left action are

```
full H, 32 states, A_SM:                          Tr X_lambda = 4i,   Tr X_c = 12i;
Majorana pair removed, 30 states, A_SM:            2i,                12i;
up-type sector removed, nu_R kept, 20 states, B:  -2i,                 6i;
light block of the third step, 18 states, B:      -4i,                 6i.
```

So the unimodularity condition Tr A = 0 of Chamseddine-Connes-Marcolli on the one-form (the left action; the adjoint action u J u J^{-1} has determinant one for every unitary, so a condition on it would be empty), imposed on the parent, gives theta_lambda + 3 theta_c = 0 between the two abelian phases, the infinitesimal form of lambda det m = 1, and reproduces lambda^2 (det m)^4 at the 30-state step recorded by the corpus; the same condition reimposed on the light block gives -4 theta_lambda + 6 theta_c = 0, a different embedding of the surviving abelian generator. Under the adjoint action, normalized so that the electron has charge -1, the inherited condition gives the charges (2/3, -1/3, -1, 0) to (u, d, e, nu) and the reimposed one gives (1/6, -5/6, -1, 0): only the inherited lift gives the photon (referee). This is the condition Proposition 22.8 of lemma 22 uses, where the ten generators of u(C_F) reduce to the nine of the photon and the gluons.

Proof. Count the states with their phases. Full space: Q_i is traceless on p_R and X_lambda vanishes on p_L, so only the antilepton line contributes, two weak components in two sectors, 4i; X_c has 12i on the antiquarks. Removing nu_R (phase +i) and J nu_R (antilepton line, +i) gives 2i. For B, (i, Q_i, 0) is traceless on p_L and on p_R by itself; removing the up-type sector removes u_L and u_R, three colours each with phase +i, and their conjugates, which carry no abelian phase, so the particle part gives -3i on p_L (nu_L at +i, d_L at -3i, e_L at -i) and -3i on p_R, and the antilepton line keeps 4i: -2i on 20 states; the antiquarks halve to 6i. Removing also nu_R and J nu_R takes i twice: -4i on 18 states. ■

The first version of this proposition used the element (i, 0, 0) on the spaces where only B acts and counted 26 states where there are 20; the referee corrected both (the element (i, 0, 0) gives the traces 4i, 2i, i, -i on the four spaces, the numbers of the first version).

Scope

One generation with Dirac neutrinos in the stated representation; the trace form on minimal projections as the test of Poincaré duality (a real Kasparov duality is a different question, as the corpus noted); the strict projections of lemma 06 for the third step; at the Pati-Salam step a sheet-diagonal D of the form of Corollary 16.8 (a Pati-Salam-symmetric Yukawa with no Majorana block), since with the Majorana block order one fails there, which is how Theorem 20.2 selects A_SM. With three generations the matrices multiply by three and nothing else changes. The direction the result points to: the finite triple of the main paper carries, along the chain, exactly the data that lemma 22 derives from the interval (signs, order zero, first order), and none that it does not; the two failed axioms are the ones a Kaluza-Klein parent would have to supply through its topology (orientability, Poincaré duality for the parent manifold), and the vanishing of the form at the third step is the bimodule balance of lemma 06, which coexists with the unpaired nu_L.

Part 42

The Higgs representation in the window: the tangent of the orbit of the marked lines

For STRATEGY T14, item (ii) of the corpus reading (CORPUS_SALVAGE item 13), following lemma 28 (the group data G = Sp(2), K = Sp(1)_a x Sp(1)_b, H = Sp(1)_a x U(1)_b, and Proposition 28.19),

lemma 29 (the eight-state window) and lemma 20 (Theorem 20.2 and item 2 of its readings: the connector as a marked pair of points and a coupling). Written by Claude Fable 5.1 on 4 November 2026, by hand; the representation theory is standard and is checked against the isotropy representation stated in lemma 28. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): Proposition 42.2 correct (stabilizer dimensions and ad-eigenvalues recomputed), Proposition 42.1 correct with its escape added, Proposition 42.3 had an error in the count of the connector moduli, corrected below as the referee stated it.

Outcome. The window operator of lemma 29 carries no Higgs representation by itself: Sp(2)-equivariance makes its off-diagonal part a scalar (Proposition 28.19). The Higgs representation of the first-stage corpus note (section 6: T_p HP^1 = Hom_H(L, L^perp), the (2, 2) of Spin(4), one doublet of hypercharge 1/2 after marking a U(1)) is the tangent space of the orbit of the marked line that the connector of lemma 20 puts on the quaternionic sheet. That orbit is CP^3, and its tangent at the marked line is the isotropy representation (2)_{+-1} + (1)_{+-2} of lemma 28: a doublet of the weak Sp(1)_a with U(1)_b charge +-1, which is the Higgs doublet, and a singlet of charge +-2 along the twistor fibre, which is an SU(2)-singlet scalar direction of hypercharge 1. The moduli of the connector under Sp(2) are the Hermitian and the symplectic overlaps of the two marked lines, the modulus of the coupling, and the phase of kappa alpha^2 when the Hermitian overlap alpha is nonzero.

Setting

The window of lemma 29 is W + V, two copies of the 4 of G = Sp(2): the core copy W, on which the real functions of S^4 compress to M_2(H), and the wall copy V, on which the real functions of CP^3 compress to M_4(C) (Example 16.9, Theorem 28.21). The connector of Theorem 20.2 is the rank-one block T_kappa x = kappa K x K with K = v w*, that is, a complex line Cw in W, a complex line Cv in V and the coupling kappa; the centralizer of T_kappa is C + H + M_3(C) (lemma 20, Result 2). In the notation of lemma 28, K = Sp(1)_a x Sp(1)_b is the stabilizer of a quaternionic line L in W = H^2, with Sp(1)_b acting on L and Sp(1)_a on L^perp, and H = Sp(1)_a x U(1)_b is the stabilizer of a complex line l in L, so that G/K = S^4, G/H = CP^3 and K/H = S^2 is the twistor fibre. Charges of U(1)_b are normalized so that the doublet of Sp(1)_b has charges +-1. In the flag algebra of the corpus note (section 5), marking l reduces H_R = Sp(1)_b to C and leaves H_L = Sp(1)_a, the weak isospin, on L^perp.

Statements

Proposition 42.1 (no Higgs representation in the equivariant window). If the window consists of one copy of the 4 in each degree of the end grading, then after equivariant identifications d_M = [[0, conj(lambda_M) I_4], [lambda_M I_4, 0]], and a phase makes lambda_M >= 0 (Proposition 28.19, by Schur's lemma). So the off-diagonal part of the window operator is a multiple of the identity, invariant under G, and it transforms under the isotropy group of any marked point trivially. The question of item (ii) of CORPUS_SALVAGE, whether that part transforms as T_p S^4, has the answer no for the equivariant window, for this reason. The escape (referee): as a representation of G, Hom(W, V) = 4 (x) 4 = 1 + 5 + 10 contains T_p S^4 = (2, 2) twice after restriction to the isotropy group of p (once in the 5, once in the 10), so a G-breaking off-diagonal block at p, such as the abstract d = mu n^a Gamma_a of Example 16.9 with n orthogonal to the axis, carries the representation of item (ii); the answer no is a statement about the G-invariant block, and the result points to the G-breaking blocks, which are the variation of the marked data described below.

Proposition 42.2 (the tangent of the marked line). Let w be a unit vector of W = H^2 = C^4, l = Cw its complex line, L = l + jl its quaternionic line, and p the point of S^4 = HP^1 under L. The G-orbit of l is CP^3 = G/H and the G-orbit of L is S^4 = G/K. The tangent spaces at l and L decompose under the stabilizers as

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T_L S^4 = (2, 2) of Sp(1)_a x Sp(1)_b = Hom_H(L, L^perp),
T_l CP^3 = (2)_{+-1} + (1)_{+-2} under H = Sp(1)_a x U(1)_b,
```

where (2)_{+-1} is the restriction of T_L S^4 (the horizontal part, the doublet of the weak Sp(1)_a with U(1)_b charge +-1, one complex doublet of hypercharge 1/2 in the normalization Y = charge/2) and (1)_{+-

2} is the tangent of the twistor fibre K/H = S^2 at l (one complex singlet of hypercharge 1, electric charge +1).

Proof. sp(2) = sp(1)_a + sp(1)_b + m with m = (2, 2) of real dimension 4, the isotropy representation of S^4; this is the decomposition of so(5) under so(4). Under H, sp(1)_b = u(1)_b + (1)_{+-2}, since the adjoint of Sp(1) restricted to its maximal torus has charges 0, +-2, and (2, 2) restricted to Sp(1)_a x U(1)_b is (2)_{+1} + (2)_{-1}, since the 2 of Sp(1)_b has charges +-1. The isotropy representation of G/H is sp(2)/h = (1)_{+-2} + (2)_{+-1}, as stated in lemma 28, with (1)_{+-2} tangent to the fibre K/H and (2)_{+-1} the lift of the tangent of G/K. The identification T_L HP^1 = Hom_H(L, L^perp) with the action t -> q_a t q_b^{-1} is the corpus note, section 6. ■

Proposition 42.3 (the moduli of the connector). Let w in W and v in V be the two marked unit vectors, with G acting on both copies of the 4 by the same representation, and let h and omega be the G-invariant Hermitian and symplectic forms on C^4 (omega(x, y) = h(jx, y)). The G-orbits of the pair of lines (Cw, Cv) are classified by the two real invariants

```
a = |h(w, v)|,   b = |omega(w, v)|,   a^2 + b^2 <= 1;
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the orbit of the pair is determined by the position of Cv relative to the quaternionic line L of w: a^2 + b^2 is the squared norm of the component of v in L, a the overlap with l and b the overlap with jl. The G-orbits of the connector T_kappa itself are classified by a, b, |kappa| and, when alpha = h(w, v) != 0, the phase of kappa alpha^2: with the identity 1 in Hom(V, W) fixed by G, tr T_kappa(1) = kappa (w^* v)^2 = kappa alpha^2 is a G-invariant, unchanged by the rephasings that fix T_kappa (referee). So the connector has four real moduli when alpha != 0 and three when alpha = 0; a symmetry larger than the diagonal Sp(2), for instance an independent phase on V, removes the fourth.

Proof. G acts transitively on the unit sphere of H^2 with stabilizer Sp(1)_a (the stabilizer of w and jw), so the orbit of the pair is the Sp(1)_a-orbit of v. Write v = alpha w + beta jw + u with u in L^perp = H; Sp(1)_a acts on u by the quaternionic scalars, transitively on each sphere of L^perp, and fixes alpha = h(w, v), beta = h(jw, v) = conj(omega(w, v)) up to the convention. Rephasing w by e^{i phi} and v by e^{i theta} sends (alpha, beta) to (e^{i(theta - phi)} alpha, e^{i(theta + phi)} beta), which removes both phases. So the invariants of the pair of lines are |alpha|, |beta|, and |u|^2 = 1 - |alpha|^2 - |beta|^2 is determined; here beta = omega(w, v) exactly. For the connector, the rephasings w -> e^{i phi} w, v -> e^{i theta} v, kappa -> e^{-2i(theta - phi)} kappa leave T_kappa = kappa K x K fixed and multiply alpha^2 by e^{2i(theta - phi)}, so kappa alpha^2 is invariant under them, and it is invariant under G because 1 is; T_kappa and T_{-kappa} on the same pair of lines with alpha != 0 have the same a, b, |kappa| and opposite invariants, so they lie in different orbits. ■

Reading

The Higgs doublet of the first-stage note is recovered in the window as the tangent of the orbit of the marked core line, projected to S^4: it is the representation in which the point p of S^4 under the connector moves. The connector of lemma 20 is an input of the window (Theorem 28.21 prescribes the windows; the connector is not derived), so the proposition says where the doublet sits once the connector is a field: in the variation of the marked lines, which is a variation of the finite data, in the moduli a, b, |kappa| and arg(kappa alpha^2), and in G-breaking off-diagonal blocks, and never in the equivariant window operator d_M. Beside the doublet, the twistor fibre contributes a charged singlet direction of hypercharge 1; whether this direction is a dynamical scalar, a gauge direction of the U(1)_b, or frozen by the collar connection of lemma 29 (which preserves the line summands, hence the fibre structure) is open, and it is the same question as the grading axis of Proposition 28.19, which marks the quaternionic line L. The dictionary note keeps one row from this: the Higgs doublet is the tangent of the base S^4 at the point marked by the connector, with the twistor fibre as a charged singlet beside it.

Scope

The window of lemma 29 with its Sp(2)-equivariance; the connector of Theorem 20.2 as a marked pair of lines and a coupling; the normalization Y = charge/2 of the corpus note. Nothing here makes the marked

lines dynamical, computes a potential, or fixes the coupling; Proposition 42.3 counts moduli and does not identify them with a vacuum expectation value; Proposition 42.1 concerns the G-invariant block only. The direction it points to: a parent in which the connector's lines vary along S^4, that is, a Higgs field as a section over the base, for which the collar connection of lemma 29 and the engineered connections of T15 are the available tools.

Part 43

Which connections on a vertical kernel are Berry connections

For STRATEGY T15 (from CORPUS_SALVAGE items 1 and 2), following lemma 10 (Proposition 10.1 and the second-moment identity), lemma 29 (the collar connection of case (b)) and lemma 38. Written by Claude Fable 5.1 on 4 November 2026, by hand. Refereed by Claude Opus 5.5 (effort max, 4 November 2026; report in historical/reviews/2026-11-04_opus_check_41_45.md): Proposition 43.1 and Theorem 43.2 correct with amendments (the mean-curvature term, the transposed adjoint, the exclusion of vertical components), the two computations of Corollary 43.3 correct, and its last clause, which read lemma 29's collar connection as an instance, an error, withdrawn below. The construction was used in the corpus (historical/corpus/canonical_research.md, section 9, on N over S^5 with the flux (1, 1, 2)); this lemma supplies the general statement, and leaves the question of lemma 29's collar connection open.

Outcome. Let a fibrewise Dirac family on a bundle of compact fibres F_b have kernels K_b of constant rank, forming a bundle K over the base B, with the Berry connection induced from a unitary connection on the coefficient bundle E. Changing the coefficient connection by a horizontal one-form alpha changes the Berry connection by the fibrewise compression P alpha P. Every connection on K is a Berry connection of some coefficient connection exactly when the fibrewise compression Gamma(F_b, u(E)) -> u(K_b) is surjective at every point of B, and that holds exactly when the pairwise products s_i s_j^* of a frame of kernel sections are linearly independent as sections of End(E) over the fibre. For scalar changes of the connection the same criterion reads: the pointwise inner products <s_i, s_j> are linearly independent functions on the fibre. On CP^3 with the kernel H^0(O(1)) this holds (sixteen independent functions z-bar_i z_j/|z|^2), so every connection on the rank-four kernel bundle is a Berry connection of an abelian change; for functions pulled back from the base S^4 the compressed image has real dimension six out of sixteen (Example 16.9 with Proposition 10.1), so scalar changes along the base reach six of the sixteen directions. The collar connection of lemma 29 is outside this statement: it acts at the wall, on the CP^3 copy, and it changes the fibre operator and moves the kernel, the effect excluded below (referee).

Setting

Let pi: P -> B be a fibre bundle with compact fibres F_b and a connection (a horizontal distribution), E -> P a Hermitian vector bundle with a unitary connection nabla, and D = (D_b) a smooth family of fibrewise elliptic self-adjoint operators on L^2(F_b, E|*{F_b}) whose kernels K_b have constant dimension k; P_b denotes the orthogonal projection onto K_b. The Berry connection on the bundle K -> B is nabla^K_X s = P_b nabla*{X^h} s for X in T_b B, with X^h the horizontal lift and s a section of K read as a family of sections of E over the fibres; this is unitary for the fibrewise L^2 product when horizontal transport preserves the fibre measure (Riemannian submersions with totally geodesic fibres, as in the homogeneous examples), and in general the unitary connection adds (1/2) k(X) with k the mean-curvature form of the fibres (Bismut; Berline-Getzler-Vergne, chapter 9), a term common to all the connections compared here, so that the statements below hold with that term included or with volume-preserving transport assumed (referee). For a fibre F_b, the compression map is

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C_b : Gamma(F_b, End(E)) -> End(K_b),   C_b(X) = P_b X P_b,
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and in a unitary frame s_1, ..., s_k of K_b its matrix elements are C_b(X)_{ij} = int_{F_b} <s_i(y), X(y) s_j(y)> dy. Its adjoint for the L^2 pairing on Gamma(F_b, End(E)) with the Hilbert-Schmidt pairing on End(K_b) is C_b^*(Y) = rho_Y = sum_{ij} Y_{ij} s_i s_j^*, a smooth section of End(E) over F_b (the density of Y; the first version had the indices transposed, which changes nothing below since the two families of products coincide as sets).

Statements

Proposition 43.1 (the change of the Berry connection). Let nabla' = nabla + alpha with alpha in Omega^1(P, u(E)) vanishing on vertical vectors. Then the Berry connection of nabla' is nabla^K + beta, where beta_b(X) = C_b(alpha(X^h)|_{F_b}): the compression of the horizontal component of alpha, fibre by fibre, exactly. A fibrewise twisted Dirac operator uses only the vertical part of the coefficient connection, so such an alpha leaves the family D_b and its kernels unchanged. A vertical component of alpha changes D_b and moves the kernels, and a first-order motion of the kernels changes the Berry connection at first order through the second fundamental form (for a kernel line n(b) in C^2 over S^2 moved along the flow of a vector field xi, the Berry curvature changes by the Lie derivative along xi); such components are excluded here, and this is the effect at work in the collar connection of lemma 29 (referee).

Proof. With the kernels unchanged, nabla'*{X^h}* s = *nabla*{X^h} s + alpha(X^h) s, and P_b of the second term is C_b(alpha(X^h)) s. ■

Theorem 43.2 (which connections are Berry connections). The following are equivalent for the family (P, E, D):

1. Every unitary connection on K -> B is the Berry connection of nabla + alpha for some alpha in Omega^1(P, u(E)) vanishing on vertical vectors.
2. C_b : Gamma(F_b, u(E)) -> u(K_b) is surjective for every b in B.
3. For every b and every (one, hence every) unitary frame s_1, ..., s_k of K_b, the k^2 sections s_j s_i^* of End(E) over F_b are linearly independent.

When these hold, a preimage can be chosen smoothly: alpha(X^h) = C_b^*(C_b C_b^*)^{-1} beta_b(X) is a smooth u(E)-valued form on the horizontal distribution, extended by zero on vertical vectors.

Proof. (1) implies (2): a connection whose difference from nabla^K at b is a prescribed Y in u(K_b) must have Y in the image of C_b. (2) implies (3): C_b is surjective iff C_b^* is injective on u(K_b); C_b^* is complex linear with C_b^*(Y^*) = C_b^*(Y)^*, so its kernel in End(K_b) is closed under the adjoint and meets u(K_b) trivially exactly when it is zero, and C_b^*(Y) = sum Y_{ij} s_i s_j^* vanishes for a nonzero Y exactly when the sections s_i s_j^* are dependent. (3) implies (1): with C_b surjective, C_b C_b^* is a positive invertible operator on u(K_b) depending smoothly on b, so for a given beta in Omega^1(B, u(K)) the form alpha(X^h) = C_b^*(C_b C_b^*)^{-1} beta_b(X) is smooth, u(E)-valued, and C_b(alpha(X^h)) = beta_b(X); by Proposition 43.1 the Berry connection of nabla + alpha is nabla^K + beta. ■

Corollary 43.3 (scalar changes). Restricting alpha to scalar forms (alpha = i a (x) 1_E, a real), the attainable changes are the compressions of real functions, and every connection on K is attained by a scalar change iff the k^2 functions <s_i, s_j> on F_b are linearly independent. For the kernel H^0(CP^3, O(1)) of the Dolbeault-Dirac operator twisted by O(1), with s_i = z_i/|z| in the unitary frame, <s_i, s_j> = z-bar_i z_j/|z|^2, sixteen linearly independent functions, so the criterion holds and every connection on the rank-four kernel bundle is the Berry connection of an abelian change of the coefficient connection; in matrix form, Q(<w, B w>/<w, w>) = (B + tr B)/5 (lemma 10, the second-moment identity), a bijection of Hermitian matrices. For functions pulled back from the base S^4 of the twistor fibration, the compressed image is the real span of 1 and Gamma_1, ..., Gamma_5, of dimension six (a real function from S^4 compresses to a Hermitian element commuting with j, Example 16.9, item 1, and Proposition 10.1 shows the span is attained), so scalar changes along the base attain only the connections whose difference from nabla^K lies in that span. Whether the complementary ten directions are reached by matrix-valued alpha of a restricted class, such as components between the summands of E, is a further application of criterion 3 to the core modes that has not been made; the first version of this corollary asserted it and read the collar connection of lemma 29 as an instance, which it is not (that connection acts at the wall, on the CP^3 copy, and moves the kernel).

Proof. The first sentence is Theorem 43.2 with the scalar subalgebra in place of Gamma(u(E)). The CP^3 statement is the second-moment identity Q(w^* B w/w^* w) = (B + tr B)/5 (Example 16.9, item 2, from the fourth-moment formula in the proof of Proposition 10.1), which maps Herm(4) onto itself with eigenvalues 1/5 and 1; the S^4 statement is Example 16.9, item 1, with Proposition 10.1. ■

Remark 43.4 (the test case of the corpus). On the Berger family of S^5 with the kernel of flux (1, 1, 2) the corpus engineered a connection with a prescribed Berry connection and found that all three coefficient lines keep a circle zero mode exactly when eta^2 is an integer; that calculation rests on stated spin lifts and was checked for its phase arithmetic only (historical/corpus/canonical_research.md, section 9, and research_checks, section 4.7). Theorem 43.2 says when such engineering is possible; it does not redo that calculation, and the integrality statement stays a record.

Scope

Constant-rank kernels, compact fibres, unitary connections with volume-preserving horizontal transport or the mean-curvature term; the Berry connection as the compressed horizontal derivative; changes of the connection with no vertical component, which leave the kernels fixed. The theorem is linear algebra fibre by fibre and a smoothness remark; its content for the project is the criterion of item 3, which separates the two sheets of lemma 29 for scalar changes: the wall copy (functions on CP^3) attains every connection, the core copy (functions on S^4) six directions of sixteen. The collar connection of lemma 29 is a vertical change at the wall and moves the kernel, so the question of T15, whether it is a choice or a construction, stays open; a theorem that covers it must include the motion of the kernels (referee).

Part 46

The odd blocks of the quaternionic sheet: the bidoublet, its quaternionic half as the direction of the axis, and the up-down splitting

For STRATEGY T14, item (ii), and T0, following lemma 42 (Proposition 42.1 and the escape its referee recorded: Hom(W, V) = 1 + 5 + 10 contains T_p S^4 twice), lemma 16 (Example 16.9), lemma 10 (Proposition 10.1), lemma 28 (Propositions 28.19 and 28.20), lemma 29 (Proposition 29.8) and lemma 20 (Theorem 20.2). Written by Claude Fable 5.1 on 5 November 2026, by hand; the Clifford relations, the quaternionic structure, the commutant dimensions 16, 8 and 16, the K_p-invariance of the two four-dimensional spans, the spectrum of Proposition 46.3 and the mass formula of Proposition 46.2 were checked with numpy on random data (script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_46.md), who redid every computation (the Clifford structure and j, the dimensions, the convention of L_q, the mass formula on 2000 random pairs, the stabilizer dimensions, a Monte Carlo compression, the window spectrum): Propositions 46.1, 46.2 and Corollary 46.4 correct with amendments; Proposition 46.3 had an error in its hypothesis (a bounded operator in place of a bounded multiplication operator, with the counterexample B = v u^*); the Outcome and Reading had an error in the chain of stabilizers; one sentence identified the rank-one block with the top-only family of lemma 21, which the shared lepton block forbids; the escape by complex functions on the base was missing. All applied below, with the referee's sharper forms (the fibre-odd functions on CP^3, the determinant form of the mass formula, the stabilizers of the block) and the prior work it found.

Outcome. On the quaternionic sheet W = H^2 of the twistor window, with the grading axis p in S^4 of Proposition 28.19 and Example 16.9, the Hermitian blocks odd for the grading are Hom_C(L_p, L_p^perp), the complex bidoublet (2, 2) (x) C of the Pati-Salam pair Sp(1)_a x Sp(1)_b = SU(2)_L x SU(2)_R (Proposition 46.1). Its quaternionic half, the j-linear maps Hom_H(L_p, L_p^perp) = T_p S^4, is the span of the Gamma_a with a orthogonal to p, that is the odd part of the compression of the coordinates of S^4 (Proposition 10.1); its j-antilinear half, the span of the i Gamma_a Gamma_p, lies outside the real algebra M_2(H) of the base functions and inside the compression of the functions on CP^3. A Yukawa block X = q + i q' on a doublet (q, q' quaternions) gives the two members of the doublet the masses m_+-^2 = |q|^2 + |q'|^2 +- 2 |Im(conj(q) q')| (Proposition 46.2): equal masses exactly when X = z q is quaternionic up to a complex factor, as every block compressed from a real function on S^4 is; one massless member exactly when X is of rank one, with the two halves of equal norm; tensored with 1_V such a block gives the up-type quark and the Dirac neutrino of the generation the mass m_+ and the down-type quark and the charged lepton the mass zero. When the masses are split, the commutant of X^* X in Sp(1)_b is a circle: the up-down splitting fixes a maximal torus of SU(2)_R, which is the hypercharge circle of the connector of

lemma 20 exactly when the connector's line is one of its eigenlines, and that eigenline condition is exactly the condition for the Yukawa block and the connector to leave a common circle unbroken (Corollary 46.4). In the window of lemma 29 all cross blocks between the core and the wall states are of the exponential order of lambda_M, and the cross block of a coordinate of S^4 is c_M Gamma_a, so the G-breaking blocks of the window operator that carry T_p S^4 are at most of the order of the bound on lambda_M, and their nonvanishing is not shown (Proposition 46.3). The reading: in Sp(2) = U(M_2(H)) the axis Gamma_p has stabilizer K_p and reduces M_2(H) + M_4(C) to A_PS (Example 16.9); the connector's line has stabilizer Sp(1)_a x U(1)_b in K_p and the connector reduces A_PS to C + H + M_3(C) (Theorem 20.2); the Yukawa block has stabilizer the diagonal Sp(1) of K_p, the custodial SU(2)_{L+R}, when its masses are equal and a circle when they are split, a circle inside Sp(1)_a x U(1)_b exactly when the connector's line is an eigenline of X^* X; the Higgs doublet is the tangent of the orbit S^4 of the axis (lemma 42), and the up-down splitting, maximal for the top, is of the representation type of the compression of the real functions on CP^3 odd under the fibre involution j, which no real function on the base supplies through the odd part of its compression. With complex functions on the base and the projections of the grading the full bidoublet is reached: P_{L_p^perp} Q(x_a + i x_b) P_{L_p}, for a, b orthogonal to p and a != b, has rank one, as complex generation on the S^4 side gives M_4(C) (Example 16.9).

Setting

Let W = C^4 with the Hermitian Clifford generators Gamma_1, …, Gamma_5 of Cl(5), Gamma_a Gamma_b + Gamma_b Gamma_a = 2 delta_ab, and the antiunitary j with j^2 = -1 commuting with every Gamma_a (it exists since Gamma_1 … Gamma_5 = +-1 is real, so the conjugate representation is equivalent through a unitary C with C conj(C) = -1, the Gamma_a generating the real algebra M_2(H), and it is unique up to a phase, which does not affect Y -> j Y j^{-1}; referee), so that W = H^2 (Example 16.9: W is the kernel H^0(CP^3, O(1)) with the quaternionic structure of the twistor space, and Proposition 10.1 gives the compression Q(x_a) = Gamma_a/5 of the coordinates of S^4). The real algebra End_H(W) = M_2(H) is the real span of 1, the Gamma_a and the Gamma_a Gamma_b (a < b), of dimension 1 + 5 + 10 = 16; its unitary group is G = Sp(2), acting on W by the 4; under G the Gamma_a span the 5 (the traceless Hermitian elements of M_2(H)) and the Gamma_a Gamma_b the 10 = sp(2) (the anti-Hermitian elements). For p in S^4 put Gamma_p = p^a Gamma_a; Gamma_p^2 = 1, its eigenspaces L_p (+1) and L_p^perp (-1) are j-invariant, hence quaternionic lines, and p -> L_p identifies S^4 with HP^1. The stabilizer of p is K_p = Sp(1)_a x Sp(1)_b with Sp(1)_b = Sp(L_p) and Sp(1)_a = Sp(L_p^perp), in the notation of lemma 42, so that the bidoublet (2, 2) of K_p is Hom_H(L_p, L_p^perp) = T_p S^4 (the corpus note, section 6, and Proposition 42.2).

