Supplemental Material to "Finite spectral triples as towerless intervals"

Draft, 3 November 2026

This is the supplement of the paper *Finite spectral triples as towerless intervals*. Its parts keep the numbers of the full Supplemental Material, so that a citation of a part in the paper refers to the part of the same name here.

Parts contained. Lemma parts: 05, 06, 07, 08, 09, 10, 11, 12, 13, 14, 15, 16, 18, 19, 20, 21. Other parts: H, L, R1 (H is the history of the field, L the literature check, R1 Counterexample R1 and S the statements of results).

The full Supplemental Material, with all parts of the project, exists as `supplemental.pdf` and `supplemental_mobile.pdf` in the same folder as this file. A reference inside a part to a part not listed here points to the full version.

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Part 05

The geometric identification exists: the minimal grading-preserving spectral truncation of a suitable chiral interval is exactly Connes–Lott, with

$$|m| = \frac{\pi}{2l}, \qquad d_{\text{trunc}} = \frac{2l}{\pi}.$$

The truncated endpoint states are smeared states on the original interval. Moreover, this interval has no zero mode, which limits the physical interpretation of both extremes.

Part 1. Deconstructed chain

Statement. There is a natural family with opposite endpoint chiralities, an exact Connes–Lott member at $N = 2$, and endpoint distance converging to l. The answer depends on the representation of the site algebra; the familiar lattice-distance anomaly is not universal. (Parts 1 and 2 were refereed by Claude Opus 5.5 on 5 November 2026, report in historical/reviews/2026-11-05_opus_check_05_21_34.md: correct as written, every formula recomputed, with the scope of the three convergence statements made explicit at the end of the Statement of Part 2.)

Hypotheses. We consider complex even spectral triples, without imposing additional real-structure or KO-dimension axioms. Let $N \geq 2$, $n = N - 1$, and

$$A_N = \mathbb{C}^N, \qquad H_N = \text{span}\{L_1, \dots, L_{N-1}, R_2, \dots, R_N\}.$$

Thus interior sites carry both components, site 1 only L, and site N only R. Define

$$\pi_N(a) = \text{diag}(a_1, \dots, a_{N-1}, a_2, \dots, a_N), \qquad \Gamma_N = \text{diag}(I_n, -I_n).$$

If S_n is the lower shift, set

$$B_N = h_N(I_n - S_n), \qquad D_N = \begin{pmatrix} 0 & B_N \\ B_N^* & 0 \end{pmatrix}.$$

These are forward/backward differences:

$$(D_N\psi)_{L_i} = h_N(R_{i+1} - R_i), \qquad (D_N\psi)_{R_j} = h_N(L_{j-1} - L_j),$$

with the missing components interpreted as

$$R_1 = 0, \qquad L_N = 0.$$

There are nearest-neighbour couplings and onsite L-R terms at interior sites.

For compatibility with the spectral truncation in Part 3, choose

$$\theta_N = \frac{\pi}{4N - 2}, \qquad h_N = \frac{\pi}{4l \sin\theta_N} \sim \frac{N}{l}. \tag{1}$$

This fixes the smallest positive eigenvalue to $\pi/(2l)$ at every N.

Proof.

The representation is faithful, $[\Gamma_N, \pi_N(a)] = 0$, and $\Gamma_N D_N = -D_N \Gamma_N$. At $N = 2$,

$$A_2 = \mathbb{C}^2, \quad H_2 = \mathbb{C}^2, \quad D_2 = \begin{pmatrix} 0 & h_2 \\ h_2 & 0 \end{pmatrix}, \quad h_2 = \frac{\pi}{2l}.$$

This is exactly the stated two-point triple; a diagonal unitary supplies any desired phase of m.

