Supplemental Material to "The real structure of a massless block: $SO(30)$ in the finite geometry"

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This is the supplement of the paper *The real structure of a massless block: $SO(30)$ in the finite geometry.* Its parts keep the numbers of the full Supplemental Material, so that a citation of a part in the paper refers to the part of the same name here.

Parts contained. Lemma parts: 06, 18, 19, 21, 22, 23, 31, 32, 40, 35, 36, 38.

The full Supplemental Material, with all parts of the project, exists as `supplemental.pdf` and `supplemental_mobile.pdf` in the same folder as this file. A reference inside a part to a part not listed here points to the full version.

Contents

Part 06 **2**

Part 18 **12**

Part 19 **35**

Part 21 **40**

Part 22 **47**

Part 23 **59**

Part 31 **74**

Part 32 **86**

Part 40 **101**

Part 35 **114**

Part 36 **121**

Part 38 **136**

Part 06

The proposed uniqueness is false: **the same KO-6 triple can admit both reductions**. Moreover, under the stated definition, the strict heavy-limit triple is a doubling of KO-7, although it is not a doubling of KO-5. The distinction between the full real Hilbert space and its physical particle subspace is essential.

1. Lemma — Product convention and necessary correction of the "Clifford point."

Under ordinary spectral-triple conventions, a graded triple on `C^2` is not itself an odd KO-1 or KO-7 triple. The appropriate interpretation is an odd point

$$P_k = (\mathbb{C}, \mathbb{C}, 0, K), \qquad k = 1 \text{ or } 7,$$

together with the auxiliary `C^2` introduced by the odd-times-odd product. For real algebras, use the point algebra `R`, so that the product algebra remains `A`.

I use the following products, with the standard antilinear tensor product of real structures.

For even $\times$ even,

$$D = D_1 \otimes 1 + \gamma_1 \otimes D_2, \qquad \gamma = \gamma_1 \otimes \gamma_2,$$

$$J = J_1 \otimes (J_2 \gamma_2^b), \qquad b = (1 - \epsilon_1'')/2.$$

For even $\times$ odd, use the same formula for `D`, no grading, and

$$J = (J_1 \gamma_1^a) \otimes J_2, \qquad (-1)^a = \epsilon_1'' \epsilon_2'.$$

Odd $\times$ even is obtained by transporting the even-first construction through the tensor flip.

For odd × odd,

$$H = H_1 \otimes H_2 \otimes \mathbb{C}^2, \quad D = D_1 \otimes 1 \otimes \sigma_1 + 1 \otimes D_2 \otimes \sigma_2,$$
$$\gamma = 1 \otimes 1 \otimes \sigma_3,$$

$$J = J_1 \otimes J_2 \otimes (BK),$$

where

$(\epsilon_1', \epsilon_2')$	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
B	σ_1	1	σ_3	$-i\sigma_2$.

This agrees, up to harmless phases, with Dąbrowski–Dossena, equation (25), Table 6.

Proof. In the odd-odd case,

$$B\sigma_1 = \epsilon_1'\sigma_1 B, \qquad B\sigma_2 = -\epsilon_2'\sigma_2 B.$$

The second minus sign compensates for complex conjugation of `sigma_2`. Consequently `JD=DJ`; computing `J^2` and `J gamma` gives the signs of the sum degree. The grading twists in the other two formulas follow by the same calculation. Order zero and first order pass to these products.

Thus the intended doubling has Hilbert space `H_0 ⊗ C^2`, with the displayed grading on the auxiliary factor. All subsequent statements use this corrected interpretation and the zero-Dirac point. ■

2. Theorem — Exact doubling criterion in KO-dimension six.

Let `(A,H,D,J,gamma)` be a finite KO-6 triple. For `s=±1`, the following are equivalent:

1. It is a product `T_n × P_k`, where
$$(s, n, k) = (+1, 7, 7) \quad \text{or} \quad (-1, 5, 1).$$
2. There is a complex-linear unitary `U` satisfying
$$U^2 = 1, \quad U\gamma = -\gamma U,$$
$$[U, a] = [U, a^o] = [U, D] = 0, \quad UJ = sJU,$$
where `a^o=Ja^\*J^{-1}`.

Write `H_L=H_{\gamma=+1}`, `H_R=H_{\gamma=-1}`, and

$$D = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}.$$

Equivalently, there is an A-bimodule unitary V:H_L→H_R such that

$$V^*M = (V^*M)^*, \qquad (V^*C)^2 = s, \quad C = J|_{H_L} : H_L \longrightarrow H_R.$$

Here C and V^*C are antilinear.

Proof. Given U, identify

$$H_L \oplus H_L \longrightarrow H, \qquad (x, y) \longmapsto x + Uy.$$

Put

$$H_0 = H_L, \qquad d = (UD)|_{H_L}, \qquad j = (UJ)|_{H_L}.$$

The assumptions give

$$d = d^*, \qquad j^2 = s, \qquad jd = s\,dj.$$

Hence (j^2,jd/dj) is (+,+) for KO-7 and (-,-) for KO-5. In these coordinates,

$$a = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$J(x, y) = (sjy, jx).$$

These are precisely the products in Result 1: B=σ_1 for (7,7) and B=-iσ_2 for (5,1).

The descended opposite action is

$$ja_0^*j^{-1} = a^o|_{H_L}.$$

Since [d,a_0]=U[D,a]|_{H_L}, order zero and first order descend. Conversely, the product has U=1⊗σ_1, with all the required properties.

Finally,

$$U = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}.$$

Its commutation with D is equivalent to self-adjointness of V^*M, while UJ=sJU is equivalent to (V^*C)^2=s.

If one prefers an anticommuting convention, replacing U by i gamma U exchanges [U,D]=0 with {U,D}=0 and preserves s. ■

In particular, ordinary equality of left and right module multiplicities is only part of the criterion; compatibility with D and J must also be checked.

3. Counterexample — J does not select a unique reduction; electrodynamics admits both.

Consider the electrodynamics triple in the basis `(e_L,e_R,bar e_L,bar e_R)`. After a phase change compatible with `A,J,gamma`, its matrices are

$$\rho(a,b) = \mathrm{diag}(a,a,b,b), \qquad \gamma = \mathrm{diag}(1,-1,-1,1),$$

$$J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} K, \qquad D = m \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad m \geq 0.$$

These are the input data of van den Dungen–van Suijlekom, §4.1, equations (4.1)–(4.2).

Define

$$U_+ = \mathrm{diag}(\sigma_1, \sigma_1), \qquad U_- = \mathrm{diag}(\sigma_1, -\sigma_1).$$

Proof. Both involutions reverse the grading and commute with the algebra, opposite algebra, and `D`. Direct multiplication gives

$$U_+ J = J U_+, \qquad U_- J = -J U_-.$$

Thus the same triple, with the same `J`, admits both factorizations.

On `H_0=span(e_L,bar e_R)`, the two descended triples have

$$d_+ = mI_2, \qquad j_+ = \sigma_1 K$$

and

$$d_- = \mathrm{diag}(m,-m), \qquad j_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K.$$

They have KO-degrees seven and five, respectively. This also holds at `m=0`. ■

4. Lemma — Exact neutral-sector criterion, including Majorana masses.

Suppose a KO-6 sector has

$$H = H_+ \oplus H_-, \qquad \dim H_+ = \dim H_- = r,$$

and both algebra actions are the same scalar character throughout this sector. Then:

- a KO-7 reduction always exists;

- a KO-5 reduction exists exactly when **every singular value of `M`, including zero, has even multiplicity**.

Proof. Choose an orthonormal basis of `H_+` and its `J`-image as the basis of `H_-`. Then

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} K.$$

The equation `JD=DJ` says `M^T=M`. Takagi factorization, implemented by a basis change preserving this form of `J`, reduces `M` to

$$\Sigma = \mathrm{diag}(\mu_1, \dots, \mu_r), \qquad \mu_i \geq 0.$$

Taking `V=I` satisfies Result 2 with `s=+1`.

For `s=-1`, the descended antiunitary `j` squares to `-1` and commutes with `d^2=D^2|_{H_+}`. Every eigenspace of `d^2` must therefore have even complex dimension. These dimensions are exactly the singular-value multiplicities.

Conversely, pair equal entries of `Sigma` and choose `V=σ_2` on each pair. Then `V` is a skew-symmetric Hermitian unitary commuting with `Sigma`; it satisfies the criterion with `s=-1`. ■

A single neutral Weyl state and its conjugate therefore **are** a KO-7 doubling, even without a Dirac partner. They cannot be a KO-5 doubling. The same statement holds for a single Majorana pair, whether its mass vanishes or not.

5. Proposition — The strict one-generation heavy limit is a KO-7 doubling, and KO-5 is obstructed.

Here is an explicit convention for the Standard Model representation. Set

$$F = \mathbb{C} \oplus \mathbb{C}^3, \qquad H_p = (\mathbb{C}^2 \otimes F)_L \oplus (\mathbb{C}^2 \otimes F)_R,$$
$$H = H_p \oplus H_{\bar{p}}.$$

For `a=(lambda,q,m)`, put

$$Q_\lambda = \mathrm{diag}(\lambda, \bar{\lambda}), \qquad B_{\lambda,m} = \mathrm{diag}(\lambda, m),$$

$$\rho_p(a) = (q \otimes I_4) \oplus (Q_\lambda \otimes I_4),$$
$$\rho_{\bar{p}}(a) = (I_2 \otimes B_{\lambda,m}) \oplus (I_2 \otimes B_{\lambda,m}).$$

Use

$$J(p,\bar{p}) = (K\bar{p}, Kp), \qquad \gamma = \mathrm{diag}(I_8, -I_8, -I_8, I_8).$$

On particles, the Dirac block is

$$S_p = \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix},$$

where `Y` has entries `y_nu,y_e,y_u I_3,y_d I_3` on the corresponding species. The antiparticle block is `bar S_p`; the additional Majorana block connects `nu_R` with `J nu_R`. This realizes the bimodule and mass-block structure in CCM, Proposition 2.4 and Theorem 2.21.

Assume precisely the strict projection described in canonical §3: remove `nu_R,J nu_R` and the complete up-type Dirac sector, and restrict to

$$B = \{(\lambda, Q_\lambda, m)\} \simeq \mathbb{C} \oplus M_3(\mathbb{C}).$$

The surviving space consists of `d_L,d_R,e_L,e_R,nu_L` and their conjugates, with `D_light=PSP`.

Proof. The surviving down-quark and electron masses pair identical `B`-bimodules. After making their masses nonnegative, each sector has both involutions from Result 3, with the identity on color.

On

$$N = \mathrm{span}(\nu_L, J\nu_L),$$

both algebra actions are the scalar character `lambda`, and `PSP|_N=0`. Thus

$$\gamma_N = \sigma_3, \qquad J_N = \sigma_1 K, \qquad U_N = \sigma_1$$

gives a KO-7 reduction. Combining this with `U_+` on the charged sectors gives a global KO-7 reduction.

A KO-5 reduction is impossible: the full space has dimension eighteen, so its descended space would have dimension nine, which cannot carry an antiunitary squaring to `-1`. More specifically, the neutral bimodule is inequivalent to the charged bimodules, so any permitted `U` preserves it; Result 4 already obstructs KO-5 there. ■

Thus the anticipated claim "the neutral sector prevents any doubling" is false under the definition in Question 1.

6. Proposition — Modifications and the distinction from physical vectorlikeness.

For the reduced algebra `B` above, the following conclusions hold.

Surviving neutral content	Odd reductions of the full light triple
No neutral sector	Both KO-5 and KO-7
`nu_L,nu_R` and conjugates, pure Dirac mass	Both KO-5 and KO-7
Only `nu_L,J nu_L`	KO-7 only
A single isolated Majorana pair	KO-7 only
General neutral Dirac–Majorana block	Always KO-7; KO-5 exactly under Result 4

Proof. The charged sectors admit both reductions by Result 3. Pure Dirac neutrinos have the same four-state construction, so adding or retaining `nu_R` restores both possibilities, including when its Dirac mass is zero.

For one generation with Dirac mass `m` and right Majorana mass `R`, the neutral symmetric matrix can be written

$$M_N = \begin{pmatrix} 0 & m \\ m & R \end{pmatrix}.$$

Phase changes make `m,R` nonnegative. Its singular values are

$$\mu_{\pm} = \frac{\sqrt{|R|^2 + 4|m|^2} \pm |R|}{2}.$$

They coincide exactly when `R=0`. Hence any nonzero right Majorana mass obstructs KO-5 in this one-generation neutral sector, but never KO-7. Treating that sector separately leaves a charged KO-5 reduction; it does not turn the complete direct sum into a KO-5 reduction. ■

There is an additional physically meaningful condition absent from Question 1. Let `Q` be `+1` on particles and `-1` on antiparticles.

No-go theorem under the extra hypothesis `[U,Q]=0`. An unpaired `nu_L` prevents doubling.

Proof. Such a `U` must map particle states of positive grading to particle states of negative grading within the

same bimodule. No particle `nu_R` exists. The neutral involution used in Result 5 instead maps `nu_L` to its conjugate and violates `[U,Q]=0`. ■

This is why physical chirality and doubling of the complete real triple are different tests.

7. Proposition — Three generations and arbitrary flavor mixing.

Assume the reduced algebra acts trivially on generation indices.

If all three right-neutrino Majorana pairs are removed and only the top Dirac sector is made heavy, the light space has

$$96 - 6 - 12 = 78$$

complex dimensions. If all three up-type sectors are removed, it has dimension fifty-four. In either case, with three surviving left neutrinos and `D_light=PSP`, the light triple admits KO-7 and excludes KO-5.

Dropping the remaining neutral sector, or retaining three right neutrinos with purely Dirac neutrino masses, permits both reductions.

Proof. Each surviving charged species has equal left/right flavor multiplicities. Singular-value decomposition of its mass matrix uses bimodule unitaries; conjugate transformations on antiparticles preserve `J`. Consequently every charged sector reduces to copies of Result 3, irrespective of mass rank or flavor mixing.

The three unpaired neutral states give `r=3`, `M_N=0` in Result 4, excluding KO-5 and admitting KO-7. Equivalently, the proposed descended spaces have odd dimensions thirty-nine and twenty-seven.

With three right neutrinos retained, the general neutral matrix is

$$M_N = \begin{pmatrix} 0 & Y_\nu^T \\ Y_\nu & R \end{pmatrix}, \qquad R^T = R.$$

For `R=0`, every singular value of `Y_nu` appears twice, so both reductions exist. For general `R`, KO-7 always exists, while KO-5 exists exactly when every singular value of this six-by-six matrix has even multiplicity. Thus

nonzero Majorana masses are not an unconditional obstruction in several generations; the exact obstruction is the stated multiplicity condition. ■

8. No-go theorem — The massless Standard Model triple is not a doubling.

Let the algebra remain the full

$$A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

with the representation of Result 5, for `g≥1` generations. Then no involution satisfying Result 2 exists, for any admissible `D`. In particular this applies at `v=0`, when only the Majorana block remains.

Proof. Consider the central projection

$$e_{\mathbb{H}} = (0, 1_{\mathbb{H}}, 0).$$

Its represented range lies entirely in the particle left-handed sector:

$$\dim \rho(e_{\mathbb{H}})H_+ = 8g, \qquad \dim \rho(e_{\mathbb{H}})H_- = 0.$$

An algebra-linear grading-reversing unitary would identify these spaces, which is impossible.

In physical representation-ring notation,

$$V_L = (3,2)_{1/6} \oplus (1,2)_{-1/2},$$

$$V_R = (3,1)_{2/3} \oplus (3,1)_{-1/3} \oplus (1,1)_{-1} \oplus (1,1)_0.$$

Thus `I_G=[V_L]-[V_R]≠0`. For the complete real graded Hilbert space, the relevant gauge character is

$$[H_+] - [H_-] = I_G - \overline{I_G},$$

not merely `I_G`. It too is nonzero: the coefficient of `(3,2)\_{1/6}` is `g`. Either this full-character calculation or the central-projection argument proves the obstruction. ■

Consequence for the 10 -> 9 claim

9. Proposition — What dimension reduction is actually established.

The heavy projection itself preserves KO-degree six, as canonical §3 explicitly states. Hence its spacetime product still has degree `4+6=10 mod 8`.

A subsequent choice of a KO-5 reduction removes a degree-one Clifford factor and gives the formal representative `4+5=9`. Such a reduction exists for the charged truncation or the pure-Dirac completion, but **both also admit KO-7 reductions**. Their fixed `J` does not select nine.

For the strict light spectrum with one or three unpaired left neutrinos, **KO-5 is obstructed; only KO-7 is available among the proposed reductions**. This gives the formal representative eleven. If particle preservation is additionally required, that strict light spectrum admits neither reduction.

Proof. These are Results 2–7. For completeness, the spacetime real structures are

$$J_{4+5} = (J_M\gamma_M) \otimes j_5, \qquad J_{4+7} = J_M \otimes j_7.$$

Thus even the odd reductions require different product conventions. No finite-dimensional operation here changes heat dimension: a finite internal factor remains spectrally zero-dimensional. Consequently the claimed loss of one dimension is, at most, a chosen Clifford-factor removal—not a proved metric collapse, and not a consequence of vectorlike charged matter alone. ■ —

Independent numerical check (Claude, 2 Nov 2026)

We used the electrodynamics data of Result 3, with m = 0.7, J = [[0, I_2],[I_2, 0]] K and gamma = diag(1,-1,-1,1). The triple has J^2 = 1, JD = DJ and J gamma = -gamma J, so its KO-dimension is 6. Both U_+ = diag(sigma_1, sigma_1) and U_- = diag(sigma_1, -sigma_1) satisfy U^2 = 1 and U gamma = -gamma U. Both commute with random algebra elements, their opposites and D; and U_+ J = J U_+, U_- J = -J U_-. On H_0 = span(e_L, bar e_R), U_+ descends to j^2 = +1 with jd = dj (KO-7) and U_- to j^2 = -1 with jd = -dj (KO-5). For No-go 8 we also rechecked by hand the dimension count dim rho(e_H) H_+ = 8g, dim rho(e_H) H_- = 0 in the representation of Result 5.

Terminology. Here "doubling" means the factorization of an even triple as an odd triple times a Clifford point (the doubling of an odd triple in Dabrowski-Dossena's

product construction). It is unrelated to the "fermion doubling" of the noncommutative Standard Model (Lizzi-Mangano-Miele-Sparano). Prior work on products and on J versus J gamma: ../LITERATURE.md, section D; the factorization criterion and the 9/11 results were not found in print.

Referee note

Terra 6.1 (3 Nov 2026; report) checked Result 3, Lemma 4, Proposition 5, Result 7, No-go theorem 8 and the no-go under [U, Q] = 0: all correct. One amendment to Result 7: "arbitrary flavor mixing" means arbitrary matrices within the displayed species blocks, with the heavy projection chosen onto the specified top Dirac sector; mixing between inequivalent species is outside the displayed Dirac operator. Escapes: for KO-5, neutral-sector pairing or a symmetry enforcing even multiplicities; for No-go 8, a change of algebra or representation, the even description, and boundary or defect realizations of chirality; for the no-go under [U, Q] = 0, a matching particle bimodule of opposite grading.

Part 18

T5b: vector masses in the spectral action at fixed cutoff

Prompt: prompts/18 (STRATEGY T5b). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5 with Python (numpy, scipy), by computations independent of the kernel derived here: a direct heat trace with constant a reproduces equation (2) for five choices of (m_1, m_2), to about 0.03%; a direct heat trace on Fourier modes with a_2(x_1) = a cos(q x_1), including the F_{12} and [T, V] vertices, reproduces the kernel (37) at three values of (m, q), to 0.1-0.3%, the size of the discretization error. The closed forms (16)-(19), the bound (28), the limits (27), the neutral matrix (30) recomputed from Standard Model charges, and the identities behind (31) also check. The largest value of the Taylor mass is M_W^2 = 0.275 Lambda^2, at m_t^2 = 2.02 Lambda^2. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): all ten numbered results are correct, with small amend-

ments incorporated below. The referee added Propositions 11 to 14 and Remark 15, which widen the scope in both directions; the ratio in Proposition 14 was added by Claude.

Scope. Within the three-generation, zero-Majorana, top-only family at fixed cutoff, these results refute an infinitely heavy W, Z, t limit for the stated quadratic mass definitions. They do not rule out other finite data (Proposition 14), a different action or constraint, a joint limit with Lambda -> infinity, or a quantum effective theory. They do not show unbroken SU(2) x U(1) on the light space (Remark 15).

For the heat cutoff, on $\mathbb{R}^4$, we obtain the exact quadratic vector form without expanding in m/Λ. In the three-generation Standard Model finite representation with only the top Dirac mass nonzero, both the mass defined by the q^0/q^2 Taylor ratio and the unique positive time-like pole mass satisfy $M_W \sim \Lambda^2/m_t$ as $m_t/\Lambda \to \infty$ at fixed Λ. The broken vectors consequently do not decouple by becoming heavy, although the heavy fermion contribution disappears from the gauge coefficients. These statements concern the classical bosonic quadratic action about a prescribed constant background; moreover, the positive Laplace class considered below does not itself select a finite nonzero stationary background when radial variations of that background are allowed.

1. Theorem (exact quadratic kernel).

Let

$$D_A = -i\gamma^\mu(\partial_\mu + iA_\mu) + \gamma_5 V, \qquad V = V_0 = V_0^\dagger,$$

on $\mathbb{R}^4$, with Hermitian Euclidean gamma matrices, $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$, $\gamma_5^2 = 1$, and four spin components. Let the finite trace be the literal trace on H_F, including every stated multiplicity, without a subsequent projection; this agrees with the bosonic trace of Chamseddine-Connes-Marcolli, which counts particles, antiparticles and four spin components. Take A smooth and compactly supported.

We use the vacuum-subtracted action, or equivalently its quadratic variation, since the constant-background trace

on $\mathbb{R}^4$ has infinite volume. Write

$$A_\mu(q) = a_\mu(q)T, \qquad T = T^\dagger, \qquad q^\mu a_\mu(q) = 0,$$
$$\int_q = \int \frac{d^4q}{(2\pi)^4}.$$

Diagonalize V_0:

$$V_0 e_i = m_i e_i, \qquad m_i \in \mathbb{R}.$$

The signed eigenvalues are needed; their squares alone do not determine the answer.

For $S_t = \operatorname{Tr} e^{-tD_A^2}$, define

$$b(u) = u(1-u), \quad d_{ij} = m_i - m_j,$$

$$I_t(a,b;q^2) = \int_0^1 e^{-t[(1-u)a+ub+u(1-u)q^2]}\,du.$$

Then

$$S_t^{(2)} = \int_q |a_T(q)|^2\, C_{t,T}(q^2),$$

where

$$\boxed{C_{t,T}(q^2) = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2 \left[\left(\frac{2}{t} + d_{ij}^2 + q^2\right) I_t(m_i^2, m_j^2; q^2) - \frac{e^{-tm_i^2} + e^{-tm_j^2}}{t}\right].} \tag{1}$$

For several generators, replace $|T_{ij}|^2$ by $\operatorname{Re}((T_a)_{ij}(T_b)_{ji})$.

In particular, writing $C(q^2) = C_0 + q^2 C_1 + O(q^4)$, we obtain

$$\boxed{C_{0,t} = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2 \left[\left(\frac{2}{t} + d_{ij}^2\right) I_t(m_i^2, m_j^2; 0) - \frac{e^{-tm_i^2} + e^{-tm_j^2}}{t}\right],} \tag{2}$$

and

$$\boxed{C_{1,t} = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2 \int_0^1 e^{-t[(1-u)m_i^2+um_j^2]}\big[1 - 2b(u) - td_{ij}^2 b(u)\big]\,du.} \tag{3}$$

These formulas also apply, after integration over $t = s/\Lambda^2$, to

$$f(z) = \int_{(0,\infty)} e^{-sz}\, d\nu(s), \quad 0 < f(0) = \nu((0,\infty)) < \infty,$$
$$\int (s^{-2} + s^2)\, d\nu(s) < \infty, \tag{4}$$

where ν is positive. These hypotheses suffice for the Euclidean action density and the Taylor coefficients used below. They do not, by themselves, guarantee analytic continuation to every negative q^2.

Proof. With $\nabla_\mu = \partial_\mu + iA_\mu$,

$$D_A^2 = -\nabla^2 + V_0^2 - \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu} + \gamma^\mu\gamma_5[A_\mu, V_0].$$

Let $P_0 = -\partial^2 + V_0^2$ and write $D_A^2 = P_0 + P_1 + P_2$, according to degree in A. Duhamel expansion and cyclicity give

$$S_t^{(2)} = -t\,\mathrm{Tr}(e^{-tP_0}P_2) + \frac{t^2}{2}\int_0^1 \mathrm{Tr}\,\left(e^{-(1-u)tP_0}P_1 e^{-utP_0}P_1\right) du. \tag{5}$$

The linear vertex from j, p to $i, p+q$ is

$$T_{ij}\left[(2p+q)\cdot a + \gamma^{\mu\nu}q_\mu a_\nu + (m_j - m_i)\gamma^\mu\gamma_5 a_\mu\right].$$

Completing the momentum square produces

$$(4\pi t)^{-2}e^{-t[(1-u)m_i^2 + um_j^2 + b(u)q^2]}.$$

For transverse a, the averaged spin trace of the vertex product is

$$4\left(\frac{2}{t} + d_{ij}^2 + q^2\right)|a|^2.$$

The P_2 term contributes

$$-\frac{1}{4\pi^2 t}\sum_{ij}|T_{ij}|^2 e^{-tm_i^2}|a|^2.$$

Symmetrizing in i, j proves (1). Differentiation proves (2)–(3).

The divided-difference structure is explicit:

$$I_t(a,b;0) = \frac{e^{-ta} - e^{-tb}}{t(b-a)}, \qquad I_t(a,a;0) = e^{-ta}, \tag{6}$$

and

$$\int_0^1 u(1-u)e^{-t[(1-u)a+ub]}\,du = \frac{1}{t^2}\partial_a\partial_b I_t(a,b;0). \tag{7}$$

Thus (2) uses a first divided difference, while (3) also uses its mixed derivative. The full momentum dependence follows from the convergent series

$$I_t(a,b;q^2) = \sum_{n=0}^{\infty} \frac{(-q^2/t)^n}{n!}\partial_a^n\partial_b^n I_t(a,b;0). \tag{8}$$

Repeated-node values are obtained continuously.

For completeness, the mass coefficient for (4) can be written entirely in dimensionless variables. Put

$$r_i = m_i/\Lambda, \quad x_i = r_i^2, \quad z(u) = (1-u)x_i + ux_j,$$
$$\Phi_1(z) = \int s^{-1}e^{-sz}\,d\nu(s).$$

The bracket replacing that in (2) is

$$\Lambda^2\left[2\int_0^1 \Phi_1(z(u))\,du + (r_i-r_j)^2\int_0^1 f(z(u))\,du - \Phi_1(x_i) - \Phi_1(x_j)\right]. \tag{9}$$

The corresponding bracket in (3) is

$$\int_0^1 \left[(1-2b)f(z) + (r_i-r_j)^2 b f'(z)\right]du. \tag{10}$$

Each unweighted integral is a divided difference of an antiderivative. The moment assumptions justify the integrations and differentiations. □

On a finite flat torus, (5) remains exact, but the Gaussian momentum integral must be replaced by the spin-structure-dependent lattice sum. Explicitly, for a transverse polarization a, its two-vertex contribution per unit volume is

$$\frac{2t^2}{\mathrm{Vol}}\sum_{ij,p}|T_{ij}|^2\int_0^1 e^{-t[(1-u)(p^2+m_i^2)+u((p+q)^2+m_j^2)]}$$
$$\left(|(2p+q)\cdot a|^2 + (d_{ij}^2+q^2)|a|^2\right)du,$$

and the quadratic-vertex contribution is

$$-\frac{4t}{\mathrm{Vol}}\sum_{ij,p}|T_{ij}|^2 e^{-t(p^2+m_i^2)}|a|^2.$$

Consequently, the continuum formulas below are statements on $\mathbb{R}^4$, or in the infinite-volume limit. A finite torus can have additional holonomy dependence, including for constant commuting connections.

2. Lemma (gauge invariance and the Goldstone direction).

Let scalar fluctuations include the gauge orbit of V_0. Then the nonzero vector coefficient C_0 in (2) is compatible with gauge invariance.

For one generator, parameterize the orbit by

$$V(x) = e^{i\pi(x)T} V_0 e^{-i\pi(x)T}.$$

With our conventions,

$$a_\mu \longmapsto a_\mu - \partial_\mu \alpha, \qquad \pi \longmapsto \pi + \alpha.$$

The invariant combination is

$$B_\mu = a_\mu + \partial_\mu \pi. \tag{11}$$

Proof. Gauge covariance gives

$$D_{A^U,V^U} = U D_{A,V} U^{-1}.$$

Applying $U = e^{-i\pi T}$ transforms V to V_0 and a to B. Hence

$$S[a, e^{i\pi T} V_0 e^{-i\pi T}] = S[B, V_0]. \tag{12}$$

This establishes the Goldstone completion directly, without a derivative expansion.

For comparison, the longitudinal vector kernel is obtained from (1) by replacing the q^2 in its prefactor by $(1-2u)^2 q^2$ inside the integral. If $[T, V_0] = 0$, its integral vanishes by

$$\int_0^1 \left[2 + tq^2(1-2u)^2\right] e^{-tq^2 u(1-u)}\, du = 2.$$

For a broken generator the longitudinal kernel need not vanish; it belongs to the invariant combination (11).

If V_0 is not stationary, a Cartesian scalar Hessian alone need not annihilate the linear gauge-orbit tangent. The quadratic term in $e^{i\pi T} V_0 e^{-i\pi T}$, contracted with the scalar

tadpole, must also be retained. Equation (12) includes that term. Transverse vectors do not mix quadratically with scalar fluctuations on this translation-invariant background. □

3. Corollary (the asymptotic normalization).

Let all finite eigenvalues be small relative to Λ, let f satisfy (4), and let Q below be positive definite on the chosen generator space (otherwise pass to the quotient by its null space). Define

$$Q_{ab} = \mathrm{Re}\,\mathrm{Tr}(T_a T_b), \qquad K_{ab} = \mathrm{Re}\,\mathrm{Tr}([T_a, V_0]^\dagger [T_b, V_0]).$$

At leading order,

$$C_0 = \frac{f(0)}{8\pi^2} K, \qquad C_1 = \frac{f(0)}{12\pi^2} Q,$$

and therefore

$$\boxed{\mathcal{M}^2_{\mathrm{Taylor}} = \frac{3}{2} Q^{-1/2} K Q^{-1/2}.} \tag{13}$$

This reproduces Proposition 13.5.

For the full three-generation Standard Model representation,

$$Q_{11} = 12, \qquad K_{11} = \sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2),$$

so

$$\boxed{8M_W^2 = \sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2).} \tag{14}$$

Proof. In (2), the orbital expression

$$\frac{2}{t} I_t(a, b; 0) - \frac{e^{-ta} + e^{-tb}}{t}$$

starts at order $t(a-b)^2$. The leading term is consequently d_{ij}^2. In (3), the leading integral is $\int_0^1 (1 - 2u(1-u))du = 2/3$, proving (13).

For $T_1 = \sigma_1/2$ on each left doublet, one generation of particles contributes 2 to Q_{11}. Charge conjugates and three generations give 12. A doublet with Dirac masses

m_u, m_d contributes $\frac{1}{2}(m_u^2 + m_d^2)$ to the particle commutator trace, before color multiplicity. This proves (14).

Equivalently, the gauge coefficient is

$$\frac{f(0)g_2^2}{2\pi^2} = \frac{1}{4}.$$

For comparison with CCM's $\operatorname{Tr} F(D/\Lambda)$, take $F(v) = f(v^2)$; then $F(0) = f(0)$. The normalization and mass constraint are precisely equations (4.10) and (4.16), with the latter stated in Lemma 4.6 of the checked preprint. CCM describes the relation as "valid at unification scale." Chamseddine–Connes–Marcolli, pp. 37–39. □

The leading bound $M^2 \le 6\|V_0\|^2$ in Lemma 14 follows from (13) and $K \le 4\|V_0\|^2 Q$. Formula (13), rather than an exact finite-cutoff identity, is its applicable approximation.

4. Proposition (the exact charged-vector Taylor mass).

Let the finite data be the full three-generation Standard Model representation, including charge conjugates. Let the Majorana matrix and every Dirac mass except one up-type color triplet be zero. Write

$$m = \mu\Lambda, \qquad x = \mu^2, \qquad r = e^{-x}.$$

Let $f(z) = e^{-z}$. Define

$$H(y) = \int_0^1 e^{-yu(1-u)}\, du,$$

$$g(y) = (2+y)H(y) - 2,$$

$$h_x(y) = (2+x+y)\int_0^1 e^{-xu-yu(1-u)}\, du - 1 - e^{-x}. \quad (15)$$

Then the full charged-vector kernel is

$$\boxed{C_W(q^2) = \frac{\Lambda^2}{4\pi^2}\left[\frac{9}{2}g(y) + \frac{3}{2}h_x(y)\right], \qquad y = q^2/\Lambda^2.} \quad (16)$$

Define

$$A(x) = h_x(0) = \frac{2[1-(1+x)e^{-x}]}{x},$$

$$w(x) = 3\int_0^1 u^2 e^{-xu}\,du = \frac{3[2-(x^2+2x+2)e^{-x}]}{x^3}. \quad (17)$$

Continuous values are understood at $x = 0$. The particle-trace kinetic coefficient is

$$K_W(x) = \frac{9}{2} + \frac{3}{2}w(x) > 0.$$

The Taylor mass, defined by $M^2_{W,\mathrm{Taylor}} = C_W(0)/C'_W(0)$, is

$$\boxed{\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{9A(x)}{4K_W(x)} = \frac{3[1-(1+x)e^{-x}]}{x[3+w(x)]}.} \quad (18)$$

Its limiting behavior is

$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{3}{8}x - \frac{23}{128}x^2 + O(x^3), \qquad \frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} \sim \frac{1}{x} \quad (x\to\infty). \quad (19)$$

Consequently,

$$\frac{m}{M_{W,\mathrm{Taylor}}} \longrightarrow \sqrt{\frac{8}{3}} \quad (\mu\to 0), \qquad \frac{m}{M_{W,\mathrm{Taylor}}} \sim \mu^2 \quad (\mu\to\infty). \quad (20)$$

Proof. On the particle space, $Q_{11} = 6$. The selected quark doublet contributes $3/2$, while the remaining doublets contribute $9/2$. Diagonalizing the massive left-right block produces eigenvalues $+m, -m$. The charged generator connects these to the massless down states; every such pair has $d_{ij}^2 = m^2$. Formula (1) therefore gives h_x for the selected doublet and g for the remaining doublets. Charge conjugation supplies the factor two in (16).

Integration by parts gives

$$h_x(y) = 2\int_0^1 u\,[x + 2y(1-u)]e^{-xu-yu(1-u)}\,du, \quad (21)$$

and

$$\boxed{h'_x(y) = 2\int_0^1 u^2[1-y(1-u)]e^{-xu-yu(1-u)}\,du.} \quad (22)$$

The boundary term proving (22) is $\partial_u[u(1-u)e^{-xu-yu(1-u)}]$. Thus

$$h'_x(0) = \frac{2}{3}w(x), \qquad g'(0) = \frac{2}{3}.$$

This proves (18). Expanding the elementary expressions at zero and taking their large-x limits proves (19)–(20). □

The same Taylor result extends to every f in (4). Put

$$\ell = f(0), \qquad w_f(x) = \int w(sx)\, d\nu(s),$$
$$A_f(x) = \int \frac{A(sx)}{s}\, d\nu(s).$$

Then

$$\boxed{\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{9A_f(x)}{4\left(\frac{9}{2}\ell + \frac{3}{2}w_f(x)\right)}.} \tag{23}$$

All denominators are positive, and dominated convergence gives

$$M^2_{W,\mathrm{Taylor}} \sim \frac{3}{8}m^2 \quad (x \to 0),$$
$$\boxed{\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} \sim \frac{\int s^{-2}\, d\nu(s)}{\ell x} \quad (x \to \infty).} \tag{24}$$

Indeed, $A(z)/z \to 1$, $w(z) \to 1$ at zero, while $zA(z) \to 2$, $w(z) \to 0$ at infinity.

If the other Dirac masses are merely small and nonzero, the answer depends on those additional parameters and is given by (1). In particular, the vanishing limit in (19) assumes they are zero: masses of surviving light doublets can leave a residual vector mass.

5. Theorem (the charged-vector pole).

Under the heat-cutoff hypotheses of Proposition 4, define a timelike pole by analytic continuation of the quadratic inverse propagator:

$$C_W(-M^2_{W,\mathrm{pole}}) = 0. \tag{25}$$

For every $x > 0$, this equation has exactly one solution with $M^2_{W,\mathrm{pole}} > 0$. It is simple and has positive residue in the convention that the Euclidean propagator near the pole is a positive multiple of $(q^2 + M^2_{W,\mathrm{pole}})^{-1}$.