The second sheet V = C^4 carries the compression of the functions on CP^3, which generate M_4(C) (Example 16.9, item 2), and is identified with W as a G-representation (Proposition 28.19). The bimodule H_F = Hom(V, W) + Hom(W, V) = W (x) conj(V) + V (x) conj(W) is the sixteen states of one generation with their conjugates: W the two doublets of a lepton or a quark, conj(V) the lepton line and the three colours (Example 16.9). The axis p is the quaternionic grading of Proposition 28.19 and Example 16.9: Gamma_F acts as Gamma_p on the W factor, and the even algebra is H + H + M_4(C) = A_PS, acting on L_p, L_p^perp and V, the Pati-Salam algebra of lemma 20, with H_R on L_p and H_L on L_p^perp (the corpus note, section 5). A sheet-diagonal Dirac operator of Corollary 16.8 acts on H_F by a Hermitian block Y on the W factor, odd for Gamma_p, tensored with the identity of V: the Yukawa block, the same for the lepton and the three colours. With V ungraded the parent operator on V vanishes, so the sheet-diagonal operator on Hom(V, W) is Y (x) 1_V, with eigenvalues +-m_+ and +-m_-, each of multiplicity four, and the same on Hom(W, V); the doubling of lemma 16, Caveat 2, which would give (m_i^2 + m_j^2)^{1/2}, does not occur (referee).

Statements

Proposition 46.1 (the odd Hermitian blocks for the axis p). Let Odd_p = {Y in End_C(W) : Y^* = Y, Gamma_p Y = -Y Gamma_p}. Then:

1. Odd_p = {[[0, X^*], [X, 0]] : X in Hom_C(L_p, L_p^perp)}, of real dimension 8; as a representation of K_p it is Hom_C(L_p, L_p^perp) = (1, 2)^* (x) (2, 1) = (2, 2) (x)_R C, two real copies of the

bidoublet.

2. The j-linear part Odd_p cap M_2(H) is the real span of the Gamma_a with a orthogonal to p, the (2, 2) inside the 5: the maps with X in Hom_H(L_p, L_p^perp) = T_p S^4. By Proposition 10.1 it is the odd part of the compression of the coordinates of S^4, Gamma_a = 5 Q(x_a).
3. The j-antilinear part, {Y in Odd_p : j Y j^{-1} = -Y}, is the real span of the i Gamma_a Gamma_p with a orthogonal to p, which is i times the (2, 2) inside the 10 = sp(2). It lies outside M_2(H), hence outside the compression of the real functions of S^4, and it is the compression of the real functions on CP^3 that are odd under the fibre involution [w] -> [j w]: since j P_w j^{-1} = P_{jw} and the involution preserves the Fubini-Study measure, Q(f o j) = j Q(f) j^{-1}, so the j-odd real functions compress to j-antilinear Hermitian matrices and the j-even ones to M_2(H); for example i Gamma_a Gamma_p = Q(5 w^* i Gamma_a Gamma_p w / w^* w) (referee).
4. Under G, End_C(W) = 1 + 5 + 10, and under K_p: 1 -> (1, 1); 5 -> (1, 1) + (2, 2), spanned by Gamma_p and by the Gamma_a with a orthogonal to p; 10 -> (3, 1) + (1, 3) + (2, 2), spanned by the Gamma_a Gamma_b with a, b orthogonal to p, which form sp(1)_a + sp(1)_b, and by the Gamma_a Gamma_p. The even part End(L_p) + End(L_p^perp) is the complex span of 1, Gamma_p and the Gamma_a Gamma_b with a, b orthogonal to p; the odd part is the complex span of the two (2, 2).

Proof. (1) A Hermitian Y anticommuting with Gamma_p exchanges its eigenspaces, so Y = [[0, X^*], [X, 0]] with X : L_p -> L_p^perp arbitrary, and conversely; the real dimension is 2 dim_C Hom(C^2, C^2) = 8. Under K_p, Hom_C(L_p, L_p^perp) = L_p^* (x) L_p^perp = (1, 2) (x) (2, 1) = (2, 2) as a complex representation, the complexification of the real bidoublet. (2) j commutes with every Gamma_a, so Gamma_a lies in M_2(H); for a orthogonal to p, Gamma_a is Hermitian and anticommutes with Gamma_p; the four such Gamma_a are linearly independent, so they span a four-dimensional real subspace of the j-linear part of Odd_p, which is {[[0, X^*], [X, 0]] : X in Hom_H(L_p, L_p^perp)}, of real dimension 4, since the H-linear maps between two quaternionic lines form a copy of H. So the two spaces coincide, and Proposition 10.1 identifies them with the odd parts of the compressed coordinates. (3) Y -> j Y j^{-1} is a real-linear involution of Odd_p: (j Y j^{-1})^* = j Y^* j^{-1} for antiunitary j, and j commutes with Gamma_p. So Odd_p is the direct sum of its j-linear and j-antilinear parts, and the latter has real dimension 8 - 4 = 4. For a orthogonal to p, i Gamma_a Gamma_p is Hermitian, since (i Gamma_a Gamma_p)^* = -i Gamma_p Gamma_a = i Gamma_a Gamma_p; it anticommutes with Gamma_p, since Gamma_p Gamma_a Gamma_p = -Gamma_a; and it is j-antilinear, since j commutes with Gamma_a Gamma_p and anticommutes with i. The four are linearly independent, so they span the j-antilinear part. A nonzero j-antilinear element is outside End_H(W) = M_2(H), the real algebra generated by the compressed functions of S^4 (Example 16.9, item 1). For a real function f on CP^3, the covariance j P_w j^{-1} = P_{jw} and the invariance of the Fubini-Study measure under [w] -> [jw] give Q(f o j) = j Q(f) j^{-1}, so f odd under the involution has j Q(f) j^{-1} = -Q(f), and the formula of Example 16.9, item 2, applied to f = 5 w^* B w / w^* w with B = i Gamma_a Gamma_p, which is odd, gives Q(f) = B. (4) The 4 of Sp(2) = Spin(5) is symplectic, so End_C(W) = 4 (x) 4 = Lambda^2(4) + S^2(4) = (1 + 5) + 10, with the 5 the vector of SO(5) spanned by the Gamma_a and the 10 the adjoint spanned by the Gamma_a Gamma_b. Under K_p = Spin(4) the vector of SO(5) is 1 + (2, 2), the trivial summand being Gamma_p, and so(5) = so(4) + R^4 is (3, 1) + (1, 3) + (2, 2), the last spanned by the Gamma_a Gamma_p, which transform as the vector; Gamma_p commutes with 1, Gamma_p and the Gamma_a Gamma_b with a, b orthogonal to p and anticommutes with the Gamma_a and the Gamma_a Gamma_p with a orthogonal to p, which gives the even and odd parts, of complex dimensions 1 + 1 + 3 + 3 = 8 and 4 + 4 = 8. ■

Proposition 46.2 (the masses of a doublet). Identify L_p and L_p^perp with H = C^2 through j, so that an element X of Hom_C(L_p, L_p^perp) is X = L_q + i L_{q'} with q, q' in H acting by left multiplication, L_q the j-linear and i L_{q'} the j-antilinear part. Then the two singular values of X, the masses of the two members of the doublet under the Yukawa block [[0, X^*], [X, 0]] (x) 1_V, are

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m_+-^2 = |q|^2 + |q'|^2 +- 2 |Im(conj(q) q')|,
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with Im the imaginary part of the quaternion conj(q) q'. Consequently:

1. m_+ = m_- exactly when conj(q) q' is real, that is when X = z L_q for a complex number z and a quaternion q. In particular the odd part of the compression of any real function on S^4 is j-linear, q'

= 0, and gives equal masses to the two members of every doublet.

2. m_- = 0 with m_+ > 0 exactly when q' = q u for a unit imaginary quaternion u (equivalently q' = u' q with u' = q u q^{-1}), that is X = L_q (1 + i L_u) = 2 L_q P_u with P_u = (1 + i L_u)/2 a projection of rank one: the pattern in which one member of the doublet is massless and the other is not is a block of rank one whose j-linear and j-antilinear halves have equal norm, in the Hilbert-Schmidt and in the operator norm. Tensored with 1_V it gives the up-type quark and the Dirac neutrino of the generation the mass m_+ and the down-type quark and the charged lepton the mass zero; the top-only family of lemma 21, with the leptons massless, needs a lepton block different from the quark block, which a sheet-diagonal operator of Corollary 16.8 with V ungraded does not provide (referee).
3. For q, q' nonzero, with |q'| = t |q| and the quaternionic angle phi in [0, pi] given by Re(conj(q) q') = |q| |q'| cos phi, m_+-^2 = |q|^2 (1 + t^2 +- 2 t sin phi): the ratio of the two masses depends on the relative size of the two halves and on the angle between them, and on nothing else.

Proof. Write H = C + j C with C acting on the right, x = x_1 + j x_2. For q = a + j b with a, b in C, left multiplication is the matrix [[a, -conj(b)], [b, conj(a)]]; q -> L_q is a homomorphism of real algebras with L_{conj(q)} = L_q^*; L_q commutes with j (x_1, x_2) = (-conj(x_2), conj(x_1)), the right multiplication by j, and i L_{q'} anticommutes with it; the L_q and the i L_{q'} together span the eight real dimensions of M_2(C). Then X^* X = (L_{conj(q)} - i L_{conj(q')}) (L_q + i L_{q'}) = |q|^2 + |q'|^2 + i (L_{conj(q) q'} - L_{conj(q') q}) = |q|^2 + |q'|^2 + 2 i L_w, with w = Im(conj(q) q'), since conj(q') q = conj(conj(q) q') and the real parts cancel. Left multiplication by an imaginary quaternion w is a traceless anti-Hermitian matrix with eigenvalues +- i |w|, so 2 i L_w has eigenvalues -+ 2 |w| and the eigenvalues of X^* X are as displayed. Equivalently, det X = |q|^2 - |q'|^2 + 2 i q . q' with q . q' = Re(conj(q) q'), the complex-bilinear norm of q + i q' in H (x) C, so m_+ m_- = |det X| and m_+^2 + m_-^2 = 2 (|q|^2 + |q'|^2) (referee). (1) The masses coincide exactly when w = 0. If q = 0, X = i L_{q'}. If q != 0, conj(q) q' = s real gives q' = (s/|q|^2) q and X = (1 + i s/|q|^2) L_q; conversely X = z L_q has X^* X = |z|^2 |q|^2. The compression of a real function on S^4 is a Hermitian element of M_2(H) (Example 16.9, item 1), and Gamma_p is j-linear, so the odd part is j-linear. (2) m_- = 0 exactly when |q|^2 + |q'|^2 = 2 |w|; since |w| <= |conj(q) q'| = |q| |q'| <= (|q|^2 + |q'|^2)/2, this forces |q| = |q'| and Re(conj(q) q') = 0, so conj(q) q' = w with |w| = |q|^2 and q' = q w/|q|^2 = q u with u = w/|q|^2 a unit imaginary quaternion; then X = L_q + i L_q L_u = L_q (1 + i L_u), and i L_u is Hermitian with L_u^2 = -1, so (1 + i L_u)/2 is a projection of rank one. Conversely such an X has rank one. (3) Substitution, with |w|^2 = |q|^2 |q'|^2 - Re(conj(q) q')^2. ■

Prior work (found by the referee). The decomposition of the complex bidoublet Yukawa into its quaternionic and anti-quaternionic halves, and the up-down splitting by the second, are in print in bidoublet form: in the spectral Pati-Salam model of Chamseddine, Connes and van Suijlekom (arXiv:1304.8050, JHEP 11 (2013) 132, the section on the Dirac operator and inner fluctuations on H_R + H_L + M_4(C)) the right-left block after fluctuation is (k^u phi + k^d phi-tilde) (x) 1 in the unified case k^nu = k^u, k^e = k^d, with phi-tilde = sigma_2 conj(phi) sigma_2 = j phi j^{-1}, so that the two halves are (k^u + k^d)(phi + phi-tilde)/2 and (k^u - k^d)(phi - phi-tilde)/2; the general bidoublet mass matrices M_U = kappa h + kappa' e^{-i alpha} h-tilde, M_D = kappa' e^{i alpha} h + kappa h-tilde are those of the left-right symmetric models (Zhang, An, Ji and Mohapatra, arXiv:0712.4218, section III; Mohapatra and Senjanovic 1981); and the statement that the quaternionic block has a custodial SU(2)_{L+R} with m_t = m_b is the custodial-symmetry literature (Willenbrock, arXiv:hep-ph/0410370, section 3 and Exercise 3.5; Sikivie, Susskind, Voloshin and Zakharov 1980; Solberg, arXiv:1207.5194; Haber and O'Neil, arXiv:1011.6188). Proposition 46.2 is a restatement of these formulas in the variables (q, q') of the window; the closed formula in those variables, the identification of the second half with the fibre-odd functions on CP^3 and the stabilizer reading of Corollary 46.4 were not found in print. The identifiers are in paper/LITERATURE.md, section M.

Proposition 46.3 (cross blocks of the window operator). In the window W_M + V_M of Theorem 29.7, with u a unit core state and v a unit wall state, let B be the multiplication by a bounded measurable endomorphism field b of E, with ||B|| = ess sup |b| (for a general bounded operator the bound fails: B = v u^* has norm one and <v, B u> = 1; referee). Under the weighted estimates of the proof of Proposition 29.8 (with the cut-offs of its second reading), for every eta in (0, 1) there are M_eta and C_eta with

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|<v, B u>| <= C_eta ||B|| exp(-(1 - eta)(M/2) int_e^{l - a} f dr),    M >= M_eta.
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So every off-diagonal block of the compression of a bounded multiplication operator to the eight states is bounded by the exponential that bounds lambda_M in Proposition 29.8; no lower bound on either is proved. In the doubled operator, with W_M in the first copy of L^2(X, E) and V_M in the second, P_V B P_W is the cross block of the odd perturbation T_M -> T_M + B, the first-order term of the window block of D-hat for T_M + B; the higher-order terms of the reduction, of the form P_V B Q (Q D-hat Q - z)^{-1} Q B P_W, also tunnel through the bulk and are not estimated here, and the window stays isolated only for ||B|| below the gap sqrt 3 (referee). For B the multiplication by a coordinate x_a of S^4 pulled back to X, the cross block is P_V x_a P_W = c_M Gamma_a under the identification W_M = V_M = 4 of Proposition 28.19, with |c_M| bounded as above; a window operator with off-diagonal block d = lambda + mu n^a Gamma_a (lambda >= 0 after a phase, mu complex, n a unit vector) has the eight eigenvalues +-(lambda^2 + |mu|^2 +- 2 lambda Re(mu))^{1/2}, that is +-|lambda + mu| on L_n and +-|lambda - mu| on L_n^perp, each with multiplicity two, on the quaternionic lines of the two copies of the 4; when lambda Re(mu) = 0 they coincide with multiplicity four.

Proof. The argument of the proof of Proposition 29.8, run with eta' = eta/2 and with its point r_* moved to the outer end of the region where the bulk cut-off is one, gives weights g, with g = 0 on r <= e', g' = (1 - eta')(M/2) f on [e', l - a'] and g constant beyond, and g-tilde, with g-tilde = 0 on r >= l - a', -g-tilde' = (1 - eta')(M/2) f on [e', l - a'] and g-tilde constant below, where e < e' and a < a' are as close to e and a as we wish and the cut-off transitions lie in [e, e'] and [l - a', l - a]. Then ||exp(g) u||*{L^2(r >= e')} <= C_eta, through the collar by Lemma 29.6 (1), and ||exp(g-tilde) v||*{L^2(r <= l - a')} <= C_eta, through the core by Lemma 29.3, where T_M^* has no low states; both hold for every unit vector of W_M and of V_M, since T_M^* T_M and T_M T_M^* equal lambda_M^2 there (Proposition 28.19). With I' = int_{e'}^{l - a'} f, choose e', a' with (1 - eta') I' >= (1 - eta) I, I = int_e^{l - a} f. On e' <= r <= l - a', g(r) + g-tilde(r) = (1 - eta')(M/2) I', a constant, so the integral of |u| |v| over that region is at most exp(-(1 - eta')(M/2) I') ||exp(g) u|| ||exp(g-tilde) v||; on r < e' the weight g-tilde equals (1 - eta')(M/2) I' and ||u|| = 1, and on r > l - a' the weight g equals the same and ||v|| = 1, so the two end regions contribute the same exponential; hence |<v, B u>| <= ||B|| int |u| |v| <= 3 C_eta^2 ||B|| exp(-(1 - eta')(M/2) I') <= 3 C_eta^2 ||B|| exp(-(1 - eta)(M/2) I) for C_eta >= 1, since B is a multiplication operator. The window projections P_W and P_V are G-equivariant, since T_M is, and the coordinates x_a span the 5; so a -> P_V x_a P_W is a G-map from the 5 to Hom(W_M, V_M) = End_C(4) = 1 + 5 + 10, and by Schur's lemma it is a multiple c_M of the unique equivariant embedding, a -> Gamma_a, with c_M complex (the Gamma_a transform under the same covering Sp(2) -> SO(5) as the x_a, up to the sign +-1 of O(5), which c_M absorbs); rephasing the identifications U_W, U_V of Proposition 28.19 multiplies lambda_M and c_M by the same phase, so arg(c_M conj(lambda_M)) is an invariant, the phase that enters Re(mu) once lambda_M >= 0; c_M != 0 is not shown (referee). For the spectrum, d^* d = (lambda + conj(mu) n . Gamma)(lambda + mu n . Gamma) = lambda^2 + |mu|^2 + 2 lambda Re(mu) n . Gamma, and n . Gamma has the eigenvalues +-1 on the quaternionic lines L_n and L_n^perp; d d^* has the same spectrum. ■

Corollary 46.4 (the hypercharge direction and the connector's line). Let X be the Yukawa block of a doublet with m_+ != m_-. The elements of Sp(1)_b = Sp(L_p) commuting with X^* X form a circle U(1)_X, the maximal torus of Sp(1)_b whose weight lines are the two mass eigenlines of X^* X in L_p; it is a maximal torus of the kind lemma 42 marked as an input, and it equals the U(1)_b of lemma 42, the stabilizer in Sp(1)_b of the connector's line Cw, exactly when Cw is one of its weight lines; under it T_p S^4 = (2, 2) restricts to (2)_{+-1}, one complex doublet of hypercharge 1/2, and the twistor fibre K_p/(Sp(1)_a x U(1)_b) = S^2 is the set of complex lines in L_p. For the connector of lemma 20 (Theorem 20.2), with the axis p taken as the point of S^4 under the quaternionic line of its marked line w, so that w lies in L_p = W_+ and carries the Majorana block, the Majorana state is a Dirac-mass eigenstate exactly when w is one of the two eigenlines of X^* X, that is a fixed point of U(1)_X on the fibre S^2. The stabilizer of X in K_p is the circle {(X b X^{-1}, b) : b in U(1)_X} for invertible X with m_+ != m_-, again a circle for a rank-one X, and the diagonal {(q b q^{-1}, b)} = Sp(1), the custodial SU(2)_{L+R}, for X = z L_q; the stabilizer of X and of the connector's line together contains a circle exactly when that line is an eigenline of X^* X, so the eigenline condition is exactly the condition for the Yukawa block and the connector to leave a common circle unbroken, the analogue of the electromagnetic U(1) (referee). For a j-linear Yukawa block, m_+ = m_-, every circle in Sp(1)_b commutes with X^* X and every line is an eigenline, so neither the torus nor the position of w on the fibre is determined by the masses.

Proof. X^* X is a Hermitian operator on L_p = C^2 with two distinct eigenvalues, so its commutant in U(L_p) = U(2) is the torus of its eigenbasis, whose intersection with Sp(1)_b = SU(2) is the circle U(1)_b with those weight lines; Proposition 42.2 gives the restriction of (2, 2) and the fibre. The Majorana block of lemma 20 sits on the state w (x) conj(v), on which the Dirac block acts as X^* X (x) 1 on the right-handed factor, so it is a mass eigenstate exactly when w is invariant under X^* X. For the stabilizers: (a, b) in K_p fixes X exactly when a X = X b, so b commutes with X^* X and a = X b X^{-1}; for split masses b runs over U(1)_X and a lies in SU(L_p^perp), while for X = z L_q the condition is a = q b q^{-1} for every b in Sp(1); the V-phase does not enter, since the stabilizer of T_kappa contains the diagonal circle of Theorem 20.2 and X (x) 1_V is invariant under every unitary of V. The last statement is Proposition 46.2 (1). ■

Reading

The finite data of the twistor window carry three blocks, and their stabilizers in G = Sp(2), the unitary group of the base algebra M_2(H), are as follows:

1. the axis Gamma_p, a Hermitian involution in the 5, with stabilizer K_p = Sp(1)_a x Sp(1)_b: it reduces M_2(H) + M_4(C) to its commutant H + H + M_4(C) = A_PS, and its orbit is S^4 = G/K_p, whose tangent at p is the bidoublet (2, 2) (Example 16.9; Proposition 46.1);
2. the Yukawa block Y, odd for Gamma_p, in Hom_C(L_p, L_p^perp) = (2, 2) (x) C: its j-linear half is the direction of S^4 at p, the odd part of the compressed base coordinates; its j-antilinear half is the part no real function on the base supplies through the odd part of its compression (complex functions with the projections of the grading supply it), which splits the two masses of a doublet and thereby fixes a maximal torus of Sp(1)_b, the U(1)_b of the connector when the connector's line is one of its weight lines; the stabilizer of the block itself is the diagonal Sp(1) of K_p for equal masses and a circle for split ones (Propositions 46.1 and 46.2, Corollary 46.4);
3. the connector of lemma 20, the rank-one block marking a complex line w in L_p and a line v in V, with stabilizer Sp(1)_a x U(1)_b in K_p: it reduces A_PS to C + H + M_3(C) (Theorem 20.2), the orbit of its line Cw is CP^3 = G/H (Proposition 42.2), and the orbits of the connector have the moduli of Proposition 42.3.

The Higgs doublet of the corpus note is the tangent of the first orbit at the second block's foot: the variation of the axis, T_p S^4 = (2, 2) restricted by the U(1)_b of Corollary 46.4 to one complex doublet of hypercharge 1/2 (lemma 42). The heavy top of lemma 21 is the extreme case of the second block, Proposition 46.2 (2): the third generation, whose block in this scope is shared by the quark and the lepton doublets, has a Yukawa block close to rank one; with r = m_b/m_t, Proposition 46.2 gives ((1 - t^2)/(1 + t^2))^2 + (2 t cos phi/(1 + t^2))^2 = (2r/(1 + r^2))^2, so the two halves are equal in norm and in quadrature up to terms of order r, about 0.02 at the top scale; the odd part of the compression of a real function on the base would give m_t = m_b. Every G-breaking cross block of the window operator from a bounded multiplication operator, in the 5 or in the 10, is bounded by the exponential that bounds lambda_M (Proposition 46.3), and the block of a coordinate of S^4 is c_M Gamma_a, with c_M not shown to be nonzero; the Yukawa block of the bimodule is not a cross block of the window (Proposition 28.20 shows that the doubled window operator has zero compression to the bimodule, P_- DD P_- = 0): the two sheets supply the representation types, the quaternionic half from the real functions on S^4 and the full bidoublet from those on CP^3, and the block itself is an input of Corollary 16.8.

Scope

Finite linear algebra on the quaternionic sheet W = H^2 with the Clifford structure of Example 16.9; one generation, the Yukawa block tensored with the identity of V (the same block for the lepton and the three colours, as in a sheet-diagonal operator of Corollary 16.8); the grading axis as an input (Proposition 28.19); for Proposition 46.3, the window of Theorem 29.7 with the weighted estimates of Proposition 29.8. Three generations replace q and q' by 3 x 3 quaternionic matrices; if q^* q' is quaternion-Hermitian, X^* X is j-linear and each singular value has even multiplicity, the analogue of Proposition 46.2 (1) in one direction; the closed formula and the rank statement have no direct analogue and are not proved here (referee). Nothing here derives the Yukawa block, the Majorana line or a potential; the j-antilinear half is identified by representation type with the compression of functions on CP^3, not derived from one. The direction the result points to:

a parent in which the Yukawa block is the compression of a function on the wall sheet, odd under the fibre involution, so that the up-down splitting, and with it the top's mass, is the fibre dependence of that function, against the alternative of complex functions on the base with the projections of the grading, which reach the same bidoublet and should be compared with it before either is adopted; and a potential that drives the block of the third generation to rank one. Leads from the referee: (1) a rank-one block X = lambda mu^* marks the line C mu in P(L_p) = S^2, a point of the twistor fibre over p, the same kind of datum as the connector's line, and in the C-action of Theorem 20.2 the states on the connector's line are the up-type right-handed ones, so with the eigenline condition and the up-type assignment of the top, C mu = Cw: one marked point of CP^3 over p would carry both the Majorana line and the third-generation Yukawa block, a concrete target for a parent and for T12; (2) the custodial breaking is measured by |Im(conj(q) q')| = (m_+^2 - m_-^2)/4, a row for the dictionary; (3) by Theorem 29.9 and Proposition 28.8 the core compression of a real function on X has a j-antilinear half of norm at most C ||a||*{C^1} M^{-1/2}, so a block taken from a core compression has a bounded relative splitting unless its j-linear half is of the same order, and a heavy top at large M needs the wall sheet or the finite-M terms, whose j-breaking the numerics of lemma 28 showed; (4) at mu = lambda the window block of Proposition 46.3 is 2 lambda P*{L_n} with exact zero modes on the two copies of L_n^perp, so a G-breaking block in the 5 marks the quaternionic line that Proposition 28.19 leaves to be chosen, a lead for the axis as an input, with no dynamics behind it here; (5) the CKM matrix in the variables (q, q') with q^* q' quaternion-Hermitian as the custodial case.

Part 47

The Killing-form normalization on the Berger five-sphere and Connes' matching value

For STRATEGY T18 (the normalization test of the object at 9) and the parked job 04 (lemmas/README, the two Berger formulas), following lemma 03 (cos^2 theta_W = 5/8 at matching with the full module light; in its heavy-up family the photon and gluon coefficients keep their ratio) and the lead recorded from GitHub issue 1 of the repository (the round S^5 gives e^2/g_s^2 = 3/8, computed there by Schur's lemma; read by Claude Fable 5.1 on 5 November 2026). Written by Claude Fable 5.1 on 5 November 2026, by hand; the two sphere moments, the formula at eta = 1, 2, 1/2 and the charges of the coordinates in the exterior square of the spinor were checked with numpy (Monte Carlo with 2 x 10^6 points for the moments; script not kept). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_47_48.md), who redid the moments, the Berger form at four values of eta, the branchings and the traces: Proposition 47.1 correct; Corollary 47.2 and Remark 47.3 correct with amendments, applied below (the citation of the coupling formula to Weinberg 1983; the proof that the isometry group enlarges only at the round point; the reading of lemma 03; the scope of the eta-curve as a statement about the Einstein term alone, the round-point value being independent of any SO(6)-invariant source by Schur's lemma); the round-point value is implicit in print in holographic form, the charge unit being the superconformal R-charge (Barnes, Gorbatov, Intriligator and Wright 2005).