Ordering the basis as

$$L_1, R_2, L_2, R_3, \dots, L_{N-1}, R_N$$

makes D_N a path adjacency matrix with alternating signs. A diagonal unitary removes those signs. The usual sine eigenvectors of a path therefore give

$$\operatorname{spec}(D_N) = \left\{ 2h_N \cos \frac{j\pi}{2N-1} : 1 \le j \le 2N-2 \right\} = \{\pm\lambda_{N,r} : 1 \le r \le N-1\},$$

where

$$\boxed{\lambda_{N,r} = 2h_N \sin((2r-1)\theta_N).} \tag{2}$$

Consequently,

$$\boxed{K_N(t) = \operatorname{Tr} e^{-tD_N^2} = 2 \sum_{r=1}^{N-1} e^{-4th_N^2 \sin^2((2r-1)\theta_N)}.} \tag{3}$$

For a real site function a, the onsite terms commute with $\pi_N(a)$, while

$$B_N \pi_R(a) - \pi_L(a) B_N = h_N \operatorname{diag}(a_2 - a_1, \dots, a_N - a_{N-1}).$$

Hence

$$\|[D_N, \pi_N(a)]\| = h_N \max_i |a_{i+1} - a_i|.$$

The triangle inequality bounds $|a_j - a_i|$ by $|j-i|/h_N$; the function $a_i = (i-1)/h_N$ attains this bound. Thus

$$\boxed{d_N(i,j) = \frac{|i-j|}{h_N}.} \tag{4}$$

In particular,

$$\boxed{d_N(1,N) = \frac{4l(N-1)}{\pi} \sin \frac{\pi}{4N-2} \longrightarrow l.} \tag{5}$$

One can instead choose the geometric normalization

$$h_N = \frac{N-1}{l}. \tag{6}$$

Then $d_N(1,N) = l$ exactly for every N, with the same continuum limit. However, its $N = 2$ mass is $1/l$, rather than the continuum lowest eigenvalue $\pi/(2l)$. Equations (1) and (6) distinguish spectral calibration from metric calibration at finite resolution.

Two other natural choices clarify the literature:

- **One-component staggered path.** Take $A = \mathbb{C}^M$, $H = \mathbb{C}^M$, alternating grading, and $D = h$ times path adjacency. Opposite endpoint chiralities require even M. Its spectrum is $2h\cos(j\pi/(M+1))$. For $k = |i-j|$,

$$\boxed{d(i,j) = \begin{cases} (k+1)/(2h), & k \text{ odd}, \\ \sqrt{k(k+2)}/(2h), & k \text{ even}. \end{cases}} \tag{7}$$

In particular $d(1, M) = M/(2h)$. Thus $h = M/(2l)$ gives endpoint distance l, whereas $h = M/l$ gives $l/2$.

Here is a direct proof of (7). Restrict the commutator to the connecting subpath and extend the function constantly outside it. After removing phases, its norm is that of weighted adjacency, with dimensionless edge weights $w_i = h|\Delta a_i|$. If its two bipartition sizes are p, q, the off-diagonal block C satisfies

$$\sum_i w_i = \mathbf{1}_p^T C \mathbf{1}_q \le \sqrt{pq}\,\|C\|.$$

For $p = q$, alternating weights $1, 0, 1, \dots, 1$ attain the bound. For $p = q + 1$, the weights

$$\frac{(q, 1, q-1, 2, \dots, 1, q)}{\sqrt{q(q+1)}}$$

attain it: the normalized constant vectors on the two bipartitions are singular vectors with singular value 1, and positivity gives the norm 1. This proves (7).

These are the nonadditive distances studied by Bimonte–Lizzi–Sparano, §5 and given explicitly by Atzmon, Eq. (30). Reduction to the connecting path also agrees with Iochum–Krajewski–Martinetti, Proposition 4.

- **Vertex-edge incidence Dirac.** Let $\partial : \mathbb{C}^N \to \mathbb{C}^{N-1}$ be the path difference matrix and use

$$D_{\text{inc}} = h \begin{pmatrix} 0 & \partial^* \\ \partial & 0 \end{pmatrix}.$$

Its spectrum and heat trace are

$$\{0\} \cup \left\{ \pm 2h \sin \frac{k\pi}{2N} : 1 \le k < N \right\},$$

$$K_{\text{inc}}(t) = 1 + 2 \sum_{k=1}^{N-1} e^{-4th^2 \sin^2(k\pi/(2N))}.$$

These follow by substituting the cosine eigenvectors into the path Laplacian $\partial^*\partial$. Representing a site function on each edge by its value at the tail gives distance $|i-j|/h$. But H has dimension $2N-1$, has a zero mode, and both endpoint vertex components have the same grading. At $N = 2$, its spectrum is $0, \pm\sqrt{2}h$, so it is not Connes–Lott.