Let $z(x) = M^2_{W,\mathrm{pole}}/\Lambda^2$. Its exact specification is the unique positive root

$$\boxed{\frac{9}{2}g(-z) + \frac{3}{2}h_x(-z) = 0.} \tag{26}$$

Furthermore,

$$z(x) \sim \frac{3}{8}x \quad (x \to 0), \qquad z(x) \sim \frac{1}{x} \quad (x \to \infty), \tag{27}$$

and both mass definitions obey

$$\boxed{0 < M^2_{W,\mathrm{Taylor}},\ M^2_{W,\mathrm{pole}} \leq \frac{\Lambda^2}{e}.} \tag{28}$$

Proof. Let

$$F_x(y) = \frac{9}{2}g(y) + \frac{3}{2}h_x(y).$$

Equation (22) implies $h'_x(y) > 0$ for $y \leq 0$, and also $g'(y) > 0$, since $g = h_0$. Hence F_x is strictly increasing on that half-axis. We have

$$F_x(0) = \frac{3}{2}A(x) > 0.$$

As $y \to -\infty$, the prefactors in (15) become negative and both integrals grow exponentially, so $F_x(y) \to -\infty$. This proves existence, uniqueness, simplicity, and the residue sign.

For $y \leq 0$, equation (22) gives $g'(y) \geq 2/3$, and therefore

$$F'_x(y) \geq 3.$$

The mean-value theorem yields

$$z(x) \leq \frac{1}{2}A(x).$$

The Taylor denominator gives the same bound for $M^2_{W,\mathrm{Taylor}}/\Lambda^2$. Finally,

$$A(x) = 2\int_0^1 xu\, e^{-xu}\, du \leq \frac{2}{e},$$

proving (28).

As $x \to 0$, the bound implies $z(x) \to 0$, and $F'_x(y) \to 4$ near the root. As $x \to \infty$, $z(x) \to 0$, $h'_x(y) \to 0$ near the root, and $F'_x(y) \to 3$. Together with $F_x(0) \sim 3x/2$ and $F_x(0) \sim 3/x$, respectively, these prove (27). □

The pole mass governs the linearized dispersion relation of this analytically continued quadratic action. The Taylor mass governs its two-derivative truncation; they need

not coincide near the cutoff. We assert no closed-form inversion of (26), no classification of complex poles, and no Lorentzian unitarity theorem.

The pole proof also extends to positive measures supported in a compact subset of $(0,\infty)$, using

$$F_{f,x}(y) = \int \frac{1}{s}\left[\frac{9}{2}g(sy) + \frac{3}{2}h_{sx}(sy)\right] d\nu(s).$$

It gives the analogues of (24) for the pole mass. For the larger class (4), only the Euclidean and Taylor conclusions have been established here; its moment assumptions alone do not justify that continuation.

6. Proposition (unbroken coefficients, neutral vectors, and weak angles).

Retain the heat-cutoff Standard Model family of Proposition 4. Normalize color generators by $\mathrm{tr}_{\mathbf{3}}\, t_a t_b = \frac{1}{2}\delta_{ab}$, and use $Q = T_3 + Y$. Define the full-trace coefficients by

$$S^{(2)}_{\text{unbroken}} = \int \left(\kappa_{\text{em}} F_{\text{em}}^2 + \kappa_s \sum_a G_a^2\right).$$

Then

$$\boxed{\begin{aligned} &\kappa_{\text{em}} = \frac{1}{24\pi^2}\frac{16}{3}(5+r), \qquad \kappa_s = \frac{1}{24\pi^2}2(5+r), \\ &\frac{\kappa_{\text{em}}}{\kappa_s} = \frac{8}{3}. \end{aligned}} \tag{29}$$

Both coefficients approach 5/6 of their all-massless values.

Let K denote the particle-trace neutral q^2 coefficient in coordinates (A_3, A_Y). Then

$$\boxed{\begin{aligned} K_{33} &= \frac{21}{4} + \left(\frac{3}{4} - \frac{3x}{8}\right) r, \\ K_{YY} &= \frac{103}{12} + \left(\frac{17}{12} - \frac{3x}{8}\right) r, \\ K_{3Y} &= -\frac{1}{4} + \left(\frac{1}{4} + \frac{3x}{8}\right) r. \end{aligned}} \tag{30}$$

The matrix K, the charged coefficient K_W, and the unbroken coefficients are positive for every $x \geq 0$.

Define the neutral-current angle from the current of the massive direction after diagonalizing the two-derivative neutral kinetic form:

$$J_Z \;\propto\; T_3 - \sin^2\theta_{\mathrm{NC}}\, Q.$$

Then

$$\boxed{\cos^2\theta_{\mathrm{NC}} = \frac{K_{YY} + K_{3Y}}{K_{33} + 2K_{3Y} + K_{YY}} = \frac{5}{8}.} \tag{31}$$

A different continuation, namely

$$\cos^2\theta_{\mathrm{match}} := \frac{K_{YY}}{K_W + K_{YY}},$$

satisfies

$$\boxed{\cos^2\theta_{\mathrm{match}}(0) = \frac{5}{8}, \qquad \cos^2\theta_{\mathrm{match}}(\infty) = \frac{103}{157}.} \tag{32}$$

Thus these definitions cease to describe one common electroweak angle.

Proof. Photon and color generators commute with V_0. Their coefficients therefore follow from (3) with $d_{ij} = 0$ on each invariant mass block. The charge traces give (29).

A Dirac species of mass m, with neutral charges q_L^a, q_R^a, contributes to K

$$e^{-x}\left[q_L^a q_L^b + q_R^a q_R^b - \frac{x}{2}(q_L^a - q_R^a)(q_L^b - q_R^b)\right]. \tag{33}$$

Applying (33) to the three top colors, and retaining all other states, proves (30).

Write

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad B_\infty = \begin{pmatrix} 21/4 & -1/4 \\ -1/4 & 103/12 \end{pmatrix},$$
$$U = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 17/12 \end{pmatrix}.$$

Then

$$K = B_\infty + rU - \frac{3xr}{8} vv^T.$$

We have $B_\infty \geq 5I$, $U \geq 0$, and $xr \leq 1/e$. Hence

$$K \geq \left(5 - \frac{3}{4e}\right) I > 0.$$

Positivity of K_W follows from (17).

The neutral mass direction is v, while the photon direction is $(1,1)$. The kinetic-orthogonal massive field is proportional to $K^{-1}v$, giving the first expression in (31). Directly,

$$K_{33}+K_{3Y}=5+r, \qquad K_{YY}+K_{3Y}=\frac{5}{3}(5+r),$$

which proves (31). Taking limits proves (32). □

The neutral mass also tends to zero. More precisely,

$$\boxed{\frac{M^2_{Z,\mathrm{Taylor}}}{\Lambda^2}=\frac{9}{4}xr\,v^TK^{-1}v\sim\frac{2}{3}xe^{-x}\quad(x\to\infty).} \tag{34}$$

For $x>0$ there is likewise one positive massive neutral pole, besides the photon pole. To prove this, put $B=B_\infty+rU$. The exact neutral kernel, apart from the common factor $\Lambda^2/(4\pi^2)$, is

$$B\,g(y)+\frac{3}{2}xr\,H(y)vv^T. \tag{35}$$

After congruence by $B^{-1/2}$, its massive eigenvalue is

$$g(y)+\beta H(y), \qquad \beta=\frac{3}{2}xr\,v^TB^{-1}v>0.$$

Its timelike root obeys

$$z-2+\frac{2}{H(-z)}=\beta. \tag{36}$$

The left side increases from zero to infinity, because

$$\frac{d}{dz}\left(z-2+\frac{2}{H(-z)}\right)=1-\frac{2\,\mathbb{E}_z[b]}{H(-z)}\geq\frac{1}{2},$$

where $\mathbb{E}_z$ uses density proportional to $e^{zb(u)}$. The pole is therefore unique and has positive residue. Since $v^TB_\infty^{-1}v=8/27$,

$$\frac{M^2_{Z,\mathrm{pole}}}{\Lambda^2}\sim\frac{2}{3}xe^{-x}.$$

In particular, M_W^2/M_Z^2 does not continue to equal $5/8$.

For every f in (4), equations (29)–(33) extend by replacing 1 by $\ell=f(0)$, r by $f(x)$, and xr by $-xf'(x)$.

Positivity follows by averaging the positive heat-cutoff matrices. The ratio 8/3, the neutral-current value 5/8, and the matching limit 103/157 persist. At the endpoint both neutral masses vanish, so the distinguished massive neutral direction—and hence its angle—requires continuation from finite x.

7. Counterexample (positive cutoff does not ensure positive broken kinetic coefficients).

Let $H_F = \mathbb{C}^2$,

$$V_0 = m\sigma_3, \qquad T = \sigma_1, \qquad x = m^2/\Lambda^2,$$

with the same spacetime and trace conventions. For the heat cutoff,

$$C_T(q^2) = \frac{\mathrm{Tr}(T^2)\Lambda^2}{8\pi^2} e^{-x} \left[(2 + 4x + y)H(y) - 2\right]. \quad (37)$$

Its q^2 Taylor coefficient is proportional to

$$e^{-x}(1 - x).$$

It vanishes at $x = 1$ and becomes negative for $x > 1$. Thus the two-derivative truncation has a wrong-sign vector kinetic term in this example.

Nevertheless, the full quadratic form has a unique positive real timelike pole with positive residue, satisfying

$$\boxed{4x < \frac{M_{\mathrm{pole}}^2}{\Lambda^2} < 4x + 2.} \quad (38)$$

Consequently $M_{\mathrm{pole}} \sim 2m$ diverges at fixed cutoff.

Proof. The generator connects the eigenvalues $m, -m$, so $d_{ij}^2 = 4m^2$ while both squared eigenvalues equal m^2. Substitution in (1) proves (37). Its Taylor ratio is

$$M_{\mathrm{Taylor}}^2 = \frac{6m^2}{1 - x},$$

where this is a positive mass only for $x < 1$.

The pole equation is (36) with $\beta = 4x$. The monotonicity proof there gives uniqueness and the residue sign. Evaluating (37) at $y = -4x$ gives $2H(-4x) - 2 > 0$, whereas at $y = -4x - 2$ it gives -2; this proves (38).

For general f satisfying (4), the Taylor kinetic coefficient in this example is proportional to

$$f(x) + xf'(x) = \frac{d}{dx}[xf(x)]. \tag{39}$$

It must be negative somewhere: $xf(x) > 0$ for $x > 0$, but $xf(x) \to 0$ at infinity under (4). A nonnegative derivative everywhere would contradict these facts. □

This counterexample concerns arbitrary finite data; the top-only Standard Model family is outside it. It also distinguishes a wrong-sign **Taylor truncation** from a negative-residue pole of the **full** form factor. We have not proved absence of complex-pole or other nonlocal problems in either model.

8. Counterexample (heavy fermions do not dynamically select the centralizer).

Let the data be those of Proposition 4, and let

$$\mathfrak{g}_{V_0} = \{X \in \mathfrak{g}_{\mathrm{SM}} : [X, V_0] = 0\}.$$

For nonzero top mass this is the represented unbroken gauge algebra $\mathfrak{su}(3) \oplus \mathfrak{u}(1)_{\mathrm{em}}$. The limit $x \to \infty$ at fixed Λ does **not** produce a theory containing only these vector directions by quadratic mass decoupling.

Proof. For fixed y, equation (21) implies

$$h_x(y) \longrightarrow 0.$$

Consequently

$$\boxed{C_W(q^2) \longrightarrow \frac{\Lambda^2}{4\pi^2}\frac{9}{2}g(q^2/\Lambda^2).} \tag{40}$$

For positive y,

$$g(y) = 4y \int_0^1 u(1-u)e^{-yu(1-u)}\,du > 0.$$

Thus a charged generator outside $\mathfrak{g}_{V_0}$ retains a finite nonzero quadratic kinetic form and a nonvanishing propagator at every fixed positive momentum. Its mass approaches zero. Equation (35) similarly leaves the positive matrix $B_\infty g(y)$, including the additional neutral direction. These facts contradict decoupling of the broken

vectors through diverging masses or vanishing propagators. □

A narrower decoupling statement is true. If we **restrict the connection from the outset** to the centralizer, the finite mass blocks are invariant and the heat trace factorizes. On the heavy block its contribution carries e^{-sm^2/Λ^2}, which tends to zero. The restricted action therefore converges to the light-block spectral action. This proves decoupling of a finite fermion block under that restriction; it does not prove dynamical elimination of the other gauge fields.

The amendment to Lemma 03 Part C is therefore:

- Its transverse kinetic weights, including $w_L(x,0) = w(x)$, and its two weak-angle formulas are correct under their stated conventions.
- They must be accompanied by the mass coefficient (2), including the quadratic-vertex subtraction.
- A centralizer-only endpoint does not follow from those weights. In the present exact quadratic family, the broken-vector masses vanish while their remaining kinetic coefficients are positive.

9. No-go theorem (finite radial vacuum for the positive Laplace class).

Let f satisfy (4), and suppose the allowed scalar configurations include

$$V = cV_*, \qquad c > 0, \qquad V_* \neq 0,$$

with $A = 0$. Let the action be the literal spectral action, without an added scalar potential or a constraint fixing c. Then no finite $c > 0$ is stationary along this radial direction.

Proof. On $\mathbb{R}^4$, the constant-background action density is

$$\mathcal{U}(c) = \frac{\Lambda^4}{4\pi^2} \int s^{-2} \operatorname{Tr}_{H_F} e^{-sc^2 V_*^2/\Lambda^2}\, d\nu(s).$$

Differentiating gives

$$\boxed{\mathcal{U}'(c) = -\frac{c\Lambda^2}{2\pi^2} \int s^{-1} \operatorname{Tr}_{H_F} \left(V_*^2 e^{-sc^2 V_*^2/\Lambda^2} \right) d\nu(s) < 0.} \tag{41}$$

The inequality is strict because $V_* \neq 0$ and ν is nonzero and positive. The same argument applies on a finite flat torus before performing its positive momentum sum. □

Thus the masses above are conditional quadratic-background quantities. In the top-only family, the ordinary Higgs radial variation supplies the direction used in this theorem. A finite symmetry-breaking minimum of the polynomial asymptotic scalar potential cannot automatically be regarded as a minimum of this full positive-Laplace spectral action.

10. Corollary (comparison of heavy limits).

Let the background family and mass definitions be those above.

At fixed Λ, with $\mu \to \infty$:

quantity, heat cutoff	behavior
m_t	$\mu\Lambda \to \infty$
$M_{W,\mathrm{Taylor}},\ M_{W,\mathrm{pole}}$	$\sim \Lambda/\mu \to 0$
$M_{Z,\mathrm{Taylor}},\ M_{Z,\mathrm{pole}}$	$\sim \sqrt{2/3}\,\Lambda\mu e^{-\mu^2/2} \to 0$
m_t/M_W	$\sim \mu^2 \to \infty$

(42)

At fixed $0 < \mu < \infty$, with $\Lambda \to \infty$, all four vector masses in (42) instead grow linearly with Λ, as does m_t. A controlled asymptotic family can also satisfy

$$\Lambda \to \infty, \qquad \mu \to 0, \qquad \mu\Lambda \to \infty;$$

then $M_W, m_t \to \infty$ while

$$m_t/M_W \to \sqrt{8/3}.$$

Proof. The first assertions follow from (19), (27), and (34)–(36). At fixed positive μ, their dimensionless mass functions are fixed and positive, so multiplication by Λ proves the joint-limit assertions. The final assertion follows from the small-μ limits. □

Accordingly, the obstruction has three distinct scopes. The CCM identity cannot be extrapolated to $\mu \to \infty$ at fixed cutoff because it belongs to the asymptotic truncation. The exact top-only quadratic action obstructs a fixed-cutoff heavy-W limit for the heat cutoff under

both mass definitions, and for class (4) under the Taylor definition. A family of actual nonzero stationary vacua of the unmodified positive-Laplace action encounters the separate obstruction (41). Counterexample 7 excludes extending the first of these exact mass conclusions to arbitrary finite representations.

11. Proposition (the radial derivative for general f; referee). On flat $\mathbb{R}^4$, for a sufficiently integrable real cutoff f, with λ_j^2 the eigenvalues of V_*^2,

$$\mathcal{U}'(c) = -\frac{c\Lambda^2}{2\pi^2} \sum_j \lambda_j^2 \int_{c^2\lambda_j^2/\Lambda^2}^{\infty} f(z)\, dz. \tag{43}$$

For nonnegative f, a finite $c > 0$ is stationary exactly when every tail associated with a nonzero λ_j vanishes. So if f has nonzero support arbitrarily far out, there is no finite stationary point on this ray; if f has compact support ending at R, there is a stationary plateau once every nonzero $c^2\lambda_j^2/\Lambda^2 \geq R$. Sign-changing cutoffs can have cancellations. A nonconstant cutoff flat near zero lies outside (4), since a nonzero positive measure gives $f'(0) < 0$; (43) covers the nonnegative ones. A radial path with an additional fixed noncommuting mass matrix need not have the form cV_*. □

12. Proposition (the top-only Taylor mass for general f; referee). For nonnegative smooth compactly supported f with $f(0) > 0$, the formulas of Proposition 4 hold with

$$A_f(x) = \frac{2}{x} \int_0^x z f(z)\, dz, \qquad w_f(x) = \frac{3}{x^3} \int_0^x z^2 f(z)\, dz,$$

by approximating f with its derivatives by real combinations of e^{-nz}. Hence

$$\frac{M_{W,\mathrm{Taylor}}^2}{\Lambda^2} \sim \frac{\int_0^\infty z f(z)\, dz}{f(0)\, x}.$$

This includes cutoffs flat near zero. For a nonanalytic cutoff there need be no analytic continuation defining the pole. □

13. Proposition (several heavy doublets; referee). For the heat cutoff, let P be the total particle-trace weight of complete massless doublets, and let heavy doublet i, of weight $d_i > 0$, have one Dirac member of mass

m_i and one massless member. Then

$$C_W(y) = \frac{\Lambda^2}{4\pi^2}\Big(Pg(y) + \sum_i d_i h_{x_i}(y)\Big),$$
$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{3}{2}\,\frac{\sum_i d_i A(x_i)}{P + \sum_i d_i w(x_i)}.$$

For finitely many doublets, if $P > 0$ and all $x_i \to \infty$, the Taylor mass tends to zero. If $P = 0$ and all $x_i = x$, then $M^2_{W,\mathrm{Taylor}} \sim m^4/(2\Lambda^2)$, the form factor has a unique positive pole with $M^2_{W,\mathrm{pole}} \sim m^2/2$, and the whole unnormalized kernel tends to zero at every fixed Euclidean momentum. The divergent Taylor and pole scales differ, and neither alone establishes a healthy limiting gauge theory. □

14. Proposition (the fully massive family; referee; the complete proof by Astra 6 follows the sketch). Give both members of every Standard Model doublet the same Dirac mass m, with no Majorana masses, and let z_W, z_Z be the charged and massive neutral poles in units of Λ^2. The charged form factor is proportional to $e^{-x}[g(y) + 2xH(y)]$, and

$$2x < z_W < 2x + 2, \qquad \tfrac{16}{5}x < z_Z < \tfrac{16}{5}x + 2.$$

So both pole masses diverge with m at fixed cutoff, and

$$\frac{M^2_{W,\mathrm{pole}}}{M^2_{Z,\mathrm{pole}}} \longrightarrow \frac{5}{8} \qquad (x \to \infty),$$

as also for $x \to 0$, where $z_W \sim 3x$ and $z_Z \sim \frac{24}{5}x$. In between the ratio differs, reaching about 0.66 near $x = 3$ (Python, solving the pole equations). The kernel tends to zero at every fixed Euclidean momentum, and the charged Taylor kinetic coefficient is proportional to $e^{-x}(1 - x/2)$, negative for $x > 2$.

Proof. The species with equal mass have eigenvalues $\pm m$. The charged generator connects them with $d^2 = 0$ and $d^2 = 4m^2$ in equal weights, which gives the charged form. For the neutral generators the diagonal charge has $d = 0$ and the off-diagonal charge $(q_L - q_R)/2 = T_3 v/2$ has $d^2 = 4m^2$; on the particle trace the kernel is $e^{-x}[\mathrm{diag}(6, 10)g + 12xHvv^T]$, so $\beta = 12x\, v^T \mathrm{diag}(6, 10)^{-1} v = \frac{16}{5}x$, and for the charged direction $\beta = 2x$. Equation (36) gives the bounds, since $H(-z) > 1$. □

Complete proof (Astra 6, 3 November 2026; written here from its validation report, item 11, so that the lemma is self-contained). Take the heat cutoff, three generations with right-handed neutrinos, no Majorana masses, and one common positive Dirac mass m for every species. Let $x = m^2/\Lambda^2$, $y = q^2/\Lambda^2$, $H(y) = \int_0^1 e^{-yu(1-u)}\,du$ and $g(y) = (2+y)H(y) - 2$. Every mass eigenvalue is $+m$ or $-m$, so the exact quadratic kernel of Theorem 1 for a generator entry is proportional to $e^{-x}[g(y) + ((m_i - m_j)^2/\Lambda^2)H(y)]|T_{ij}|^2$. A charged generator connects equal and opposite mass signs with equal weights, the average of $(m_i - m_j)^2$ is $2m^2$, and the charged kernel is proportional to $e^{-x}[g(y) + 2xH(y)]$. In the neutral particle trace, with $v = (1,-1)^T$, the charge and mass-commutator traces give $Q = \mathrm{diag}(6,10)$ and $C_{\mathrm{neutral}}(y) \propto e^{-x}[Qg(y) + 12xH(y)vv^T]$, whose determinant factors as $\det Q\, g(y)[g(y) + 12x\, v^TQ^{-1}v\, H(y)]$; since $v^TQ^{-1}v = 1/6 + 1/10 = 4/15$, the massive neutral factor has coefficient $16x/5$. Charge conjugation multiplies the common normalization and moves no root. Both massive pole equations therefore have the form $(2 + \beta - z)H(-z) = 2$ with $\beta_W = 2x$ and $\beta_Z = 16x/5$. Let $h(z) = H(-z)$ and $F(z) = z - 2 + 2/h(z)$. For $z \geq 0$, $h(z) \geq 1$ and $0 < h'(z) = \int_0^1 u(1-u)e^{zu(1-u)}\,du \leq h(z)/4$, so $F'(z) = 1 - 2h'(z)/h(z)^2 \geq 1 - 1/(2h(z)) \geq 1/2$; with $F(0) = 0$ and $F(z) \to \infty$, each $\beta > 0$ has a unique positive solution, at which $h(z) > 1$, so $z = \beta + 2 - 2/h(z)$ gives $\beta < z < \beta + 2$: the displayed bounds. For small z, $h(z) = 1 + z/6 + O(z^2)$ and $F(z) = \frac{2}{3}z + O(z^2)$, so $z_W \sim 3x$, $z_Z \sim 24x/5$ and the ratio tends to $5/8$; for $x \to \infty$ the bounds give $z_W = 2x + O(1)$, $z_Z = 16x/5 + O(1)$, again $5/8$. At fixed Euclidean momentum every kernel entry is e^{-x} times a function growing at most linearly in x, hence tends to zero. Finally $H'(0) = -1/6$ and $g'(0) = 2/3$, so the charged Taylor kinetic coefficient is proportional to $e^{-x}(\frac{2}{3} - \frac{x}{3}) = \frac{2}{3}e^{-x}(1 - x/2)$, negative for $x > 2$. A quadrature and root calculation (Python 3, SciPy, not kept) gave $z_W = 7.4996$, $z_Z = 11.3746$, ratio 0.6593 at $x = 3$. □

This is a counterexample to any general no-go for diverging pole locations at fixed cutoff. It has no light fermions, and the physical quality of that limit is open.

15. Remark (the light space does not carry un-

broken $SU(2) \times U(1)$; referee). In the top-only limit the broken masses vanish, but this does not restore the electroweak group on the whole light space. The light space retains b_L after removing t_L. If $P_{\rm light}$ projects off the top mass block, then on b_L we have $P_{\rm light} T_1 P_{\rm light} = P_{\rm light} T_2 P_{\rm light} = 0$ and $P_{\rm light} T_3 P_{\rm light} = -\frac{1}{2}$, which do not obey the $SU(2)$ relations. Correspondingly $K_W = 9/2$, $K_{33} = 21/4$, $K_{3Y} = -1/4$ in the limit, while a linearly realized unbroken action would have equal weak coefficients and no mixing. The remaining complete massless doublets carry the ordinary representation, and their contribution is gauge invariant. If both members of every removed doublet are heavy, the retained light space can be invariant under the full group; a full limiting theory still needs control of the remaining fields, normalizations, interactions and vacuum. "The heavy fermion leaves the loop" is shorthand for the suppression in this classical trace.

The light-space statement should be read together with the separate role of the top (lemma 21): among single charged species only an up-type quark can be removed while keeping cos^2 theta_NC = 5/8 and the ratio 8/3, and removing it leaves the five other quarks and the leptons, on which SU(3) x U(1)_em acts with b_L paired with b_R.

16. Remark (the CCM relation as the small-mass form of a weighted sum rule; Claude). For the heat cutoff and doublets with one massive member, Proposition 13 reads

$$8\kappa M^2_{W,\mathrm{Taylor}} = \sum_f N_f m_f^2 \, \rho(m_f^2/\Lambda^2),$$
$$\rho(x) = \frac{A(x)}{x} = \frac{2[1-(1+x)e^{-x}]}{x^2},$$

with $N_f = 2d_f$ and $\kappa = (P + \sum_i d_i w(x_i))/6$. As $x \to 0$, $\rho, \kappa \to 1$, which is the CCM relation (14). The relation is therefore the leading term of an expansion, and its range is $m \ll \Lambda$. For the top alone, $\kappa = (3+w)/4$; m/M_W departs from $\sqrt{8/3}$ by 5% at $m \approx 0.45\Lambda$, and M_W peaks at about 0.52Λ at $m \approx 1.42\Lambda$ (Python). The maximum is unique, at $x = 2.024$ with $M_W/\Lambda = 0.5242$ (referee, by a sign argument and Python). A massive diagonal contribution to the trace is exponentially suppressed above the cutoff, while the mixed loop with a massless partner

has the algebraic $\rho \sim 2/x^2$ at large x; a vacuum with no other mass leaves the W mass with no source. Whether the classical action is meaningful for $m > \Lambda$ is a separate question, on which these formulas do not decide. In the top-only heat-cutoff family, the ratio differs from $\sqrt{8/3}$ for every $x > 0$, since $3 + w - 4\rho$ vanishes at zero and is increasing, so the limit with M_W and m_t diverging together and $m_t/M_W \to \sqrt{8/3}$ needs $\Lambda \to \infty$ faster than m_t (Corollary 10). This necessity is for that family: with one complete massless lepton doublet and one member of each other doublet of common mass m, the weighted sum equals the unweighted CCM sum at some $x_* > 0$ (referee).

As algebraic checks, we used **Python 3 with SymPy** to verify the integration-by-parts identities, the displayed small-x Taylor series, and $v^T B_\infty^{-1} v = 8/27$. The arguments above provide the proofs; these computations introduce no fitted constants.

Consequence for T5b. At fixed cutoff, the specified top-only Standard Model spectral action does not realize an infinitely heavy W, Z, t family: the top grows while both vector masses tend to zero, for the heat cutoff and the stated quadratic mass definitions. A joint limit with $\Lambda \to \infty$ permits all three background mass scales to diverge, but neither centralizer-only decoupling nor a stationary Higgs vacuum follows from that scaling. These conclusions concern the classical bosonic action and contain no quantum corrections or claims about the physical Standard Model at those scales. Other finite data behave differently: with every doublet heavy and no light fermions, the pole masses diverge at fixed cutoff with M_W^2/M_Z^2 -> 5/8 (Proposition 14). With a massless complete doublet retained, the Taylor mass tends to zero (Proposition 13). So in the families treated here, a heavy W and Z at fixed cutoff is found only without a massless complete doublet. Light nonzero masses are not treated: they leave a residual vector mass that depends on them.

Part 19

Inner fluctuations of the two-point data without order one

Step 1 of the next steps in the background note, for STRATEGY T0b. Proved by Claude Opus 5.5 without Astra or Fable; every identity checked with numpy on random data (section "Numerical check"). Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): Theorem 19.1 and the comparison with arXiv:1304.7583 are correct; Corollary 19.2 item 3 and the Wilson phase after the table were wrong as first written, and these and the amendments are incorporated below.

Outcome. The truncated two-point data satisfy order zero and fail order one, for each natural real structure. So the inner fluctuations of Chamseddine-Connes-van Suijlekom (arXiv:1304.7583), with their quadratic term, are the applicable ones. The result factorizes. With the inherited pointwise real structure, the fluctuated operator is m |Phi|^2 sigma_1: a real gauge-neutral radial field, with trivial gauge group. With the reflection real structure, which has the KO-6 signs, the link is m Phi^2: all of C is reached, and the gauge group U(1) acts with charge 2. Along a gauge orbit the quadratic term keeps the modulus of the link fixed. This parallels the fixed singular value of the covariant projection of lemma 14, which the fixed compression of lemma 13 lacks. So the first-order failure of Counterexample 08.4 points to the framework of arXiv:1304.7583 for these truncated data.

Setting

Let A = C + C act diagonally on H = C^2, with D = m sigma_1, m > 0, and gamma = sigma_3. For a real structure J = C K put x-hat = J x J^-1. We use two choices.

- **J_0 = K** (or sigma_3 K, the other sign of lemma 08), inherited from pointwise reality on the interval. Then x-hat = conj(x) on A, and J_0 commutes with gamma.
- **J_r = sigma_1 K**, inherited from the reflection real structure sigma_1 R_l K of Counterexample 08.4. Then diag(x_1, x_2)-hat = diag(conj(x_2),

conj(x_1)), J_r^2 = 1, J_r D = D J_r and J_r gamma = -gamma J_r: the KO-6 signs.

Both satisfy order zero, since A is commutative and x-hat lies in A. Both fail order one: for e = diag(1, 0), [[D, e], e-hat] = m sigma_1 for J_0 and -m sigma_1 for J_r.

For a universal one-form A = sum_j a_j delta(b_j), the fluctuated operator of arXiv:1304.7583 is

```
D' = D + A_(1) + A~_(1) + A_(2),
A_(1) = sum a_j [D, b_j],   A~_(1) = sum a_j-hat [D,
 ↪  b_j-hat] = eps J A_(1) J^-1,   A_(2) = sum
 ↪  a_j-hat [A_(1), b_j-hat],
```

with J D J^-1 = eps D. Gauge transformations are U = u J u J^-1 for unitary u in A, and U D(A) U* = D(gamma_u(A)), where gamma_u(A) = u delta(u*) + u A u*.

Put

```
phi_1 = sum a_{j,1} (b_{j,2} - b_{j,1}),     phi_2 =
 ↪  sum a_{j,2} (b_{j,1} - b_{j,2}),
```

so that A_(1) = m [[0, phi_1], [phi_2, 0]], which is self-adjoint exactly when phi_2 = conj(phi_1). The pair (phi_1, phi_2) takes every value in C^2, as a = diag(x, y) and b = diag(0, 1) show.

The fluctuated operator

Theorem 19.1. For every universal one-form,

```
J_0:   D' = m [[0, |1 + phi_1|^2], [|1 + phi_2|^2,
 ↪  0]],
J_r:   D' = m [[0, (1 + phi_1)(1 + conj(phi_2))],
 ↪  [(1 + phi_2)(1 + conj(phi_1)), 0]].
```

For a self-adjoint form, with Phi = 1 + phi_1,

```
J_0:   D' = m |Phi|^2 sigma_1,          J_r:   D' =
 ↪  m [[0, Phi^2], [conj(Phi)^2, 0]].
```

Proof. For diagonal b, [X, b] for off-diagonal X = [[0, x], [y, 0]] is [[0, x (b_2 - b_1)], [y (b_1 - b_2), 0]].

For J_0: a-hat and b-hat are the conjugates. Then A~*(1) = m [[0, conj(phi_1)], [conj(phi_2), 0]], and the first row of A*(2) is m phi_1 sum conj(a_{j,1}) (conj(b_{j,2}) - conj(b_{j,1})) = m phi_1 conj(phi_1); the second row is m phi_2 conj(phi_2). So the (1,2) entry of D' is m (1

+ phi_1 + conj(phi_1) + |phi_1|^2) = m |1 + phi_1|^2, and likewise (2,1).

For J_r: a-hat = diag(conj(a_2), conj(a_1)) and b-hat = diag(conj(b_2), conj(b_1)). Then A~*(1) = sigma_1 conj(A*(1)) sigma_1 = m [[0, conj(phi_2)], [conj(phi_1), 0]], and the first row of A_(2) is m phi_1 sum conj(a_{j,2}) (conj(b_{j,1}) - conj(b_{j,2})) = m phi_1 conj(phi_2), and the second is m phi_2 conj(phi_1). This gives the stated factorization. ■

Corollary 19.2 (gauge action and image). Let u = diag(e^{i alpha}, e^{i beta}) and psi = alpha - beta. Under gamma_u, Phi -> e^{i psi} Phi.

1. For J_0: U = 1. The image of the self-adjoint forms is {t sigma_1 : t >= 0}, so the link is real, and t is a gauge-neutral field. The value t = 0 is reached at Phi = 0.
2. For J_r: U = diag(e^{i psi}, e^{-i psi}), and the link m Phi^2 goes to e^{2 i psi} m Phi^2. So U(1) acts with charge 2 in the variable of the link, and the image is all of C, with Phi -> Phi^2 two-to-one away from 0. The image statement is pointwise: a globally prescribed smooth link need not have a global smooth square root.
3. The gauge orbit of the zero form has Phi = e^{i psi}. It gives D' = D for J_0, whose operator orbit is the single point D, and link m e^{2 i psi} for J_r. Without A_(2), the link is m (2 cos psi - 1) for J_0, of modulus m (1 - psi^2 + O(psi^4)), although U = 1; and m (2 e^{i psi} - 1) for J_r, of modulus m (5 - 4 cos psi)^{1/2} = m (1 + psi^2 + O(psi^4)), off the orbit of D.

Proof. gamma_u(0) has phi_1 = u_1 conj(u_2) - 1 = e^{i psi} - 1. The rest is direct. ■

Item 3 shows, in the smallest example, why the quadratic term is needed: it restores gauge covariance.

Comparison with the parent

Proposition 19.3. Consider the chiral interval at M = 0, with m = pi/(2l), and the links of lemmas 13 and 14.

1. *With no real structure*, the finite fluctuation is m (1 + phi_1), all of C, while the fixed compression of a

gauge field gives only the line m + i B (Proposition 13.2).

2. *With pointwise reality*, the parent has no scalar fluctuation at all: the algebra C(I) is commutative with J f J^-1 = conj(f), it satisfies order one, and A + J A J^-1 = 0 for a scalar one-form. The truncated data do have a fluctuation, the neutral radial field t. So the finite fluctuation has a gauge-neutral radial parameter, while the specified pointwise-real parent has no surviving self-adjoint scalar inner fluctuation. This does not establish a dynamical scalar field or its action, and it says nothing about parent fields in other models. Compression is not multiplicative, so there is no canonical identification of finite potentials with parent potentials.
3. *With the reflection real structure*, the parent satisfies order one, since J_r f J_r^-1 = conj(f) composed with the reflection commutes with [D, g]. One finds J_r D_{A_5} J_r^-1 = D_{A_5 o R}, with R the reflection, so that the symmetrized field A_5 + A_5 o R survives, and the fixed compression gives m + i B^s with B^s = 2 B, because the weight sin(pi x/l) is even about l/2. The modulus along a pure-gauge field behaves as follows.

object	modulus of the link along a gauge orbit
covariant projection (lemma 14, Theorem 3)	constant: lambda W_l^* has fixed singular value
finite data with A_(2) (Corollary 19.2)	constant: m
fixed compression (Proposition 13.2), with B the integral of the symmetrized coefficient for J_r	(m^2 + B^2)^{1/2}, growing at second order along paths where B has a linear term
finite data with J_r, linear terms only	m (5 - 4 cos psi)^{1/2}, growing at second order

Proof. Item 1 is lemma 13. For item 2, a represented self-adjoint parent one-form is E h(x) with h = i a and a real, and K (i E a) K^-1 = -i E a, so the real fluctuation cancels. An individual commutator need not change sign, since K [D, x] K^-1 = [D, x] = E: self-adjointness is the

step that matters. For item 3, K changes the sign of i A_5, the reflection changes the sign of the derivative and composes A_5 with R, and sigma_1 E sigma_1 = -E for E = [[0, 1], [-1, 0]]. The table collects Theorem 3 of lemma 14, Corollary 19.2 and Proposition 13.2. ■

The table states a parallel and no identification. The variables differ: for constant unsymmetrized A_5 = a, the transported link is m e^{i a l} = m + i (pi a/2) + O(a^2), with Wilson phase a l, against m + i (2 a/pi) in the compression. For the reflection-real fluctuation from the same a, these become m e^{2 i a l} and m + i (4 a/pi). The phase of a link depends on the frame, while its modulus does not. In the parent, the covariance defect is repaired by moving the projection, and in the finite data by the quadratic term.