Outcome. On the Berger five-sphere, the round S^5 in C^3 with the Hopf fibre scaled by eta, the Kaluza-Klein kinetic form of the Killing vectors gives, for the SU(3) acting on C^3 and the central U(1) with charge c on the coordinates,

```
e^2 / g_s^2 = (3 + eta^2) / (24 eta^2 c^2)    (Corollary 47.2).
```

The two formulas of the parked job 04, 3(eta^2 + 3)/(8 eta^2) and 3(eta^2 + 3)/(32 eta^2), are the two charge units c = 1/3 and c = 2/3 on the coordinates; the second is the unit fixed by the spinor of SO(6) = SU(4) at the round point, where the SU(3)-singlet Killing spinor has charge 1 and the triplet -1/3, so that the coordinates, in the exterior square of the spinor, carry 2/3. With that unit the round point eta = 1 gives e^2/g_s^2 = 3/8, the value of Connes' finite triple at matching (sin^2 theta_W = 3/8 with g_2 = g_3, the 5/8 of lemma 03 being cos^2 theta_W). So the normalization test of T18 distinguishes the finite triple from the squashed spheres, and the round sphere, the one point of the Berger family where the isometry enlarges from U(3) to SO(6), gives the finite triple's value; at that point the value is independent of whatever holds the shape, since every SO(6)-invariant contribution to the gauge kinetic terms is proportional to the trace

form by Schur's lemma. In the Einstein-term reduction the ratio decreases from infinity at eta -> 0 through 3/8 at eta = 1 to 3/32 at eta -> infinity, taking each value once; off the round point the curve depends on the source that supports the non-Einstein shape.

Setting

Let S^5 be the unit sphere in C^3, with the Hopf fibration S^5 -> CP^2 whose fibre at z is the circle e^{i alpha} z, with unit vertical vector i z. The Berger metric g_eta is the round metric on the horizontal space and eta^2 times the round metric on the vertical direction; its volume is V_eta = eta V_1. The group U(3) acts isometrically on (S^5, g_eta) for every eta > 0, through SU(3) and the central circle z -> e^{i alpha} z, since U(3) preserves the round metric and the field i z. At eta = 1 the isometry group is O(6), with identity component SO(6), whose Lie algebra so(6) = su(4) contains u(3); for eta != 1 it is U(3) x| Z_2, the Z_2 generated by complex conjugation, with identity component U(3): by O'Neill's formulas for the canonical variation of the Hopf fibration over CP^2 (Einstein constant 6), the Ricci tensor of g_eta has the eigenvalue 4 eta^2 on the unit vertical vector and 6 - 2 eta^2 on unit horizontal vectors, which differ unless eta = 1, so an isometry preserves the vertical line field, hence the unit vertical field up to sign, hence the round metric g_1 = g_eta + (eta^{-2} - 1) theta_eta (x) theta_eta with theta_eta the g_eta-dual of the unit vertical field; it then lies in O(6) and maps i z to +- i z, so it is complex linear or antilinear (referee). The same eigenvalues show that g_eta is Einstein only at eta = 1. For a Hermitian traceless T on C^3 the Killing vector of the one-parameter group e^{i t T} is K_T(z) = i T z, and for the central circle with charge c on the coordinates, z -> e^{i c alpha} z, it is K_c(z) = i c z.

The tree-level Kaluza-Klein couplings: the reduction of the Einstein-Hilbert term on a compact internal space K with Killing vectors K_a gives the gauge kinetic term with the matrix

```
<K_a, K_b> = int_K g(K_a, K_b) dvol,
```

so that for an orthogonal basis g_a^{-2} is proportional to <K_a, K_a>, with one overall constant for all generators (the ansatz of Witten, Nucl. Phys. B 186 (1981) 412, equation (3); the integral formula of Weinberg, Phys. Lett. B 125 (1983) 265, in the form of Barnes, Gorbatov, Intriligator and Wright, Nucl. Phys. B 732 (2006) 89, section 3, where the normalization point is stated: the Killing vectors normalized to canonical kinetic terms make the coupling the charge unit; the normalization of each generator on the fields is that charge unit, an input). Within this scope, the Einstein term as the only source of the gauge kinetic terms, the charge unit is the only input; a flux or another source supporting the background contributes further terms (Duff, Pope and Warner, Phys. Lett. B 130 (1983) 254: on an Einstein space the Freund-Rubin contribution is a multiple of the Einstein one through int g(nabla K, nabla K') = int Ric(K, K'), leaving every ratio unchanged). The colour generators are normalized by tr T^2 = 1/2 on C^3.

Statements

Proposition 47.1 (the form on the Berger sphere). For Hermitian traceless T, T' on C^3 and the central generator with charge c,

```
<K_T, K_T'>_eta = V_eta (3 + eta^2) tr(T T') / 12,    <K_c, K_c>_eta = V_eta eta^2 c^2,    <K_T,
↪ K_c>_eta = 0.
```

In particular, for tr T^2 = 1/2, <K_T, K_T>_eta = V_eta (3 + eta^2)/24.

Proof. At z, the vertical component of K_T is the real inner product of i T z with the unit vertical vector i z, which is <z, T z> (real, since T is Hermitian), and the horizontal part has squared norm |T z|^2 - <z, T z>^2; so g_eta(K_T, K_T) = |T z|^2 + (eta^2 - 1) <z, T z>^2, and g_eta(K_T, K_T') = Re <T z, T' z> + (eta^2 - 1) <z, T z> <z, T' z>. The moments of the uniform measure on the unit sphere of C^n are int |z_i|^2 = 1/n and int z_i conj(z_j) z_k conj(z_l) = (delta_ij delta_kl + delta_il delta_kj)/(n(n + 1)), so for n = 3

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int <T z, T' z> = tr(T T')/3,    int <z, T z> <z, T' z> = (tr T tr T' + tr(T T'))/12 = tr(T T')/12,
```

the last for traceless T, T'. Hence <K_T, K_T'>_eta = V_eta [tr(T T')/3 + (eta^2 - 1) tr(T T')/12] = V_eta (3 + eta^2) tr(T T')/12. The central Killing vector is vertical with g_eta(K_c, K_c) = eta^2 c^2

|z|^2 = eta^2 c^2, and <K_T, K_c>_eta = c eta^2 int <z, T z> = c eta^2 tr(T)/3 = 0. ■

Corollary 47.2 (the coupling ratio). With g_s the coupling of the SU(3) generators normalized by tr T^2 = 1/2 and e the coupling of the central circle with charge c on the coordinates,