Thus the quoted integer deconstruction tower belongs to the incidence construction. Opposite chiral boundary conditions produce the shifted tower (2).

Status: PROVED

Part 2. Continuum limit

Statement. The chain above converges to an interval Dirac operator in three precise senses: ordered low eigenvalues, heat traces locally uniformly for $t > 0$, and pure-state metric spaces in Gromov–Hausdorff distance. It also has a controlled one-dimensional heat window. Here spectral convergence means that each eigenvalue lambda_{N,r} increases to (2r-1) m_1 at fixed r, so that the counting function on [-E, E] converges for every E outside spec(D_l); the Hilbert spaces differ and no resolvent convergence is asserted. The heat traces converge uniformly on [eps, infinity) for every eps > 0. The Gromov-Hausdorff statement concerns the pure-state spaces with their Connes metrics, at distance at most l/(N-1) (referee, 5 November 2026).

Hypotheses. On $H_l = L^2([0, l], \mathbb{C}^2)$, define

$$D_l = \begin{pmatrix} 0 & \partial_x \\ -\partial_x & 0 \end{pmatrix}, \qquad \operatorname{Dom} D_l = \{(L, R) \in H^1 : R(0) = 0,\ L(l) = 0\}, \tag{8}$$

with scalar multiplication by $C^\infty([0,l])$ and grading $\Gamma = \mathrm{diag}(1,-1)$.

These are opposite local chiral conditions. Reflection parity is Γ at 0 and $-\Gamma$ at l; their composition gives antiperiodicity on the doubled interval.

Proof.

Integration by parts gives the boundary form

$$[\overline{L}\,\widetilde{R} - \overline{R}\,\widetilde{L}]_0^l.$$

The conditions in (8) form a maximal isotropic boundary subspace, so D_l is self-adjoint. Its resolvent is compact by the compact embedding $H^1 \hookrightarrow L^2$.

Solving $D_l\psi = \lambda\psi$ gives

$$\psi_\lambda(x) = c(\cos\lambda x, \sin\lambda x), \qquad \cos(\lambda l) = 0.$$

Therefore

$$\boxed{\mathrm{spec}(D_l) = \left\{ \pm\frac{(r-\frac{1}{2})\pi}{l} : r = 1,2,... \right\}.} \tag{9}$$

The kernel is zero. Squaring gives mixed Neumann–Dirichlet conditions on L, and Dirichlet–Neumann conditions on R.

For scalar f,

$$\|[D_l, f]\| = \|f'\|_\infty, \qquad d_l(x,y) = |x-y|. \tag{10}$$

Under normalization (1), with $m_1 = \pi/(2l)$,

$$\lambda_{N,r} = m_1 \frac{\sin((2r-1)\theta_N)}{\sin\theta_N} \longrightarrow (2r-1)m_1. \tag{11}$$

This proves low-eigenvalue convergence. Since

$$\sin y \geq \frac{2y}{\pi} \quad (0 \leq y \leq \pi/2),$$

we also have

$$\lambda_{N,r} \geq \frac{2r-1}{l}. \tag{12}$$

A summable Gaussian majorant now proves, uniformly on every $t \in [\varepsilon, T]$,

$$K_N(t) \longrightarrow K_l(t) = 2\sum_{r=1}^{\infty} e^{-t((r-\frac{1}{2})\pi/l)^2}. \tag{13}$$

Poisson summation gives

$$\boxed{K_l(t) = \frac{l}{\sqrt{\pi t}}\left(1 + 2\sum_{q=1}^{\infty}(-1)^q e^{-q^2 l^2/t}\right).} \tag{14}$$

In particular, $K_l(t) \sim l/\sqrt{\pi t}$ as $t/l^2 \to 0$.