Reading

Spectral compression can break order one, and it can also break order zero: on the circle with J = K, compression to the zero mode preserves both, while compression to the modes -1, 0, 1 sends e^{ix} to a truncated shift S, with [S, S*] != 0, so that order zero fails. Where a represented algebra, a preserved real structure and order zero hold, as here, the framework of arXiv:1304.7583 applies to the truncated data as they stand. So the first-order failure of Counterexample 08.4 has an escape, and this calculation is a first use of it. Two points were open. Whether the parallel in the table extends to an exact statement for a fluctuated parent and its truncation: this holds on gauge orbits, by Corollary 22.6 of lemma 22, and remains open for the nonunitary part. And whether the neutral radial parameter t is related to the singlet of the grand-symmetry model (Devastato-Lizzi-Martinetti, where sigma comes from a Majorana-operator fluctuation in a larger algebra). This is an analogy only: in arXiv:1304.7583 the quadratic term does not generate new scalar fields, and here too it depends on the same two parameters. An identification would need a map of algebras, representations, fields and gauge actions.

Numerical check

Python (numpy), run on 3 Nov 2026 in a temporary directory, not kept. On random complex a_j, b_j, with m

= 1.3: the three real structures have the stated signs; the formulas of Theorem 19.1 hold to about 10^-15, without assuming self-adjointness; D(gamma_u(0)) = U D U* holds to 10^-16 for both real structures, and fails by 4.5 for the sample when A_(2) is dropped; U D(A) U* = D(gamma_u(A)) holds for a general form; the values of [[D, e], e-hat] are as stated.

Part 21

The separate role of the top: removal identities and counting

Prompted by the observation that the top has a separate role, which matters more than the loss of an electroweak partner for b_L. Proposition 21.1 and the counts below are exact calculations by Claude Opus 5.5, in Python with exact fractions. The readings in the last section are proposals, and nothing there is derived. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): one error, in Proposition 21.2, and several amendments, incorporated below. Proposition 21.4 and Corollary 21.5 (added 4 Nov 2026) were refereed by Claude Opus 5.5 (effort max, 5 Nov 2026; report): the counts correct, two amendments (the group SO(6(n_u + n_d)); the SU(3)^3 trace) and one error in the window clause, which held in reading (a) only; all applied.

Outcome. The spectral action detects the top: in the exact quadratic action of lemma 18, among single charged species only an up-type quark can be removed while keeping both the neutral-current value cos^2 theta_NC = 5/8 and the ratio 8/3 of the photon and gluon coefficients. Removing it leaves 84 of the 96 states of the finite Hilbert space. Those 84 are the seed of an SO(30), inside SO(32), whose adjoint contains the charged scalar counts of three generations, and 84 is also the number of on-shell components of the three-form of eleven-dimensional supergravity.

Removal identities

For a Dirac species with charges q_L, q_R in the coordinates (T_3, Y), equation (33) of lemma 18 gives a contribution to the neutral kinetic matrix K with weight

e^{-x}. The off-diagonal charge is (q_L - q_R)/2 = (T_3/2)(1, -1), which is orthogonal to the photon direction (1, 1). So the combinations K_33 + K_3Y and K_YY + K_3Y depend on x only through the factor e^{-x} of each massive species, and cos^2 theta_NC = (K_YY + K_3Y)/(K_33 + 2 K_3Y + K_YY) is determined by the sums of T_3 Q and Y Q.

Proposition 21.1. Per species, with colour multiplicity, on the particle trace for the first two columns and with charge conjugates and both chiralities for the last two:

species	sum T_3 Q	sum Y Q	Q^2 coefficient	colour coefficient
up-type quark	1	5/3	16/3	2
down-type quark	1/2	1/6	4/3	2
charged lepton	1/2	3/2	4	0
neutrino	0	0	0	0
three generations	6	10	32	12

Hence the ratios of the whole, 5/3 and 8/3, are those of an up-type quark alone. As one species is removed, that is, as its mass tends to infinity at fixed cutoff in the setting of lemma 18:

removed	cos^2 theta_NC	photon/gluon coefficient
up-type quark	5/8	8/3
down-type quark	59/92	46/15
charged lepton	17/28	7/3

So among single charged species only an up-type quark preserves both values. For an up-type quark they hold at every x, as in Proposition 6 of lemma 18. Within one generation, the down-type quark and charged lepton together contribute exactly what the up-type quark contributes: (1, 5/3; 16/3, 2).

Proof. Arithmetic with the Standard Model charges, Y = Q - T_3. The last statement is direct addition. ■

The scope is that of lemma 18: classical bosonic trace, with these definitions of the coefficients and of the angle, three generations, and the other species with the mass assignments of that calculation. Other definitions of the weak angle do not stay at 5/8 (Proposition 6 there). At the endpoint where the neutral mass vanishes, the massive neutral direction is no longer distinguished, so its angle is read as the limit from finite mass, as lemma 18 states.

The state count in the action

Proposition 21.2. With three generations and nu_R, dim H_F = 96 (Chamseddine-Connes-Marcolli). With only an up-type Dirac mass m and no Majorana mass, the constant-background density of lemma 18 has finite trace Tr exp(-V_0^2/Lambda^2) = 84 + 12 e^{-x} for the heat cutoff. So the coefficient of the vacuum density moves from 96 to 84 along the radial direction, with no finite nonzero stationary point on this free radial ray (No-go theorem 9 and Proposition 11 of lemma 18). This concerns the literal flat-space action with no added potential or constraint. Compact support can give a plateau; sign-changing cutoffs, additional terms, constraints and paths with an additional fixed noncommuting mass lie outside it. At m = 0 the derivative is zero.

Proof. The top block has 3 colours, 2 chiralities and charge conjugates, so 12 states, each with V_0^2 = m^2. ■

This also gives the top-only family a place in the radial problem, as the endpoint of a descending direction. A radial direction giving other species masses descends further, so this is the endpoint for that direction only.

SO(30)

Let the five quarks other than the top, in three colours, with their conjugates, form the vector 30 = (5, 3) + (5-bar, 3-bar) of SU(5) x SU(3) inside SO(30). The adjoint, of dimension 435, is

```
Lambda^2(30) = (24, 8) + (24, 1) + (1, 8) + (1, 1) +
↪ (10, 6) + (15, 3-bar) + conjugates of the last
↪ two.
```

With Q = 2/3 for u, c and Q = -1/3 for d, s, b, that is, Q = diag(3/5, 3/5, -2/5, -2/5, -2/5) + (1/15) I_5, which includes the U(1) commuting with SU(5), the charges are:

representation	charge: number
(15, 3-bar)	-2/3: 6; +1/3: 6; +4/3: 3
(10, 6)	-2/3: 3; +1/3: 6; +4/3: 1
(24, 1) + (1, 1)	+1: 6; -1: 6; 0: 13

Three generations of scalar partners in a supersymmetric Standard Model with nu_R need, for antisquarks in the 3-bar, 6 of charge -2/3 and 6 of charge +1/3, and for sleptons 6 of charge +1, 6 of charge -1 and 12 neutral real states. So (15, 3-bar) and (24, 1) + (1, 1) contain exactly the charged counts. What is left over in them is three triplets of charge +4/3 and one neutral state. The counting of 15 diquarks and 25 mesons, with the charge +4/3 states, and the remark that the top is unable to form them, go back to an early preprint (hep-ph/0512065), which finds neither SO(30) nor a role for SO(32). What SO(30) adds is one group whose adjoint contains both families. It also sits in SO(32), with 496 -> (435, 1) + (1, 1) + (30, 2) under SO(30) x SO(2), which hints at a role for SO(32); we have not explored this. These are representation and charge counts: identifying the entries with physical scalar partners needs a field construction, spin and chirality assignments, interactions and dynamics.

Proposition 21.4 (the counting forces the top out and leaves the charge 4/3 half-filled). Let n_u up-type and n_d down-type light quarks in three colours, with their conjugates, span the vector of SO(6(n_u + n_d)), whose 6(n_u + n_d) real dimensions are those of (n_u + n_d, 3) + conjugate. In the colour antitriplet of the adjoint, the symmetric diquarks S^2(n_u + n_d) (x) 3-bar have n_d(n_d + 1)/2 states of charge -2/3, n_u n_d of charge +1/3 and n_u(n_u + 1)/2 of charge +4/3. Three generations of antisquarks need 6 and 6 in the first two charges, and the only solution in nonnegative integers is (n_u, n_d) = (2, 3). The charge +4/3 count is then 3. With the top included, (n_u, n_d) = (3, 3) gives (6, 9, 6), and the top would need 36 = 30 + 6 real

dimensions rather than the 32 of SO(32), whose extra plane is colour-neutral.

Proof. n_d(n_d + 1)/2 = 6 gives n_d = 3, then n_u n_d = 6 gives n_u = 2, and 2 x 3/2 = 3. For n_u = n_d = 3 the three counts are 6, 9, 6. A coloured quark adds 6 real dimensions to the vector; 32 - 30 = 2. ■

Corollary 21.5 (the charge-4/3 sector). Under the bijection of Proposition 21.4 between the (6, 6) part and the antisquarks of three generations, the three antitriplets of charge +4/3 are half of what a chiral multiplet pairing would need: a Dirac quark of charge 4/3 in each generation needs two complex scalars, so three generations need 6. If the three scalars are read as superpartners, their three fermions are left-handed Weyl fields in (3-bar, 4/3), SU(2) singlets, and are anomalous on their own (with T(3) = 1/2 and A(3) = 1, the SU(3)^3, SU(3)^2 U(1), U(1)^3 and gravitational U(1) traces are -3, 2, 64/3 and 12. The meson sector, six complex scalars of charge +-1 and neutral ones, cancels none of them for any chirality of its fermions: they are colour singlets, their cubic charge sum is an integer, and their linear charge sum has absolute value at most 6. With any other chirality assignment of the three multiplets, the SU(3)^3 trace is odd, so the exotic fermions stay anomalous). The readings compatible with the counting are: (a) the three scalars have no fermion partner, as the Higgs doublet has none in the Standard Model, so supersymmetry is a property of the (6, 6) part alone; (b) the top is included, which completes the exotic to 6 and breaks the +1/3 count to 9. In the window of the interval parent (lemma 38, Proposition 11.4, the shadow B_ij -> (m_i - m_j) B_ij, for distinct masses), the same-flavour diquarks drop out and the antitriplet counts become (3, 6, 1) in reading (a). One state of charge +4/3 remains, the combination of u and c antisymmetric in flavour, with coefficient proportional to m_u - m_c, in the (Lambda^2 5, 3bar) of S^2 V. In reading (b), with six flavours, the counts become (3, 9, 3), and three states of charge +4/3 remain, uc, ut and ct. In both readings the window has 3 states of charge -2/3, so the match with the antisquarks in Proposition 21.4 holds in the parent and fails in the window. This records a conflict between the count of three generations and the anomaly of the exotic sector; the premise of the whole counting is that five light quarks generate every

squark and slepton as a composite, scalars that the finite geometry of Connes et al. does not contain.

Provenance. Stated by the user and Claude Fable 5.1 in conversation on 4 November 2026; the arithmetic checked by hand. The remark that the top cannot form the diquarks is from hep-ph/0512065; the chirality reading of the charge-4/3 states as an additional quark per generation is in the abstract of the EPJC paper (Rivero 2024), where the half-multiplet problem is stated.

Direction (referee, 5 Nov 2026). The selection of (2, 3) belongs to the parent: with the flavour-antisymmetric counts n_d(n_d - 1)/2, n_u n_d, n_u(n_u - 1)/2, which are those of the window and of the sign s = -1 (USp(30)), the condition (6, 6) has no integer solution (n_d = 4 forces n_u = 3/2), so five light quarks with the top excluded are a property of Lambda^2 V with walls reducing to M_15(C) (lemma 38, Theorem 11.3), and the question is whether that wall condition can be derived. The anomaly is a parity: in reading (a) the three exotic antitriplets give an odd SU(3)^3 trace, which no partners for the rest of the SO(30) adjoint cancel (octets contribute 0, sextets multiples of 7); in reading (b) the six charge-4/3 states pair vectorlike but the three extra charge-1/3 antitriplets td, ts, tb leave an odd trace again. In both readings an odd number of antitriplets lies beyond the antisquarks, 3 = 15 - 12 and 9 = 21 - 12, so a supersymmetric reading of the diquark sector needs partnerless scalars or an even excess. With four generations the symmetric counting has no solution, since n_d(n_d + 1)/2 = 8 has no integer root.

The two families of 84

Proposition 21.3. Count four states per Dirac fermion and colour, and take Dirac neutrinos. Then

```
96 = 84 + 12:   five quarks and all leptons (60 + 12
 ↪  + 12), and the top (12);
96 = 84 + 12:   all electrically charged fermions
 ↪  (72 + 12), and the neutrinos (12).
```

The two families of 84 overlap in 72 states, and each is the other with the top and the neutrinos exchanged. Without nu_R, the first family has 78 states and the second stays at 84. In eleven dimensions, the three-form

C_3 has C(9, 3) = 84 on-shell components, and its dual C_6 has C(9, 6) = 84. The 84 in H_F is a complex finite-Hilbert-space dimension, and the three-form count is a real bosonic polarization count, so an identification needs a map of representations and degrees of freedom.

Readings, as proposals

These are proposals to test, with no derivation behind them.

1. **Seiberg duality.** For SU(3) with five flavours, N_f = 5 lies in the conformal window 3 N_c/2 < N_f < 3 N_c, and the magnetic dual is SU(N_f - N_c) = SU(2) with five flavours and a 5 x 5 gauge-singlet meson chiral superfield coupled by W = M q q-tilde (N. Seiberg, Nucl. Phys. B 435 (1995) 129, arXiv:hep-th/9411149, checked by the referee). The removal of the top is what gives N_f = 5, and SU(2) is the magnetic group there. To become a statement this needs an N = 1 supersymmetric theory with the required massless matter and dynamics, for example an emergent one, with an operator and anomaly map. Five-flavour ordinary QCD alone does not supply the duality, and the magnetic SU(2) needs a separate identification with any electroweak group.
2. **Branes.** The first family of 84 could relate to C_3 or to an M2 or M5 brane, and the second family to the dual object. The exchange of the top and the neutrinos would then be the content of the duality on the Standard Model side. The counts match, and equality of the number is insufficient: a representation and dynamics must relate the finite fermionic states to the bosonic form or brane degrees of freedom. The pairing of item 1 is one candidate. The first family also needs Dirac neutrinos for its 84, and the second does not.
3. **Relation to the odd labels.** The removal of the top leaves the five quarks and the leptons, on which SU(3) x U(1)_em acts with b_L paired with b_R. So the loss of the electroweak partner of b_L has no role at the label 9, where SU(2) is absent. A statement here needs the required involution on this finite sector, including its action on D, J, the algebra and its opposite; the vectorlike charges alone do not

establish it, and the neutral states still need treatment.

Numerical check

Python with exact fractions, run on 3 Nov 2026 in a temporary directory, not kept: all tables above, and the dimension $435 = 224 + 1 + 105 + 105$ with $105 = 60 + 45$ and $224 = 192 + 24 + 8$.

Part 22

The finite Dirac operator as a boundary condition on an interval

For STRATEGY T0, T0b, T10 and T13, following lemma 12 and lemma 20. Proved by Claude Opus 5.5, with the identities checked in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): two errors and one gap as first written, and amendments; all are incorporated below.

Outcome. Let (H_F, D_F, J_F, Gamma_F) be finite data of signs (+, +, -), and let l ||D_F|| < pi/2. Then D_F, with J_F and Gamma_F, is exactly the lowest spectral window of the pure derivative on an interval of length l, with no bulk mass and no Yukawa term, on the domain fixed by the boundary coupling tan(l D_F). So all the eigenvalues of D_F, Yukawa and Majorana blocks alike, are quantized internal momenta k with tan(k l) equal to a singular value of the coupling. The input is the finite triple: the content is that a consistent parent exists, with these properties:

- Preservation of the domain forces the observables at the wall into C_F, the commutant of D_F in A_F, while A_F acts freely in the bulk.
- The parent satisfies order one for any finite data with order zero.
- The compression is the exact average Q_l(a) = (2l)^-1 int_{-l}^{l} e^{ixD_F} a e^{-ixD_F} dx, so it tends to the finite representation as l -> 0, with defects of order l^2.
- The scalar inner fluctuations of the parent are pure gauge under its allowed endpoint gauge group. On the moving window, they act by the gauge orbit of

D_F through the transport along the interval. This reproduces the quadratic term of Chamseddine-Connes-van Suijlekom on the orbit, and it leaves the singular values fixed, so the radial mode of the Higgs field is absent from this parent.

With the family X(l) = tan(l M) and l -> 0, the couplings of the light fermions vanish as m l at fixed mass, the tower decouples, and the broken gauge directions go to pi/(2l). The parent algebra is matrix-valued, so T0 for a commutative parent remains open.

Setting

Let H_F = H_+ (+) H_- be finite-dimensional, Gamma_F = diag(1, -1), let c : H_+ -> H_- be antiunitary, and put J_F(a, b) = (c^-1 b, c a), so that J_F^2 = 1 and J_F Gamma_F = -Gamma_F J_F. Let D_F = [[0, M*], [M, 0]] with M : H_+ -> H_-. Then J_F D_F = D_F J_F is equivalent to M = c M* c. By the Takagi factorization, in bases related by c, we may write M = V |M| with a unitary V satisfying V = c V* c.

Let l > 0 with l ||D_F|| < pi/2, and define X : H_- -> H_+ by

```
tan(l D_F) = [[0, X], [X\*, 0]],   that is,   X =
↪  tan(l |M|) V\*.
```

Since J_F commutes with real functions of D_F, c X = X* c^-1. Conversely, every X with c X = X* c^-1 arises from exactly one such D_F, namely D_F = l^-1 arctan of the block operator. In the limit, X = l M* + O(l^3).

The parent. On H_l = L^2([0, l]; C^2) (x) (H_+ (+) H_-), write psi = (Q_L, Q_R; U_L, U_R), with Q valued in H_+ and U in H_-. Let (D psi)_L = psi_R' and (D psi)_R = -psi_L' on both fields, the operator of lemma 12 with no bulk mass, on the domain

```
Q_R(0) = 0,   U_L(0) = 0,   Q_R(l) = X U_R(l),
↪  U_L(l) = -X\* Q_L(l).
```

Call it D_X. Put Gamma^ psi = (Q_L, -Q_R; U_L, -U_R) and J^ psi = (c^-1 U_R, -c^-1 U_L; -c Q_R, c Q_L). Let (E psi)_L = psi_R and (E psi)_R = -psi_L, so that D_X = E d/dx on its domain.

In the Standard Model form of the classification, with E

= Hom(V, W) and W = W_+ (+) W_-, one takes H_+ = E_+ (+) E_-* and H_- = E_- (+) E_+*, with c the adjoint map. At X = 0, the Q-fields have boundary type (a) and the U-fields type (b), in the terms of lemma 09; for X != 0 the modes continue from those zero modes, which generally lift. A single matrix X encodes the Yukawa blocks and the Majorana block together. Its entries need not keep the original block pattern, since the tangent brings in higher powers of D_F: for M = [[a, b], [b, 0]] real symmetric, X_22 = (l^3/3) a b^2 + O(l^5).

The window

Theorem 22.1. D_X is self-adjoint with compact resolvent, and J^^2 = 1, J^ D_X = D_X J^, J^ Gamma^ = -Gamma^ J^, Gamma^ D_X = -D_X Gamma^. Let R(x) = exp(x Gamma_F D_F), and define W : H_F -> H_l so that for phi = (a, b), the L-components (Q_L, U_L) of W phi at x are l^{-1/2} R(x)(a, 0), and the R-components (Q_R, U_R) are l^{-1/2} R(x)(0, b). Then W is an isometry and

```
D_X W = W D_F,    J^ W = W J_F,    Gamma^ W = W
↪ Gamma_F.
```

With sigma_k the singular values of M and theta_k = l sigma_k, the signed spectrum of D_X is (n pi +- theta_k)/l, n in Z, with multiplicities added when values coincide; a zero singular value gives a zero mode of multiplicity two. So the spectrum of |D_X| consists of the sigma_k and of (n pi +- theta_k)/l for n >= 1, and ran W is the spectral subspace of |D_X| <= ||D_F||, and the next level is at least (pi - l ||D_F||)/l.

Proof. Self-adjointness is Lemma 12.1. Since Gamma_F D_F is anti-Hermitian, R(x) is unitary, and with C(x) = cos(x |M|), C'(x) = cos(x |M*|), S(x) = V sin(x |M|),

```
R(x)(a, 0) = (C(x) a, -S(x) a),    R(x)(0, b) =
↪ (S(x)\* b, C'(x) b).
```

Isometry follows from the unitarity of R(x) and the orthogonality of the L- and R-components. The conditions at 0 hold since S(0) = 0. At l, using V* cos(l |M*|) = cos(l |M|) V*, we get X C'(l) = S(l)* and X* C(l) = S(l), which are the conditions at the wall. For the derivative, the L-part of D W phi is d/dx R(x)(0, b) = R(x) Gamma_F D_F (0, b) = R(x)(M* b, 0), and the R-part

is -d/dx R(x)(a, 0) = R(x)(0, M a); together these are W D_F phi. The relations for Gamma^ are direct. For J^, use c |M| c^-1 = |M*| and V = c V* c, which give c^-1 C' = C c^-1 and c S(x)* = S(x) c^-1. The signs of the parent follow in the same way, using c X = X* c^-1 for the domain. For the spectrum, constant unitaries on Q and on U commute with E d/dx and bring X to diag(tan theta_k), which reduces to Example 12.5 in each channel. ■

Corollary 22.2 (conjugated form). D_X = R (D_0 + E Gamma_F D_F) R*, with R acting pointwise on the pair (Q, U) and D_0 the uncoupled operator, X = 0. So after this unitary change of domain, D_F is a constant matrix connection, odd for the grading, with relative endpoint rotation R(l). This is equation (21) of lemma 20, for general D_F.

Proof. D(R phi) = R(D phi + E R* R' phi), and R* R' = Gamma_F D_F. For phi in the uncoupled domain, the computation at the wall in the proof above shows that R phi satisfies the coupled conditions. ■

So in this family a fermion mass is a momentum: D_X has no mass term, and the light eigenvalue k = theta/l is fixed by tan(k l) = tan(theta), a singular value of X. In the finite description, the condition relating k to l is lost and the value appears as free data in D_F.

The parent as a real spectral triple

Let (A_F, H_F, D_F, J_F, Gamma_F) be finite data as above, with an even representation pi = pi_+ (+) pi_- of a unital finite-dimensional real or complex *-algebra A_F, faithful, unital and satisfying order zero, with pi°(b) = J_F pi(b)* J_F^-1. Order one is not assumed. Let

```
C_F = {a in A_F : [pi(a), D_F] = 0},    A_l = {f in
 ↪  C^infinity([0, l], A_F) : f(l) in C_F},
```

with f acting pointwise by pi_+(f(x)) on Q and pi_-(f(x)) on U.

Theorem 22.3.

1. A function f in C^infinity([0, l], A_F) preserves Dom D_X if and only if f(l) lies in C_F.

2. (A_l, H_l, D_X, J^, Gamma^) is a real even spectral triple, with [D_X, pi(f)] = E pi(f') and J^ pi(g)* J^^-1 = pi°(g) pointwise.
3. It satisfies order zero and order one, with signs (+, +, -).

So A_F acts freely in the bulk, and the boundary condition restricts the observables at the wall to the commutant of D_F. In the Standard Model data with nonzero Yukawa blocks, this is the algebra {(lambda, lambda, m)} of the photon and gluons; with vanishing Yukawa blocks the commutant is larger. Order one holds in this parent whatever the finite data, for its pointwise algebra: the finite condition appears only on the truncation, and arbitrary compressions need not preserve either order condition.

Proof. Item 1: at l, the traces Q_L(l) and U_R(l) are free, and applying f to the conditions gives pi_+(f(l)) X = X pi_-(f(l)) and pi_-(f(l)) X* = X* pi_+(f(l)), as in Result 3 of lemma 20. That is [pi(f(l)), tan(l D_F)] = 0, equivalent to [pi(f(l)), D_F] = 0 because l ||D_F|| < pi/2 makes D_F a function of tan(l D_F). At 0 the conditions are preserved since pi is even. Item 2: the commutator is the Leibniz rule. Conjugation by J^ acts on (Q, U) as J_F does, and pi(g) acts identically on the two spin components, which gives the second formula. This preserves the domain since J_F commutes with D_F. Item 3: pi(f(x)) and pi°(g(x)) commute at each x by finite order zero, and E commutes with both; so [E pi(f'), pi°(g)] = E [pi(f'), pi°(g)] = 0. ■

This is consistent with Counterexample 4 of lemma 20, whose finite data A_F = C + C, D_F = m sigma_1 fail order one while the parent satisfies it.

The compression

Let Q_l(T) = W* T W for a bounded operator T on H_l. For T(x) a bounded function with values in the even operators on H_F, acting pointwise and identically on both spin components, we write T for the multiplication operator.

Theorem 22.4.

```
Q_l(T) = l^-1 int_0^l cos(x ad_{D_F})(T(x)) dx,
 ↪  Q_l(E T) = -l^-1 int_0^l sin(x ad_{D_F})(T(x))
 ↪  dx.
```

For constant a in A_F,

```
Q_l(a) = sinc(l ad_{D_F})(pi(a)) = (2l)^-1
 ↪  int_{-l}^{l} e^{ix D_F} pi(a) e^{-ix D_F} dx,
```

so in an eigenbasis of D_F with eigenvalues mu_i, the matrix elements are pi(a)_{ij} sin(l (mu_i - mu_j))/(l (mu_i - mu_j)). Moreover:

1. Q_l(A_l) = Q_l(C^infinity([0, l], A_F)), so the image of the parent algebra contains Q_l(a) for every a in A_F.
2. ||Q_l(a) - pi(a)|| <= (l^2/6) |||[D_F, [D_F, pi(a)]]|| <= (2/3) l^2 ||D_F||^2 ||a||.
3. Q_l([D_X, f]) = [D_F, Q_l(f)] for f in A_l.
4. With a' = Q_l(a) and b' = J_F Q_l(b)* J_F^-1, |||[a', b']|| <= (8/3) l^2 ||D_F||^2 ||a|| ||b||, and |||[[D_F, a'], b'] - [[D_F, pi(a)], pi°(b)]|| <= (16/3) l^2 ||D_F||^3 ||a|| ||b||.

So as l -> 0 with D_F fixed, D_F, J_F and Gamma_F stay exact, and the compressed observables converge to pi(A_F) with rate l^2. The finite order conditions hold in the limit exactly when they hold for the finite data.

Proof. From the definition of W, the expectation of T splits into the L- and R-parts, so Q_l(T) is l^-1 int of the Gamma_F-even part of R(x)* T(x) R(x). Put K = Gamma_F D_F. For even Z, ad_K Z = Gamma_F [D_F, Z], which is odd, and ad_K (Gamma_F [D_F, Z]) = -[D_F, [D_F, Z]]. So the even part of e^{-xK} T e^{xK} is cos(x ad_{D_F})(T). Likewise, E exchanges the L- and R-parts with a sign, so Q_l(E T) is l^-1 int of Gamma_F times the odd part of R* T R, and that odd part is -Gamma_F sin(x ad_{D_F})(T). The formula for constant a follows from cos(x ad_D)(Z) = (e^{ixD} Z e^{-ixD} + e^{-ixD} Z e^{ixD})/2.

Item 1: the map f -> Lambda(Q_l(f)), for a linear functional Lambda, is integration against a smooth density. If it vanishes on A_l, which contains all f of compact support in [0, l), the density vanishes on [0, l), hence everywhere by continuity. Item 2: by Taylor's formula with integral remainder, ||cos(x ad_D)(Z) - Z|| <= (x^2/2) ||ad_D^2 Z||, since cos(x ad_D) has norm at most 1 by

the average formula; averaging over x gives l^2/6. Item 3: P commutes with D_X. It also follows from the two formulas by integration by parts, where the boundary term at l vanishes because f(l) is in C_F. Item 4: J_F commutes with D_F, so b' is the same average of pi°(b). Expand both commutators with the bound of item 2 and ||[D_F, Z]|| <= 2 ||D_F|| ||Z||. ■

The convergence is a statement about the maps Q_l, on the operator systems, in the setting of Connes and van Suijlekom (arXiv:2004.14115, sections 1 and 3). The generated algebra is not continuous at l = 0. An analytic example, from the referee: let M = [[0, 0, 1], [0, 0, 1], [1, 1, 0]], D = [[0, M], [M, 0]], with A_F = C^3 acting diagonally and equally on the two blocks. With P_1, P_2 the first two channel projections, [Q_l(P_1), Q_l(P_2)]_{12} = -l^2/3 + O(l^4), so the compressions of commuting projections generate a noncommutative algebra for small nonzero l. On the one-generation Standard Model data, a numerical closure with tolerance 10^-7 gives complex dimension 15 for the algebra generated by pi(B), with B = C + H + M_3(C), and 30 for Q_l(B) at l = 0.3; these are numerical values with no certified rank.

Scalar fluctuations

Theorem 22.5. Let h be smooth with values in the anti-Hermitian elements of A_F. The self-adjoint one-forms of the parent are exactly the E pi(h). Let u' = -h u, u(0) = 1, and let w = u(l), a unitary of A_F. Then

```
D_X + E pi(h) = pi(u) D_{X_w} pi(u)\*,      X_w =
↪  pi_+(w)\* X pi_-(w),
```

and the window of D_{X_w} carries pi(w)* D_F pi(w). With the real structure, for A = E pi(h) we have J^ A J^^-1 = E J_F pi(h) J_F^-1, and D_X + A + J^ A J^^-1 has the window U_w* D_F U_w, with U_w = pi(w) J_F pi(w) J_F^-1, and this commutes with J_F. Moreover, with u~(x) = u(x) w*, which lies in A_l since u~(l) = 1, and U~ = pi(u~) J_F pi(u~) J_F^-1,

```
D_X + A + J^ A J^^-1 = U~ D_X U~\*.
```

So every scalar connection in this model is gauge equivalent to the original parent under the allowed endpoint gauge group (referee).

Proof. A one-form is sum f_1 [D_X, f_2] = E pi(sum f_1 f_2'), and E pi(h) is self-adjoint exactly when h is anti-Hermitian, since E is anti-Hermitian and commutes with pi(h) and pi is faithful. Every smooth such h arises: with b(x) = int_l^x h, we have b(l) = 0, so b is in A_l, and [D_X, b] = E pi(h). Next, pi(u)* D pi(u) = D + E pi(u* u'), and pi(u)* E pi(h) pi(u) = E pi(u* h u), so the sum is D with no connection term. At 0 the conditions are unchanged. At l, the substitution psi = pi(u) phi gives the conditions with X_w. Then tan(l pi(w)* D_F pi(w)) has the block X_w. For general w, this operator need not commute with J_F, so we use the spectrum and the isometry of Theorem 22.1 without its real-structure part. For the real fluctuation, J^ commutes with E, u(x) is replaced by pi(u) J_F pi(u) J_F^-1, using finite order zero, and U_w commutes with J_F, so the real window follows. The last formula follows by differentiating, with the endpoint condition. ■

Corollary 22.6 (the quadratic term as transport). By Chamseddine-Connes-van Suijlekom (arXiv:1304.7583, section III), for unitary v in A_F, with U = v J v J^-1, one has U D U* = D + A_(1) + A~*(1)* + *A*(2), for the universal form v delta(v*). Taking v = w*, on the gauge orbit the fluctuation of the parent, which satisfies order one and has no quadratic term, gives on the moving window exactly the fluctuated operator of that framework (equations (10)-(11) of section III.2 there), quadratic term included, for finite data that fail order one. This answers the first open point of lemma 19 on these gauge orbits and with the moving window. For the two-point data with J = sigma_1 K and w = diag(e^{i alpha}, e^{i beta}), it gives the upper link m e^{-2 i (alpha - beta)}, which is m e^{2 i psi} of Corollary 19.2 with psi defined from v = w*. Theorem 22.5 and the covariant projection of lemma 14 share a transport principle. The fixed compression, as in lemma 13, is a different object: by Theorem 22.4, it gives Q_l(E pi(h)) = -[D_F, l^-1 int_0^l x pi(h) dx] + O(l^3 ||D_F||^3 ||h||) for h fixed, so that for constant h it is -(l/2)[D_F, h], while the first derivative of the transported operator is -l [D_F, h]. The difference is the contribution of the moving spectral projection.

No-go theorem 22.7 (the radial mode). *Hypotheses:* the parent above, on one interval, with fixed X, no

bulk masses, and scalar inner fluctuations from the even algebra A_l, with the moving spectral window. *Conclusion:* the fluctuations leave all singular values of the finite Dirac operator fixed. In the usual Standard Model representation with nonzero Yukawa data, the normalized shifted Higgs quaternion runs through the unit quaternions, a three-dimensional gauge orbit, with no radial mode. Degenerate data can have a smaller orbit. A fixed compression can show changes in singular values at second order, so the window has to be named.

Proof. Theorem 22.5: D_F and pi(w)* D_F pi(w) are unitarily equivalent. The same holds with U_w. ■

Known escapes.

These lie outside the hypotheses, and each needs its own construction.

- The coupling X as a field on the wall, with its own kinetic term and potential.
- Connections odd for the grading, as in Corollary 22.2 and lemma 23: the fluctuation K -> K + delta K changes the singular values.
- Bulk masses that do not commute with the gauge action.
- Two or more internal dimensions, where a connection has curvature, although curvature alone does not prove a radial mode.
- An additional composite or other degree of freedom; a quantum effective action on the same configuration space does not by itself add a radial coordinate.
- A fluctuation of the finite data alone, as with the radial parameter of lemma 19, which is not the moving window.

Direction. On the interval, the even connection is pure gauge up to its boundary effect, so the finite orbit is all it can reach. The physical Higgs modulus must then be sought at the wall, in odd directions, or in more dimensions. At the 10 -> 9 endpoint this agrees with T10, where the Higgs is frozen.

The gauge sector and the 10 -> 9 family

Proposition 22.8. Consider the product of M^4 with this parent, with algebra of functions f(y, x) valued in A_F and f(y, l) in C_F. Then the gauge one-forms sum

f_1 d_mu f_2 take values in C_F at x = l. Assume, in addition, a flat product, a positive five-dimensional Yang-Mills kinetic form, the trivial background, unrestricted endpoint variation in the allowed directions, and no additional boundary kinetic or mass terms. Then the directions in the Lie algebra of C_F have masses n pi/l, n >= 0, and the orthogonal directions have masses (n - 1/2) pi/l, n >= 1. The physical gauge algebra comes from the skew-adjoint elements through the real gauge representation, with its kernel and any unimodularity condition.

Proof. The first statement is direct evaluation at x = l. The rest is the interval spectrum in the gauge A_5 = 0, with Neumann conditions at 0, and with Neumann or Dirichlet conditions at l. ■

In the Standard Model data with nonzero Yukawas, M_W = M_Z = pi/(2l). The Lie algebra of C_F has dimension 10 before unimodularity; with the unimodularity condition of Chamseddine-Connes-Marcolli (arXiv:hep-th/0610241, section 2.5), the massless modes are the photon and the gluons. These are the boundary conditions of Higgsless models (Csaki, Grojean, Murayama, Pilo and Terning, Phys. Rev. D 69 (2004) 055006), and the fermion formula tan(m L) = M_D L is equation (5.6) of Csaki, Grojean, Hubisz, Shirman and Terning (Phys. Rev. D 70 (2004) 015012); paper/LITERATURE.md, section C. Boundary spectral geometry for the Standard Model has been studied by Chamseddine and Connes (arXiv:1008.3980, section 5). What is added here, as far as our searches went, is the combination: the real structure, the finite triple as an exact window, the wall algebra C_F, the compression formula and the fluctuations. A wider claim of novelty would need a fuller comparison.

Corollary 22.9 (the 10 -> 9 family). Fix a block M_0 for the light fermions and a block Theta/l on an orthogonal channel, compatible with J_F and the grading, with 0 < Theta < pi/2, for the top and if wanted for the Majorana block. Let D_F(l) have M(l) = M_0 (+) Theta/l, and let l -> 0. Then:

1. X(l) has the light block tan(l |M_0|) V_0* = l M_0* + O(l^3) and a heavy block with singular values tan(Theta): the light couplings vanish as m l at fixed mass m.
2. The heavy channel has mass Theta/l, the broken

gauge directions have mass pi/(2l), and the other levels diverge.