```
e^2 / g_s^2 = <K_T, K_T>_eta / <K_c, K_c>_eta = (3 + eta^2) / (24 eta^2 c^2).
```

1. c = 1/3, the unit in which the coordinates carry charge 1/3 and the colour-singlet cubic z_1 z_2 z_3 carries charge 1 (the singlet Killing spinor then carries 1/2), gives 3(eta^2 + 3)/(8 eta^2); c = 2/3 gives 3(eta^2 + 3)/(32 eta^2): the two formulas of the parked job 04, which belong to two charge units of one computation. In general the spinor triplet carries -c/2 and the singlet 3c/2.
2. At the round point the Killing spinors form the 4 of SU(4) = Spin(6), which restricts to U(3) as 3_q + 1_{-3q}; with the singlet at charge 1 the triplet has charge -1/3 (the charges of the down quark and the positron, up to the overall sign), Q = diag(-1/3, -1/3, -1/3, 1), and the coordinates of C^3, which span one of the two triplets of the vector 6 = Lambda^2(4) = 3_{2/3} + conj(3)_{-2/3} (which one depends on the orientation of the complex structure; c^2 does not), carry c = 2/3, the charge of the up quark. With this unit, e^2/g_s^2 = 3(eta^2 + 3)/(32 eta^2), equal to 3/8 at eta = 1.
3. At eta = 1 the same value follows from Schur's lemma alone: the form is ad-invariant under so(6) = su(4), which is simple, so it is proportional to the trace form in the 4, and tr Q^2 = 3/9 + 1 = 4/3 against tr T^2 = 1/2 gives e^2/g_s^2 = (1/2)/(4/3) = 3/8.
4. In the Einstein-term reduction, the function 3(eta^2 + 3)/(32 eta^2) = 3/32 + 9/(32 eta^2) decreases from infinity at eta -> 0 to 3/32 at eta -> infinity, so each value above 3/32 is taken at exactly one shape, and 3/8 at the round point only. Off the round point the metric is not Einstein, so the shape needs a supporting source whose contribution to the kinetic terms changes the curve: with a term lambda int Ric_eta(K, K) added, int Ric_eta(K_T, K_T) = V tr T^2 [(6 - 2 eta^2)/4 + eta^4/3] and int Ric_eta(K_c, K_c) = 4 V eta^4 c^2, the ratio stays 3/8 at eta = 1 for every lambda and falls below 3/32 at large eta for lambda = 2 (referee).

Proof. Items 1 and 2 are substitutions in Proposition 47.1 and the branching of the 4 and of Lambda^2(4) under U(3) in SU(4). Item 3: a U(3)-invariant symmetric form on so(6) that is also invariant under the rest of SO(6), which holds at eta = 1 since SO(6) acts isometrically, is ad(so(6))-invariant, and so(6) is simple. Item 4: the derivative of (eta^2 + 3)/eta^2 = 1 + 3/eta^2 is negative. ■

Remark 47.3 (the dictionary). Connes' finite triple fixes at matching g_3 = g_2 and sin^2 theta_W = 3/8 (lemma 03: cos^2 theta_W = 5/8 at matching with the full module light; in its heavy-up family the neutral-current angle stays 5/8 and the photon and gluon coefficients keep their ratio, so e^2/g_3^2 = 3/8 holds there too, while the charged-W/hypercharge matching ratio moves), so e^2 = (3/8) g_3^2 there; directly from the traces of lemma 03, Part B, per generation the colour trace is 2 and tr Q^2 = 8/3 + 2/3 + 2 = 16/3, so e^2/g_3^2 = 2/(16/3) = 3/8, which is also the SU(5) value, the lemma's Q being the electric charge on the first four entries of the 5 = (d, d, d, e^+, conj(nu)). The round S^5 gives the same ratio of the colour and electromagnetic couplings, at tree level, with the charge unit of its own Killing spinors. So the test proposed in T18, whether the object at 9 fixes the gauge normalization, distinguishes the finite triple from the squashed spheres, and the round sphere gives the finite triple's value; the round point is the point of the Berger family where the isometry algebra enlarges from su(3) + u(1) to su(4), an enhanced-symmetry point in the language of candidate 5 of T18, the matching value sits there, and there it is independent of the source holding the shape. In the Einstein-term reduction the values along the family lie in (3/32, infinity), and the direction of squashing that lowers the ratio is eta > 1, the fibre stretched. The value 3/8 depends only on the embedding u(3) in su(4) with the 4 = 3_{-1/3} + 1_1; the same embedding carries the B - L reading of Pati-Salam, where the 4 is 3_{1/3} + 1_{-1} and Q is -(B - L), and the superconformal R-symmetry of AdS_5 x S^5. At any enhanced point where su(3) + u(1) sits in a simple algebra g the ratio is tr_V T^2 / tr_V Q^2 for any representation V of g, so the matching value appears exactly when the embedding gives Q the SU(5) index, tr Q^2 = (8/3) tr T^2 (referee).

Prior work (found by the referee). The formula on the Berger sphere was not found in print (searches on squashed five-spheres and on the weak angle from Kaluza-Klein theory before 1992). The round-point value is implicit in holographic form: Barnes, Gorbatov, Intriligator and Wright (Nucl. Phys. B 732 (2006) 89, arXiv:hep-th/0507146, sections 3 and 6) give for IIB on S^5 the coefficients tau_SO(6) = N^2 and tau_RR

= 4N^2/3, with the U(1)_R Killing vector of norm squared 4/9, which is the central generator with c = 2/3 (the holomorphic 3-form at charge 2); with tau_SU(3) = N^2/2 this gives tau_SU(3)/tau_RR = 3/8, and a field-theory check gives the same value from N = 4 super Yang-Mills. So the charge unit of Corollary 47.2 is the superconformal R-charge. Related: Ezawa, Koh and Rajpoot, Phys. Lett. B 147 (1984) 284, Pati-Salam unification from an SU(4) x SU(2) isometry (the B - L reading); Hu and Shen, Phys. Lett. B 178 (1986) 373, the weak angle from a deformed three-sphere, the S^3 analogue of this family. The identifiers are in paper/LITERATURE.md, section F.

Scope

Tree-level Kaluza-Klein reduction of the Einstein-Hilbert term, with no other contribution to the gauge kinetic terms and no threshold corrections; the Berger family of U(3)-invariant metrics on S^5; the charge unit as an input, fixed here by the SU(4) spinor at the round point and continued along the family. The identification of the SU(3) acting on C^3 with colour and of the central circle with electromagnetism is the reading of the 9 end of T18 (a five-dimensional internal space with isometry su(3) + u(1)); the same embedding carries the B - L and the R-symmetry readings, and the value fixes the index of the embedding and leaves the reading open. Off the round point the Berger metric is not Einstein, so the shape needs a supporting source, whose contribution to the gauge kinetic terms is outside the statement; the value 3/8 at eta = 1 is unaffected by any SO(6)-invariant contribution, by Schur's lemma. The matter content, the running from the matching scale and the physical value of e^2/g_s^2 are outside the statement. The direction the result points to: the round point as the matching point of the dictionary, where both sides agree and the value is source-independent, and the squashing eta as the parameter that the finite triple does not have; for the other geometric families of T18 the question reduces to an index computation, whether the embedding of su(3) + u(1) at the enhanced point gives Q the SU(5) index, and the methods of Barnes, Gorbatov, Intriligator and Wright compute such ratios on other Sasaki-Einstein five-manifolds with flavour symmetry (referee).

Part 48

The spin lift of the Weyl action and the quaternions: the real Standard Model algebra from two orbit types

For STRATEGY T16 (second half: the real form H), following lemma 45 (Proposition 45.1, Theorem 45.2 and Remark 45.3 as corrected by its referee: the corpus obtains C + H at the orbit type (a, a, a) and not at the fixed point, where the lift of the inversion is central) and lemma 16 (Theorem 16.7, Corollary 16.8). Written by Claude Fable 5.1 on 5 November 2026, by hand, from lemma 45 and the representation theory of the binary dihedral group, without consulting the corpus construction (historical/corpus, canonical section 10, which the referee of lemma 45 cited for the same orbit type); the character table of Dic_3 and the Frobenius-Schur types were checked by hand. Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_47_48.md), who built an explicit Cl(6), the lifted groups with their element orders, the spinor decompositions, the real spans and their centres, the fibres of the spinorial crossed product and the bimodule axioms: Propositions 48.1 and 48.2 correct with amendments (one step of the proof; the displayed matrix of the standard representation; the lift group is also S_3 x Z_4; the spin structure on A is forced); Theorem 48.3 items 1 and 2 correct with amendments (the corner at O_1 against the quotient at O_2 is a choice; C + H is a real subalgebra, not a real form); item 3 had an error in its last sentence and overread the selection; the Outcome and Remark 48.4 misdescribed lemma 45's complex algebra; all applied below, with the referee's table of orbit types and its leads.

Outcome. The Weyl group W = S_3 x Z_2 of SU(3) x SU(2) acts on T^6 = A x E by isometries, and its lift to the spinors is the group W-tilde = (Dic_3 x Z_4)/Z_2 of order 24, isomorphic to S_3 x Z_4: the transpositions lift to elements of order four, which is the obstruction rs_N20 of the corpus, the inversion of E lifts to a central element of square -1, and the twisted copy of S_3 that pairs each odd permutation with the inversion lifts homomorphically (Proposition 48.1). The algebra that acts on the spinors at an orbit is the spinorial part of the real group algebra of the lifted stabilizer, the part where -1 acts as -1, each

of whose irreducible representations occurs in the spinor fibre. At the orbit type (a, a, a), 3a = 0, with a point of E that is not 2-torsion, the stabilizer is S_3, its lift is Dic_3, and the spinorial part of R[Dic_3] = R + R + M_2(R) + C + H is C + H: the pair of characters in which the lifted transposition acts as +-i, and the faithful two-dimensional representation, which is quaternionic (Proposition 48.2). At the fixed point the central lift of the inversion acts as +-i on every spinorial representation, so the spinorial part is C + C + M_2(C), with no quaternionic summand even over R (lemma 45 used the same point with the complex untwisted group algebra, from which it selected C + M_2(C)); at the three-point orbit the spinorial part is C + C, two conjugate pairs of characters, each pair a copy of M_3(C) after Green's imprimitivity. The spinorial condition fixes the algebra on each of the six orbit types, and the orbit type (a, a, a) is the only one whose stabilizer algebra has a quaternionic summand, as the three-point orbit is the only one with matrix size three. So the two-sheet bimodule of Theorem 45.2, with its first sheet moved from the fixed point to the orbit type (a, a, a), carries the real algebra C + H + M_3(C), the Standard Model algebra, the two orbit types being forced by that target (Theorem 48.3). What remains of T16: a Dirac operator from the geometry (item (b) of Remark 45.3), a real structure from the geometry in place of the adjoint flip of Corollary 16.8, and the choices listed in Remark 48.4.

Setting

Let E be an elliptic curve with its flat metric, A = {(z_1, z_2, z_3) in E^3 : z_1 + z_2 + z_3 = 0}, isometric to a flat four-torus with the metric induced from E^3, and X = A x E. The Weyl group W = S_3 x Z_2 acts by isometries: S_3 permutes the z_i, and the generator iota of Z_2 acts on E by z -> -z. The tangent space of A at any point is a (x) R^2, with a = {x in R^3 : sum x_i = 0} the standard real representation rho of S_3 and R^2 the tangent space of E; the tangent space of E is R^2. Fix the spin structure of X induced from the trivial spin structures of the factors (periodic spinors); the periodic spin structure is the only one on A invariant under S_3 (S_3 acts on A_2/2A_2 = F_2^2 as GL_2(F_2), with no nonzero invariant functional), and the four spin structures of E are invariant under the inversion and give the same lifted stabilizers (referee). The spinor bundle is X x Sigma_6 with Sigma_6 = C^8, and an affine isometry g with constant differential dg in SO(6) acts on spinors through either of the two lifts of dg to Spin(6) in Cl(6), which differ by the sign. Write B-tilde = C(X, R) x W-tilde for the real crossed product by the lift group W-tilde, the preimage of dW in Spin(6), acting on L^2(X, Sigma_6) with -1 in W-tilde acting as -1 on spinors; let B_spin be its quotient by the ideal generated by u_{-1} + 1, the part that acts on the spinors; since -1 is central in W-tilde and acts trivially on X, u_{-1} is central, B_spin is the direct summand B-tilde (1 - u_{-1})/2, and it is the twisted crossed product of C(X, R) by W for the cocycle of the extension W-tilde -> W (referee). For a finite group S-tilde with a central element -1 of order two, the spinorial part of its real group algebra R[S-tilde] is the quotient by u_{-1} + 1, the sum of the simple summands on which -1 acts as -1. The orbit types of W on X and the fibres of the complex crossed product are those of lemma 45, Proposition 45.1.

Statements

Proposition 48.1 (the lift of the Weyl action). (i) For a transposition tau in S_3, d tau = rho(tau) (x) 1 on a (x) R^2 is the rotation by pi in the two-plane n (x) R^2, with n the normal of the reflection rho(tau); both of its lifts to Spin(4) are the products +- f_1 f_2 of an orthonormal basis of that plane, of order four. For a three-cycle c, d c is the rotation by 2 pi/3 in two orthogonal two-planes, and one of its lifts has order three. The lift of S_3 to Spin(4) is the binary dihedral group Dic_3 = <a, x | a^6 = 1, x^2 = a^3, x a x^{-1} = a^{-1}> of order 12, and no homomorphism S_3 -> Spin(4) lifts d: this is the corpus statement rs_N20, either preimage of a transposition squares to -1. (ii) d iota = -1 on the tangent space of E lifts to iota-tilde = +- e_5 e_6 in Spin(2), with iota-tilde^2 = -1, which commutes with the lift of S_3. (iii) The lift group is W-tilde = (Dic_3 x Z_4)/Z_2, of order 24, with Z_4 generated by iota-tilde and the Z_2 identifying the two elements -1. It is isomorphic to S_3 x Z_4: the elements a^k x iota-tilde are involutions, since (x iota-tilde)^2 = x^2 iota-tilde^2 = 1, so the twisted copy {(sigma, iota^{sgn sigma})} of S_3 in W lifts homomorphically (tau iota is the rotation by pi of the four-dimensional subspace (n (x) R^2) + T E, whose lift squares to +1, as in the T^4/Z_2 orbifold), and W-tilde is the product of that lift with ; so rs_N20 holds for S_3 and for every subgroup of W that contains a transposition without the inversion (referee). The lifted stabilizers are Dic_3 at an orbit of stabilizer S_3, W-tilde at the fixed point, and (Z_4

x Z_4)/Z_2, of order 8, generated by the lifts of a transposition and of iota, at the three-point orbit of lemma 45.

Proof. (i) rho(tau) is a reflection of a, rho(tau) = 1 - 2 n n^T, so rho(tau) (x) 1 = 1 - 2 P with P the orthogonal projection onto the two-plane n (x) R^2: the identity on the orthogonal complement and -1 on the plane, the rotation by pi in it. In Cl(4) the rotation by pi in the plane of the orthonormal vectors f_1, f_2 lifts to +- f_1 f_2, and (f_1 f_2)^2 = -f_1^2 f_2^2 = -1. rho(c) is the rotation by 2 pi/3 of a, so rho(c) (x) 1 is that rotation in each of the two planes a (x) e_1 and a (x) e_2; with orthonormal bases g_1, g_2 and h_1, h_2 of the two planes, the lift exp((pi/3)(g_1 g_2 + h_1 h_2)) has cube exp(pi g_1 g_2) exp(pi h_1 h_2) = (-1)(-1) = 1, of order three. The preimage of rho(S_3) (x) 1 in Spin(4) is a group of order 12 containing a cyclic subgroup of order 6 generated by a = (lift of c of order 3) times (-1), and an element x of order four with x^2 = -1 = a^3; x c-tilde x^{-1} has order three and lies above tau c tau^{-1} = c^{-1}, so it is the order-three lift c-tilde^{-1}, hence x a x^{-1} = x (-c-tilde) x^{-1} = -c-tilde^{-1} = a^{-1}; this is the presentation of Dic_3, which has a single involution, -1, and so differs from Z_2 x S_3 and from the abelian groups of order 12. A lift S_3 -> Spin(4) would send tau to an element of order two above d tau, and both candidates have order four. (ii) The rotation by pi of the plane of E lifts to +- e_5 e_6, of square -1; e_5 e_6 is an even element in the Clifford algebra of the E-directions and commutes with the even elements in the Clifford algebra of the A-directions. (iii) The lifts of S_3 and of Z_2 commute and share the element -1, which gives the central product; the stabilizers are read off from lemma 45 and from the fact that -1 lies in every lifted stabilizer. ■

Proposition 48.2 (the spinorial parts of the real group algebras). (i) The irreducible representations of Dic_3 are the four characters of its abelianization Z_4, x -> i^k and a -> (-1)^k for k = 0, 1, 2, 3, and two representations of dimension two, a -> diag(zeta, conj(zeta)) and x -> [[0, zeta^3], [1, 0]], that is [[0, -1], [1, 0]] for zeta a primitive sixth root of unity (faithful) and [[0, 1], [1, 0]] for zeta a primitive third root (the standard representation of S_3). Their Frobenius-Schur types are real for k = 0 and k = 2 and for the standard representation, complex for the conjugate pair k = 1, 3, and quaternionic for the faithful representation, so

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R[Dic_3] = R + R + M_2(R) + C + H,    of real dimension 1 + 1 + 4 + 2 + 4 = 12.
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The spinorial part, where -1 = x^2 = a^3 acts as -1, is C + H: the pair k = 1, 3, on which the lifted transposition x acts as +-i, and the faithful representation. Each spinorial irreducible representation of Dic_3, of W-tilde and of the lift of <tau, iota> occurs in Sigma_6 (Sigma_6 restricted to Dic_3 is 2(chi_1 + chi_3) + 2 V_f, with chi_1 + chi_3 and V_f in opposite chiralities of the A-directions), so the spinorial part is the algebra generated on the spinor fibres over the orbit (referee). (ii) At the fixed point, the spinorial irreducible representations of W-tilde are the spinorial representations of Dic_3 tensored with the two characters iota-tilde -> +-i, all of complex type, and the spinorial part of R[W-tilde] is C + C + M_2(C), with no quaternionic summand: in the description W-tilde = S_3 x Z_4 with -1 = iota-tilde^2, it is R[S_3] (x)_R C. (iii) At the three-point orbit, the lifted stabilizer (Z_4 x Z_4)/Z_2 has four spinorial characters, x -> +-i and iota-tilde -> +-i, forming two conjugate pairs, and the spinorial part of its real group algebra is C + C.

Proof. (i) The commutator subgroup of Dic_3 is <a^2>, of order three, since x a x^{-1} a^{-1} = a^{-2}, so the abelianization has order four and is generated by the image of x, with a mapping to x^2 = a^3; this gives the four characters, with a -> (-1)^k. The two-dimensional representations are induced from the characters a -> zeta of = Z_6 with zeta != +-1; zeta and conj(zeta) give the same representation, so there are two, and the sum of squares 4 x 1 + 2 x 4 = 12 shows the list is complete. The character k = 2 is the sign of S_3 and k = 0 the trivial one; k = 1, 3 are conjugate and take the value +-i, so their type is complex. The faithful representation lies in SU(2), with x^2 = -1, so its Frobenius-Schur indicator is -1 and its type is quaternionic; the standard representation of S_3 is real. The real group algebra is the sum over the irreducible real representations of the matrix algebras over their commutants, R, C or H, which gives the display; -1 acts as -1 exactly on the characters with k odd and on the faithful representation. (ii) W-tilde is a central product, so an irreducible representation of W-tilde is an irreducible representation of Dic_3 tensored with a character of = Z_4 agreeing on -1; the spinorial ones pair the spinorial representations of Dic_3 with iota-tilde -> +-i. A representation on which a central element acts as i or -i is of complex type,

since a real or quaternionic structure would commute with the central element and conjugate i to -i. So the characters give C + C and the faithful representation tensored with +-i gives M_2(C). (iii) The group is abelian, generated by x and iota-tilde with x^2 = iota-tilde^2 = -1; its spinorial characters are the four sign choices, in the two conjugate pairs (i, i), (-i, -i) and (i, -i), (-i, i), each pair a copy of C in the real group algebra. ■

Theorem 48.3 (the real two-sheet bimodule over the two orbit types). Let O_1 be an orbit of type (a, a, a), 3a = 0, with a point of E that is not 2-torsion, of size two with stabilizer S_3, and O_2 the three-point orbit of lemma 45. Then:

1. The fibre of B_spin at O_1 is M_2(C + H), and its corner at one point of the orbit is the real algebra C + H, with C the character pair of Proposition 48.2 (i) and H the faithful representation; the fibre at O_2 is M_3(C + C), and one conjugate pair of characters of the stabilizer gives the real algebra M_3(C). The first sheet uses the corner at one point of O_1 and the second the full quotient at O_2; this is a choice, like the choice of characters (with corners on both sheets the algebra would be C + H + C, with full fibres on both M_2(C) + M_2(H) + M_3(C)).

2. The data of Theorem 45.2 with V_X = C + C^2, the real algebra A_{F,X} = C + H acting by the character on C and by left multiplication on C^2 = H, the grading G = (1, -1), central in C + H, and d_X = D_v; V_Y = C^3 with A_{F,Y} = M_3(C), gamma_Y = 1, d_Y = 0, give through the finite content of Corollary 16.8 the 18-state bimodule H_F = Hom(V_Y, V_X) + Hom(V_X, V_Y) with the KO-6 signs, order zero, order one, nonzero one-forms and the real orientability relation, now over the real algebra

   A_F = C + H + M_3(C),

   the Standard Model algebra, with the intersection form on minimal projections [[0, 0, 1], [0, 0, -2], [-1, 2, 0]] of the corpus (rs_N6), skew of rank 2.

3. The spinorial condition fixes the algebra on each orbit type. The six orbit types of W on X, with the spinorial parts of the lifted stabilizer algebras and the fibres, are: trivial stabilizer, orbit of size 12, R, M_12(R); a transposition, size 6, C, M_6(C); the inversion, size 6, C, M_6(C); S_2 x Z_2 (O_2), size 3, C + C, M_3(C + C); S_3 (O_1), size 2, C + H, M_2(C + H); S_3 x Z_2 (the fixed points), size 1, C + C + M_2(C); there is no stabilizer of type Z_3 alone, since a three-cycle fixes only the diagonal points, which S_3 fixes (referee). So the orbit type (a, a, a) is the only one whose stabilizer algebra has a quaternionic summand, and there the spinorial part is exactly C + H; the three-point orbit is the only one with matrix size three. Given the target C + H + M_3(C), the two orbit types are forced. The module V_X = C + C^2 keeps one of the two conjugate characters, a choice of submodule of the spinor fibre chi_1 + chi_3 + V_f; on the whole fibre C acts by (lambda, conj(lambda)) and no choice is made. At O_2 the choice is between the two conjugate pairs of Proposition 48.2 (iii). At the fixed point the spinorial part is C + C + M_2(C), of complex type in every summand; lemma 45 used the same point with the untwisted complex group algebra, from which it selected C + M_2(C).

Proof. (1) Green's imprimitivity for the finite orbit, as in Proposition 45.1 (ii), holds with real coefficients and with the twisted action: C(O, R) x_{lift} W is M_{|O|} of the spinorial part of R[S-tilde] for the lifted stabilizer S-tilde, the functions on the orbit acting by diagonal matrices and the lifted group elements by permutation matrices twisted by S-tilde; the corner at a point is the stabilizer algebra. Proposition 48.2 gives the stabilizer algebras. (2) C + H is a real *-subalgebra of the complex algebra C + M_2(C) of Theorem 45.2, acting on the same V_X, whose complex span is C + M_2(C) (a real form of C + M_2(C) would have real dimension 5; C + H has 6); the grading G = (1, -1) lies in it and is central; Theorem 16.7 and Corollary 16.8 are formal for any finite even representation (Remark 16.5), and their conclusions, the signs, the order conditions, the one-forms and the orientability relation, hold for the real algebra as for its complexification, since they are stated for the represented operators. The intersection form over the real algebra, with the minimal projection of H of rank two, is the corpus form, as the referee of lemma 45 computed. (3) Proposition 48.2. ■

Remark 48.4 (what this settles and what it leaves). Settled: the real form H of the second summand, from the quaternionic type of the faithful representation of the binary dihedral lift, at the orbit type the

corpus used, reached here from representation theory alone; the absence of a quaternionic summand at the fixed point in the spin sector; the forcing of the two orbit types by the target, since the spinorial condition fixes the algebra on each orbit type and the quaternionic summand occurs only at stabilizer S_3, which is the positive form of item (c) of Remark 45.3 for the orbit types. Left: item (b) of Remark 45.3, a Dirac operator from the geometry in place of the inserted D_v, for which the torus Dirac operator with the W-tilde action on spinors is now in place (and the twisted S_3 acts without the cocycle), and the compression question of lemma 20, Result 7; a real structure from the geometry, since the J here is the adjoint flip of Corollary 16.8 between the two sheets, an input of the two-sheet construction (lemma 16, Caveat 4), and a derivation from the regular bimodule of the crossed product is open; the choices: the corner at O_1 against the quotient at O_2, one character or the whole fibre at O_1, the pair at O_2; and the physical identification of the orbit O_1, a pair of points of T^6 exchanged by the inversion of E at a 3-torsion configuration of A, against the fixed point of lemma 45. Leads from the referee: on the whole spinor fibre at O_1, Sigma_4 = chi_1 + chi_3 + V_f with C acting as (lambda, conj(lambda)) on the two characters and H on the doublet, in opposite chiralities of the A-directions, the action of C + H on the right-handed singlets and the left-handed doublet of one quark generation; with V_X = Sigma_4, G the A-chirality (1, 1, -1, -1) and V_Y = C^3, Corollary 16.8 gives a 24-state bimodule over C + H + M_3(C), the bimodule form of one quark generation, with all axioms (checked numerically by the referee) and intersection form [[0, 0, 2], [0, 0, -2], [-2, 2, 0]], of which the 18-state bimodule is the restriction to one character; the grading G of Theorem 48.3 is the A-chirality e_1 e_2 e_3 e_4 restricted to the stabilizer algebra, which lies in the image of R[Dic_3] on the spinors, so the grading is available from the geometry; a lepton sheet would need an orbit whose spinorial algebra is C, such as the corner at a point with stabilizer a transposition or the inversion, untested.

Scope

Flat T^6 with the periodic spin structure and the isometric action of W; real crossed products by the lift group, with the spinorial quotient; finite groups and finite orbits, where Green's imprimitivity is elementary; the Dirac operator D_v and the grading as inputs, as in lemma 45. The direction: with the lift group acting on the spinors of T^6, the compression of the torus Dirac operator to the two selected sheets is the next computation of T16, and the Frobenius-Schur type of a lifted stabilizer's spinorial representations is the rule that decides the real form of each summand, the rule that also decides the orthogonal, symplectic and unitary factors of orientifold gauge groups (Blum and Intriligator 1997, section 5, where the representations of Dic_3 are sorted by type with the spinorial ones pseudo-real or complex; paper/LITERATURE.md, section N).

Part 49

The quaternionic sheet as the spinor fibre of the four-torus: the chirality as the axis and the lifted transposition as the hypercharge

For STRATEGY T14 and T16, joining lemma 46 (the odd blocks of the sheet W = H^2 with axis p) and lemma 48 (the spin lift of the Weyl action and the real algebra C + H at the orbit type (a, a, a)), following lemma 16 (Example 16.9), lemma 42 (the isotropy representation of the marked point) and lemma 20 (Theorem 20.2). Proposition 49.5 is the lead that the referee of lemma 48 recorded and checked numerically (historical/reviews/2026-11-05_opus_check_47_48.md, New insights, items 2 and 3), written here with its proof. Written by Claude Fable 5.1 on 5 November 2026, by hand; the Clifford algebra of the four-torus with its chirality and quaternionic structure, the lifts of a transposition and of a three-cycle, their action on the two chiralities, the real dimension 6 of the algebra they generate, and the complex dimensions 2 and 0 of the Yukawa blocks invariant under the lifted transposition and under the whole lifted stabilizer were checked with numpy (script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_49.md), who rebuilt the Clifford algebra, the lifts, the invariant blocks, the stabilizers and the 24-state bimodule with all its axioms: Propositions 49.1, 49.2, 49.4 and 49.5 correct with amendments (the orientation that makes Sigma_R the line of H_R; the order-six lifts; the eigenline condition for lemma 42's torus; the unitary group {+-1} of R; a nonzero block in 49.5 and the condition for the restriction to Theorem 48.3), Proposition 49.3 with one phrase in error (the invariance group of the diagonal blocks is the circle of the lifted transposition on both chiralities, the

algebra C_F of Chamseddine, Connes and Marcolli, and no block is invariant under the circle on Sigma_R alone), four sentences of the Outcome and Reading overread; all applied, with the referee's identifications in the spectral Standard Model and its leads.

Outcome. The spinor fibre Sigma_4 = Sigma_R + Sigma_L of the flat four-torus A of lemma 48, with the Clifford action of its tangent space and its chirality Gamma, is the quaternionic sheet W = H^2 of Example 16.9 with the axis p at Gamma: the chirality is the grading axis, the two chiralities are the quaternionic lines L_p and L_p^perp, Spin(4) is the Pati-Salam pair Sp(1)_a x Sp(1)_b, and the Clifford action of the tangent space of A is the bidoublet Hom_H(L_p, L_p^perp) = T_p S^4, the Higgs directions of lemmas 42 and 46 (Proposition 49.1). The lifted stabilizer Dic_3 of the orbit type (a, a, a) acts on one chirality, Sigma_R, through its abelianization Z_4 and on the other, Sigma_L, through its faithful representation, so the real algebra it generates is C + H inside the Pati-Salam H_R + H_L = End_H(Sigma_R) + End_H(Sigma_L), with C the Cartan subalgebra of H_R generated by the lifted transposition, which acts on Sigma_R as (i, -i) (Proposition 49.2): the circle of the lifted transposition is a maximal torus of Sp(1)_R, the T_3R circle, which is the U(1)_b that lemma 42 marked by hand exactly when the connector's line is one of its two weight lines, and the torus that Corollary 46.4 reads from the masses exactly when the mass eigenlines are its weight lines; and the whole fibre carries the action (lambda, conj(lambda)) of C on the two right-handed singlets and of H on the left-handed doublet of the spectral Standard Model. The Yukawa blocks commuting with the lifted transposition, acting on both chiralities, are exactly the diagonal ones, X = diag(y_1, y_2) in the weight bases, the Y_u, Y_d of that model, with quaternionic half (y_1 + conj(y_2))/2 and anti-quaternionic half (y_1 - conj(y_2))/(2i); commuting with that circle is the condition of Chamseddine, Connes and Marcolli that the finite Dirac operator commute with their subalgebra C_F, the photon condition; no nonzero block is invariant under the circle on Sigma_R alone, and none under the whole lifted stabilizer, since the three-cycles act freely on Sigma_L (Proposition 49.3). Along the orbit types of A the lifted stabilizer algebras are C + H at the nine S_3-fixed points, C on the mirror planes of the transpositions, where the generated algebra is the circle of the lifted transposition acting on both chiralities, the C_F of Chamseddine, Connes and Marcolli restricted to the first sheet and the unbroken circle of Corollary 46.4, and R elsewhere: unitary groups U(1) x SU(2), U(1) and {+-1} (Proposition 49.4). With the whole fibre as the first sheet, Corollary 16.8 gives a 24-state bimodule over C + H + M_3(C), the quark summand H_F^(3) of Chamseddine, Connes and Marcolli for one generation, with its grading the chirality of A and their printed pairing (Proposition 49.5).

Setting

Let A = a (x) R^2 be the flat four-torus of lemma 48, with tangent space R^4 = a (x) R^2 at every point, a the standard representation of S_3, and the periodic spin structure, the only one invariant under S_3 (lemma 48, Setting). Its spinor fibre is Sigma_4 = C^4 = Sigma_+ + Sigma_-, the two half-spin representations of Spin(4) = Sp(1)+ *x Sp(1)_-, each a quaternionic line H on which its Sp(1) acts by left multiplication and the other trivially; the quaternionic structure j, the right multiplication by the unit quaternion j on each line, commutes with Spin(4), with the Clifford multiplication c(v) by vectors v of R^4, and with the chirality Gamma = c(e_1) c(e_2) c(e_3) c(e_4), which is +1 on Sigma+* and -1 on Sigma_-. We orient A so that the lifts of order three of the three-cycles act trivially on Sigma_+; for the complex orientation of A inherited from E, with Gamma = c(e_1) c(e_2) c(e_3) c(e_4) in an oriented orthonormal basis, they act trivially on Sigma_- (the generator of the rotation of a three-cycle is an anti-self-dual two-form for that orientation), so the orientation used here is the opposite one, and with the complex orientation the axis is p = -e_5; with this convention Sigma_R = Sigma_+ = L_p, which carries Sp(1)_b and H_R as in lemmas 42 and 46 (referee). The real structure of the spin geometry of A, with J_M^2 = -1, J_M D = D J_M and J_M Gamma = Gamma J_M, anticommutes with the Hermitian c(v) and equals j Gamma up to a phase; it agrees with j up to sign on each chirality, so H_R + H_L and C + H are the same for both, while on the odd blocks the roles are exchanged, c(v) being J_M-antilinear and i c(v) Gamma J_M-linear; the words quaternionic and anti-quaternionic below refer to j, the structure of Example 16.9 (referee). The circle of the summand C below is the T_3R circle; lemma 42 calls it the hypercharge with Y = charge/2, which is exact on the Higgs, where B - L = 0, and the hypercharge of the spectral Standard Model is its unimodular combination with the U(1) of M_3(C). Put Gamma_a = c(e_a) for a = 1, ..., 4 and Gamma_5 = Gamma: five Hermitian operators with Gamma_a Gamma_b + Gamma_b Gamma_a = 2 delta_ab, commuting with j, whose

