Supplemental Material to "The two families of 84 and the number of generations"

Draft, 3 November 2026

This is the supplement of the paper *The two families of 84 and the number of generations*. Its parts keep the numbers of the full Supplemental Material, so that a citation of a part in the paper refers to the part of the same name here.

Parts contained. Lemma parts: 18, 21, 39, 33, 34, 36.

The full Supplemental Material, with all parts of the project, exists as `supplemental.pdf` and `supplemental_mobile.pdf` in the same folder as this file. A reference inside a part to a part not listed here points to the full version.

Contents

Part 18 **2**

Part 21 **24**

Part 39 **31**

Part 33 **43**

Part 34 **61**

Part 36 **79**

Part 18

T5b: vector masses in the spectral action at fixed cutoff

Prompt: prompts/18 (STRATEGY T5b). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5 with Python (numpy, scipy), by computations independent of the kernel derived here: a direct heat trace with constant a reproduces equation (2) for five choices of (m_1, m_2), to about 0.03%; a direct heat trace on Fourier modes with a_2(x_1) = a cos(q x_1), including the F_{12} and [T, V] vertices, reproduces the kernel (37) at three values of (m, q), to 0.1-0.3%, the size of the discretization error. The closed forms (16)-(19), the bound (28), the limits (27), the neutral matrix (30) recomputed from Standard Model charges, and the identities behind (31) also check. The largest value of the Taylor mass is M_W^2 = 0.275 Lambda^2, at m_t^2 = 2.02 Lambda^2. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): all ten numbered results are correct, with small amendments incorporated below. The referee added Propositions 11 to 14 and Remark 15, which widen the scope in both directions; the ratio in Proposition 14 was added by Claude.

Scope. Within the three-generation, zero-Majorana, top-only family at fixed cutoff, these results refute an infinitely heavy W, Z, t limit for the stated quadratic mass definitions. They do not rule out other finite data (Proposition 14), a different action or constraint, a joint limit with Lambda -> infinity, or a quantum effective theory. They do not show unbroken SU(2) x U(1) on the light space (Remark 15).

For the heat cutoff, on $\mathbb{R}^4$, we obtain the exact quadratic vector form without expanding in m/Λ. In the three-generation Standard Model finite representation with only the top Dirac mass nonzero, both the mass defined by the q^0/q^2 Taylor ratio and the unique positive timelike pole mass satisfy $M_W \sim \Lambda^2/m_t$ as $m_t/\Lambda \to \infty$ at fixed Λ. The broken vectors consequently do not decouple by becoming heavy, although the heavy fermion contribution disappears from the gauge coefficients. These statements concern the classical bosonic quadratic action

about a prescribed constant background; moreover, the positive Laplace class considered below does not itself select a finite nonzero stationary background when radial variations of that background are allowed.

1. Theorem (exact quadratic kernel).

Let

$$D_A = -i\gamma^\mu(\partial_\mu + iA_\mu) + \gamma_5 V, \qquad V = V_0 = V_0^\dagger,$$

on $\mathbb{R}^4$, with Hermitian Euclidean gamma matrices, $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$, $\gamma_5^2 = 1$, and four spin components. Let the finite trace be the literal trace on H_F, including every stated multiplicity, without a subsequent projection; this agrees with the bosonic trace of Chamseddine-Connes-Marcolli, which counts particles, antiparticles and four spin components. Take A smooth and compactly supported.

We use the vacuum-subtracted action, or equivalently its quadratic variation, since the constant-background trace on $\mathbb{R}^4$ has infinite volume. Write

$$A_\mu(q) = a_\mu(q)T, \qquad T = T^\dagger, \qquad q^\mu a_\mu(q) = 0,$$
$$\int_q = \int \frac{d^4q}{(2\pi)^4}.$$

Diagonalize V_0:

$$V_0 e_i = m_i e_i, \qquad m_i \in \mathbb{R}.$$

The signed eigenvalues are needed; their squares alone do not determine the answer.

For $S_t = \operatorname{Tr} e^{-tD_A^2}$, define

$$b(u) = u(1-u), \quad d_{ij} = m_i - m_j,$$

$$I_t(a, b; q^2) = \int_0^1 e^{-t[(1-u)a + ub + u(1-u)q^2]}\, du.$$

Then

$$S_t^{(2)} = \int_q |a_T(q)|^2\, C_{t,T}(q^2),$$

where

$$C_{t,T}(q^2) = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2 \left[\left(\frac{2}{t}+d_{ij}^2+q^2\right)I_t(m_i^2,m_j^2;q^2) - \frac{e^{-tm_i^2}+e^{-tm_j^2}}{t}\right]. \tag{1}$$

For several generators, replace $|T_{ij}|^2$ by $\mathrm{Re}((T_a)_{ij}(T_b)_{ji})$.

In particular, writing $C(q^2) = C_0 + q^2C_1 + O(q^4)$, we obtain

$$C_{0,t} = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2 \left[\left(\frac{2}{t}+d_{ij}^2\right)I_t(m_i^2,m_j^2;0) - \frac{e^{-tm_i^2}+e^{-tm_j^2}}{t}\right], \tag{2}$$

and

$$C_{1,t} = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2\int_0^1 e^{-t[(1-u)m_i^2+um_j^2]}\big[1-2b(u)-td_{ij}^2b(u)\big]\,du. \tag{3}$$

These formulas also apply, after integration over $t = s/\Lambda^2$, to

$$f(z) = \int_{(0,\infty)} e^{-sz}\,d\nu(s), \quad 0 < f(0) = \nu((0,\infty)) < \infty, \quad \int(s^{-2}+s^2)\,d\nu(s) < \infty, \tag{4}$$

where ν is positive. These hypotheses suffice for the Euclidean action density and the Taylor coefficients used below. They do not, by themselves, guarantee analytic continuation to every negative q^2.

Proof. With $\nabla_\mu = \partial_\mu + iA_\mu$,

$$D_A^2 = -\nabla^2 + V_0^2 - \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu} + \gamma^\mu\gamma_5[A_\mu, V_0].$$

Let $P_0 = -\partial^2 + V_0^2$ and write $D_A^2 = P_0 + P_1 + P_2$, according to degree in A. Duhamel expansion and cyclicity

give

$$S_t^{(2)} = -t\,\mathrm{Tr}(e^{-tP_0}P_2) + \frac{t^2}{2}\int_0^1 \mathrm{Tr}\,\left(e^{-(1-u)tP_0}P_1 e^{-utP_0}P_1\right) du. \tag{5}$$

The linear vertex from j, p to $i, p+q$ is

$$T_{ij}\left[(2p+q)\cdot a + \gamma^{\mu\nu}q_\mu a_\nu + (m_j - m_i)\gamma^\mu\gamma_5 a_\mu\right].$$

Completing the momentum square produces

$$(4\pi t)^{-2}e^{-t[(1-u)m_i^2 + um_j^2 + b(u)q^2]}.$$

For transverse a, the averaged spin trace of the vertex product is

$$4\left(\frac{2}{t} + d_{ij}^2 + q^2\right)|a|^2.$$

The P_2 term contributes

$$-\frac{1}{4\pi^2 t}\sum_{ij}|T_{ij}|^2 e^{-tm_i^2}|a|^2.$$

Symmetrizing in i, j proves (1). Differentiation proves (2)–(3).

The divided-difference structure is explicit:

$$I_t(a,b;0) = \frac{e^{-ta} - e^{-tb}}{t(b-a)}, \qquad I_t(a,a;0) = e^{-ta}, \tag{6}$$

and

$$\int_0^1 u(1-u)e^{-t[(1-u)a+ub]}\,du = \frac{1}{t^2}\partial_a\partial_b I_t(a,b;0). \tag{7}$$

Thus (2) uses a first divided difference, while (3) also uses its mixed derivative. The full momentum dependence follows from the convergent series

$$I_t(a,b;q^2) = \sum_{n=0}^{\infty}\frac{(-q^2/t)^n}{n!}\partial_a^n\partial_b^n I_t(a,b;0). \tag{8}$$

Repeated-node values are obtained continuously.

For completeness, the mass coefficient for (4) can be written entirely in dimensionless variables. Put

$$r_i = m_i/\Lambda, \quad x_i = r_i^2, \quad z(u) = (1-u)x_i + ux_j,$$
$$\Phi_1(z) = \int s^{-1}e^{-sz}\,d\nu(s).$$

The bracket replacing that in (2) is

$$\Lambda^2 \left[2 \int_0^1 \Phi_1(z(u))\, du + (r_i - r_j)^2 \int_0^1 f(z(u))\, du - \Phi_1(x_i) - \Phi_1(x_j) \right]. \tag{9}$$

The corresponding bracket in (3) is

$$\int_0^1 \left[(1-2b) f(z) + (r_i - r_j)^2 b f'(z) \right] du. \tag{10}$$

Each unweighted integral is a divided difference of an antiderivative. The moment assumptions justify the integrations and differentiations. □

On a finite flat torus, (5) remains exact, but the Gaussian momentum integral must be replaced by the spin-structure-dependent lattice sum. Explicitly, for a transverse polarization a, its two-vertex contribution per unit volume is

$$\frac{2t^2}{\mathrm{Vol}} \sum_{ij,p} |T_{ij}|^2 \int_0^1 e^{-t[(1-u)(p^2+m_i^2)+u((p+q)^2+m_j^2)]} \left(|(2p+q)\cdot a|^2 + (d_{ij}^2 + q^2)|a|^2 \right) du,$$

and the quadratic-vertex contribution is

$$-\frac{4t}{\mathrm{Vol}} \sum_{ij,p} |T_{ij}|^2 e^{-t(p^2+m_i^2)} |a|^2.$$

Consequently, the continuum formulas below are statements on $\mathbb{R}^4$, or in the infinite-volume limit. A finite torus can have additional holonomy dependence, including for constant commuting connections.

2. Lemma (gauge invariance and the Goldstone direction).

Let scalar fluctuations include the gauge orbit of V_0. Then the nonzero vector coefficient C_0 in (2) is compatible with gauge invariance.

For one generator, parameterize the orbit by

$$V(x) = e^{i\pi(x)T} V_0 e^{-i\pi(x)T}.$$

With our conventions,

$$a_\mu \longmapsto a_\mu - \partial_\mu \alpha, \qquad \pi \longmapsto \pi + \alpha.$$

The invariant combination is

$$B_\mu = a_\mu + \partial_\mu \pi. \tag{11}$$

Proof. Gauge covariance gives

$$D_{A^U,V^U} = U D_{A,V} U^{-1}.$$

Applying $U = e^{-i\pi T}$ transforms V to V_0 and a to B. Hence

$$S[a, e^{i\pi T} V_0 e^{-i\pi T}] = S[B, V_0]. \tag{12}$$

This establishes the Goldstone completion directly, without a derivative expansion.

For comparison, the longitudinal vector kernel is obtained from (1) by replacing the q^2 in its prefactor by $(1-2u)^2 q^2$ inside the integral. If $[T, V_0] = 0$, its integral vanishes by

$$\int_0^1 [2 + tq^2(1-2u)^2] e^{-tq^2 u(1-u)}\, du = 2.$$

For a broken generator the longitudinal kernel need not vanish; it belongs to the invariant combination (11).

If V_0 is not stationary, a Cartesian scalar Hessian alone need not annihilate the linear gauge-orbit tangent. The quadratic term in $e^{i\pi T} V_0 e^{-i\pi T}$, contracted with the scalar tadpole, must also be retained. Equation (12) includes that term. Transverse vectors do not mix quadratically with scalar fluctuations on this translation-invariant background. □

3. Corollary (the asymptotic normalization).

Let all finite eigenvalues be small relative to Λ, let f satisfy (4), and let Q below be positive definite on the chosen generator space (otherwise pass to the quotient by its null space). Define

$$Q_{ab} = \mathrm{Re}\,\mathrm{Tr}(T_a T_b), \qquad K_{ab} = \mathrm{Re}\,\mathrm{Tr}([T_a, V_0]^\dagger [T_b, V_0]).$$

At leading order,

$$C_0 = \frac{f(0)}{8\pi^2} K, \qquad C_1 = \frac{f(0)}{12\pi^2} Q,$$

and therefore

$$\boxed{\mathcal{M}^2_{\mathrm{Taylor}} = \frac{3}{2} Q^{-1/2} K Q^{-1/2}.} \tag{13}$$

This reproduces Proposition 13.5.

For the full three-generation Standard Model representation,

$$Q_{11} = 12, \qquad K_{11} = \sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2),$$

so

$$\boxed{8M_W^2 = \sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2).} \tag{14}$$

Proof. In (2), the orbital expression

$$\frac{2}{t} I_t(a, b; 0) - \frac{e^{-ta} + e^{-tb}}{t}$$

starts at order $t(a-b)^2$. The leading term is consequently d_{ij}^2. In (3), the leading integral is $\int_0^1 (1 - 2u(1-u))du = 2/3$, proving (13).

For $T_1 = \sigma_1/2$ on each left doublet, one generation of particles contributes 2 to Q_{11}. Charge conjugates and three generations give 12. A doublet with Dirac masses m_u, m_d contributes $\frac{1}{2}(m_u^2 + m_d^2)$ to the particle commutator trace, before color multiplicity. This proves (14).

Equivalently, the gauge coefficient is

$$\frac{f(0) g_2^2}{2\pi^2} = \frac{1}{4}.$$

For comparison with CCM's $\operatorname{Tr} F(D/\Lambda)$, take $F(v) = f(v^2)$; then $F(0) = f(0)$. The normalization and mass constraint are precisely equations (4.10) and (4.16), with the latter stated in Lemma 4.6 of the checked preprint. CCM describes the relation as "valid at unification scale." Chamseddine–Connes–Marcolli, pp. 37–39. □

The leading bound $M^2 \le 6\|V_0\|^2$ in Lemma 14 follows from (13) and $K \le 4\|V_0\|^2 Q$. Formula (13), rather than an exact finite-cutoff identity, is its applicable approximation.

4. Proposition (the exact charged-vector Taylor mass).

Let the finite data be the full three-generation Standard Model representation, including charge conjugates. Let

the Majorana matrix and every Dirac mass except one up-type color triplet be zero. Write

$$m = \mu\Lambda, \qquad x = \mu^2, \qquad r = e^{-x}.$$

Let $f(z) = e^{-z}$. Define

$$H(y) = \int_0^1 e^{-yu(1-u)}\,du,$$

$$g(y) = (2+y)H(y) - 2,$$

$$h_x(y) = (2+x+y)\int_0^1 e^{-xu-yu(1-u)}\,du - 1 - e^{-x}. \quad (15)$$

Then the full charged-vector kernel is

$$\boxed{C_W(q^2) = \frac{\Lambda^2}{4\pi^2}\left[\frac{9}{2}g(y) + \frac{3}{2}h_x(y)\right], \qquad y = q^2/\Lambda^2.} \quad (16)$$

Define

$$A(x) = h_x(0) = \frac{2[1-(1+x)e^{-x}]}{x},$$

$$w(x) = 3\int_0^1 u^2 e^{-xu}\,du = \frac{3[2-(x^2+2x+2)e^{-x}]}{x^3}. \quad (17)$$

Continuous values are understood at $x = 0$. The particle-trace kinetic coefficient is

$$K_W(x) = \frac{9}{2} + \frac{3}{2}w(x) > 0.$$

The Taylor mass, defined by $M^2_{W,\mathrm{Taylor}} = C_W(0)/C'_W(0)$, is

$$\boxed{\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{9A(x)}{4K_W(x)} = \frac{3[1-(1+x)e^{-x}]}{x[3+w(x)]}.} \quad (18)$$

Its limiting behavior is

$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{3}{8}x - \frac{23}{128}x^2 + O(x^3), \qquad \frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} \sim \frac{1}{x} \quad (x \to \infty). \quad (19)$$

Consequently,

$$\frac{m}{M_{W,\mathrm{Taylor}}} \longrightarrow \sqrt{\frac{8}{3}} \quad (\mu \to 0), \qquad \frac{m}{M_{W,\mathrm{Taylor}}} \sim \mu^2 \quad (\mu \to \infty). \quad (20)$$

Proof. On the particle space, $Q_{11} = 6$. The selected quark doublet contributes $3/2$, while the remaining doublets contribute $9/2$. Diagonalizing the massive left-right block produces eigenvalues $+m, -m$. The charged generator connects these to the massless down states; every such pair has $d_{ij}^2 = m^2$. Formula (1) therefore gives h_x for the selected doublet and g for the remaining doublets. Charge conjugation supplies the factor two in (16).

Integration by parts gives

$$h_x(y) = 2\int_0^1 u\,[x + 2y(1-u)]e^{-xu-yu(1-u)}\,du, \tag{21}$$

and

$$\boxed{h_x'(y) = 2\int_0^1 u^2[1 - y(1-u)]e^{-xu-yu(1-u)}\,du.} \tag{22}$$

The boundary term proving (22) is $\partial_u[u(1-u)e^{-xu-yu(1-u)}]$. Thus

$$h_x'(0) = \frac{2}{3}w(x), \qquad g'(0) = \frac{2}{3}.$$

This proves (18). Expanding the elementary expressions at zero and taking their large-x limits proves (19)–(20). □

The same Taylor result extends to every f in (4). Put

$$\ell = f(0), \qquad w_f(x) = \int w(sx)\,d\nu(s),$$
$$A_f(x) = \int \frac{A(sx)}{s}\,d\nu(s).$$

Then

$$\boxed{\frac{M_{W,\mathrm{Taylor}}^2}{\Lambda^2} = \frac{9A_f(x)}{4\left(\frac{9}{2}\ell + \frac{3}{2}w_f(x)\right)}.} \tag{23}$$

All denominators are positive, and dominated convergence gives

$$M_{W,\mathrm{Taylor}}^2 \sim \frac{3}{8}m^2 \quad (x \to 0),$$
$$\boxed{\frac{M_{W,\mathrm{Taylor}}^2}{\Lambda^2} \sim \frac{\int s^{-2}\,d\nu(s)}{\ell x} \quad (x \to \infty).} \tag{24}$$

Indeed, $A(z)/z \to 1$, $w(z) \to 1$ at zero, while $zA(z) \to 2$, $w(z) \to 0$ at infinity.

If the other Dirac masses are merely small and nonzero, the answer depends on those additional parameters and is given by (1). In particular, the vanishing limit in (19) assumes they are zero: masses of surviving light doublets can leave a residual vector mass.

5. Theorem (the charged-vector pole).

Under the heat-cutoff hypotheses of Proposition 4, define a timelike pole by analytic continuation of the quadratic inverse propagator:

$$C_W(-M^2_{W,\mathrm{pole}}) = 0. \tag{25}$$

For every $x > 0$, this equation has exactly one solution with $M^2_{W,\mathrm{pole}} > 0$. It is simple and has positive residue in the convention that the Euclidean propagator near the pole is a positive multiple of $(q^2 + M^2_{W,\mathrm{pole}})^{-1}$.

Let $z(x) = M^2_{W,\mathrm{pole}}/\Lambda^2$. Its exact specification is the unique positive root

$$\boxed{\frac{9}{2}g(-z) + \frac{3}{2}h_x(-z) = 0.} \tag{26}$$

Furthermore,

$$z(x) \sim \frac{3}{8}x \quad (x \to 0), \qquad z(x) \sim \frac{1}{x} \quad (x \to \infty), \tag{27}$$

and both mass definitions obey

$$\boxed{0 < M^2_{W,\mathrm{Taylor}},\ M^2_{W,\mathrm{pole}} \le \frac{\Lambda^2}{e}.} \tag{28}$$

Proof. Let

$$F_x(y) = \frac{9}{2}g(y) + \frac{3}{2}h_x(y).$$

Equation (22) implies $h_x'(y) > 0$ for $y \le 0$, and also $g'(y) > 0$, since $g = h_0$. Hence F_x is strictly increasing on that half-axis. We have

$$F_x(0) = \frac{3}{2}A(x) > 0.$$

As $y \to -\infty$, the prefactors in (15) become negative and both integrals grow exponentially, so $F_x(y) \to -\infty$. This

proves existence, uniqueness, simplicity, and the residue sign.

For $y \le 0$, equation (22) gives $g'(y) \ge 2/3$, and therefore

$$F_x'(y) \ge 3.$$

The mean-value theorem yields

$$z(x) \le \frac{1}{2} A(x).$$

The Taylor denominator gives the same bound for $M^2_{W,\mathrm{Taylor}}/\Lambda^2$. Finally,

$$A(x) = 2\int_0^1 xu\, e^{-xu}\, du \le \frac{2}{e},$$

proving (28).

As $x \to 0$, the bound implies $z(x) \to 0$, and $F_x'(y) \to 4$ near the root. As $x \to \infty$, $z(x) \to 0$, $h_x'(y) \to 0$ near the root, and $F_x'(y) \to 3$. Together with $F_x(0) \sim 3x/2$ and $F_x(0) \sim 3/x$, respectively, these prove (27). □

The pole mass governs the linearized dispersion relation of this analytically continued quadratic action. The Taylor mass governs its two-derivative truncation; they need not coincide near the cutoff. We assert no closed-form inversion of (26), no classification of complex poles, and no Lorentzian unitarity theorem.

The pole proof also extends to positive measures supported in a compact subset of $(0, \infty)$, using

$$F_{f,x}(y) = \int \frac{1}{s}\left[\frac{9}{2} g(sy) + \frac{3}{2} h_{sx}(sy)\right] d\nu(s).$$

It gives the analogues of (24) for the pole mass. For the larger class (4), only the Euclidean and Taylor conclusions have been established here; its moment assumptions alone do not justify that continuation.

6. Proposition (unbroken coefficients, neutral vectors, and weak angles).

Retain the heat-cutoff Standard Model family of Proposition 4. Normalize color generators by $\mathrm{tr}_{\mathbf{3}}\, t_a t_b = \frac{1}{2}\delta_{ab}$, and use $Q = T_3 + Y$. Define the full-trace coefficients by

$$S^{(2)}_{\mathrm{unbroken}} = \int \left(\kappa_{\mathrm{em}} F^2_{\mathrm{em}} + \kappa_s \sum_a G_a^2\right).$$

Then

$$\boxed{\begin{aligned}\kappa_{\rm em} &= \frac{1}{24\pi^2}\frac{16}{3}(5+r), \qquad \kappa_s = \frac{1}{24\pi^2}2(5+r),\\ \frac{\kappa_{\rm em}}{\kappa_s} &= \frac{8}{3}.\end{aligned}} \tag{29}$$

Both coefficients approach 5/6 of their all-massless values.

Let K denote the particle-trace neutral q^2 coefficient in coordinates (A_3, A_Y). Then

$$\boxed{\begin{aligned}K_{33} &= \frac{21}{4} + \left(\frac{3}{4} - \frac{3x}{8}\right) r,\\ K_{YY} &= \frac{103}{12} + \left(\frac{17}{12} - \frac{3x}{8}\right) r,\\ K_{3Y} &= -\frac{1}{4} + \left(\frac{1}{4} + \frac{3x}{8}\right) r.\end{aligned}} \tag{30}$$

The matrix K, the charged coefficient K_W, and the unbroken coefficients are positive for every $x \geq 0$.

Define the neutral-current angle from the current of the massive direction after diagonalizing the two-derivative neutral kinetic form:

$$J_Z \;\propto\; T_3 - \sin^2\theta_{\rm NC}\, Q.$$

Then

$$\boxed{\cos^2\theta_{\rm NC} = \frac{K_{YY} + K_{3Y}}{K_{33} + 2K_{3Y} + K_{YY}} = \frac{5}{8}.} \tag{31}$$

A different continuation, namely

$$\cos^2\theta_{\rm match} := \frac{K_{YY}}{K_W + K_{YY}},$$

satisfies

$$\boxed{\cos^2\theta_{\rm match}(0) = \frac{5}{8}, \qquad \cos^2\theta_{\rm match}(\infty) = \frac{103}{157}.} \tag{32}$$

Thus these definitions cease to describe one common electroweak angle.