3. The massless gauge sector is C_F. Let P_L be the projection reducing D_F(l) to the fixed light block D_L. For constant a, P_L Q_l(a) P_L = sinc(l ad_{D_L})(P_L pi(a) P_L), with ||P_L Q_l(a) P_L - P_L pi(a) P_L|| <= (2/3) l^2 ||D_L||^2 ||pi(a)||, also when pi(a) mixes light and heavy channels. The limiting light compression need not be a representation, since products can pass through the heavy subspace. The full estimate of Theorem 22.4 does not apply, because l ||D_F(l)|| tends to Theta. For example, with D_F(l) = sigma_1 (x) diag(m, Theta/l) and pi(a) = 1 (x) sigma_1, one has Q_l(a) = sinc(Theta - l m) pi(a), which tends to sinc(Theta) pi(a).
4. With l = pi/(2 M_W), we get X_q ~ (pi/2) m_q/M_W, which has the form of y_q = sqrt(2) m_q/v -> 0 at fixed mass, as the coupling vanishes while the mass stays. Identifying X_q with the four-dimensional Yukawa coupling is a hypothesis.
5. The ratio is m_t/M_W = 2 Theta/pi < 1, and M_W/M_Z = 1.

So the 10 -> 9 proposal is realized at this level as a limit l -> 0, with W, Z and top at the tower scale and the light fermions keeping explicit masses. Item 5 does not reach m_t/M_W = sqrt(8/3), which belongs to the interior. This is a statement in the stated family: lowest level, no bulk mass, strict wall algebra. Bulk masses for the top, and a changed gauge wall condition, are directions beyond it.

A numerical comparison for the interior (Python). A separate model, with a finite wall scalar and Neumann conditions at 0, lies outside Proposition 22.8 and No-go theorem 22.7. In it, the mass of a broken direction satisfies t tan t = s, with t = M l and s = g^2 v^2 l^2/4 (for the Z, s/cos^2 theta_0), which has the form of M_k tan(M_k pi R) = -V in Csaki et al. (arXiv:hep-ph/0305237, section on mixed boundary conditions, read in the LaTeX source), with the sign convention of V there. If g'^2/g^2 = 3/5 is taken at all scales, M_W^2/M_Z^2 goes from 5/8 as s -> 0 to 1 as s -> infinity. It equals the measured 0.777

at M_W l = 1.03, with 1/l = 78 GeV and a next W level at 280 GeV; whether collider searches allow this would need the couplings, branching fractions and an applicable limit, which we have not analyzed. The top needs M_W l < 0.96 for m_t/M_W = sqrt(8/3), or M_W l < 0.73 for the measured ratio, in the lowest level with no bulk mass. So in the flat tree-level toy without bulk masses or running, the top and gauge requirements are incompatible. It does show the interior interpolating between the spectral-action value 5/8 and the endpoint value 1. The directions are bulk masses, warping and running.

What is input and what is derived

The finite data (A_F, H_F, D_F, J_F, Gamma_F) are all inputs, so the existence of the window is close to a reparametrization: X = tan(l D_F). The derived statements are the wall algebra C_F for that input, the order conditions of the parent, the compression formula with its rate, the form of the fluctuations, and the gauge spectrum under the stated assumption. The window identifies D_F, J_F and Gamma_F exactly, while the represented algebra is obtained through compression and is not an exact representation of A_F at l > 0. In particular, this does not select A_F. T0 asks for a commutative parent: a natural next step is a two-stage parent, compressing functions on S^4 and CP^3 at level one to get A_F-valued functions on the interval, and then taking this window, where the difficulty is the same interface problem as after Result 7 of lemma 20.

Numerical check

By Claude Opus 5.5, in Python (numpy and scipy), 3 Nov 2026, in a temporary directory, not kept.

- Random complex symmetric M on C^5, l ||M|| = 1.3: D u_+(a) = u_-(M a) and D u_-(b) = u_+(M* b) by finite differences, to about 10^-9; boundary conditions and J^ to about 10^-15; norm of W to 10^-13.
- Q_l(a) = sinc(l ad)(a), by quadrature, to 2 x 10^-8. The formula for Q_l(E T), with a nonconstant T, to 10^-14.

- One-generation Standard Model data with Yukawas and the rank-one Majorana block, with signs, order zero and order one exact on B = C + H + M_3(C), and commutant of D in B of real dimension 20. At l = 0.1 and 0.03, ||Q_l(a) - a + (l^2/6)[D, [D, a]]|| is 10^-5 and 4 x 10^-8, and the order-one defect of Q_l is 1.5 x 10^-4 and 1.2 x 10^-5, which scales as l^2.
- The dimensions of the generated algebras quoted above, 15 and 30, from a numerical closure with tolerance 10^-7, not retained and not certified; at smaller l the numerical rank depends on the tolerance.
- The roots of t tan t = s, and the values quoted in the comparison for the interior.
- On the two-point data with J = sigma_1 K: D + A_(1) + A~*(1)* + *A*(2) = U D U* to 4 x 10^-16, and the link of U_w* D U_w equal to m e^{-2 i psi}.
- The bounds of Theorem 22.4, items 2 and 4, on 30 random samples of the one-generation data: largest ratios to the bound 0.33, 10^-4 and 10^-3.

Part 23

Two gradings on an interval: masses as principal angles, the Higgs modulus, and exact W and Z masses

For STRATEGY T0b, T5b, T10 and T13, following lemma 22. Proved by Claude Opus 5.5, with the identities checked in Python (section "Numerical check"). Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): four errors and one gap as first written, and amendments, all incorporated below. The simpler form of the mass formulas, and the proof of item 3 of Theorem 23.6, are by the referee.

Outcome. A self-adjoint boundary condition for the derivative on an interval is a grading at each wall. The whole finite Dirac operator is then the relative position of two gradings: its singular values are the principal angles between them, divided by l, and

```
D_F = (1/2l) Gamma_0 log(Gamma_1 Gamma_0).
```

With a bulk prealgebra, this gives four results.

1. Aligned walls compress a prealgebra stable under the grading to its even part, with the grading inherited, which is the step that Example 16.9 had to add by hand.
2. Odd scalar fluctuations move the second grading, so they change the singular values and supply the radial mode that No-go theorem 22.7 excludes for even fluctuations. For M_2(H) the field space is S^4, and the Higgs modulus is half the geodesic distance between the points chosen by the two walls.
3. With a wall connector on one channel and a five-dimensional Yang-Mills action with common normalization, the masses are exact: sin^2(M_W l) = sin^2(2 theta)/2 and sin^2(M_Z l) = (4/5) sin^2(2 theta), for universal fermions of mass theta/l. At small theta this gives M_W^2 = 2 m^2 and M_Z^2 = (16/5) m^2, the constants in the lower bounds of the fully massive family of lemma 18, with M_W^2/M_Z^2 tending to 5/8 at both ends.
4. The heat-cutoff spectral action on the interval is a Jacobi theta function of theta, decreasing on [0, pi/2]. This family lies outside the hypotheses of the radial no-go of lemma 18: the radial direction is compact, with a stationary point at m = pi/(2l), the Connes-Lott value, where M_W = M_Z = 0.

Everything finite is still an input, and the parent algebra is matrix-valued. For a commutative parent, lemma 24 finds the window exactly, with spectral walls, while direct compression of scalar functions gives only R + R on the S^4 factor. The formulas for gauge masses depend on the assumed action. The relation to measured values fails in the flat toy, as stated at the end.

Two gradings

Let H be a finite-dimensional Hilbert space, and let Gamma_0, Gamma_1 be self-adjoint involutions on H. On L^2([0, l]; C^2) (x) H, with psi = (psi_L, psi_R), let D = E d/dx, where (E psi)_L = psi_R and (E psi)_R = -psi_L, on the domain

```
Gamma_0 psi_L(0) = psi_L(0),   Gamma_0 psi_R(0) =
 ↪  -psi_R(0),   Gamma_1 psi_L(l) = psi_L(l),
 ↪  Gamma_1 psi_R(l) = -psi_R(l).
```

Call it D(Gamma_0, Gamma_1), and let Gamma^ act as +1 on psi_L and -1 on psi_R.

Theorem 23.1.

1. D(Gamma_0, Gamma_1) is self-adjoint with compact resolvent, and it anticommutes with Gamma^.
2. Suppose Gamma_0 and Gamma_1 have eigenspaces of equal dimensions, and that -1 is not an eigenvalue of Gamma_1 Gamma_0. Let theta_k in [0, pi/2) be the principal angles between the +1 eigenspaces. Then K = (1/2l) log(Gamma_1 Gamma_0), with the principal branch, is anti-Hermitian, Gamma_0 K Gamma_0 = -K, and D_F = Gamma_0 K is self-adjoint, odd for Gamma_0, with singular values theta_k/l on the +1 eigenspace. Moreover Gamma_1 = e^{lK} Gamma_0 e^{-lK}.
3. With H_+, H_- the eigenspaces of Gamma_0, and X the H_- -> H_+ block of tan(l D_F), as in lemma 22, D(Gamma_0, Gamma_1) = D_X, and the spectral part of Theorem 22.1 gives the spectrum and the window, with no real structure needed. Conversely, every D_F odd for Gamma_0 with l ||D_F|| < pi/2 arises from exactly one such Gamma_1.
4. Let an antiunitary J_1 satisfy J_1^2 = -1, J_1 Gamma_i J_1^-1 = -Gamma_i for i = 0, 1. Then (J^ psi)_L = J_1 psi_R, (J^ psi)_R = -J_1 psi_L defines a real structure of signs (+, +, -) for D(Gamma_0, Gamma_1). With J_F = Gamma_0 J_1 up to sign, J_F^2 = 1, J_F Gamma_0 = -Gamma_0 J_F and J_F D_F = D_F J_F, which are the signs of lemma 22.

So J_F itself does not anticommute with Gamma_1 in general: J_F Gamma_1 J_F^-1 = -Gamma_0 Gamma_1 Gamma_0, the reflection of Gamma_1 through Gamma_0.

Proof. Item 1: the boundary form is [<psi_R, phi_L> - <psi_L, phi_R>] evaluated between 0 and l, which vanishes on the domain because L-values and R-values lie in orthogonal eigenspaces at each wall, and these are half of the boundary data.

Item 2: U = Gamma_1 Gamma_0 is unitary and

Gamma_0 U Gamma_0 = Gamma_0 Gamma_1 = U^-1, so K is anti-Hermitian and anticommutes with Gamma_0. The eigenvalues of U are e^{+-2 i theta_k}, by the standard two-projection decomposition, and K then has eigenvalues +-i theta_k/l. Since e^{lK} Gamma_0 e^{-lK} = e^{2lK} Gamma_0 = U Gamma_0 = Gamma_1, the last formula follows.

Item 3: in the notation of Theorem 22.1, R(l) = e^{lK} carries H_+ to the graph {(C a, -S a)} = graph(-X*), and H_- to graph(X). These are the +1 and -1 eigenspaces of Gamma_1, which is the condition at the wall. The converse uses the same formula, with uniqueness from injectivity of the exponential on the stated range.

Item 4: J^ commutes with E by direct computation, and J^^2 = -J_1^2 = 1. The domain is preserved since J_1 exchanges the eigenspaces of each Gamma_i. For the last statement, J_1 U J_1^-1 = U, so as J_1 is antilinear, J_1 K J_1^-1 = K in the sense of the principal branch, and J_F = Gamma_0 J_1 then satisfies J_F K J_F^-1 = -K and commutes with D_F = Gamma_0 K. ■

When theta_k = pi/2 the channel has type (c) of lemma 09, with no zero mode and lowest level pi/(2l), the Connes-Lott mass of Theorem A in the paper. There the principal logarithm and the isolated window cease to be unique.

The bulk prealgebra

Let P be a *-subalgebra of End(H), which need not commute with Gamma_0. Let

```
A(Gamma_0, Gamma_1) = {f in C^infinity([0, 1], P) :
 ↪  [f(0), Gamma_0] = 0, [f(1), Gamma_1] = 0},
```

acting pointwise and identically on both spin components.

Theorem 23.2.

1. A function f in C^infinity([0, 1], P) preserves the domain exactly when it lies in A(Gamma_0, Gamma_1). The commutators are [D, f] = E f', and Gamma^ commutes with every f, including those with values odd for Gamma_0.
2. Under the hypotheses of Theorem 23.1, item 2, with W the isometry of Theorem 22.1, Q(T) = l^-1

int_0^1 E_0(R(x)* T(x) R(x)) dx, where E_0(Z) = (Z + Gamma_0 Z Gamma_0)/2 is the conditional expectation onto the even part.

3. *Aligned walls.* If Gamma_1 = Gamma_0, then D_F = 0, the window consists of constant L-values in H_+ and constant R-values in H_-, and Gamma^ restricts on it to Gamma_0. For an ambient constant multiplier a in P, Q(a) = E_0(a), and the image of the boundary algebra A(Gamma_0, Gamma_0) is E_0(P), by compactly supported profiles. If Gamma_0 P Gamma_0 = P, then E_0(P) is the even part of P, so the compressions of the prealgebra are exactly its even part, with the grading inherited from the even parent operator Gamma^. Without that hypothesis this can fail: for H = C^2, Gamma_0 = sigma_z and P generated by sigma_x + sigma_z, E_0(P) = span{1, sigma_z}, while the even part of P is C.
4. *Order conditions.* Let J_1 be as in Theorem 23.1 and let P° = {J_1 b* J_1^-1 : b in P} commute with P. Then the parent has order zero and order one for A(Gamma_0, Gamma_1).

Proof. Item 1: as in Theorem 22.3, using that the conditions at a wall say that L- and R-values lie in the eigenspaces of Gamma_i. Item 2: as in the proof of Theorem 22.4. Item 3 is the case R = 1. Item 4: [D, f] = E f', and pointwise commutation gives both conditions. ■

Item 3 evades the inherited-grading obstruction of Result 11 of lemma 20, because hypothesis 4 there fails: for odd a, here Q(a) = 0, while there the coordinates are retained as Q(x_a) = Gamma_a/5. So at the level of a matrix-valued parent this addresses the question left in step 3 of the framework note: the prealgebra is the bulk algebra, the even algebra is what the window sees, and the grading is a boundary condition. The commutative-parent and retained-coordinate questions remain open, and Theorem 4 of lemma 24 shows a real difference: scalar functions compressed directly give R + R on the S^4 factor, and H + H needs the matrix algebra generated at the intermediate stage.

Remark 23.3 (the twist). On H = E (+) E*, with J(x, y) = (y*, x*), the operator J_1 = Gamma_0 J

is the one that anticommutes with both gradings. For an odd anti-Hermitian h in P, J_1 h J_1^-1 is y -> +y h on E*, where J alone gives y -> -y h. So the opposite representation b° = J_1 b* J_1^-1 incorporates on the W-factor the grading automorphism sigma(b) = gamma_W b gamma_W. In this parent the twist comes from the requirement that J_1 preserve both boundary conditions. Twisted spectral triples for the grand symmetry (Devastato, Lizzi and Martinetti, JHEP 01 (2014) 042, arXiv:1304.0415; Devastato and Martinetti, Math. Phys. Anal. Geom. 20 (2017) 2, arXiv:1411.1320, Eqs. (1.8)-(1.11)) use a different automorphism: their rho exchanges left and right spinorial indices and algebra summands, while conjugation by gamma_W is inner on M_2(H) and fixes its centre. So this is a structural analogy between involutive automorphisms, and an identification needs a comparison of the representations, opposites and first-order conditions.

Odd fluctuations and the radial mode

Theorem 23.4. Let h be smooth with values in the anti-Hermitian elements of P, and let u' = -h u, u(0) = 1, w = u(l). Then

```
D(Gamma_0, Gamma_1) + E h = u D(Gamma_0, w\* Gamma_1
↪  w) u\*.
```

So scalar fluctuations move the second grading along its orbit under the unitary group of P. If h has an odd part, the principal angles between Gamma_0 and w* Gamma_1 w change in general.

Proof. u* D u = D + E u* u', and u* E h u = E u* h u, so the connection cancels. Since u(0) = 1, the wall at 0 is unchanged. At l, psi = u phi gives the stated grading for phi. ■

This lies outside No-go theorem 22.7, whose hypotheses include the even algebra: an even connection gives a w commuting with Gamma_0, and the angles are kept, while an odd connection changes them. For a real fluctuation, add the conjugate one-form: D + E h + J^(E h)J^^-1, with generator of the form (L_F, R_F) in Example 23.5. The invariant is a double coset. With H_i = Stab(Gamma_i) in the connected unitary group G of P, a gauge transformation removing h has the form u(x)

c, and it preserves both endpoints exactly when c is in H_0 and w c in H_1, that is, when w is in H_1 H_0. So the scalar moduli of the parent are the double cosets H_1 G / H_0, the relative position of the two gradings. This is with the moving window, as in lemma 22.

Example 23.5 (M_2(H) and S^4). Let W = H^2 = C^4, with the Clifford generators Gamma_a of lemma 16, and for a unit vector p in R^5 put Gamma_p = p_a Gamma_a. Each grading of W with one-dimensional quaternionic +1 and -1 eigenspaces is such a Gamma_p, so these gradings form S^4 = Sp(2)/(Sp(1) x Sp(1)), the quaternionic projective line. For two points, Gamma_q Gamma_p = p . q + (p wedge q), with eigenvalues e^{+- i d}, where d is the geodesic distance; so all the principal angles equal theta = d/2.

Take H = Hom(V, W) (+) Hom(W, V), J(x, y) = (y*, x*), Gamma_0(x, y) = (gamma x, -y gamma) with gamma = Gamma_{p_0}, and K(x, y) = (theta/l)(F x, y F), with F an odd anti-Hermitian element of M_2(H), F^2 = -1. Then:

- D_F = Gamma_0 K has all singular values equal to theta/l: a universal Dirac mass, for all colours.
- The second grading is Gamma_1(x, y) = (Gamma_p x, -y Gamma_{p'}), where p is at distance 2 theta from p_0 and p' is its mirror image through p_0. J_1 = Gamma_0 J satisfies Theorem 23.1, item 4, while J alone does not.
- The coordinates (theta, q) on S^4, with F determined by the unit quaternion q, are the polar coordinates of a Higgs quaternion: theta is the radial coordinate, bounded by pi/2, and q gives three angular directions, undefined at the poles. At theta = pi/2 the finite logarithm and the lowest window are degenerate. Quotienting by the stabilizer leaves the radial interval. A canonically normalized field needs the scalar kinetic term, which we have not computed.

With bulk masses, by Theorem 12.6, the light masses are about (tan(theta)/l) Phi_Q Phi_U when small compared with the tower thresholds, with a universal matrix before flavour data; general finite-angle eigenvalues need the secular equation. This field space is the coset of the minimal composite Higgs model (Agashe, Contino and

Pomarol, Nucl. Phys. B 719 (2005) 165, sections 2-3, with their h/f_pi corresponding to 2 theta). Their construction uses a warped interval and additional fermion and boundary data, so the shared coset does not identify the actions or potentials. The appearance here of the S^4 of the twistor window is an observation, and no identification between the two roles of S^4 is made.

The weak bosons

Take P = M_2(H) (+) M_4(C), acting on E = Hom(V, W) on the left and on the right. Assume the following.

- A five-dimensional Yang-Mills field valued in sp(2) (+) su(4), with the trace form Tr_W + Tr_V, so with one coupling (the normalization of the spectral action, g_3 = g_2 = sqrt(5/3) g_Y).
- At x = 0: Neumann for the stabilizer of p_0, Dirichlet for its complement. At x = l: Neumann for the stabilizer of p_l and of a connector, that is, the a with [a_W, Gamma_p] = 0, a_W w = lambda w and a_V v = lambda v, as in Results 2 and 5 of lemma 20, for a line w in the +1 eigenspace of Gamma_p and a line v in V; Dirichlet for the complement.
- The connector is prescribed gauge-breaking wall data here. If it is instead a term in the fermion boundary operator, as for a unit vector e in the +1 eigenspace of Gamma_1, f = J_1 e, K_c = |f><e| - |e><f| and Gamma_{1,c} = e^{eta K_c} Gamma_1 e^{-eta K_c}, then J_1 still anticommutes with Gamma_{1,c}, the parent order conditions are kept, and the affected fermion tower changes, so the universal fermion formula must be recalculated for that channel.

Theorem 23.6. With theta the angle above, the modes that are massless for all theta are exactly nine, the photon and the eight gluons, with three additional electroweak zero modes at theta = 0 and theta = pi/2. The lowest charged and neutral massive modes satisfy

```
sin^2(M_W l) = sin^2(2 theta)/2,
sin^2(M_Z l) = (4/5) sin^2(2 theta),
```

equivalently tan^2(M_W l) = sin^2(2 theta)/(1 + cos^2(2 theta)) and tan^2(M_Z l) = 4 sin^2(2 theta)/(1

+ 4 cos^2(2 theta)).

The photon has $1/e^2 = (8/3)/g^2$. Moreover:

1. As theta -> 0, with m = theta/l: M_W^2 -> 2 m^2 and M_Z^2 -> (16/5) m^2, so M_W^2/M_Z^2 -> 5/8.
2. As theta -> pi/2: M_W, M_Z -> 0, with M_W^2/M_Z^2 -> 5/8, while m -> pi/(2l).
3. At theta = pi/4: M_W l = pi/4, tan(M_Z l) = 2, and M_W^2/M_Z^2 = ((pi/4)/arctan 2)^2 = 0.5032..., the global minimum. So M_W^2/M_Z^2 stays in [0.5032, 0.625), with 5/8 as a limit at the ends, where the ratio itself is undefined.
4. m/M_W goes from 1/sqrt(2) at theta -> 0 to infinity at theta -> pi/2. It equals sqrt(8/3) at theta = 1.0589.

Proof. In the gauge A_5 = 0 the field is free, with cos(k x) on the Neumann directions at 0 and sin(k x) on the Dirichlet ones, and the condition at l holds in the frame rotated by Ad exp(theta F). The generators split according to the frequency of Ad F. Write L, R for su(2) on W_- and W_+, S = (L + R)/sqrt(2), A = (L - R)/sqrt(2), and O for the odd generators; Ad F has frequency 0 on S, and frequency 2 on the plane (A, O). So S = s cos kx, A = alpha cos kx, O = beta sin kx.

At l, in the rotated frame, A' = cos(2 theta) A + sin(2 theta) O and O' = -sin(2 theta) A + cos(2 theta) O, with O' = 0 at l, which gives beta = alpha tan(2 theta) cot(k l), and so A'(l) = alpha cos(k l)/cos(2 theta).

Charged directions. The connector forces R^+- to vanish at l, with L^+- Neumann. So S(l) = A'(l), and d/dx (S + A') = 0 at l. Substituting gives sin^2(kl) (1 + cos^2(2 theta)) = sin^2(2 theta) cos^2(kl).

Neutral directions. Add B = b cos kx for the generator T_15 = diag(1, 1, 1, -3)/(2 sqrt 6) of su(4). With Y = sqrt(3/5) R + sqrt(2/5) B and Z' = -sqrt(2/5) R + sqrt(3/5) B, the connector forces Z' to vanish at l, with L and Y Neumann. The Neumann conditions give b = -sqrt(3) s, and Z' = 0 gives s = alpha/(4 cos(2 theta)). Substituting in the condition for L gives sin^2(kl) (1/4 + cos^2(2 theta)) = sin^2(2 theta) cos^2(kl).

Massless modes and limits. At k = 0, a zero mode is a

constant in the intersection of the two stabilizers. This consists of a_W = diag(q, q) on W_+ and W_-, identified by F, with q preserving the line of w, together with a_V = lambda (+) m, and with the trace condition this has dimension 1 + 8. In it, L = R and B/L = sqrt(2/3), which gives 1/e^2 = (1 + 1 + 2/3)/g^2. The limits follow by expanding. A second derivation, from the referee, uses the principal angles between the Neumann subspaces at the two walls: for each charged direction the squared sine of the angle is sin^2(2 theta)/2, and in the neutral sector the single nonzero value is (4/5) sin^2(2 theta); it also covers cos(2 theta) = 0, where the divisions above need a limit. For item 3, with a = sin(2 theta), the ratio is [arcsin(a/sqrt 2)/arcsin(2a/sqrt 5)]^2, and this decreases with a on (0, 1], since x/(sqrt(1 - x^2) arcsin x) increases. ■

The constants in item 1 are the values ||[D_F, a]||^2/||a||^2 in the Hilbert-Schmidt norm, for universal mass m, for a = L^+- and for the direction orthogonal to the photon and Z', which has A-component of squared length 4/5. These are the coefficients of the lower bounds for the poles of the fully massive family in Proposition 14 of lemma 18 (2 x < z_W, (16/5) x < z_Z); the small-x pole coefficients there are 3 and 24/5. The limit 5/8 is common to both, with different behaviour: the finite-action poles grow in the heavy limit of lemma 18, while the interval masses here tend to zero at the antipodal point. The spectral-action Taylor mass for universal m is M_W^2 = 3 m^2, which exceeds the value here by the factor 3/2 of Proposition F in the paper: there the mass matrix carries the relative heat coefficients of the action, and here we have assumed a five-dimensional Yang-Mills action, which gives the commutator norm alone. A direct matching claim needs the full parent spectral action, with its tower and boundary conditions.

Item 2 gives a geometric form of the behaviour that lemma 18 found: heavy fermions with light vectors. At antipodal points Gamma_p = -Gamma_{p_0}, so both walls have the same stabilizer in sp(2) again, and the fermion channels have type (c) with mass pi/(2l).

The spectral action on the interval

Proposition 23.7. For the heat cutoff and universal theta, with s = l Lambda, the trace over a channel is

```
Tr_channel exp(-D^2/Lambda^2) = 2 sum_{n in Z}
 ↪  exp(-(n pi + theta)^2/s^2).
```

This is proportional to the Jacobi function theta_3(theta, exp(-s^2)). It is strictly decreasing on [0, pi/2], with a maximum at theta = 0 and a minimum at theta = pi/2, where it is stationary, and it is symmetric under theta -> pi - theta. In the towerless regime s -> 0 at fixed x = theta^2/s^2, it tends to 2 e^{-x}, the form of the vacuum density in lemma 18 and Proposition 21.2.

Proof. The eigenvalues in a channel are +-(n pi + theta)/l, by Theorem 22.1, and the sum is over Z after combining the signs. By Poisson summation the sum equals (s/sqrt(pi)) sum_{k in Z} exp(-k^2 s^2) e^{2 i k theta}, and the Jacobi triple product writes this as a constant times the product over n >= 1 of (1 + 2 q^{2n-1} cos(2 theta) + q^{4n-2}), with q = exp(-s^2). Each factor is positive and strictly decreasing in theta on [0, pi/2]. ■

In the form of the Jacobi function, the trace is (2 s/sqrt(pi)) theta_3(theta, exp(-s^2)). This family lies outside the hypotheses of No-go theorem 9 of lemma 18, which concerns finite internal data and a free ray V = c V_*: here the operator family is different, the radial direction is a compact orbit of relative gradings, and the tower, which the finite description omits, supplies a second branch with a stationary point at theta = pi/2. In the present family the literal action has its minimum at antipodal gradings, m = pi/(2l), with M_W = M_Z = 0 by Theorem 23.6, where the lowest eigenspace is degenerate. This is a vacuum with massless vectors, so it is not the physical point. The sign of the term agrees with the fermion loop in the Hosotani mechanism, where gauge-boson loops enter with the opposite sign. So an interior minimum could arise from that competition, which the classical spectral action does not contain. We have not computed it.

The one-loop potential in the same family

From the exact spectra, the products over the towers are, relative to theta = 0 and in Euclidean momentum x = p l,

```
fermion channel:  1 + sin^2(theta)/sinh^2(x),
charged vectors:  1 + sin^2(2 theta)/(2 sinh^2(x)),
neutral vector:   1 + (4/5) sin^2(2
 ↪  theta)/sinh^2(x).
```

So in the gauge-boson and fermion-loop form of the Hosotani mechanism, assuming the usual one-loop formula with three polarizations per vector, and with n_f = 2 x (number of physical Dirac species),

```
8 pi^2 l^4 V(theta) = int_0^infinity x^3 dx [ 3
 ↪  log(1 + s^2/(2 sinh^2 x)) + (3/2) log(1 + (4/5)
 ↪  s^2/sinh^2 x) - n_f log(1 + sin^2(theta)/sinh^2
 ↪  x) ],
```

with s = sin(2 theta). A Dirac top with three colours has n_f = 6, which needs a separately specified fermion sector, since the universal connection gives masses to all: one generation has n_f = 16 and three have n_f = 48. With I = int x^3/sinh^2 x dx = (3/2) zeta(3) and I' = int x^3/cosh^2 x dx = (9/8) zeta(3), near theta = 0 the bracket is (10.8 - n_f) I theta^2 + o(theta^2), and with epsilon = pi/2 - theta, near theta = pi/2 it is V(pi/2) + (10.8 I + n_f I') epsilon^2 + o(epsilon^2), where we have written V for the left side.

Proposition 23.8 (referee). In the massless flat family with this content, for every n_f > 0, theta = pi/2 is the unique global minimum on [0, pi/2], since the gauge logarithms are nonnegative and vanish there, and the negative fermion logarithm is minimized at sin^2(theta) = 1. For n_f >= 10.8, V is strictly decreasing on the interior. Indeed, with t = sin^2(theta) and J(z) = int_0^infinity x^3 dx/(sinh^2 x + z), the derivative with respect to t is, up to a positive factor,

```
(1 - 2t) [6 J(2t(1 - t)) + (24/5) J((16/5) t (1 -
 ↪  t))] - n_f J(t).
```

For t <= 1/2, both arguments in the bracket are at least t, and J decreases; for t > 1/2 both contributions have the required sign.

For 0 < n_f < 10.8, on grids of theta (Python), for n_f

in {1, 2, 3, 4, 6, 8, 10}, V has one interior maximum, at about 0.33 for n_f = 8, and no interior minimum. Theorem 9 of lemma 24, by Astra 6, proves this for all n_f in that range: exactly one interior stationary point, a strict maximum, using a log-concavity property of the integral.

These are negative results in an exact scope: one interval, no bulk masses, fermions of frequency 1 under F, this gauge content, one loop, this condition at the connector wall. As a formal leading approximation, not uniform at x = 0, V has the form B sin^2(2 theta) - A sin^2(theta) with A, B > 0, whose interior stationary point cos(2 theta) = A/(4B) is a maximum. *Bulk masses.* With bulk masses, the fermion product becomes cos^2(theta) + sin^2(theta) G(x), with G = (w_Q coth w_Q - c_Q)(w_U coth w_U - c_U)/x^2 and w = (c^2 + x^2)^{1/2} in the secular form of lemma 12. For a = c_Q + c_U != 0, G = 1 - a/x + O(x^-2), so the potential has angle-dependent ultraviolet divergences, and a finite value depends on a regulator and subtraction: as the referee showed, an earlier scan of general pairs here did not establish a potential, and it is withdrawn. The exceptional case c_U = -c_Q = c has a common bulk mass on the two mixed components and G = 1 + w^2/(x^2 sinh^2 w), which decays exponentially. Keeping the flat gauge products as a separate tree-level assumption, a scan for c in {0, 1/2, 1, 3/2, 2, 3, 5}, n_f in {1, 3, 6, 8, 10, 16, 48}, on a grid of 721 values of theta, finds no interior minimum (Python). General bulk masses need a gauge-invariant completion, a regulator and wall counterterms.

So an interior minimum needs another change: a negative coefficient of sin^2(2 theta), as from fermions of frequency 2 under F, which the bimodule E = Hom(V, W) does not contain; general bulk masses or other gauge form factors, as with warping; a wall scalar with its own potential; or tree-level terms at the walls. With h/f_pi = 2 theta, the leading flat form has the angular form alpha cos(h/f_pi) - beta sin^2(h/f_pi) of Agashe, Contino and Pomarol (section 2.1, Eqs. (19)-(23)) with spinorial SO(5) fermions, with alpha = A/2 and beta = -B, up to a constant. Their form factors can give the sign and competition needed for an interior minimum, so the flat results here do not exclude that construction. This points away from a pure flat Wilson line, toward those changed

form factors or a wall scalar with its own potential, for the physical interior.

Comparison with measured values in the flat toy

These are numerical comparisons (Python), with no running and no bulk masses, in lowest levels, using the Particle Data Group 2026 values quoted by the referee, m_t = 172.60 GeV as a direct-measurement mass, M_W = 80.3625 GeV and M_Z = 91.1879 GeV, so m_t/M_W = 2.1478 and M_W^2/M_Z^2 = 0.7767. The ratio m_t/M_W reaches sqrt(8/3) at theta = 1.0589 and the measured value at theta = 1.1626, where M_W l = 0.541, so 1/l is about 148.5 GeV. There, M_W^2/M_Z^2 = 0.581. The wall-scalar model of lemma 22 covers [0.625, 1], and this family covers [0.5032, 0.625); in the wall-scalar model, fitting the measured vector ratio gives M_W l = 1.028, where the lowest no-bulk-mass fermion channel has m/M_W < 1.528. So neither flat family fits both ratios on its own, and the flat one-loop potential above does not select an interior point. Whether a tower at 1/l of this size is allowed would need the couplings and an applicable experimental limit, which we have not analyzed. The directions are bulk masses for the light fermions, warping, running of the couplings, and families combining a Wilson line with a wall scalar. These negative statements hold in the flat tree-level toy only.

Open questions

1. The interior vacuum: the effective potential with a wall scalar, or with warping, or with general bulk masses, with a gauge-invariant completion, a regulator and wall counterterms. The flat cases above have none on the grids.
2. The relation between the two roles of S^4: the base of the twistor window, and the field space of the Higgs quaternion.
3. A commutative parent: functions on S^4 x I and CP^3 x I, with the gradings as local boundary conditions.
4. The factor 3/2 between the five-dimensional normalization and the spectral action: whether the spectral action on the parent itself, with its tower, reproduces the formula of Theorem 23.6.

5. The comparison with twisted spectral triples, Remark 23.3.
6. The contribution of a real connector to the fermion tower, the degenerate antipodal window, and the scalar kinetic normalization needed for a canonically normalized Higgs field.

Numerical check

By Claude Opus 5.5, in Python (numpy and scipy), 3 Nov 2026, in a temporary directory, not kept.

- Random complex symmetric M on C^4 with l ||M|| = 1.25: Gamma_1 is an involution, Gamma_1 = exp(2 l K) Gamma_0; D_F = (1/2l) Gamma_0 log(Gamma_1 Gamma_0) to 3 x 10^-15; the principal angles over l equal the singular values of M to five digits.
- Sp(2) x SU(4), of dimension 25, with the Neumann and Dirichlet subspaces built from the stated stabilizers: the two-point boundary problem was solved as a linear system on a grid in k l. For theta = 0.2, pi/4, 1.06 and 1.3, the two lowest nonzero roots are (0.279, 0.356), (0.785, 1.107), (0.647, 0.868), (0.373, 0.479), equal to the formulas of Theorem 23.6 to three digits, with other roots at pi/2 and above. The intersection of the two stabilizers has dimension 9 at each theta tested.
- Small theta: M_W^2/m^2 = 2.000 and M_Z^2/m^2 = 3.200.
- The universal connection on E (+) E*: J D = D J, Gamma_0 D = -D Gamma_0, all |eigenvalues| equal to theta/l, Gamma_1 = (left Gamma_p, right mirror), J_1^2 = -1, J_1 Gamma_1 = -Gamma_1 J_1, while J alone fails; J_1 h J_1^-1 is right multiplication by +h on E*, for odd h.
- The theta function: monotone on a grid for s = 0.3, 1, 2, with the towerless value at s = 0.3 equal to e^{-x} to five digits; Poisson summation and the triple product to 10^-16.
- The ratio M_W^2/M_Z^2 on a grid: range [0.5032, 0.6250], symmetric under theta -> pi/2 - theta to 10^-15, minimum at pi/4. At theta = 1.1625, M_W l = 0.541, and at theta = 1.0589, M_W l = 0.648.
- The one-loop integrals by adaptive quadrature

(scipy), and by Gauss-Legendre quadrature on 400 nodes for the scan with c_U = -c_Q. The referee recomputed the derivative with scipy, found one sign change for each n_f in {1, 2, 3, 4, 6, 8}, with the stationary angle about 0.3382 at n_f = 8, and checked I = (3/2) zeta(3).