real span together with the Gamma_a Gamma_b is End_H(Sigma_4) = M_2(H). The unitary group of End_H(Sigma_4) is Sp(2), containing Spin(4) as the stabilizer of Gamma_5; under Spin(4) the vector R^4 is the (2, 2) and acts, through c, as the odd Hermitian j-linear maps Sigma_+ -> Sigma_- and back. The lifted stabilizer of the orbit type (a, a, a) is Dic_3 = <a, x> in Spin(4), x the lift of a transposition, of order four, and c-tilde = -a the lift of order three of a three-cycle (lemma 48, Proposition 48.1). Sigma_6 restricted to Dic_3 is twice Sigma_4 (lemma 48), so the whole fibre of T^6 carries two copies of everything below, with the spinors of E as multiplicity space, on which T E and the inversion act; c(e_5) and c(e_6) anticommute with the c(e_a) and commute with the chirality of A, outside the Cl(5) below.

Statements

Proposition 49.1 (the sheet as the spinor fibre). (Sigma_4, Gamma_1, …, Gamma_5, j) is the quaternionic sheet W = H^2 of Example 16.9 and lemma 46 with the axis p = e_5: Gamma_p is the chirality, L_p = Sigma_+ and L_p^perp = Sigma_-, the stabilizer K_p = Sp(1)_b x Sp(1)_a is Spin(4) = Sp(1)+ *x Sp(1)_-, and the bidoublet Hom_H(L_p, L_p^perp) = T_p S^4 of lemma 42 is the Clifford action of the tangent space of A: v -> c(v) restricted to Sigma+* is a Spin(4)-equivariant isomorphism R^4 -> Hom_H(Sigma_+, Sigma_-). The even algebra H + H of Example 16.9 for this axis is End_H(Sigma_+) + End_H(Sigma_-), the Pati-Salam pair acting on the two chiralities, and the odd Hermitian blocks of lemma 46 are the j-linear c(v) and the j-antilinear i c(v) Gamma.

Proof. Cl(4) acts on Sigma_4 irreducibly with c(v)^2 = |v|^2 and c(v)^* = c(v) for the Hermitian metric of the spinors, and the chirality Gamma anticommutes with every c(v), is Hermitian and squares to one; so Gamma_1, …, Gamma_5 generate Cl(5) on C^4, and Example 16.9 and lemma 46 apply with these generators. The real algebra End_H(Sigma_4) has real dimension 16 and contains the 16 linearly independent elements 1, Gamma_a, Gamma_a Gamma_b, so it is their real span, M_2(H); Spin(4) is the subgroup of Sp(2) = U(End_H(Sigma_4)) that commutes with Gamma_5, since the even Clifford algebra of R^4 is Spin(4)'s and the stabilizer of Gamma_5 in Sp(2) is the unitary group of the even algebra H + H, which is Sp(1) x Sp(1). The map v -> c(v)|*{Sigma+}* is Spin(4)-equivariant by the covariance of Clifford multiplication, injective, and both sides have real dimension four, Hom_H between two quaternionic lines being a copy of H. The rest is lemma 46, Proposition 46.1, with Gamma_p = Gamma. ■

The identification is the isomorphism of Cl(5)-modules with quaternionic structure that matches c(e_a) with Gamma_a for a = 1, …, 4 and Gamma with Gamma_5, unique up to a phase by Schur's lemma. Composing it with an element of Sp(2) carries the axis to any point of S^4: the axis is fixed by the four-torus, whose symmetry at a point is Spin(4), and it remains an input of the twistor window, whose symmetry is Sp(2) (referee).

Proposition 49.2 (the lifted stabilizer inside the sheet). (i) The lifts of order three of the three-cycles act trivially on one chirality, which we call Sigma_R, and by rotations of order three on the other, Sigma_L; the lifts of order six act as -1 on Sigma_R and with order six on Sigma_L; the lift x of a transposition acts on both with eigenvalues +-i, and the three transpositions give the same circle on Sigma_R up to sign. So Dic_3 -> Sp(1)_R has image the cyclic group Z_4 generated by x|*{Sigma_R}, and Dic_3 -> Sp(1)_L is the faithful embedding of the binary dihedral group in SU(2). (ii) The real algebra generated by Dic_3 on Sigma_4 is C + H, with C = R[x|_{Sigma_R}] = span_R(1, x|*{Sigma_R}) acting on Sigma_R as (lambda, conj(lambda)) in the eigenbasis of x, and H = End_H(Sigma_L) the real span of the faithful representation: the part acting on the first sheet of the Standard Model subalgebra A_F = {(lambda, q_L, lambda, m)} of Chamseddine, Connes and Marcolli (arXiv:hep-th/0610241, Proposition 2.11 and Definition 2.14, where every embedding of C in H is conjugate to this one), inside the Pati-Salam H_R + H_L of Example 16.9, with C the Cartan subalgebra of H_R generated by the lifted transposition; in their model the same lambda also acts on the lepton line of the second sheet. (iii) Consequently the circle generated by the lifted transposition x is a maximal torus of Sp(1)_R, the same for the three transpositions. It is the U(1)_b of lemma 42 exactly when the connector's line Cw is one of its two weight lines, and it is the torus U(1)_X of Corollary 46.4 exactly when X^* X has distinct eigenvalues with the weight lines of x as eigenlines, as for the blocks of Proposition 49.3 with |y_1| != |y_2|; its two weight lines in Sigma_R are the two right-handed singlets on which C acts by lambda and conj(lambda).

Proof. (i) Spin(4) = Sp(1)+ x *Sp(1)_- acts on R^4 = H by v -> p v conj(q). Identify a with C by a complex structure, so that R^4 = a (x) R^2 = C (x) R^2 = C^2 = H; the differential of a three-cycle, the rotation by 2 pi/3 of a tensored with the identity of R^2, is the complex scalar e^{2 pi i/3} on C^2, a rotation by the same angle and in the same sense in two orthogonal planes, which is left (or right) multiplication by a unit complex number: an element of one Sp(1) factor only. Its lift c-tilde of order three therefore acts trivially on the chirality of the other factor, Sigma_R, and on Sigma_L by a rotation of order three; the lift -c-tilde acts as -1 on Sigma_R and with order six on Sigma_L. The differential of a transposition is the rotation by pi in the two-plane n (x) R^2 (lemma 48, Proposition 48.1), with eigenvalues (-1, -1, 1, 1). The lift x has x^2 = -1 (lemma 48), and -1 = (-1, -1) in Sp(1)+ x Sp(1)_-, so each component u of x satisfies u^2 = -1: both components are unit imaginary quaternions, with eigenvalues +-i on their chiralities. As a cross-check, an element of one factor alone acts on H by left or right multiplication by a unit quaternion, an isoclinic rotation with the same angle in two orthogonal planes, while d tau has the angles pi and 0 (for instance (i, i) acts as v -> i v i^{-1}, the identity on the complex line of 1 and i and -1 on that of j and k, a rotation by pi in one plane). (ii) The representation of Dic_3 on Sigma_R + Sigma_L is spinorial, -1 acting as -1, and faithful on the spinorial part C + H of R[Dic_3] (lemma 48, Proposition 48.2, where Sigma_R carries the character pair chi_1 + chi_3, since x acts as (i, -i) and a as -1, and Sigma_L carries the faithful representation V_f); so the image of R[Dic_3] is C + H, with the C acting on Sigma_R through span_R(1, x) = C as (lambda, conj(lambda)) in the eigenbasis of x, and the H acting on Sigma_L as the real span of the finite subgroup Dic_3 of Sp(1), which is all of H since Dic_3 contains 1, the imaginary unit u and an element outside the complex line of u. The even algebra of Example 16.9 for the axis Gamma is End_H(Sigma_R) + End_H(Sigma_L) = H_R + H_L (Proposition 49.1), and span_R(1, u') is a maximal commutative subalgebra of H_R, a Cartan subalgebra. (iii) The unitary group of the C of (ii) is the circle {cos t + u' sin t} in Sp(1)_R, the maximal torus with weight lines the eigenlines of u' = x|_{Sigma_R}; lemma 42 marks such a torus by hand, and Corollary 46.4 identifies it from a split Yukawa block when the connector's line is one of its weight lines. ■

Proposition 49.3 (the invariant Yukawa blocks). Identify Sigma_R and Sigma_L with H = C + j C so that x acts on both as left multiplication by i (possible by conjugation in Sp(1)_R and Sp(1)_L). The odd Hermitian blocks [[0, X^*], [X, 0]] with X in Hom_C(Sigma_R, Sigma_L) that commute with x are exactly those with X = diag(y_1, y_2) in the bases (1, j) of the two lines, a complex two-dimensional space: the blocks Y_u, Y_d of the spectral Standard Model. Commuting with x is commuting with the circle of x acting on both chiralities, whose real span on Sigma_4 is the image of the subalgebra C_F = {(lambda, lambda, 0)} with which Chamseddine, Connes and Marcolli require the finite Dirac operator to commute (arXiv:hep-th/0610241, Definition 2.20 and Remark 2.19: the photon remains massless). No nonzero odd block is invariant under the circle of x on Sigma_R alone, the circle of the summand C. In the variables of lemma 46, X = L_a + i L_c with a = (y_1 + conj(y_2))/2 and c = (y_1 - conj(y_2))/(2i), both complex numbers, and the mass formula of Proposition 46.2 reads |a|^2 + |c|^2 +- 2 |Im(conj(a) c)| = |y_1|^2, |y_2|^2; the quaternionic blocks commuting with x are those with y_2 = conj(y_1), equal masses with conjugate couplings. No nonzero odd block commutes with the whole of Dic_3, nor with the lift of order three of a three-cycle alone: a Yukawa block breaks the S_3 of the symmetric point, and on the mirror of tau, where the lifted stabilizer is , the blocks commuting with it are those above; the escapes are a fibre larger than Sigma_4 (the E factor) or another grading.

Proof. With x acting as L_i on both lines, X L_i = L_i X means that X preserves the eigenlines of L_i, which are C and j C, so X is diagonal in the basis (1, j); conversely every diagonal X commutes with L_i. For a, c complex, L_a = diag(a, conj(a)) and i L_c = diag(i c, i conj(c)) in that basis, so X = diag(a + i c, conj(a) + i conj(c)) and y_1 = a + i c, y_2 = conj(a) + i conj(c), which inverts as stated; the singular values of a diagonal matrix are |y_1|, |y_2|, and the formula of Proposition 46.2 with q = a, q' = c complex gives |a|^2 + |c|^2 +- 2 |Im(conj(a) c)| = |a +- i c|^2 in the appropriate order. The quaternionic case is c = 0, that is y_2 = conj(y_1). If X commutes with the lift c-tilde of a three-cycle, which acts as the identity on Sigma_R and without fixed vectors on Sigma_L, then X = c-tilde X maps into the fixed space of c-tilde on Sigma_L, which is zero. ■

Proposition 49.4 (the orbit types of A and the unbroken groups). The stabilizer in S_3 of a point of A is S_3 at the nine points (a, a, a) with 3a = 0; the group generated by one transposition on the mirror

z_i = z_j of that transposition, off the nine points; and trivial elsewhere. The real algebras generated on Sigma_4 by the lifted stabilizers are C + H, C and R, with C = span_R(1, x) acting on both chiralities as (lambda, conj(lambda)) in the weight bases of x; their unitary groups are U(1) x SU(2), U(1) and {+-1}, the image of the kernel of Spin(4) -> SO(4). The algebra of the middle type is the restriction to Sigma_4 of the subalgebra C_F of Chamseddine, Connes and Marcolli (Proposition 49.3), and its circle is the common unbroken circle of Corollary 46.4: it is the stabilizer, in K_p = Spin(4), of every diagonal block diag(y_1, y_2) with |y_1| != |y_2| together with a weight line of x in Sigma_R. For a displacement delta of the point along the mirror plane of tau, the Clifford element c(delta) is a quaternionic odd block commuting with x, of the form L_a with a complex, that is y_2 = conj(y_1).

Proof. A permutation fixes (z_1, z_2, z_3) exactly when it permutes equal coordinates; with z_1 + z_2 + z_3 = 0, the three coordinates are equal exactly when 3 z_1 = 0, nine points of the 3-torsion of E, and two are equal on the mirror z_i = z_j. The lifted stabilizers are Dic_3, = Z_4 and the trivial group, whose spinorial real algebras on Sigma_4 are C + H (Proposition 49.2), span_R(1, x) = C acting on both chiralities by x = (i, -i) in its weight bases, and R; the unitary groups follow. The stabilizer in Sp(1)_R x Sp(1)_L of a diagonal block X = diag(y_1, y_2) with distinct moduli is the circle {(X b X^{-1}, b)} with b in the torus of x (Corollary 46.4), and X b X^{-1} = b for diagonal X and b in that torus, so the circle is {(b, b)}, the circle of x acting on both chiralities, which also fixes the weight lines. For delta in the fixed plane of d tau, x c(delta) x^{-1} = c(d tau delta) = c(delta), so c(delta) is a j-linear odd block commuting with x: by Proposition 49.3, L_a with c = 0. ■

Proposition 49.5 (the quark generation on the whole fibre). Apply the finite content of Corollary 16.8 with V_X = Sigma_4, A_{F,X} = C + H acting as in Proposition 49.2, gamma_X = Gamma (the chirality of A, +1 on Sigma_R and -1 on Sigma_L, with the orientation of the Setting; for the complex orientation gamma_X = -Gamma), d_X a nonzero odd Hermitian block of Proposition 49.3, and V_Y = C^3 with A_{F,Y} = M_3(C), gamma_Y = 1, d_Y = 0. Then H_F = Hom(V_Y, V_X) + Hom(V_X, V_Y) has complex dimension 24, carries the KO-6 signs, order zero, order one, nonzero one-forms and the real orientability relation of Corollary 16.8, over the real algebra C + H + M_3(C); it is the quark sector of one generation of the spectral Standard Model (the summand H_F^(3) of Chamseddine, Connes and Marcolli, arXiv:hep-th/0610241, Proposition 2.30, whose pairing they print as the matrix below; their nondegenerate pairing, Corollary 2.31, uses the lepton class as well), C acting on the two right-handed singlets by (lambda, conj(lambda)), H on the left-handed doublet, M_3(C) on colour, with the grading the chirality of A. Its intersection form on the minimal projections 1_C, 1_H, e_11 is [[0, 0, 2], [0, 0, -2], [-2, 2, 0]], skew of rank 2. The 18-state Hilbert space of Theorem 48.3, with its algebra, real structure and grading, is the sub-bimodule on one weight line of x in Sigma_R; D_F preserves it exactly when X vanishes on the other weight line (y_2 = 0 for the line of y_1), and then restricts to D_v with v = (y_1, 0) in the weight basis of Sigma_L.

Proof. As in Theorem 48.3 (2): the data are a finite even representation of the real algebra C + H, which is a real *-subalgebra of End_C(Sigma_4), with the central grading Gamma (1 on Sigma_R, where C acts, and -1 on Sigma_L, where H acts), so Theorem 16.7 and Corollary 16.8 apply formally (Remark 16.5), and their conclusions hold for the real algebra as for its complex span. The dimension is 2 x 4 x 3 = 24. The action is that of the spectral Standard Model on the quark sector: the bimodule Hom(C^3, Sigma_4) = Sigma_4 (x) conj(C^3) is a right-handed up-type and down-type singlet (the two weight lines of x, with lambda and conj(lambda)) and a left-handed doublet, each in three colours, with the conjugates in the other summand. The intersection form is <e, f> = tr(Gamma a_e) tr(b_f) - tr(b_e) tr(Gamma a_f) as in Theorem 48.3, with tr(Gamma 1_C) = 2, tr(Gamma 1_H) = -2, tr(b_{e_11}) = 1. The subspace on one weight line of x in Sigma_R, (line + Sigma_L) (x) conj(C^3) with its conjugate, has dimension 18 and is invariant under the algebra, J and Gamma_F; D_F maps Sigma_L to Sigma_R by X^*, whose image is all of Sigma_R when y_1 y_2 != 0, so D_F preserves the subspace exactly when X vanishes on the other weight line. ■

Reading

In the four-torus model the three inputs that the twistor picture left as data, the grading axis (Proposition 28.19, Example 16.9), the marked torus U(1)_b in Sp(1)_b (lemma 42) and the real form of the second summand (lemma 45), are supplied by the spinor fibre of A with its Weyl stabilizer: the axis is the chirality

of A, fixed up to the orientation; the torus is the circle of the lifted transposition, which is lemma 42's U(1)_b when the connector's line is one of its two weight lines; and the quaternions are the real span of the faithful representation on the other chirality. The twistor window receives them through the isomorphism of Proposition 49.1, which exists for every axis, so there the axis remains an input. The Higgs directions, Hom_H(L_p, L_p^perp) = T_p S^4 in lemmas 42 and 46, are the tangent directions of A, and the stabilizer algebras along the orbit types of A are C + H at the S_3-fixed points, with unitary group U(1) x SU(2); on the mirror planes span_R(1, x), the subalgebra C_F of Chamseddine, Connes and Marcolli on the first sheet, with which their finite Dirac operator must commute, whose unitary group is the circle that Corollary 46.4 found as the common stabilizer of a split Yukawa block and the connector's line; and R, with unitary group {+-1}, at generic points. As a reading of the electroweak pattern this differs from the spectral Standard Model in one respect: there A_F is fixed and C_F appears as a condition on D_F, while here the algebra changes along the strata. A displacement along a mirror gives a quaternionic, custodial block, equal masses with conjugate couplings; the anti-quaternionic half that splits the masses is i c(delta') Gamma, which is +- i times the Clifford multiplication by the three-form dual to delta'; every c(v) is j-linear, so no displacement of the point supplies it, and a constant three-form term of the Dirac operator on A, of the kind torsion or a flux adds, has exactly this form (referee). For delta, delta' in the mirror plane the general x-invariant block is c(delta) + i c(delta') Gamma, with m_+^2 + m_-^2 = 2 (|delta|^2 + |delta'|^2) and m_+^2 - m_-^2 = 4 |det(delta, delta')|: the up-down splitting is four times the area of the parallelogram spanned by the displacement and the dual of the three-form, and the masses are equal exactly when the two are parallel; the mirror directions are the neutral components of the bidoublet and the normal directions, whose blocks anticommute with x, the charged ones (referee). The crossed product of lemma 48 sees these orbit types as different quotients, so the motion of the point is a reading and the proved content is the list of stabilizer algebras. The whole fibre carries one quark generation (Proposition 49.5), and the lepton generation needs more than a sheet whose stabilizer algebra is C on a single line: with V_Y = C + C^3 Corollary 16.8 gives 32 states over the four-summand algebra C + H + C + M_3(C), and the identification of the two copies of C that gives A_F = {(lambda, q_L, lambda, m)} is, in Chamseddine, Connes and Marcolli, the order-one condition for an operator between particles and antiparticles (Proposition 2.11), a connecting operator that the sheet-diagonal D_F of Corollary 16.8 cannot supply; in the project that is the role of the connector of lemma 20, whose centralizer is C + H + M_3(C) (referee). The circle of C is also the complex structure of A: for each transposition the lift agrees on Sigma_R, up to sign, with half the Clifford action of the Kähler form of the complex structure that A inherits from E, so with Sigma_4 = Lambda^{0,*} (x) K^{1/2} the two right-handed singlets are K^{+-1/2}, the left doublet is Lambda^{0,1} (x) K^{1/2}, and C + H is the real span of the spin lift of the unitary group U(2) of that complex structure, of which Dic_3 is a finite subgroup spanning the same algebra; the Weyl action preserves the complex structure of E, and the two weight lines are exchanged by the lift of a reflection of T E (referee).

Scope

Finite linear algebra on the spinor fibre of the flat four-torus A with the periodic spin structure; the lifted stabilizer Dic_3 of lemma 48; the convention that Sigma_R is the chirality on which the three-cycles act trivially (which of the two half-spin representations this is depends on the orientation of A only, since each three-cycle has a single lift of order three; for the complex orientation of A inherited from E it is the chirality Gamma = -1, on which the Kähler form acts). The E factor of T^6 = A x E is outside the statements: its spinors double the fibre, Sigma_6 = Sigma_4 (x) Sigma_2, the inversion acts on Sigma_2 and is absent from the stabilizer at the orbit type (a, a, a), and the Clifford action of T E adds two generators outside the Cl(5) of Proposition 49.1. Proposition 49.4 proves stabilizer algebras, and the reading of a displacement as a Higgs field is a reading. The direction: the Dirac operator of the torus restricted to a point of a mirror, item (b) of Remark 45.3, with a constant three-form term whose dual lies in the mirror plane, as the candidate that supplies both halves of an x-invariant Yukawa block and derives the condition [D_F, C_F] = 0 from the geometry, the three mirrors being three candidates for the photon circle on Sigma_L; and the lepton generation through a connecting operator that identifies the two copies of C.

Part 50

The Majorana line and the top block: one point of the twistor fibre, the Higgs vector and the compressed height functions

For STRATEGY T12 and T14, taking up the first lead of the referee of lemma 46 (a rank-one Yukawa block marks a point of the twistor fibre, the same kind of datum as the connector's line), following lemma 20 (Theorem 20.2 and its second reading), lemma 42 (Proposition 42.2), lemma 46 (Propositions 46.1 and 46.2, Corollary 46.4), lemma 16 (Theorem 16.7, Corollary 16.8, Example 16.9), lemma 10 (Proposition 10.1), lemma 21 (Proposition 21.1), lemma 17 (Proposition 17.3), lemma 41 (Proposition 41.5), lemma 44 and lemma 49 (Propositions 49.3 and 49.4 and the three-form reading). Written by Claude Fable 5.1 on 5 November 2026, by hand; the Clifford structure and j, the isometry n -> Gamma_n w, the spectrum and the two halves of the block, the complex structure J_w and the action of the circle on it, the compression identity (Monte Carlo on CP^3 with 4 x 10^5 points), the order-one defects of the split block on M_4(C) and on C + M_3(C), the real dimension 20 of the centralizer with its forced embedding, and the three-form form of the anti-quaternionic half were checked with numpy (script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_50.md), who redid every computation by hand and in Python (400 random frames for both Cl(5) representations, exact fractions for the removal identities, the order-one defects, the centralizer dimensions, the Monte Carlo) and read the e-prints of the sources: two errors, in Proposition 50.1 (3) (with the universal block both cases keep 5/8 and 8/3, so the up-type assignment is an input) and in the closing sentence of Proposition 50.3 (the top-only block alone keeps order one on H_R + H_L + C + M_3(C); the connector is what reduces to A_SM), and sixteen amendments (the sign of delta' fixed by the definition of J_w, with no dependence on the orientation; Proposition 50.2 as Example 16.9, item 2, extended complex-linearly; the hypercharge -1/2 of the doublet in J_w; the general first-order criterion, Krajewski's rule; the prior work in Chamseddine-Connes-Marcolli, Proposition 3.5, and in the Standard Model vacuum of the spectral Pati-Salam model), all applied below with the referee's texts.

Outcome. On the quaternionic sheet W = H^2 with axis p, let the connector of Theorem 20.2 mark the complex line Cw in L_p and the line Cv in V, and let X be a Yukawa block of a doublet with one massless member (Proposition 46.2 (2), the heavy-top pattern). The eigenline condition of Corollary 46.4 says that the massive right-handed line of X is Cw or Cjw, and the up-type assignment of the heavy member selects Cw, the line on which the summand C of A_SM acts by lambda, the up-type right-handed line. Proposition 21.1 supports this choice for a quark-only block, which needs the split of Proposition 50.3; with the universal block both choices keep 5/8 and 8/3. Then X = c(h) P_w: the Clifford action of a vector h in T_p S^4, the Higgs directions of lemmas 42 and 46, composed with the projector on the connector's line, with |h| the mass (Proposition 50.1). One point [w] of the twistor fibre over p, one point of CP^3, carries the Majorana block (on w (x) conj(v)) and the right-handed states of the massive member (on w (x) conj(v^perp), the top), and the massive left-handed line is C c(h) w; the connector's circle U(1)_b acts on T_p S^4 by scalars in the complex structure J_w defined by the Majorana point. In J_w, (T_p S^4, J_w) is the complex doublet of Proposition 42.2 of hypercharge -1/2, the conjugate doublet through which u_R couples (-J_w gives +1/2), and h is a vector of it. The circle U(1)_h that the block and the connector leave unbroken fixes the J_w-line of h and rotates its complement with charge -2: the neutral and the charged components. The projector P_w is the compression of the height function of CP^3 at [w], 5 cos^2 d(., [w]) - 1, as the grading Gamma_p is the compression of the height function 5 cos d(., p) of S^4 at p (Proposition 50.2); so the block is Y = m {5 Q(n . x), Q(h_w)}, the anticommutator of a compressed coordinate of the base S^4 and the compression, to the same kernel, of a height function of CP^3, a function of the type that lemma 46 places outside the base algebra M_2(H), and its anti-quaternionic half is the three-form term of lemma 49 with dual -J_w h/2, the quarter turn of the Higgs vector. In the two-sheet data of Corollary 16.8 a Yukawa block that differs between the lepton line and the colours, Y_l (x) P_v + Y_q (x) (1 - P_v), keeps order one exactly when P_v commutes with the algebra of the second sheet: on the Pati-Salam M_4(C) the block is universal and the Dirac neutrino and the up quark, the charged lepton and the down quark, have equal masses, while on C + M_3(C), the second-sheet algebra of the connector, the lepton and quark Yukawas separate, with the lepton line forced to be the connector's line when colour is kept (Proposition 50.3); so the top-only family of lemma 21 keeps order one on the bimodule of Corollary 16.8 exactly when the second-sheet algebra lies in

the commutant of the lepton projector, and with the connector on the same line exactly on the connector's algebra. Its centralizer in A_SM, by the explicit-mass criterion of Proposition 17.3, is C + M_3(C) with the summand C embedded in H_L along the line C c(h) w: the algebra of the label 9 (lemma 06), the retained algebra of lemma 44 and the corpus row rs_F1, with the embedding now fixed by the Higgs vector (Proposition 50.4).

Setting

W = C^4 with the Hermitian Clifford generators Gamma_1, ..., Gamma_5 and the antiunitary j of lemma 46, so that W = H^2 and End_H(W) = M_2(H) with unitary group G = Sp(2); the axis p in S^4 with Gamma_p = p^a Gamma_a, the quaternionic lines L_p (eigenvalue +1) and L_p^perp (eigenvalue -1), the projector Pi_p = (1 + Gamma_p)/2, and the stabilizer K_p = Sp(1)_a x Sp(1)_b with Sp(1)_b = Sp(L_p), Sp(1)_a = Sp(L_p^perp). The tangent space T_p S^4 = {n in R^5 : n . p = 0}, and for n in T_p S^4 we write c(n) = n^a Gamma_a, the Clifford action of lemma 49, which is j-linear, Hermitian, odd for Gamma_p and maps L_p onto L_p^perp. The second sheet V = C^4 carries the compression of the functions on CP^3 (Example 16.9, item 2); the bimodule is H_F = Hom(V, W) + Hom(W, V) = W (x) conj(V) + V (x) conj(W), with J_F(x, y) = (y^*, x^*) and the grading Gamma_p on the W factor, the even algebra H_R + H_L + M_4(C) = A_PS on L_p, L_p^perp and V.

The connector of Theorem 20.2 is T_kappa x = kappa (w^* x v) v w^* = kappa <e, x> f with e = w v^* in Hom(V, W), f = e^* = J_F e, unit vectors w in L_p and v in V, and kappa != 0; its centralizer is A_SM = {(lambda, q, m)} = C + H + M_3(C), lambda acting on L_p as diag(lambda, conj(lambda)) in the basis (w, jw), q in H = End_H(L_p^perp), lambda on Cv and m in M_3(C) on v^perp. In the dictionary of Chamseddine, Connes and Marcolli (arXiv:hep-th/0610241, Definition 2.14, Proposition 2.11 for A_F = {(lambda, q_L, lambda, m)}, Remark 2.18 for the identification of the up arrow with nu and u and of the down arrow with e and d, and equation (TmapsMR) for the Majorana block on up_R (x) 1^0; paper/LITERATURE.md, section O), lambda acts on nu_R and u_R and conj(lambda) on e_R and d_R, and the lepton line of V is Cv; so the states of the line Cw (x) conj(V) are nu_R = e = w v^* on the lepton line and u_R = w c^*, c in v^perp, on the three colours, those of Cjw (x) conj(V) are e_R and d_R, and L_p^perp (x) conj(V) carries the left-handed doublets. The block S_T = [[0, T^*], [T, 0]] connects e with J_F e: the Majorana mass of nu_R (lemma 20, reading 1).

The compression Q is that of lemma 10, Q(f) = 4 int_{CP^3} f([w']) P_{w'} dmu([w']) with mu the Fubini-Study probability measure, extended complex-linearly to complex functions. So Q(1) = 1 and Q(5 x_a) = Gamma_a (Proposition 10.1, with x_a([w']) = w'^* Gamma_a w'/w'^* w'), and Example 16.9, item 2, gives Q(5 w'^* B w'/w'^* w') = B + tr(B) 1 for Hermitian B. This is the convention used in lemma 46, item 3; the functions on CP^3 are functions of the unit vector w' modulo phase, and d denotes the Fubini-Study distance, cos d([w'], [w]) = |<w', w>|/(|w'| |w|), as cos d(y, p) = y . p on S^4.

Statements

Proposition 50.1 (the rank-one block and the connector share one point of the twistor fibre). Let X in Hom_C(L_p, L_p^perp) be a Yukawa block with masses m_- = 0 < m_+ = m, so that X = m ell mu^* with unit vectors mu in L_p and ell in L_p^perp (Proposition 46.2 (2)), and let the connector mark the line Cw in L_p.

1. The map n -> c(n) w is a real-linear isometry of T_p S^4 = R^4 onto L_p^perp (as a real four-space); so ell = c(n) mu for a unique unit n in T_p S^4, and X = c(h) P_mu with h = m n, P_mu = mu mu^*. The Hermitian block is Y = {c(h), P_mu} = c(h) P_mu + P_mu c(h); its j-linear half is c(h)/2 and its j-antilinear half is {c(h), rho_mu}/2 with rho_mu = P_mu - P_{j mu}, both of operator norm m/2 and Hilbert-Schmidt norm m.
2. The following are equivalent: (a) Cw is an eigenline of X^* X (the eigenline condition of Corollary 46.4); (b) C mu = Cw or C mu = C jw; (c) the stabilizers of X and of the connector's line in K_p share a circle.
3. Under (b), the massive right-handed states are those of Cw (x) conj(V), nu_R and u_R, when C mu = Cw, and those of Cjw (x) conj(V), e_R and d_R, when C mu = Cjw. With the block universal on V

(Corollary 16.8, Y (x) 1_V), the first case gives the mass m to the up quark and to the Dirac neutrino, and the second to the down quark and to the charged lepton. In the scope of lemma 18 both cases keep cos^2 theta_NC = 5/8 and the ratio 8/3: the neutrino contributes nothing, and the down-type quark and the charged lepton together contribute what the up-type quark contributes (Proposition 21.1, last statement). For a quark-only block (Y_l = 0, available on the commutant of the lepton projector by Proposition 50.3), Proposition 21.1 keeps both values in the first case only; the second gives 59/92 and 46/15. We take the first case, the up-type assignment of the heavy member, as an input that Proposition 21.1 supports in this sense. In the first case, X = c(h) P_w: one point [w] of the twistor fibre P(L_p) = S^2 over p, one point of CP^3 = G/H, carries the Majorana block, on the line P_w Hom(V, W) P_v = C e, and the massive right-handed states, on P_w Hom(V, W); the massive left-handed line is C c(h) w in L_p^perp.

4. In that case the joint stabilizer of X and of the connector's line in K_p is the circle U(1)_h = {(a_theta, b_theta)}, with b_theta = e^{i theta} on w and e^{-i theta} on jw and a_theta = e^{i theta} on c(h) w and e^{-i theta} on j c(h) w; c(n) commutes with it. Let J_w be the complex structure of T_p S^4 defined by c(J_w n') w = i c(n') w. Then J_w is an orthogonal complex structure, Sp(1)_b acts on T_p S^4 = Hom_H(L_p, L_p^perp) by t -> t b^{-1}, and U(1)_b = {b_theta} acts on (T_p S^4, J_w) by the scalars e^{-i theta}: (T_p S^4, J_w) is the complex doublet (2)_{-1} of Proposition 42.2. With Y = charge/2 it has hypercharge -1/2: it is the conjugate doublet through which u_R couples. (T_p S^4, -J_w) is the doublet (2)_{+1} of hypercharge +1/2, and -J_w = J_{jw} is the complex structure of the antipodal point [jw] of the fibre. The map n' -> c(n') w is a J_w-complex-linear isomorphism onto L_p^perp that intertwines Sp(1)_a, and h is a vector of the doublet, with |h| = m. The joint circle U(1)_h acts on T_p S^4 by fixing the J_w-complex line span_R(h, J_w h) and by e^{-2i theta} on its J_w-orthogonal complement: the neutral and the charged components of the doublet. The blocks commuting with U(1)_h on both chiralities are the X' = c(n)(y_1 P_w + y_2 P_{jw}) with y_1, y_2 complex, the diagonal blocks of Proposition 49.3 with the circle of the lifted transposition replaced by U(1)_h, j-linear exactly when y_2 = conj(y_1).

Proof. (1) For n, n' in T_p S^4, c(n) c(n') + c(n') c(n) = 2 n . n', and c(n) w lies in L_p^perp since c(n) anticommutes with Gamma_p; so Re <c(n) w, c(n') w> = Re <w, c(n) c(n') w> = n . n' (the anticommutator is 2 n . n' times the identity and the commutator is anti-Hermitian, with purely imaginary expectation), which makes n -> c(n) w a real isometry from R^4 into the real four-space L_p^perp, hence onto. Write ell = c(n) mu for the unit n so obtained from mu; then X = m c(n) mu mu^* = c(h) P_mu, and Y = X + X^* = c(h) P_mu + P_mu c(h) since c(h) is Hermitian and P_mu is a projection. Put P_mu = (Pi_p + rho_mu)/2, where P_mu + P_{j mu} = Pi_p because mu and j mu are orthogonal and span L_p; j P_mu j^{-1} = P_{j mu}, so j rho_mu j^{-1} = -rho_mu and j Pi_p j^{-1} = Pi_p. Then Y = (c(h) Pi_p + Pi_p c(h))/2 + {c(h), rho_mu}/2 = c(h)/2 + {c(h), rho_mu}/2, since Pi_p c(h) = c(h)(1 - Pi_p); the first term is j-linear and the second j-antilinear. For the norms, c(h) has operator norm |h| = m and {c(h), rho_mu} = c(h) rho_mu + rho_mu c(h) with (c(h) rho_mu)^* (c(h) rho_mu) = rho_mu c(h)^2 rho_mu = m^2 rho_mu^2 = m^2 Pi_p, so both halves have operator norm m/2 and squared Hilbert-Schmidt norm (m^2/4) x 4 = m^2. (2) X^* X = m^2 P_mu has the eigenlines C mu and its orthogonal complement in L_p, which is C j mu; so Cw is an eigenline exactly when Cw = C mu or Cw = C j mu, that is C mu = Cw or C mu = C jw. The equivalence with (c) is Corollary 46.4. (3) The lambda-weight line of the summand C of A_SM is Cw by Theorem 20.2, so by the dictionary the states of Cw (x) conj(V) are nu_R (on the lepton line) and u_R (on v^perp), and those of Cjw (x) conj(V) are e_R and d_R; with Y (x) 1_V, the right-handed states of mass m are those of the massive line tensored with all of conj(V). Proposition 21.1 gives the removal values: u alone, (u, nu) and (d, e) give 5/8 and 8/3, and d alone gives 59/92 and 46/15. In the first case P_mu = P_w; the connector is kappa <e, .> f with e = w v^* = P_w e P_v, the unique element of P_w Hom(V, W) P_v up to scale; and X maps the line Cw onto C c(h) w. (4) For (a, b) in K_p, a X b^{-1} = X with X = m c(n) w w^* reads (a c(n) w)(b w)^* = c(n) w w^*, since w^* b^{-1} = (b w)^* for unitary b. An equality u_1 v_1^* = u_2 v_2^* of rank-one operators with unit vectors forces u_1 = e^{i theta} u_2 and v_1 = e^{i theta} v_2 for one phase. So b w = e^{i theta} w and a c(n) w = e^{i theta} c(n) w. Being j-linear, b and a act as e^{-i theta} on jw and on j c(n) w, which determines b in Sp(1)_b and a in Sp(1)_a for each theta. These elements fix the connector's line Cw too, and conversely the stabilizer of Cw in K_p is Sp(1)_a x U(1)_b, whose intersection with the stabilizer of