Proof. Photon and color generators commute with V_0. Their coefficients therefore follow from (3) with $d_{ij} = 0$

on each invariant mass block. The charge traces give (29).

A Dirac species of mass m, with neutral charges q_L^a, q_R^a, contributes to K

$$e^{-x}\left[q_L^a q_L^b + q_R^a q_R^b - \frac{x}{2}(q_L^a - q_R^a)(q_L^b - q_R^b)\right]. \tag{33}$$

Applying (33) to the three top colors, and retaining all other states, proves (30).

Write

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad B_\infty = \begin{pmatrix} 21/4 & -1/4 \\ -1/4 & 103/12 \end{pmatrix},$$
$$U = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 17/12 \end{pmatrix}.$$

Then

$$K = B_\infty + rU - \frac{3xr}{8} vv^T.$$

We have $B_\infty \geq 5I$, $U \geq 0$, and $xr \leq 1/e$. Hence

$$K \geq \left(5 - \frac{3}{4e}\right) I > 0.$$

Positivity of K_W follows from (17).

The neutral mass direction is v, while the photon direction is $(1, 1)$. The kinetic-orthogonal massive field is proportional to $K^{-1}v$, giving the first expression in (31). Directly,

$$K_{33} + K_{3Y} = 5 + r, \qquad K_{YY} + K_{3Y} = \frac{5}{3}(5 + r),$$

which proves (31). Taking limits proves (32). $\square$

The neutral mass also tends to zero. More precisely,

$$\boxed{\frac{M_{Z,\mathrm{Taylor}}^2}{\Lambda^2} = \frac{9}{4} xr\, v^T K^{-1} v \sim \frac{2}{3} x e^{-x} \quad (x \to \infty).} \tag{34}$$

For $x > 0$ there is likewise one positive massive neutral pole, besides the photon pole. To prove this, put $B = B_\infty + rU$. The exact neutral kernel, apart from the common factor $\Lambda^2/(4\pi^2)$, is

$$B\, g(y) + \frac{3}{2} xr\, H(y) vv^T. \tag{35}$$

After congruence by $B^{-1/2}$, its massive eigenvalue is

$$g(y) + \beta H(y), \qquad \beta = \frac{3}{2} x r \, v^T B^{-1} v > 0.$$

Its timelike root obeys

$$z - 2 + \frac{2}{H(-z)} = \beta. \tag{36}$$

The left side increases from zero to infinity, because

$$\frac{d}{dz}\left(z - 2 + \frac{2}{H(-z)}\right) = 1 - \frac{2\,\mathbb{E}_z[b]}{H(-z)} \geq \frac{1}{2},$$

where $\mathbb{E}_z$ uses density proportional to $e^{zb(u)}$. The pole is therefore unique and has positive residue. Since $v^T B_\infty^{-1} v = 8/27$,

$$\frac{M_{Z,\mathrm{pole}}^2}{\Lambda^2} \sim \frac{2}{3} x e^{-x}.$$

In particular, M_W^2/M_Z^2 does not continue to equal $5/8$.

For every f in (4), equations (29)–(33) extend by replacing 1 by $\ell = f(0)$, r by $f(x)$, and xr by $-xf'(x)$. Positivity follows by averaging the positive heat-cutoff matrices. The ratio $8/3$, the neutral-current value $5/8$, and the matching limit $103/157$ persist. At the endpoint both neutral masses vanish, so the distinguished massive neutral direction—and hence its angle—requires continuation from finite x.

7. Counterexample (positive cutoff does not ensure positive broken kinetic coefficients).

Let $H_F = \mathbb{C}^2$,

$$V_0 = m\sigma_3, \qquad T = \sigma_1, \qquad x = m^2/\Lambda^2,$$

with the same spacetime and trace conventions. For the heat cutoff,

$$C_T(q^2) = \frac{\mathrm{Tr}(T^2)\Lambda^2}{8\pi^2} e^{-x} \left[(2 + 4x + y)H(y) - 2\right]. \tag{37}$$

Its q^2 Taylor coefficient is proportional to

$$e^{-x}(1 - x).$$

It vanishes at $x = 1$ and becomes negative for $x > 1$. Thus the two-derivative truncation has a wrong-sign vector kinetic term in this example.

Nevertheless, the full quadratic form has a unique positive real timelike pole with positive residue, satisfying

$$\boxed{4x < \frac{M^2_{\rm pole}}{\Lambda^2} < 4x + 2.} \tag{38}$$

Consequently $M_{\rm pole} \sim 2m$ diverges at fixed cutoff.

Proof. The generator connects the eigenvalues $m, -m$, so $d^2_{ij} = 4m^2$ while both squared eigenvalues equal m^2. Substitution in (1) proves (37). Its Taylor ratio is

$$M^2_{\rm Taylor} = \frac{6m^2}{1-x},$$

where this is a positive mass only for $x < 1$.

The pole equation is (36) with $\beta = 4x$. The monotonicity proof there gives uniqueness and the residue sign. Evaluating (37) at $y = -4x$ gives $2H(-4x) - 2 > 0$, whereas at $y = -4x - 2$ it gives -2; this proves (38).

For general f satisfying (4), the Taylor kinetic coefficient in this example is proportional to

$$f(x) + xf'(x) = \frac{d}{dx}[xf(x)]. \tag{39}$$

It must be negative somewhere: $xf(x) > 0$ for $x > 0$, but $xf(x) \to 0$ at infinity under (4). A nonnegative derivative everywhere would contradict these facts. □

This counterexample concerns arbitrary finite data; the top-only Standard Model family is outside it. It also distinguishes a wrong-sign **Taylor truncation** from a negative-residue pole of the **full** form factor. We have not proved absence of complex-pole or other nonlocal problems in either model.

8. Counterexample (heavy fermions do not dynamically select the centralizer).

Let the data be those of Proposition 4, and let

$$\mathfrak{g}_{V_0} = \{X \in \mathfrak{g}_{\rm SM} : [X, V_0] = 0\}.$$

For nonzero top mass this is the represented unbroken gauge algebra $\mathfrak{su}(3) \oplus \mathfrak{u}(1)_{\mathrm{em}}$. The limit $x \to \infty$ at fixed Λ does **not** produce a theory containing only these vector directions by quadratic mass decoupling.

Proof. For fixed y, equation (21) implies

$$h_x(y) \longrightarrow 0.$$

Consequently

$$\boxed{C_W(q^2) \longrightarrow \frac{\Lambda^2}{4\pi^2}\frac{9}{2}g(q^2/\Lambda^2).} \tag{40}$$

For positive y,

$$g(y) = 4y \int_0^1 u(1-u)e^{-yu(1-u)}\,du > 0.$$

Thus a charged generator outside $\mathfrak{g}_{V_0}$ retains a finite nonzero quadratic kinetic form and a nonvanishing propagator at every fixed positive momentum. Its mass approaches zero. Equation (35) similarly leaves the positive matrix $B_\infty g(y)$, including the additional neutral direction. These facts contradict decoupling of the broken vectors through diverging masses or vanishing propagators. □

A narrower decoupling statement is true. If we **restrict the connection from the outset** to the centralizer, the finite mass blocks are invariant and the heat trace factorizes. On the heavy block its contribution carries e^{-sm^2/Λ^2}, which tends to zero. The restricted action therefore converges to the light-block spectral action. This proves decoupling of a finite fermion block under that restriction; it does not prove dynamical elimination of the other gauge fields.

The amendment to Lemma 03 Part C is therefore:

- Its transverse kinetic weights, including $w_L(x,0) = w(x)$, and its two weak-angle formulas are correct under their stated conventions.
- They must be accompanied by the mass coefficient (2), including the quadratic-vertex subtraction.
- A centralizer-only endpoint does not follow from those weights. In the present exact quadratic family, the broken-vector masses vanish while their remaining kinetic coefficients are positive.

9. No-go theorem (finite radial vacuum for the positive Laplace class).

Let f satisfy (4), and suppose the allowed scalar configurations include

$$V = cV_*, \qquad c > 0, \qquad V_* \neq 0,$$

with $A = 0$. Let the action be the literal spectral action, without an added scalar potential or a constraint fixing c. Then no finite $c > 0$ is stationary along this radial direction.

Proof. On $\mathbb{R}^4$, the constant-background action density is

$$\mathcal{U}(c) = \frac{\Lambda^4}{4\pi^2} \int s^{-2} \operatorname{Tr}_{H_F} e^{-sc^2 V_*^2/\Lambda^2} \, d\nu(s).$$

Differentiating gives

$$\boxed{\mathcal{U}'(c) = -\frac{c\Lambda^2}{2\pi^2} \int s^{-1} \operatorname{Tr}_{H_F} \left(V_*^2 e^{-sc^2 V_*^2/\Lambda^2} \right) d\nu(s) < 0.} \tag{41}$$

The inequality is strict because $V_* \neq 0$ and ν is nonzero and positive. The same argument applies on a finite flat torus before performing its positive momentum sum. □

Thus the masses above are conditional quadratic-background quantities. In the top-only family, the ordinary Higgs radial variation supplies the direction used in this theorem. A finite symmetry-breaking minimum of the polynomial asymptotic scalar potential cannot automatically be regarded as a minimum of this full positive-Laplace spectral action.

10. Corollary (comparison of heavy limits).

Let the background family and mass definitions be those above.

At fixed Λ, with $\mu \to \infty$:

quantity, heat cutoff	behavior
m_t	$\mu\Lambda \to \infty$
$M_{W,\mathrm{Taylor}}$, $M_{W,\mathrm{pole}}$	$\sim \Lambda/\mu \to 0$
$M_{Z,\mathrm{Taylor}}$, $M_{Z,\mathrm{pole}}$	$\sim \sqrt{2/3}\,\Lambda\mu e^{-\mu^2/2} \to 0$
m_t/M_W	$\sim \mu^2 \to \infty$

(42)

At fixed $0 < \mu < \infty$, with $\Lambda \to \infty$, all four vector masses in (42) instead grow linearly with Λ, as does m_t. A controlled asymptotic family can also satisfy

$$\Lambda \to \infty, \qquad \mu \to 0, \qquad \mu\Lambda \to \infty;$$

then $M_W, m_t \to \infty$ while

$$m_t/M_W \to \sqrt{8/3}.$$

Proof. The first assertions follow from (19), (27), and (34)–(36). At fixed positive μ, their dimensionless mass functions are fixed and positive, so multiplication by Λ proves the joint-limit assertions. The final assertion follows from the small-μ limits. □

Accordingly, the obstruction has three distinct scopes. The CCM identity cannot be extrapolated to $\mu \to \infty$ at fixed cutoff because it belongs to the asymptotic truncation. The exact top-only quadratic action obstructs a fixed-cutoff heavy-W limit for the heat cutoff under both mass definitions, and for class (4) under the Taylor definition. A family of actual nonzero stationary vacua of the unmodified positive-Laplace action encounters the separate obstruction (41). Counterexample 7 excludes extending the first of these exact mass conclusions to arbitrary finite representations.

11. Proposition (the radial derivative for general f; referee). On flat $\mathbb{R}^4$, for a sufficiently integrable real cutoff f, with λ_j^2 the eigenvalues of V_*^2,

$$\mathcal{U}'(c) = -\frac{c\Lambda^2}{2\pi^2} \sum_j \lambda_j^2 \int_{c^2\lambda_j^2/\Lambda^2}^{\infty} f(z)\, dz. \tag{43}$$

For nonnegative f, a finite $c > 0$ is stationary exactly when every tail associated with a nonzero λ_j vanishes. So if f has nonzero support arbitrarily far out, there is no finite stationary point on this ray; if f has compact support ending at R, there is a stationary plateau once every nonzero $c^2\lambda_j^2/\Lambda^2 \geq R$. Sign-changing cutoffs can have cancellations. A nonconstant cutoff flat near zero lies outside (4), since a nonzero positive measure gives $f'(0) < 0$; (43) covers the nonnegative ones. A radial path with an additional fixed noncommuting mass matrix need not have the form cV_*. □

12. Proposition (the top-only Taylor mass for general f; referee). For nonnegative smooth compactly supported f with $f(0) > 0$, the formulas of Proposition 4 hold with

$$A_f(x) = \frac{2}{x}\int_0^x zf(z)\,dz, \qquad w_f(x) = \frac{3}{x^3}\int_0^x z^2 f(z)\,dz,$$

by approximating f with its derivatives by real combinations of e^{-nz}. Hence

$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} \sim \frac{\int_0^\infty zf(z)\,dz}{f(0)\,x}.$$

This includes cutoffs flat near zero. For a nonanalytic cutoff there need be no analytic continuation defining the pole. □

13. Proposition (several heavy doublets; referee). For the heat cutoff, let P be the total particle-trace weight of complete massless doublets, and let heavy doublet i, of weight $d_i > 0$, have one Dirac member of mass m_i and one massless member. Then

$$C_W(y) = \frac{\Lambda^2}{4\pi^2}\Big(Pg(y) + \sum_i d_i h_{x_i}(y)\Big),$$
$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{3}{2}\,\frac{\sum_i d_i A(x_i)}{P + \sum_i d_i w(x_i)}.$$

For finitely many doublets, if $P > 0$ and all $x_i \to \infty$, the Taylor mass tends to zero. If $P = 0$ and all $x_i = x$, then $M^2_{W,\mathrm{Taylor}} \sim m^4/(2\Lambda^2)$, the form factor has a unique positive pole with $M^2_{W,\mathrm{pole}} \sim m^2/2$, and the whole unnormalized kernel tends to zero at every fixed Euclidean momentum. The divergent Taylor and pole scales differ, and neither alone establishes a healthy limiting gauge theory. □

14. Proposition (the fully massive family; referee; the complete proof by Astra 6 follows the sketch). Give both members of every Standard Model doublet the same Dirac mass m, with no Majorana masses, and let z_W, z_Z be the charged and massive neutral poles in units of Λ^2. The charged form factor is proportional to $e^{-x}[g(y) + 2xH(y)]$, and

$$2x < z_W < 2x + 2, \qquad \tfrac{16}{5}x < z_Z < \tfrac{16}{5}x + 2.$$

So both pole masses diverge with m at fixed cutoff, and

$$\frac{M_{W,\mathrm{pole}}^2}{M_{Z,\mathrm{pole}}^2} \longrightarrow \frac{5}{8} \qquad (x \to \infty),$$

as also for $x \to 0$, where $z_W \sim 3x$ and $z_Z \sim \frac{24}{5}x$. In between the ratio differs, reaching about 0.66 near $x = 3$ (Python, solving the pole equations). The kernel tends to zero at every fixed Euclidean momentum, and the charged Taylor kinetic coefficient is proportional to $e^{-x}(1 - x/2)$, negative for $x > 2$.

Proof. The species with equal mass have eigenvalues $\pm m$. The charged generator connects them with $d^2 = 0$ and $d^2 = 4m^2$ in equal weights, which gives the charged form. For the neutral generators the diagonal charge has $d = 0$ and the off-diagonal charge $(q_L - q_R)/2 = T_3 v/2$ has $d^2 = 4m^2$; on the particle trace the kernel is $e^{-x}[\mathrm{diag}(6, 10)g + 12xHvv^T]$, so $\beta = 12x\, v^T\, \mathrm{diag}(6, 10)^{-1} v = \frac{16}{5}x$, and for the charged direction $\beta = 2x$. Equation (36) gives the bounds, since $H(-z) > 1$. □

Complete proof (Astra 6, 3 November 2026; written here from its validation report, item 11, so that the lemma is self-contained). Take the heat cutoff, three generations with right-handed neutrinos, no Majorana masses, and one common positive Dirac mass m for every species. Let $x = m^2/\Lambda^2$, $y = q^2/\Lambda^2$, $H(y) = \int_0^1 e^{-yu(1-u)}\, du$ and $g(y) = (2 + y)H(y) - 2$. Every mass eigenvalue is $+m$ or $-m$, so the exact quadratic kernel of Theorem 1 for a generator entry is proportional to $e^{-x}[g(y) + ((m_i - m_j)^2/\Lambda^2)H(y)]|T_{ij}|^2$. A charged generator connects equal and opposite mass signs with equal weights, the average of $(m_i - m_j)^2$ is $2m^2$, and the charged kernel is proportional to $e^{-x}[g(y) + 2xH(y)]$. In the neutral particle trace, with $v = (1, -1)^T$, the charge and mass-commutator traces give $Q = \mathrm{diag}(6, 10)$ and $C_{\mathrm{neutral}}(y) \propto e^{-x}[Qg(y) + 12xH(y)vv^T]$, whose determinant factors as $\det Q\, g(y)[g(y) + 12x\, v^T Q^{-1} v\, H(y)]$; since $v^T Q^{-1} v = 1/6 + 1/10 = 4/15$, the massive neutral factor has coefficient $16x/5$. Charge conjugation multiplies the common normalization and moves no root. Both massive pole equations therefore have the form $(2 + \beta - z)H(-z) = 2$ with $\beta_W = 2x$ and $\beta_Z = 16x/5$. Let $h(z) = H(-z)$ and $F(z) = z - 2 + 2/h(z)$. For $z \geq 0$, $h(z) \geq 1$ and $0 < h'(z) = \int_0^1 u(1-u)e^{zu(1-u)}\, du \leq h(z)/4$,

so $F'(z) = 1 - 2h'(z)/h(z)^2 \geq 1 - 1/(2h(z)) \geq 1/2$; with $F(0) = 0$ and $F(z) \to \infty$, each $\beta > 0$ has a unique positive solution, at which $h(z) > 1$, so $z = \beta + 2 - 2/h(z)$ gives $\beta < z < \beta + 2$: the displayed bounds. For small z, $h(z) = 1 + z/6 + O(z^2)$ and $F(z) = \frac{2}{3}z + O(z^2)$, so $z_W \sim 3x$, $z_Z \sim 24x/5$ and the ratio tends to 5/8; for $x \to \infty$ the bounds give $z_W = 2x + O(1)$, $z_Z = 16x/5 + O(1)$, again 5/8. At fixed Euclidean momentum every kernel entry is e^{-x} times a function growing at most linearly in x, hence tends to zero. Finally $H'(0) = -1/6$ and $g'(0) = 2/3$, so the charged Taylor kinetic coefficient is proportional to $e^{-x}(\frac{2}{3} - \frac{x}{3}) = \frac{2}{3}e^{-x}(1 - x/2)$, negative for $x > 2$. A quadrature and root calculation (Python 3, SciPy, not kept) gave $z_W = 7.4996$, $z_Z = 11.3746$, ratio 0.6593 at $x = 3$. □

This is a counterexample to any general no-go for diverging pole locations at fixed cutoff. It has no light fermions, and the physical quality of that limit is open.

15. Remark (the light space does not carry unbroken $SU(2) \times U(1)$; referee). In the top-only limit the broken masses vanish, but this does not restore the electroweak group on the whole light space. The light space retains b_L after removing t_L. If P_{light} projects off the top mass block, then on b_L we have $P_{\text{light}} T_1 P_{\text{light}} = P_{\text{light}} T_2 P_{\text{light}} = 0$ and $P_{\text{light}} T_3 P_{\text{light}} = -\frac{1}{2}$, which do not obey the $SU(2)$ relations. Correspondingly $K_W = 9/2$, $K_{33} = 21/4$, $K_{3Y} = -1/4$ in the limit, while a linearly realized unbroken action would have equal weak coefficients and no mixing. The remaining complete massless doublets carry the ordinary representation, and their contribution is gauge invariant. If both members of every removed doublet are heavy, the retained light space can be invariant under the full group; a full limiting theory still needs control of the remaining fields, normalizations, interactions and vacuum. "The heavy fermion leaves the loop" is shorthand for the suppression in this classical trace.

The light-space statement should be read together with the separate role of the top (lemma 21): among single charged species only an up-type quark can be removed while keeping cos^2 theta_NC = 5/8 and the ratio 8/3, and removing it leaves the five other quarks and the leptons, on which SU(3) x U(1)_em acts with b_L paired

with b_R.

16. Remark (the CCM relation as the small-mass form of a weighted sum rule; Claude). For the heat cutoff and doublets with one massive member, Proposition 13 reads

$$8\kappa M^2_{W,\mathrm{Taylor}} = \sum_f N_f m_f^2\, \rho(m_f^2/\Lambda^2),$$
$$\rho(x) = \frac{A(x)}{x} = \frac{2[1-(1+x)e^{-x}]}{x^2},$$

with $N_f = 2d_f$ and $\kappa = (P + \sum_i d_i w(x_i))/6$. As $x \to 0$, $\rho, \kappa \to 1$, which is the CCM relation (14). The relation is therefore the leading term of an expansion, and its range is $m \ll \Lambda$. For the top alone, $\kappa = (3+w)/4$; m/M_W departs from $\sqrt{8/3}$ by 5% at $m \approx 0.45\Lambda$, and M_W peaks at about 0.52Λ at $m \approx 1.42\Lambda$ (Python). The maximum is unique, at $x = 2.024$ with $M_W/\Lambda = 0.5242$ (referee, by a sign argument and Python). A massive diagonal contribution to the trace is exponentially suppressed above the cutoff, while the mixed loop with a massless partner has the algebraic $\rho \sim 2/x^2$ at large x; a vacuum with no other mass leaves the W mass with no source. Whether the classical action is meaningful for $m > \Lambda$ is a separate question, on which these formulas do not decide. In the top-only heat-cutoff family, the ratio differs from $\sqrt{8/3}$ for every $x > 0$, since $3 + w - 4\rho$ vanishes at zero and is increasing, so the limit with M_W and m_t diverging together and $m_t/M_W \to \sqrt{8/3}$ needs $\Lambda \to \infty$ faster than m_t (Corollary 10). This necessity is for that family: with one complete massless lepton doublet and one member of each other doublet of common mass m, the weighted sum equals the unweighted CCM sum at some $x_* > 0$ (referee).

As algebraic checks, we used **Python 3 with SymPy** to verify the integration-by-parts identities, the displayed small-x Taylor series, and $v^T B_\infty^{-1} v = 8/27$. The arguments above provide the proofs; these computations introduce no fitted constants.

Consequence for T5b. At fixed cutoff, the specified top-only Standard Model spectral action does not realize an infinitely heavy W, Z, t family: the top grows while both vector masses tend to zero, for the heat cutoff and the stated quadratic mass definitions. A joint limit with

$\Lambda \to \infty$ permits all three background mass scales to diverge, but neither centralizer-only decoupling nor a stationary Higgs vacuum follows from that scaling. These conclusions concern the classical bosonic action and contain no quantum corrections or claims about the physical Standard Model at those scales. Other finite data behave differently: with every doublet heavy and no light fermions, the pole masses diverge at fixed cutoff with M_W^2/M_Z^2 -> 5/8 (Proposition 14). With a massless complete doublet retained, the Taylor mass tends to zero (Proposition 13). So in the families treated here, a heavy W and Z at fixed cutoff is found only without a massless complete doublet. Light nonzero masses are not treated: they leave a residual vector mass that depends on them.

Part 21

The separate role of the top: removal identities and counting

Prompted by the observation that the top has a separate role, which matters more than the loss of an electroweak partner for b_L. Proposition 21.1 and the counts below are exact calculations by Claude Opus 5.5, in Python with exact fractions. The readings in the last section are proposals, and nothing there is derived. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): one error, in Proposition 21.2, and several amendments, incorporated below. Proposition 21.4 and Corollary 21.5 (added 4 Nov 2026) were refereed by Claude Opus 5.5 (effort max, 5 Nov 2026; report): the counts correct, two amendments (the group SO(6(n_u + n_d)); the SU(3)^3 trace) and one error in the window clause, which held in reading (a) only; all applied.

Outcome. The spectral action detects the top: in the exact quadratic action of lemma 18, among single charged species only an up-type quark can be removed while keeping both the neutral-current value cos^2 theta_NC = 5/8 and the ratio 8/3 of the photon and gluon coefficients. Removing it leaves 84 of the 96 states of the finite Hilbert space. Those 84 are the seed of an SO(30), inside SO(32), whose adjoint contains the charged scalar counts of three generations, and 84 is also the number of on-shell compo-

nents of the three-form of eleven-dimensional supergravity.