Part 31

The interval of H as an eleventh dimension

Lemma 31. Written as file 01 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README). Propositions 1.1, 1.4 and 1.5 are proved here; Proposal 1.3 is a dictionary to be tested. The joint-limit family of the section of that name is from the Astra 6 audit of 3 Nov 2026 (historical/reviews/2026-11-03_astra_audit_d11.md, §3 and §4), restated here in circle language and checked. Attempted by Claude (Anthropic), 3 Nov 2026; checks in Python, checks_11d.py, sections F, F2 and H. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

Referee note (GPT-6.1 Terra, 4 Nov 2026). Proposition 1.1 and the joint family's Weyl constant are independently checked; the unfolded circle is scalar, and even extensions of arbitrary smooth interval functions need not be smooth at the fixed points. See the referee report. Proposition 1.5 is amended at its statement below.

Referee note on Proposition 1.4 (GPT-6.1 Terra). The endpoint value, minimum and small-angle coefficient are independently checked under the stated one-loop hypotheses; see the report.

Correction (3 Nov 2026). There was an error in the first version of Proposition 1.4: it read the potential of lemma 24 as the whole energy of the walls, but that potential is taken relative to theta = 0 and omits the theta-independent Casimir energy. The corrected statement is below; the reading "the walls attract" is withdrawn. The first version of Proposal 1.3 also placed the W at the bottom of the tower, M_W ~ 1/(2R), along every path to v = 0; after the audit (its §2.2), that holds only on the scale-free path of Proposition 1.5.

Setting

Hypothesis H of STRATEGY.md reads the two sheets of the finite geometry as the minimal truncation of an interval I_l with opposite chiral boundary conditions, with l ~ 1/M_W. Its three regimes carry the labels 9, 10 and 11. These are KO labels. At the 11 endpoint (v -> 0, l -> infinity) the heat dimension is 4 + 1 = 5, since the finite factor F keeps heat dimension 0 for every v (canonical §2.1). To read 11 as a metric dimension, the KO-6 factor must itself be resolved into a six-dimensional space, which gives the count

```
11 = 4 (spacetime) + 1 (the interval) + 6 (a space
↪  resolving F).
```

The files of this folder take the six-dimensional space to be B_6 = CP^2 x S^2, the base of Witten's spaces (02), and study two ways of completing 4 + 6 to eleven dimensions (03). This file treats the "1", and the path along which v -> 0 opens it.

Proposition 1.1 (the chiral interval is the antiperiodic circle)

Statement. Let D_l act on H_l = L^2([0, l], C^2) as in lemma 05, eq. (8): D_l(L, R) = (R', -L'), with R(0) = 0 and L(l) = 0, grading Gamma = diag(1, -1) and the scalar functions C^infinity([0, l]). Let S_R be the circle of length 2l, that is of radius R = l/pi, with coordinate x in [-l, l] and antiperiodic functions u(x + 2l) = -u(x). The map

```
(W(L, R))(x) = (L(|x|) + i sgn(x) R(|x|)) / sqrt 2
```

is a unitary H_l -> L^2_antiperiodic(S_R) with

1. W D_l W^* = -i d/dx on its antiperiodic domain;
2. W Gamma W^* = P, the reflection (Pu)(x) = u(-x);
3. W f W^* = multiplication by f(|x|), so the algebra goes to the reflection-even functions;
4. the walls x = 0 and x = l go to the two fixed points x = 0 and x = l ~ -l of the reflection.

Hence spec D_l = {(k + 1/2)/R : k in Z}, each eigenvalue simple, which is (9) of lemma 05, and the Connes-Lott mass of the minimal truncation is m = pi/(2l) = 1/(2R).

Proof. The inverse map is L(x) = (u(x) + u(-x))/sqrt 2, R(x) = (u(x) - u(-x))/(i sqrt 2) for x in [0, l], so W is a bijection, and |u(x)|^2 + |u(-x)|^2 = |L(x)|^2 + |R(x)|^2 gives unitarity. For H^1 data, u is continuous at 0 exactly when R(0) = 0, and u(l) + u(-l) = sqrt 2 L(l), so the antiperiodic matching u(-l) = -u(l) is the condition L(l) = 0. On x > 0, -i (L + iR)' = R' + i(-L'), the image of (R', -L') = D_l(L, R). On x < 0, u(x) = (L(-x) - i R(-x))/sqrt 2 and -i u'(x) = (R'(-x) - i(-L'(-x)))/sqrt 2, which is again the image of (R', -L'). Gamma(L, R) = (L, -R) maps to L(|x|) - i sgn(x) R(|x|) = u(-x). Items 3 and 4 are immediate. The antiperiodic spectrum of -i d/dx on a circle of length 2l is pi(2k + 1)/(2l) = (k + 1/2)/R. ■

So the two sheets of H are the two fixed points of a reflection on a circle whose spinors are antiperiodic, and the grading of the finite geometry is that reflection. With finite data D_F supplied as in lemma 22, the parent unfolds to -i d/dx + D_F on the periodic circle of length 2l (lemma 25, Proposition 25.1), and the angle theta interpolates between that circle (theta = 0) and the antiperiodic circle of Proposition 1.1 (theta = pi/2): D_F is a constant connection, a Wilson line, and the wall angles theta_k = l mu_k (mu_k the eigenvalues of D_F) are its holonomy phases.

The joint family: weak masses below a descending tower

At fixed l, the vector masses of lemma 23 vanish as theta -> 0 and also as theta -> pi/2, while the tower spacing pi/l stays fixed; vanishing masses alone open no dimension (audit §2.2). The audit gives a family in which they do.

Proposition (Astra 6, audit §3; restated). Let D_F(epsilon) = epsilon D_* and l_epsilon = l_* epsilon^{-alpha}, 0 < alpha < 1, with wall data X = tan(l D_F) as in lemma 22. Then the exact window is W^* D W = epsilon D_, *the rest of the spectrum lies at /lambda/ >= pi/l_epsilon - epsilon //D_||*, the compression error is at most (2/3) l_^2 //D_||^2 epsilon^{2(1-alpha)} ||a|| (Theorem 22.4, item 2), and for alpha < beta < 1 a cut-off E_epsilon = E_* epsilon^beta contains exactly the r = dim H_F finite states, while at every fixed E > 0 the number of states N_epsilon(E) grows like l_epsilon E, which is proportional to epsilon^{-alpha}.

In the circle language of lemma 25 (the periodic circle, on which D_F is a Wilson line; the window is its n = 0 mode) this reads: the radius R_epsilon = l_epsilon/pi goes to infinity, and the Wilson-line angles theta_epsilon = l_epsilon epsilon mu_k go to zero like epsilon^{1-alpha}. The hierarchy is

```
m_weak ~ epsilon  <<  E_epsilon ~ epsilon^beta  <<
↪  1/R_epsilon ~ epsilon^alpha  ->  0.
```

In the five-dimensional Yang-Mills toy of lemma 23 the vector poles along this family are M_W^2 = 2 m^2 (1 + O(theta^2)) and M_Z^2 = (16/5) m^2 (1 + O(theta^2)), so M_W^2/M_Z^2 -> 5/8 (audit §3.1, from Theorem 23.6).

Check (Claude). The parent space of lemma 22 is L^2([0, l]; C^2) ⊗ H_F, of fibre dimension 2r. By Theorem 22.1 its signed spectrum is (n pi +- theta_k)/l, n in Z, two progressions of spacing pi/l for each of the r/2 singular values, so N(E) = 2 r l E/pi + O(r), the constant of the audit (the scalar circle of Proposition 1.1 gives 2 E l/pi; with fibre H_F the count is 2 r l E/pi). The window, gap and compression statements follow from Theorems 22.1 and 22.4 as stated. (Python, section H: exact spectra for random singular values.)

The audit also gives a localized family (its §4): a bulk mass mu_epsilon = mu_* epsilon^alpha and mu l = (1 - alpha) log(1/epsilon) + c_0, with light mass m ~ 2 mu_* e^{-c_0} epsilon, gap ~ mu_* epsilon^alpha, and the exact tail identity e^{-2 kappa l} = (m/(mu + kappa))^2. Its table of roots is reproduced in section H (Python, bisection on tanh s = s/c and tan y = y/c): the

four light masses and next masses agree with the audit to the digits printed there.

Referee note (GPT-6.1 Terra, 4 Nov 2026). The localized asymptotics and tail identity are independently checked analytically; the length includes a logarithmic factor. The numerical table was not recomputed. See the referee report.

Proposal 1.3 (the circle of H as the M-theory circle)

Take the circle of Proposition 1.1 to be the eleventh direction of M-theory over a ten-dimensional type IIA background, so R = g_s l_s. Then, along the joint family:

H	circle of Proposition 1.1	M-theory / string side (recalled, to check)
v -> 0 along the joint family, label 11	R -> infinity, Wilson-line angles -> 0, m_weak R -> 0	strong coupling: D0-brane bound states of mass n/(g_s l_s) become light and are the momentum modes of the eleventh dimension (Witten 1995); the weak masses lie below this tower
v finite, label 10, two sheets	R finite, fixed points of a reflection	M-theory on an interval with two walls (Hořava-Witten); here with walls of opposite chirality
v -> infinity, label 9, sheets merge	R -> 0, antiperiodic spinors	a circle with antiperiodic fermions shrinking: bulk fermions get masses >= 1/(2R); compare type 0 strings, which have no bulk spacetime fermions (Bergman-Gaberdiel 1999), and the walls of opposite chirality of Fabinger-Hořava 2000

Each row matches a statement already in the corpus: the

11 endpoint needs the tower (canonical §2.1, lemma 05 Part 4), and at the 9 endpoint the bare interval keeps no light fermion, so light fermions must come from another sector (STRATEGY, T1 and T3). On the string side the same happens: the light fermions of a type 0 theory come from open strings on branes. The proposal therefore places the light fermions of the 9 endpoint on branes or walls, and 03 develops a brane origin for them.

Tests.

1. Along the joint family m_weak/M_KK ~ epsilon^{1 - alpha} -> 0, so the weak scale is far below 1/(g_s l_s); the scale-free path of Proposition 1.5, with M_W R fixed, is the other possibility and keeps no finite window.
2. Gauge couplings. A five-dimensional gauge field with constant profile on the interval has g_4^{-2} = l/g_5^2 = pi R/g_5^2 (audit §7.2 for the general volume formula), so at fixed g_5 the four-dimensional couplings vanish as R -> infinity, and M_W = g v/2 could vanish through g. The limit sought keeps g finite, which needs g_5^2 ~ R, and this must be shown for the vectors in question before the masses are read as restoration of the symmetry.
3. The path must be selected by an action: Proposition 1.5 shows what the flat interval alone selects.

Proposition 1.4 (the angle-dependent one-loop term at the vacuum angle)

Statement. Under the hypotheses of Theorem 9 of lemma 24 (flat interval, zero bulk masses, frequency-one fermions, the stated gauge towers and multiplicities, no wall potential), the angle-dependent part of the one-loop potential, taken relative to theta = 0 as in lemma 23, has its minimum at theta = pi/2 with value

```
V(pi/2) - V(0) = - 93 zeta(5) n_f / (512 pi^2 l^4),
```

and near theta = 0 it is (3 zeta(3)/(16 pi^2)) (54/5 - n_f) theta^2/l^4 + o(theta^2). At theta = pi/2 the exact vector poles of lemma 23 give M_W = M_Z = 0, with the fermion at the tower scale m = pi/(2l).

Proof. In the notation of lemma 24, V = $\mathcal{V}$/(8 pi^2 l^4), with F(0) = 0, and $\mathcal{V}$(1) = -n_f F(1) since the gauge

terms vanish at t = 1. With 1 + 1/sinh^2 x = coth^2 x and log coth x = 2 sum over odd k of e^{-2kx}/k,

```
F(1) = 2 int_0^infinity x^3 log coth x dx = 4 sum_{k
 ↪ odd} (1/k) 3!/(2k)^4 = (3/2) sum_{k odd} k^{-5}
 ↪ = (3/2)(31/32) zeta(5) = 93 zeta(5)/64.
```

Near t = 0, $\mathcal{V}$'(0) = (6 + 24/5 - n_f) J(0) with J(0) = int x^3/sinh^2 x dx = (3/2) zeta(3), the coefficient already given in lemma 23. Python: F(1) = 1.506786, J(0) = 1.803085, series against quadrature to 10^-6. ■

Reading. The θ-independent Casimir energy of the towers, a_0/l^4, is not part of this potential and is not computed in lemmas 23 or 24; the direction in which l moves depends on the sign of a_0 - 93 zeta(5) n_f/(512 pi^2). The minimum over the angle is the point with massless W and Z at fixed l, which opens no dimension (lemma 23 already notes that it is not the physical point).

Proposition 1.5 (a scale-free potential with an isolated stationary angle fixes M_W l)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The derivative identity is correct, and an isolated stationary angle gives $l \propto 1/m$. The universal conclusion has a flat-direction counterexample $G(\theta) = \theta^4$; for the displayed one-loop determinant the small-angle obstruction is recovered when $n_f \neq 54/5$. See the referee report.

Statement. Let V(l, theta) = l^{-4} G(theta) with G of class C^1, the form of every one-loop potential of a flat interval whose only scales are l and the wall angles (lemmas 23 and 24, with a_0 included in G).

1. If the angle is free, V is monotone in l at every fixed theta with G(theta) != 0. At the minimizing angle theta_min the length runs away: to l -> infinity if G(theta_min) > 0, to l -> 0 if G(theta_min) < 0. Stationary points exist only on the set G = G' = 0.
2. If the angle is tied to a fixed mass, theta = l m (as for X = tan(l M) in lemma 22), the stationary lengths satisfy theta G'(theta) = 4 G(theta) at theta = l m. So l = theta_*/m for a root theta_* that does not depend on m, and M_W l, M_Z l (functions of theta alone by Theorem 23.6) are the same at every m.

Hence, when the roots of theta G' - 4 G are isolated, a stationary branch has l = theta_*/m: the path l ~ 1/M_W of H's original statement, alpha = 1 in the scaling of the joint family (Remark 10.7 of 10 for a small fixed ratio). A root interval, as for G = theta^4 near 0, makes every alpha < 1 stationary and selects none (referee amendment). The joint family with alpha < 1 needs a second scale in the action: a bulk mass (the localized family, G(l m, l mu)), the cutoff of the spectral action (lemma 25, G(theta, l Lambda)), curvature, flux or a wall term.

Proof. d/dl [l^{-4} G(l m)] = l^{-5} (theta G'(theta) - 4 G(theta)) at theta = l m. For item 1, d/dl V = -4 l^{-5} G(theta) and d/dtheta V = l^{-4} G'(theta). ■

Amendment (4 Nov 2026). The window can separate on the scale-free path, with a fixed small ratio, if the light mass is exponentially small in a stabilized shape parameter, and a_0 is now computed. See 10, Remark 10.7.

Consequence for the modulus target of the audit (§7.1). The audit proposes V = V_*(a r^{-p} + b epsilon^2 r^q) with a, b, p, q > 0, whose minimum r ~ epsilon^{-2/(p+q)} gives alpha = 2/(p + q) < 1 when p + q > 2. In the flat one-loop family the epsilon-dependent term near theta = 0 is (3 zeta(3)/(16 pi^2)) (54/5 - n_f) m^2/l^2, so q = -2, and with the Casimir term p = 4 the stationary point has l ~ 1/m, alpha = 1, in agreement with item 2. A term growing with l at fixed v, as the ansatz requires, is absent from the flat family and must come from one of the second scales listed above.

What remains open here

- Whether the identification of Proposal 1.3 survives the gauge sector: lemmas 13, 14 and 25 tie M_W to the tower with model-dependent coefficients, which here become a relation between M_W, g_s and l_s, and test 2 above requires g_5^2 ~ R.
- The theta-independent Casimir coefficient a_0 of the flat family, which decides the runaway direction of Proposition 1.5, item 1.
- A potential with a second scale that selects alpha < 1: the localized family of the audit is the first candidate, with G(l m, l mu) computable from the exact secular equations of lemma 09.

- A fermion with both a fibre momentum and a base twist: on a nontrivial circle bundle (Witten spaces, file 02) the antiperiodic shift corresponds to a spin structure whose fibre circle does not lift the circle action. Which Witten spaces carry such a structure is an open item for the next job.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 1.4. *From the referee report on the 11D line, section 'Proposition 1.4: the one-loop endpoint and small-angle coefficient', transferred verbatim in substance on 4 Nov 2026.*

Hypotheses: those of Theorem 9 of lemma 24, including n_f > 0 and its normalization. The result is relative to theta $= 0$ and leaves the theta-independent Casimir term open.

Let

```
F(t) = int_0^infinity x^3 log(1 + t/sinh^2 x) dx,
 ↪  t = sin^2 theta,
```

so that, by equation (22) of lemma 24, $\mathcal{V}$(t) $= 8$ pi^2 l^4 V(theta) $= 3$ F(2t(1 - t)) + (3/2) F((16/5) t(1 - t)) - n_f F(t). For every x > 0 the integrand increases strictly with t, so F is strictly increasing, and F(0) $= 0$. The two gauge terms are nonnegative and vanish at t $= 1$, where their arguments vanish; the fermion term -n_f F(t) is strictly decreasing for n_f > 0. Hence t $= 1$ is the unique global minimum.

Endpoint value. At t $= 1$, $1 + 1/\sinh^2 x = \coth^2 x$. With q $= e^{-2x}$, coth x $= (1 + q)/(1 - q)$, so

```
log coth x = log(1 + q) - log(1 - q) = 2 sum_{k >=
 ↪  1, k odd} e^{-2kx}/k,     int_0^infinity x^3
 ↪  e^{-2kx} dx = 3!/(2k)^4.
```

All terms are positive, so the series may be integrated term by term, and

```
F(1) = 2 int_0^infinity x^3 log coth x dx = 4 sum_{k
↪  odd} (1/k) 3!/(2k)^4 = (3/2) sum_{k odd} k^{-5}
↪  = (3/2)(1 - 2^{-5}) zeta(5) = 93 zeta(5)/64.
```

Since $\mathcal{V}(0) = 0$ and $\mathcal{V}(1) = -n_f F(1)$, division by 8 pi^2 l^4 gives the stated endpoint value V(pi/2) - V(0) = -93 zeta(5) n_f/(512 pi^2 l^4).

Small-angle coefficient. Differentiation under the integral at t = 0 is allowed because x^3/sinh^2 x is integrable on (0, infinity). With 1/sinh^2 x = 4 e^{-2x}/(1 - e^{-2x})^2 = 4 sum_{n >= 1} n e^{-2nx},

```
F'(0) = int_0^infinity x^3/sinh^2 x dx = 4 sum_{n >=
↪  1} n 3!/(2n)^4 = (3/2) sum_{n >= 1} n^{-3} =
↪  (3/2) zeta(3),
```

which is J(0) in the proof of Proposition 1.4. The two gauge arguments 2t(1 - t) and (16/5) t(1 - t) have derivatives 2 and 16/5 at t = 0, with prefactors 3 and 3/2, so

```
V'(0) = (3 * 2 + (3/2)(16/5) - n_f) F'(0) = (54/5 -
↪  n_f)(3/2) zeta(3).
```

With t = sin^2 theta = theta^2 + o(theta^2), V(theta) - V(0) = $\mathcal{V}'(0)$ theta^2/(8 pi^2 l^4) + o(theta^2) = (3 zeta(3)/(16 pi^2))(54/5 - n_f) theta^2/l^4 + o(theta^2), the claimed coefficient. The exact vector formulas of lemma 23 vanish at theta = pi/2, where the lowest fermion mass is pi/(2l).

Counterexample (referee) to the universal conclusion of Proposition 1.5. *From the referee report on the 11D line, section 'Proposition 1.5: scale-free potentials and the joint path', transferred verbatim in substance on 4 Nov 2026.*

Verdict of the referee: an error in the universal conclusion, and correct with an amendment for isolated stationary angles. The derivative identities in the proof hold. The universal conclusion, that every stationary family has constant angle and that a second scale is necessary for the existence of a joint family, fails by the counterexample below.

At fixed m > 0 put theta = l m. Then

```
d/dl V(l, l m) = l^{-5} (theta G'(theta) - 4
↪  G(theta)).
```

If a continuous stationary branch chooses an isolated positive root theta_, *then theta_* is constant along the branch and l = theta_*/m. A root interval gives a flat direction instead.

Counterexample. Let G(theta) = theta^4 on a neighbourhood 0 <= theta < delta, with a C^1 extension to the angle domain. There

```
V(l, l m) = l^{-4} (l m)^4 = m^4
```

is independent of l. For any 0 < alpha < 1, the family m = epsilon, l = epsilon^{-alpha} has theta = l m = epsilon^{1 - alpha} < delta for all sufficiently small epsilon, so it is stationary, and l m -> 0. The vector mass formula of Theorem 23.6 gives M_W l ~ sqrt 2 l m -> 0. The potential leaves this path undetermined and does not dynamically select it. A smooth extension can also be chosen with G(theta) >= theta^4 on the rest of the fundamental interval; then V(l, l m) >= m^4, with equality on l m < delta, and the example is a flat family of minima.

The specified flat one-loop determinant. By Proposition 1.4, with a_0 the theta-independent Casimir coefficient,

```
G(theta) = a_0 + c theta^2 + o(theta^2),    theta
↪ G'(theta) - 4 G(theta) = -4 a_0 - 2 c theta^2 +
↪ o(theta^2),
```

where c = (3 zeta(3)/(16 pi^2))(54/5 - n_f) is proportional to 54/5 - n_f (the expansion of G' follows from that of $\mathcal{V}$', since $\mathcal{V}$ is C^1 at t = 0). If c != 0, there is no sequence of positive stationary angles tending to zero, whether or not a_0 = 0: for a_0 != 0 the right side tends to -4 a_0, and for a_0 = 0 it equals -2 c theta^2 (1 + o(1)). In particular this covers every integer n_f under the displayed determinant. The tuned case c = 0 requires its higher-order term to be checked. The comparison with the audit's ansatz with q > 0, in the paragraph on the modulus target, is correct: the displayed quadratic contribution is l^{-4} c (l m)^2 = c m^2/l^2.

Free angle. For a free angle, d/dl V = -4 l^{-5} G(theta); a nonzero minimized G produces the stated runaway, and a zero minimum leaves the length flat. At an endpoint of a constrained angle interval, the stationarity condition on G' must be understood one-sided.

Scope and direction. The corrected fixed-angle result concerns isolated stationary roots of a scale-free C^1 potential. A flat modulus, a tuned higher-order potential, or additional dimensionful terms leave other families possible. A mechanism that selects a joint path with finite gauge couplings remains to be constructed.

Independent proof of the asymptotics of the localized family (audit, its §4). *From the referee report on the 11D line, section 'Audit section 4 and file 01: localized joint family', transferred verbatim in substance on 4 Nov 2026.*

The asymptotic light mass, the gap, the localization ratio and the exact tail identity follow analytically from the interval equations. The localized length includes a logarithmic factor, and so differs from the pure-power length of the boundary-angle family above. The numerical table of section H was not recomputed by the referee.

Let c = mu l, s = kappa l, and s = c tanh s, the light-mode condition kappa = mu tanh(kappa l). For c > 1 there is a positive solution, and as c -> infinity it has s -> infinity and s/c = tanh s -> 1. Moreover

```
0 < c - s = c (1 - tanh s) = 2c e^{-2s}/(1 +
 ↪  e^{-2s}) <= 2c e^{-2s} -> 0.
```

Therefore

```
m = mu sech s = 2 mu e^{-s}/(1 + e^{-2s}) ~ 2 mu
 ↪  e^{-s} ~ 2 mu e^{-c}.
```

With mu = mu_* epsilon^alpha and c = (1 - alpha) log(1/epsilon) + c_0, e^{-c} = e^{-c_0} epsilon^{1 - alpha}, which gives m ~ 2 mu_* e^{-c_0} epsilon and l = c/(mu_* epsilon^alpha). The localization width is 1/kappa, so its ratio to l is 1/s ~ 1/c.

The first oscillatory root y_1 = k_1 l of tan y = y/c satisfies pi < y_1 < 3 pi/2, since y/c > 0 is small for large c. The gap above the light pair is therefore

```
Delta = sqrt(mu^2 + k_1^2) = mu sqrt(1 + (k_1
 ↪  l/c)^2) ~ mu = mu_* epsilon^alpha.
```

Finally, s = c tanh s reads mu = kappa coth s, and with it m = mu sech s = kappa/sinh s. Then

```
m/(mu + kappa) = 1/(sinh s (coth s + 1)) = 1/(cosh s
↪ + sinh s) = e^{-s},    m^2 = kappa^2/sinh^2 s =
↪ kappa^2 (coth^2 s - 1) = (mu - kappa)(mu +
↪ kappa),
```

and the first identity squared is e^{-2 kappa l} = (m/(mu + kappa))^2. This proves both displayed tail identities.

Part 32

Witten spaces as the hypercharge quotient of the five-dimensional representation

Lemma 32. Written as file 02 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README). Theorem 2.1 and Propositions 2.2 to 2.4 and 2.6 are proved here, Theorem 2.1 and Proposition 2.6 conditionally as the referee notes say; Proposition 2.5 is a negative result in a stated scope, refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5. Attempted by Claude (Anthropic), 3 Nov 2026, with a remark from the steer of the same day on the mixing of u and d. Checks: checks_11d.py, sections A to D, Python with exact fractions. Referee pending.

Referee note (GPT-6.1 Terra, 4 Nov 2026). The exclusion in Proposition 2.5 has an independent analytic proof. The required-bidegrees lemma and the general assertion that u^c, d^c must have the same flux image require a common ten-dimensional chirality; the exclusion itself also holds without that extra assumption. See the referee report; other reviewed items are noted separately.

Referee note on Theorem 2.1 and Propositions 2.2, 2.4 (GPT-6.1 Terra). The quotient and tensor-centre calculations are independently checked. The cited M^{111} identification is conditional on the recalled fibration; the original quotient metric needs rescaling to be

Sasaki-Einstein, and the action on holomorphic functions is dual to the action on U. See the report.

Referee note on Proposition 2.3 (GPT-6.1 Terra). The Gysin, Smith normal form and spin calculations are independently checked; the cited Kreck-Stolz paper was unavailable. See the report.

Setting

Witten's spaces M^{pqr} = G/H, with G = SU(3) x SU(2) x U(1) and H = SU(2) x U(1) x U(1), are the seven-dimensional spaces of least dimension with the isometry group of the Standard Model, so 4 + 7 = 11 (Witten 1981; paper/LITERATURE.md §F). Castellani, D'Auria and Fré gave them Einstein metrics; Killing spinors exist only for p = q, where the AdS4 vacuum has N = 2 and the space is M^{111} (Fabbri, Fré, Gualtieri, Termonia, hep-th/9903036). Fabbri et al. (hep-th/9907219, §4) write the cone over M^{111} as the Kähler quotient of C^5 by a circle acting with weights 2 on a triplet U and -3 on a doublet V, with D-term 2|U|^2 = 3|V|^2, and M^{111} as a circle bundle over P^2 x P^1. In later notation it is M^{3,2}.

These weights are six times the hypercharges of the 5-bar of SU(5). The rest of the file develops that observation.

Theorem 2.1 (M^{111} is the hypercharge quotient of the 5-bar)

Statement. Let 5bar = C^3 + C^2 carry the Standard Model action on the left-handed fields d^c in (3bar, 1)_{1/3} and L in (1, 2)_{-1/2}; write U for d^c and V for L, so 6Y = 2 on U and -3 on V. Let mu_Y = (2|U|^2 - 3|V|^2)/6 be the moment map of hypercharge, and

```
Y_{3,2} = (mu_Y^{-1}(0) ∩ S^9) / U(1)_Y.
```

1. Y_{3,2} is a smooth, simply connected seven-manifold, the circle bundle of O(3, 2) over CP^2 x CP^1 (up to orientation), and its cone is the Kähler quotient C^5 // U(1)_Y of Fabbri et al.
2. The cone is Calabi-Yau (Gorenstein, sum of weights 0) because tr Y = 0 on the 5-bar, that is, because U(1)_Y lies in SU(5).
3. The group acting effectively on Y_{3,2} through the quotient is (U(3) x U(2))/U(1)_Y. Its subgroup

S(U(3) x U(2))/U(1)_Y is PSU(3) x SO(3), and the overall phase of U(5) supplies one further circle, the Reeb (R-symmetry) direction.

4. The functions on Y_{3,2} that are polynomial in U, V and their conjugates have U(1)_Y-charge zero; the holomorphic ones are spanned by the monomials of degree (3k, 2k) in (U, V), whose span as tensors is Sym^{3k}(3bar) x Sym^{2k}(2) and as functions its dual Sym^{3k}(3) x Sym^{2k}(2): for k = 1 the functions form the (10, 3) of Fabbri et al.

Proof. On mu_Y^{-1}(0) ∩ S^9 we have |U|^2 = 3/5 and |V|^2 = 2/5, a product of round spheres S^5 x S^3. The circle acts by (lambda^2 U, lambda^{-3} V); a fixed point would need lambda^2 = lambda^3 = 1, so lambda = 1 and the action is free. The quotient of S^5 x S^3 by the full torus T^2 of phases of U and V is CP^2 x CP^1, and Y_{3,2} -> CP^2 x CP^1 is a principal bundle for T^2/U(1)_Y. A character lambda_U^m lambda_V^n of T^2 is trivial on U(1)_Y exactly when 2m - 3n = 0, so the characters of T^2/U(1)_Y are generated by (m, n) = (3, 2), whose line bundle is O(3) ⊠ O(2) up to the sign convention; this gives item 1 and item 4. Simple connectivity follows from the exact homotopy sequence of U(1) -> S^5 x S^3 -> Y_{3,2}. Item 2: the sum of weights is 3 * 2 + 2 * (-3) = 0. Item 3: an element (g, h) of U(3) x U(2) acts trivially on the quotient exactly when (g, h) = (lambda^2, lambda^{-3}), because mu_Y^{-1}(0) spans C^5. The intersection of SU(3) x SU(2) with U(1)_Y consists of lambda^6 = 1, generated by (e^{2 pi i/3} I_3, -I_2), the centre Z_3 x Z_2. ■

So the isometry group of the Witten space with Killing spinors, SU(3) x SU(2) x U(1) (Fabbri et al.), is the centralizer of hypercharge in U(5), divided by hypercharge. The SU(3) x SU(2) part is the Standard Model one; the remaining circle is the R-symmetry, and hypercharge itself has been quotiented. Witten assigned the U(1) of G to hypercharge; in this realization hypercharge is a different U(1) (03, Proposal 3.1).

Proposition 2.2 (uniqueness of the count 7)

Statement. (a) Among the quotients Y_{a,b} = (S^5 x S^3)/U(1), with weights b on C^3 and -a on C^2 and gcd(a, b) = 1, a, b > 0, the cone is Calabi-Yau only

for (a, b) = (3, 2). (b) For a complex representation of dimension n and a free circle action, (S^{2n-1} ∩ mu^{-1}(0))/U(1) has dimension 2n - 3; this is 7 only for n = 5. Among representations of SU(5), with hypercharge as the circle, only the 5 and the 5-bar give a seven-dimensional quotient.

Proof. (a) The Calabi-Yau condition is 3b - 2a = 0. (b) Dimension count: 2n - 1 for the sphere, minus one for the moment map, minus one for the circle. The irreducible representations of SU(5) of dimension 5 are the 5 and the 5-bar. ■

Witten's count reads: 11 = 4 + (2 * 5 - 3). The seven internal dimensions come from the smallest representation of SU(5), with hypercharge as the circle. For the 10 the quotient has dimension 17, and for the 5 + 10 it has 27.

Proposition 2.3 (topology of Y_{a,b})

Statement. Let Y_{a,b} -> B = CP^2 x CP^1 be the circle bundle with Euler class a h_1 + b h_2, a != 0. Then H^2(Y) = Z + Z/gcd(a, b), b_3(Y) = 0, H^4(Y) is a group of order a^2, cyclic when gcd(a, b) = 1, and Y is spin exactly when a is odd and b is even. For (a, b) = (3, 2): H^2 = Z, H^4 = Z/9, spin. The pullback H^2(B) = Z^2 -> H^2(Y) = Z is (x, y) -> 2x - 3y up to sign: the two divisor classes h_1 = [{l(U) = 0}] and h_2 = [{m(V) = 0}] go to 2 and -3, the values of 6Y on d^c and L.

Proof. The Gysin sequence H^{k-2}(B) -> H^k(B) -> H^k(Y) -> H^{k-1}(B) -> H^{k+1}(B), with the first map the cup product with c = a h_1 + b h_2 and H^*(B) = Z[h_1, h_2]/(h_1^3, h_2^2). In degree 2 the cokernel of 1 -> (a, b) is Z^2/(a, b). In degree 4 the map H^2(B) -> H^4(B) has matrix [[a, 0], [b, a]] in the bases (h_1, h_2), (h_1^2, h_1 h_2), with determinant a^2 and Smith invariants (gcd(a, b), a^2/gcd(a, b)); since H^3(B) = 0 this gives H^4(Y), and the same matrix is injective, so H^3(Y) = 0. For spin: w_2(B) = h_1 mod 2, and the kernel of H^2(B; Z/2) -> H^2(Y; Z/2) is spanned by c mod 2; so w_2(Y) = 0 exactly when (1, 0) = (a, b) mod 2. For (3, 2), Z^2/(3, 2) -> Z by (x, y) -> 2x - 3y is an isomorphism. ■ (Python: Smith invariants and the spin condition, section C.)

The product member CP^2 x S^2 x S^1 of Witten's family (a = b = 0) is not spin.

Proposition 2.4 (the Z_6 of the Standard Model)

Statement. (a) Every monomial in U and V, and every state of the exterior algebra Λ^*(5bar), satisfies 2t + 3d + 6Y = 0 mod 6, with t the triality (1 for 3, -1 for 3bar) and d = 2j mod 2. (b) The U(1)_Y-neutral monomials have t = 0 and d = 0. (c) Every tensor field on Y_{3,2} carries a representation of (U(3) x U(2))/U(1)_Y, on which SU(3) x SU(2) acts through PSU(3) x SO(3). So no tensor Kaluza-Klein mode on Y_{3,2}, on a background invariant under the displayed action, is a colour triplet or a weak doublet; spinors are open.

Proof. (a) holds for U (t = -1, d = 0, 6Y = 2) and V (t = 0, d = 1, 6Y = -3) and is additive. (b) U^alpha V^beta is neutral when 2 alpha = 3 beta, so 3 | alpha and 2 | beta. (c) The effective group of Theorem 2.1 acts on the frame bundle and on every tensor bundle, and its intersection with SU(3) x SU(2) is the centre Z_6 (Theorem 2.1, item 3). ■

(a) is the statement that the Standard Model group is (SU(3) x SU(2) x U(1))/Z_6 = S(U(3) x U(2)). In the quotient realization the Z_6 is the kernel of SU(3) x SU(2) -> Isom(Y_{3,2}). States with fractional hypercharge are those charged under the quotient circle, and they cannot be functions or tensors on Y_{3,2}. For spinors the isometry group may lift to a double cover; whether fermionic Kaluza-Klein modes on Y_{3,2} can carry half-integer weak isospin or triality is open (next job, item 2 of the README).

Chiral zero modes on the base with abelian fluxes

Witten's argument makes Kaluza-Klein fermions on an odd-dimensional space vectorlike, for gauge fields acting by isometries. The even base B_6 = CP^2 x CP^1 escapes it: abelian fluxes on B_6 give chiral four-dimensional fermions through the index of the twisted Dirac operator. This section asks which fluxes give one generation.

Setting. Ten-dimensional Weyl fermions on M^4 x

CP^2 x CP^1, each with definite charges under some U(1) factors with fluxes on B_6. Since CP^2 is not spin, a fermion is a spin^c spinor, $\Lambda^{0,*} \otimes K^{1/2} \otimes N$ with N the flux line bundle of its charges, and the four-dimensional chiral content is the virtual representation H^*(O(alpha, beta)) with (alpha, beta) = (-3/2, -1) + pairing of the charges with the fluxes. The cohomology is H^0(O(alpha)) = Sym^alpha(3bar) for alpha >= 0, H^2(O(alpha)) = Sym^{-alpha-3}(3) for alpha <= -3, and zero otherwise on CP^2; on CP^1, H^0(O(beta)) = Sym^beta(2) for beta >= 0 and H^1(O(beta)) = Sym^{-beta-2}(2) for beta <= -2. Each Standard Model Weyl field must come from one ten-dimensional field, from the whole cohomology of one line bundle.

Lemma (required bidegrees). Writing P_s = (alpha + 3/2, beta + 1) for the pairing of the charges of field s (up to the common sign of charge and chirality):

field	P_s	note
Q (3, 2)	(-5/2, 2)	H^2(O(-4)) x H^0(O(1))
u^c, d^c (3bar, 1)	(5/2, 1)	H^0(O(1)) x H^0(O(0))
L (1, 2)	(+-3/2, 2)	
e^c, nu^c (1, 1)	(+-3/2, 1)	

The second entry is 2j + 1 and the first is +-5/2 on colour triplets and +-3/2 on singlets. (Python, section D, by enumeration of the line bundles with these cohomologies and the Riemann-Roch check.)