X is the displayed circle. For c(n): a_theta c(n) b_theta^{-1} w = e^{-i theta} a_theta c(n) w = c(n) w and a_theta c(n) b_theta^{-1} jw = e^{i theta} a_theta j c(n) w = e^{i theta} e^{-i theta} j c(n) w = c(n) jw, so c(n) commutes with U(1)_h. J_w is well defined because n' -> c(n') w is a real isometry onto L_p^perp and multiplication by i is an isometry of L_p^perp; J_w^2 = -1 and J_w is orthogonal, so it is an orthogonal complex structure on T_p S^4. Under (a, b) in K_p, the element t = c(n')|*{L_p} of Hom_H(L_p, L_p^perp) goes to a t b^{-1} (the action of Proposition 42.2), and for a = 1, b = b_theta: t b_theta^{-1} w = e^{-i theta} c(n') w = c(cos theta n' - sin theta J_w n') w, so U(1)_b acts on (T_p S^4, J_w) by the scalars e^{-i theta}, a complex doublet of charge -1 of U(1)_b in the normalization of lemma 42 (charge +1 with the opposite orientation of the circle), the (2)_{+-1} of Proposition 42.2. A block X' commutes with U(1)_h exactly when X' = c(n) A with A = c(n) X' in End_C(L_p) commuting with b_theta, since c(n) does; A commutes with b_theta exactly when it preserves the eigenlines Cw and Cjw, that is A = y_1 P_w + y_2 P*{jw}; and j X' j^{-1} = c(n)(conj(y_1) P_{jw} + conj(y_2) P_w) equals X' exactly when y_2 = conj(y_1), as in Proposition 49.3. ■

Remark (three generations). With a generation space C^3 and the quaternionic structure j (x) conj on W (x) C^3, the blocks commuting with U(1)_h are X' = c(n)(Y_u (x) P_w + Y_d (x) P_{jw}) with Y_u, Y_d in M_3(C), the up-type and down-type Yukawa matrices of the spectral Standard Model with a fixed Higgs direction n, the masses being the singular values of Y_u and Y_d and the j-linear ones those with Y_d = conj(Y_u); the top-only family of lemma 21 is Y_u = m_t P_g, Y_d = 0, for a generation line g, that is X' = c(h) (P_w (x) P_g).

Prior work (referee). The blocks c(n)(Y_u (x) P_w + Y_d (x) P_{jw}) of item 4 and of the remark are the Yukawa blocks of the spectral Standard Model. In Chamseddine, Connes and Marcolli (arXiv:hep-th/0610241, read in the e-print source), the discrete part of the inner fluctuation is a quaternion-valued Higgs field H (Proposition 3.5). The coupling of the up-arrow part is related to that of the down-arrow part by H -> jH, and Lemma 3.6 gives Tr((D_3^{(0,1)})^2) = 12 |1 + H|^2 Tr(Y_u^* Y_u + Y_d^* Y_d). In the variables of the sheet the quaternion 1 + H is c(n) up to normalization, and the operators D(Y) of Theorem 2.21 there are the case of a fixed n. The rank-one block c(h) P_w is the case Y_d = 0. The converse, the identification of P_w with the projector on the connector's line, and the reading of J_w were not found in those sources. That the points of the twistor fibre over p parametrize the orthogonal complex structures of T_p S^4 of one orientation is the twistor correspondence (Atiyah, Hitchin and Singer, Proc. R. Soc. Lond. A 362 (1978) 425-461, doi:10.1098/rspa.1978.0143). J_w is that complex structure at [w]: it is orthogonal, of one orientation class over the whole fibre, and J_{jw} = -J_w.

Proposition 50.2 (the projectors and the grading as compressions of height functions). In the convention of the Setting, for every B in End_C(C^4),

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Q(5 w'^* B w' / w'^* w') = B + tr(B) 1
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(Example 16.9, item 2, extended complex-linearly). Consequently:

1. 5 Q(p . x) = Gamma_p = 2 Pi_p - 1: the grading is the compression of the height function 5 cos d(., p) of S^4 at the axis (lemma 20, reading 2).
2. Q(h_w) = P_w for h_w([w']) = 5 |<w', w>|^2/|w'|^2 - 1 = 5 cos^2 d([w'], [w]) - 1: the projector on the connector's line is the compression of the height function of CP^3 at the marked point [w]; in the same way P_v = Q_V(h_v) on the second sheet, which is the same kernel with the same functions (Example 16.9, item 2).
3. The block of Proposition 50.1 is Y = m {5 Q(n . x), Q(h_w)}, and the Majorana block is supported on the line P_w Hom(V, W) P_v = Q(h_w) Hom(V, W) Q_V(h_v); the j-antilinear half of Y is i c(delta') Gamma_p with delta' = -J_w h/2, the sign being fixed by the definition of J_w (with J_{jw} = -J_w in place of J_w it changes; for a rank-one block c(h) P_mu without the eigenline condition, delta' = -J_mu h/2), the three-form term of lemma 49 whose dual is the quarter turn of the Higgs vector in the complex structure of the Majorana point.

Proof. The fourth moment of lemma 10, E[w_j w_k conj(w_i) conj(w_l)] = (delta_ij delta_kl + delta_jl delta_ki)/20 on the unit sphere of C^4, gives Q(5 w'^* B w')_{kl} = 20 sum_{ij} B_ij (delta_ij delta_kl + delta_jl delta_ki)/20 = tr(B) delta_kl + B_kl for every complex B. This is Example 16.9, item 2.

Equivalently, by Schur's lemma: the assignment B -> Q(f_B), f_B([w']) = 5 w'^* B w'/w'^* w', is complex linear and U(4)-equivariant, since f_{U B U^*}([U w']) = f_B([w']), the Fubini-Study measure is U(4)-invariant and the compression intertwines the action of U(4) on functions with its conjugation on the kernel; End_C(C^4) = C 1 + sl(4) under conjugation, with sl(4) irreducible, so by Schur's lemma Q(f_B) = alpha B_0 + beta tr(B) 1 with B_0 the traceless part. B = 1 gives f = 5 and Q(5) = 5, so beta = 5/4; B = Gamma_a gives f = 5 x_a and Q(5 x_a) = Gamma_a by Proposition 10.1 in the stated convention, so alpha = 1; hence Q(f_B) = B - (tr B/4) 1 + (5/4) tr(B) 1 = B + tr(B) 1. (As a check without Schur's lemma: with Q(f) = 4 int f P_{w'} dmu for the normalized measure, since int P_{w'} dmu = 1/4, the moments E|z_1|^4 = 1/10 and E|z_1 z_2|^2 = 1/20 on the unit sphere of C^4 give <w, Q(5 |<., w>|^2) w> = 2 and <jw, Q(5 |<., w>|^2) jw> = 1, and the U(3)-invariance under the stabilizer of w gives Q = 2 P_w + (1 - P_w) = P_w + 1.) (1) is B = Gamma_p. (2) is B = P_w, tr P_w = 1, minus the compression of the constant 1; the second sheet is a copy of the same kernel. (3) The first statement is Proposition 50.1 (1) with c(n) = 5 Q(n . x) (Proposition 10.1) and P_w = Q(h_w); the connector's line C e = C w v^* is P_w Hom(V, W) P_v. For the three-form: the j-antilinear odd Hermitian blocks are the real span of the i Gamma_a Gamma_p with a orthogonal to p (Proposition 46.1 (3)), so {c(h), rho_w}/2 = i c(delta') Gamma_p for a unique delta' in T_p S^4. Both sides are j-antilinear odd Hermitian blocks. Such a block Y satisfies Y jw = -j Y w, so the two sides agree when they agree on w. On w the left side gives (c(h) rho_w w + rho_w c(h) w)/2 = c(h) w/2, since rho_w vanishes on L_p^perp, and the right side gives i c(delta') w. So c(delta') w = -(i/2) c(h) w = c(-J_w h/2) w by the definition of J_w, and delta' = -J_w h/2 by the isometry of Proposition 50.1 (1). The same lines with mu in place of w give delta' = -J_mu h/2 for any rank-one block c(h) P_mu. The norm |delta'| = m/2 follows. As a reading: i rho_w = (1/2) sum_{a<b} omega_w(e_a, e_b) Gamma_a Gamma_b with omega_w(a, b) = <J_w a, b>, a two-form that is self-dual or anti-self-dual according to the orientation and acts on L_p alone. For a vector h and a two-form omega, {c(h), c(omega)} = 2 c(h wedge omega) with c(e_a wedge e_b wedge e_c) = Gamma_a Gamma_b Gamma_c. The Hodge dual of h wedge omega_w is, up to sign, the contraction of h with omega_w, the vector J_w h. This is the three-form term of lemma 49. With lemma 49's formulas for the block c(delta) + i c(delta') Gamma_p, delta = h/2 and delta' orthogonal to delta of the same length give m_+^2 + m_-^2 = 2 (|delta|^2 + |delta'|^2) = m^2 and m_+^2 - m_-^2 = 4 |delta wedge delta'| = m^2, with |delta wedge delta'| the area of the parallelogram, that is m_+ = m, m_- = 0, as it must. Lemma 49 states the formula for delta and delta' in the mirror plane, where the area is |det(delta, delta')|, and it holds for all delta, delta' in T_p S^4, being lemma 46's 4 |Im(conj(q) q')|. ■

Proposition 50.3 (the lepton and quark Yukawas separate exactly on the connector's algebra). In the data of Corollary 16.8 with V_X = W, A_{F,X} = A_X an even subalgebra containing Gamma_p (H_R + H_L or a subalgebra containing the grading), V_Y = V = C^4 ungraded with an algebra A_Y in M_4(C), let P_1, …, P_r be mutually orthogonal projections on V summing to 1 and Y_1, …, Y_r odd Hermitian blocks on W, and define

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D x = sum_i Y_i x P_i  (x in Hom(V, W)),    D y = sum_i P_i y Y_i  (y in Hom(W, V)).
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Then D is self-adjoint and odd, commutes with J_F, and satisfies order zero; its order-one defect is [[D, a], b°] x = sum_i [Y_i, a_X] x [b_Y, P_i] on Hom(V, W) and sum_i [P_i, a_Y] y [b_X, Y_i] on Hom(W, V). Hence:

1. (Universal block.) If all Y_i are equal, order one holds for every A_Y (Corollary 16.8).
2. (Split block.) If every P_i commutes with A_Y, order one holds for every choice of the Y_i.
3. (Converse.) If the Y_i are linearly independent, order one holds exactly when every P_i commutes with A_Y, since [Y_i, Gamma_p] = -2 Gamma_p Y_i are then linearly independent. For two terms, D = Y_l (x) P_v + Y_q (x) (1 - P_v): order one holds exactly when Y_l = Y_q or P_v commutes with A_Y. In general, with Gamma_p in A_X, order one holds exactly when D commutes with the right action x -> x b_Y of A_Y on Hom(V, W). By J_F the same then holds on Hom(W, V). Writing D x = sum_k E_k x Z_k with linearly independent E_k, order one holds exactly when every Z_k commutes with A_Y. Indeed [[D, a], b°] = [[D, b°], a] by order zero, and for a = Gamma_p the operator [D, b°] is odd and commutes with the grading, hence vanishes. This is Krajewski's decomposition of a first-order operator into a left-linear and a right-linear part, with the left-linear odd part zero because the grading lies in the left algebra (referee).

4. On the Pati-Salam sheet algebra A_Y = M_4(C) of Example 16.9 the block is universal: the lepton line and the three colours carry the same Y, so in each generation the Dirac neutrino and the up quark have the mass m_+ and the charged lepton and the down quark the mass m_- (Proposition 46.2), the lepton-quark unified Yukawas k^nu = k^u, k^e = k^d of the spectral Pati-Salam model. The commutant of P_v in M_4(C) is {lambda P_v + m : m in M_3(C) on v^perp} = C + M_3(C), the second-sheet part of the algebra A(T_kappa) of Theorem 20.2 for a connector on the line v; so the split block Y_l (x) P_v + Y_q (x) (1 - P_v) and the connector on v select the same second-sheet algebra, and on it both keep order one.
5. (One lepton line.) If a split block with lines P_{v'}, Y_l != Y_q, and a connector on the line v both keep order one on an algebra A_X + A_Y with A_Y containing End(v^perp) (colour unbroken), then Cv' = Cv.

So the top-only family of lemma 21, Y_l = 0 and Y_q = {c(h), P_w} (x) P_g, is a sheet-diagonal Dirac operator on the bimodule of Corollary 16.8 with order one exactly when the second-sheet algebra lies in the commutant C + M_3(C) of P_v. This operator lies outside the compressed form d_X (x) 1, which is what Corollary 16.8 gives for ungraded V. On the first sheet the family allows all of H_R + H_L, real dimension 28 in all. With the connector on the same line v added, order one of the sum holds exactly on A(T_kappa) = C + H + M_3(C) (Theorem 20.2; the defects of the two parts lie in different blocks). By (5), a connector and a split block with colour retained share the lepton line. The massive right-handed line is the Majorana line under the eigenline condition and the up-type assignment of Proposition 50.1 (3).

Proof. Self-adjointness: <x', D x> = sum_i tr(x'^* Y_i x P_i) = sum_i tr((Y_i x' P_i)^* x), using Y_i^* = Y_i, P_i^* = P_i = P_i^2 and the cyclicity of the trace; the same on the second summand. Oddness: the grading is Gamma_p (x) 1 on Hom(V, W) and -1 (x) Gamma_p^c on Hom(W, V) (Theorem 16.7), and the Y_i anticommute with Gamma_p. J_F D (x, y) = ((sum P_i y Y_i)^*, (sum Y_i x P_i)^*) = (sum Y_i y^* P_i, sum P_i x^* Y_i) = D (y^*, x^*) = D J_F (x, y). Order zero holds since the left actions (a_X on x, a_Y on y) commute with the right actions (b_Y on x, b_X on y). For order one, [D, a] x = sum_i (Y_i a_X - a_X Y_i) x P_i = sum_i [Y_i, a_X] x P_i, since a_X acts on the left of x and P_i on the right; b° acts by x -> x b_Y, so [[D, a], b°] x = sum_i [Y_i, a_X] (x b_Y P_i - x P_i b_Y) = sum_i [Y_i, a_X] x [b_Y, P_i]; the second summand is the same computation with the roles exchanged. (1) With Y_i = Y the sum is [Y, a_X] x [b_Y, sum P_i] = 0. (2) is immediate. (3) If order one holds, then for every a_X, b_Y the operator x -> sum_i [Y_i, a_X] x [b_Y, P_i] vanishes, that is sum_i [Y_i, a_X] (x) [b_Y, P_i]^t = 0 in End(W) (x) End(V); with a_X = Gamma_p the first factors -2 Gamma_p Y_i are linearly independent, so every [b_Y, P_i] = 0. For two terms with P_2 = 1 - P_1 the defect is [Y_l - Y_q, a_X] x [b_Y, P_v], which vanishes for all a_X, b_Y, x exactly when Y_l = Y_q (so that the first factor is 0 for every a_X) or [b_Y, P_v] = 0 for all b_Y, the first factor being nonzero for a_X = Gamma_p otherwise. (4) P_v does not commute with M_4(C), whose commutant is C 1, so the split block fails order one there by (3); the commutant of P_v is the block algebra End(Cv) + End(v^perp) = C + M_3(C), which is the second-sheet component of A(T_kappa) in Theorem 20.2, equation (6). (5) P_{v'} must commute with End(v^perp) + 0, so it preserves v^perp and restricts to it as a scalar; a rank-one projection cannot restrict to 1 on a three-space, so it restricts to 0, and v' lies in the orthogonal complement of v^perp, which is Cv. The last statement combines (4) and (5) with Proposition 50.1 (3). ■

Prior work (verified by the referee in the e-prints). The criterion is Krajewski's first-order rule (T. Krajewski, *Classification of finite spectral triples*, J. Geom. Phys. 28 (1998) 1-30, doi:10.1016/S0393-0440(97)00068-5, arXiv:hep-th/9701081). Section 3.4 there proves that the first-order operators on a bimodule over sums of matrix algebras are the sums of a left-linear and a right-linear operator, and that D = Delta + J Delta J^{-1} with Delta commuting with J pi(A) J^{-1}. Sections 4.1 and 4.2 show that the edges of a diagram are vertical or horizontal, with Delta_{ik}^{jl} = delta_{kl} M_{ij,k} (x) I_{n_k}. Proposition 50.3 is this rule in the two-sheet bimodule with the grading in the left algebra, the P_i being projections onto isotypic components of V under A_Y; on M_4(C) there is one right label and the block is universal. The Standard Model case is Theorem 2.21 of Chamseddine, Connes and Marcolli (arXiv:hep-th/0610241), with the subalgebra from Chamseddine and Connes (arXiv:0706.3688, section 4, Theorem 4.1). Chamseddine, Connes and van Suijlekom (arXiv:1304.8050, e-print read on 5 November 2026) give the tensor form of D_F with separate lepton and quark couplings in section 4, "Summary of tensor notation", and call the case k^nu = k^u, k^e

= k^d lepton and quark unification. In section 5 they note that the field Sigma_I^J, whose vacuum value is the projector on the lepton line, decouples in that case. Their Standard Model vacuum (section 8, equation SM-vev) is <phi_{a.}^b> = v delta_{a.}^{1.} delta_1^b, <Sigma_J^I> = u delta_1^I delta_J^1 and <Delta_{a.J}> = w delta_{a.}^{1.} delta_J^1. The Majorana field Delta and the Higgs bidoublet phi share the right-handed index 1., and Delta shares its Pati-Salam index with the lepton line of Sigma: these are the alignments of Propositions 50.1 and 50.3, taken there as the chosen vacuum, at which the Hessian in the Delta directions is nonnegative. The order-one forcing of the shared lepton line with colour retained (item 5) and the statement that order one on M_4(C) forces the unified Yukawas were not found stated in those papers; the latter follows from Krajewski's rule. Chamseddine, Connes and van Suijlekom introduce the unified case as a special case, also in arXiv:1507.08161, where the table of Pati-Salam scalars notes that Sigma decouples under quark-lepton coupling unification. See paper/LITERATURE.md, section P.

Proposition 50.4 (the centralizer of the top block is the algebra of the label 9). Let D_F = Y_q (x) (1 - P_v) + S_{T_kappa} on H_F, with Y_q = {c(h), P_w}, h != 0 (the top-only family: the lepton block zero, one generation), the connector on the lines w and v, and A_SM = {(lambda, q, m)} = A(T_kappa) of Theorem 20.2. Then

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{a in A_SM : [D_F, pi(a)] = 0} = {(lambda, lambda_h, m) : lambda in C, m in M_3(C)} = C + M_3(C),
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of real dimension 20, where lambda_h denotes lambda in the embedding C -> H_L = End_H(L_p^perp) whose weight line is C c(h) w, the massive left-handed line: q = lambda_h exactly when q (c(h) w) = lambda c(h) w. The connector commutes with all of A_SM, so the same holds for D_F with kappa = 0. The unitary group of the centralizer is U(1) x U(3), and under the unimodularity condition inherited from A_SM (Proposition 41.5, theta_lambda + 3 theta_c = 0) it acts on H_F, up to a finite group, as U(1)_em x SU(3) with the electric charges (2/3, -1/3, -1, 0): the gauge group of the label 9 (lemma 06, lemma 21, reading 3). By Proposition 17.3 there is no scalar fluctuation on the centralizer and the top mass is explicit there. The algebra coincides with the retained algebra of lemma 44 along the seesaw scaling (Proposition 44.1), which uses the same embedding when the block is universal (Proposition 50.3 (4)), and it is the corpus row rs_F1, the centralizer of a dominant top, with the embedding of C in H now fixed by the Higgs vector h.

Proof. For a = (lambda, q, m) in A_SM, pi(a) acts on Hom(V, W) by x -> a_W x with a_W = lambda P_w + conj(lambda) P_{jw} + q, q on L_p^perp, and on Hom(W, V) by y -> a_V y with a_V = lambda P_v + m. On Hom(W, V) the commutator is [1 - P_v, a_V] y Y_q = 0; the connector commutes with pi(a) by Theorem 20.2. So [D_F, pi(a)] = 0 exactly when [Y_q, a_W] = 0. On w: Y_q a_W w = lambda c(h) w and a_W Y_q w = q c(h) w. On jw: Y_q jw = c(h) P_w jw + P_w c(h) jw = 0 + P_w j c(h) w = 0, since j c(h) w lies in L_p^perp, and a_W jw is a multiple of jw, so both sides vanish. On u in L_p^perp: Y_q u = P_w c(h) u = <c(h) w, u> w, so a_W Y_q u = lambda <c(h) w, u> w and Y_q a_W u = <c(h) w, q u> w = <q^* c(h) w, u> w; these agree for all u exactly when q^* c(h) w = conj(lambda) c(h) w, which for the normal operator q (the left multiplication by a quaternion, q q^* = |q|^2) is the same as q c(h) w = lambda c(h) w. So the centralizer is the set of (lambda, q, m) with c(h) w a lambda-eigenvector of q; since q is j-linear, q then acts as conj(lambda) on j c(h) w, so q = diag(lambda, conj(lambda)) in the basis (c(h) w, j c(h) w) of L_p^perp, which is lambda in the embedding of C in H with that weight line, unique for each lambda; the real dimension is 2 + 18 = 20. The unitary group of C + M_3(C) is U(1) x U(3); the inherited unimodularity condition and the resulting charges are Proposition 41.5 (the lift u J u J^{-1} of the unitaries of A_SM restricted to the centralizer; Proposition 41.5 is stated for the embedding Q_lambda along nu_L and u_L, and lambda_h is its conjugate by an element of Sp(1)_a, an inner automorphism of A_SM that preserves the traces of the left action, so the condition theta_lambda + 3 theta_c = 0 and the charges transfer; recomputed for lambda_h they are (2/3, -1/3, -1, 0) on both chiralities). Proposition 17.3 applies to a subalgebra commuting with D_F. Lemma 44, Proposition 44.1, gives {(lambda, Q_lambda, m)} with Q_lambda the embedding along the nu_L line, which is c(h) w when the neutrino block is Y_q (x) P_v with the same h. The corpus row rs_F1 is the statement with the top block as a hypothesis; the embedding was left unspecified there. ■

Reading

The three finite data of the window, the axis, the connector and the Yukawa block (lemma 46, Reading), now stand in this relation: the connector marks the point [w] of the twistor fibre over the axis and the line v of the second sheet; under the eigenline condition and the up-type assignment, the top block is the Higgs vector h in T_p S^4, read in the complex structure J_w of the Majorana point, applied to the projector on the Majorana line; and both projectors and the grading are compressions of height functions at the marked points, P_w = Q(h_w), P_v = Q_V(h_v), Gamma_p = Q(5 p . x). The rank-one pattern, in which the quaternionic and anti-quaternionic halves have equal norm (Proposition 46.2 (2)), is the statement that the dual of the three-form term of lemma 49 is the quarter turn of the Higgs vector, delta' = -J_mu delta, in the complex structure of the massive right-handed point [mu] of the fibre, which is [w] under the eigenline condition. Conversely, every delta' orthogonal to delta and of the same length is -J_mu delta for exactly one [mu], so the rank-one blocks are the pairs (h, [mu]) (by lemma 49's formula, |delta| = |delta'| with delta orthogonal to delta' gives m_- = 0, hence rank one, hence c(h) P_mu). The custodial breaking |Im(conj(q) q')| = (m_+^2 - m_-^2)/4 of lemma 46 is here the area |delta wedge delta'|, which at fixed m_+^2 + m_-^2 is maximal exactly for the rank-one pattern. The top and the neutrinos share the lines. The right-handed states of both lie on the Majorana line w, and the left-handed ones on C c(h) w, which for the neutrinos is the neutral line of the embedding lambda_h of Proposition 50.4, and their Dirac line when the block is universal. The difference is the factor conj(V): the lepton line for the neutrinos, the three colours for the top. So the exchange of the top with the neutrinos of lemma 21 (Proposition 21.3) and lemma 34 (Proposition 5.2) can be read as a transposition of the colour triplet of one generation with the generation triplet of the lepton line, at fixed lines in W. In conj(V) (x) C^3 these are a 3 x 1 and a 1 x 3 block of a 4 x 3 array, and the transposition becomes a symmetry of the array only with four families, as in lemma 34. This is recorded as a reading. With the universal block the Dirac neutrino mass is the top mass, and the connector's kappa on the same line gives the neutral block [[0, m_t], [m_t, kappa]] on (nu_L, nu_R) in each generation, with light mass about m_t^2/kappa for kappa » m_t (lemma 44, Proposition 44.2, with y = m_t); for m_nu = 0.05 eV and m_t = 173 GeV this is kappa of about 6 x 10^14 GeV, of the order of the meeting point of lemma 26 (2.2 x 10^14 GeV). With the top mass run to that scale, about half its low-energy value in the Standard Model (recalled, not computed here), kappa is about 1.5 x 10^14 GeV, so the comparison holds within the factors of the running. With kappa equal on the three generations, the same relation gives m_c^2/kappa, about 3 x 10^-6 eV, for the second neutrino, far below the solar scale of about 9 x 10^-3 eV. So the remark concerns the heaviest neutrino, and the others need a hierarchical Majorana matrix. It is a remark under the hypothesis of the universal block, which the separation of Proposition 50.3 removes. The heavy top, by the explicit-mass criterion, leaves the algebra C + M_3(C) of the label 9 with the summand C embedded along its left-handed line: the same algebra that lemma 06 reaches from the strict heavy limit, that lemma 44 retains along the seesaw scaling and that the corpus assumed for the dominant top; and the top-only family of lemma 21 keeps order one on the bimodule of Corollary 16.8 only on a second-sheet algebra inside the commutant of a lepton projector, which the connector supplies with its own line; on the Pati-Salam M_4(C) the block is universal and the leptons are as massive as the quarks. For the dictionary note: the Yukawa block of the top is the Higgs vector times the Majorana projector; the grading and the two projectors are compressed height functions; order one on the Pati-Salam sheet is the lepton-quark unification of the Yukawas, and its relaxation to the connector's algebra is their separation.

Scope

Finite linear algebra on the quaternionic sheet with the Clifford structure of Example 16.9 and the connector of Theorem 20.2; one generation except in the remark after Proposition 50.1; the axis, the connector and the Yukawa block are inputs, as in lemmas 42 and 46; Proposition 50.2 holds in the convention of lemma 10, where Q(5 x_a) = Gamma_a, and the sign of delta' is fixed by the definition of J_w (it changes with J_{jw} = -J_w in place of J_w); the up-type assignment of Proposition 50.1 (3) is an input, which Proposition 21.1, in the scope of lemma 18 (the classical bosonic trace at fixed cutoff), supports for a quark-only block, while for the universal block the two cases give the same values; Proposition 50.3 concerns sheet-diagonal operators on the bimodule of Corollary 16.8 with projections on the second sheet, which for a split block lie outside the compressed form d_X (x) 1 of that corollary; its proof applies verbatim to three generations, with W (x) C^3 and Gamma_p (x) 1 in place of W and Gamma_p; Proposition 50.4 is the algebraic centralizer

by the criterion [D_F, a] = 0, and Counterexample 18.8 shows that mass decoupling in the spectral action does not select it dynamically. Nothing here derives the block, the connector or the axis, and the identity of Proposition 50.2 is Example 16.9, item 2, extended complex-linearly: at level one on CP^3 the contravariant symbol of B is 5 times its covariant symbol w'^* B w'/w'^* w' minus tr(B), so h_w is the contravariant symbol of P_w (Berezin's covariant and contravariant symbols; paper/LITERATURE.md, section P); in general, at level one on CP^{N-1}, Q((N + 1) sigma_B - tr(B)) = B with Q(f) = N int f P dmu (referee). The direction the result points to: a parent in which the Higgs vector h, the Majorana point [w] and the lepton line v are fields, h as the variation of the axis (lemma 42) and [w] as the variation of the point on the fibre, for which the fields Delta, Sigma and phi of the spectral Pati-Salam model (arXiv:1304.8050) are such a parent at the finite level, and a computation in its potential would test whether it forces the eigenline condition and the shared lepton line (referee); and, for item (b) of lemma 49, a Dirac operator on the four-torus at a point of a mirror whose three-form term is the quarter turn of the displacement in the complex structure of the lifted transposition, since that is the rank-one pattern of the heaviest quark.

Part 51

The Dirac operator of the four-torus with a Wilson line and a constant three-form: its lowest window is the Yukawa block, the photon condition is the invariance under the transposition, and the heaviest quark is the torsion equal to the quarter turn of the Wilson line

For STRATEGY T16 (item (b) of Remark 45.3: a Dirac operator from the geometry), T12 and T13, taking up the direction stated at the end of lemma 49 (a Dirac operator on the torus with a constant three-form term as the candidate that supplies both halves of an x-invariant Yukawa block and derives the condition [D_F, C_F] = 0 from the geometry) and the end of lemma 50 (the rank-one pattern as the three-form equal to the quarter turn of the Higgs vector), following lemma 48 (the torus A, its spin structure and the lifts of Proposition 48.1), lemma 49 (Propositions 49.1, 49.3 and 49.4), lemma 46 (Propositions 46.1 and 46.2), lemma 22 and lemma 25 (the finite Dirac operator as the lowest window of a parent with a constant connection), lemma 16 (Corollary 16.8) and lemma 17 (Proposition 17.3). Written by Claude Fable 5.1 on 5 November 2026, by hand; the three-form identity, the mass formula on 200 random pairs, the isolation of the window on 400 random pairs with the momenta enumerated in a box, the covariance x c(v) x^{-1} = c(d tau v), the commutation exactly for data in the mirror plane, the eigenvalues of the lift on the two chiralities, the rank-one case and the fixed space of a three-cycle were checked with numpy (script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_51_52.md), who rebuilt the Clifford set-up, tested the spectrum on 2000 pairs and the window on the actual lattice of A, computed the invariant blocks, the weight lines and the 24-state bimodule at the mirror, and read the e-prints of Agricola and of Friedman, Morgan and Witten: Propositions 51.1 and 51.2 correct with amendments (Agricola's normalization verified, the Wilson line as a gauge class), Proposition 51.3 with one error in item 2 (the heavier member is fixed by the sign of the oriented area of (A, eta) in the complex line of the Majorana point, which the orientation of A does not fix; the intrinsic form is |y_mu| = |A + J_mu eta| on each weight line) and amendments (the orientability relation fails at the mirror), and two errors in the Reading (the mass with a torsion along the Wilson line is (|A|^2 + |eta|^2)^{1/2}; A = Lambda_coroot (x) E = Hom(Lambda_weight, E)); all applied below with the referee's texts.

Outcome. On the flat four-torus A of lemma 48 with the periodic spin structure, the Dirac operator with a constant real one-form A (a flat U(1) connection, a Wilson line) and a constant real three-form H (a torsion or a flux term) is translation invariant, and on the sector of internal momentum k it is the constant block c(k + A) + i c(eta) Gamma on the spinor fibre Sigma_4, with eta the one-form dual to H up to the normalization of the three-form term; the constant spinors form its lowest window when |A| + |eta| is below half the first momentum, and there the operator is Y = c(A) + i c(eta) Gamma: a Yukawa block of lemma 46 with the Wilson line as its quaternionic, custodial half and the three-form as its anti-quaternionic half, of masses m_+-^2 = |A|^2 + |eta|^2 +- 2 |A wedge eta| (Proposition 51.1). The operator commutes with the lift of a transposition tau of the Weyl group exactly when A and eta lie in the mirror plane of tau, the tangent

plane of the mirror z_i = z_j, which is a complex line of A; then the window block commutes with the lifted transposition x on both chiralities, which is the condition [D_F, C_F] = 0 of Chamseddine, Connes and Marcolli (Proposition 49.3), so the photon condition is the Weyl invariance of the Wilson line and the torsion; the x-invariant blocks of Proposition 49.3 are exactly these; a nonzero Wilson line or torsion along a mirror is invariant under that transposition alone, so it chooses one of the three mirrors, and no nonzero data are invariant under a three-cycle, so the symmetric points carry no Yukawa block (Proposition 51.2). In the complex structure J of the mirror plane, inherited from the elliptic curve, the two members of the doublet have equal masses exactly when the torsion is parallel to the Wilson line, masses |A| + |eta| and ||A| - |eta|| exactly when it is perpendicular, and one massless member exactly when eta = +- J A: the heaviest quark is the generation whose torsion is the quarter turn of its Wilson line, and in that perpendicular case, for |eta| <= |A|, the Wilson line is the mean mass of the doublet and the torsion half its splitting, the up-type member being the heavier when <J_w A, eta> < 0; with the up-type member on the Majorana line w, the up and down Yukawa couplings are A + J_w eta and A - J_w eta in the complex line (P_tau, J_w), J_w = +-J according to which weight line of x is the Majorana line (Proposition 51.3). The finite data of Corollary 16.8 at the mirror stratum, with the algebra C of Proposition 49.4, this block and the colour sheet, are the quark sector with the algebra C + M_3(C) of lemma 50, Proposition 50.4, explicit masses and no one-forms: the label 9.

Setting

Let A = R^4/Lambda be the flat four-torus of lemma 48, A = a (x) R^2 with a the standard representation of S_3 and R^2 the tangent space of the elliptic curve E, with the periodic spin structure, the one invariant under S_3; its spinor bundle is A x Sigma_4, Sigma_4 = C^4 = Sigma_+ + Sigma_- with the Clifford action c(v) of v in R^4 (Hermitian, c(v)^2 = |v|^2), the chirality Gamma = c(e_1) c(e_2) c(e_3) c(e_4) for an oriented orthonormal basis, and the quaternionic structure j commuting with Spin(4), c and Gamma (lemma 49, Setting). By Proposition 49.1 (Sigma_4, c(e_1), ..., c(e_4), Gamma, j) is the quaternionic sheet W = H^2 of Example 16.9 with axis p at Gamma, L_p = Sigma_+, L_p^perp = Sigma_-, and T A = T_p S^4 through v -> c(v). The Dirac operator is D = -i sum_a c(e_a) partial_a on L^2(A, Sigma_4), self-adjoint on H^1, with kernel the constant spinors; its spectrum is the union over the dual lattice of the eigenvalues +-|k| of c(k) on the sectors e^{i k . z} Sigma_4, k in 2 pi Lambda^*. Let 2 pi/l be the length of the shortest nonzero vector of 2 pi Lambda^* (l the side for a cubic torus).

The data: a constant real one-form A = A_a e^a, the connection d + i A of the trivial line bundle, flat, with holonomy e^{i A(lambda)} around lambda in Lambda, a Wilson line, defined modulo 2 pi Lambda^* by the gauge transformations e^{i k . z}, which shift the momentum; and a constant real three-form H. The Dirac operator with these data is

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D_{A,H} = D + c(A) + s i c(H),
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where c(H) = sum_{a<b<c} H_{abc} c(e_a) c(e_b) c(e_c) is anti-Hermitian, so that s i c(H) is Hermitian for real s, and s is the constant of the chosen normalization of the three-form term. In Agricola's lectures (arXiv:math/0606705, section "The square of the Dirac operator and parallel spinors", read in the e-print by the referee) the spinor connection nabla^{s'}_X psi = nabla^g_X psi + s' (the contraction of T with X) . psi has torsion 4 s' T and Dirac operator D^{s'} = D^g + 3 s' T. In the Hermitian convention used here this is s = 3 s'. So s = 3/4 for the Dirac operator of the connection with torsion H (s' = 1/4), and s = 1/4 for the connection with torsion H/3 (s' = 1/12). The second is the rescaling Bismut used for his local index theorem, and on naturally reductive spaces it is Kostant's cubic Dirac operator. The coupling of a three-form flux to spinors gives another constant. The value of s enters none of the statements, provided s != 0. In four dimensions c(e_a) c(e_b) c(e_c) = +- Gamma c(e_d) for {a, b, c, d} = {1, 2, 3, 4} (for instance c(e_1) c(e_2) c(e_3) = Gamma c(e_4)), so for every constant three-form H there is a unique eta in R^4, eta = +- s *H with the sign fixed by the orientation conventions, such that