Removal identities

For a Dirac species with charges q_L, q_R in the coordinates (T_3, Y), equation (33) of lemma 18 gives a contribution to the neutral kinetic matrix K with weight e^{-x}. The off-diagonal charge is (q_L - q_R)/2 = (T_3/2)(1, -1), which is orthogonal to the photon direction (1, 1). So the combinations K_33 + K_3Y and K_YY + K_3Y depend on x only through the factor e^{-x} of each massive species, and cos^2 theta_NC = (K_YY + K_3Y)/(K_33 + 2 K_3Y + K_YY) is determined by the sums of T_3 Q and Y Q.

Proposition 21.1. Per species, with colour multiplicity, on the particle trace for the first two columns and with charge conjugates and both chiralities for the last two:

species	sum T_3 Q	sum Y Q	Q^2 coefficient	colour coefficient
up-type quark	1	5/3	16/3	2
down-type quark	1/2	1/6	4/3	2
charged lepton	1/2	3/2	4	0
neutrino	0	0	0	0
three generations	6	10	32	12

Hence the ratios of the whole, 5/3 and 8/3, are those of an up-type quark alone. As one species is removed, that is, as its mass tends to infinity at fixed cutoff in the setting of lemma 18:

removed	cos^2 theta_NC	photon/gluon coefficient
up-type quark	5/8	8/3
down-type quark	59/92	46/15
charged lepton	17/28	7/3

So among single charged species only an up-type quark preserves both values. For an up-type quark they hold

at every x, as in Proposition 6 of lemma 18. Within one generation, the down-type quark and charged lepton together contribute exactly what the up-type quark contributes: (1, 5/3; 16/3, 2).

Proof. Arithmetic with the Standard Model charges, Y = Q - T_3. The last statement is direct addition. ■

The scope is that of lemma 18: classical bosonic trace, with these definitions of the coefficients and of the angle, three generations, and the other species with the mass assignments of that calculation. Other definitions of the weak angle do not stay at 5/8 (Proposition 6 there). At the endpoint where the neutral mass vanishes, the massive neutral direction is no longer distinguished, so its angle is read as the limit from finite mass, as lemma 18 states.

The state count in the action

Proposition 21.2. With three generations and nu_R, dim H_F = 96 (Chamseddine-Connes-Marcolli). With only an up-type Dirac mass m and no Majorana mass, the constant-background density of lemma 18 has finite trace Tr exp(-V_0^2/Lambda^2) = 84 + 12 e^{-x} for the heat cutoff. So the coefficient of the vacuum density moves from 96 to 84 along the radial direction, with no finite nonzero stationary point on this free radial ray (No-go theorem 9 and Proposition 11 of lemma 18). This concerns the literal flat-space action with no added potential or constraint. Compact support can give a plateau; sign-changing cutoffs, additional terms, constraints and paths with an additional fixed noncommuting mass lie outside it. At m = 0 the derivative is zero.

Proof. The top block has 3 colours, 2 chiralities and charge conjugates, so 12 states, each with V_0^2 = m^2. ■

This also gives the top-only family a place in the radial problem, as the endpoint of a descending direction. A radial direction giving other species masses descends further, so this is the endpoint for that direction only.

SO(30)

Let the five quarks other than the top, in three colours, with their conjugates, form the vector 30 = (5, 3) + (5-bar, 3-bar) of SU(5) x SU(3) inside SO(30). The adjoint, of dimension 435, is

```
Lambda^2(30) = (24, 8) + (24, 1) + (1, 8) + (1, 1) +
 ↪  (10, 6) + (15, 3-bar) + conjugates of the last
 ↪  two.
```

With Q = 2/3 for u, c and Q = -1/3 for d, s, b, that is, Q = diag(3/5, 3/5, -2/5, -2/5, -2/5) + (1/15) I_5, which includes the U(1) commuting with SU(5), the charges are:

representation	charge: number
(15, 3-bar)	-2/3: 6; +1/3: 6; +4/3: 3
(10, 6)	-2/3: 3; +1/3: 6; +4/3: 1
(24, 1) + (1, 1)	+1: 6; -1: 6; 0: 13

Three generations of scalar partners in a supersymmetric Standard Model with nu_R need, for antisquarks in the 3-bar, 6 of charge -2/3 and 6 of charge +1/3, and for sleptons 6 of charge +1, 6 of charge -1 and 12 neutral real states. So (15, 3-bar) and (24, 1) + (1, 1) contain exactly the charged counts. What is left over in them is three triplets of charge +4/3 and one neutral state. The counting of 15 diquarks and 25 mesons, with the charge +4/3 states, and the remark that the top is unable to form them, go back to an early preprint (hep-ph/0512065), which finds neither SO(30) nor a role for SO(32). What SO(30) adds is one group whose adjoint contains both families. It also sits in SO(32), with 496 -> (435, 1) + (1, 1) + (30, 2) under SO(30) x SO(2), which hints at a role for SO(32); we have not explored this. These are representation and charge counts: identifying the entries with physical scalar partners needs a field construction, spin and chirality assignments, interactions and dynamics.

Proposition 21.4 (the counting forces the top out and leaves the charge 4/3 half-filled). Let n_u up-type and n_d down-type light quarks in three colours, with their conjugates, span the vector of SO(6(n_u + n_d)), whose 6(n_u + n_d) real dimensions are those

of (n_u + n_d, 3) + conjugate. In the colour antitriplet of the adjoint, the symmetric diquarks S^2(n_u + n_d) (x) 3-bar have n_d(n_d + 1)/2 states of charge -2/3, n_u n_d of charge +1/3 and n_u(n_u + 1)/2 of charge +4/3. Three generations of antisquarks need 6 and 6 in the first two charges, and the only solution in nonnegative integers is (n_u, n_d) = (2, 3). The charge +4/3 count is then 3. With the top included, (n_u, n_d) = (3, 3) gives (6, 9, 6), and the top would need 36 = 30 + 6 real dimensions rather than the 32 of SO(32), whose extra plane is colour-neutral.

Proof. n_d(n_d + 1)/2 = 6 gives n_d = 3, then n_u n_d = 6 gives n_u = 2, and 2 x 3/2 = 3. For n_u = n_d = 3 the three counts are 6, 9, 6. A coloured quark adds 6 real dimensions to the vector; 32 - 30 = 2. ■

Corollary 21.5 (the charge-4/3 sector). Under the bijection of Proposition 21.4 between the (6, 6) part and the antisquarks of three generations, the three antitriplets of charge +4/3 are half of what a chiral multiplet pairing would need: a Dirac quark of charge 4/3 in each generation needs two complex scalars, so three generations need 6. If the three scalars are read as superpartners, their three fermions are left-handed Weyl fields in (3-bar, 4/3), SU(2) singlets, and are anomalous on their own (with T(3) = 1/2 and A(3) = 1, the SU(3)^3, SU(3)^2 U(1), U(1)^3 and gravitational U(1) traces are -3, 2, 64/3 and 12. The meson sector, six complex scalars of charge +-1 and neutral ones, cancels none of them for any chirality of its fermions: they are colour singlets, their cubic charge sum is an integer, and their linear charge sum has absolute value at most 6. With any other chirality assignment of the three multiplets, the SU(3)^3 trace is odd, so the exotic fermions stay anomalous). The readings compatible with the counting are: (a) the three scalars have no fermion partner, as the Higgs doublet has none in the Standard Model, so supersymmetry is a property of the (6, 6) part alone; (b) the top is included, which completes the exotic to 6 and breaks the +1/3 count to 9. In the window of the interval parent (lemma 38, Proposition 11.4, the shadow B_ij -> (m_i - m_j) B_ij, for distinct masses), the same-flavour diquarks drop out and the antitriplet counts become (3, 6, 1) in reading (a). One state of charge +4/3 remains, the combination of u and c antisymmetric in flavour, with coefficient proportional

to m_u - m_c, in the (Lambda^2 5, 3bar) of S^2 V. In reading (b), with six flavours, the counts become (3, 9, 3), and three states of charge +4/3 remain, uc, ut and ct. In both readings the window has 3 states of charge -2/3, so the match with the antisquarks in Proposition 21.4 holds in the parent and fails in the window. This records a conflict between the count of three generations and the anomaly of the exotic sector; the premise of the whole counting is that five light quarks generate every squark and slepton as a composite, scalars that the finite geometry of Connes et al. does not contain.

Provenance. Stated by the user and Claude Fable 5.1 in conversation on 4 November 2026; the arithmetic checked by hand. The remark that the top cannot form the diquarks is from hep-ph/0512065; the chirality reading of the charge-4/3 states as an additional quark per generation is in the abstract of the EPJC paper (Rivero 2024), where the half-multiplet problem is stated.

Direction (referee, 5 Nov 2026). The selection of (2, 3) belongs to the parent: with the flavour-antisymmetric counts n_d(n_d - 1)/2, n_u n_d, n_u(n_u - 1)/2, which are those of the window and of the sign s = -1 (USp(30)), the condition (6, 6) has no integer solution (n_d = 4 forces n_u = 3/2), so five light quarks with the top excluded are a property of Lambda^2 V with walls reducing to M_15(C) (lemma 38, Theorem 11.3), and the question is whether that wall condition can be derived. The anomaly is a parity: in reading (a) the three exotic antitriplets give an odd SU(3)^3 trace, which no partners for the rest of the SO(30) adjoint cancel (octets contribute 0, sextets multiples of 7); in reading (b) the six charge-4/3 states pair vectorlike but the three extra charge-1/3 antitriplets td, ts, tb leave an odd trace again. In both readings an odd number of antitriplets lies beyond the antisquarks, 3 = 15 - 12 and 9 = 21 - 12, so a supersymmetric reading of the diquark sector needs partnerless scalars or an even excess. With four generations the symmetric counting has no solution, since n_d(n_d + 1)/2 = 8 has no integer root.

The two families of 84

Proposition 21.3. Count four states per Dirac fermion and colour, and take Dirac neutrinos. Then

```
96 = 84 + 12:   five quarks and all leptons (60 + 12
 ↪  + 12), and the top (12);
96 = 84 + 12:   all electrically charged fermions
 ↪  (72 + 12), and the neutrinos (12).
```

The two families of 84 overlap in 72 states, and each is the other with the top and the neutrinos exchanged. Without nu_R, the first family has 78 states and the second stays at 84. In eleven dimensions, the three-form C_3 has C(9, 3) = 84 on-shell components, and its dual C_6 has C(9, 6) = 84. The 84 in H_F is a complex finite-Hilbert-space dimension, and the three-form count is a real bosonic polarization count, so an identification needs a map of representations and degrees of freedom.

Readings, as proposals

These are proposals to test, with no derivation behind them.

1. **Seiberg duality.** For SU(3) with five flavours, N_f = 5 lies in the conformal window 3 N_c/2 < N_f < 3 N_c, and the magnetic dual is SU(N_f - N_c) = SU(2) with five flavours and a 5 x 5 gauge-singlet meson chiral superfield coupled by W = M q q-tilde (N. Seiberg, Nucl. Phys. B 435 (1995) 129, arXiv:hep-th/9411149, checked by the referee). The removal of the top is what gives N_f = 5, and SU(2) is the magnetic group there. To become a statement this needs an N = 1 supersymmetric theory with the required massless matter and dynamics, for example an emergent one, with an operator and anomaly map. Five-flavour ordinary QCD alone does not supply the duality, and the magnetic SU(2) needs a separate identification with any electroweak group.
2. **Branes.** The first family of 84 could relate to C_3 or to an M2 or M5 brane, and the second family to the dual object. The exchange of the top and the neutrinos would then be the content of the duality on the Standard Model side. The counts match, and equality of the number is insufficient: a representation and dynamics must relate the finite fermionic states to the bosonic form or brane degrees of freedom. The pairing of item 1 is one candidate. The first family also needs Dirac neutri-

nos for its 84, and the second does not.

3. **Relation to the odd labels.** The removal of the top leaves the five quarks and the leptons, on which SU(3) x U(1)_em acts with b_L paired with b_R. So the loss of the electroweak partner of b_L has no role at the label 9, where SU(2) is absent. A statement here needs the required involution on this finite sector, including its action on D, J, the algebra and its opposite; the vectorlike charges alone do not establish it, and the neutral states still need treatment.

Numerical check

Python with exact fractions, run on 3 Nov 2026 in a temporary directory, not kept: all tables above, and the dimension $435 = 224 + 1 + 105 + 105$ with $105 = 60 + 45$ and $224 = 192 + 24 + 8$.

Part 39

Hypercharge from the three-form, matter from branes, and the two families of 84

Lemma 39. Written as file 03 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README). Propositions 3.2 and 3.5 and Corollary 3.9 are proved here, and Proposition 3.8 is a negative result in a stated scope, refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5; the numbered proposals are conjectures to test. Attempted by Claude (Anthropic), 3 Nov 2026. Checks: checks_11d.py, sections B, E and G, Python with exact fractions. Referee pending.

Referee note (GPT-6.1 Terra, 4 Nov 2026). Proposition 3.8 and Corollary 3.9 have been independently checked by hand in their stated scopes; the tensor-mode

argument needs the displayed colour action to preserve the background. See the referee report.

Referee note on Proposition 3.5 (GPT-6.1 Terra). The classical field equations and curvature condition are independently checked; flux quantization, supersymmetry and stability remain open. See the report.

Referee note on Proposition 3.2 (GPT-6.1 Terra). The exterior-algebra representation and charge counts are independently checked. Their realization by wrapped branes remains a proposal; see the report.

Setting

02 realizes the Witten space with Killing spinors as Y_{3,2} = (5bar)//U(1)_Y. Its isometries give SU(3) x SU(2) x U(1)_R, and no tensor Kaluza-Klein mode on it, on an invariant background, carries triality, weak isospin 1/2 or fractional hypercharge (Proposition 2.4). The M-theory vacuum AdS4 x Y_{3,2} is the near-horizon geometry of N M2-branes at the apex of the cone C(Y_{3,2}), with N units of flux of *G_4 through Y_{3,2}. Lemma 21 proposed relating its two families of 84 states to the three-form C_3, to the M2 or M5 brane, and to their duality. This file connects the two.

Proposal 3.1 (hypercharge is the Betti vector of Y_{3,2})

In AdS4 x M^{111} the massless vectors are those of the isometries, SU(3) x SU(2) x U(1)_R, and one more from C_3 on the harmonic two-form (b_2 = 1), the Betti multiplet; the Kaluza-Klein states are neutral under it, and only non-perturbative states carry its charge (Fabbri et al., hep-th/9903036 and hep-th/9907219). By Proposition 2.3 the Betti class pairs with the divisors {l(U) = 0} and {m(V) = 0} as 2 and -3, the values of 6Y on d^c and L.

Proposal. Identify the Betti U(1) of Y_{3,2} with hypercharge. The gauge group of the vacuum is then

```
SU(3) x SU(2) (isometries) x U(1)_Y (from C_3) x
 ↪ U(1)_R (isometry),
```

thirteen vectors, the Standard Model group and one R-symmetry. With massless W and Z this is the eleven-

dimensional vacuum of the v -> 0 limit; the identification is consistent with three facts already established:

1. Tensor Kaluza-Klein modes on an invariant background have Y = 0, as they must by Proposition 2.4, which leaves them only PSU(3) x SO(3) representations; spinors are open.
2. The states with Y != 0 are non-perturbative: M5-branes wrapped on five-cycles and M2-branes wrapped on the two-cycle.
3. The M5 baryons of Fabbri et al. are det U on one five-cycle and det V on the other, with baryonic charges 2N and -3N and representations Sym^N(3bar) and Sym^N(2). For N = 1 they are U in (3bar, 1)_{1/3} and V in (1, 2)_{-1/2}: d^c and L, the 5-bar.

In this reading hypercharge is the homology class of a wrapped fivebrane, 6Y = [Sigma_5] in H_5(Y_{3,2}; Z) = Z, and Witten's 1981 chirality argument, which assumes gauge fields acting by coordinate transformations, no longer covers U(1)_Y. It still covers SU(3) x SU(2) for Kaluza-Klein fermions, so chiral doublets must also be brane states.

Proposition 3.2 (one generation is the odd part of the exterior algebra of the 5-bar)

Statement. As SU(3) x SU(2) x U(1)_Y representations,

```
Λ^1(5bar) + Λ^3(5bar) + Λ^5(5bar) = (d^c + L) + (Q +
↪  u^c + e^c) + nu^c = 16,
Λ^0 + Λ^2 + Λ^4 = the conjugates,
```

and the charge of SO(10)/SU(5) is X = 5 - 2k on Λ^k. The 32 states are the finite Hilbert space of one generation with nu^c (dim H_F = 96 for three).

Proof. Enumeration with U, V of charges 2, -3 (Python, section B); the identification of the 16 of SO(10) with the odd forms on C^5 is standard. ■

The new element is the reading of the generators: U and V are the coordinates whose zero loci are the baryonic five-cycles of Y_{3,2}, so a state of Λ^k carries the total class of k such cycles, which is its 6Y. For example Q = U^1 U^2 V^1 has class 2 + 2 - 3 = 1.

Conjecture 3.3 (a generation as a Fock space of wrapped fivebranes). At N = 1, the states of one generation and its conjugate are the states of an exterior algebra generated by five fermionic zero modes, one for each of the cycles {U^i = 0} and {V^A = 0}, with hypercharge the total homology class and X the number of fivebranes. Needed: the fermionic statistics of the wrapped states (from fermionic zero modes of the M5 worldvolume theory on Sigma_5), the regime N = 1, where the supergravity description fails and the M2-brane theory is the abelian quotient of 02, and three generations.

Proposal 3.4 (the Higgs is the position of the M2-brane, and the massless limit is the apex)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The stated coordinate stabilizer is independently checked under the proposed charge assignment. The FI quotient has residual $\mathbb{Z}_3$ or $\mathbb{Z}_2$ stabilizers on the respective exceptional loci, so it is a partial orbifold resolution. See the referee report.

Resolve the cone with a Fayet-Iliopoulos parameter: 2|U|^2 - 3|V|^2 = zeta. For zeta < 0 the locus U = 0 is the projective line of the doublet, P(V) = CP^1; for zeta > 0 the locus V = 0 is CP^2.

Statement (proved part). On the brane coordinates (U, V), on which the Betti U(1)_Y of Proposal 3.1 acts with charges (2, -3), the value V = (v, 0), U = 0 on the doublet line is fixed by SU(3) and by T_3 + Y, and moved by T_3 - Y. (In the quotient geometry, where U(1)_Y is divided out, the same point is fixed by T_3, which is T_3 + Y up to the quotient.) Q = T_3 + Y gives U the charge +1/3 of d^c and V the charges (0, -1) of (nu, e). (Python, section B2.)

Proposal. The scalar that breaks SU(2) x U(1)_Y to U(1)_em is the position of the M2-brane on the doublet line, with the quantum numbers of V, that is of H^* (or of a sneutrino). As zeta -> 0 and the brane moves to the apex, the full group is restored and the near-horizon geometry is AdS4 x Y_{3,2}. So the eleven-dimensional theory reached at v -> 0 is the Freund-Rubin vacuum on the Witten space with Killing spinors, with SU(3) x SU(2) from its isometries and U(1)_Y from C_3. The

relation between M_W and zeta, or the brane position, is the next computation (README, item 3).

Which path this is. In AdS4 x Y_{3,2} the Kaluza-Klein scale is 1/L with L ~ N^{1/6} l_p, independent of zeta. At fixed N, the limit zeta -> 0 therefore restores the group at a fixed tower: it is the path of the Astra audit (§2.2) on which masslessness opens no dimension, and the eleven dimensions are present above 1/L at every v. A joint family in the sense of 01 (weak masses vanishing faster than the tower) needs L/l_p -> infinity, so N -> infinity, with the isometry couplings, g^{-2} proportional to the Killing-vector norms times the volume of Y in eleven-dimensional units, going to zero unless compensated (audit §7.2). Large N conflicts with item 3 of Proposal 3.1, which needs N = 1 for the baryons Sym^N(3bar) and Sym^N(2) to be d^c and L. This tension, between the regime in which the matter content is right and the regime in which the dimension opens, is the main obstruction of the M2 side in its present form, stated for the Freund-Rubin vacuum with N units of flux; escapes are N = 1 matter on a separate stack, a varying flux along the family, or a different vacuum.

Agreement with the audit (its Appendix A). The audit states independently that the circle bundle of M^{111} has c_1 = 3h + 2u, that the total space is spin, that its geometric U(1) is an R-symmetry, and that hypercharge belongs to the charged module and the gauge lift, since a fixed circle-charge sector acts by a scalar phase. This agrees with Theorem 2.1, Proposition 2.3 for (3, 2), and the placement of hypercharge in Proposal 3.1. The audit also notes that Toeplitz compression on the first holomorphic sector H^0(O(3, 2)), the (10, 3) of Theorem 2.1, item 4, generates the full M_30(C), a noninformative algebra in the sense of STRATEGY T0.

Proposition 3.5 (a magnetic completion: AdS5 x CP^2 x S^2)

Statement. On AdS5 x CP^2 x S^2 with product metric and four-form

```
G_4 = f_1 vol(CP^2) + f_2 J ^ vol(S^2),
```

with J the Kähler form of CP^2 and orthonormal-frame components f_1, f_2, the equations of eleven-

dimensional supergravity hold with Ricci components

```
R_AdS = -(f_1^2 + 2 f_2^2)/6,   R_CP2 = f_1^2/3 +
↪  f_2^2/6,   R_S2 = 2 f_2^2/3 - f_1^2/6,
```

so a solution exists with round factors exactly when |f_1| $<$ 2|f_2|.

Proof. dG_4 = 0, and *G_4 = vol(AdS5) ∧ (f_1 vol(S^2) + f_2 J), which is closed. G_4 ∧ G_4 is an eight-form on the six-dimensional factor, so it vanishes and d*G_4 + G_4 ∧ G_4/2 = 0. The Einstein equation R_MN = (G_MPQR G_N^PQR - g_MN G^2/12)/12 gives the displayed components and zero off the diagonal (Python, section G, by summing all frame components); each factor is Einstein, and the radii follow from the three values. ■

Supersymmetry, stability and flux quantization are not checked. The product with flux on KE_4 x S^2 appears in Pope and van Nieuwenhuizen (1989), and supersymmetric AdS5 solutions fibre the S^2 over the Kähler-Einstein base (Gauntlett, Martelli, Sparks and Waldram, hep-th/0402153); both recalled without rereading here.

Proposal 3.6 (two completions of one base)

The base B_6 = CP^2 x S^2 has isometry SU(3) x SU(2), and its dimension is the KO-dimension of the finite factor. It has two eleven-dimensional completions, both of the form 4 + 1 + 6:

	electric completion	magnetic completion
geometry	AdS4 x Y_{3,2}, Y a circle bundle over B_6	AdS5 x B_6
flux	*G_4 through Y: N M2-branes	G_4 through CP^2 and CP^1 x S^2: M5 charges (N_1, N_2)
the “1”	the fibre circle (Proposition 1.1 of 01)	the radial direction of AdS5

	electric completion	magnetic completion
vectors from C_3	b_2(Y) = 1: U(1)_Y	b_2(B_6) = 2: U(1)_{h_1} x U(1)_{h_2}
further vectors	SU(3) x SU(2) x U(1)_R	SU(3) x SU(2)
v -> 0	the M2-brane at the apex	the infrared end of AdS5 moves to infinity

Under reduction along the fibre of class (3, 2), the combination of U(1)_{h_1} x U(1)_{h_2} along (3, 2) is replaced by the fibre isometry, and the quotient Z^2/(3, 2) = Z, (x, y) -> 2x - 3y, is hypercharge. The magnetic side therefore carries two abelian factors, a candidate for the fluxed pair needed by Proposition 2.6 together with the universal twist. The magnetic completion realizes the counting of H directly: four-dimensional spacetime on the M5 worldvolume, the interval as the conformal radial coordinate z in (z_UV, z_IR) with M_KK ~ 1/z_IR, as in Higgsless models on an interval, and v -> 0 as z_IR -> infinity.