Proposition 2.5 (no generation from fluxes along hypercharge and B - L) *Scope note (4 Nov 2026, after the referee).* The statement on equal images and the linearity of 2j + 1 assume a common ten-dimensional chirality for the fields. The conclusion holds without it: in the span of 6Y and 3(B - L) the charges of L are -3 times those of Q, so their second pairings are P_2(L) = -3 P_2(Q), against |P_2| = 2 on both doublets.

Referee note on the third-charge search (GPT-6.1 Terra, 4 Nov 2026). The bounded search below has an independent analytic proof. The same argument extends to any real third charge and independent source chiralities; this stronger extension was checked by Claude Sonnet 5.5 (second check: the vectors y, b, p, w have rank

4 for every sign pattern), so it stands as a no-go in the same scope with two model identifiers. See the referee report.

Statement. In the setting above, with the canonical spin^c root and fluxes only in U(1) directions spanned by Y and B - L (any flux quanta), no choice reproduces one generation, with or without nu^c. More generally, the pairing must give u^c and d^c the same image, and 2j + 1 = (2, 1, 1, 2, 1, 1) on (Q, u^c, d^c, L, e^c, nu^c) must be a linear function of the fluxed charges.

Proof. The pairing M is linear in the charges, and M c_s = P_s with c_s the charges of the left-handed field s. With only Y and B - L fluxed, c_u - c_d = (-6, 0) in the coordinates (6Y, 3(B - L)), so M vanishes on the hypercharge direction and P_s = 3(B - L)_s m for one vector m. The second entries 2 on Q and 1 on u^c would then need 3(B - L) to take values in the ratio 2 : 1 on (Q, u^c), against 1 : -1. In general, 2j + 1 is not in the span of 6Y = (1, -4, 2, -3, 6, 0) and 3(B - L) = (1, -1, -1, -3, 3, 3): from Q and u^c the coefficients are -1 and 3, and then d^c gives -5 against 1. ■ An exhaustive search adding a third integer charge z in [-3, 3]^6 to Y and B - L finds no solution either (Python, section D).

Scope and escapes. Abelian fluxes on CP^2 x CP^1 with the Fubini-Study complex structure; one ten-dimensional Weyl field per Standard Model field; canonical spin^c root; Dolbeault index as the chiral content. Known escapes: nonabelian bundles (an SU(2)_R doublet bundle), several ten-dimensional fields per four-dimensional field, a changed common twist, orbifolds and walls, and non-Kaluza-Klein (brane) states. The result points to the next proposition.

Proposition 2.6 (one generation from a universal charge, the grading and B - L) Referee note (GPT-6.1 Terra, 4 Nov 2026). The index calculation works with the stated cohomology convention. For the colour action of the earlier quotient, reverse the first-component coefficients to obtain the labelled colour representations; global flux quantization remains to be specified. See the referee report.

Statement. Let the fluxed charges be F (one on every field: a change of the common spin^c twist), chi (+1 on

Q and L, -1 on u^c, d^c, e^c, nu^c) and B - L. Then one generation with nu^c is the exact index content for the pairing

```
P^{(1)} = c_0 chi + c_1 (B - L),   P^{(2)} = (3F +
 ↪  chi)/2,
```

with (c_0, c_1) = (-9/4, -3/4) or (-3/2, -3). The bidegrees are Q: (-4, 1); u^c, d^c: (1, 0); L: (-3, 1) or (0, 1); e^c, nu^c: (0, 0) or (-3, 0). Without chi, or without B - L, there is no solution.

Proof. Substitution, and the two failures by the span test (Python, section D2). ■

Reading.

1. u^c and d^c see the same line bundle O(1, 0), and so do e^c and nu^c. The flux treats them as SU(2)_R doublets with a common internal profile; hypercharge Y = T_3R + (B - L)/2 enters through T_3R, which carries no flux. This is the left-right pattern of Pati-Salam, produced here by the index.
2. chi is the finite grading gamma_F on the particle basis. The flux therefore couples to the grading, an axial abelian connection on B_6; in the language of the finite geometry this is a fluctuation proportional to gamma_F, the structure of the two gradings of lemma 23.
3. **Mass and charge bases.** Because u^c and d^c share one zero-mode profile, overlap integrals on B_6 cannot separate them; their masses come from data that break SU(2)_R (Higgs or wall couplings), and the basis that diagonalizes the masses is set by those data, independently of the basis in which the charges are diagonal. The steer of 3 Nov 2026 remarks that u and d should still mix, since mass eigenvalues are not charge eigenvalues, and that at the 9 label, with the left-handed interactions off, charge eigenvalues can coincide. Proposition 2.6 puts the coincidence in the flux charges: equal flux charges for u^c and d^c, unequal hypercharges through T_3R. At the 9 label the electroweak part of the algebra is removed (lemma 21, STRATEGY T2), and what distinguishes u^c from d^c in the light theory is then the vectorlike charge alone. Making this precise needs the SU(2)_R-breaking

wall data and the 9D reduction of this index; listed as item 4 of the README.

What this file establishes, and the direction

Referee note (GPT-6.1 Terra, 4 Nov 2026). The M^{111} identification is conditional on the recalled fibration and metric normalization. The flux examples establish a particular construction and scoped obstructions; a general necessity for the grading flux remains open. See the wording audit.

The Witten space with Killing spinors is the hypercharge quotient of the 5-bar (Theorem 2.1), the count 7 is fixed by SU(5) (Proposition 2.2), and the Z_6 of the Standard Model is the kernel of the isometry action (Proposition 2.4). The price is that hypercharge is no longer an isometry, and no tensor Kaluza-Klein mode on an invariant background carries triality, weak isospin 1/2 or fractional hypercharge. Kaluza-Klein chirality on the base needs fluxes that see the grading and leave T_3R free (Propositions 2.5 and 2.6). Both results point to states outside the Kaluza-Klein spectrum of Y_{3,2} for the quarks and leptons: 03 takes them to be wrapped branes.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 2.5 (the exclusion). *From the referee report on the 11D line, section 'Proposition 2.5: hypercharge and B - L fluxes', transferred verbatim in substance on 4 Nov 2026.*

Hypotheses: one abelian line bundle per field, the canonical formal shift, fluxes in the span of Y and B - L. The exclusion holds even if the ten-dimensional Weyl fields have different chiralities.

Let M_2 be the component of the flux pairing M along CP^1, and use the charges c = (6Y, 3(B - L)). A weak doublet occurs for a line bundle O(beta) on CP^1 only when beta = 1 or beta = -3, which give H^0(O(1)) = 2

and H^1(O(-3)) = 2 respectively. The formal K^{1/2} shift on CP^1 is -1, so M_2(c) = beta + 1 and |M_2(c)| = 2 for each doublet. This remains true if a right-handed mode is conjugated to the left-handed convention, since conjugation reverses the sign of the pairing.

For the left-handed doublets,

```
c_Q = (1, 1),   c_L = (-3, -3) = -3 c_Q.
```

Linearity gives M_2(c_L) = -3 M_2(c_Q), so |M_2(c_L)| = 3 |M_2(c_Q)|, whereas both absolute values must be 2. This contradiction excludes a generation with or without nu^c, for arbitrary flux quanta in the two specified directions. It uses neither the colour component nor a choice of signs for the individual doublets.

Amendment to the required-bidegrees lemma. On CP^2 x CP^1, K^{1/2} is a formal notation: K = O(-3, -2) has no line-bundle square root. A genuine spin^c structure and its charged twists must give integral final bidegrees. With a common ten-dimensional chirality, Serre duality relates a right-handed conjugate to the sign-reversed pairing, and the convention P^{(2)} = 2j + 1 of the table is consistent. If the ten-dimensional chiralities can be chosen independently, the table is incomplete. For example, O(-4, -3) has H^3 = H^2(O(-4)) ⊗ H^1(O(-3)) = (3, 2) and pairing (-5/2, -2); a suitable ten-dimensional chirality makes this a left-handed Q, and (-5/2, -2) is neither the table entry (-5/2, 2) nor its overall negative.

Likewise, with independent chiralities and independent flux directions, u^c can arise from H^0(O(1, 0)) = 3bar, while d^c arises by conjugating a right-handed H^2(O(-4, 0)) = 3 with opposite abelian charges. Their pairings in the left-handed charge convention are (5/2, 1) and (5/2, -1). The equality of their images, and the unsigned linear formula P^{(2)} = 2j + 1, are therefore conditional on the common-chirality convention; the direct doublet obstruction above uses neither.

Scope and direction. This concerns the Dolbeault cohomology of a single abelian line bundle for each field on CP^2 x CP^1, with the canonical formal shift and fluxes in the span of Y and B - L. Nonabelian bundles, additional charges or a common twist, multiple source fields, walls, orbifolds and brane states remain possible directions. This is an independent check of the scoped ob-

struction and makes no claim about those alternatives.

Independent proof of the third-charge exclusion in Proposition 2.5. *From the referee report on the 11D line, section 'Proposition 2.5: adding a third charge', transferred verbatim in substance on 4 Nov 2026.*

Status as the referee stated it: the argument proves the original bounded search, a third integer charge in [-3, 3]^6 under the common-chirality convention. The same argument covers an arbitrary real third charge and independently chosen ten-dimensional chiralities; the referee stated that extension as a conjecture awaiting an independent check by another model, and the referee note at Proposition 2.5 records that check by Claude Sonnet 5.5.

Write, on (Q, u^c, d^c, L, e^c, nu^c),

```
y = 6Y = (1, -4, 2, -3, 6, 0),    b = 3(B - L) = (1,
↪  -1, -1, -3, 3, 3).
```

(Here b is a charge vector, unrelated to the Euler-class coefficient b of Proposition 2.3.) For the stated colour convention the necessary CP^2 pairings are

```
p = (-5/2, 5/2, 5/2, p_L, p_e, p_nu),   |p_L| = 3/2,
```

and the CP^1 pairings have the form

```
w = (2 s_Q, s_u, s_d, 2 s_L, s_e, s_nu),   s_i in
↪  {+1, -1}.
```

The signs allow independently chosen source chiralities. Reversing the colour convention reverses p and leaves the argument unchanged.

Suppose that p and w both lie in the span of y, b and z, for some third charge z. The Q and L entries of y and of b have ratio -3, so every vector of span(y, b) has L entry equal to -3 times its Q entry, while |w_L/w_Q| = 1; hence w lies outside span(y, b). Then span(y, b, w) has dimension three and lies in span(y, b, z), so the two spans coincide, and

```
p = a y + b_0 b + t w
```

for real a, b_0, t. Apply the combination 2Q + u^c + d^c, the linear form v -> 2 v_Q + v_u + v_d. It vanishes on y (2 - 4 + 2 = 0), on b (2 - 1 - 1 = 0) and on p (-5 + 5/2 + 5/2 = 0), and takes the value 4 s_Q + s_u + s_d on w. So

```
0 = t (4 s_Q + s_u + s_d).
```

The factor in parentheses is nonzero, since |4 s_Q| = 4 > 2 >= |s_u + s_d|; hence t = 0. Equality of the u^c and d^c entries of p = a y + b_0 b, -4a - b_0 = 2a - b_0, gives a = 0, and their common value -b_0 = 5/2 gives b_0 = -5/2. The resulting L entry is -3 b_0 = 15/2, contradicting |p_L| = 3/2. The same four fields suffice without nu^c.

Scope and direction. This concerns the one-line-bundle-per-field Dolbeault construction, the specified Y and B - L charges and the canonical formal shift. A fourth charge direction, a changed common twist, nonabelian bundles, multiple source fields, walls, orbifolds and brane states remain directions to investigate.

Independent proof of Proposition 2.3. *From the referee report on the 11D line, section 'Proposition 2.3: topology of the circle bundle', transferred verbatim in substance on 4 Nov 2026.*

The calculation is independent of the specific M^{111} identification. The cited Kreck-Stolz paper was unavailable to the referee, so no direct comparison with that paper is claimed.

Let B = CP^2 x CP^1, H^*(B; Z) = Z[h_1, h_2]/(h_1^3, h_2^2), and let the Euler class be e = a h_1 + b h_2, with a != 0. The Gysin sequence gives

```
H^2(Y) = Z^2/<(a, b)> ~= Z ⊕ Z/g,   g = gcd(a, b).
```

The map H^2(B) -> H^4(B), cup product with e, sends h_1 to a h_1^2 + b h_1 h_2 and h_2 to a h_1 h_2 (since h_2^2 = 0); in the bases (h_1, h_2) and (h_1^2, h_1 h_2) its matrix is

```
[[a, 0], [b, a]].
```

Its determinant is a^2 != 0, so its kernel is zero, and since H^3(B) = 0 the Gysin sequence gives H^3(Y) = 0. The greatest common divisor of its entries is g, so its Smith invariants are g and a^2/g, and H^4(Y), its cokernel, is Z/g ⊕ Z/(a^2/g): a group of order a^2, cyclic when g = 1.

Spin. The vertical tangent line is trivial, spanned by the generator of the circle action, so w_2(Y) = pi^* w_2(B) = pi^*(h_1 mod 2), using w_2(CP^2) = h_1 mod 2 and

w_2(CP^1) = 0. The mod-two Gysin sequence H^0(B; Z/2) -> H^2(B; Z/2) -> H^2(Y; Z/2), whose first map is the cup product with e mod 2, shows that the kernel of the pullback on H^2 is the span of e mod 2. Thus Y is spin exactly when h_1 ≡ e mod 2, that is, when a is odd and b is even. For (a, b) = (3, 2), the primitive homomorphism (x, y) -> 2x - 3y is onto Z and has kernel generated by (3, 2), so it identifies H^2(Y) with Z; and H^4(Y) ≅ Z/9.

Independent proofs for Theorem 2.1 and Propositions 2.2 and 2.4: freeness, the neutral-monomial congruence, the dual action and the Sasaki-Einstein normalization. *From the referee report on the 11D line, section 'Theorem 2.1 and Propositions 2.2 and 2.4: quotient and tensor action', transferred verbatim in substance on 4 Nov 2026.*

Hypotheses: the identification with M^{111} is conditional on the recalled description of M^{111} as the circle bundle of class 3h_1 + 2h_2; the cited paper could not be accessed by the referee.

Freeness and the bundle. On the moment level in the unit sphere, |U|^2 = 3/5 and |V|^2 = 2/5. Both factors are nonzero, so the stabilizer of a point under (lambda^2 U, lambda^{-3} V) has lambda^2 = lambda^3 = 1, hence lambda = 1, and the action is free. The quotient has dimension 9 - 1 - 1 = 7 and is simply connected by the homotopy sequence of U(1) -> S^5 x S^3 -> Y_{3,2}. The residual character of the two Hopf phases has primitive weight (3, 2); its circle bundle over CP^2 x CP^1 therefore has Euler class ±(3h_1 + 2h_2). Principal circle bundles over this base are classified by their Euler classes, which gives the conditional identification with M^{111}. The zero sum 3(2) + 2(-3) = 0 gives the Gorenstein holomorphic cone with trivial canonical bundle; a Ricci-flat metric needs a separate argument.

The congruence of Proposition 2.4. The congruence 2t + 3d + 6Y ≡ 0 (mod 6) holds on U, where (t, d, 6Y) = (-1, 0, 2), and on V, where (t, d, 6Y) = (0, 1, -3), and extends additively to tensor products and to conjugates, which reverse all three quantities. For a monomial in U, V and their conjugates, let Delta_U be the number of factors U minus the number of factors Ubar, and Delta_V likewise for V. Neutrality under U(1)_Y reads

```
2 Delta_U - 3 Delta_V = 0,
```

so 3 | Delta_U and 2 | Delta_V. The triality of the monomial is -Delta_U ≡ 0 (mod 3) and d ≡ Delta_V ≡ 0 (mod 2): zero triality and integer weak isospin. This extends item (b) of Proposition 2.4 from the holomorphic monomials U^alpha V^beta to all monomials in U, V and their conjugates.

Dual action. If U is the vector representation 3bar and projective space parametrizes its lines, the conventional pullback action on holomorphic coordinate functions is the dual 3: functions of degree 3k transform as Sym^{3k}(3). Treating U as formal field components gives Sym^{3k}(3bar) instead. The convention must be fixed consistently when identifying the 10 versus the 10bar and the later cohomology table.

Sasaki-Einstein normalization. With the standard unit-sphere quotient metric, the two transverse Fubini-Study factors have relative scales 3/5 and 2/5. CP^2 is the base of the Hopf circle on the sphere of radius sqrt(3/5), with Ricci constant 6/(3/5) = 10, and CP^1 is the base of the Hopf circle on the sphere of radius sqrt(2/5), a round sphere of radius sqrt(2/5)/2 with Ricci constant 4/(2/5) = 10; so each has Ricci constant 10. A seven-dimensional Sasaki-Einstein metric needs transverse Ricci constant 8. The Sasaki metric obtained by the standard transverse homothety of factor 5/4 has the same underlying circle bundle and the required Einstein normalization. Thus the existence of the Sasaki-Einstein metric on that manifold is compatible with the quotient, and the initial round-sphere quotient metric is identified with it only after the rescaling.

Independent proof of Proposition 2.6, with the colour convention. *From the referee report on the 11D line, section 'Proposition 2.6: flux assignments for one generation', transferred verbatim in substance on 4 Nov 2026.*

Verdict of the referee: correct with amendments, conditional on global flux quantization. The bidegrees and the two linear solutions are correct with the cohomology convention of the file. With the colour action on the quotient fixed as in Theorem 2.1, that convention is conjugated and the first-component coefficients must be reversed.

The pairings. The charges of the left-handed fields are F = 1 on all; chi = +1 on Q, L and -1 on u^c, d^c, e^c, nu^c; and B - L = 1/3, -1/3, -1/3, -1, 1, 1 on Q, u^c, d^c, L, e^c, nu^c. For (c_0, c_1) = (-9/4, -3/4), direct substitution in P^{(1)} = c_0 chi + c_1 (B - L) and P^{(2)} = (3F + chi)/2 gives

```
P_Q = (-5/2, 2),   P_{u^c, d^c} = (5/2, 1),   P_L =
 ↪  (-3/2, 2),   P_{e^c, nu^c} = (3/2, 1);
```

for instance P_Q^{(1)} = -9/4 - 1/4 and P_L^{(1)} = -9/4 + 3/4. For (c_0, c_1) = (-3/2, -3), the quark pairings are unchanged (-3/2 - 1 = -5/2 and 3/2 + 1 = 5/2), and the first components for L and for e^c, nu^c are +3/2 and -3/2 respectively. Subtracting the formal shift (3/2, 1) yields exactly the listed integral bidegrees. On CP^2, the nonzero cohomology of a line bundle O(k) is H^0(O(k)) for k >= 0, or H^2(O(k)) ≅ H^0(O(-k - 3))^* for k <= -3; CP^1 has the analogous H^0/H^1 formula. So O(-4, 1) gives (3, 2), O(1, 0) gives 3bar, O(-3, 1) and O(0, 1) give (1, 2), and O(0, 0) and O(-3, 0) give a singlet, in the convention of the Setting. All listed choices have even total Dolbeault degree (0 or 2), so the common-chirality convention gives the indicated chiral index.

The colour convention. If the base is P(3bar), as in the quotient of Theorem 2.1, then H^0(O(1)) = 3 and H^2(O(-4)) = 3bar, and the listed quark bundles carry the conjugates of the labelled colour representations. Replacing (c_0, c_1) by (9/4, 3/4) or (3/2, 3), with the second component unchanged, reverses the first pairings and exchanges O(1) with O(-4) for the quarks; this gives the labelled Q and u^c, d^c with the colour action of the quotient. Alternatively, the original table is valid for P(3), with the corresponding conjugate colour convention.

Without chi. The required weak-doublet pairings have absolute value 2. Write P^{(2)} = alpha F + beta (B - L), so P_Q^{(2)} = alpha + beta/3 and P_L^{(2)} = alpha - beta. If the two signs for Q and L agree, then beta = 0 and |alpha| = 2, and u^c gets |P^{(2)}| = 2, the wrong absolute value for a weak singlet. If they disagree, say alpha + beta/3 = 2 and alpha - beta = -2, then beta = 3, alpha = 1, and u^c gets P^{(2)} = 1 - 1 = 0.

Without B - L. Q and L have identical charges F = 1, chi

$= +1$, so their pairings coincide, while they require colour pairings of absolute values $5/2$ and $3/2$ respectively.

These obstructions use only absolute values, so they also allow independent source chiralities.

Global data. The final line-bundle degrees are integral, but K^{1/2} itself is formal and the displayed coefficients include fractional flux components. An actual background requires a specified global gauge group, charge lattice, spin^c determinant and compatible flux quantization. The index calculation alone establishes neither those data nor the physical interpretation of the grading as a gauge connection.

Part 40

Dualities between nine and eleven dimensions, adapted to the spectral triple

Lemma 40. Written as file 07 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, task C). Attempted by Claude (Anthropic), 4 Nov 2026. Propositions 7.1 to 7.3 are proved here by hand; the string-theory statements of section 1 are recalled without rereading; the readings are proposals. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

1. The dualities (recalled)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The duality descriptions are conditional on the recalled sources. The M2 wrapping number and the auxiliary

torus flux are separately specified; a map identifying them remains open. See the referee report.

1. **M-theory on T^2 and type IIB on S^1** (Schwarz, hep-th/9508143; Aspinwall, hep-th/9508154). The modulus tau of the torus is the IIB coupling and SL(2, Z) is IIB S-duality; the IIB radius scales as a negative power of the area A of the torus. M2-branes wrapped on T^2 are IIB momentum states, and momenta on T^2 are (p, q)-strings wound on S^1. Eleven dimensions at A -> infinity, ten (IIB) at A -> 0, nine in between.
2. **Sixteen supercharges, rank 17** (Polchinski and Witten, hep-th/9510169; Hořava and Witten, hep-th/9510209). The two heterotic strings on S^1 are T-dual through Wilson lines; their type I' form is an interval with two orientifold planes O8 and sixteen D8-branes whose positions are the Wilson-line eigenvalues; at strong coupling, M-theory on an interval times a circle, with E8 walls.
3. **Sixteen supercharges, lower rank.** The CHL string (rank 9) and M-theory on the Möbius strip; a rank-1 component related to the Klein bottle (Aharony, Komargodski and Patir, hep-th/0702195); further components with discrete theta angles (Montero and Parra de Freitas, arXiv:2209.03361).
4. **Light-cone M-theory.** On a light-like circle with momentum N, M-theory is the U(N) quantum mechanics of nine Hermitian matrices (Banks, Fischler, Shenker and Susskind, hep-th/9610043; Susskind). On T^2 with a light-like component of C_3, the torus becomes noncommutative and SL(2, Z) acts by Morita equivalence (Connes, Douglas and Schwarz, hep-th/9711162).

2. What each one offers the finite geometry

duality	9	11	the finite triple
1	the nine-dimensional theory	large torus	the lowest Landau level of a torus with flux p: algebra M_p(C), J from charge conjugation with doubling (Proposition 7.3); towers = Landau levels; p = wrapped M2-branes = IIB momentum
2	type I' interval, or heterotic on S^1	Hořava-Witten on S^1	two sheets = two orientifold walls; D_F = Wilson lines = D8 positions (Propositions 6.1, 7.1, 7.2); A_F from Chan-Paton algebras; J from the orientifold
3	rank-reduced nine-dimensional theories	M-theory on non-orientable surfaces	J as an orientation reversal; the reflection with antiperiodicity of Proposition 1.1 is the Möbius-type ingredient
4	nine transverse matrices, Spin(9)	large N	A_F as the commutant of block-diagonal matrices: stacks of 1, 2, 3 give C + M_2(C) + M_3(C); H_F in off-diagonal blocks; D_F = Γ^i ad(X^i), masses = separations

Duality 2 is the one with an exact meeting point with the corpus, through the angle of the top channel.

3. Proposition 7.1 (the angle of a channel lives on an interval)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The spectral quotient and endpoint multiplicities are independently checked. Theorem 22.1 uses the strict interior; the antiperiodic endpoint is a limit, with a degenerate window. Equality of one channel spectrum does not by itself identify full physical backgrounds. See the referee report.

Statement. In the parent of lemma 22, a channel with singular value σ of the Yukawa block and angle θ = l σ has the signed spectrum {(n pi + θ)/l, (n pi − θ)/l : n in Z} (Theorem 22.1). As a set, this depends only on the class of θ in R modulo the group generated by θ -> −θ and θ -> θ + pi, whose fundamental domain is [0, pi/2]. At the endpoints:

1. θ = 0: the spectrum is n pi/l with a zero mode of multiplicity two (periodic);
2. θ = pi/2: the spectrum is (n + 1/2) pi/l, each value twice (antiperiodic), two copies of the spectrum of the chiral interval of Proposition 1.1; it is the limit X -> infinity of the wall data X = tan(θ), where the lowest eigenspace is degenerate, and in lemma 23 the point of antipodal gradings, where M_W = M_Z = 0 at fixed l with the fermion at pi/(2l).

Proof. The set is invariant under θ -> −θ (it exchanges the two progressions) and θ -> θ + pi (it shifts n). The orbit of θ meets [0, pi/2] in one point. The endpoint values are substitutions; for θ -> pi/2 the wall conditions Q_R(l) = X U_R(l), U_L(l) = −X^* Q_L(l) tend to U_R(l) = Q_L(l) = 0. The statements at θ = pi/2 are those of lemma 23 (Theorem 23.6 and the reading after Proposition 23.7). ■

So the parameter space of one channel of the finite Dirac operator, for a given parent, is an interval with two distinguished ends, and lemma 22's restriction l ||D_F|| < pi/2 is the choice of the fundamental domain.

4. Proposition 7.2 (the far end is the displaced D8-brane on the other wall)

Referee note (GPT-6.1 Terra, 4 Nov 2026). At $a = 1/2$ the full $SO(32)$ centralizer is $S(O(30) \times O(2))$,

whose identity component is $SO(30)\times SO(2)$. The stated one-loop minimum uses the hypotheses of lemma 24 and $n_f > 0$; the D8 dictionary remains a proposal. (Second check.) In Spin(32)/Z_2 the component of determinant -1 sends the lift of W_{1/2} to its negative, so the heterotic centralizer is connected; a -> a + 1 holds for SO(32) matrices, and the lift has period 2. See the referee report.

Statement. Put a = θ/pi in [0, 1/2]. By Proposition 6.1 the channel spectrum is that of the states (30)_{±1} of SO(32) on a circle of radius R = l/pi with the Wilson line W_a = 1_30 ⊕ R(2 pi a). The interval [0, 1/2] is the moduli interval of one Wilson-line eigenvalue modulo a -> −a and a -> a + 1; the unbroken group is SO(32) at a = 0 and SO(30) x SO(2) for 0 < a <= 1/2; at a = 1/2 the holonomy is −1 on the plane of SO(2), W_{1/2} = diag(1_30, −1_2), and the (30)_{±1} tower is (n + 1/2)/R, the antiperiodic tower of Proposition 1.1. The one-loop minimum of lemma 24, Theorem 9, θ = pi/2, is the point a = 1/2.

Proof. Proposition 6.1 and Proposition 7.1 with θ = pi a; the commutant of W_a in SO(32) for 0 < a < 1/2 is SO(30) x SO(2), and at a = 1/2 its identity component is still SO(30) x SO(2), since only two eigenvalues are −1. Lemma 24, Theorem 9, item 1, places the minimum at t = sin^2 θ = 1. ■

Reading (proposal). In type I' the Wilson line W_a is fifteen D8-branes, with images, on one orientifold plane and the sixteenth at distance proportional to a from it (proportional only at constant dilaton; see 10, Proposition 10.1); a = 1/2 puts it on the other orientifold plane (recalled: hep-th/9711098, hep-th/0108135). The dictionary of file 06 then reads: the top angle is the position of the top's D8-brane on the type I' interval, the two ends of the fundamental domain are the two orientifold walls, and the one-loop vacuum of the flat family of lemmas 23 and 24 carries the top brane to the far wall, where the tower of the stack-brane strings is antiperiodic. In supersymmetric type I' the D8 positions are flat directions; the potential of lemma 24 belongs to the non-supersymmetric flat family, the setting in which a force on the brane is expected.

Under the strong-coupling duality of item 2 of section 1,

the D8 position is a Wilson-line eigenvalue of the heterotic string on S^1, and in eleven dimensions an E8 holonomy around the circle on a Hořava-Witten wall. So the same datum, the top angle, is a brane position at 9 and a holonomy at 11.

5. Proposition 7.3 (a torus with flux, doubled, has the colour form)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The signs, kernel dimensions and generated matrix corner check out, with a spin-line twist for nontrivial spin structures. The displayed action is degenerate and nonunital on the doubled Hilbert space, so the claimed unital finite triple needs an extra scalar summand. See the referee report.

Statement. Let T^2 be a flat torus of area A with a spin structure, L a line bundle of degree p >= 1 with a connection of constant curvature, and Lbar = L^{-1} with the conjugate connection. On H = L^2(S ⊗ L) ⊕ L^2(S ⊗ Lbar) put D = D_L ⊕ D_Lbar, gamma = gamma_S ⊕ gamma_S, and J(ψ, χ) = (Cχ, −Cψ), where C is the charge conjugation of the spin structure (antiunitary, C^2 = −1, C D = D C, C gamma_S = −gamma_S C, mapping S ⊗ Lbar to S ⊗ L). Let C^infinity(T^2) act by multiplication on the first summand only. Then:

1. J^2 = 1, J D = D J and J gamma = −gamma J, the signs of KO-dimension six, and the algebra satisfies order zero;
2. ker D = H^0(L) ⊕ H^1(Lbar) ≅ C^p ⊕ conj(C^p), with gamma = +1 on the first summand and −1 on the second; the other eigenvalues are the Landau levels, of order (p/A)^{1/2}, which go to infinity as A -> 0 at fixed p;
3. the compressions of the functions to the kernel generate M_p(C) on C^p ⊕ 0, so the window is the finite triple (M_p(C), C^p ⊕ conj(C^p), 0, J, gamma), with KO-6 signs and order zero: M_p(C) acts on the first summand and, through J, by its opposite on the second.

Proof. (1) J^2(ψ, χ) = (C(−Cψ), −C(Cχ)) = (ψ, χ) since C^2 = −1; JD = DJ and J gamma = −gamma J follow from the properties of C; order zero holds because the algebra acts on the first summand and J b

J^{-1} on the second. (2) On a torus the spinor bundle is Λ^{0,*} up to a flat factor, so ker D_L = H^0(L) ⊕ H^1(L) = C^p ⊕ 0 by Riemann-Roch in genus one, and ker D_Lbar = 0 ⊕ H^1(Lbar) ≅ conj(C^p) by Serre duality; the Landau levels of a constant field of total flux p on area A scale as (p/A)^{1/2}. (3) The compression of a character e_{m,n} to H^0(L) is a nonzero multiple of a clock-and-shift Weyl operator U^m V^n with UV = e^{2 pi i/p} VU (Berezin-Toeplitz quantization of the torus; recalled; in the theta-function basis the compression is a Gaussian factor times the Weyl word, checked for p = 2 to 5 in the second check), and these operators span M_p(C). ■

The construction is the torus analogue of lemma 16 (a doubled parent with restricted algebra and a J covering the flip; Example 16.9 for the twistor case), and its window at p = 3 has the form of the colour sector of the Standard Model triple: M_3(C) on the quarks and its opposite on the antiquarks. Berezin-Toeplitz quantization of the torus is known; the KO-6 doubling is the corpus device.

Reading for duality 1 (proposal). The area A -> 0 at fixed flux keeps the window and sends the tower to infinity, the towerless limit; A -> infinity brings the Landau levels down. In M-theory on T^2, flux and wrapping number of M2-branes on T^2 are IIB momentum states, so the dimension p of the window counts IIB momentum on the nine-dimensional circle, and the SL(2, Z) of the torus acts on the family by Morita equivalence of the noncommutative tori of Connes, Douglas and Schwarz when a C_3 background is on. Whether p = 3 for colour has a reason in this picture is open.

6. A remark on duality 4 (known)

For U(N) matrices X^i (i = 1, ..., 9) that are block-diagonal, X^i = ⊕*a* *x_a^i* *1*{n_a} with distinct points x_a in R^9, the commutant of all X^i is ⊕*a* *M*{n_a}(C), and the fermion operator Γ^i ad(X^i) on the adjoint, with the sixteen-component spinor of Spin(9), vanishes on the diagonal blocks and has eigenvalues ±|x_a − x_b|, each with multiplicity 8 n_a n_b per ordered block (a, b), that is 16 n_a n_b per unordered pair. With stacks of sizes 1, 2, 3 the commutant is C + M_2(C)

+ M_3(C), the complex form of A_F, the off-diagonal blocks are bifundamentals as in the Krajewski diagrams of the finite geometry, and the masses are the distances between stacks; coincident stacks restore U(N), the analogue of massless W and Z, and large N is the eleven-dimensional limit. A quaternionic factor H needs an orientifold projection on the two-stack, and chirality needs more than the nine Hermitian matrices.

6b. The three labels in type I' variables (proposal)

Correction (4 Nov 2026). The length l of H and the type I' length are T-dual, and the limit at label 11 below ends at ten-dimensional SO(32). See 10, Proposition 10.3.

Hypothesis H has two parameters for the top channel: the interval length l and the angle θ = l m_t in [0, pi/2] (Proposition 7.1). In the type I' reading of Proposition 7.2 they are the length of the interval and the position of the top's D8-brane as a fraction of it. Then:

- **label 9:** θ = pi/2 at fixed l, the brane on the far orientifold wall: the top at the tower scale pi/(2l) (lemma 22, Corollary 22.9: W, Z and top at the tower scale in the 10 -> 9 family), the antiperiodic tower, and the one-loop minimum of the flat family (lemma 24);
- **label 10:** 0 < θ < pi/2, the brane between the walls, a window below the tower;
- **label 11:** θ -> 0 together with l -> infinity, the joint family of 01 (θ_epsilon ~ epsilon^{1 − alpha}): the brane returns to the stack, SO(32) is restored on that channel, and the interval opens.

In this reading the 9 and 11 endpoints are the two ends of the brane's path, the far wall and the stack, and the second scale required by Proposition 1.5 must fix how l grows as the brane returns. The flat one-loop potential pushes the brane to the far wall (label 9); reaching label 11 needs a term that pulls it back to the stack while the interval grows.

7. What this adds, and what remains open

Referee note (GPT-6.1 Terra, 4 Nov 2026). The brane and E_8-holonomy readings below are proposals.

Proposition 7.3's displayed algebra action is nonunital; the corrected finite datum is in the referee report. Proposition 1.5 also permits flat directions.

Added: the angle of the top channel lives on an interval whose ends are the two orientifold walls of type I' (7.1, 7.2); the corpus's one-loop vacuum puts the top brane on the far wall, the SO(32) Wilson line diag(1_30, −1_2); the same datum is an E8 holonomy in the eleven-dimensional frame; a torus with flux gives the colour form of the finite triple with KO-6 signs (7.3).

Open: - A nine-dimensional spectral triple with sixteen D8 positions as the eigenvalues of D_F on the type I' interval, and its strong-coupling limit with E8 walls; the configuration 15 + 1 lies at the SO(32) end, while E8 needs seven D8-branes at an orientifold plane with divergent coupling. - The quaternionic factor H from a torus with even flux and a real structure of quaternionic type, and the full A_F from several tori or from a rank-three bundle with flux. - The lower-rank nine-dimensional components (duality 3), whose J-like orientation reversal may supply the real structure of the finite triple.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 7.1. *From the referee report on the 11D line, section 'Proposition 7.1: channel-angle fundamental domain', transferred verbatim in substance on 4 Nov 2026.*

The stated fundamental domain classifies the signed channel spectrum, including its endpoint multiplicities. The spectrum modulo pi/l consists of the two residues ±θ/l. Equality of two such multisets is equivalent to θ' ≡ ±θ (mod pi): if θ' = ±θ + k pi the progressions are relabelled, and conversely the residue sets {±θ'} and {±θ} modulo pi must agree. This gives the unique representative in [0, pi/2]. In the interior the two progressions are disjoint, since n pi + θ = m pi - θ would need 2θ ∈ pi Z. At zero they coincide on pi Z/l; at pi/2 they coincide

on (Z + 1/2) pi/l. Thus every endpoint eigenvalue has multiplicity two, including the zero at the first endpoint.

The channel has two copies of the scalar antiperiodic circle spectrum at the second endpoint. Theorem 22.1 assumes the strict window l ||D_F|| < pi/2; θ = pi/2 is the corresponding boundary-condition limit, and its lowest level is degenerate. The massless electroweak vectors at that endpoint come from the specified two-grading model of lemma 23 and do not follow from the spectrum of an isolated channel. The endpoints require separate domains or limits, and the quotient of one spectrum need not identify complete physical backgrounds.