```
s i c(H) = i c(eta) Gamma,
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and, for s != 0, every eta arises from a unique H. All statements below are in terms of eta. By Proposition 46.1, i c(eta) Gamma is the general j-antilinear odd Hermitian block and c(A) the general j-linear one.

The Weyl group S_3 acts on A by the permutations of the coordinates (z_1, z_2, z_3) of E^3 restricted to z_1 + z_2 + z_3 = 0; a transposition tau acts on T A = a (x) R^2 by rho(tau) (x) 1, the reflection of a in the hyperplane orthogonal to its normal n_tau, so d tau is the rotation by pi in the plane n_tau (x) R^2 with fixed plane P_tau = n_tau^perp (x) R^2, the tangent plane of the mirror {z_i = z_j}, which is a complex line for the complex structure J = 1 (x) J_E that A inherits from E; the lift of tau to Spin(4) is x = +- c(f_1) c(f_2) for an orthonormal basis of n_tau (x) R^2, of order four, acting on both chiralities with eigenvalues +-i (lemma 48, Proposition 48.1; lemma 49, Proposition 49.2), and tau-tilde psi(z) = x psi(tau^{-1} z) is its action on spinor fields, which is well defined for the periodic spin structure. A three-cycle acts by a rotation of order three of a tensored with the identity of R^2.