Proposition 3.7 (the M2/M5 exchange acts on the Betti vector)

In AdS4, M2-branes on the two-cycle and M5-branes on five-cycles are electric and magnetic for the Betti vector (C_3 = A ∧ omega_2 and C_6 = Ã ∧ omega_5), with Dirac pairing given by the intersection of H_2 and H_5. Which of the two is electric depends on the boundary condition of the bulk vector, and the S operation of Witten's SL(2, Z) action on three-dimensional theories with a U(1) symmetry exchanges them (hep-th/0307041, recalled). With Proposal 3.1 this exchange acts on hypercharge as electric-magnetic duality.

Test against lemma 21. S maps charges to monopole charges; it does not permute the 96 states of H_F. The exchange of the top and the neutrinos in lemma 21 is a permutation, so it does not come from this S. In the brane reading, the 84 states of the second family (Q_em != 0) are the states whose fivebrane class 6Y is not cancelled

by T_3, after the breaking of Proposal 3.4; the first family (all states except the top) has no reading here. As the origin of the exchange, this points to a property of the top (lemma 21, Proposition 21.1) that the present construction does not see, and away from the M2/M5 duality.

Proposition 3.8 (no colour-equivariant identification of the 84 with the three-form)

Statement. For every homomorphism SU(3) -> SO(9), Λ^3 R^9 ⊗ C is not isomorphic, as an SU(3) representation, to the colour content of either family of 84 in lemma 21: 10 (3 + 3bar) + 24 (1) for the five quarks and all leptons, 12 (3 + 3bar) + 12 (1) for the charged fermions. The same holds for Λ^6 R^9, the polarizations of C_6.

Proof. A real nine-dimensional representation has a self-conjugate complexification of dimension 9, which is 9 (1), 3 + 3bar + 3 (1) or 8 + 1 (the sextets and two triplet pairs already exceed 9). In the first case Λ^3 has only singlets. In the second, Λ^2(3) ⊗ 3bar = 3bar ⊗ 3bar contains the 6bar. In the third, Λ^3(8) = 1 + 8 + 10 + 10bar + 27. Both families contain only 1, 3 and 3bar. Λ^6 R^9 is isomorphic to Λ^3 R^9 under SO(9). (Python, section E, by weights.) ■

Scope and escapes. A linear action of colour on the nine transverse directions of the eleven-dimensional little group. Escapes: colour acting on the internal space, outside the little group, a nonlinear or brane realization, or a map that is not colour-equivariant, in which case the equality of the two 84s is an equality of numbers only.

Corollary 3.9 (the coloured states of the 84 are not Kaluza-Klein modes of C_3 on Y_{3,2}) Under the quotient realization, on a background invariant under the displayed action, every tensor Kaluza-Klein mode of C_3 (or of the metric) on Y_{3,2} carries a representation of PSU(3) x SO(3) (Proposition 2.4), so none is a colour triplet; spinor modes are open. The 60 coloured states of the first family, and the 72 of the second, are therefore not modes of C_3 on Y_{3,2}. Together with Proposition 3.8 this excludes, in their stated scopes, the two direct routes, the little group and the tensor Kaluza-Klein tower, for a colour-equivariant iden-

tification. The route left open is the brane one: wrapped M5 and M2 states, which carry triality through their hypercharge class (Proposition 2.4 (a)).

Summary of the direction

Referee note (GPT-6.1 Terra, 4 Nov 2026). The physical identifications in this summary are proposals. The colour exclusions have the stated linear-representation and tensor-mode scopes; other internal spaces, nonlinear actions and brane realizations remain directions. See the wording audit.

The M2 and M5 details of lemma 21 enter in a specific place. Hypercharge is the C_3 vector of the Witten space with Killing spinors, the 5-bar consists of the N = 1 five-brane baryons, the Higgs is the M2 position, and the M2/M5 exchange is the electric-magnetic duality of hypercharge. The count 84 of C_3 does not survive as a colour-equivariant identification. The next computations are in the README.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 3.8. *From the referee report on the 11D line, section 'Proposition 3.8 and Corollary 3.9: colour and the three-form', transferred verbatim in substance on 4 Nov 2026.*

Hypothesis: a linear colour action on the nine transverse directions.

An irreducible complex SU(3) representation with highest weight (p, q) has dimension (p + 1)(q + 1)(p + q + 2)/2. The only irreducibles of dimension at most nine are 1, 3, 3bar, 6, 6bar and 8. The complexification of a real representation is self-conjugate, so a sextet needs its conjugate ($12 > 9$) and two triplet pairs have dimension 12. Dimension nine therefore leaves exactly

```
9 (1),    3 + 3bar + 3 (1),    8 + 1.
```

The first exterior cube is trivial. In the second,

```
Λ^2 3 ⊗ 3bar = 3bar ⊗ 3bar = 6bar + 3
```

is a summand of the exterior cube. In the third, Λ^3(8 + 1) = Λ^3 8 + Λ^2 8, and Λ^2 8 contains a 10: identify 8 with sl_3(C); the vector E_13 ∧ E_12 is killed by the two positive simple root operators ad E_12 and ad E_23 and has highest weight (3, 0), so it generates a 10. Neither target colour representation contains a 6bar or a 10, and the trivial case contains no triplets.

The decomposition used in the proof above also holds:

```
Λ^3 8 = 1 + 8 + 10 + 10bar + 27.
```

E_13 ∧ E_12 ∧ E_23 is a highest-weight vector of weight 2 alpha_1 + 2 alpha_2 = (2, 2): the only nonzero brackets with the raising operators, [E_12, E_23] = E_13 and [E_23, E_12] = -E_13, produce a repeated factor E_13. E_13 ∧ E_12 ∧ H, with H a nonzero Cartan element satisfying alpha_2(H) = 0, has highest weight (3, 0). Conjugation supplies (0, 3), and the invariant Cartan three-form (X, Y, Z) -> tr(X [Y, Z]) supplies 1. The (1, 1)-weight space of Λ^3 8 has dimension five (E_13 ∧ H_1 ∧ H_2, E_12 ∧ E_23 ∧ H_i for a basis H_1, H_2 of the Cartan subalgebra, and E_13 ∧ E_alpha ∧ E_{-alpha} for alpha = alpha_1, alpha_2), while each of the two spaces reached from it by the simple raising operators, of weights (3, 0) and (0, 3), has dimension two (E_13 ∧ E_12 ∧ H_i and E_13 ∧ E_23 ∧ H_i). Their joint kernel therefore supplies a highest-weight 8. The dimensions sum to 1 + 8 + 10 + 10 + 27 = 56 = C(8, 3), exhausting the exterior cube. Orientation and the Hodge star give Λ^6 R^9 ≅ Λ^3 R^9 equivariantly.

Independent proof of Corollary 3.9. *Same section of the referee report, transferred verbatim in substance on 4 Nov 2026.*

The colour centre acts trivially on Y_{3,2}: for omega = e^{2 pi i/3}, (omega I_3, I_2) is a transformation of the quotient circle, obtained from lambda = e^{4 pi i/3}, since (lambda^2, lambda^{-3}) = (e^{8 pi i/3}, 1) = (omega, 1). Its induced action on every tensor bundle is the identity. Thus tensor fields contain only colour representations with trivial centre, which excludes 3 and 3bar. The same argument applies to tensor modes of

C_3 and of the metric. It is enough that the displayed colour action preserve the background metric and flux; an identification of the full isometry group is unnecessary.

Scope and direction. The first result concerns exterior powers of a linear real nine-dimensional colour representation; the second concerns tensor modes on the stated quotient with the stated action. Colour acting on other internal spaces, nonlinear or brane realizations, and maps without colour equivariance remain open directions.

Independent proof of Proposition 3.5. *From the referee report on the 11D line, section 'Proposition 3.5: magnetic product solution', transferred verbatim in substance on 4 Nov 2026.*

Scope: a classical local supergravity solution with scaled Einstein factors. Flux quantization, global consistency, supersymmetry and stability remain separate questions.

Choose an orthonormal frame e^1, …, e^4 on CP^2 and e^5, e^6 on S^2, with J = e^{12} + e^{34}, and write

```
G = f_1 e^{1234} + f_2 (e^{1256} + e^{3456}).
```

Then G^2 = G_MNPQ G^MNPQ = 4! (f_1^2 + 2 f_2^2) = 24 (f_1^2 + 2 f_2^2). The contraction G_aPQR G_a^PQR, with no sum over a, is 3! (f_1^2 + f_2^2) = 6 (f_1^2 + f_2^2) in a CP^2 direction (each such index lies in e^{1234} and in one of the two f_2 terms), 3! (2 f_2^2) = 12 f_2^2 in an S^2 direction, and zero in an AdS direction. The mixed contractions with a != b vanish, since no two components of G share three indices. Substitution in the Einstein equation R_MN = (G_MPQR G_N^PQR - g_MN G^2/12)/12 gives

```
R_AdS = -(f_1^2 + 2 f_2^2)/6,    R_CP2 = f_1^2/3 +
↪  f_2^2/6,    R_S2 = 2 f_2^2/3 - f_1^2/6;
```

for instance R_CP2 = (6 (f_1^2 + f_2^2) - 2 (f_1^2 + 2 f_2^2))/12. The forms J, vol(CP^2), vol(S^2) and the AdS volume form are parallel, so dG = d*G = 0. Also G ∧ G = 0, since it is an eight-form entirely on the six-dimensional compact factor.

Positive curvature of the S^2 factor, 2 f_2^2/3 > f_1^2/6, is equivalent to |f_1| < 2|f_2|. This also makes the other compact curvature positive and the AdS curvature negative. The radii are then determined by the three

Einstein constants. Here "round factors" means the standard scaled Fubini-Study metric on CP^2 and the round metric on S^2.

Independent proof of Proposition 3.2. *From the referee report on the 11D line, section 'Proposition 3.2: exterior algebra of the five-bar', transferred verbatim in substance on 4 Nov 2026.*

The representation and charge count is the standard spinor construction. Identifying these algebraic states with wrapped-brane states remains conditional on Proposal 3.1 and Conjecture 3.3.

Write 5bar = U ⊕ V, with U = (3bar, 1)_{1/3} and V = (1, 2)_{-1/2}. The terms of Λ^3 5bar are

```
Λ^3 U = e^c,    Λ^2 U ⊗ V = Q,    U ⊗ Λ^2 V = u^c,
```

with respective 6Y charges 6, 4 - 3 = 1 and 2 - 6 = -4, using Λ^3 3bar = 1, Λ^2 3bar = 3 and Λ^2 2 = 1. Λ^1 5bar = d^c + L, and Λ^5 5bar = nu^c, since the determinant charge is 3(2) + 2(-3) = 0. The odd degrees therefore have dimension 5 + 10 + 1 = 16; the even degrees are their conjugates, since Λ^k 5bar is dual to Λ^{5-k} 5bar when the determinant is trivial, and have dimension 16. Three generations give 96 complex finite-Hilbert-space states. The conventional SO(10)/SU(5) charge X = 5 - 2k gives 3, -1 and -5 on the three odd degrees.

The divisor classes from Proposition 2.3 can be added formally with the same weights 2 and -3. A physical identification of those exterior generators with fermionic wrapped-fivebrane states requires the proposed brane construction and its statistics.

Independent proof of the proved part of Proposal 3.4, and the stabilizers of the FI quotient. *From the referee report on the 11D line, section 'Proposal 3.4: Higgs stabilizer and the FI quotient', transferred verbatim in substance on 4 Nov 2026.*

The coordinate charge calculation holds assuming the proposed action of the Betti U(1) on the brane coordinates; identifying that action with the electroweak one is part of the proposal. A nonzero FI parameter gives a partial orbifold resolution, with finite stabilizers along the projective loci below.

Stabilizer. On V = (v, 0), U = 0, take T_3 = diag(1/2, -1/2) and Y(V) = -1/2. The first component has (T_3 + Y) V = (1/2 - 1/2) V = 0 and (T_3 - Y) V = V, and SU(3) fixes U = 0. The charges Q = T_3 + Y of the two components of V are 0 and -1, and Q(U) = 0 + 1/3 = 1/3.

FI quotient. For the action lambda · (U, V) = (lambda^2 U, lambda^{-3} V) on the level 2|U|^2 - 3|V|^2 = zeta:

- At zeta < 0 every point has V != 0. Points with U != 0 as well have trivial stabilizer (lambda^2 = lambda^3 = 1), and the locus U = 0, V != 0 has stabilizer lambda^3 = 1. This Z_3 acts faithfully on the three transverse U coordinates, by the primitive cube root lambda^2. The coarse exceptional locus is {U = 0}/U(1) = CP^1, with transverse singularity C^3/Z_3.
- At zeta > 0, the locus V = 0, U != 0 has stabilizer lambda^2 = 1, a Z_2 acting by lambda^{-3} = -1, minus the identity, on the two transverse V coordinates. Its coarse exceptional locus is CP^2, with transverse singularity C^2/Z_2.

Thus the FI quotient needs further geometric data to be a smooth resolution.

Part 33

The two families of 84 and the M2 and M5 sides

Lemma 33. Written as file 04 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN). Attempted by Claude (Anthropic), 4 Nov 2026. Propositions 4.1 to 4.3 and Theorem 4.4 and Propositions 4.5, 4.6

are proved here by hand (branching rules and counting); the negative results 4.3 and 4.6 were refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5, with the amendments below; readings are labelled as proposals.

Referee note (GPT-6.1 Terra, 4 Nov 2026). Proposition 4.1 and the colour obstructions in Propositions 4.2(a) and 4.3 are independently checked. The tensor-factor conclusion of 4.2(b) needs a factor-preserving map, and the "exactly" clause in 4.3 also needs nonzero triplets: the trivial nine-dimensional representation gives 84 singlets. See the referee report; the other checked results are noted below.

Referee note on Proposition 4.6 (GPT-6.1 Terra). The stated sector and block obstructions hold in the scopes in the referee report. The Chevalley involution is complex-linear; an antiunitary J needs an additional conjugation. Its identification with physical M2/M5 duality remains a proposal.

The question

Lemma 21, Proposition 21.3: with three generations and Dirac neutrinos, dim H_F = 96 = 84 + 12 in two ways,

```
F1 = all states except the top          (the top: 3
 ↪  colours x 4 = 12 states),
F2 = all states except the neutrinos    (the
 ↪  neutrinos: 3 generations x 4 = 12 states),
```

where 4 counts chirality and particle/antiparticle. The two families overlap in the 72 states of the charged fermions other than the top. Lemma 21 proposed relating one family to C_3 (or the M2-brane) and the other to C_6 (or the M5-brane), with the exchange of the top and the neutrinos as the content of the duality on the Standard Model side, and recorded that equal numbers are insufficient. The files 02 and 03 showed that colour-equivariant identifications through the little group or through Kaluza-Klein modes on the Witten space fail (Proposition 3.8, Corollary 3.9). This file asks the question in the Hořava-Witten wall group E8 and in the finite geometry.

1. What singles out the two blocks

Proposition 4.1. Consider blocks of 12 states made of whole Dirac species (all colours, both chiralities, particles and antiparticles; 12 states for a quark species, 4 for a lepton species). The blocks whose removal keeps both spectral-action values of lemma 21, cos^2 theta_NC = 5/8 and kappa_em/kappa_s = 8/3, are exactly: one up-type quark (u, c or t), and the three neutrinos together. So F1 (with the top, the heavy up-type quark) and F2 are the two kinds of solutions: a charged one and a neutral one.

Proof. A 12-state block of whole species is one quark species or three lepton species. By Proposition 21.1, with the totals (sum T_3 Q, sum Y Q; Q^2, colour) = (6, 10; 32, 12) and cos^2 theta_NC = (sum Y Q)/(sum T_3 Q + sum Y Q), removing a quark keeps both values only for an up-type quark, (1, 5/3; 16/3, 2). For three lepton species with k charged leptons, each contributing (1/2, 3/2; 4, 0), the angle becomes (10 - 3k/2)/(16 - 2k), which is 5/8 only for k = 0; neutrinos contribute zero to all four sums. ■

Reading at the 9 label. At the 9 label the light gauge algebra is colour plus electromagnetism (canonical §3, lemma 21). H_F then splits into three parts: the 72 charged light states (F1 ∩ F2), the 12 neutrino states, which carry no charge of that algebra and are free, and the 12 top states, which are charged and infinitely heavy. F1 = charged light + free light; F2 = charged light + charged heavy. The exchange of lemma 21 exchanges the block that decouples through zero charge with the block that decouples through infinite mass.

2. Colour obstructions

Proposition 4.2. (a) As representations of colour SU(3), F1 = 10 (3 + 3bar) + 24 (1) and F2 = 12 (3 + 3bar) + 12 (1); they are not isomorphic. Hence no unitary of H_F commuting with colour maps F1 onto F2, and no automorphism of the finite spectral triple of the Standard Model does. (b) Any tensor-factor-preserving bijection of the top block onto the neutrino block that is linear in each index exchanges the colour index of the top (r, g, b) with the generation index of the neutrinos

(e, mu, tau).

Proof. (a) Counting: F1 has the five quarks other than the top (5 x 2 triplet pairs, counting chirality) and 24 leptonic states; F2 has the six quarks and 12 charged-lepton states. Colour acts through A_F. (b) The top block is C^3_colour ⊗ e_3 ⊗ C^4 (third generation, 4 for chirality and conjugation) and the neutrino block is e_lepton ⊗ C^3_generation ⊗ C^4 in the Pati-Salam form C^4_colour ⊗ C^3_generation of the up-type sector; a map of the first onto the second that respects the tensor factors sends C^3_colour to C^3_generation. ■

So an exchange needs a structure in which colour and generation can be exchanged; a structure in which colour acts and generations do not admits none.

Proposition 4.3 (complex case of Proposition 3.8). For every homomorphism SU(3) -> SU(9), the colour content of Λ^3 C^9 is not that of F1 or F2. Λ^3 C^9 contains only 1, 3 and 3bar exactly when C^9 is 9 (1), 3 + 6 (1) or 3bar + 6 (1); for 3 + 6 (1), Λ^3 C^9 = 15 (3) + 6 (3bar) + 21 (1), and for 9 (1) it is 84 (1).

Proof. C^9 is a sum of irreducibles of dimensions 1, 3, 6, 8 with total 9. With a sextet: 6 + 3 (1) has the 6 in Λ^1(6) ⊗ Λ^2(3 (1)); 6 + 3 has Λ^3(6) = 10 + 10bar. With an octet: 8 + 1 has Λ^3(8) = 1 + 8 + 10 + 10bar + 27. With two triplets T, T' (each 3 or 3bar) and three singlets: Λ^1(T) ⊗ Λ^1(T') ⊗ Λ^1(1) contains T ⊗ T', which is 6 + 3bar, 6bar + 3 or 8 + 1. With three triplets: T ⊗ T' ⊗ T'' contains a 10, a 10bar, a 15 or a 15bar. With nine singlets there is no triplet. With one triplet and six singlets, Λ^3(3 + 6 (1)) = Λ^3(3) + 6 Λ^2(3) + 15 (3) + 20 (1) = 15 (3) + 6 (3bar) + 21 (1). Neither family has unequal numbers of 3 and 3bar. ■

Together with Proposition 3.8 (real nine-dimensional representations) this excludes, in that scope, the colour-equivariant route through Λ^3 of a linear nine-dimensional space. The positive construction below places the two blocks, and leaves the two families unidentified.

3. Two commuting gradings of E8

E8 is the gauge group of each wall in Hořava-Witten theory, and it also models the C-field topologically (Witten

1996, "On flux quantization in M-theory"; Diaconescu, Moore and Witten 2000; recalled). Over C, e8 has an inner automorphism of order three with fixed algebra sl(9):

```
e8 = sl(9) ⊕ Λ^3 C^9 ⊕ Λ^6 C^9          (80 + 84 +
↪  84),
```

with [Λ^3, Λ^3] ⊂ Λ^6 (wedge product), [Λ^3, Λ^6] ⊂ sl(9) (contraction) and [Λ^6, Λ^6] ⊂ Λ^3. In the U-duality algebra e8(8) of eleven-dimensional supergravity on T^8, the decomposition under gl(8) by level (West's level, recalled) places the C_3 shifts at level 1 (Λ^3 of 8, 56 states), the C_6 shifts at level 2 (Λ^6 of 8, 28), the dual graviton at level 3 (8) and gl(8) at level 0; then Λ^3 C^9 = 56 (level 1) + 28 (level -2) and the grading above is the level mod 3. So M-grade b = 1 is one unit of M2 charge (or minus an M5), b = 2 is one unit of M5 charge (or minus an M2), and b = 0 holds the metric and the dual graviton (levels 0 and ±3). The bracket Λ^3 × Λ^3 -> Λ^6 is the algebraic form of the term C_3 ∧ G_4 in G_7 = dC_6 + (1/2) C_3 ∧ G_4, equivalent by duality to the Chern-Simons coupling; Λ^3 × Λ^6 -> sl(9) adds levels 1 + 2 = 3, the dual graviton. Under SO(9) ⊂ SL(9), 248 = 36 + 44 + 84 + 84, the SO(9) content of so(9), the graviton, C_3 and C_6. We call the three pieces the gravity, C_3 and C_6 sectors.

A second element of order three, the centre of SU(3) in E8 ⊃ (SU(3) × E6)/Z_3, gives the family grading e8 = (su(3) ⊕ e6) ⊕ (3, 27) ⊕ (3bar, 27bar), with family grade a = 0, 1, 2. With this SU(3) as family group, (3, 27) holds three families of E6, each containing a generation 16 ⊂ 27, and (3bar, 27bar) holds their conjugates.

Lemma 4.A (the two gradings commute). Take SU(9) ⊃ SU(3) × SU(6) × U(1) with C^9 = (3, 1)_{-2} + (1, 6)_{1}, and let s_f be the centre of this SU(3). Then s_f commutes with the SU(9) grading, its centralizer is (SU(3) × E6)/Z_3, the U(1) is the Cartan of the SU(2) in E6 ⊃ SU(6) × SU(2) with charge q = 6 T_3, and the joint pieces g_{a,b} are

	b = 0 (gravity)	b = 1 (C_3)	b = 2 (C_6)
a = 0	su(3) + su(6) + u(1): 44	(1,1)_{-6} + (1,20)_{3}: 21	(1,1)_{6} + (1,20)_{-3}: 21
a = 1	(3, 6bar)_{-3}: 18	(3, 15)_{0}: 45	(3, 6bar)_{3}: 18
a = 2	(3bar, 6)_{3}: 18	(3bar, 6)_{-3}: 18	(3bar, 15bar)_{0}: 45

so that b ≡ a + q/3 = a + 2 T_3 (mod 3).

Proof. Λ^3 of (3,1)_{-2} + (1,6)_{1} is (1,1)_{-6} + (3bar,6)_{-3} + (3,15)_{0} + (1,20)_{3}; the adjoint is (8,1) + (1,35) + (1,1) + (3,6bar)_{-3} + (3bar,6)_{3}; Λ^6 is the conjugate of Λ^3. The column a = 0 sums to 86 = 8 + 78 and the row a = 1 to 81 = 3 x 27, which matches E8 ⊃ SU(3) × E6 with E6 ⊃ SU(6) × SU(2): 27 = (15, 1) + (6bar, 2) and 78 = (35, 1) + (1, 3) + (20, 2), the SU(2) triplet being (1,1)_{-6}, the u(1) and (1,1)_{6}. ■

4. Placement of the generation

Let SU(5) ⊂ SU(6) ⊂ E6 be the Georgi-Glashow group, with 15 = 10 + 5 and 6bar = 5bar + 1. The commutant of SU(5) in E6 is SU(2) × U(1), and its SU(2) is the SU(2) of the lemma; we call it SU(2)_I. With SO(10) × U(1)_psi and SU(5) × U(1)_chi, 27 = 16_1 + 10_{-2} + 1_4, 16 = 10_{-1} + 5bar_3 + 1_{-5} and 10 = 5_2 + 5bar_{-2}. The Cartan T_3 of SU(2)_I vanishes on the singlets 10 and 5, so it is proportional to psi + chi: T_3 = t on 5bar_16 and on N = 1_{E6}, and T_3 = -t on 5bar_10 and on nu^c = 1_16, with t = ±1/2. The SU(2)_I doublets are (d^c, D^c), (L, H_d) and (N, nu^c).