Independent proof of Proposition 7.2, with the full centralizer at a = 1/2. *From the referee report on the 11D line, section 'Proposition 7.2: Wilson-line endpoint and one-loop minimum', transferred verbatim in substance on 4 Nov 2026.*

For 0 < a < 1/2, the nonreal conjugate eigenvalues e^{±2 pi i a} of the rotation force a commuting orthogonal transformation to preserve the two-plane of R(2 pi a) and its complement, and to act on the plane by an element of O(2) commuting with a rotation by an angle other than 0 and pi, that is, by a rotation. Its centralizer in SO(32) is therefore SO(30) x SO(2). At a = 1/2 the holonomy is diag(1_30, -1_2); a commuting orthogonal transformation preserves both eigenspaces and is otherwise arbitrary on them, so the full centralizer is

```
S(O(30) x O(2)) = {(A, B) ∈ O(30) x O(2) : det A
↪  det B = 1},
```

with identity component SO(30) x SO(2) and a second component of pairs with det A = det B = -1. At a = 0 the centralizer is SO(32). Conjugation by a reflection in each subspace, an element of SO(32), sends a to -a, and a -> a + 1 leaves this SO(32) holonomy unchanged.

Minimum. For the determinant of lemma 24, write t = sin^2 θ. Its two gauge terms are nonnegative and vanish at t = 1; the fermion term -n_f F(t) is strictly decreasing for n_f > 0. Thus the unique global minimum is t = 1, that is a = 1/2. This conclusion assumes the flat massless determinant, its stated field content and wall conditions, and no additional angle-dependent term. It does not derive a force on a supersymmetric type I'

brane, and the D8 interpretation is conditional on the proposed dictionary.

Independent proof of Proposition 7.3, items 1 and 2, and of the generation step in item 3. *From the referee report on the 11D line, section 'Proposition 7.3: doubled torus and the finite algebra', transferred verbatim in substance on 4 Nov 2026.*

Zero modes. Let S_0 be the flat spin line of the chosen spin structure, with S_0^2 ≅ O, so that S ⊗ L = Λ^{0,*}(L ⊗ S_0). The two zero-mode spaces are

```
H^0(L ⊗ S_0)    and    H^1(L^{-1} ⊗ S_0).
```

The first has dimension p: a negative-degree line bundle has no holomorphic sections, so by Serre duality (K trivial on the torus) H^1(L ⊗ S_0) = 0, and Riemann-Roch on a genus-one curve gives χ(L ⊗ S_0) = p. For the same reason H^0(L^{-1} ⊗ S_0) = 0, and Serre duality identifies H^1(L^{-1} ⊗ S_0) with the dual of H^0(L ⊗ S_0^{-1}) = H^0(L ⊗ S_0), hence with its complex conjugate after choosing the inner product. With the stated orientation these have opposite gradings. The untwisted formula of item 2, H^0(L) ⊕ H^1(Lbar), is literal for the trivial spin structure; the dimensions hold for all four.

J and the levels. The assumed charge conjugation C intertwines the two coupled Dirac operators and reverses the grading. Consequently the displayed J satisfies J^2 = 1, JD = DJ and J gamma = -gamma J. In a constant field B = 2 pi p/A, the nonzero Dirac levels have magnitudes sqrt(2|B| n), n >= 1, in the standard Dirac normalization.

Compression. For a reciprocal lattice vector k, split the position operator as x = R + , with guiding-centre part R and oscillator part , which commute with each other and satisfy [_1, _2] = -[R_1, R_2], of modulus 1/|B|. Then e^{ik · x} = e^{ik · R} e^{ik · }, and the oscillator-vacuum matrix element of e^{ik · } is e^{-|k|^2/(4|B|)}. So projecting multiplication by e^{ik · x} onto the lowest Landau level gives

```
P e^{ik·x} P = e^{-|k|^2/(4|B|)} e^{ik·R} P,
```

the nonzero Gaussian factor times a guiding-centre translation. For primitive reciprocal vectors k_1, k_2, with |k_1 x k_2| = (2 pi)^2/A and 1/|B| = A/(2 pi p), the two

translations U = e^{ik_1 · R} P and V = e^{ik_2 · R} P are unitary on the lowest Landau level and form a clock-and-shift pair, UV = e^{±2 pi i/p} VU. Their p^2 products U^m V^n, 0 <= m, n < p, span M_p(C): the commutation relation forces tr(U^m V^n) = 0 unless m ≡ n ≡ 0 (mod p), so the p^2 products are orthogonal for the trace inner product. Flat spin and connection holonomies change only phases, and leave the nonzero Gaussian factor. Compression generates this algebra, and the compression map itself is not multiplicative on the function algebra.

Counterexample (referee) to the unital finite triple of Proposition 7.3, item 3, and the unital repair. *Same section of the referee report, transferred verbatim in substance on 4 Nov 2026.*

On V ⊕ Vbar, with V = H^0(L ⊗ S_0) ≅ C^p, the generated representation is (a) = a ⊕ 0. Thus (1) = 1_V ⊕ 0, and both the parent and the finite representations are degenerate on the second summand. They do not have the unital representation required in the finite-data setting of lemma 22. In particular, for p = 3, no complex-linear unital M_3(C) representation on C^3 ⊕ conj(C^3) can satisfy order zero with a unital opposite action: the two commuting unital actions would make this six-dimensional space a module for M_3(C) ⊗ M_3(C)^op ≅ M_9(C), whose nonzero modules have dimensions divisible by nine.

One repair is to unitize the represented corner:

```
A = M_p(C) ⊕ C,     (a, ) = a ⊕  I_{Vbar}.
```

In conjugate bases J (b,)^* J^{-1} = I_V ⊕ b^T, which commutes with (a,). This yields a unital finite KO-6 datum with D = 0 and order zero. It has an additional scalar summand; deriving the full Standard Model algebra and its representation remains open.

Second-check derivation (Claude Sonnet 5.5, 4 November 2026)

Spin lift of W_{1/2} and connectedness of the heterotic centralizer (Proposition 7.2). *From the second check on the 11D line (historical/reviews/2026-11-04_sonnet_second_check_11d.md, batch 4, item B2),*

adopted in the note at Proposition 7.2 and written out on 4 Nov 2026.

Let G_1, …, G_32 generate the Clifford algebra of R^32, with G_i G_j + G_j G_i = 2 _ij, and let pi: Spin(32) -> SO(32) be the covering map, under which g~ acts on vectors by v -> g~ v g~^{-1}. The rotation by angle 2 pi a in the plane of the coordinates 31 and 32 lifts to

```
W~_a = exp(pi a G_31 G_32) = cos(pi a) + sin(pi a)
 ↪  G_31 G_32,
```

using (G_31 G_32)^2 = -1. At a = 1/2 the lift of W_{1/2} = diag(1_30, -1_2) is W~ = G_31 G_32, up to the sign of the covering. The product of the reflections in e_1 and in e_31 lies in the component of S(O(30) x O(2)) with det A = det B = -1, and lifts to g~ = G_1 G_31. Since G_1 anticommutes with G_31 and with G_32, it commutes with their product, and

```
g~ W~ g~^{-1} = G_1 G_31 (G_31 G_32) G_31 G_1 = G_1
 ↪  G_32 G_31 G_1 = G_32 G_31 = -W~.
```

Every other element of that component is the product of this one with an element of SO(30) x SO(2), whose lifts are products of exp(φ G_i G_j/2) with i, j <= 30 or {i, j} = {31, 32}, all commuting with G_31 G_32. So every lift of the determinant -1 component conjugates W~ to -W~. The computation involves only G_1, G_31 and G_32 and holds in every even dimension.

Let = G_1 G_2 … G_32 be the volume element. It is central in Spin(32), ^2 = 1, != ±1, and pi( ) = -1. The heterotic group is the quotient of Spin(32) by {1, } (or by {1, - } for the other chirality convention; the argument is the same). The class of g~ centralizes the class of W~ exactly when g~ W~ g~^{-1} ∈ {W~, W~}. The case W~ would give pi(g~) W pi(g~)^{-1} = -W, which is impossible, since W and -W have the eigenvalue -1 with multiplicities 2 and 30. So pi(g~) lies in S(O(30) x O(2)). On the determinant -1 component g~ W~ g~^{-1} = -W~, and -1 differs from 1 and from . Hence the centralizer of the class of W~ in Spin(32)/Z_2 is the image of the preimage of SO(30) x SO(2) in Spin(32). That preimage is connected, since it contains the path exp(φ G_1 G_2/2), φ ∈ [0, 2 pi], from 1 to -1. So the heterotic centralizer is connected. Finally W~_{a+1} = -W~_a: the SO(32) holonomy has period 1 in a, and its

lift, in Spin(32) and in Spin(32)/Z_2, has period 2.

Part 35

The sign of lemma 06 as an orientifold type, and E8 walls from the configuration 15 + 1

Lemma 35. Written as file 08 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, task C, next step of 07). Attempted by Claude (Anthropic), 4 Nov 2026. Proposition 8.1 is proved (it adds a commutant statement to lemma 06); Proposition 8.2 is proved from Proposition 7.1; Remark 8.3 rests on recalled string-theory rules and on the dictionary of 06; readings are proposals. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

1. Setting

In type I' the two ends of the interval are orientifold planes. A stack of n Chan-Paton labels (k D8-branes with their images, n = 2k) on an orientifold plane of type O^- carries SO(n), and on one of type O^+ carries USp(n) (recalled; the physical O8 planes of the SO(32) web are O8^-). Lemma 06 factors a KO-6 finite triple into an odd one through a grading-reversing unitary involution U, commuting with the algebra and with D, with U J = s J U; s = −1 gives KO-5 (label 9) and s = +1 gives KO-7 (label 11), and the antiunitary j induced on H_+ satisfies j^2 = s and commutes with d^2 = D^2|{H+} (lemma 06, proof of Result 2 and the check of the electrodynamics data). STRATEGY T2c: with a realistic seesaw sector only s = +1 occurs.

2. Proposition 8.1 (the factorization sign fixes orthogonal or symplectic symmetry)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The antiunitary commutants are independently checked. They need not preserve d or the remaining data, and the Chan-Paton identification is a proposal. See the referee report.

Statement. Let E be an eigenspace of d^2 in H_+, of dimension n. The unitaries of E that commute with j form O(n) if s = +1, and USp(n) if s = −1, in which case n is even.

Proof. j preserves E because it commutes with d^2. An antiunitary with j^2 = +1 is a real structure: E = E_R ⊗ C with E_R its fixed points, and a unitary commutes with j exactly when it preserves E_R, which gives O(n). An antiunitary with j^2 = −1 is a quaternionic structure: E is a right H-module of quaternionic dimension n/2, and the unitaries commuting with j form USp(n). ■

The even dimension for s = −1 is lemma 06's own statement (Kramers degeneracy of the singular values). The proposition adds the symmetry group.

Reading (proposal). The groups are those of Chan-Paton labels on an orientifold plane: s = +1 with O^-(SO), s = −1 with O^+ (USp). So the two odd factorizations of the modern triple, which carry the labels 11 and 9, have the form of the two kinds of orientifold planes. The seesaw sector selects s = +1 (T2c), which is the O^-type of the SO(32) web used in 06 and 07: the physical neutrino sector and the type I' walls of the dictionary agree in type. Two refinements from the second check: the symmetries of the pair (j, d) are smaller, O(n_+) x O(n_-) for s = +1 and U(n/2) for s = -1, since for s = -1 the antiunitary j swaps the +-mu eigenspaces of d; and there is a tension of labels, since here label 9 goes with O^+ while section 6b of 07 puts label 9 on a far O8^-wall (the two are the KO sign and the position of the top's brane, see 10).

3. Proposition 8.2 (wall channels come with doubled multiplicity)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The stated count is for one signed eigenvalue. The lowest $|D|$ multiplicity is $2k$ in the interior and at zero, and $4k$ at $\pi/2$; interpreting the signed count as Chan-Paton labels requires further data. See the referee report.

Statement. In the parent of lemma 22, k channels at a common angle θ contribute, at the eigenvalue θ/l of the window, an eigenspace of dimension k if $0 < \theta < \mathrm{pi}/2$ and of dimension 2k at the ends $\theta = 0$ and $\theta = \mathrm{pi}/2$.

Proof. By Proposition 7.1 each channel has the spectrum {(n pi ± θ)/l}; the value θ/l occurs once (n = 0, sign +) for $0 < \theta < \mathrm{pi}/2$, and twice at the ends: at $\theta = 0$ as the zero mode of multiplicity two, at $\theta = \mathrm{pi}/2$ as $(0 + \mathrm{pi}/2)/l = (\mathrm{pi} - \mathrm{pi}/2)/l$. ■

Reading (proposal). k D8-branes in the bulk carry k Chan-Paton labels and k D8-branes on an orientifold plane carry 2k, with their images; the parent reproduces this count, with the ends of the angle interval as the walls. With Proposition 8.1, a stack of channels at a wall has a symmetry of type SO(2k) or USp(2k), as a stack of D8-branes on an O8^- or O8^+ plane. The rank-reduced nine-dimensional theory with one O8^- and one O8^+ (duality 3 of file 07) would then be a parent whose two walls carry opposite signs s: one sheet of type 11 and the other of type 9.

4. Remark 8.3 (E8 walls from 15 + 1; conditional)

Correction (4 Nov 2026). The far-wall group is reached at the string-scale radius R = R_*, with the tower fixed, and only its charged states are D0-branes. See 10.

Referee note (GPT-6.1 Terra, 4 Nov 2026). The brane and slope arithmetic is independently checked conditional on the recalled rule. Wall enhancement requires tuning $z = 0$; for $7 + 7$ the two bulk positions must sum to L. The E_2 identification with the physical electroweak sector needs further data. The cited primary sources were unavailable to the referee; the slope rule and the E_{n+1} table were confirmed against Polchinski-Witten and Bergman-Gaberdiel-Lifschytz in the second

check. See the referee report.

Conditions: the dictionary of 06 (fifteen D8-branes for the five light quarks in three colours, the sixteenth for the top) and the recalled rule that an O8^- plane with n <= 7 D8-branes on it, at infinite coupling, carries E_{n+1} (E_4 = SU(5), E_5 = SO(10), E_6, E_7, E_8), while n >= 8 admits no such point (Seiberg, hep-th/9608111; Bergman, Gaberdiel and Lifschytz, hep-th/9711098; recalled).

1. Dilaton bookkeeping: the dilaton is e^{phi} = z^{−5/6} with z piecewise linear along the interval (conformal coordinate; Polchinski and Witten, hep-th/9510169) and slope proportional to 8 − n_left, where n_left counts the D8-branes to the left (O8^-carrying −8 units), so the coupling grows toward the wall with fewer branes; this reproduces the constant dilaton of 8 + 8, the SU(2) at the empty wall of the SO(32) point (the self-dual radius of the heterotic string) and 7 + 7 for E8 x E8 (extrapolated from the single-wall rule; neither Polchinski-Witten nor Bergman-Gaberdiel-Lifschytz states it). In the configuration 15 + 1 the wall with fifteen branes has no E point (15 > 7), and the coupling can diverge only at the far wall. With the top's brane on the far wall (the label-9 configuration of Proposition 7.2, n = 1) the far wall enhances to E_2 = SU(2) x U(1), the electroweak group; with the top's brane in the bulk it is empty and enhances to E_1 = SU(2). The light-quark wall keeps SO(30).
2. If branes move in colour triples (a light flavour is three D8-branes), a wall holds 3j branes, or 3j + 1 with the top's brane. The reachable wall groups are E_1, E_2 (the top's brane alone), E_4 = SU(5) (one flavour), E_5 = SO(10) (one flavour and the top's brane), E_7 (two flavours) and E_8 (two flavours and the top's brane); E_6 (five branes) is not reachable.
3. Hence an E8 wall needs the top's brane together with two light flavours, and two E8 walls are impossible with one top brane: E8 x E8 requires breaking a colour triple.

In this reading the top is what completes E8 at a wall (7 = 2 x 3 + 1), and with colour triples the proposed E8 x

E7 arrangement has seven branes at one wall, six at the other and a bulk triple at x_b, while two E8 walls require relaxing the colour-triple rule (referee replacement): with that arrangement, the slopes are proportional to +1 on (0, x_b) and −2 on (x_b, L), so z vanishes at both walls when x_b = 2L/3. The E_n series also contains E_3 = SU(3) x SU(2) (two branes at a wall), which colour triples do not reach.

Reading (proposal). At strong coupling the wall that holds the top's brane in the label-9 configuration carries SU(2) x U(1), and the wall of the five light quarks in three colours carries SO(30). The electroweak group then appears at the top's wall, where the M-theory circle opens locally, and the quarks' flavour group at the other wall.

5. What remains open

- The bulk symmetry: k D8-branes in the bulk carry U(k), while Proposition 8.1 gives O or USp on every eigenspace of d^2; the relation between the window's degenerate levels in the bulk and U(k) needs the parent's own structures in addition to j.
- The dilaton: a spectral-triple counterpart of the type I' dilaton profile (a coupling varying along the interval, with slope set by the brane charge to one side) and the infinite-coupling points of Remark 8.3. Proposition 9.1 of 09 writes the profile as a trace of a function of |D_F|, only implicitly after 10, Proposition 10.1, and the gradient is not the second scale in the scope of Propositions 10.5 and 10.6 there.
- The physical status of E8 x E7 with colour triples, and whether the top's brane at an E8 wall has a counterpart in the finite geometry.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 8.1, with its caution. *From the referee report on the 11D line, section 'Proposition 8.1: antiunitary commutants', trans-*

ferred verbatim in substance on 4 Nov 2026.

On each finite-dimensional eigenspace E of d^2, the commutant of j in the unitary group of E is O(n) for j^2 = 1, and USp(n) for j^2 = -1, with even complex dimension n in the latter case.

For j^2 = 1, the fixed points of j form a real subspace E_R with E = E_R ⊕ i E_R, and the inner product is real on E_R, since <jv, jw> = conj<v, w>. A real orthonormal basis of E_R is an orthonormal basis of E and identifies j with complex conjugation; the unitaries commuting with complex conjugation are the real orthogonal matrices, O(n). For j^2 = -1, a unit vector v and jv are orthogonal: antiunitarity gives <jv, j^2 v> = conj<v, jv> = <jv, v>, and j^2 v = -v gives <jv, j^2 v> = -<jv, v>, so <jv, v> = 0. The span of v and jv is j-invariant, and so is its orthogonal complement; induction gives a quaternionic orthonormal basis v_1, jv_1, ..., v_m, jv_m, with n = 2m. The unitaries commuting with j are then the compact symplectic group USp(n).

Caution. This commutant is defined using j and the d^2-eigenspace alone. It need not commute with d, with the represented algebra, or with other physical structures. For example, on C^2 let j be complex conjugation and d = diag(1, -1): the commutant on the d^2 = 1 space is O(2), and a generic rotation does not commute with d. Moreover O(n) contains disconnected transformations, while SO(n) is its identity component for n >= 2. The orientifold comparison therefore requires an explicit Chan-Paton representation and compatibility with the rest of the data.

Independent proof of Proposition 8.2, with the multiplicities of |D|. *From the referee report on the 11D line, section 'Proposition 8.2: endpoint channel multiplicities', transferred verbatim in substance on 4 Nov 2026.*

For k independent channels at the same angle, the signed eigenvalue theta/l has multiplicity k in the interior and 2k at either endpoint. For 0 < theta < pi/2, only n = 0 in the positive progression gives +theta/l: (n pi + theta)/l = theta/l forces n = 0, and (n pi - theta)/l = theta/l forces theta = n pi/2, outside the interior. At zero both progressions have n = 0. At pi/2 the positive

n = 0 and negative n = 1 branches coincide. This proves the claimed signed count.

The multiplicity of the corresponding lowest eigenvalue of |D| is 2k in the interior (the values ±theta/l, k each), 2k at zero, and 4k at pi/2 (the values ±pi/(2l), 2k each). At zero the two interior signed eigenvalues simply coalesce; at pi/2 the next level also meets the window. Thus the signed count alone does not establish a doubling of independent physical Chan-Paton labels or an orientifold projection.

Independent proof of the arithmetic of Remark 8.3. *From the referee report on the 11D line, section 'Remark 8.3: wall bookkeeping and enhancements', transferred verbatim in substance on 4 Nov 2026.*

The slope and brane-count arithmetic below is conditional on the recalled type I' rules; the referee could not access the primary sources, and the second check later confirmed the slope rule and the E_{n+1} table against Polchinski-Witten and Bergman-Gaberdiel-Lifschytz (note at the remark). An enhancement requires tuning the coupling to a zero of z, and identifying the resulting E_2 factor with the physical electroweak sector requires additional data.

Assume z' = c (8 - N), with c > 0, N(x) the number of D8-branes to the left of x, and z > 0 in the open interval.

- *8 + 8.* N = 8 in the interior, so the interior slope is zero.
- *16 + 0.* z' = -8c, and the empty far wall reaches z = 0 only when z(0) = 8cL.
- *7 + 7.* With the other two branes at x_1 < x_2 in (0, L), the slopes are c, 0 and -c on (0, x_1), (x_1, x_2) and (x_2, L). Both walls have z = 0 precisely when

  ```
  z(0) = 0,    x_1 + x_2 = L,
  ```

  since z(L) = z(0) + c x_1 - c (L - x_2). Then z is positive in the interior. Thus "7 + 7 for E8 x E8" includes a coupling and a position condition.
- *Isolated zeros.* With n branes at a wall where z = 0, the slope next to the wall is c (8 - n) at x = 0 and c (n - 8) at x = L, so z > 0 inside requires n <=

7: an isolated zero with a positive interior requires at most seven branes at that wall under this slope rule.

- *15 + 1.* With the single brane at L, the slope is c (8 - 15) = -7c, so the far-wall zero is at z(0) = 7cL. Assuming the recalled E_{n+1} rule gives E_2, with Lie algebra su(2) ⊕ u(1), at that tuned wall. The rule alone does not determine the Standard Model charges, couplings, global group or embedding.

Colour triples. If every light flavour moves as a triple and there is a single top brane, a wall holds 3j or 3j + 1 branes, so the allowed wall counts up to seven are 0, 1, 3, 4, 6, 7. Only a wall containing the top can have seven (7 = 2 x 3 + 1); two such walls require relaxing the triple constraint or the single-top assumption. For seven branes at 0, six at L, and the remaining triple at x_b, N = 7 on (0, x_b) and N = 10 on (x_b, L), so the slopes are +c and -2c. With z(0) = z(L) = 0, c x_b - 2c (L - x_b) = 0 gives x_b = 2L/3, and

```
z(x) = c x              for 0 <= x <= 2L/3,
z(x) = 2c (L - x)       for 2L/3 <= x <= L.
```

The resulting E8 x E7 interpretation is conditional on the recalled enhancement rule and the proposed brane dictionary.

Part 36

The type I' dilaton as a spectral function, the SO(32) of the EPJC paper, and two more dualities

Lemma 36. Written as file 09 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line of research (README, PLAN, task C). Attempted by Claude (Anthropic), 4 Nov 2026, af-

ter the steer of the same day (the coincident-brane route is overused; the type I' dilaton and the colour-flavour idea are worth pursuing; compare with the SO(32) of the EPJC paper). Proposition 9.1 is proved from the recalled type I' slope rule; Proposition 9.2 is proved from representation theory and the recalled orientifold projections; Proposition 9.3 is proved (linear algebra and Connes's sign table); the comparison of section 1 is read from the arXiv text of the paper through a summarizing fetch; the rest are proposals. Refereed by GPT-6.1 Terra and checked a second time by Claude Sonnet 5.5 (historical/reviews/2026-11-04_terra_check_11d.md, historical/reviews/2026-11-04_sonnet_second_check_11d.md).

Correction (same day): the first version stated the slope rule for e^{−phi}. The piecewise-linear function is z, with e^{phi} = z^{−5/6}; Proposition 9.1 holds as stated with z in place of e^{−phi}, and infinite coupling is z = 0.

1. The SO(32) of A. Rivero, Eur. Phys. J. C 84 (2024) 1058 (arXiv:2407.05397)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The branching algebra is independently checked. The archived original abstract also states an additional chiral charge-+4/3 quark per generation. The full paper was unavailable for direct comparison, so its detailed attribution remains pending. See the referee report.

That paper uses the adjoint of SO(32) to classify the scalars of a supersymmetric Standard Model with three generations, as composites: pairs of quarks as colour triplets and quark-antiquark pairs as singlets and octets, with three extra scalars of charge +4/3. Its pattern is SO(2n) ⊃ SO(n_1) x SU(n_2) x SU(3) x U(1) with 2n = n_1 + 6 n_2, which for SO(32) and n_2 = 5 gives SO(2) x SU(5) x SU(3) x U(1): the 30 = 6 n_2 are the five light quarks u, c, d, s, b ("turtles") in three colours with conjugates, and the top is the non-participating "elephant". Its explicit route also passes through SO(32) ⊃ SU(16) x U(1), with 496 = 1 + 120 + 120bar + 255 (quarks from the 120s, leptons from the 255), and it mentions open strings, orientifolds and an SU(9) for M2 and M5.

Reading of the LaTeX source (Claude Sonnet 5.5, 4 Nov 2026; the earlier comparison used a summarizing fetch). The paper uses colour singlets for mesons and colour antitriplets for diquarks, and keeps only (1, 24, 1), (1, 15, 3bar) and (1, 15bar, 3) of its decomposition; colour octets and sextets lie outside it. The top is the favoured non-binding quark ("we have not a concrete argument"), and the postulates force the non-binding quark to be up-type. The chain is SU(16) x U(1) to SU(15) to SU(5) x SU(3), with SO(30) and SO(32) reached by "adding singlets". The identification of the top with the SO(2) factor, the O8$\hat{}$- plane and the Chan-Paton reading belong to these files, and "orientifold" in the paper means planar orientifold QCD. The footnote on the 84 + 12 split and SU(9) for M2 and M5 anticipates Proposition 21.3 and 04. The abstract specifies a chiral +4/3 quark per generation; the body has three scalars of charge +4/3 and leaves their fermion partner open. The paper gives (15, 3bar) as the antisymmetric square of one fundamental without stating chirality, so the question of Proposition 9.3 stays open; its remark on odd-parity diquarks is consistent with that proposition. The paper's selection of an up-type quark by its postulates and lemma 21's by the spectral-action values are independent arguments with the same outcome.

Coincidence. The chain SO(32) ⊃ SO(2) x SO(30) ⊃ SO(2) x SU(15) x U(1) ⊃ SO(2) x SU(5) x SU(3) x U(1), the 30 as five light quarks in three colours, and the top as the factor SO(2), are the same as lemma 21 and the dictionary of 06. Lemma 21 records the branching 496 -> (435, 1) + (1, 1) + (30, 2) and leaves SO(32) unexplored; the EPJC paper explores it through composites, and the files 06 to 09 through type I' branes. What the 11D files add: the type I' realization of the SO(2) as the top's D8-brane and of the 30 as the Chan-Paton labels of fifteen D8-branes on an O8$\hat{}$- wall (06); the top angle as the brane's position and the one-loop vacuum on the far wall (Propositions 7.1, 7.2); the orientifold type O$\hat{}$- matched by the seesaw sign (Proposition 8.1); the far wall's enhancement to SU(2) x U(1) at infinite coupling (Remark 8.3); the dilaton (Proposition 9.1); the diquark counts as strings between the stack and its image, which need the O$\hat{}$- type (Proposition 9.2); and the real-structure condition under which these counts arise, as connection

components along the interval, outside the finite inner fluctuations (Proposition 9.3). In type I' language the paper's other chain, SU(16) x U(1), is the configuration with all sixteen D8-branes coincident in the bulk, and the chain with SO(2) x SO(30) is the configuration with fifteen on a wall and one apart; the second is the one the corpus selects. The SU(9) of M2 and M5 in the paper is the grading of e8 used in 04.

2. Proposition 9.1 (the type I' coupling function is a trace of a function of the finite Dirac operator)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The integration is independently checked conditional on the recalled slope rule. The trace is over sixteen selected channels, and the full physical D_F Hilbert space enters only through a specified projection onto them; direct comparison with the cited source was unavailable to the referee. (Second check, against Polchinski-Witten.) The formula z(L) = z(0) - c (8L - sum x_k) is exact in the positions; the dictionary x_k = (2L/pi) l m_k holds only for constant z, and in general theta_k = (pi/2) W(x_k)/W(L) with W = int z^{-1/3} (10, Proposition 10.1). In the profile 15 + 1 the endpoint values 7cL and 8cL stand, and theta_t = pi/4 gives 7.358 cL in place of 7.5 cL. See the referee report.

Setting. Type I' on an interval [0, L], in the conformal coordinate x, with sixteen D8-branes at x_1, ..., x_16 and O8^- planes at the ends. Recalled rule (Polchinski and Witten, hep-th/9510169): the dilaton is e^{phi} = z^{−5/6} for a function z that is continuous and piecewise linear in x, with slope c (8 − N(x)), where N(x) is the number of D8-branes to the left of x and c > 0; the coupling diverges where z vanishes. Dictionary (06, 07): x_k = (2L/pi) θ_k with θ_k = l m_k the angles of sixteen channels of the finite Dirac operator (fifteen light-quark channels with colour, and the top).

Statement.

1. z(x) = z(0) + c Σ_k min(x/2, x_k − x/2) for x in [0, L], so z is z(0) plus c Tr g_x(l |D_F|) over the sixteen channels, with g_x(θ) = min(x/2, (2L/pi) θ − x/2): a trace of a function of the finite Dirac operator, as the spectral action is.
2. At the far wall, z(L) = z(0) − c (8L − Σ_k x_k).

3. Infinite coupling at the far wall ($z(L) = 0$) holds exactly when $z(0) = (2cL/pi)(4\,pi - l\,\Sigma_k\ m_k)$: the coupling at the light-quark wall is fixed by the sum of the sixteen channel masses, the trace of |D_F| on these channels.

Proof. *∫0^x N(y) dy = Σ_k (x − x_k)+*, so $z(x) = z(0) + c\,(8x - \Sigma_k\,(x - x_k)+) = z(0) + c\,\Sigma_k\,(x/2 - (x - x_k)+)$, and $x/2 - (x - x_k)_+ = \min(x/2, x_k - x/2)$. Item 2 is $x = L$ with $\Sigma_k\,(L - x_k) = 16L - \Sigma\ x_k$. Item 3 sets $z(L) = 0$ and substitutes $x_k = (2L/pi)\ l\ m_k$. ■

Correction (4 Nov 2026). The dictionary $x_k = (2L/pi)\ theta_k$ is exact only for constant z; the map from angles to positions is nonlinear. See 10, Proposition 10.1.

Reading (proposal). With the light quarks near $\theta = 0$, $\Sigma_k\ m_k \approx m_t$ and the condition is $z(0) \approx (2cL/pi)(4\ pi - \theta_t)$: 7cL at label 9 ($\theta_t = pi/2$) and 8cL at label 11 ($\theta_t \to 0$). The coupling therefore varies along the interval on a scale set by the spectrum of D_F, which is the kind of second scale that Proposition 1.5 requires to select a joint family; whether the type I' coupling can play that role in the corpus's flat family is the next question. Since the slope decreases with x, z is concave, so $z > 0$ on $[0, L)$ once $z(0) > 0$ and $z(L) = 0$; positivity of $z(0)$ gives $l\ \Sigma_k\ m_k < 4\ pi$, a bound on the trace of |D_F| in units of the tower scale, met when only the top is heavy ($\theta_t <= pi/2$).

3. Proposition 9.2 (diquarks as strings between the stack and its image)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The representation and charge counts are independently checked, conditional on the recalled projections. The shifts describe the specified gauge-field channels; Λ^2 of a line excludes that component only, and leaves the other open-string states and the strings to other branes. See the referee report.

Setting. V = C^5 ⊗ C^3, five light flavours in three colours, with Q = 2/3 on u, c and Q = −1/3 on d, s, b; the 30 of lemma 21 is V ⊕ Vbar. Fifteen D8-branes on an O8 plane carry the Chan-Paton space 30 (branes and images), with group SO(30) on an O8^- and USp(30)

on an O8^+, and U(15) when the stack is moved off the plane (recalled).

Statement.

1. Under U(15), the adjoint of SO(30) is $\Lambda^2 V \oplus \Lambda^2 \overline{V} \oplus (V \otimes \overline{V})$ and the adjoint of USp(30) is $S^2 V \oplus S^2 \overline{V} \oplus (V \otimes \overline{V})$. The summands $\Lambda^2 V$ or $S^2 V$ and their conjugates are the strings between the stack and its image; $V \otimes \overline{V}$ are the strings with both ends on the stack.
2. $\Lambda^2 V = (S^2 5, \overline{3}) \oplus (\Lambda^2 5, 6)$ and $S^2 V = (S^2 5, 6) \oplus (\Lambda^2 5, \overline{3})$.
3. In the colour antitriplet, $\Lambda^2 V$ has the charge counts ($-2/3$: 6, $+1/3$: 6, $+4/3$: 3) of lemma 21, and $S^2 V$ has ($-2/3$: 3, $+1/3$: 6, $+4/3$: 1). So the antisquark counts of three generations recorded in lemma 21 (6 of charge $-2/3$ and 6 of charge $+1/3$ in the $\overline{3}$) occur among the stack-image strings of an O8^- plane, while those of an O8^+ plane have the counts (3, 6, 1).
4. With the angles $\theta_k = l\, m_k$ of 06, a string from channel i to the image of channel j has the spectrum $(n\pi \pm (\theta_i + \theta_j))/l$, and a string from channel i to channel j has $(n\pi \pm (\theta_i - \theta_j))/l$, $n \in \mathbb{Z}$.
5. A single brane has no string to its own image on an O8^- plane (Λ^2 of a line vanishes) and one on an O8^+ plane.

Proof. 1: $30 = V \oplus \overline{V}$ as a U(15) module, $\Lambda^2(V \oplus \overline{V}) = \Lambda^2 V \oplus \Lambda^2 \overline{V} \oplus (V \otimes \overline{V})$ and likewise for S^2; the orientifold projection keeps the antisymmetric part for SO and the symmetric part for USp, and the mixed summand $V \otimes \overline{V}$ is the U(15) adjoint. 2: $\Lambda^2(A \otimes B) = (S^2 A \otimes \Lambda^2 B) \oplus (\Lambda^2 A \otimes S^2 B)$ and $S^2(A \otimes B) = (S^2 A \otimes S^2 B) \oplus (\Lambda^2 A \otimes \Lambda^2 B)$, with $\Lambda^2 \overline{3} = 3$. 3: split $5 = 2 \oplus 3$ by charge; $S^2 5 = S^2 2 \oplus (2 \otimes 3) \oplus S^2 3$ has dimensions 3, 6, 6 with charges 4/3, 1/3, $-2/3$, and $\Lambda^2 5 = \Lambda^2 2 \oplus (2 \otimes 3) \oplus \Lambda^2 3$ has dimensions 1, 6, 3 with the same charges. 4: Proposition 6.1 with charges $(+1, +1)$ and $(+1, -1)$ under $U(1)_i \times U(1)_j$, whose holonomy is $e^{\{2\pi i (a_i \pm a_j)\}}$, and $\theta = \pi a$. 5: dimensions. ■

Reading (proposal). (11, Theorem 11.3 and Proposition 11.4, qualify the selection below: the window sees the counts (3, 6, 1) also at $s = +1$, through the commuta-

tor with the masses, and parent scalars with the counts (6, 6, 3) need M_15(C) at both walls.) The diquark counts of lemma 21 and of the EPJC paper are the stack-image strings of an O8^- plane, the type that Proposition 8.1 associates with the factorization sign s = +1 selected by the seesaw sector (T2c); the O8^+ alternative gives the wrong number of up-type antisquarks. So the seesaw sign and the scalar counts select the same orientifold type. Two features follow from the projection. Diquark masses go with the sum θ_i + θ_j and meson masses with the difference θ_i − θ_j, so when the stack leaves the wall the diquarks become massive and the mesons stay massless: the diquarks are tied to the orientifold. And the top's single brane has no string to its own image, while its strings to the stack, the (30, 2), have masses near m_t (Proposition 6.1); in the EPJC paper the top's exclusion is an input, and here the O8^- projection with one brane for the top removes the top-top channel only (item 5); the top-to-stack strings (30, 2) remain, with masses near m_t, so the rest of the exclusion is a mass statement, as in the paper.

4. Proposition 9.3 (the real structure decides between the two diquark counts)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The stated sign constraint excludes a nonzero antisymmetric block in the three KO labels. The signs permit all of S^2 V (second check: B ranges over exactly S^2 V for eps eps' = +1, with the diagonal block free); the algebra, grading and order conditions of a triple then select a subspace. Distinct-multiplet blocks remain related to their transposes. See the referee report.