Statements

Proposition 51.1 (the lowest window of the torus with a Wilson line and a torsion). D_{A,H} commutes with the translations, and on the sector e^{i k . z} Sigma_4 it is the constant matrix

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Y_k = c(k + A) + i c(eta) Gamma,
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an odd Hermitian block of lemma 46 with j-linear half c(k + A) and j-antilinear half i c(eta) Gamma, whose eigenvalues are +-m_+(k) and +-m_-(k), each once, with

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m_+-(k)^2 = |k + A|^2 + |eta|^2 +- 2 |(k + A) wedge eta|,
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|v wedge eta| = (|v|^2 |eta|^2 - (v . eta)^2)^{1/2}. If |A| + |eta| < pi/l, the constant spinors are the lowest spectral window of D_{A,H}: the four eigenvalues +-m_+(0), +-m_-(0) lie in [-(|A| + |eta|), |A| + |eta|] and every other eigenvalue has absolute value at least 2 pi/l - |A| - |eta|, which is larger. The compression of D_{A,H} to that window is exact and equals

```
Y = c(A) + i c(eta) Gamma,
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the Yukawa block of lemma 46 on the sheet Sigma_4 = W of Proposition 49.1, with q <-> A and q' <-> eta in the variables of Proposition 46.2 and |Im(conj(q) q')| = |A wedge eta|. The tower above the window has the masses m_+-(k), k != 0, the Wilson line shifting the momenta and the torsion splitting each level into two.

Proof. The coefficients of D_{A,H} are constant, so it commutes with the translations z -> z + z_0 and preserves each sector e^{i k . z} Sigma_4, where -i partial_a acts as k_a; so D_{A,H} e^{i k . z} psi = e^{i k . z} (c(k + A) + s i c(H)) psi = e^{i k . z} Y_k psi. Y_k is Hermitian, odd for Gamma (c(v) anticommutes with Gamma, and i c(eta) Gamma anticommutes with Gamma), c(k + A) commutes with j and i c(eta) Gamma anticommutes with it. For the spectrum put v = k + A and X = Y_k restricted to L_p = Sigma_+, X : L_p -> L_p^perp; on L_p, Gamma = 1, and

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X^* X = (c(v) - i Gamma c(eta)) (c(v) + i c(eta) Gamma) = c(v)^2 + c(eta)^2 + i (c(v) c(eta) -
↪  c(eta) c(v)) Gamma = |v|^2 + |eta|^2 + 2 i c(v wedge eta),
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restricted to L_p, where c(v wedge eta) = (c(v) c(eta) - c(eta) c(v))/2 is the Clifford action of the two-form v wedge eta; we used Gamma c(eta) = -c(eta) Gamma and Gamma = 1 on L_p. The two-form v wedge eta is simple of norm |v wedge eta|, so c(v wedge eta) squares to -|v wedge eta|^2 and is traceless on each chirality (it is a multiple of c(f_1) c(f_2) for an orthonormal pair, which commutes with Gamma and has trace zero on each chirality), with eigenvalues +-i |v wedge eta| on L_p; hence X^* X has the eigenvalues m_+-^2 as displayed, and Y_k, with X^* X on L_p and X X^* on L_p^perp of the same spectrum, has the eigenvalues +-m_+, +-m_-, each once. For the window: m_+(0) <= |A| + |eta| by the triangle inequality for the operator norm of Y_0 = c(A) + i c(eta) Gamma, and for k != 0, m_-(k) >= |k + A| - |eta| >= 2 pi/l - |A| - |eta|, since m_-(k)^2 >= (|k + A| - |eta|)^2 (the smaller eigenvalue of X^* X is at least (|v| - |eta|)^2 because |v wedge eta| <= |v| |eta|) and |k| >= 2 pi/l; the inequality |A| + |eta| < 2 pi/l - |A| - |eta| is the hypothesis. The compression to the invariant subspace of the constant spinors is the restriction, Y_0 = Y. The identification with lemma 46 is Proposition 49.1, with the j-linear and j-antilinear halves matched by Proposition 46.1, and Proposition 46.2 gives the same masses. The map v -> c(v)|_{L_p} identifies R^4 isometrically with Hom_H(L_p, L_p^perp) = {L_q}, so q = q(A) and q' = q(eta) for one real-linear

isometry q : R^4 -> H. For quaternions, Re(conj(q) q') is the Euclidean inner product and |conj(q) q'| = |q| |q'|. Hence |Im(conj(q) q')|^2 = |q|^2 |q'|^2 - (q . q')^2 = |A|^2 |eta|^2 - (A . eta)^2 = |A wedge eta|^2. ■

Proposition 51.2 (the photon condition is the invariance under the transposition). Let tau be a transposition with mirror plane P_tau and lift x, and tau-tilde its action on spinor fields. Then

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tau-tilde D_{A,H} tau-tilde^{-1} = D + c(d tau A) + i c(d tau eta) Gamma,
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so D_{A,H} commutes with tau-tilde exactly when d tau A = A and d tau eta = eta, that is exactly when A and eta lie in P_tau. Since A is defined modulo 2 pi Lambda^*, D_{A,H} commutes with tau-tilde up to a gauge transformation e^{i k . z} exactly when d tau A - A lies in 2 pi Lambda^* and d tau eta = eta. Under the window hypothesis |A| < pi/l of Proposition 51.1, |d tau A - A| <= 2 |A| < 2 pi/l, which is below the length of every nonzero vector of 2 pi Lambda^*. So d tau A = A, and the two conditions agree. In that case the window block Y = c(A) + i c(eta) Gamma commutes with x on both chiralities, that is with the algebra C = span_R(1, x) of the mirror stratum (Proposition 49.4), which is the restriction to the first sheet of the subalgebra C_F of Chamseddine, Connes and Marcolli, so that [D_F, C_F] = 0, the condition that keeps the photon massless (Proposition 49.3), holds; conversely the blocks of Proposition 49.3 commuting with x are exactly the Y with A, eta in P_tau, a real four-dimensional family (two complex Yukawa couplings). A nonzero A, or a nonzero eta, in P_tau is fixed by tau and by no other element of S_3, since P_tau meets P_tau' in 0 for tau' != tau and a three-cycle fixes no nonzero vector of a (x) R^2; so a nonzero Wilson line or torsion breaks S_3 to the stabilizer of one mirror, and D_{A,H} commutes with the lift of the whole of S_3 exactly when A = eta = 0. So a first sheet at a symmetric point, whose algebra C + H requires invariance under the whole lifted stabilizer, carries no Yukawa block from these data, as Proposition 49.3 found for the invariant blocks. For v in the normal plane P_tau^perp = n_tau (x) R^2, x c(v) x^{-1} = -c(v): these are the components of the Higgs directions T A = T_p S^4 that anticommute with x, the charged ones of lemma 49.

Proof. tau is a linear isometry of R^4 preserving Lambda, so it acts on A, and tau-tilde preserves the periodic spin structure. The Dirac operator D commutes with every isometry lifted to the spinors, since c(d tau v) = x c(v) x^{-1} and partial transforms by d tau: tau-tilde D tau-tilde^{-1} = D. For the constant terms, tau-tilde c(A) tau-tilde^{-1} = x c(A) x^{-1} = c(d tau A), and tau-tilde i c(eta) Gamma tau-tilde^{-1} = i c(d tau eta) x Gamma x^{-1} = i c(d tau eta) Gamma, since x lies in Spin(4) and d tau has determinant one, so that x commutes with Gamma. The difference D_{A,H} - tau-tilde D_{A,H} tau-tilde^{-1} = c(A - d tau A) + i c(eta - d tau eta) Gamma vanishes exactly when both terms vanish, the j-linear and j-antilinear parts being independent (Proposition 46.1), that is when A and eta are fixed by d tau, whose fixed space is P_tau. The window block is the restriction of D_{A,H} to the constant spinors, on which tau-tilde acts as x; so it commutes with x. The real span of x on Sigma_4 with the identity is the algebra C of Proposition 49.4, and Proposition 49.3 identifies the commutation with x on both chiralities with the condition [D_F, C_F] = 0. For the converse count, the blocks commuting with x are X = diag(y_1, y_2) of complex dimension two (Proposition 49.3), real dimension four, and the Y with A, eta in P_tau form a real four-dimensional space of such blocks (A and eta in a two-plane; the map (A, eta) -> c(A) + i c(eta) Gamma is injective, by the injectivity of c and Proposition 46.1), so the two spaces coincide. P_tau = n_tau^perp (x) R^2 and P_tau' = n_tau'^perp (x) R^2 with n_tau, n_tau' linearly independent in the two-dimensional a, so n_tau^perp and n_tau'^perp are distinct lines of a and the planes meet in 0; a three-cycle acts on a by a rotation of order three, with no fixed vector, so its action on a (x) R^2 has none. For v in n_tau (x) R^2, d tau v = -v, and x c(v) x^{-1} = c(d tau v) = -c(v). ■

Proposition 51.3 (the rank-one pattern: the torsion as the quarter turn of the Wilson line). Let A and eta lie in the mirror plane P_tau, a complex line for the complex structure J of A inherited from E, and let phi in [0, pi] be the angle between them. Then m_+-^2 = |A|^2 + |eta|^2 +- 2 |A| |eta| sin phi, and:

1. m_+ = m_- exactly when eta is parallel to A (sin phi = 0), the custodial case of Proposition 46.2 (1): a torsion along the Wilson line gives equal masses to the two members of the doublet.
2. m_+ = |A| + |eta| and m_- = ||A| - |eta|| exactly when eta is perpendicular to A, that is eta in R J A. Then (m_+ + m_-)/2 = max(|A|, |eta|) and (m_+ - m_-)/2 = min(|A|, |eta|): for |eta| <= |A| the Wilson line is the mean mass of the doublet and the torsion half its splitting. Put the up-type

member on the Majorana line w, a weight line of x (Proposition 49.2 (iii); the up-type assignment of Proposition 50.1 (3) is an input). Then the up-type mass is |A + J_w eta| (item 4). So m_up = |A| + |eta| and m_down = |A| - |eta| when eta is a positive multiple of -J_w A, and the two masses are exchanged when eta is a positive multiple of J_w A. Which member is heavier is the sign of <J_w A, eta>, the oriented area of (A, eta) in the complex line (P_tau, J_w), a datum of the pair (A, eta).

3. m_- = 0 exactly when eta = +- J A: the heaviest quark, with its massless partner in this block, is the generation whose torsion dual is the quarter turn of its Wilson line in the complex structure of the elliptic curve; then m_+ = 2 |A| = 2 |eta|, and Y = {c(2A), P_mu} with mu the weight line of x whose complex structure is J_mu = -+J, so that eta = -J_mu A, lemma 50's delta' = -J_mu h/2 with h = 2A. It is the block c(h) P_w of lemma 50 exactly when the Majorana line w is mu, that is J_w = -+J for eta = +-J A.
4. For a weight line mu of x in Sigma_R, with complex structure J_mu = +-J, the coupling of Proposition 49.3 on mu has modulus |A + J_mu eta|. With the up-type member on the Majorana line w, y_u = A + J_w eta and y_d = A - J_w eta in the complex line (P_tau, J_w), up to a common phase and the conjugation of Proposition 49.3. In a complex coordinate z of P_tau in which J_w acts as i, y_u = z(A) + i z(eta) and conj(y_d) = z(A) - i z(eta). Hence |y_u|^2 = |A|^2 + |eta|^2 - 2 <J_w A, eta> and |y_d|^2 = |A|^2 + |eta|^2 + 2 <J_w A, eta>, with <J_w A, eta> = +-|A| |eta| sin phi. In the coordinate given by J these formulas hold when J_w = J, and they hold with i replaced by -i when J_w = -J. So the sign is fixed by which weight line of x is the Majorana line, an input, and no other convention enters.

The finite data of Corollary 16.8 with V_X = Sigma_4, A_{F,X} = C = span_R(1, x) (the mirror stratum, Proposition 49.4), d_X = Y with A, eta in P_tau, and V_Y = C^3 with M_3(C), d_Y = 0, are a 24-state bimodule over C + M_3(C) with the KO-6 signs, order zero and order one; every one-form vanishes, since C commutes with Y, so the masses are explicit (Proposition 17.3) and the algebra is the centralizer of the block (the centralizer of Y in C + H is span_R(1, x) for generic, rank-one and parallel data; checked by the referee), the algebra of the label 9 of lemma 50, Proposition 50.4; the orientability relation fails there: on the first summand every sum of products pi(a) b° lies in span_R(1, x) (x) M_3(C), which acts on the +i weight lines of x in Sigma_R and in Sigma_L by the same matrix, while Gamma_F = Gamma (x) 1 is 1 on the first and -1 on the second (referee).

Proof. In P_tau, |A wedge eta| = |A| |eta| sin phi, and the formula is Proposition 51.1. (1) sin phi = 0. (2) sin phi = 1 gives m_+-^2 = (|A| +- |eta|)^2; the perpendicular of A in the two-plane P_tau is R J A, J being orthogonal on P_tau; conversely m_+ = |A| + |eta| forces sin phi = 1 when A, eta are nonzero. The mass on the line w is |A + J_w eta| by item 4, which gives the assignment for each sign of <J_w A, eta>. (3) m_- = 0 exactly when |A|^2 + |eta|^2 = 2 |A| |eta| sin phi, which forces |A| = |eta| and sin phi = 1; then eta = +- J A, and Y = c(A) + i c(+- J A) Gamma; in lemma 50, Proposition 50.2 (3), the rank-one block c(h) P_w has j-linear half c(h)/2 and j-antilinear half i c(delta') Gamma with delta' = -J_w h/2, so A = h/2 and eta = delta' = -J_w A, that is J_w = -+ J, the sign according to which weight line of x carries the heavy member, since J_{jw} = -J_w (lemma 50, Proposition 50.2 (3)), and the two weight lines of x in Sigma_R are the two points of the twistor fibre with J_mu = J and J_mu = -J, since x|*{Sigma_R} = +-c(omega_J)/2 with omega_J the Kähler form of J (lemma 49, Reading; checked by the referee). (4) By Proposition 49.4 the Clifford element c(A) of a vector of the mirror plane is a block L_a commuting with x, with a complex, and A -> a(A) is a real-linear isometry P_tau -> C, complex linear for J_mu with mu the line on which x acts as i in the identification of Proposition 49.3, and i c(eta) Gamma = i L_c with c the image of eta; Proposition 49.3 gives y_1 = a + i c and conj(y_2) = a - i c, with |y_1|^2 and |y_2|^2 as displayed, Im(conj(a) c) being +- the area |A| |eta| sin phi. For the finite data, Theorem 16.7 applies formally to the real subalgebra C of End_C(Sigma_4) (Remark 16.5); the one-forms [D_F, pi(a)] are [Y, a_X] (x) 1 on Hom(V_Y, V_X) and vanish for a_X in span(1, x) by Proposition 51.2; Proposition 17.3 then gives the explicit masses; the centralizer statement is Proposition 50.4 with the summand C of A_SM embedded along the weight lines of x; Gamma acts as (1, -1) on the chiralities while x acts with eigenvalues +-i on both, so Gamma lies outside span_R(1, x), the hypothesis gamma*{V_X} in A_{F,X} of Corollary 16.8 fails, and the relation itself fails, since the products pi(a) b° act on the two +i weight lines by one matrix and Gamma_F by opposite signs. ■

Reading

Lemma 22 and lemma 25 read every finite Dirac operator as the lowest window of a derivative on an interval, unfolded to a circle with a constant connection; the present lemma is the four-torus form of that reading with the sheet of lemma 49 as the fibre: the custodial half of the Yukawa block is a Wilson line, the mass of a doublet without torsion is |A|, the internal momentum shifted by the holonomy, and a torsion along the Wilson line gives both members the mass (|A|^2 + |eta|^2)^{1/2} (the mechanism of Hosotani, recorded for the interval in lemma 40 and lemma 20, reading 3), and the up-down splitting, which no Wilson line and no displacement supplies, is a constant three-form, the torsion or flux term of the Dirac operator. Along a mirror of the Weyl group the two data make the two complex Yukawa couplings of a generation, y_u = A + i eta and conj(y_d) = A - i eta in the complex coordinate of the mirror, and the photon condition of Chamseddine, Connes and Marcolli, imposed there on D_F by hand, is the invariance of the Wilson line and the torsion under the transposition whose mirror carries them. The heaviest quark is the extreme in which the torsion is the quarter turn of the Wilson line; for the third generation in the perpendicular case the ratio |eta|/|A| = (m_t - m_b)/(m_t + m_b) is about 0.97 with the running masses at the scale m_t (m_t about 163 GeV, m_b about 2.8 GeV; 0.95 with m_t = 173 GeV and m_b = 4.2 GeV). For the second generation (m_c - m_s)/(m_c + m_s) is about 0.84 at a common scale (m_c/m_s about 11.8; 0.86 with m_c(m_c) = 1.27 GeV and m_s(2 GeV) = 0.093 GeV). The first generation has m_d > m_u, so in the perpendicular case its torsion has the opposite orientation relative to J_w A, with |eta|/|A| = (m_d - m_u)/(m_d + m_u) about 0.37. The lighter generations are further from the rank-one pattern. (Running values recalled, not computed here; ratios at a common scale are independent of the QCD running.) The symmetric points of A, where the algebra is C + H, carry no Yukawa block and the quark sector is massless with the full algebra; the mirror, where the algebra is C, carries the block with explicit masses, no one-forms and the algebra C + M_3(C) of the label 9: the two strata are the two ends of the 10 -> 9 reduction of the quark sector, as a reading, with the choice of mirror the choice of the photon circle among three (lemma 49). A lead, from the geometry of A: A = Lambda_coroot (x) E = Hom(Lambda_weight, E), the triples (z_1, z_2, z_3) of points of E with z_1 + z_2 + z_3 = 0. Its quotient by the Weyl group is the moduli space of semistable SU(3)-bundles on E, equivalently of flat SU(3)-connections on E. Sources: Friedman, Morgan and Witten, arXiv:hep-th/9701162, read in the e-print by the referee: section 2.1 (V = N_1 + … + N_n with N_1 (x) … (x) N_n = O, the moduli space P H^0(E, O(np)) = CP^{n-1}); section 3.1 ((T x T)/W for simply connected G); and section 4, "A note on flat connections" (Hom(L', E) = L' (x) E with L'' the weight lattice). For general G the quotient is a weighted projective space (Looijenga, Invent. Math. 38 (1976) 17-32, doi:10.1007/BF01390167). For SU(3) it is CP^2, the corpus quotient of lemma 45. A point of A is then a flat SU(3)-connection on E with holonomies in the maximal torus. A generic point has centralizer the maximal torus U(1)^2. A point of a mirror z_i = z_j off the nine symmetric points has centralizer S(U(2) x U(1)) = U(2). The nine symmetric points, z_1 = z_2 = z_3 with 3 z_1 = 0, are the pairs of holonomies in the centre Z_3 x Z_3, with centralizer SU(3); the Wilson line A of this lemma is a flat U(1)-connection on A, a different object, and whether the two can be identified through a duality of the torus is open. For three generations the block needs flavour-dependent data, for instance a Wilson line in a Cartan subalgebra of a flavour group with three weights q_g A and a flavour-dependent three-form; a torsion common to the three generations is excluded by the masses for any Wilson lines A_g. In every generation m_+^2 + m_-^2 = 2 (|A_g|^2 + |eta|^2) >= 2 |eta|^2, so the first generation bounds |eta| by about 4 MeV. The third has m_t^2 - m_b^2 = 4 |A_3 wedge eta| <= 4 |A_3| |eta| with |A_3|^2 <= (m_t^2 + m_b^2)/2, which needs |eta| of at least (m_t^2 - m_b^2)/(2 sqrt(2) (m_t^2 + m_b^2)^{1/2}), about 61 GeV. In the perpendicular case with |eta| <= |A_g| the splitting would be the common value 2 |eta| (referee).

Scope

The flat four-torus with the periodic spin structure and constant data A, H; the window statement needs |A| + |eta| < pi/l, and the compression is exact because the window is a reducing subspace, so no estimate is involved; the Wilson line is a gauge class modulo 2 pi Lambda^*, which under the window hypothesis changes nothing in Proposition 51.2; the constant s != 0 of the three-form term (verified in Agricola's lectures) and the sign of eta depend on conventions and enter no statement; the sign that does enter, which weight line of x is the Majorana line, is an input (Proposition 51.3); the equivariance statement uses the lift of

the transposition as an isometry of the whole torus, and the finite data at the mirror stratum are those of Corollary 16.8 with the algebra of Proposition 49.4, as an identification of data, the crossed product of lemma 48 seeing the orbit types as quotients; the one-generation form with one Wilson line and one three-form, the three-generation form being left as a direction; the E factor of T^6, whose spinors double the fibre and whose own Wilson line c(A_E) is even for the chirality of A, is outside the statements; the reading of the two strata as the ends of the 10 -> 9 reduction and the moduli-space lead are readings. Nothing here derives A or H from an action, and no potential selects eta = +- J A. Two remarks of the referee, in their scope: in heterotic flux compactifications the Dirac operator of the gauge fermions uses the connection with torsion one third (Kimura and Yi, arXiv:hep-th/0605247, abstract), the value s = 1/4 here, which would enter a flux-quantization condition relating |eta| to the integer fluxes of H over the three-cycles of A; and on the compact flat torus a constant H != 0 solves no leading-order NS-NS equation, since the trace of R_ij + 2 nabla_i nabla_j phi - (1/4) H_ikl H_j^kl = 0 integrates to (1/4) int |H|^2 = 0, so the direction below needs a source or another background (Lie groups with parallel torsion, where nabla^{+-1/4} are flat; brane or orientifold sources; higher orders), a remark not checked by a second model and leaving the statements, which take A and H as data, unchanged. The direction the result points to: a potential or a quantization condition on the torus that relates the three-form to the Wilson line (a flux quantization of H over three-cycles of A, with the Wilson line on the dual cycle); the E factor, where a Wilson line gives an even block, the candidate for the connecting operator of the lepton generation; and the identification of the point of A, a flat SU(3)-connection on E, with the Wilson line on A.

Part 52

The lepton generation in three views: the lepton line of the two sheets with the connector, the Majorana spinor of nine dimensions, and the partner of the flavour adjoint

For STRATEGY T16 (the lepton generation, the lead at the end of lemma 49: a connecting operator identifying the two copies of C), T12 (the scalar partners of lemma 21), T2c and T13 (the nine-dimensional reading and the masses as internal momenta), following lemma 20 (Theorems 20.1 and 20.2), lemma 16 (Theorem 16.7, Corollary 16.8), lemma 49 (Propositions 49.1, 49.2 and 49.5), lemma 50 (Proposition 50.3), lemma 51 (Proposition 51.1), lemma 41 (Lemma 41.1 and Theorem 41.2), lemma 42 (Proposition 42.2), lemma 06 (Lemma 4 and Proposition 7), lemma 44 (Remark 44.4) and lemma 21 (the SO(30) section and Proposition 21.3). The three views were named by the user on 5 November 2026 (plain Connes; partner to the scalars of the 24 with 6 positive, 6 negative and 12 neutral; the Dirac operator of a Kaluza-Klein reduction from nine dimensions). Written by Claude Fable 5.1 on 5 November 2026, by hand; the existence of the Majorana conjugation of Spin(1, 8) on an explicit Clifford representation, its commutation with the spin generators, the exchange of the two chiral eigenspaces, the commutant dimensions 32, 16 and 2 (of Spin(1, 3), Spin(5) and their product on C^16; the referee's values, the first draft recorded 4 for the first) that give the decomposition (2, 1) (x) 4 + (1, 2) (x) 4, and the lepton pairing were checked with numpy (script not kept, by the rule of the project). Refereed by Claude Opus 5.5 (effort max, 5 November 2026; report in historical/reviews/2026-11-05_opus_check_51_52.md), who rebuilt the 32-state bimodule and Cl(1, 8) with its Majorana and charge-conjugation matrices and read the e-print of Chamseddine, Connes and Marcolli: Propositions 52.1 and 52.3 correct with amendments (the dictionary needs the eigenline condition; the rank-one defect formula of the connector; the pairing nondegenerate over Q and equal to their Proposition 2.30 (3); the 3 against the 3-bar), Proposition 52.2 with one error in statement (iii) (a nine-dimensional Majorana field carries exactly one generation only on its self-conjugate sector, where the mass operator is i c(eta) Gamma with the equal masses |eta|; the sectors of nonzero momentum pair into a generation and a mirror, so the quantized momenta enter only there and the contrast with the quark masses falls) and one in its proof (the bilinear vanishes for anticommuting components), with amendments (two irreducible modules; e_R^c on w and nu_R^c on jw; the scope of the Majorana-mass statement; the factorizations); all applied below with the referee's texts.

Outcome. The lepton generation is the quark generation of Proposition 49.5 with the three colours replaced by one line, in three settings that agree on its content. In the two-sheet data with the first sheet the spinor

fibre Sigma_4 and the second sheet the lepton line Cv beside the colour C^3, the lepton sub-bimodule has eight states and is preserved by the algebra, the real structure, the grading, the split Yukawa block of Proposition 50.3 and the connector of Theorem 20.2, which lives entirely inside it; the four-summand algebra C + H + C' + M_3(C) of the two sheets is reduced by the connector's order-one condition to the Standard Model algebra {(lambda, q, lambda, m)} exactly when the connector's line is a weight line of the lifted transposition, the two copies of C being identified, and to R + H + M_3(C) otherwise; the trace-form pairing of the lepton module over C + H is [[0, 2], [-2, 0]], nondegenerate over Q, the restriction of the matrix printed by Chamseddine, Connes and Marcolli (Proposition 2.30), while over the three-summand algebra the lepton and quark modules together give the skew form of rank two of lemma 41 (Proposition 52.1). A Majorana spinor of nine-dimensional Minkowski space, sixteen real components, is under Spin(1, 3) x Spin(5) the complex space (2, 1) (x) 4: four left-handed Weyl spinors of the four-dimensional spacetime in the spinor fibre W = H^2 of the five-dimensional internal space, which is the sheet of Proposition 49.1; under Sp(1)_a x Sp(1)_b it is the left-handed doublet and the right-handed pair, under Sp(1)_a x U(1)_b the doublet of U(1)_b charge zero and the two singlets of U(1)_b charge +-1 (T_3R = +-1/2), one lepton generation with nu_R; its nine-dimensional mass bilinear vanishes, so its masses are the internal Dirac operator on the fibre, which on A x S^1 with the torsion of lemma 51 is c(k) + i c(eta) Gamma + k_5 Gamma on the sector of momentum (k, k_5), shifted by the spin structure, since a Majorana field carries no U(1) charge and admits no Wilson line. The self-conjugate sector carries exactly the generation, with the masses |eta| of both members, and the other sectors pair into a generation and a mirror with the masses of Proposition 51.1; a Majorana mass for nu_R alone is absent from Lorentz-invariant and constant bulk data, wall data in that scope, and selects the KO-7 factorization for one generation (Proposition 52.2). As representations of U(1)_em, three lepton generations with nu_R (24 states) and the complexified adjoint of the flavour SU(5) of the five light quarks (lemma 21) are isomorphic, with multiplicities 12, 6 and 6, the adjoint decomposing under SU(2)_u x SU(3)_d x U(1)_em as (3, 1)_0 + (1, 8)_0 + (1, 1)_0 + (2, 3-bar)_{+1} + (2-bar, 3)_{-1}, so that the charged states are (up flavour) x (down flavour) in number as the charged leptons are (chirality) x (generation), and the neutral ones span the flavour gauge algebra (Proposition 52.3).

Setting

The two-sheet data of Corollary 16.8 with V_X = Sigma_4, the spinor fibre of the four-torus A with the Clifford action, the chirality Gamma and the quaternionic structure j (Proposition 49.1: the sheet W = H^2 with axis p at Gamma, L_p = Sigma_R, L_p^perp = Sigma_L), A_{F,X} = C + H the algebra of the lifted stabilizer at a symmetric point (Proposition 49.2: C = span_R(1, x) acting on Sigma_R as (lambda, conj(lambda)) in the weight basis of the lifted transposition x (ordered (w, jw) when Cw is a weight line), H = End_H(Sigma_L)), and V_Y = V = Cv + C^3 = C^4 with A_{F,Y} = C' + M_3(C) = {lambda' P_v + m}, the second-sheet algebra of a connector on the line v (Theorem 20.2, Proposition 50.3 (4)). The bimodule is H_F = Hom(V, V_X) + Hom(V_X, V) = Sigma_4 (x) conj(V) + V (x) conj(Sigma_4), of complex dimension 32, with J_F(x, y) = (y^*, x^*) and Gamma_F = Gamma on the first summand and -Gamma^c on the second. The connector is T_kappa x = kappa <e, x> f with e = w v^*, f = e^*, w a unit vector of Sigma_R. The finite Dirac operators considered are D_F = D_Y + S_{T_kappa}, with D_Y x = Y_l x P_v + Y_q x P_v^perp and D_Y y = P_v y Y_l + P_v^perp y Y_q (Proposition 50.3). The dictionary of Chamseddine, Connes and Marcolli is as in lemma 50: lambda on nu_R and u_R, conj(lambda) on e_R and d_R, lambda on the antileptons, m on the antiquarks.

For the second view, Spin(1, 8) is the spin group of nine-dimensional Minkowski space R^{1,8} with signature (+, -, ..., -), its Dirac spinor is C^16, and the Majorana spinors are the fixed points of an antilinear C with C^2 = 1 commuting with the Clifford action (which exists in signature (1, 8), where t - s = -7 = 1 mod 8); M^4 x K_5 with K_5 a spin five-manifold has the structure group Spin(1, 3) x Spin(5) of its frame bundle at a point, Spin(5) = Sp(2) acting on the spinor fibre C^4 = H^2 of K_5 by the 4. For the third view, the five light quarks u, c, d, s, b of lemma 21 span the 5 of a flavour SU(5) with the electric charge Q = diag(2/3, 2/3, -1/3, -1/3, -1/3), whose commutant in SU(5) is S(U(2)_u x U(3)_d), and the 24 is the complexified adjoint sl(5, C) with the adjoint action.

Statements

Proposition 52.1 (the lepton line of the two sheets). (i) H_F = H_l + H_q with H_l = Sigma_4 (x) conj(Cv) + Cv (x) conj(Sigma_4), of complex dimension 8, and H_q = Sigma_4 (x) conj(C^3) + C^3 (x) conj(Sigma_4), of dimension 24 (the quark generation of Proposition 49.5); both are invariant under the left and the opposite actions of C + H + C' + M_3(C), under J_F and Gamma_F, under D_Y, and under the connector, whose vectors e and f lie in H_l; so every D_F above is block diagonal in H_l + H_q. In the dictionary, which item (ii) makes available when Cw is a weight line of x, H_l carries nu_R = w (x) conj(v) and e_R = jw (x) conj(v). These are the two weight lines of x tensored with the lepton line, w being the Majorana line. H_l also carries the left-handed doublet Sigma_L (x) conj(v) and the conjugates of all three. Here lambda, the eigenvalue of the summand C on w, acts on nu_R, conj(lambda) on e_R, q on the doublet and lambda' on the antileptons v (x) conj(Sigma_4).

(ii) The largest subalgebra of C + H + C' + M_3(C) on which D_F = D_Y + S_{T_kappa}, kappa != 0, satisfies order one is its intersection with A(T_kappa) of Theorem 20.2: it is the Standard Model algebra A_SM = {(lambda, q, lambda, m)}, in which the two copies of C are identified, exactly when the connector's line Cw is a weight line of x (the eigenline condition of Proposition 49.2 (iii)), and it is {(r, q, r, m) : r real} = R + H + M_3(C) otherwise. On A_SM the lepton module H_l is the summand H_F^(1) of one generation of the spectral Standard Model (Chamseddine, Connes and Marcolli, Proposition 2.30), with the Majorana block of the connector on nu_R and the Yukawa block Y_l of the leptons.

(iii) The trace-form pairing <p, p'> = Tr_{H}(Gamma_F p p'^o) of lemma 41 on H_l, for the minimal projections p_1 = (1_C, 0, 1_{C'}, 0), p_2 = (0, 1_H, 0, 0), p_3 = (0, 0, 0, e_11) of A_SM, is [[0, 2, 0], [-2, 0, 0], [0, 0, 0]]: over the lepton algebra C + H, the summands acting on H_l, the pairing [[0, 2], [-2, 0]] is nondegenerate over Q; its determinant is 4, so over Z it is not unimodular, and the trace-form Poincaré duality holds rationally for the lepton generation over the algebra that acts on it; the matrices on H_l and H_q are those printed by Chamseddine, Connes and Marcolli (arXiv:hep-th/0610241, Proposition 2.30 (3), per generation), and over the three summands their Corollary 2.31 obtains nondegeneracy from the pair of classes of H_F^(1) and H_F^(3), their Remark 2.32 adding that KO-homology is generated by two elements; on H_q it is [[0, 0, 2], [0, 0, -2], [-2, 2, 0]] (Proposition 49.5), and on H_F the sum [[0, 2, 2], [-2, 0, -2], [-2, 2, 0]], skew of rank two, the form of Theorem 41.2 at the Standard Model step up to the sign convention.

Proof. (i) The left actions are a_X (x) 1 on the first summand and a_Y (x) 1 on the second, the opposite actions 1 (x) (b_Y^*)^c and 1 (x) (b_X^*)^c (Theorem 16.7); P_v and P_v^perp commute with A_{F,Y} = C' + M_3(C), so Sigma_4 (x) conj(Cv) and Sigma_4 (x) conj(C^3) are invariant under the left action of A_{F,X} (which acts on the first factor) and under the opposite action of A_{F,Y} (which preserves Cv and C^3), and likewise for the second summand; J_F exchanges Sigma_4 (x) conj(Cv) with Cv (x) conj(Sigma_4), and Gamma_F acts on the first factors; D_Y x = Y_l x P_v + Y_q x P_v^perp maps x = xi (x) conj(v) to Y_l xi (x) conj(v), and x = xi (x) conj(c) to Y_q xi (x) conj(c) for c in C^3; e = w v^* lies in Sigma_4 (x) conj(Cv) and f = v w^* in Cv (x) conj(Sigma_4), and T_kappa and T_kappa^* have range in the lines C f and C e. The dictionary assignments are those of lemma 50, Setting, with Sigma_R = L_p the lambda-weight line of the summand C on w (Proposition 49.2 (ii)). (ii) By Theorem 20.1 and Theorem 20.2, D_Y satisfies order one on the whole of C + H + C' + M_3(C) (Proposition 50.3 (4)), and the defects of D_Y and of S_{T_kappa} lie in different blocks, so order one of the sum is order one of each. Write a = (a_c, q_a, lambda'*a, m_a) for an element of the four-summand algebra, with a_c its action on Sigma_R. The order-one defect of S*{T_kappa} on Hom(V, Sigma_4) -> Hom(Sigma_4, V) is the rank-one map x -> kappa <(a_c^* w - conj(lambda'_a) w) v^*, x> v (conj(lambda'_b) w - b_c^* w)^*, and the other block is its adjoint (checked by the referee on random data). On a *-subalgebra B with order one, a = b gives a_c^* w = conj(lambda'_a) w for every a in B, hence a_c w = lambda'_a w by a -> a^*. Conversely, these conditions make the defect vanish. So the largest *-subalgebra with order one is {a : a_c w = lambda'*a w}, the intersection with A(T_kappa) = {(lambda_1 on w, conj(lambda_1) on jw) + q + (lambda_1 on v) + m} of Theorem 20.2, the condition on v holding automatically in the four-summand algebra (Theorem 20.1 alone gives the inclusion in some A(R), and the rank-one form gives A(T_kappa) itself); an element (lambda on Sigma_R in the weight basis of x, q, lambda' on v, m) of the four-summand algebra lies there exactly*

*when lambda' = lambda_1 and the action of C on Sigma_R, diag(lambda, conj(lambda)) in the weight basis of x, equals diag(lambda_1, conj(lambda_1)) in the basis (w, jw). If Cw is a weight line of x the two bases coincide up to order, and the condition is lambda' = lambda (if Cw is the weight line on which x acts as -i, the condition reads lambda' = conj(lambda_x) in the parameter lambda_x of the +i line; in the eigenvalue lambda of the summand C on w it reads lambda' = lambda in both cases, so the algebra is {(lambda on w, conj(lambda) on jw, q, lambda on v, m)}, which is A_SM with nu_R and u_R on Cw; the parameter of A_SM is the eigenvalue on the Majorana line, and the dictionary of item (i) uses it); if Cw is not a weight line, a matrix diagonal in both bases is scalar, so lambda = conj(lambda) = lambda_1 = lambda' is real. The identification with H_F^(1) is the dictionary, with the Majorana block S{*T_kappa} on the line C e = C nu_R (lemma 20, reading 1) and Y_l (x) P_v the lepton Yukawa block. (iii) On H_l the pairing is <p, p'> = tr(Gamma a_p) tr(b_{p'}) - tr(b_p) tr(Gamma a_{p'}), with a the first-sheet part and b the second-sheet part restricted to Cv, as in Theorem 48.3 and Proposition 49.5; tr(Gamma a_{p_1}) = tr(Gamma|Sigma_R) = 2, tr(b_{p_1}) = 1; tr(Gamma a_{p_2}) = -2, tr(b_{p_2}) = 0; a_{p_3} = 0 and b_{p_3} = 0 on Cv. The 2 x 2 block on (p_1, p_2) has determinant 4. On H_q the second-sheet traces are tr(b_{p_3}) = 1 and tr(b_{p_1}) = tr(b_{p_2}) = 0 (lambda P_v vanishes on C^3), which gives the form of Proposition 49.5; the sum is as displayed, with the sign convention of lemma 49, opposite to that of lemma 41. ■

Proposition 52.2 (the Majorana spinor of nine dimensions is a lepton generation). (i) The Dirac spinor C^16 of Spin(1, 8) restricted to Spin(1, 3) x Spin(5) is (2, 1) (x) 4 + (1, 2) (x) 4, where (2, 1) and (1, 2) are the two Weyl spinors of Spin(1, 3), conjugate to each other, and 4 is the spinor of Spin(5) = Sp(2), quaternionic. A Majorana conjugation C (antilinear, C^2 = 1, commuting with the Clifford action of R^{1,8}, hence with Spin(1, 8)) exists and exchanges the two summands; so the real Majorana spinor R^16 is, as a complex vector space with the action of Spin(1, 3) x Spin(5), the first summand (2, 1) (x) 4: four left-handed Weyl spinors of M^4 in the 4 of Sp(2).

(ii) On M^4 x K_5 the 4 is the spinor fibre W = H^2 of K_5 at a point, with Sp(2) its structure group; for K_5 = A x S^1 it is the sheet of Proposition 49.1 with c(e_5) = +-Gamma, the grading axis of the sheet being the circle direction. Under K_p = Sp(1)_a x Sp(1)_b it is (2, 1) + (1, 2), the left-handed doublet and the pair of right-handed singlets written as left-handed conjugates, and under Sp(1)_a x U(1)_b it is (2)_0 + (1)_{+1} + (1)_{-1}, the U(1)_b charges of Proposition 42.2 (T_3R = +-1/2 with Y = charge/2): one lepton generation with nu_R, the weak isospin Sp(1)_a and the T_3R circle, in the two-sheet data the quark generation is the same fibre tensored with the colour sheet C^3 (Proposition 49.5); in nine dimensions a colour triplet is a complex field, whose left-handed content (2, 1) (x) 4 (x) (3 + 3-bar) is a quark generation and its mirror.

(iii) The Lorentz-invariant mass bilinear of a Majorana spinor vanishes in dimension 8k + 1 (the corpus row rs_F3, Alvarez-Gaumé and Witten), so the four-dimensional masses of the generation are the eigenvalues of the internal Dirac operator on the fibre. A Majorana field carries no U(1) charge and admits no Wilson line, so on A x S^1 its internal momentum is a vector (k, k_5) of the dual lattice shifted by the spin structure. A single Majorana field admits the torsion of lemma 51, since the bilinear of Gamma^{abc} is antisymmetric in signature (1, 8) and the coupling survives. With it, the mass operator on the sector of momentum (k, k_5) is c(k) + i c(eta) Gamma + k_5 Gamma, with squared masses m_+-(k)^2 + k_5^2, m_+-(k)^2 = |k|^2 + |eta|^2 +- 2 |k wedge eta| (Proposition 51.1). The Majorana condition relates the sector (k, k_5) to (-k, -k_5). The self-conjugate sector (k, k_5) = 0 exists for the periodic spin structure on A x S^1. It carries one lepton generation, with mass operator i c(eta) Gamma: the Yukawa block of that generation is purely anti-quaternionic, and the two members of the doublet have the equal masses |eta| (Proposition 46.2 with q = 0). A pair of sectors (k, k_5), (-k, -k_5) with (k, k_5) != 0 carries an unconstrained complex field: a lepton generation and a mirror generation. These are four Dirac fermions, with the masses (m_+(k)^2 + k_5^2)^{1/2} and (m_-(k)^2 + k_5^2)^{1/2} each twice. On these levels the custodial half c(k) is a quantized internal momentum. The Lorentz-invariant bulk mass is absent (rs_F3). Constant three-form backgrounds with one leg on S^1 add to the self-conjugate sector the terms i c(beta) Gamma, beta a two-form on A; their mass bilinears omega(., i c(beta) Gamma .), with omega the Spin(5)-invariant form, give nu_R^c and e_R^c Majorana entries of equal modulus (checked by the referee). So a Majorana mass for nu_R

alone, the one that conserves the electric charge, is absent from Lorentz-invariant and constant bulk data. In that scope it is wall data in the sense of Remark 44.4 (the connector of Proposition 52.1), and with it the neutral sector of one generation admits only the KO-7 factorization (lemma 06, Lemma 4: the singular values of [[0, m_D], [m_D, M]] are distinct for M != 0). With Dirac masses only, the neutral singular values come in pairs and both factorizations exist (lemma 06, Proposition 7 with R = 0). With three generations and a general Majorana matrix, KO-5 exists exactly under the multiplicity condition of Proposition 7.

Proof. (i) Cl(1, 8) is the graded tensor product of Cl(1, 3) and Cl(0, 5), and each of its two irreducible complex modules, distinguished by the sign of the volume element, is C^16; the two restrict to equivalent representations of Spin(1, 8), and C^16 is the tensor product of the Dirac spinor C^4 of Spin(1, 3) and the spinor C^4 of Spin(5), on which the even subalgebras act factorwise; under Spin(1, 3) the Dirac spinor is (2, 1) + (1, 2), the eigenspaces of the chirality gamma_5 = gamma_0 gamma_1 gamma_2 gamma_3, which squares to -1 in this signature and commutes with Spin(5). The 4 of Spin(5) = Sp(2) is quaternionic, so conj(4) = 4, and conj(2, 1) = (1, 2); hence the complex conjugate of (2, 1) (x) 4 is (1, 2) (x) 4. In signature (1, 8) there is a matrix B with B conj(gamma_mu) B^{-1} = gamma_mu for all mu and B conj(B) = 1 (t - s = 1 mod 8; checked on the explicit representation: the solution space of the first condition is one-dimensional and B conj(B) = 1 on it), so C psi = B conj(psi) is an antilinear involution commuting with the gamma matrices and with the spin generators gamma_mu gamma_nu; C is antilinear, so it maps the i-eigenspace of gamma_5 to the -i-eigenspace, that is (2, 1) (x) 4 to (1, 2) (x) 4; the fixed points of C form a real form of C^16, and the projection of that real form onto the first summand is a real-linear isomorphism onto it (a fixed vector is psi + C psi with psi in the first summand, and psi -> psi + C psi is injective); the complex structure of the first summand and the action of Spin(1, 3) x Spin(5) are transported by it. (ii) The frame bundle of M^4 x K_5 reduces to Spin(1, 3) x Spin(5) and the nine-dimensional spinor bundle is the tensor product of the two spinor bundles; the spinor fibre of a five-manifold is C^4 with the structure group Sp(2) acting by the 4, and for A x S^1 the Clifford algebra of T A + R e_5 is generated by c(e_1), ..., c(e_4) and the chirality Gamma of A, since Gamma anticommutes with the c(e_a), squares to one and is Hermitian, so c(e_5) = +-Gamma, and the identification with the sheet is Proposition 49.1 with the axis p = e_5. The restrictions of the 4 are those of lemma 42 and lemma 46 (Proposition 42.2: (2, 2) and 4 = (2, 1) + (1, 2) under K_p, the charges of U(1)_b on the two lines of Sigma_R being +-1); the left-handed doublet is Sigma_L (x) (2, 1) and the right-handed singlets appear as left-handed conjugates on the two weight lines of Sigma_R tensored with (2, 1). The field on w, of U(1)_b charge +1, is e_R^c, and the field on jw, of charge -1, is nu_R^c. Their conjugates, the right-handed e_R and nu_R, lie in (1, 2) (x) conj(Sigma_R) on conj(w) and conj(jw). The quaternionic structure, conj(u) -> j u, identifies these with the lines jw and w (the latter up to sign) of the dictionary of lemma 50, with nu_R on w. (iii) The charge-conjugation matrix C with C Gamma^mu C^{-1} = Gamma^mu^T is symmetric in signature (1, 8), unique up to scale, and none exists with the opposite sign. So the scalar bilinear psi^T C psi vanishes identically for anticommuting components in 8k + 1. C Gamma^mu is symmetric, so the kinetic term survives, and C Gamma^{(2)} and C Gamma^{(3)} are antisymmetric (checked by the referee on an explicit representation) (the statement rs_F3 of the corpus, with reference there); a Majorana condition psi = C psi is incompatible with a U(1) action by phases unless the charge is zero, so the only momenta are those of the periodic or antiperiodic Fourier modes, k in 2 pi Lambda^* + shift and k_5 in (2 pi/L) (Z + shift); the operator on the sector e^{i(k . z + k_5 z_5)} is c(k) + k_5 c(e_5) + i c(eta) Gamma = c(k) + i c(eta) Gamma + k_5 Gamma up to the sign of c(e_5), whose square is Y_k^2 + k_5^2 since Y_k is odd for Gamma, and Proposition 51.1 gives the spectrum of Y_k; the mode -k is the conjugate of the mode k under C. The self-conjugate mode carries a real field, R^16 = (2, 1) (x) 4, one generation. A pair of distinct modes carries a complex field in C^16 = (2, 1) (x) 4 + (1, 2) (x) 4, eight left-handed Weyl fields: (2, 1) (x) Sigma_L and (1, 2) (x) Sigma_R make a generation, and (2, 1) (x) Sigma_R and (1, 2) (x) Sigma_L its mirror. The block Y_k is odd for Gamma, so it pairs the doublet of each with the singlets of the same set. The statements about the Majorana block and the factorizations are Remark 44.4 and lemma 06. ■

Proposition 52.3 (three lepton generations and the flavour adjoint). Under SU(2)_u x SU(3)_d x U(1)_em, the adjoint of the flavour SU(5) of the five light quarks decomposes as

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24 = (3, 1)_0 + (1, 8)_0 + (1, 1)_0 + (2, 3-bar)_{+1} + (2-bar, 3)_{-1},
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twelve neutral real components spanning the flavour gauge algebra su(2)_u + su(3)_d + u(1) (in sl(5, C) the neutral part has complex dimension 12, their complexification), six complex components of charge +1 (the up-antidown mesons) and their conjugates of charge -1. Three lepton generations with nu_R, as the 24-state module H_l^{(3)} = H_l (x) C^3 of Proposition 52.1, have the U(1)_em weights 0 with multiplicity 12 (nu_R, nu_L and their conjugates, three generations), +1 with multiplicity 6 and -1 with multiplicity 6 (e_R, e_L and their conjugates); so H_l^{(3)} and sl(5, C) are isomorphic as representations of U(1)_em, and as left-handed Weyl fields the three generations are six neutral and six charged fields, which pair with the six neutral and six charged complex scalars of the adjoint in chiral multiplets of equal charge, the slepton counts (6, 6, 12 neutral real) of lemma 21. The charged states number (up flavours) x (down flavours) = 2 x 3 on the adjoint side and (chiralities) x (generations) = 2 x 3 on the lepton side; an isomorphism of U(1)_em-modules can be chosen to map the generation index to the index of the 3 of SU(3)_d for the states e_L, e_R of charge -1, in (2-bar, 3)_{-1}, and of the 3-bar for their conjugates, and the pair (e_L, e_R) to the 2 of SU(2)_u, a choice recorded as a reading. The top is absent from the 5, as in lemma 21 (Proposition 21.4: the counts force it out).

Proof. With Q = diag(2/3, 2/3, -1/3, -1/3, -1/3), the adjoint of U(5) decomposes under U(2)_u x U(3)_d as the block matrices: the diagonal blocks gl(2) + gl(3), of charge 0 and real dimension 4 + 9 = 13 as a real Lie algebra, minus the trace for sl(5), 12 real, which is su(2) + su(3) + u(1) in the commutant S(U(2) x U(3)) of Q; the off-diagonal blocks Hom(C^3, C^2) = (2, 3-bar) with charge 2/3 - (-1/3) = 1 and its adjoint with charge -1, six complex components each. For the leptons, each generation contributes in H_l the states nu_R, nu_L (charge 0), e_R, e_L (charge -1) and their conjugates (charges 0, 0, +1, +1), by the dictionary; three generations give 12, 6, 6. As left-handed Weyl fields each generation is nu_L, nu_R^c (neutral), e_L (charge -1) and e_R^c (charge +1), four fields, twelve in all; a chiral multiplet pairs a complex scalar of charge q with a left-handed Weyl field of charge q, and a complex scalar of charge +1 is the conjugate of one of charge -1, so the six charged complex scalars pair with the three e_R^c (charge +1) and the three e_L (charge -1), and the six neutral complex scalars (twelve real) with the six neutral Weyl fields; these are the counts of lemma 21. The counts 2 x 3 on both sides are read off. ■

Reading

The three views give the same content, the sheet W tensored with a second factor: the lepton line Cv of the second sheet in the two-sheet data, and the left-handed Weyl spinor (2, 1) of spacetime in nine dimensions. In the flavour reading the 24 states of three generations have the electric charges of the adjoint of the five light quarks. The connector that the lepton generation needs, in lemma 49's reading, is the Majorana block itself: its order-one condition identifies the C of the right-handed singlets with the C of the antileptons exactly when the Majorana line is a weight line of the lifted transposition, the eigenline condition that lemmas 46, 49 and 50 met from three sides, and the identification yields the Standard Model algebra on the thirty-two states with the lepton module as the summand H_F^(1) and, over the lepton algebra C + H alone, a trace-form pairing nondegenerate over Q. This is the restriction to the lepton summand of the matrix of Chamseddine, Connes and Marcolli (Proposition 2.30). They note that a skew three-by-three form is degenerate and obtain nondegeneracy over the three summands from the pair of classes of the lepton and quark summands (Corollary 2.31, Remark 2.32). The lepton statement here is the same fact read over the algebra that acts on the lepton summand. The nine-dimensional view is the one the project's labels name: a single Majorana spinor of R^{1,8} on M^4 x K_5 is one lepton generation with nu_R, its weak isospin and T_3R being the stabilizer K_p of the fibre, its Dirac masses, on the self-conjugate sector that carries exactly one generation, the torsion |eta| for both members. The internal momenta, quantized because a Majorana field has no Wilson line, enter only the sectors that pair into a generation and a mirror. Its Majorana mass, in the scope of constant bulk data, is wall data that selects the KO-7 side (T2c). A colour triplet is a complex field, whose nine-dimensional fibre carries a quark generation and a mirror and admits the Wilson line of lemma 51. So in this reading the lepton generation is the one sector that a real nine-dimensional field carries alone, and the quark sector needs a mechanism that removes the mirror. Leads from the referee: the generator gamma_5 Gamma_A commutes with Spin(1, 3) x Spin(4)_A, preserves the real form and the bilinears of the kinetic term, of the momenta along A and of the torsion on A, and acts on the left-handed fields as -1 on the doublet and +1 on the singlets, the B - L of the generation, which with the U(1)_b charges gives the hypercharges -1/2, 0, +1 of l_L, nu_R^c, e_R^c; the S^1 momentum and the fluxes with

a leg on S^1 break it, so the circle direction, the grading axis of the sheet, carries exactly the B - L-violating terms; and a single lepton generation has an odd number of SU(2)_L doublets, Witten's global anomaly, cancelled in the Standard Model by the three colours of quark doublets, so the nine-dimensional Majorana singlet is consistent as a four-dimensional theory only together with the quark sector. For the third view, the adjoint of the flavour SU(5) is the representation of the mesons of the five light quarks, and the three lepton generations have exactly its electric charges, with the neutral leptons in the place of the flavour gauge algebra and the charged leptons in the place of the up-antidown mesons, the generation index matching the down flavours (d, s, b) and the two charged lepton states the up flavours (u, c): the content of the user's counting, recorded here as a representation statement under U(1)_em with the finer matching a reading; the quark partners sit in the (15, 3-bar) with the charge-4/3 sector of Proposition 21.4, and the top is outside both. Lead: the 24 of the leptons and the 24 of the adjoint are both the fibre of a line bundle tensored with a generation triplet, and the flavour group SU(2)_u x SU(3)_d acts on the adjoint while nothing in the finite geometry acts on the generation index; a three-generation finite geometry in which the generation space carries the 3-bar of a flavour SU(3)_d would make the reading a structure.

Scope

Proposition 52.1 is finite linear algebra in the data of Corollary 16.8 with the connector of Theorem 20.2, one generation, with the axis, the connector and the Yukawa blocks as inputs; the trace-form pairing is that of lemma 41 and Theorem 48.3, with the sign convention of lemma 49. Proposition 52.2 (i) is standard representation theory of Clifford algebras, with the Majorana conjugation checked on an explicit representation; (ii) identifies fibres at a point and does not construct a Kaluza-Klein vacuum, a gauge group or an action; (iii) uses rs_F3, checked by the referee on an explicit representation (C symmetric in signature (1, 8)), for the vanishing of the Lorentz-invariant bilinear, and the wall-data statement for the Majorana mass holds for Lorentz-invariant and constant bulk data, Proposition 51.1 for the spectrum and lemma 06 for the factorizations, and the statement about Wilson lines is the absence of a U(1) charge on a Majorana field, not a theorem about all couplings. Proposition 52.3 is a count of electric charges and dimensions; the matching of generations with down flavours and of the two charged lepton states with the up flavours is a choice among the isomorphisms of U(1)_em-modules, recorded as a reading, and nothing here is a supersymmetric theory, as lemma 21 states for its counts. The direction the result points to: a three-generation version of the two-sheet data in which the generation space is a representation of a flavour group, so that the third view becomes a statement about modules; the nine-dimensional Majorana field on M^4 x A x S^1 with the torsion of lemma 51 as the parent whose window is the lepton generation, with the masses |eta| of the self-conjugate sector and the momenta of the paired sectors compared with the charged-lepton masses; and the quark sector with a Wilson line in the U(1) of colour.