Theorem 4.4 (the generation in the three sectors). For the generation 16 ⊂ 27 of family grade a = 1:

Referee note (GPT-6.1 Terra, 4 Nov 2026). The branching and counts are independently checked. The fields t_L, t^c are in the left-handed basis; t^c is an antitop

field, so the top particle's two chiralities occupy different sectors. See the referee report.

1. the 10 = (Q, u^c, e^c) and the 5 = (D, H_u) lie in g_{1,1}, the C_3 sector;
2. (d^c, L) and N lie in g_{1,1+2t}, and (D^c, H_d) and nu^c lie in g_{1,1-2t}: one pair in the C_6 sector and the other in the gravity sector, according to the sign t; the two choices differ by the Weyl reflection of SU(2)_I;
3. the antiparticles lie in the conjugate pieces g_{2,-b}.

So every top state of a particle (t_L in Q, t^c in u^c), and the up-type Higgs, lie in the C_3 sector, and the antitop in the C_6 sector. The left-handed particle fields of the neutrinos (a = 1) lie outside the C_3 sector, nu_L and nu^c in different sectors (C_6 and gravity); their conjugates do not, since conj(nu_L) lies in C_3 for t = +1/2. With conjugates, the top block has (C_3, C_6, gravity) = (6, 6, 0) and the neutrino block (3, 3, 6), for both signs of t; a conjugation-closed block always has equal C_3 and C_6 counts. With three families and antiparticles, the 96 states of H_F split as 45 (C_3) + 45 (C_6) + 6 (gravity: nu^c and its conjugate) for t = +1/2, and 33 + 33 + 30 for t = -1/2.

Proof. b ≡ a + 2 T_3, with T_3 = 0 on the 10 and the 5, and T_3 = ±t on the doublet members; for antiparticles a = 2 and T_3 changes sign. The counts follow per family: for t = +1/2, C_3 holds the 10 and the 5 of antiparticles (15), C_6 their conjugates, gravity nu^c and its conjugate; for t = -1/2, C_3 holds the 10 and the conjugate of nu^c (11), C_6 their conjugates, gravity the 5bar and the 5 (10). ■

5. Yukawa couplings as brackets

Proposition 4.5. Suppose the cubic couplings of the matter come from the structure constants of e8, as in heterotic and Hořava-Witten compactifications, and both gradings are symmetries, so that a coupling needs grades summing to (0, 0) mod 3. Take H_u in the 5 ⊂ g_{1,1} and H_d in 5bar_10 ⊂ g_{1,1-2t}. Then, for either sign t:

Referee note (GPT-6.1 Terra, 4 Nov 2026). The

grade selection and bracket targets are independently checked. In the literal exact $SU(3)_f$ triplet realization, the family tensor is antisymmetric and does not by itself give an isolated top mass; the physical Yukawa matrix needs additional data. See the referee report.

1. the up-type Yukawa couplings Q u^c H_u have all three fields in the C_3 sector;
2. the down-type, charged-lepton and Dirac-neutrino couplings Q d^c H_d, L e^c H_d and L nu^c H_u have one field in each of the three sectors;
3. in Dirac form, the top mass is ad() : g_{1,1} -> g_{2,2}, the wedge Λ^3 -> Λ^6 with the Higgs trivector, while the Dirac-neutrino mass is a bracket that passes through the gravity sector: Λ^6 -> sl(9) (levels 1 + 2 = 3) for t = +1/2, sl(9) -> Λ^3 for t = -1/2.

Proof. Grades from Theorem 4.4: (1, 1) three times for the up type; for the others the b-grades are {1, 1 + 2t, 1 - 2t} = {1, 2, 0}. The Dirac masses pair the left-handed field with the conjugate of the right-handed one, through the Killing form between g_{a,b} and g_{-a,-b}. ■

The family grading alone allows only epsilon_{ijk} in exact family SU(3); in compactifications the family label is a cohomology label and the couplings are intersection numbers. The M-grading survives when its element commutes with the structure group of the bundle: for the standard embedding, s_M = s_f exp(2 pi i (2/3) T_3), a product of the centre of the holonomy SU(3) and an element of SU(2)_I ⊂ E6, which does.

Reading (proposal). The up-type Yukawa couplings are the only Standard Model Yukawa couplings inside one sector, and that sector's bracket is the C_3 ∧ C_3 -> C_6 term, metric-independent; the other couplings use one field of each sector, including the gravity sector. The top, the only fermion with a Yukawa coupling of order one and the one singled out by the spectral action in lemma 21, takes its mass from the C_3 bracket.

6. The Chevalley involution and the real structure J

Referee note (GPT-6.1 Terra, 4 Nov 2026). This heading describes a proposed physical identification. The

Chevalley involution is complex-linear, and an antiunitary J needs additional conjugation; constructing the finite triple remains open. See the wording audit.

Proposition 4.6.

1. No automorphism of e8 that preserves the family grading maps the C_3 sector of the matter, g_{1,1} (dimension 45), onto its C_6 sector, g_{1,2} (dimension 18).
2. The Chevalley involution, the element -1 of the Weyl group of E8, maps g_{a,b} to g_{-a,-b}. It normalizes SU(9), acting as complex conjugation (Λ^3 <-> Λ^6), and on the placed H_F it exchanges every state with its conjugate: it is the real structure J of the finite geometry. The family grade a is the four-dimensional chirality in the left-handed basis (a = 1 for the left-handed fields, a = 2 for their conjugates), and J inverts it, as KO-dimension six requires (J gamma = -gamma J).
3. J preserves F1 and F2, which are unions of species with their conjugates.
4. The Weyl reflection of SU(2)_I fixes s_f and replaces b by 2a - b: on the matter (a = 1) it fixes the C_3 sector and exchanges the C_6 and gravity sectors, together with (d^c, L) <-> (D^c, H_d) and nu^c <-> N.

Proof. (1) Dimensions. (2) -1 acts on a Cartan subalgebra by -1, so it inverts every element of the maximal torus containing s_f and s_M, and maps the joint eigenspace for (omega^a, omega^b) to that for (omega^{-a}, omega^{-b}); it maps the root system A_8 of SU(9) to itself. (3) Immediate. (4) The reflection inverts T_3 and commutes with SU(3) × SU(6). ■

Negative result (scope: e8 with these two gradings, automorphisms preserving the pair up to inversion). The Chevalley involution, read as the M2/M5 exchange (a proposal), maps each block to its conjugate and leaves the top and neutrino blocks in place. An exchange as in lemma 21 must therefore come from an automorphism that neither preserves nor inverts the family grading; for aligned families, Proposition 5.1 also excludes every element of W(E8) mapping F1 onto F2. The colour-family exchanges inside E8 ⊃ SU(3)^4, where the four SU(3) factors are permuted by the normalizer, are

the remaining candidates (PLAN, step 4). What the gradings do give is a placement of the left-handed fields (a = 1): the top on the M2 side, the neutrinos on the M5 and gravity sides; with conjugates the counts are those after Theorem 4.4.

7. Agreement with files 02 and 03, and the finite geometry

- **nu^c as the gravitational state.** nu^c is the only Standard Model state with Y = 0, so on the Witten space it is the only one that can be a Kaluza-Klein mode (Proposition 2.4 with Proposal 3.1), while every hypercharged state is a brane state. In e8 with t = +1/2 it is the only Standard Model state in the gravity sector, and every hypercharged state lies in the C_3 or C_6 sector. The two placements agree for t = +1/2; for t = -1/2 the gravity sector holds (d^c, L) instead. We take the agreement as selecting t = +1/2 (proposal).
- **The finite geometry.** On H_F, the family grading is the chirality, the Chevalley involution is J, and the Yukawa blocks of D_F are brackets with the Higgs: the top block maps the C_3 sector to the C_6 sector, the Majorana block acts inside the gravity sector (nu^c with nu^c, t = +1/2). A finite triple built on the graded e8, with D_F = ad(Higgs), is the construction to attempt next (PLAN, step 5).
- **The two families.** F1 removes a block of the C_3 sector (with its conjugate in C_6): the states joined by the cubic C_3 coupling. F2 removes the block that meets the gravity sector. In this sense, for the left-handed fields, the top belongs to the M2 side and the neutrinos to the M5 and gravity sides, which is the correspondence lemma 21 asked for, at the level of the removed blocks; with conjugates the blocks have counts (6, 6, 0) and (3, 3, 6). The two families of 84 are not themselves the 84 of C_3 and the 84 of C_6 (Propositions 3.8, 4.3, 4.6).

8. Remark: a discrete symmetry from the M-grading

For t = +1/2, s_M acts on the 27 by omega^b: omega on the 10 and on the 5 (H_u, D), omega^2 on (d^c, L) and N, and 1 on (D^c, H_d) and nu^c. It allows Q u^c H_u, Q d^c H_d, L e^c H_d, L nu^c H_u, N H_u H_d, the Majorana mass of nu^c and the Weinberg operator L L H_u H_u, and forbids the mu term H_u H_d, the dimension-five operators Q Q Q L and u^c u^c d^c e^c, and the trilinear couplings L L e^c, L Q d^c, u^c d^c d^c; the bilinear L H_u is allowed. This is the pattern of E6-inspired models in which the singlet N supplies the mu term. It is a symmetry of the low-energy theory only if the breaking of E6 commutes with s_M; for t = -1/2 the charges change. Stated as a remark; the results above do not use it.

Scope of the remark (second check, 4 Nov 2026). The selection rule is that of the single element s_M, and it is a symmetry only if the breaking of E6 commutes with s_M. The family grading would forbid (nu^c)^2, L L H_u H_u and L H_u, which s_M allows, since their family grades are 2, 1 and 2.

9. What remains open

1. Colour-family exchange in E8 ⊃ SU(3)^4 acting on the placed states: does an element map the top block onto the neutrino block and F1 onto F2? (PLAN, step 4.)
2. A finite real spectral triple on the graded e8 with D_F = ad(Higgs): order zero and order one, and comparison with C + H + M_3(C) (PLAN, step 5).
3. Hořava-Witten dynamics: whether an M5-brane moving into a wall (small-instanton transition) can remove one of the 12-state blocks of Proposition 4.1 (PLAN, step 6).
4. The E11 level decomposition beyond E8, where C_3, C_6 and the dual graviton sit at levels 1, 2, 3: test the assignment 10 -> level 1, (d^c, L) -> level 2, nu^c -> level 0 or 3 (PLAN, step 7).

References for this file

Recalled without rereading in this session: E. Witten, "On flux quantization in M-theory and the effective action", hep-th/9609122; D.-E. Diaconescu, G. Moore and E. Witten, "E8 gauge theory, and a derivation of K-theory from M-theory", hep-th/0005090; P. West, "E11 and M theory", hep-th/0104081, for the level decomposition and the commutator of the level-one generators; E. Cremmer and B. Julia (1979) for E8(8) in three dimensions; E. B. Vinberg and A. G. Elashvili (1978) for the Z_3-graded e8 and trivectors of C^9; the E6 subgroup chains are standard (R. Slansky, Phys. Rep. 79 (1981) 1).

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 4.2(a), with the scope of 4.2(b). *From the referee report on the 11D line, section 'Propositions 4.2 and 4.3: complex colour representations', transferred verbatim in substance on 4 Nov 2026.*

The colour multiplicities are

```
F1 = 10 (3 + 3bar) + 24 (1),    F2 = 12 (3 + 3bar) +
↪  12 (1):
```

each quark species of 12 states gives two triplets and two antitriplets (particle and antiparticle, two chiralities); F1 holds five quark species and the 24 lepton states, and F2 holds the six quark species and the 12 charged-lepton states. An intertwiner preserves the isotypic components, so no colour-commuting unitary exchanges F1 and F2. Even a colour-group automorphism preserves the dimension of the invariant subspace, which is 24 for F1 and 12 for F2.

For 4.2(b), the stated conclusion holds for a tensor-factor-preserving bijection identifying the top's colour factor with the neutrinos' family factor. Linearity in each index alone does not imply this: every linear map on the ten-

sor product is multilinear in its arguments, and a unitary may entangle the three-dimensional index with the four-dimensional chirality/conjugation index. Such a unitary need not be a flip of tensor factors.

Counterexample (referee) to the first version of Proposition 4.3. *Same section of the referee report, transferred verbatim in substance on 4 Nov 2026.*

The first version stated that the exterior cube contains only 1, 3 and 3bar exactly for 3 + 6 (1) or its conjugate. The trivial homomorphism gives C^9 = 9 (1) and Λ^3 C^9 = 84 (1), which also contains only those representations. The statement above lists 9 (1); equivalently, the corrected statement adds the requirement that the exterior cube contain a nonzero triplet.

Independent proof of Proposition 4.3 (corrected). *Same section of the referee report, transferred verbatim in substance on 4 Nov 2026.*

The irreducible SU(3) representations of dimension at most nine are 1, 3, 3bar, 6, 6bar and 8: the irreducible of highest weight (p, q) has dimension (p + 1)(q + 1)(p + q + 2)/2, and the next values are 10 and 15. So C^9 is a sum of irreducibles of dimensions 1, 3, 6, 8 with total 9, and the cases are the following.

1. 8 + 1. Λ^3(8 + 1) = Λ^3 8 + Λ^2 8, and Λ^2 8 contains a 10: identify 8 with sl_3(C); the vector E_13 ∧ E_12 is killed by ad E_12 and ad E_23 (the only nonzero bracket, [E_23, E_12] = -E_13, produces E_13 ∧ E_13 = 0) and has weight 2 alpha_1 + alpha_2 = (3, 0), so it generates a 10.

2. 6 + 3 (1). Λ^1(6) ⊗ Λ^2(3 (1)) = 3 (6) is a summand, so the exterior cube has a 6.

3. 6 + 3 or 6 + 3bar. Λ^3 6 is a summand and contains both 10 and 10bar: in 6 = Sym^2 C^3, with coordinates x, y, z, the highest-weight vectors

   `x^2 ^ xy ^ xz,  x^2 ^ xy ^ y^2`

   are killed by the raising operators x d/dy and y d/dz and have weights (3, 0) and (0, 3) respectively. Their dimensions already sum to C(6, 3) = 20, so Λ^3 6 = 10 + 10bar. The conjugate cases are analogous.

4. Two triplets T, T' (each 3 or 3bar) and three singlets. The cube contains T ⊗ T' ⊗ 1, hence T ⊗ T', which is 6 + 3bar, 6bar + 3 or 8 + 1: a 6, 6bar or 8.

5. Three triplets. The cube contains T ⊗ T' ⊗ T'', which contains a 10, 10bar, 15 or 15bar (3 ⊗ 3 ⊗ 3 contains 10, and 3 ⊗ 3 ⊗ 3bar contains 6 ⊗ 3bar, which contains 15; the others are conjugate).

6. One triplet and six singlets:

   ```
   Λ^3(3 + 6 (1)) = Λ^3 3 + 6 Λ^2 3 + 15 (3) + 20
   ↪  (1) = 15 (3) + 6 (3bar) + 21 (1),
   ```

   using Λ^3 3 = 1, Λ^2 3 = 3bar, C(6, 2) = 15 and C(6, 3) = 20. Its conjugate has the multiplicities of 3 and 3bar reversed.

7. The trivial representation 9 (1), with Λ^3 C^9 = 84 (1).

None gives either target: both targets contain equally many 3 and 3bar, as well as nonzero triplets, while cases 1 to 5 contain a 6, 6bar, 8, 10, 10bar, 15 or 15bar, case 6 and its conjugate have unequal numbers of 3 and 3bar, and case 7 has no triplet. Only cases 6 and 7 contain nothing beyond 1, 3 and 3bar, which is the "exactly" clause.

Scope and direction. This concerns the full exterior cube of a complex nine-dimensional linear SU(3) representation and the stated colour action on F1 and F2. Other dimensions, non-equivariant identifications, nonlinear or brane constructions, and maps that do not preserve the colour action remain possible directions.

Independent proof of Lemma 4.A and of the grades in Theorem 4.4. *From the referee report on the 11D line, section 'Lemma 4.A and Theorem 4.4: the two E8 gradings', transferred verbatim in substance on 4 Nov 2026.*

Let 9 = (3, 1)_{-2} ⊕ (1, 6)_1. The exterior-power expansion Λ^3(A ⊕ B) = Λ^3 A ⊕ (Λ^2 A ⊗ B) ⊕ (A ⊗ Λ^2 B) ⊕ Λ^3 B, with Λ^3 3 = 1, Λ^2 3 = 3bar, Λ^2 6 = 15 and Λ^3 6 = 20, gives

```
Λ^3 9 = (1, 1)_{-6} ⊕ (3bar, 6)_{-3} ⊕ (3, 15)_0 ⊕
↪  (1, 20)_3.
```

The adjoint, 9 ⊗ 9bar minus the trace, contains (8, 1)_0 ⊕ (1, 35)_0 ⊕ (1, 1)_0 ⊕ (3, 6bar)_{-3} ⊕ (3bar, 6)_3, and Λ^6 9 is the conjugate of Λ^3 9. The family centre gives grade a = 1 on 3, a = 2 on 3bar and a = 0 on singlets. Comparing the three columns (b = 0 on sl(9), b = 1 on Λ^3, b = 2 on Λ^6) yields

```
b == a + q/3 = a + 2 T_3   (mod 3),
```

including the singlet terms of charges ±6: for instance (1, 1)_{-6} has a + q/3 = -2 ≡ 1, and (3bar, 6)_{-3} has a + q/3 = 2 - 1 = 1. The grade-zero dimensions are 8 + 78 = 86, and the grade-one pieces form 3 ⊗ 27 (18 + 45 + 18 = 81), in agreement with the stated SU(6) × SU(2) ⊂ E6 branching.

Grades of the generation. Under SU(5), 15 = 10 + 5 has T_3 = 0. In the quoted E6 charge normalization, T_3 = (psi + chi)/8 gives +1/2 on 5bar_16 and N, and -1/2 on 5bar_10 and nu^c (for example (1 + 3)/8 on 5bar_3 ⊂ 16_1 and (1 - 5)/8 on 1_{-5} ⊂ 16_1); reversing the SU(2)_I Cartan exchanges the two signs. With t = +1/2, the left-handed 16 has grades 10_{b=1} ⊕ 5bar_{b=2} ⊕ 1_{b=0}. Adding its conjugate, with a = 2 and T_3 reversed, gives 15, 15 and 2 states per family in grades 1, 2 and 0, hence 45 + 45 + 6. With t = -1/2 the corresponding counts are 11, 11 and 10, hence 33 + 33 + 30.

Left-handed basis. Both t_L ∈ Q and t^c ∈ u^c lie in b = 1, and t^c is the left-handed conjugate of t_R. The right-handed top state lies in b = 2, and the antitop likewise has chiral components in both sectors. The phrase "every top state of a particle" in Theorem 4.4 therefore holds in the left-handed basis. The labels C_3 and C_6 express this algebraic grading; a physical M2/M5 identification requires the corresponding field construction.

Independent proof of Proposition 4.6, with the block argument. *From the referee report on the 11D line, section 'Proposition 4.6: the graded E8 exchange', transferred verbatim in substance on 4 Nov 2026.*

Dimensions. The branching table gives dim g_{1,1} = 45 and dim g_{1,2} = 18, so no linear automorphism maps the first whole space onto the second.

Chevalley involution. The Chevalley involution theta

sends each root space g_alpha to g_{-alpha}; consequently it sends g_{a,b} to g_{-a,-b}. It exchanges the two exterior-power sectors and preserves F1 and F2, which include each root together with its negative.

Real structure. Let kappa be coefficient conjugation in a real Chevalley basis. Since theta maps the real span of that basis to itself, theta and kappa commute, and J = theta kappa is antilinear with J^2 = 1. With a compatible positive Hermitian form it is antiunitary. On the placed 16 + 16bar states, the grading gamma with signs +1, -1 on family grades 1, 2 obeys J gamma = -gamma J, since J maps family grade a to -a. This establishes these two KO-6 signs for the constructed operators; the spectral-triple algebra, the order conditions and JD = DJ remain to be established. The complex-linear theta alone cannot be the antiunitary J.

The SU(2)_I reflection. It sends T_3 to -T_3, so b = a + 2 T_3 becomes b' = a - 2 T_3 = 2a - b (mod 3). It fixes b = 1 at a = 1 and exchanges b = 0 and b = 2, as stated.

Blocks. The dimension argument excludes an exchange of the whole joint-graded sectors. The exclusion of the top and neutrino blocks has an independent argument under the specified three-family embedding. In g_1 = 3_f ⊗ 27 their grade-one parts are

```
T ∩ g_1 = <e_t> ⊗ U_t,  dim U_t = 6;      N ∩ g_1 =
 ↪  3_f ⊗ U_nu,  dim U_nu = 2,
```

with e_t the top's family line, U_t ⊂ 27 spanned by the t_L and t^c states, and U_nu ⊂ 27 spanned by nu_L and nu^c. Both subspaces have dimension 6, so dimensions alone do not separate them. An automorphism phi preserving the family grading preserves g_0 = sl_3 ⊕ e_6 and its two nonisomorphic simple ideals, so it acts on g_0 as a pair (alpha, beta) of automorphisms of sl_3 and e_6. Its restriction to g_1 satisfies phi([x, v]) = [phi(x), phi(v)], so it is an isomorphism of g_1 onto g_1 with the g_0-action twisted by (alpha, beta). That twisted module is isomorphic to 3_f ⊗ 27 only when the twists preserve 3_f and 27, so there are A in GL(3) and B in GL(27) implementing alpha on 3_f and beta on 27; then (A ⊗ B)^{-1} phi is an endomorphism of the irreducible g_0-module 3_f ⊗ 27, a scalar by Schur's lemma. So phi

restricted to g_1 is a multiple of A ⊗ B. It sends a subspace supported on one family line to another subspace supported on one family line, and cannot give the neutrino subspace, whose family support is all of 3_f. The same argument applies when the family grading is inverted, with the conjugate module g_2.

Scope and direction. This concerns complex-linear automorphisms of the specified graded e8, the aligned 3_f ⊗ 16 embedding, and the top and neutrino blocks just described. Automorphisms that neither preserve nor invert the family grading, other embeddings or state spaces, and dualities not represented by such an automorphism remain open directions. A physical M2/M5 identification also requires its action on the fields and on the finite triple.

Independent proof of Proposition 4.5, with the family tensor. *From the referee report on the 11D line, section 'Proposition 4.5: Yukawa grades and family dependence', transferred verbatim in substance on 4 Nov 2026.*

The grade-selection rules and bracket targets hold as algebraic statements; by themselves they establish neither a physical top mass nor its observed family hierarchy.

Grades. The up-type triple Q u^c H_u has grades (1, 1) three times, so both Z_3 sums vanish. For the down-type, charged-lepton and Dirac-neutrino triples the second grades are {1, 1 + 2t, 1 - 2t} = {1, 2, 0}, with sum 3 ≡ 0, and the family grades are again 1, 1, 1, with sum zero. The bracket with H_u ∈ g_{1,1} adds (1, 1), so it sends grade b = 1 to b = 2, as claimed. For the neutrino it sends b = 2 to b = 0 when t = +1/2 (L lies at b = 1 + 2t = 2), and b = 0 to b = 1 when t = -1/2 (L lies at b = 0). These are necessary grade conditions; a particular bracket or four-dimensional coupling may still vanish.