Setting. E = V ⊕ Vbar with V as in Proposition 9.2, an antiunitary J on E that maps V onto Vbar and Vbar onto V (as particles and antiparticles are exchanged), $J^2 = \varepsilon$, and a Hermitian X on E with JX = ε' XJ, the condition that inner fluctuations obey in a KO-dimension with signs (ε, ε'). Let B be the block of X from Vbar to V, an element of V ⊗ V.

Statement.

1. B is symmetric (B in S^2V) if εε' = +1 and antisymmetric (B in Λ^2V) if εε' = −1.

2. In the KO-dimensions 5, 6, 7 of the finite part for the labels 9, 10, 11, (ε, ε') = (−1, −1), (+1, +1), (+1, +1), so εε' = +1: the diquark block of every J-compatible fluctuation lies in S^2V, whose colour antitriplet ($\Lambda^2 5$, 3bar) has the counts (−2/3: 3, +1/3: 6, +4/3: 1).
3. For a connection on a circle whose real structure is complex conjugation (KO-dimension 1, (ε, ε') = (+1, −1)), the block lies in Λ^2V, whose colour antitriplet ($S^2 5$, 3bar) has the counts (−2/3: 6, +1/3: 6, +4/3: 3) of lemma 21. The SO(32) Wilson line of Proposition 6.1 is of this type.

Proof. Choose a basis of V and let K: V -> Vbar be coordinatewise conjugation; J(v, w) = (ε K^{-1} w, K v) has $J^2 = \varepsilon$. Write X = [[A, B], [B*, D]]. The component of JX = ε' XJ that maps V to V gives ε K^{-1} B* = ε' B K, so B = εε' K^{-1} B* K^{-1} = εε' B^T in coordinates. Under the U(15)-equivariant identification of Hom(Vbar, V) with $V \otimes V$, symmetric matrices form S^2V and antisymmetric ones Λ^2V. Items 2 and 3 use Connes's sign table, ε = (+, +, −, −, −, −, +, +) and ε' = (+, −, +, +, +, −, +, +) for KO-dimensions 0 to 7, and the counts of Proposition 9.2. On the circle, D = −i d/dθ and J = conjugation give JD = −DJ, and a Hermitian connection A with JA = −AJ is i times a real antisymmetric matrix, an element of so(n), where the generator of R(2 pi a) lies. ■

Item 2 is a negative statement with the following scope: the blocks of J-compatible Hermitian operators between particles and antiparticles, for one multiplet V, in the three labels. It is the finite-geometry form of the symmetry of Majorana mass matrices, and of the Fermi-statistics constraint on a Lorentz-scalar bilinear ψ_i ψ_j of one Weyl multiplet without derivatives (recalled), under which the (15, 3bar) vanishes. Known escapes: item 3 (connection components along a circle or an interval), blocks between two distinct multiplets, which J does not constrain, and derivative couplings.

Reading (proposal). The counts of lemma 21 and of the EPJC paper, (6, 6, 3), belong to the antisymmetric block. In the corpus's setting this block arises as the component of an SO(30) connection along the circle or interval: the Wilson-line moduli of the type I' stack,

whose stack-image part is Proposition 9.2. The finite triple alone, in any of the three labels, gives the symmetric block and the counts (3, 6, 1). So the composite scalars with the counts of lemma 21 point to the interval parent of lemma 22 and its gauge field, and the selection of O8^- in Proposition 9.2 concerns these connection components. This narrows the spin gap of Propositions 6.1 and 9.2: the $\Lambda^2 V$ states are scalars as circle components of a nine-dimensional gauge field, and they come with that field's vector and gaugino partners.

5. Two more dualities between 9 and 11 (recalled; proposals)

Referee note (GPT-6.1 Terra, 4 Nov 2026). The E_1 alternatives need correction: the canonical-orthogonal root lattices distinguish $\mathbb{P}^1 \times \mathbb{P}^1$ from $dP_1 = \mathbb{F}_1$, and the usual local engineering assigns the latter to $\widetilde{E}_1$. The dP_2 and brane duality claims remain conditional on the unavailable primary sources. The Weyl action is a gauge relabelling of a holonomy, subject to the full background data. See the report on the del Pezzo comparison and Weyl action.

1. **E-type walls and del Pezzo surfaces.** The E_{n+1} symmetry of an O8^- plane with n D8-branes at infinite coupling is the symmetry of the five-dimensional theory on a D4-brane probe, which M-theory realizes on a Calabi-Yau threefold with a shrinking del Pezzo surface dP_{n+1} (Seiberg, hep-th/9608111; Morrison and Seiberg, hep-th/9609070; Intriligator, Morrison and Seiberg, hep-th/9702198). So the far wall with the top's brane (n = 1, E_2 = SU(2) x U(1)) has an eleven-dimensional description by a local dP_2, and the empty far wall (E_1 = SU(2)) by dP_1 or P^1 x P^1; the light-quark wall (n = 15) has none. The Witten base CP^2 x CP^1 of 02 is built from the same pieces (CP^2 = dP_0, and P^1 factors).
2. **Colour-flavour exchange as a T-duality.** In the heterotic string on S^1, the Weyl group of the gauge group acts on the Wilson lines as part of the T-duality group O(17, 1; Z). The colour-family transposition M of 05 is an element of the Weyl group of E8, so it is a T-duality of the nine-

dimensional E8 x E8 string, and through Hořava-Witten a symmetry of the E8 holonomy on a wall in eleven dimensions. As a Weyl element it acts as a global gauge transformation, so by itself it relabels gauge-equivalent vacua; it exchanges colour and family physically only relative to a structure that tells them apart, such as the placement of H_F in 05. The four-family requirement of Propositions 5.1 and 5.4 remains.

6. Open

Referee note (GPT-6.1 Terra, 4 Nov 2026). The finite KO signs restrict the symmetric block without producing the full representation. Light interval scalar modes, their boundary conditions and the physical string dictionary remain to be constructed; the detailed EPJC comparison is pending.

- Whether the type I' coupling, as a spectral function of D_F (Proposition 9.1), supplies the second scale of Proposition 1.5 in the flat family of lemmas 23 and 24.
- The local dP_2 of the top's wall and its relation to the Witten base CP^2 x CP^1.
- Spin: in type I' the strings of Proposition 9.2 belong to nine-dimensional vector multiplets, with one real scalar each (the relative position), the same gap as in Proposition 6.1. By Proposition 9.3 a finite triple in the labels 9 to 11 gives S^2V, so the Λ^2V scalars must come from the connection along the interval; the next computation is the inner fluctuations of the parent of lemma 22 with an SO(30) algebra, and whether its interval component carries Λ^2V with the counts of lemma 21.
- Whether the (15, 3bar) of the EPJC paper could be read as bilinears of two distinct multiplets (the paper itself gives one fundamental with itself) (for example left-handed quarks and conjugates of right-handed ones arranged into one 30), which Proposition 9.3 does not constrain.

Referee proofs (GPT-6.1 Terra, 4 November 2026)

The arguments below are those of the referee report on the 11D line, written out in the notation of this lemma; the report in historical/reviews/ stays as the provenance record.

Independent proof of Proposition 9.1, with the scope of its trace. *From the referee report on the 11D line, section 'Proposition 9.1: integrated coupling profile', transferred verbatim in substance on 4 Nov 2026.*

Hypothesis: the recalled slope rule. The Polchinski-Witten source could not be accessed by the referee; the comparison with it is the second-check note at the proposition.

Let N(x) = sum_k 1_{x_k < x}. Integrating z' = c (8 - N) almost everywhere, with z continuous, gives

```
z(x) = z(0) + c (8x - sum_{k=1}^{16} (x - x_k)_+) =
  ↪  z(0) + c sum_{k=1}^{16} min(x/2, x_k - x/2),
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since int_0^x N(y) dy = sum_k (x - x_k)+, *and x/2 - (x - x_k)+* = min(x/2, x_k - x/2) (both sides equal x/2 for x <= x_k and x_k - x/2 for x > x_k). At x = L, with every x_k <= L, this becomes

```
z(L) = z(0) - c (8L - sum_k x_k).
```

With x_k = 2 l L m_k/pi, the dictionary of the Setting (exact only for constant z, by the correction after the proof), the zero condition z(L) = 0 is z(0) = (2cL/pi)(4 pi - l sum_k m_k), as stated. The slopes c (8 - N(x)) are nonincreasing, so z is concave; if z(0) > 0 and z(L) = 0, then z(x) >= (1 - x/L) z(0) > 0 on [0, L).

Scope of the trace. The trace is over the sixteen selected channel values. The equality with a spectral trace assumes a chosen sixteen-dimensional channel space with eigenvalues m_k, or an explicitly normalized projection selecting those values. The physical D_F has additional chirality, conjugation, colour and generation multiplicities, which cannot be dropped from its full trace without specifying that projection. The assignment of the sixteen brane positions to those channels is also part of the proposal. This profile alone does not derive a stabilizing term for the interval modulus. The slope, coordinate and

e^phi-to-z conventions are those of the recalled rule.

Independent proof of Proposition 9.2, items 2 and 3, with the scope of items 4 and 5. *From the referee report on the 11D line, section 'Proposition 9.2: diquark representations and string channels', transferred verbatim in substance on 4 Nov 2026.*

Hypothesis: the recalled orientifold projections. For V = C^5 ⊗ C^3, the swap of V ⊗ V is the product of the swaps of the two tensor factors, so its eigenspaces combine the symmetric and alternating projectors on the factors:

```
Λ²V = (S²C^5 ⊗ 3bar) ⊕ (Λ²C^5 ⊗ 6),
S²V = (S²C^5 ⊗ 6) ⊕ (Λ²C^5 ⊗ 3bar),
```

with $\Lambda^2 3 = 3\text{bar}$ and $S^2 3 = 6$. Their dimensions are 15 * 3 + 10 * 6 = 45 + 60 = 105 and 15 * 6 + 10 * 3 = 90 + 30 = 120, which are C(15, 2) and C(16, 2). Splitting the flavour space into two charge-2/3 and three charge-(-1/3) directions, $S^2C^5 = S^2 2 \oplus (2 \otimes 3) \oplus S^2 3$ and $\Lambda^2C^5 = \Lambda^2 2 \oplus (2 \otimes 3) \oplus \Lambda^2 3$ give the antitriplet counts (6, 6, 3) in S^2C^5 and (3, 6, 1) in Λ^2C^5, ordered by charges (-2/3, 1/3, 4/3). The full adjoints have dimensions 2(105) + 225 = 435 = dim so(30) and 2(120) + 225 = 465 = dim usp(30).

Scope of items 4 and 5. For the assumed Wilson-line action, the antisymmetric or symmetric block has weights a_i + a_j and the mixed block has weights a_i - a_j; hence their field-theory Kaluza-Klein momenta have the displayed shifts (n pi ± (θ_i ± θ_j))/l, with θ = pi a. This calculation leaves out the rest of the open-string spectrum, which also includes oscillator states and their orientifold projections. The sum-weight component with i = j is absent in Λ^2 and present in S^2; neutral diagonal modes remain. The claim of item 5, that a single brane has no string to its own image, holds in this sector. Absence of the same-line component leaves the strings between the top brane and other branes. Moving all stack branes by the same angle makes the mixed-block differences vanish; different individual positions can give nonzero differences. Identifying the resulting Chan-Paton modes with the proposed composites is an additional physical construction. The O8^-/O8^+ projection and global tadpole conditions were not independently verified, since the primary sources were unavail-

able to the referee.

Independent proof of Proposition 9.3, item 1, with the scope of item 2. *From the referee report on the 11D line, section 'Proposition 9.3: real-structure constraint on diquark blocks', transferred verbatim in substance on 4 Nov 2026.*

Choose an antiunitary identification K: V -> Vbar and write

```
J(v, w) = (eps K^{-1} w, K v),    X = [[A, B], [B*,
↪  D]].
```

Then JX(v, w) = (ε K^{-1}(B* v + D w), K(A v + B w)) and XJ(v, w) = (ε A K^{-1} w + B K v, ε B* K^{-1} w + D K v). The V -> V component of JX = ε' XJ, the coefficient of v in the first entry, gives

```
eps K^{-1} B* = eps' B K,   hence   B = epseps'
↪  K^{-1} B* K^{-1} = epseps' B^T,
```

the last equality in coordinates in which K is coordinate-wise conjugation. Thus εε' = +1 forces B ∈ S^2V, and εε' = -1 forces B ∈ Λ^2V. For the displayed KO-5, KO-6 and KO-7 conventions the products are all +1. The zero block belongs to both subspaces, so "symmetric if εε' = +1" states the symmetry forced by the sign. For V = C^3 ⊗ C^3 the decomposition of S^2V and the counts are those of the proof of Proposition 9.2: Λ^2C^3 has multiplicities 3, 6, 1 at charges -2/3, 1/3, 4/3, and S^2C^3, hence the colour-3bar part of Λ^2V, has 6, 6, 3.

Scope of item 2. A J-compatible finite fluctuation can occupy only a subspace of S^2V; the algebra, grading and order conditions determine which blocks actually occur. For example, with D = 0 every inner one-form vanishes, despite the symmetry condition. The KO signs alone therefore exclude a nonzero antisymmetric block and leave the full symmetric scalar spectrum to be established. For a circle operator D = -i d/dx with ordinary conjugation J, JD = -DJ. A Hermitian connection satisfying the same sign is purely imaginary and antisymmetric in a real basis; the complexified so(30) representation contains Λ^2V and its conjugate. Whether these components survive the particular interval boundary conditions as light scalars requires a separate construction. For distinct multiplets, an off-diagonal block B_12 may be chosen freely, while the full constraint still relates it to B_21

= εε' B_12^T.

Scope and direction. The exclusion concerns a Hermitian J-compatible operator between one multiplet and its conjugate, with the stated signs. Distinct multiplets, other real structures or signs, derivative couplings, and circle or interval connection components remain possible directions; each requires its own field and boundary data.

Independent proof of the branching in section 1. *From the referee report on the 11D line, section 'File 09, section 1: the SO(32) comparison', transferred verbatim in substance on 4 Nov 2026.*

The alternative chain follows from 32_C = 16_{+1} ⊕ 16bar_{-1} under U(16) ⊂ SO(32): so(32)_C = Λ²32 = Λ²16 ⊕ Λ²16bar ⊕ (16 ⊗ 16bar), so

```
496 = 120_{+2} ⊕ 120bar_{-2} ⊕ 255_0 ⊕ 1_0.
```

Under SU(15), 16 = 15 ⊕ 1 and 120 = Λ²(15 ⊕ 1) = 105 ⊕ 15. With 15 = 5 ⊗ 3, the plethysm of Proposition 9.2 gives

```
105 = Λ²(5 ⊗ 3) = (15, 3bar) ⊕ (10, 6),
```

with S²5 = 15 and Λ²5 = 10, and 15 ⊗ 15bar = (5 ⊗ 5bar, 3 ⊗ 3bar) = (24, 8) ⊕ (24, 1) ⊕ (1, 8) ⊕ (1, 1). The antitriplet charge multiplicities are those checked in Proposition 9.2. These group products are Lie-algebra shorthand; global embeddings involve finite central quotients.

Scope. The SO(2) plane is a colour singlet. Assigning it to a top brane is a label in the proposed dictionary and does not by itself construct a colour-triplet top field. Chan-Paton bilinearity and matching representation counts likewise do not by themselves establish dynamical quark composites. The referee could not access the paper itself; the comparison with its text is the reading of the LaTeX source in section 1.

Counterexample (referee) to the unqualified E_1 surface identification in section 5, item 1. *From the referee report on the 11D line, section 'File 09, section 5: del Pezzo wall correspondence', transferred verbatim in substance on 4 Nov 2026.*

The two degree-eight surfaces have different root lattices orthogonal to the canonical class. On dP_1, let H^2 = 1,

E^2 = -1, H · E = 0 and K = -3H + E. If D = aH + bE is orthogonal to K, then D · K = -3a - b = 0, so b = -3a and D^2 = a^2 - b^2 = -8a^2. There is no integral class with D^2 = -2. On P^1 x P^1, with ruling classes F_1, F_2 (F_1^2 = F_2^2 = 0, F_1 · F_2 = 1) and K = -2F_1 - 2F_2, the difference F_1 - F_2 is orthogonal to K and has square -2. These surfaces therefore cannot be interchanged in the root-lattice comparison. In the usual local engineering description (recalled), P^1 x P^1 gives E_1, whereas dP_1 = F_1 gives the distinct $\tilde{E}_1$ theory.

For dP_2, with exceptional classes E_1, E_2 and K = -3H + E_1 + E_2, the class E_1 - E_2 is orthogonal to K and has square -2; the orthogonal lattice has rank two, consistent with an A_1 factor and an additional abelian direction. The dP_2 comparison is compatible with the expected A_1 root. This computation alone does not construct the corresponding five-dimensional field theory or the type I' duality, and the brane-to-geometry map remains conditional on the cited dualities.

Sources. The distinction between the two degree-eight engineering models is also stated in a secondary geometric-engineering discussion. The Seiberg and Morrison-Seiberg primary sources were unavailable to the referee for direct comparison. The occurrence of CP^2 and CP^1 in the separate six-dimensional base does not by itself identify that compactification with a local dP_2.

Independent proof of the Weyl action in section 5, item 2. *From the referee report on the 11D line, section 'File 09, section 5: Weyl action on a Wilson line', transferred verbatim in substance on 4 Nov 2026.*

A Weyl transformation of E8 acts by conjugation on an E8 holonomy and gives a lattice automorphism in the heterotic circle frame. Whether it acts as a symmetry of a fixed compactification depends on all of its background and boundary data.

Let T ⊂ E8 be a maximal torus. An element w ∈ W(E8) = N(T)/T has a representative g ∈ N(T), and the constant gauge transformation by g sends a holonomy W ∈ T to g W g^{-1} ∈ T. Thus the holonomy alone determines the same gauge orbit. The corresponding isometry w ⊕ 1 of E8 ⊕ E8 ⊕ II_{1,1} is an element of the auto-

morphism group of the nine-dimensional Narain lattice. Proposition 5.2 places the transposition M in that Weyl group.

If a bundle, a defect, an algebra embedding or a distinguished set of zero modes is held fixed, the same g need not preserve it. Transforming the entire configuration produces a gauge-equivalent description when the transformation is allowed; an exchange relative to fixed colour and family structures needs an explicit action on those structures. The root no-go applies to its stated three-slot configuration, while bundle zero modes remain an open escape.

Part 38

Inner fluctuations of the interval parent with an SO(30) algebra

Lemma 38. Written as file 11 of the eleven-dimensional line (branch 11D, by another Claude session) and moved here on 4 November 2026 with its text unchanged; its proposals keep their labels, and the plan and the unfinished files of that line are in historical/11D/. Internal references to "file 0n" of that line mean the lemma with the same content: 01 is lemma 31, 02 is 32, 03 is 39, 04 is 33, 05 is 34, 06 is in historical/11D, 07 is 40, 08 is 35, 09 is 36, 10 is 37, 11 is 38.

Part of the 11D line (README, PLAN). Attempted by Claude Opus 5.5 on 4 November 2026, to answer the question left in 09: whether the gauge field along the interval of lemma 22 carries the antisymmetric block with the counts of lemma 21, given that Proposition 9.3 excludes it from the finite triple alone. Numbers in Python (numpy), scripts not kept by the rule of the project. The Codex quota is spent, so no Terra referee is available. An independent reading by Claude Sonnet 5.5, whose amendments are applied below, is in the review. Refereed by GPT-6.1 Terra (effort max, 5 November 2026; report): the block calculations, the commutant, the one-forms and the compression formula correct; three corrections applied below, all on one point, that the representation pi is nonunital on the doubled space, so that the gauge action of Theorem 11.3 and the realization of the shadow

of Proposition 11.4 as an inner fluctuation at nonzero mass need a unital completion, which is now stated as a hypothesis.

Outcome

1. **Yes, under a condition on the walls.** With a real algebra M_30(R) on the light block, the fluctuation along the interval is an so(30) connection, and its block between particles and antiparticles lies in Λ^2 V, with the counts (6, 6, 3) in the colour antitriplet (Proposition 11.2). Massless scalars in exactly Λ^2 V + conjugate appear when both walls reduce the algebra to M_15(C), the commutant of quark number, under the Yang-Mills action with its natural boundary conditions and a gauge action supplied by a unital completion of pi (Theorem 11.3): 210 real scalar and 225 real vector components at tree level.
2. **In lemma 22's own parent the massless scalars are absent.** Its algebra is free at x = 0, so the interval component has no zero mode, for any algebra (Theorem 11.3, item 2).
3. **A generator has a massless vector or a massless scalar, never both, on the interval**, in free five-dimensional Yang-Mills at trivial background. So the whole of Λ^2(30) of lemma 21 cannot be massless scalars with colour unbroken. The antisquark counts appear as scalars, while the slepton part (24, 1) + (1, 1) appears as vectors (Theorem 11.3, item 4), in the two-U(15)-wall model at trivial background; these are tree-level field counts, not a quantum massless spectrum.
4. **The finite window sees a symmetric shadow.** The connection acts on the window, by compression, through its commutator with the masses, B_ij -> (m_i - m_j) B_ij, which maps the (S^2 5, 3bar) of Λ^2 V onto the (Λ^2 5, 3bar) of S^2 V, with counts (3, 6, 1) (Proposition 11.4). This reconciles Propositions 9.2 and 9.3. The kernel is the same-flavour diquarks. The compression is an operator identity; its realization as an inner fluctuation of the algebra at nonzero mass needs a unital completion, since for invertible M every element of the corner algebra of lemma 22 vanishes at x = l (ref-

eree).

5. **SO(30) has a precise meaning in the finite geometry:** it is the commutant of the real structure j = J U of lemma 06 on the massless light block, for the sign s = +1, and USp(30) for s = -1 (Proposition 11.1).

Setting

Let V = C^5 ⊗ C^3, with Q = 2/3 on u, c and Q = -1/3 on d, s, b. In the finite Hilbert space of the five light quarks, with colour, the chirality gamma = +1 subspace is

```
H_+ = {q_L} ⊕ {conj(q_R)} = V ⊕ Vbar,
```

of dimension 30: the left-handed Weyl fields of fifteen Dirac fermions. H_- = {q_R} ⊕ {conj(q_L)}, and J_F exchanges H_+ and H_-, with c = J_F restricted to H_+, as in lemma 22. Let U be a unitary, grading-reversing and J-related as in lemma 06, with U q_L = q_R and U conj(q_R) = s conj(q_L), so U J_F = s J_F U. Then j = U^-1 c is an antiunitary on H_+, with j^2 = s, j q_L = s conj(q_R) and j conj(q_R) = q_L. Here U only defines j: the representation pi below does not commute with U, so this is not a doubling in the sense of lemma 06, in which U commutes with the algebra.

The interval parent is that of lemma 22: D_X = E d/dx on L^2([0, l]; C^2) ⊗ (H_+ ⊕ H_-), with Q in H_+, U-components in H_-, and the wall conditions Q_R(0) = U_L(0) = 0, Q_R(l) = X U_R(l), U_L(l) = -X^* Q_L(l).

The block

Proposition 11.1 (the commutant of j). Let s = +1. In the basis V ⊕ Vbar with j(v, w) = (K^-1 w, K v), the skew-Hermitian a with j a j^-1 = a form so(30), and

```
a = [[alpha, B], [conj(B), conj(alpha)]],    alpha^*
 ↪  = -alpha,   B^T = -B,
```

so, as real vector spaces, so(30) = u(15) ⊕ (Λ^2 V)_R, where u(15) commutes with the complex structure I = (i, -i), the quark number, and the second summand has complex dimension 105 and real dimension 210, with 435 = 225 + 210; the decomposition is not a direct sum of

Lie algebras (referee). The Hermitian generator H = i a satisfies j H j^-1 = -H. For s = -1 the same computation gives usp(30) = u(15) ⊕ (S^2 V)_R, of real dimension 225 + 240 = 465.

Proof. Linear algebra, with Proposition 8.1 for the groups. For s = +1, j a j^-1 = a gives the block form, and a^* = -a gives alpha^* = -alpha and B^* = -conj(B), that is B^T = -B. Then j (i a) j^-1 = -i a. ■

So the SO(30) of lemma 21 is the symmetry of the massless block for the sign s = +1, the sign that Proposition 8.1 and STRATEGY T2c associate with label 11 and with the seesaw; and the counts of lemma 21 select the same sign:

	colour antitriplet	its charges (-2/3, +1/3, +4/3)	colour sextet
s = +1, Λ^2 V	(S^2 5, 3bar)	(6, 6, 3)	(Λ^2 5, 6)
s = -1, S^2 V	(Λ^2 5, 3bar)	(3, 6, 1)	(S^2 5, 6)

Fluctuations along the interval

Take on the light block D_F = 0, so X = 0, and let C_0, C_l be subalgebras of A = M_30(R), the real elements of End H_+. Let

```
A_l = {f in C^infinity([0, l], A) : f(0) in C_0,
 ↪  f(l) in C_l},    pi(f) = f on Q, 0 on the H_-
 ↪  components,
```

a nonunital representation on the doubled space: pi(1) is the projection onto H_+ (referee). A unital completion is pi~(f, r) = f (+) r 1 with a real scalar summand; it is used where a gauge action is needed below.

Proposition 11.2. Assume 1 lies in C_0 and C_l.

1. pi(f) preserves the domain, for every choice of C_0, C_l. Order zero and order one hold on this block.
2. The self-adjoint one-forms sum pi(f_i) [D_X, pi(g_i)] are exactly E a_x on the Q components, with a_x in C^infinity([0, l], so(30)), with no condition at the walls. The fluctuated operator is

```
D_A = E (d/dx + a_x ⊕ U a_x U^-1).
```

3. J-hat A J-hat^-1 = E (U a_x U^-1 on H_-), with sign +, as KO-dimension six requires. The block of a_x between Vbar and V is antisymmetric. The two statements agree since E is antisymmetric: the block of the Hermitian operator E a on C^2 ⊗ E is symmetric, as Proposition 9.3, item 2 requires, with B antisymmetric and the factor E carrying the other sign. So the interval component evades Proposition 9.3 through the escape in its item 3.

For D_F != 0 on this block, order one fails in the finite window: [[D_F, pi(a)], J pi(b) J^-1] has the blocks -U b U^-1 M a from H_+ to H_-, and -a M^* U b U^-1 from H_- to H_+, which vanish for all a, b only if M = 0. The corner representation still satisfies order one by the direct pointwise calculation; for invertible M its endpoint matrix corner is zero. Theorem 22.3 assumes a unital represented algebra, so a unital completion must be specified separately: for pi~(f, r) = f (+) r 1 and invertible M the endpoint condition is f(l) = r(l) 1, and the finite order-one failure remains, since with the original corner action a = b = 1 gives the off-diagonal blocks -M^* and -M (referee). So this algebra describes the massless block, and the masses need a treatment without order one, as in lemma 19.

Proof. (1) The conditions Q_R = 0 and U_L = 0 at the walls are preserved by any operator acting on the H_+ index. [D_X, pi(g)] = E g' on Q, and J-hat pi(b) J-hat^-1 acts on the H_- components only, so both commutators vanish. (2) Given T, choose tau with support inside (0, l) and the same integral as T, which is a finite sum sum f_i g_i' of functions supported inside the interval (take a compactly supported scalar g with g' = 1 on the support of tau). Take f = 1 and g = int_0^x (T - tau), which vanishes at both walls, so g lies in A_l and g' = T - tau. Self-adjointness of E T, with E skew, requires T skew. (3) J-hat E = E J-hat (direct from the formula for J-hat in Theorem 22.1), and c a c^-1 = U j a j^-1 U^-1 = U a U^-1. ■

Theorem 11.3 (zero modes of the interval component). Let k_0, k_l be the skew parts of C_0, C_l, and assume a five-dimensional Yang-Mills action for the fluctuation on a flat product with M^4, as in Proposi-

tion 22.8, with the usual gauge action, which the corner representation pi does not supply by itself (pi(u) J-hat pi(u) J-hat^{-1} = 0) and a unital completion such as pi~ does, and with the Yang-Mills boundary variational problem at trivial background, unrestricted allowed endpoint variations and no wall terms (referee).

1. The gauge group is {g : I -> O(30), g(0) in K_0, g(l) in K_l}, with det g(0) = det g(l); if K_0 or K_l contains reflections it does not act separately on SO(30). The holonomy of a_x transforms as P -> g(l) P g(0)^-1, so the moduli are the double coset K_l SO(30) / K_0, and the massless four-dimensional scalars from a_x, before the quotient by the residual four-dimensional gauge transformations, form

   ```
   the orthogonal complement of k_0 + k_l in
    ↪ so(30),
   ```

   while the vector zero modes form k_0 ∩ k_l. The action supplies the quadratic form and the natural boundary conditions; order one alone supplies no scalar kinetic term, and the complementary Dirichlet and Neumann conditions for A_x need a compatible gauge fixing.

2. In lemma 22's algebra, C_0 = A, so k_0 = so(30), and there are no such scalars, for any algebra of this kind.

3. If C_0 = C_l = M_15(C), the massless scalars are exactly Λ^2 V ⊕ Λ^2 Vbar, 210 real components with colour-antitriplet counts (6, 6, 3), and the massless vectors are u(15), 225 real components.

4. A generator in k_0 ∩ k_l has a massless vector and no massless scalar; a generator in the orthogonal complement of k_0 + k_l has a massless scalar and no massless vector; all others have neither. If the orthogonal projections onto k_0 and k_l commute (aligned walls), the towers are n pi/l in the first two cases, and (n - 1/2) pi/l for a generator in exactly one of k_0, k_l. For nonaligned walls the towers move, and the zero-mode statements are to be read as the subspaces above: k_0 spanned by T_1 and k_l by cos(phi) T_1 + sin(phi) T_2 give k l = n pi +- phi, which is the mechanism of lemma 23;

a nontrivial holonomy can change the zero-mode spaces.

5. With colour unbroken at both walls, the (1, 8) of Λ^2(30) is a vector, so Λ^2(30) cannot consist entirely of massless scalars. In item 3, the (24, 8) + (24, 1) + (1, 8) + (1, 1) of lemma 21 are vectors, and the scalars are (15, 3bar) + (10, 6) + conjugates.

Proof. (1) A gauge parameter xi(x) with xi(0) in k_0, xi(l) in k_l shifts int a_x by xi(l) - xi(0), so the linear moduli are so(30)/(k_0 + k_l), and a constant a_x in the orthogonal complement is a zero mode of the quadratic form. (2), (3): substitution, with Proposition 11.1. (4) A_mu has Neumann or Dirichlet conditions at each wall according to membership in k_wall, and a_x the opposite, so the mode functions are cos(n pi x/l), sin(n pi x/l) or their half-integer versions. (5) Item 4, and the branching of lemma 21. ■

Scope of item 4. Free five-dimensional Yang-Mills at trivial background, with the gauge action and boundary problem stated in the hypotheses; the counts are tree-level field counts and not a quantum massless spectrum. Escapes: a circle without walls, nontrivial holonomy, brane-localized terms, further internal dimensions.

Note that the choice in item 3 is an input: the domain does not force the wall algebra to M_15(C), since by Proposition 11.2, item 1, every f preserves it at D_F = 0. It says that the walls conserve quark number and the bulk does not. On a circle, without walls, every generator has both a massless vector and a massless scalar, and the whole of Λ^2(30) appears, which is the nine-dimensional adjoint scalar used in 09. A wall condition, or the reflection-even algebra of Proposition 1.1, removes half. So the two readings differ: for the stack off the O8 plane, with holonomy I, diquarks are massive and mesons have massless scalars; on the interval with parity Ad(I), diquarks have massless scalars and mesons have none.

The shadow in the finite window

Proposition 11.4. Let s = +1, M = U Lambda with Lambda real for j and symmetric, and let a_tot = a ⊕ U a U^-1 be a constant bounded perturbation, with the

window condition of Theorem 22.1. Then:

1. [D_F, a_tot] has the off-diagonal blocks U S and S U^-1, with S = [Lambda, a] real and symmetric, and [D_F, [D_F, a_tot]] = [Lambda, S] ⊕ U [Lambda, S] U^-1.

2. By Theorem 22.4, the compression of E a_tot to the window is -l^-1 int_0^l sin(x ad_{D_F})(a_tot) dx. For Lambda = diag(m_i) on flavours, on both V and Vbar, the block B_ij goes to

   ```
   - ((1 - cos(l (m_i - m_j))) / (l (m_i - m_j)))
    ↪ B_ij,    which is -(l/2)(m_i - m_j) B_ij
    ↪ to leading order, with the continuous
    ↪ value zero at equal masses,
   ```

   as a change of Lambda. The result is symmetric, as Proposition 9.3 requires of a finite J-compatible fluctuation.

3. For distinct masses, B -> (m_i - m_j) B_ij maps (S^2 5, 3bar) in Λ^2 V onto (Λ^2 5, 3bar) in S^2 V as vector spaces, with kernel the five same-flavour diquarks (five colour-antitriplet multiplets), and maps (Λ^2 5, 6) into (S^2 5, 6); distinct fixed masses break flavour SU(5), so the map is SU(5)-equivariant between the named representations only with the mass matrix read as a spurion (referee). In charges, (6, 6, 3) - (3, 0, 2) = (3, 6, 1). If the wall condition is imposed on the gauge unitaries g ⊕ U g U^-1, so that k_l is the commutant of Lambda in so(30), then with k_0 = u(15) the massless scalars of Theorem 11.3 are the same off-diagonal blocks. If it is imposed on pi(f) itself, as in the algebra of lemma 22, then k_l = 0 for invertible M.

4. For a universal mass, Lambda = m, the compression vanishes: S = 0, and M = m U is invariant under SO(30). Fifteen Dirac fermions of equal mass are thirty Majorana fermions of equal mass.

Proof. (1) Direct multiplication, using a_tot on H_- equal to U a U^-1. S is real and symmetric since a is real and antisymmetric and Lambda real and symmetric. (2) Item 1 gives sin(x ad)(a_tot) = (sin(x delta)/delta) ad(a_tot) on B_ij, with delta = m_i - m_j; integrate.

(3) In the colour antitriplet, B is symmetric in flavour, and (m_i - m_j) B_ij is antisymmetric in flavour, hence total symmetric. Colour sextets are analogous. (4) S = 0. ■ Checked in Python on a small model, with V = C^3: reality, symmetry, the blocks of item 1 and their squares. Claude Sonnet 5.5 also checked the compression by direct quadrature of W^*(E a_tot)W, with five flavours in three colours, against Theorem 22.4 and the closed block (Python, 10^-15).

Reading. The commutator [D_F, a] is the form of the Higgs field in Connes' finite geometry, and E a_x is a Kaluza-Klein component along the interval. Theorem 22.4 (3), Q_l([D_X, f]) = [D_F, Q_l(f)], says that the second compresses to the first. Indeed E a_tot = [D_X, f_tot] with f(x) = (x - l) a and f(l) = 0. For invertible M, however, every element of the corner algebra of lemma 22 vanishes at l, hence so does every coefficient sum f_i g_i' of one of its one-forms, so a nonzero constant a_tot is not an inner one-form of that algebra, and f_tot is not one of its elements; the compression identity stands as an operator calculation, and its realization as an inner fluctuation at nonzero mass needs a unital completion and a construction of the real one-form there (referee). With that construction, in this parent, to first order and for the fixed compression, a finite Higgs one-form is the shadow of a gauge field along the interval, and it inherits the symmetry type of the finite triple by the commutator with masses. The moving window of Corollary 22.6 gives -l [D_F, h] and a quadratic term (lemma 19). This shadow is tangent to the O(30) orbit of Lambda, so it gives no radial mode, in agreement with No-go theorem 22.7; the odd generators of lemma 23 are absent here since M_30(R) maps H_+ to H_+. A generator commuting with the masses has no shadow. This explains the difference between the counts (6, 6, 3) of the parent and (3, 6, 1) of the finite triple, and it replaces the alternative O8^- against O8^+ of Proposition 9.2, for these blocks, by a parent and its window.

What remains open

- Whether a boundary condition, for example a wall term that conserves quark number, can derive M_15(C) at the walls, where at present it is chosen.
- The unital completion of pi and the gauge action

it supplies, used as hypotheses in Theorem 11.3, and the construction of the nonzero-mass one-form whose compression is the shadow of Proposition 11.4 (referee).

- The counts of Theorem 11.3 are tree-level field counts in the two-U(15)-wall model at trivial background; other wall data, holonomy or further directions need separate calculations.
- The treatment of nonzero masses, where order one fails on the block: the quadratic terms of lemma 19 for this algebra.
- The sleptons: on an interval the (24, 1) + (1, 1) are vectors. Scalars in these representations need components along further internal directions, such as the base of 02.
- The colour sextets (10, 6) come with the antitriplets, and nothing removes them in this construction.
- The relation to the top: the (30, 2) of SO(32), with the second factor as the top's SO(2).