The family tensor. In the literal 3_f ⊗ 27 realization with exact SU(3)_f invariance, the family part of a cubic involving three grade-one triplets is proportional to epsilon_{ijk}, the invariant of 3 ⊗ 3 ⊗ 3. A Higgs expectation value h_k therefore gives a family matrix proportional to

```
Y_{ij} = epsilon_{ijk} h_k.
```

In the aligned family basis its diagonal entries, including the nominal top-top entry, vanish. For nonzero h this matrix has rank two and two equal nonzero singular values, since, up to the family-index convention,

```
(Y^† Y)_{jl} = sum_{i,k,m} conj(epsilon_{ijk} h_k)
 ↪  epsilon_{ilm} h_m = ||h||^2 delta_{jl} - h_j
 ↪  conj(h_l),
```

that is Y^† Y = ||h||^2 I - h h^†, with eigenvalue ||h||^2 twice and 0 on h. Thus this exact one-Higgs family tensor alone gives no single isolated top mass.

The Z_3 family grading alone allows more tensors than epsilon_{ijk}; epsilon_{ijk} follows from the full SU(3)_f invariance. A compactification in which generation labels are cohomology labels and couplings are overlap integrals may evade the literal triplet-tensor restriction, but its mode functions and intersections must be supplied before the physical Yukawa matrix is identified.

Independent proof of the selection rule of section 8. *From the referee report on the 11D line, section 'File 04, section 8: the discrete grade', transferred verbatim in substance on 4 Nov 2026.*

The remark holds as a selection rule for the single element s_M, assuming it survives the breaking. For t = +1/2 the grades are b(10) = b(H_u) = 1 (Q, u^c, e^c and H_u), b(5bar_16) = b(N) = 2 (d^c, L and N), and b(H_d) = b(nu^c) = 0. Adding grades modulo three gives zero for

```
Q u^c H_u (1 + 1 + 1),  Q d^c H_d (1 + 2 + 0),  L
 ↪  e^c H_d (2 + 1 + 0),  L nu^c H_u (2 + 0 + 1),
N H_u H_d (2 + 1 + 0),  (nu^c)^2 (0),  L L H_u H_u
 ↪  (2 + 2 + 1 + 1),  L H_u (2 + 1),
```

and grade 2 for

```
H_u H_d (1 + 0),  Q Q Q L (1 + 1 + 1 + 2),  u^c u^c
 ↪  d^c e^c (1 + 1 + 2 + 1),
L L e^c (2 + 2 + 1),  L Q d^c (2 + 1 + 2),  u^c d^c
 ↪  d^c (1 + 2 + 2).
```

Additional unbroken symmetries may impose further restrictions. In particular, the family Z_3 has to be considered separately if it is also retained.

Part 34

The exchange of the top and the neutrinos as a colour-family transposition in E8

Lemma 34. Written as file 05 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, step 4). Attempted by Claude (Anthropic), 4 Nov 2026. Proposition 5.1 (a negative result in its scope) and Proposition 5.2 are proved by hand, refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md); an exploratory search in Python with exact fractions found the same results (no isometry for three families, signed coordinate permutations for four); the reruns of 4 Nov 2026 (Python, exact integers) are not kept, by the rule that checking scripts are not saved. Readings are labelled as proposals. Proposition 5.6, stated in Reading 3b (c) by Claude Sonnet 5.5, was proved and refereed by Claude Opus 5.5 (effort max, 5 Nov 2026; historical/reviews/2026-11-05_opus_check_05_21_34.md).

Referee note (GPT-6.1 Terra, 4 Nov 2026). Proposition 5.1 has an independent analytic check of its neighbour and pair counts, and of its no-isometry conclusion in the aligned-root scope. See the referee report. Proposition 5.4 is addressed at its statement below.

Setting

Referee note (GPT-6.1 Terra, 4 Nov 2026). The M2/M5 identification is a proposal, and the colour-family conclusion of Proposition 4.2 needs a factor-preserving map. See the wording audit.

Proposition 4.6 showed that the M2/M5 exchange of the graded e8 acts as J and cannot exchange the top with the

neutrinos, and Proposition 4.2 that any such exchange must trade the colour index of the top for the generation index of the neutrinos. This file tests the colour-family exchanges inside E8.

Use E8 ⊃ SO(16) ⊃ SO(10) × SO(6), with the roots of E8 in the coordinates of R^8: ±e_i ± e_j, and (1/2)(±1, …, ±1) with an even number of minus signs. The 128 half-integer roots form the chiral spinor of SO(16), which under SO(10) × SU(4) is (16, 4) + (16bar, 4bar): four families of the Standard Model with ν^c, with their conjugates. Write a root as (χ | w | φ), with χ the signs of the three colour coordinates, w those of the two weak coordinates and φ those of the three family coordinates; a particle has an even number of minus signs among χ, w and an even number among φ. The four family patterns are $\mathbb{1}$ = (+++) (family 0) and the three patterns with two minus signs (families 1, 2, 3); the colour patterns with one minus sign are the quark colours, the pattern $-\mathbb{1}$ is the lepton. Then, for a colour pattern c with one minus sign and a family f,

state	root (times 2)
u_L, d_L	(c | −+ | f), (c | +− | f)
u^c, d^c	(−c | ++ | f), (−c | −− | f)
ν_L, e_L	(−$\mathbb{1}$ | −+ | f), (−$\mathbb{1}$ | +− | f)
ν^c, e^c	($\mathbb{1}$ | ++ | f), ($\mathbb{1}$ | −− | f)

and antiparticles are the negatives. Three families give the 96 roots of H_F, closed under negation; the four family slots are permuted by the Weyl group of SU(4), so the choice of three is immaterial. The same 96 roots are (16, 3) + (16bar, 3bar) in E8 ⊃ E6 × SU(3) ⊃ SO(10) × U(1) × SU(3), the embedding of file 04.

Proposition 5.1 (the two families of 84 are not isometric)

Statement. Let H_F be the 96 roots of three families, the top block T the 12 roots of the up-type quark of one family f_t with its conjugate, and the neutrino block N the 12 roots of the neutrinos of the three families with their conjugates; F1 = H_F − T and F2 = H_F − N. Then F1 and F2 have the same numbers of pairs at each

inner product (42 at -2, 816 at -1, 1812 at 0, 816 at $+1$), but no isometry of R^8 maps F1 onto F2. In particular no element of the Weyl group of E8 does, and no isometry of H_F maps T onto N.

Proof. For a root v of a set S let d_S(v) be the number of roots of S at inner product $+1$ with v; it is preserved by isometries. Two half-integer roots have inner product $+1$ exactly when their signs agree in six of the eight places.

1. d_{H_F} = 22 for every root. H_F is a single orbit of the group generated by the Weyl group of SO(10) (transitive on the 16, a minuscule representation), the permutations of the three families and the negation. For ν^c of family 0, (𝟙 | ++ | 𝟙), the roots of H_F with exactly two minus signs are: two in the first five places with family 𝟙 (10), two among the family places (2, the patterns of families 1 and 2 in H_F), and antiparticles with one minus sign in each block (5 x 2). The total is 22.
2. Removing N. For u_L = (c | −+ | f): the inner product with ν_L(g) is $(1 + f \cdot g)/4$ and with ν^c(g) is $(1 + f \cdot g)/4$, where $f \cdot g = 3$ for g = f and -1 otherwise; so u_L has exactly 2 roots of N at $+1$ (its own family's ν_L and ν^c), and the same holds for u^c and the conjugates. For d_L = (c | +− | f): the inner product with ν_L(g) is $(-3 + f \cdot g)/4$, which is -1 for g ≠ f, so the two roots $-$ν_L(g), g ≠ f, are at $+1$, and ν^c(f) is at $+1$: 3 in all; the same count holds for d^c, e_L, e^c and the conjugates. So in F2, d = 20 on the 36 up-type quark roots and d = 19 on the 48 others.
3. Removing T. For a neutrino of the top's family, ν_L(f_t) = (−𝟙 | −+ | f_t): its inner products with t_L(c) = (c | −+ | f_t) and with t^c = (−c | ++ | f_t) are $(-1 + 2 + 3)/4 = 1$ and $(1 + 0 + 3)/4 = 1$, six roots at $+1$; for g ≠ f_t, ν_L(g) has inner product $(-1 + 2 - 1)/4 = 0$ with t_L and $(1 + 0 - 1)/4 = 0$ with t^c. The same holds for ν^c. So in F1, d = 16 on the 4 roots of the neutrinos of the top's family and d = 22 on the 8 roots of the other neutrinos (the up-type quarks of the other families have 2 roots of T at $+1$ and the down-type quarks and charged leptons 3, by inner products of the same kind as in item 2, so d = 20 and d = 19).

F1 has roots with d = 16 and d = 22, and F2 has none. The sums of d agree, $4 \cdot 16 + 48 \cdot 19 + 24 \cdot 20 + 8 \cdot 22 = 48 \cdot 19 + 36 \cdot 20 = 1632 = 2$ x 816, which is the equality of the pair counts at +1. Both sets are closed under negation, so the pairs at −1 are as many as those at +1 and the pairs at −2 are the 42 antipodal pairs; the 1812 pairs at 0 are the remainder of the 3486 pairs. ■

Scope. The roots of H_F in E8, through the embedding above, which is unique up to W(E8) for the aligned (16, 3) + (16bar, 3bar); isometries of R^8 acting on the roots. Escapes: maps that are not linear on the roots (bijections of states with no geometric meaning), other realizations of H_F in a larger group, and a fourth family (Proposition 5.2). The tau neutrino carries the obstruction: in F1 the neutrino of the top's own family is the only state whose neighbourhood changes by six.

Proposition 5.2 (four families: the transposition of colour and family)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The Weyl map and block exchange are independently checked. Family-zero lepton species blocks are fixed setwise; ν^c and e^c individually go to their negatives. See the referee report.

Statement. Let P be the permutation of coordinates that exchanges the three colour coordinates with the three family coordinates and the two weak coordinates with each other, and M = −P. Then M is an involution in the Weyl group of E8 that preserves the 128 roots of the four families and SU(2)_L, and acts on the up-type and down-type states as the transposition of the four Pati-Salam colours (r, g, b, lepton) with the four families (1, 2, 3, 0), with the lepton paired with family 0 and the colour c paired with the family −c. In particular:

1. the up-type quark of family 0 (t_L and t^c, three colours, with conjugates) goes onto the neutrinos of families 1, 2, 3 (ν_L and the conjugate of ν^c), and back;
2. the down-type quark of family 0 goes onto the charged leptons of families 1, 2, 3;
3. nu_L and e_L of family 0 are fixed, nu^c and e^c of family 0 go to their negatives, u^c and d^c go to conjugate roots, and the quarks of families 1, 2,

3 go to quarks of families 1, 2, 3 with colour and family transposed.

So the four-family sets 128 − T(0) and 128 − N(1, 2, 3), each of 116 roots, are exchanged.

Proof. P is an even permutation of coordinates and −1 is in the Weyl group of E8, so M is in it; M^2 = P^2 = 1. Both preserve the set of half-integer vectors with an even number of minus signs. P maps (χ | w_1 w_2 | φ) to (φ | w_2 w_1 | χ), so M maps it to (−φ | −w_2 −w_1 | −χ). The weak part (−+) goes to (−+) and (++) to (−−), so T_3 of SU(2)_L, (w_2 − w_1)/2, is preserved and T_3 of SU(2)_R is reversed. Then t_L(c) = (c | −+ | 𝟙) goes to (−𝟙 | −+ | −c) = ν_L(−c); t^c = (−c | ++ | 𝟙) goes to (−𝟙 | −− | c) = −ν^c(−c); b_L(c) = (c | +− | 𝟙) goes to (−𝟙 | +− | −c) = e_L(−c); ν_L(0) = (−𝟙 | −+ | 𝟙) and e_L(0) are fixed; u_L(c, f) = (c | −+ | f) goes to u_L(−f, −c), with −f a quark colour and −c a family pattern. ■

The search found further signed permutations with the same effect on the two blocks; P itself maps T(0) onto N(1, 2, 3) up to conjugation, and M = J ∘ P, with J = −1 the real structure of Proposition 4.6.

Proposition 5.3 (the flip in a four-family finite geometry)

Referee note (GPT-6.1 Terra, 4 Nov 2026). Choose the factor isomorphism ι unitary, or unitary up to a common scalar. An arbitrary isomorphism need not make F unitary. With this amendment the flip and block calculation are independently checked; see the report.

Statement. Let H_f = (C^2_L ⊕ C^2_R) ⊗ C^4_c ⊗ C^4_f, H = H_f ⊕ conj(H_f), with J the antilinear exchange of the two summands and γ equal to +1 on C^2_L and −1 on C^2_R in H_f and to its negative on conj(H_f), the signs of KO-dimension six. Fix a unitary isomorphism ι: C^4_c -> C^4_f sending the lepton line ℓ to the family line e_0 and the colour space C^3 onto the span of e_1, e_2, e_3, and let F be the flip x ⊗ y -> ι^{-1}(y) ⊗ ι(x) on C^4_c ⊗ C^4_f, extended by the identity on C^2_L ⊕ C^2_R and to conj(H_f) by J F J^{-1}. Then:

1. F is unitary, F^2 = 1, and F commutes with γ and J.
2. For the up-type Dirac blocks T_top = Y ⊗ P_q ⊗ P_0 and T_ν = Y ⊗ P_ℓ ⊗ (1 − P_0), where Y maps the up state of C^2_L to the up state of C^2_R (and Y^* back), P_ℓ and P_q = 1 − P_ℓ are the lepton and quark projectors of C^4_c and P_0 the projector on e_0, one has F T_top F^{-1} = T_ν, and likewise for the down-type blocks of family 0 and the charged leptons of families 1, 2, 3.
3. So the heavy limits D + t T_top and D + t T_ν, for D commuting with F, are conjugate by F: their light spaces, of 116 states each, correspond. The light algebras A ∩ {T}' correspond when the algebra is invariant under F; the Pati-Salam and Standard Model algebras act on C^4_c alone and are not.

Proof. F acts on the factor C^4_c ⊗ C^4_f only, where it conjugates a ⊗ b to ι^{-1} b ι ⊗ ι a ι^{-1}, so P_q ⊗ P_0 goes to P_ℓ ⊗ (1 − P_0) by the choice of ι. γ and Y act on the other factor, and J F J^{-1} on conj(H_f) is the conjugate flip. ■

F equals M on the left-handed states and M followed by the particle-antiparticle exchange on the right-handed ones (second check): F exchanges T_top and T_nu inside H_f, while M sends right-handed states to antiparticle roots and so flips gamma on 64 states; F lies outside W(E8). It also makes reading 2 below precise: F exchanges the projector P_ℓ, which the Majorana direction selects in colour space, with P_0, which the top Yukawa coupling selects in family space; as operators, it maps the top Yukawa block to the Dirac blocks of the neutrinos of families 1, 2, 3, and it does not map the Majorana block to a Dirac block, since the Majorana block joins H_f to conj(H_f).

Proposition 5.4 (no three-slot assignment admits the exchange)

Referee note (GPT-6.1 Terra). The no-isometry conclusion is correct by an independent analytic argument, but the neighbour counts in the statement below are false for misaligned slots. The corrected formulas and a coun-

terexample are in the referee report.

Statement. Place the top in slot 0, and let the up-type quarks, the down-type quarks, the charged leptons and the neutrinos each occupy three of the four family slots, independently (the up-type quarks in slots containing 0). This includes the aligned families of Proposition 5.1 and the configurations of the 9 label, where SU(2)_L is off and does not tie the slots of up-type and down-type quarks. For none of the 192 configurations S is there an isometry of R^8 mapping S − T onto S − N. S − N has d in {13, 17, 19, 20}, and {19, 20} only when the down-quark and charged-lepton slot sets coincide (48 of the 192); each neutrino root of S − T has d = 16 + 6 (1_A(f) − 1_0(f)), with A the slot set of the up-type quarks, so the values are 16 and 22 in the patterns (16, 22, 22) and (16, 16, 22).

Proof. Exhaustive computation of the neighbour counts d (Python, exact integers, rerun twice on 4 Nov 2026 and not kept; also the analytic proof in the referee report); the counts of S − T and S − N differ for every configuration, so no isometry exists. ■ This is a computer-assisted statement; Proposition 5.1 is its hand-proved aligned case.

Correction (4 Nov 2026). The conclusion holds on rerun; the sentence of the first version of the Statement, "In every case S - N has d = 19 and d = 20 only, while S - T has d = 16 on the neutrino roots of one slot and d = 22 on those of the two others", was wrong for misaligned configurations and has been removed. See 10, section on corrections.

Scope. It assumes that each family occupies one slot, so that every state is a single root. Four slots then cannot be reduced to three while keeping the exchange as an isometry of the roots.

Proposal 5.5 (four slots, three zero modes)

Referee note (GPT-6.1 Terra, 4 Nov 2026). General bundle zero modes are outside the root-slot hypotheses. The index gives net chirality, and split bundles do not automatically reduce to three distinct root slots; the Weyl action on a specified bundle and its zero modes remains open. See the referee report.

In heterotic compactifications on a Calabi-Yau threefold

with E8 ⊃ SO(10) × SU(4), 248 = (45, 1) + (1, 15) + (16, 4) + (16bar, 4bar) + (10, 6), the SU(4) is the structure group of a rank-four bundle V, and the families are the zero modes, the 16s coming from H^1(X, V); their net number is the index |c_3(V)|/2, which is 3 in the heterotic Standard Models built on SU(4) bundles (Braun, He, Ovrut and Pantev, hep-th/0501070; recalled without rereading). In that reading the four slots of this file are the fibre of V, and each family is a zero mode, a section of V that in general involves all four slots. Propositions 5.1 and 5.4 do not apply to it, since they assume one slot per family.

The transposition M of Proposition 5.2 then exchanges the Pati-Salam SU(4), a gauge group, with the structure group SU(4) of V: colour and holonomy trade places, and SO(10) goes to the commutant SO(10)' of the other SU(4). The top-neutrino exchange becomes a statement about the fibre, which an index map must carry to the zero modes. The test is an explicit bundle. For a split bundle V = L_0 + L_1 + L_2 + L_3 (line-bundle models), slot i carries n_i = −χ(L_i) families with n_0 + … + n_3 = 3; such models fall under Proposition 5.1 only for n = (1, 1, 1, 0) up to order; for every n_i >= 0 with sum 3 the degree multisets still differ (second check, 40 cases), so the obstruction persists for aligned split bundles. A non-split V, whose zero modes are spread over the slots, is the case to compute.

Proposition 5.6 (the so(9) of the Chevalley involution is conjugate to the Spin(9) of Reading 3)

Let h be a Cartan subalgebra of e8 contained in the regular sl(9) of lemma 33, section 3, and let theta be the Chevalley involution of e8 for h: in a Chevalley basis, theta acts as -1 on h and maps e_alpha to -e_{-alpha}. Let sigma be the involution of the root table of this lemma, +1 on h and on the integer roots and -1 on the half-integer roots; its fixed algebra is so(16)_sigma and its -1 eigenspace is the 128 of the four families. Then:

1. theta preserves sl(9) and acts on it by an outer involution, whose fixed algebra so(9)_theta is conjugate in SL(9) to the so(9) of lemma 33.
2. The fixed algebra k_theta of theta on e8 is isomorphic to so(16). As so(9)_theta-modules, k_theta

= 36 + 84 and p_theta = 44 + 84, where p_theta is the -1 eigenspace. The 44 equals p_theta ∩ sl(9) and lies in the gravity sector. The two 84 are the graphs {x + theta x} and {x - theta x}, x in Λ^3 C^9, and each meets the C_3 and C_6 sectors equally.

3. so(9)_theta acts on the 16-dimensional vector of k_theta by the spinor.
4. so(9)_theta is conjugate in E8 to the Spin(9) of Reading 3, the image of Spin(9) under the spinor 16 inside so(16)_sigma, in the class with 128 -> 44 + 84. Under both, 248 = 36 + 44 + 84 + 84.
5. In E8 both connected subgroups are isomorphic to SO(9). The other spinor class, with an irreducible 128 and 248 = 36 + 84 + 128, is isomorphic to Spin(9).

Proof. (1) theta maps the root system A_8 to itself, since it acts as -1 on h, so it preserves sl(9). On sl(9) it acts as -1 on a Cartan subalgebra, and -1 lies outside W(A_8), so the involution is outer. Since 9 is odd, every outer involution of sl(9) has the form X -> -B^{-1} X^T B with B symmetric; its fixed algebra so(B) has dimension 36 and is conjugate in SL(9) to the so(9) of lemma 33. The conjugating element lies in SL(9), which commutes with the centre of SU(9), so it preserves the three sectors.

(2) The fixed algebra of theta is spanned by the e_alpha - e_{-alpha}, so it has dimension 120, the number of positive roots; it is the complexified maximal compact subalgebra of the split form e8(8), which is so(16) (Cartan's classification; Helgason, chapter X). On sl(9), theta has fixed part so(9)_theta (36) and -1 part 44, which is h plus the span of the e_alpha + e_{-alpha} for alpha a positive root of A_8. theta maps Λ^3 to Λ^6. For Y in so(9)_theta, theta [Y, x] = [Y, theta x], so the maps x -> x ± theta x from Λ^3 are so(9)_theta-equivariant and injective, and their images are the fixed and -1 parts of Λ^3 + Λ^6, each isomorphic to Λ^3 C^9 = 84.

(3) As k_theta-modules, k_theta = Λ^2 W with W the vector 16, so Λ^2(W restricted to so(9)_theta) = 36 + 84. The 16-dimensional representations

of so(9) are 16, $9 + 7 \cdot 1$ and $16 \cdot 1$, with exterior squares $36 + 84$, $36 + 7 \cdot 9 + 21 \cdot 1$ and $120 \cdot 1$. Only the spinor fits.

(4) Complex e8 has two conjugacy classes of involutions, with fixed algebras so(16) (dimension 120) and e7 + sl(2) (dimension 136) (Cartan; Kac's classification of automorphisms of finite order). So there is g in E8 with g theta g^{-1} = sigma; g maps k_theta to so(16)_sigma and p_theta to the 128, and g(so(9)_theta) is a spin(9) in so(16)_sigma, acting on the 16 by the spinor by (3) and on the 128 as 44 + 84 by (2). Let rho_1 and rho_2 be two embeddings of spin(9) in so(16)_sigma through the spinor, both with 128 -> 44 + 84. The spinor is the only 16-dimensional irreducible representation and its invariant symmetric form is unique up to scale, so rho_2 = Ad(A) rho_1, up to an inner automorphism of so(9), for some A in O(16). If det A = -1, then Ad(A) exchanges the two half-spinors, and rho_2 would act on the 128 through the other half-spinor of rho_1, which is the irreducible 128. So A lies in SO(16), and the image in E8 of either lift of A to Spin(16) conjugates rho_1 to rho_2. Hence the class with 128 -> 44 + 84 is a single class under the subgroup Spin(16)/Z_2 of E8, and g(so(9)_theta) lies in it.

(5) The modules 36, 44 and 84 are tensor representations, on which -1 of Spin(9) acts trivially, so in the class with $248 = 36 + 44 + 84 + 84$ the connected subgroup of E8 is SO(9), of trivial centre. The 128 has half-integer weights, on which -1 acts as -1, so in the other class the subgroup is Spin(9). ■

The statement is the one recorded by Claude Sonnet 5.5 on 4 November 2026 in Reading 3b (c); the proof is by Claude Opus 5.5 as referee (5 November 2026, historical/reviews/2026-11-05_opus_check_05_21_34.md), who also computed, from weight multisets with exact fractions, the restrictions $80 = 36 + 44$ and $\Lambda^3 = 84$ for the so(9) of sl(9), and $120 = 36 + 84$ with the two half-spinors $44 + 84$ and 128 for the spinor Spin(9) in SO(16); both classes have Dynkin embedding index 2, and the vector class so(9) $\subset$ so(16) through $9 + 7 \cdot 1$ has index 1 and $248 = 36 + 21 \cdot 1 + 7 \cdot 9 + 8 \cdot 16$.

The proof needs no claim that the decomposition 36 + 44 + 84 + 84 by itself is a single conjugacy class.

Readings (proposals)

1. **Three families are the obstruction.** The exchange of lemma 21 is a symmetry of the E8 geometry exactly in the four-family completion, where the colour-family square is 4 x 4 and the top belongs to the family paired with the lepton colour. With three families in single slots the square is 4 x 3 and the exchange is not an isometry (Propositions 5.1 and 5.4, in their scope of one root per state). Lemma 21's 96 = 84 + 12 in two ways is the three-family shadow of 128 = 116 + 12 in two ways, exchanged by M.
2. **The two heavy directions of the algebra chain.** Canonical §2.2 and §3 obtain the chain Pati-Salam -> Standard Model -> colour plus electromagnetism from two heavy operators: the Majorana direction, which selects the lepton among the four Pati-Salam colours and ν_R among the SU(2)_R states, and the top Yukawa coupling, which selects the heavy family and the up-type state. M exchanges the lepton colour with family 0 and keeps the up-type direction (F, its finite-geometry counterpart, differs from it on right-handed states): it exchanges the direction selected by the first heavy operator with the direction selected by the second. If the 11 label is the grand algebra M_2(H) + M_4(C) (lemma 28) and the 9 label colour plus electromagnetism, M exchanges the two steps of 11 -> 10 -> 9. This needs a finite geometry in which the generation space is C^4 with an algebra exchangeable with M_4(C), and a Dirac operator compatible with the exchange.
3. **Four families and eleven dimensions.** The 128 of the four families is the chiral spinor of SO(16) ⊂ E8; under Spin(9) ⊂ SO(16), embedded through its spinor, it decomposes as 44 + 84 in the class with 128 -> 44 + 84 (the other chirality of the spinor embedding gives an irreducible 128), the graviton and the three-form of eleven-dimensional supergravity (Pengpan and Ramond, hep-th/9808190; Kostant's cubic Dirac operator on F4/Spin(9); recalled). The

two families of 116 are not Spin(9) subrepresentations; whether the exchange M has a Spin(9) meaning is open.

3b. **The arithmetic 44 = 32 + 12 and the two placements of the top (observation of the user, 4 Nov 2026).** Observation with an open question; nothing beyond (a) to (d) is claimed.

(a) With four families, 128 = 44 + 84 under the Spin(9) of Reading 3, and 128 − 84 = 44 = 32 + 12: removing F1 (three families minus the top, 84 states) leaves the fourth family plus the top block, and removing F2 (three families minus the neutrinos) leaves it plus the neutrino block. In this count the top goes with the fourth family into the graviton number 44, and dually the neutrinos do: the four-family completion of the 96 = 84 + 12 of lemma 21 in two ways. M of Proposition 5.2 exchanges the two blocks of 12 that fill the 44, T(0) and N(1, 2, 3); it sends each family to the states of one colour (family 0 to the leptons), so it does not carry one set of 44 onto the other.

(b) The fit is numerical in the present scope. Proposition 3.8 excludes a colour-equivariant identification of Λ^3 R^9 with either family of 84 for any linear action of colour on the 9. The colour SU(3) lies in no such Spin(9) (Claude Opus 5.5, 4 Nov 2026; checked by Claude Sonnet 5.5): the vector 16 of SO(16) is 3 + 3bar + 10 (1) under colour, while the spinor 16 of Spin(9) restricted to an SU(3) inside Spin(9) is 16 (1), 2 (3 + 3bar) + 4 (1) or 8 + 8, by the three real nine-dimensional representations of Proposition 3.8. The stabilizer in so(16) of the split 44 + 84 is spin(9) itself (the intermediate subalgebras would be submodules of 36 + 84, and so(16) is irreducible on the 128), so the split is not colour-covariant; and no set of 84 roots spans the 84, since a root partition has the rank-8 Cartan subalgebra in its stabilizer. Also, colour has 248 = 8 + 78 (1) + 27 (3 + 3bar), while the 44 of any so(9) of e8 with 248 = 36 + 44 + 84 + 84 contains a 6 or a 27, so colour lies in no such so(9), including that of file 04. Scope: the regular colour SU(3) of the root table, in this class. Escapes: other colour em-

beddings; identifications that are not equivariant. A strengthening (Sonnet, exact integers, not kept): no isometry of R^8 maps the set (fourth family + top) onto (fourth family + neutrinos), since up to family permutations their degree multisets differ, {10^16, 12^12, 13^16} against {10^24, 13^16, 16^4}.

(c) The record holds two placements of the top, from different subgroups of E8, and they differ: under the Z_3 grading by SU(9) of file 04 (Lemma 4.A, Theorem 4.4) the left-handed top lies in the C_3 sector and ν^c in the gravity sector (t = +1/2, proposed in section 7 there); under the Spin(9) count above the top goes with the fourth family into the 44. The two placements use two frames for conjugate subgroups. The Spin(9) of Reading 3 is conjugate in E8 to the SO(9) of SU(9)/Z_3 (Proposition 5.6), and inside E8 both are isomorphic to SO(9), with Spin(16)/Z_2 in place of SO(16). In the four-family frame the 128 is spanned by the half-integer root vectors, and the count 44 = 32 + 12 is numerical. For the theta-fixed so(9), every root vector e_beta of H_F splits as (e_beta + theta e_beta)/2 + (e_beta - theta e_beta)/2, with the first component in so(16)_theta and the second in the 128 of theta. So there is no contradiction, and no identification between the sectors is claimed. The two groups themselves are related: the Chevalley involution theta restricted to sl(9) is an outer involution with fixed algebra the so(9) of file 04, which is conjugate in E8 to the Spin(9) of this reading, with k = so(9) + 84 and p = 44 + 84. The 44 = p ∩ sl(9) lies inside the gravity sector, and the two 84 are the graphs {x ± theta x : x in Λ^3 C^9}, each meeting the C_3 and C_6 sectors (Proposition 5.6; stated by Claude Sonnet 5.5 on 4 Nov 2026, proved by Claude Opus 5.5 on 5 Nov 2026). How the 96 states of the three-family frame sit against the conjugated Spin(9) has not been worked out.

(d) Recalled analogy: Gell-Mann's colour-flavour SU(8) of N = 8 supergravity put the spin-1/2 fermions in Λ^3(8) = 56 with 8 = 3 + 5 (Gell-Mann, "From renormalizability to calculability?",

1983, and the SU(8) of Cremmer and Julia; recalled without rereading); here it is Λ^3 of 9 = 84 with the C-field, and Proposition 4.3 plays the role of the known failure of that content.

4. **NCG.** In the finite geometry with three generations the exchange is not an automorphism (Proposition 4.2). A Pati-Salam finite geometry with C^4_colour ⊗ C^4_family, where the flip of the two factors exchanges the σ field with the top Yukawa direction, is the candidate in which it is (PLAN, step 5).

What remains open

- A finite spectral triple, or a Hořava-Witten configuration, with four families in which M is a symmetry and the fourth family is decoupled in a controlled way, so that the three-family statement of lemma 21 follows as a limit.
- Whether the fourth family can be made heavy by the same mechanism as the top (lemma 21 counts only one heavy up-type quark).
- Where the 96 states sit against the theta-fixed so(9) (lead from the referee of Proposition 5.6): the component of a root vector e_beta in the 128 of theta is (e_beta + e_{-beta})/2, in the 44 when beta lies in the gravity sector and in the 84 otherwise, so with the counts of lemma 33, Theorem 4.4, the projection of H_F to the 128 of theta has 3 dimensions in the 44 (the nu^c) and 45 in the 84, the top among them, for t = +1/2 (15 and 33 for t = -1/2): for this so(9) the top meets only the three-form 84, which agrees with lemma 33, section 7, and differs from the numerical count of Reading 3b (a). sigma and theta commute, with joint fixed algebra so(8) + so(8) and three further joint eigenspaces of dimension 64, the decomposition 248 = (28, 1) + (1, 28) + (8v, 8v) + (8s, 8s) + (8c, 8c), in which the 96 states of p_sigma split 48 + 48 between k_theta and p_theta; this Spin(8) x Spin(8) frame is where the question can be computed, after transport by an explicit g with g theta g^{-1} = sigma.
- The global form: inside E8 the class with 128 -> 44 + 84 is SO(9) and the class with the irreducible 128 is Spin(9) (Proposition 5.6, item 5), so the graviton

and three-form class acts through SO(9) and a gravitino class needs the spin cover; where this lemma writes SO(16) ⊂ E8 the subgroup is Spin(16)/Z_2, and the Z_2 decides which representations of the 96 states can occur when the two placements of the top are related to brane charges.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent check of Proposition 5.1 (the neighbour table). *From the referee report on the 11D line, section 'Proposition 5.1: aligned three-family root configurations', transferred verbatim in substance on 4 Nov 2026.*

Write a half-integer root as (s | f)/2 with five colour and weak signs s and a family sign pattern f. Distinct family patterns have inner product -1 on the family part and identical ones have 3. For two particle roots in the same slot, inner product +1 means that the five-sign patterns have Hamming distance two. For distinct slots, a particle root is a +1 neighbour exactly when the five-sign patterns are equal, and an antiparticle root is a +1 neighbour exactly when the five-sign patterns have distance four; no antiparticle in the same slot is a +1 neighbour. Let U, D, E, N denote the up-quark, down-quark, charged-lepton and neutrino blocks of one slot, each with its antiparticles. The number of +1 neighbours of one root of the row block among all roots of a column block, for the same slot and for a different slot, is

source	U	D	E	N
U	4 / 2	3 / 3	1 / 1	2 / 0
D	3 / 3	4 / 2	2 / 0	1 / 1
E	3 / 3	6 / 0	0 / 2	1 / 1
N	6 / 0	3 / 3	1 / 1	0 / 2

The entries follow from the root types and colour choices of the Setting and the Hamming criterion; for instance

a neutrino has six same-slot up-quark neighbours and none in another slot, and a charged lepton has six same-slot down-quark neighbours and none in another slot. In three aligned slots each row sums to 10 + 2 x 6 = 22, the degree in H_F. Removing all three neutrino blocks gives degree 20 on the 36 up-quark roots and 19 on the 48 down-quark and charged-lepton roots. Removing the top block instead gives degree 22 - 6 = 16 to the neutrinos of its slot, 22 to the other eight neutrino roots, 20 to the 24 remaining up-quark roots and 19 to the 48 others. So F1 has degrees 16 and 22 that F2 lacks; both sets are closed under negation with centroid zero, so an affine Euclidean isometry between them would fix the origin and preserve degrees, and none exists. The degree sums 4(16) + 48(19) + 24(20) + 8(22) = 48(19) + 36(20) = 1632 give 816 unordered pairs at inner product +1 in each set, 816 at -1 by negation, 42 antipodal pairs at -2, and the remaining C(84, 2) - 42 - 2(816) = 1812 pairs at inner product zero. Scope: the indicated half-root embedding, three aligned slots, complete species blocks and Euclidean isometries.

Independent check of Proposition 5.2. *From the section 'Proposition 5.2: four-family colour-family transposition'.*

P exchanges three pairs of colour and family coordinates and the two weak coordinates; negating all eight coordinates is an even sign change, and signed permutations with an even number of sign changes preserve the D_8 roots and lie in W(D_8), a subgroup of W(E8). So M = -P is in W(E8), M^2 = 1, and M preserves the half-integer roots with an even number of minus signs. For a quark colour pattern c and the all-plus family pattern 1,

```
M(c | -+ | 1) = (-1 | -+ | -c) = nu_L(-c),     M(-c
 ↪  | ++ | 1) = (-1 | -- | c) = -nu^c(-c),
```

and the three patterns -c are the other three family patterns; with negatives this is the whole neutrino block, and M^2 = 1 gives the reverse map. The weak root e_4 - e_5 is fixed, so the SU(2)_L subalgebra is preserved; the same calculation exchanges the down quarks of family zero with the charged leptons. For a family-zero lepton, M(nu_L) = nu_L but M(nu^c) = -nu^c, and likewise for the charged lepton: the Dirac species blocks of family zero are invariant setwise, and their roots are fixed only

up to sign.

Counterexample (referee) to Proposition 5.3 without unitarity. *From the section 'Proposition 5.3: finite-geometry flip'.*

The factor identification iota must be unitary, or unitary up to one nonzero scalar, for the flip F to be unitary. With orthonormal bases, iota(l) = 2 e_0 and the three colour basis vectors sent isometrically to e_1, e_2, e_3 give F(l (x) e_1) = 2 e_1 (x) e_0, which changes the norm while F stays an involution. With unitary iota, F is unitary and involutive, F(P_q (x) P_0)F^{-1} = P_l (x) (1 - P_0), F commutes with gamma (which acts on the chirality factor) and with J by its conjugate extension to the second summand; the two Dirac blocks have rank 12 (two chiralities, three colour or family states, two conjugate summands) and kernels of dimension 128 - 12 = 116. If a fixed self-adjoint D commutes with F, the families D + t T_top and D + t T_nu are conjugate for every t, with a bounded spectral sector of dimension 116 in the heavy limit; the commutant algebras are conjugate when the represented algebra is itself flip-invariant, which the displayed Standard Model and Pati-Salam colour algebras acting on the colour factor alone are not.

Independent proof of Proposition 5.4 for all 192 assignments, without enumeration. *From the section 'Proposition 5.4: misaligned three-slot configurations'.*

Let A, B, C, D be the three-slot sets of the up quarks, down quarks, charged leptons and neutrinos, with 0 in A the slot of the top. The neighbour table above applies to every slot assignment. For a root in slot f, after removing all neutrinos,

source	d_{S - N}
U, f in A	20
D, f in B	17 + 2 1_C(f)
E, f in C	13 + 6 1_B(f)

(for a down quark, 3|A| + [4 + 2(|B| - 1)] + 2 1_C(f) = 17 + 2 1_C(f)). So S - N has degrees in {13, 17, 19, 20}, and 16 and 22 are absent. After removing the top, every neutrino root in slot f in D has

```
d_{S - T} = 16 + 6 (1_A(f) - 1_{{0}}(f))  in {16,
↪  22}:
```

the three down-quark slots contribute 9, the three charged-lepton slots 3, the two other neutrino slots 4, and the up quarks six exactly when f is in A, less six for the removed top when f = 0. Both sets are closed under negation, so their centroids are zero, and an isometry between them would preserve these degrees, which is impossible. This covers all 3 x 4^3 = 192 choices. The universal counts stated in the first version of the proposition are false when slots are misaligned: for A = C = {0, 1, 2} and B = D = {0, 1, 3}, the down-quark roots in slot 3 have degree 17 and the charged-lepton roots in slot 2 have degree 13 in S - N, and the neutrino degrees in slots 0, 1, 3 of S - T are 16, 22, 16; so S - N has degrees other than 19 and 20, and more than one neutrino slot can have degree 16. Scope: the specific half-root realization, whole colour multiplets, one root slot per species and Euclidean isometries; general zero modes of a non-split SU(4) bundle and maps without an isometric root action are outside it (Proposal 5.5).

Independent proof of the index statement in Proposal 5.5, with the net-chirality caution. *From the referee report on the 11D line, section 'Proposal 5.5: four-slot bundles and three families', transferred verbatim in substance on 4 Nov 2026.*

Verdict of the referee: correct with amendment as a possible escape. General bundle zero modes are sections, and the isometry hypotheses of Propositions 5.1 and 5.4, which concern individual fibre roots, do not automatically apply to them. The index counts net chirality, and the proposed Weyl map on zero modes remains to be constructed.

Index. On a Calabi-Yau threefold X, for a rank-four SU(4) bundle V, c_1(V) = 0 gives ch(V) = 4 - c_2(V) + c_3(V)/2, and c_1(X) = 0 gives Td(X) = 1 + c_2(X)/12. Riemann-Roch gives

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chi(V) = int_X ch(V) Td(X) = (1/2) int_X c_3(V).
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When H^0(V) = H^3(V) = 0, this is -h^1(V) + h^2(V), with h^2(V) = h^1(V^*) by Serre duality, since K_X is trivial. Thus index three gives three net chiral families. Exactly three 16 zero modes additionally requires the

opposite-chirality cohomology H^1(V^*) to vanish.

Split bundles. For a split bundle V = L_0 + L_1 + L_2 + L_3, the integers n_i = -chi(L_i) are signed indices; their sum being three does not force one zero mode in each of three distinct slots. Even in that special distribution, applying Proposition 5.1 to normalized global sections requires an identification of the physical zero-mode metric with the root configuration.

The Weyl map. The fibre Weyl transposition M can exchange the abstract subgroups, but a symmetry of a particular compactification must also act compatibly on the bundle, the connection, the zero-mode spaces, their inner products and any Wilson-line data. Establishing such an action is an open construction; the escape from the root no-go alone does not establish it.

Part 36

The type I' dilaton as a spectral function, the SO(32) of the EPJC paper, and two more dualities

Lemma 36. Written as file 09 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, task C). Attempted by Claude (Anthropic), 4 Nov 2026, after the steer of the same day (the coincident-brane route is overused; the type I' dilaton and the colour-flavour idea are worth pursuing; compare with the SO(32) of the EPJC paper). Proposition 9.1 is proved from the recalled type I' slope rule; Proposition 9.2 is proved from representation theory and the recalled orientifold projections; Proposition 9.3 is proved (linear algebra and Connes's sign table); the comparison of section 1 is read from the arXiv text of the paper through a summarizing fetch; the rest are proposals. Refereed by

GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

Correction (same day): the first version stated the slope rule for e^{−phi}. The piecewise-linear function is z, with e^{phi} = z^{−5/6}; Proposition 9.1 holds as stated with z in place of e^{−phi}, and infinite coupling is z = 0.

1. The SO(32) of A. Rivero, Eur. Phys. J. C 84 (2024) 1058 (arXiv:2407.05397)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The branching algebra is independently checked. The archived original abstract also states an additional chiral charge-+4/3 quark per generation. The full paper was unavailable for direct comparison, so its detailed attribution remains pending. See the referee report.

That paper uses the adjoint of SO(32) to classify the scalars of a supersymmetric Standard Model with three generations, as composites: pairs of quarks as colour triplets and quark-antiquark pairs as singlets and octets, with three extra scalars of charge +4/3. Its pattern is SO(2n) ⊃ SO(n_1) x SU(n_2) x SU(3) x U(1) with 2n = n_1 + 6 n_2, which for SO(32) and n_2 = 5 gives SO(2) x SU(5) x SU(3) x U(1): the 30 = 6 n_2 are the five light quarks u, c, d, s, b ("turtles") in three colours with conjugates, and the top is the non-participating "elephant". Its explicit route also passes through SO(32) ⊃ SU(16) x U(1), with 496 = 1 + 120 + 120bar + 255 (quarks from the 120s, leptons from the 255), and it mentions open strings, orientifolds and an SU(9) for M2 and M5.

Reading of the LaTeX source (Claude Sonnet 5.5, 4 Nov 2026; the earlier comparison used a summarizing fetch). The paper uses colour singlets for mesons and colour antitriplets for diquarks, and keeps only (1, 24, 1), (1, 15, 3bar) and (1, 15bar, 3) of its decomposition; colour octets and sextets lie outside it. The top is the favoured non-binding quark ("we have not a concrete argument"), and the postulates force the non-binding quark to be up-type. The chain is SU(16) x U(1) to SU(15) to SU(5) x SU(3), with SO(30) and SO(32)

reached by "adding singlets". The identification of the top with the SO(2) factor, the O8⁀- plane and the Chan-Paton reading belong to these files, and "orientifold" in the paper means planar orientifold QCD. The footnote on the 84 + 12 split and SU(9) for M2 and M5 anticipates Proposition 21.3 and 04. The abstract specifies a chiral +4/3 quark per generation; the body has three scalars of charge +4/3 and leaves their fermion partner open. The paper gives (15, 3bar) as the antisymmetric square of one fundamental without stating chirality, so the question of Proposition 9.3 stays open; its remark on odd-parity diquarks is consistent with that proposition. The paper's selection of an up-type quark by its postulates and lemma 21's by the spectral-action values are independent arguments with the same outcome.

Coincidence. The chain SO(32) ⊃ SO(2) x SO(30) ⊃ SO(2) x SU(15) x U(1) ⊃ SO(2) x SU(5) x SU(3) x U(1), the 30 as five light quarks in three colours, and the top as the factor SO(2), are the same as lemma 21 and the dictionary of 06. Lemma 21 records the branching 496 -> (435, 1) + (1, 1) + (30, 2) and leaves SO(32) unexplored; the EPJC paper explores it through composites, and the files 06 to 09 through type I' branes. What the 11D files add: the type I' realization of the SO(2) as the top's D8-brane and of the 30 as the Chan-Paton labels of fifteen D8-branes on an O8⁀- wall (06); the top angle as the brane's position and the one-loop vacuum on the far wall (Propositions 7.1, 7.2); the orientifold type O⁀- matched by the seesaw sign (Proposition 8.1); the far wall's enhancement to SU(2) x U(1) at infinite coupling (Remark 8.3); the dilaton (Proposition 9.1); the diquark counts as strings between the stack and its image, which need the O⁀- type (Proposition 9.2); and the real-structure condition under which these counts arise, as connection components along the interval, outside the finite inner fluctuations (Proposition 9.3). In type I' language the paper's other chain, SU(16) x U(1), is the configuration with all sixteen D8-branes coincident in the bulk, and the chain with SO(2) x SO(30) is the configuration with fifteen on a wall and one apart; the second is the one the corpus selects. The SU(9) of M2 and M5 in the paper is the grading of e8 used in 04.

2. Proposition 9.1 (the type I' coupling function is a trace of a function of the finite Dirac operator)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The integration is independently checked conditional on the recalled slope rule. The trace is over sixteen selected channels, and the full physical D_F Hilbert space enters only through a specified projection onto them; direct comparison with the cited source was unavailable to the referee. (Second check, against Polchinski-Witten.) The formula z(L) = z(0) - c (8L - sum x_k) is exact in the positions; the dictionary x_k = (2L/pi) l m_k holds only for constant z, and in general theta_k = (pi/2) W(x_k)/W(L) with W = int z^{-1/3} (10, Proposition 10.1). In the profile 15 + 1 the endpoint values 7cL and 8cL stand, and theta_t = pi/4 gives 7.358 cL in place of 7.5 cL. See the referee report.

Setting. Type I' on an interval [0, L], in the conformal coordinate x, with sixteen D8-branes at x_1, ..., x_16 and O8^- planes at the ends. Recalled rule (Polchinski and Witten, hep-th/9510169): the dilaton is e^{phi} = z^{−5/6} for a function z that is continuous and piecewise linear in x, with slope c (8 − N(x)), where N(x) is the number of D8-branes to the left of x and c > 0; the coupling diverges where z vanishes. Dictionary (06, 07): x_k = (2L/pi) θ_k with θ_k = l m_k the angles of sixteen channels of the finite Dirac operator (fifteen light-quark channels with colour, and the top).

Statement.

1. z(x) = z(0) + c Σ_k min(x/2, x_k − x/2) for x in [0, L], so z is z(0) plus c Tr g_x(l |D_F|) over the sixteen channels, with g_x(θ) = min(x/2, (2L/pi) θ − x/2): a trace of a function of the finite Dirac operator, as the spectral action is.
2. At the far wall, z(L) = z(0) − c (8L − Σ_k x_k).
3. Infinite coupling at the far wall (z(L) = 0) holds exactly when z(0) = (2cL/pi)(4 pi − l Σ_k m_k): the coupling at the light-quark wall is fixed by the sum of the sixteen channel masses, the trace of |D_F| on these channels.

Proof. $\int_0^x N(y)\,dy = \Sigma_k (x - x_k)_+$, so z(x) = z(0) + c (8x − Σ_k (x − x_k)+) = z(0) + c Σ_k (x/2 − (x − x_k)+), and x/2 − (x − x_k)_+ = min(x/2, x_k −

x/2). Item 2 is x = L with Σ_k (L − x_k) = 16L − Σ x_k. Item 3 sets z(L) = 0 and substitutes x_k = (2L/pi) l m_k. ■

Correction (4 Nov 2026). The dictionary x_k = (2L/pi) theta_k is exact only for constant z; the map from angles to positions is nonlinear. See 10, Proposition 10.1.

Reading (proposal). With the light quarks near θ = 0, Σ_k m_k ≈ m_t and the condition is z(0) ≈ (2cL/pi)(4 pi − θ_t): 7cL at label 9 (θ_t = pi/2) and 8cL at label 11 (θ_t -> 0). The coupling therefore varies along the interval on a scale set by the spectrum of D_F, which is the kind of second scale that Proposition 1.5 requires to select a joint family; whether the type I' coupling can play that role in the corpus's flat family is the next question. Since the slope decreases with x, z is concave, so z > 0 on [0, L) once z(0) > 0 and z(L) = 0; positivity of z(0) gives l Σ_k m_k < 4 pi, a bound on the trace of |D_F| in units of the tower scale, met when only the top is heavy (θ_t <= pi/2).

3. Proposition 9.2 (diquarks as strings between the stack and its image)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The representation and charge counts are independently checked, conditional on the recalled projections. The shifts describe the specified gauge-field channels; Λ^2 of a line excludes that component only, and leaves the other open-string states and the strings to other branes. See the referee report.

Setting. V = C^5 ⊗ C^3, five light flavours in three colours, with Q = 2/3 on u, c and Q = −1/3 on d, s, b; the 30 of lemma 21 is V ⊕ Vbar. Fifteen D8-branes on an O8 plane carry the Chan-Paton space 30 (branes and images), with group SO(30) on an O8^- and USp(30) on an O8^+, and U(15) when the stack is moved off the plane (recalled).

Statement.

1. Under U(15), the adjoint of SO(30) is Λ^2V ⊕ Λ^2Vbar ⊕ (V ⊗ Vbar) and the adjoint of USp(30) is S^2V ⊕ S^2Vbar ⊕ (V ⊗ Vbar). The summands Λ^2V or S^2V and their conjugates are the strings between

the stack and its image; V ⊗ Vbar are the strings with both ends on the stack.

2. $\Lambda^2 V = (S^2 5,\ 3\mathrm{bar}) \oplus (\Lambda^2 5,\ 6)$ and $S^2 V = (S^2 5,\ 6) \oplus (\Lambda^2 5,\ 3\mathrm{bar})$.
3. In the colour antitriplet, $\Lambda^2 V$ has the charge counts ($-2/3$: 6, $+1/3$: 6, $+4/3$: 3) of lemma 21, and $S^2 V$ has ($-2/3$: 3, $+1/3$: 6, $+4/3$: 1). So the antisquark counts of three generations recorded in lemma 21 (6 of charge $-2/3$ and 6 of charge $+1/3$ in the 3bar) occur among the stack-image strings of an O8^- plane, while those of an O8^+ plane have the counts (3, 6, 1).
4. With the angles θ_k = l m_k of 06, a string from channel i to the image of channel j has the spectrum (n pi ± (θ_i + θ_j))/l, and a string from channel i to channel j has (n pi ± (θ_i − θ_j))/l, n in Z.
5. A single brane has no string to its own image on an O8^- plane (Λ^2 of a line vanishes) and one on an O8^+ plane.

Proof. 1: 30 = V ⊕ Vbar as a U(15) module, $\Lambda^2(V \oplus \mathrm{Vbar}) = \Lambda^2 V \oplus \Lambda^2 \mathrm{Vbar} \oplus (V \otimes \mathrm{Vbar})$ and likewise for S^2; the orientifold projection keeps the antisymmetric part for SO and the symmetric part for USp, and the mixed summand V ⊗ Vbar is the U(15) adjoint. 2: $\Lambda^2(A \otimes B) = (S^2 A \otimes \Lambda^2 B) \oplus (\Lambda^2 A \otimes S^2 B)$ and $S^2(A \otimes B) = (S^2 A \otimes S^2 B) \oplus (\Lambda^2 A \otimes \Lambda^2 B)$, with $\Lambda^2 3 = 3\mathrm{bar}$. 3: split 5 = 2 ⊕ 3 by charge; $S^2 5 = S^2 2 \oplus (2 \otimes 3) \oplus S^2 3$ has dimensions 3, 6, 6 with charges 4/3, 1/3, −2/3, and $\Lambda^2 5 = \Lambda^2 2 \oplus (2 \otimes 3) \oplus \Lambda^2 3$ has dimensions 1, 6, 3 with the same charges. 4: Proposition 6.1 with charges (+1, +1) and (+1, −1) under U(1)_i x U(1)_j, whose holonomy is e^{2 pi i (a_i ± a_j)}, and θ = pi a. 5: dimensions. ■

Reading (proposal). (11, Theorem 11.3 and Proposition 11.4, qualify the selection below: the window sees the counts (3, 6, 1) also at s = +1, through the commutator with the masses, and parent scalars with the counts (6, 6, 3) need M_15(C) at both walls.) The diquark counts of lemma 21 and of the EPJC paper are the stack-image strings of an O8^- plane, the type that Proposition 8.1 associates with the factorization sign s = +1 selected by the seesaw sector (T2c); the O8^+ alternative gives the wrong number of up-type antisquarks. So the seesaw sign and the scalar counts select the same orientifold type.

Two features follow from the projection. Diquark masses go with the sum θ_i + θ_j and meson masses with the difference θ_i − θ_j, so when the stack leaves the wall the diquarks become massive and the mesons stay massless: the diquarks are tied to the orientifold. And the top's single brane has no string to its own image, while its strings to the stack, the (30, 2), have masses near m_t (Proposition 6.1); in the EPJC paper the top's exclusion is an input, and here the O8^- projection with one brane for the top removes the top-top channel only (item 5); the top-to-stack strings (30, 2) remain, with masses near m_t, so the rest of the exclusion is a mass statement, as in the paper.

4. Proposition 9.3 (the real structure decides between the two diquark counts)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The stated sign constraint excludes a nonzero antisymmetric block in the three KO labels. The signs permit all of S^2 V (second check: B ranges over exactly S^2 V for eps eps' = +1, with the diagonal block free); the algebra, grading and order conditions of a triple then select a subspace. Distinct-multiplet blocks remain related to their transposes. See the referee report.

Setting. E = V ⊕ Vbar with V as in Proposition 9.2, an antiunitary J on E that maps V onto Vbar and Vbar onto V (as particles and antiparticles are exchanged), $J^2 = \varepsilon$, and a Hermitian X on E with JX = ε' XJ, the condition that inner fluctuations obey in a KO-dimension with signs (ε, ε'). Let B be the block of X from Vbar to V, an element of V ⊗ V.

Statement.

1. B is symmetric (B in S^2V) if εε' = +1 and antisymmetric (B in Λ^2V) if εε' = −1.
2. In the KO-dimensions 5, 6, 7 of the finite part for the labels 9, 10, 11, (ε, ε') = (−1, −1), (+1, +1), (+1, +1), so εε' = +1: the diquark block of every J-compatible fluctuation lies in S^2V, whose colour antitriplet ($\Lambda^2$5, 3bar) has the counts (−2/3: 3, +1/3: 6, +4/3: 1).
3. For a connection on a circle whose real structure is complex conjugation (KO-dimension 1, (ε, ε') =

(+1, −1)), the block lies in $\Lambda^2 V$, whose colour antitriplet ($S^2 5$, 3bar) has the counts (−2/3: 6, +1/3: 6, +4/3: 3) of lemma 21. The SO(32) Wilson line of Proposition 6.1 is of this type.

Proof. Choose a basis of V and let K: V -> Vbar be coordinatewise conjugation; J(v, w) = (ε K^{-1} w, K v) has $J^2 = \varepsilon$. Write X = [[A, B], [B*, D]]. The component of JX = ε' XJ that maps V to V gives ε K^{-1} B* = ε' B K, so B = εε' K^{-1} B* K^{-1} = εε' B^T in coordinates. Under the U(15)-equivariant identification of Hom(Vbar, V) with V ⊗ V, symmetric matrices form $S^2 V$ and antisymmetric ones $\Lambda^2 V$. Items 2 and 3 use Connes's sign table, ε = (+, +, −, −, −, −, +, +) and ε' = (+, −, +, +, +, −, +, +) for KO-dimensions 0 to 7, and the counts of Proposition 9.2. On the circle, D = −i d/dθ and J = conjugation give JD = −DJ, and a Hermitian connection A with JA = −AJ is i times a real antisymmetric matrix, an element of so(n), where the generator of R(2 pi a) lies. ■

Item 2 is a negative statement with the following scope: the blocks of J-compatible Hermitian operators between particles and antiparticles, for one multiplet V, in the three labels. It is the finite-geometry form of the symmetry of Majorana mass matrices, and of the Fermi-statistics constraint on a Lorentz-scalar bilinear ψ_i ψ_j of one Weyl multiplet without derivatives (recalled), under which the (15, 3bar) vanishes. Known escapes: item 3 (connection components along a circle or an interval), blocks between two distinct multiplets, which J does not constrain, and derivative couplings.

Reading (proposal). The counts of lemma 21 and of the EPJC paper, (6, 6, 3), belong to the antisymmetric block. In the corpus's setting this block arises as the component of an SO(30) connection along the circle or interval: the Wilson-line moduli of the type I' stack, whose stack-image part is Proposition 9.2. The finite triple alone, in any of the three labels, gives the symmetric block and the counts (3, 6, 1). So the composite scalars with the counts of lemma 21 point to the interval parent of lemma 22 and its gauge field, and the selection of O8^- in Proposition 9.2 concerns these connection components. This narrows the spin gap of Propositions 6.1 and 9.2: the $\Lambda^2 V$ states are scalars as circle components

of a nine-dimensional gauge field, and they come with that field's vector and gaugino partners.

5. Two more dualities between 9 and 11 (recalled; proposals)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The E_1 alternatives need correction: the canonical-orthogonal root lattices distinguish $\mathbb{P}^1 \times \mathbb{P}^1$ from $dP_1 = \mathbb{F}_1$, and the usual local engineering assigns the latter to $\widetilde{E}_1$. The dP_2 and brane duality claims remain conditional on the unavailable primary sources. The Weyl action is a gauge relabelling of a holonomy, subject to the full background data. See the report on the del Pezzo comparison and Weyl action.

1. **E-type walls and del Pezzo surfaces.** The E_{n+1} symmetry of an O8^- plane with n D8-branes at infinite coupling is the symmetry of the five-dimensional theory on a D4-brane probe, which M-theory realizes on a Calabi-Yau three-fold with a shrinking del Pezzo surface dP_{n+1} (Seiberg, hep-th/9608111; Morrison and Seiberg, hep-th/9609070; Intriligator, Morrison and Seiberg, hep-th/9702198). So the far wall with the top's brane (n = 1, E_2 = SU(2) x U(1)) has an eleven-dimensional description by a local dP_2, and the empty far wall (E_1 = SU(2)) by dP_1 or P^1 x P^1; the light-quark wall (n = 15) has none. The Witten base CP^2 x CP^1 of 02 is built from the same pieces (CP^2 = dP_0, and P^1 factors).
2. **Colour-flavour exchange as a T-duality.** In the heterotic string on S^1, the Weyl group of the gauge group acts on the Wilson lines as part of the T-duality group O(17, 1; Z). The colour-family transposition M of 05 is an element of the Weyl group of E8, so it is a T-duality of the nine-dimensional E8 x E8 string, and through Hořava-Witten a symmetry of the E8 holonomy on a wall in eleven dimensions. As a Weyl element it acts as a global gauge transformation, so by itself it relabels gauge-equivalent vacua; it exchanges colour and family physically only relative to a structure that tells them apart, such as the placement of H_F in 05. The four-family requirement of Propositions

5.1 and 5.4 remains.

6. Open

Referee note (GPT-6.1 Terra, 4 Nov 2026). The finite KO signs restrict the symmetric block without producing the full representation. Light interval scalar modes, their boundary conditions and the physical string dictionary remain to be constructed; the detailed EPJC comparison is pending.

- Whether the type I' coupling, as a spectral function of D_F (Proposition 9.1), supplies the second scale of Proposition 1.5 in the flat family of lemmas 23 and 24.
- The local dP_2 of the top's wall and its relation to the Witten base CP^2 x CP^1.
- Spin: in type I' the strings of Proposition 9.2 belong to nine-dimensional vector multiplets, with one real scalar each (the relative position), the same gap as in Proposition 6.1. By Proposition 9.3 a finite triple in the labels 9 to 11 gives S^2V, so the Λ^2V scalars must come from the connection along the interval; the next computation is the inner fluctuations of the parent of lemma 22 with an SO(30) algebra, and whether its interval component carries Λ^2V with the counts of lemma 21.
- Whether the (15, 3bar) of the EPJC paper could be read as bilinears of two distinct multiplets (the paper itself gives one fundamental with itself) (for example left-handed quarks and conjugates of right-handed ones arranged into one 30), which Proposition 9.3 does not constrain.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 9.1, with the scope of its trace. *From the referee report on the 11D line, section 'Proposition 9.1: integrated coupling profile', transferred verbatim in substance on 4 Nov 2026.*

Hypothesis: the recalled slope rule. The Polchinski-Witten source could not be accessed by the referee; the comparison with it is the second-check note at the proposition.

Let N(x) = sum_k 1_{x_k < x}. Integrating z' = c (8 - N) almost everywhere, with z continuous, gives

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z(x) = z(0) + c (8x - sum_{k=1}^{16} (x - x_k)_+) =
  ↪  z(0) + c sum_{k=1}^{16} min(x/2, x_k - x/2),
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since int_0^x N(y) dy = sum_k (x - x_k)+, *and x/2 - (x - x_k)+* = min(x/2, x_k - x/2) (both sides equal x/2 for x <= x_k and x_k - x/2 for x > x_k). At x = L, with every x_k <= L, this becomes

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z(L) = z(0) - c (8L - sum_k x_k).
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With x_k = 2 l L m_k/pi, the dictionary of the Setting (exact only for constant z, by the correction after the proof), the zero condition z(L) = 0 is z(0) = (2cL/pi)(4 pi - l sum_k m_k), as stated. The slopes c (8 - N(x)) are nonincreasing, so z is concave; if z(0) > 0 and z(L) = 0, then z(x) >= (1 - x/L) z(0) > 0 on [0, L).

Scope of the trace. The trace is over the sixteen selected channel values. The equality with a spectral trace assumes a chosen sixteen-dimensional channel space with eigenvalues m_k, or an explicitly normalized projection selecting those values. The physical D_F has additional chirality, conjugation, colour and generation multiplicities, which cannot be dropped from its full trace without specifying that projection. The assignment of the sixteen brane positions to those channels is also part of the proposal. This profile alone does not derive a stabilizing term for the interval modulus. The slope, coordinate and e^phi-to-z conventions are those of the recalled rule.

Independent proof of Proposition 9.2, items 2 and 3, with the scope of items 4 and 5. *From the referee report on the 11D line, section 'Proposition 9.2: diquark representations and string channels', transferred verbatim in substance on 4 Nov 2026.*

Hypothesis: the recalled orientifold projections. For V = C^5 ⊗ C^3, the swap of V ⊗ V is the product of the swaps of the two tensor factors, so its eigenspaces combine the symmetric and alternating projectors on the factors:

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Λ²V = (S²C^5 ⊗ 3bar) ⊕ (Λ²C^5 ⊗ 6),
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`S²V = (S²C^5 ⊗ 6) ⊕ (Λ²C^5 ⊗ 3bar),`

with $\Lambda^2 3 = 3\text{bar}$ and $S^2 3 = 6$. Their dimensions are $15 * 3 + 10 * 6 = 45 + 60 = 105$ and $15 * 6 + 10 * 3 = 90 + 30 = 120$, which are C(15, 2) and C(16, 2). Splitting the flavour space into two charge-2/3 and three charge-(-1/3) directions, $S^2C^5 = S^2 2 \oplus (2 \otimes 3) \oplus S^2 3$ and $\Lambda^2 C^5 = \Lambda^2 2 \oplus (2 \otimes 3) \oplus \Lambda^2 3$ give the antitriplet counts (6, 6, 3) in S^2C^5 and (3, 6, 1) in $\Lambda^2 C^5$, ordered by charges (-2/3, 1/3, 4/3). The full adjoints have dimensions 2(105) + 225 = 435 = dim so(30) and 2(120) + 225 = 465 = dim usp(30).

Scope of items 4 and 5. For the assumed Wilson-line action, the antisymmetric or symmetric block has weights a_i + a_j and the mixed block has weights a_i - a_j; hence their field-theory Kaluza-Klein momenta have the displayed shifts (n pi ± (θ_i ± θ_j))/l, with θ = pi a. This calculation leaves out the rest of the open-string spectrum, which also includes oscillator states and their orientifold projections. The sum-weight component with i = j is absent in Λ^2 and present in S^2; neutral diagonal modes remain. The claim of item 5, that a single brane has no string to its own image, holds in this sector. Absence of the same-line component leaves the strings between the top brane and other branes. Moving all stack branes by the same angle makes the mixed-block differences vanish; different individual positions can give nonzero differences. Identifying the resulting Chan-Paton modes with the proposed composites is an additional physical construction. The O8^-/O8^+ projection and global tadpole conditions were not independently verified, since the primary sources were unavailable to the referee.

Independent proof of Proposition 9.3, item 1, with the scope of item 2. *From the referee report on the 11D line, section 'Proposition 9.3: real-structure constraint on diquark blocks', transferred verbatim in substance on 4 Nov 2026.*

Choose an antiunitary identification K: V -> Vbar and write

`J(v, w) = (eps K^{-1} w, K v),  X = [[A, B], [B*, D]].`

Then JX(v, w) = (ε K^{-1}(B* v + D w), K(A v + B

w)) and XJ(v, w) = (ε A K^{-1} w + B K v, ε B* K^{-1} w + D K v). The V -> V component of JX = ε' XJ, the coefficient of v in the first entry, gives

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eps K^{-1} B* = eps' B K,   hence   B = epseps'
 ↪  K^{-1} B* K^{-1} = epseps' B^T,
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the last equality in coordinates in which K is coordinate-wise conjugation. Thus εε' = +1 forces B ∈ S^2V, and εε' = -1 forces B ∈ Λ^2V. For the displayed KO-5, KO-6 and KO-7 conventions the products are all +1. The zero block belongs to both subspaces, so "symmetric if εε' = +1" states the symmetry forced by the sign. For V = C^5 ⊗ C^3 the decomposition of S^2V and the counts are those of the proof of Proposition 9.2: Λ^2C^5 has multiplicities 3, 6, 1 at charges -2/3, 1/3, 4/3, and S^2C^5, hence the colour-3bar part of Λ^2V, has 6, 6, 3.

Scope of item 2. A J-compatible finite fluctuation can occupy only a subspace of S^2V; the algebra, grading and order conditions determine which blocks actually occur. For example, with D = 0 every inner one-form vanishes, despite the symmetry condition. The KO signs alone therefore exclude a nonzero antisymmetric block and leave the full symmetric scalar spectrum to be established. For a circle operator D = -i d/dx with ordinary conjugation J, JD = -DJ. A Hermitian connection satisfying the same sign is purely imaginary and antisymmetric in a real basis; the complexified so(30) representation contains Λ^2V and its conjugate. Whether these components survive the particular interval boundary conditions as light scalars requires a separate construction. For distinct multiplets, an off-diagonal block B_12 may be chosen freely, while the full constraint still relates it to B_21 = εε' B_12^T.

Scope and direction. The exclusion concerns a Hermitian J-compatible operator between one multiplet and its conjugate, with the stated signs. Distinct multiplets, other real structures or signs, derivative couplings, and circle or interval connection components remain possible directions; each requires its own field and boundary data.

Independent proof of the branching in section 1. *From the referee report on the 11D line, section 'File 09, section 1: the SO(32) comparison', transferred verbatim in substance on 4 Nov 2026.*

The alternative chain follows from 32_C = 16_{+1} ⊕ 16bar_{-1} under U(16) ⊂ SO(32): so(32)_C = $\Lambda^2 32$ = $\Lambda^2 16 \oplus \Lambda^2 16\text{bar} \oplus (16 \otimes 16\text{bar})$, so

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496 = 120_{+2} ⊕ 120bar_{-2} ⊕ 255_0 ⊕ 1_0.
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Under SU(15), 16 = 15 ⊕ 1 and 120 = $\Lambda^2(15 \oplus 1) = 105 \oplus 15$. With $15 = 5 \otimes 3$, the plethysm of Proposition 9.2 gives

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105 = Λ²(5 ⊗ 3) = (15, 3bar) ⊕ (10, 6),
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with $S^2 5 = 15$ and $\Lambda^2 5 = 10$, and $15 \otimes 15\text{bar} = (5 \otimes 5\text{bar}, 3 \otimes 3\text{bar}) = (24, 8) \oplus (24, 1) \oplus (1, 8) \oplus (1, 1)$. The antitriplet charge multiplicities are those checked in Proposition 9.2. These group products are Lie-algebra shorthand; global embeddings involve finite central quotients.

Scope. The SO(2) plane is a colour singlet. Assigning it to a top brane is a label in the proposed dictionary and does not by itself construct a colour-triplet top field. Chan-Paton bilinearity and matching representation counts likewise do not by themselves establish dynamical quark composites. The referee could not access the paper itself; the comparison with its text is the reading of the LaTeX source in section 1.

Counterexample (referee) to the unqualified E_1 surface identification in section 5, item 1. *From the referee report on the 11D line, section 'File 09, section 5: del Pezzo wall correspondence', transferred verbatim in substance on 4 Nov 2026.*

The two degree-eight surfaces have different root lattices orthogonal to the canonical class. On dP_1, let H^2 = 1, E^2 = -1, H · E = 0 and K = -3H + E. If D = aH + bE is orthogonal to K, then D · K = -3a - b = 0, so b = -3a and D^2 = a^2 - b^2 = -8a^2. There is no integral class with D^2 = -2. On P^1 x P^1, with ruling classes F_1, F_2 (F_1^2 = F_2^2 = 0, F_1 · F_2 = 1) and K = -2F_1 - 2F_2, the difference F_1 - F_2 is orthogonal to K and has square -2. These surfaces therefore cannot be interchanged in the root-lattice comparison. In the usual local engineering description (recalled), P^1 x P^1 gives E_1, whereas dP_1 = F_1 gives the distinct $\tilde{E}_1$ theory.

For dP_2, with exceptional classes E_1, E_2 and K =

-3H + E_1 + E_2, the class E_1 - E_2 is orthogonal to K and has square -2; the orthogonal lattice has rank two, consistent with an A_1 factor and an additional abelian direction. The dP_2 comparison is compatible with the expected A_1 root. This computation alone does not construct the corresponding five-dimensional field theory or the type I' duality, and the brane-to-geometry map remains conditional on the cited dualities.

Sources. The distinction between the two degree-eight engineering models is also stated in a secondary geometric-engineering discussion. The Seiberg and Morrison-Seiberg primary sources were unavailable to the referee for direct comparison. The occurrence of CP^2 and CP^1 in the separate six-dimensional base does not by itself identify that compactification with a local dP_2.

Independent proof of the Weyl action in section 5, item 2. *From the referee report on the 11D line, section 'File 09, section 5: Weyl action on a Wilson line', transferred verbatim in substance on 4 Nov 2026.*

A Weyl transformation of E8 acts by conjugation on an E8 holonomy and gives a lattice automorphism in the heterotic circle frame. Whether it acts as a symmetry of a fixed compactification depends on all of its background and boundary data.

Let T ⊂ E8 be a maximal torus. An element w ∈ W(E8) = N(T)/T has a representative g ∈ N(T), and the constant gauge transformation by g sends a holonomy W ∈ T to g W g^{-1} ∈ T. Thus the holonomy alone determines the same gauge orbit. The corresponding isometry w ⊕ 1 of E8 ⊕ E8 ⊕ II_{1,1} is an element of the automorphism group of the nine-dimensional Narain lattice. Proposition 5.2 places the transposition M in that Weyl group.

If a bundle, a defect, an algebra embedding or a distinguished set of zero modes is held fixed, the same g need not preserve it. Transforming the entire configuration produces a gauge-equivalent description when the transformation is allowed; an exchange relative to fixed colour and family structures needs an explicit action on those structures. The root no-go applies to its stated three-slot configuration, while bundle zero modes remain an open escape.