The finite-chain heat window has the following rigorous meaning. For every sequence $t_N > 0$ satisfying

$$\frac{t_N}{l^2} \to 0, \qquad \frac{N^2 t_N}{l^2} \to \infty, \tag{15}$$

one has

$$\boxed{K_N(t_N) \sim \frac{l}{\sqrt{\pi t_N}}, \qquad -2t_N\frac{K_N'(t_N)}{K_N(t_N)} \to 1.} \tag{16}$$

To prove this, put $u_r = \pi\sqrt{t_N}(r-\frac{1}{2})/l$. On bounded u_r-intervals, the sine argument in (11) tends uniformly to zero, so $\sqrt{t_N}\lambda_{N,r}/u_r \to 1$. The mesh tends to zero and its upper endpoint tends to infinity. Bound (12) controls the tails. Gaussian Riemann sums, and then the same sums weighted by $t_N\lambda_{N,r}^2$, prove both assertions.

Finally, identify site i with $x_i = (i-1)l/(N-1)$. Equation (4) gives

$$d_N(i,j) = \rho_N|x_i - x_j|, \qquad \rho_N = \frac{N-1}{lh_N} = 1 + O(N^{-1}).$$

The metric distortion and the grid mesh are both $O(l/N)$, proving Gromov–Hausdorff convergence to $[0,l]$.

The endpoint chirality here concerns allowed boundary components. It does not supply localized chiral zero modes.

Status: PROVED

Part 3. Minimal spectral truncation

Statement. The minimal nonzero grading-preserving spectral truncation of (8) is unitarily equivalent to Connes–Lott, with

$$\boxed{|m| = m_1 = \frac{\pi}{2l}, \qquad c = \frac{\pi}{2}.}$$

Its natural truncated endpoint states have distance $1/m_1 = 2l/\pi$. Literal endpoint evaluations do not descend through compression.

Hypotheses. Let

$$P = \mathbf{1}_{[0,E]}(|D_l|), \qquad m_1 \le E < 3m_1,$$

and give $S = P\,C([0,l])\,P$ its inherited operator-system structure. This is the compression construction described in Connes–van Suijlekom, §1.

Proof.

An orthonormal chiral basis of PH_l is

$$u_L(x) = \sqrt{\frac{2}{l}}\begin{pmatrix}\cos(\pi x/(2l))\\ 0\end{pmatrix}, \qquad u_R(x) = \sqrt{\frac{2}{l}}\begin{pmatrix}0\\ \sin(\pi x/(2l))\end{pmatrix}.$$

In this basis,

$$PD_lP = \begin{pmatrix}0 & m_1\\ m_1 & 0\end{pmatrix}, \qquad P\Gamma P = \mathrm{diag}(1,-1). \tag{17}$$

Compression of multiplication is

$$Q(f) = \begin{pmatrix}\frac{2}{l}\int_0^l f(x)\cos^2\frac{\pi x}{2l}\,dx & 0\\ 0 & \frac{2}{l}\int_0^l f(x)\sin^2\frac{\pi x}{2l}\,dx\end{pmatrix}. \tag{18}$$

Since

$$Q(1) = I, \qquad Q(\cos(\pi x/l)) = \tfrac{1}{2}\,\mathrm{diag}(1,-1),$$

its range is exactly the full diagonal algebra:

$$\boxed{S = \mathbb{C}\oplus\mathbb{C}.} \tag{19}$$

Here the compressed operator system happens to be an algebra, although Q is not multiplicative.

Evaluation on the finite-dimensional space PH_l is bounded. At 0, only u_L has a nonzero boundary value; at l, only u_R does. The normalized evaluation vectors therefore define

$$\omega_0(\mathrm{diag}(a,b)) = a, \qquad \omega_l(\mathrm{diag}(a,b)) = b.$$

Using (17),

$$\|[PD_lP, \mathrm{diag}(a,b)]\| = m_1|a-b|,$$

and hence

$$\boxed{d_{\mathrm{trunc}}(\omega_0,\omega_l) = \frac{1}{m_1} = \frac{2l}{\pi} < l.} \tag{20}$$

Minimality follows because there is no zero mode and the grading exchanges the eigenspaces at $+m_1$ and $-m_1$. A nonzero grading-preserving spectral subspace must retain both.

There is an essential distinction between these states and literal evaluations. For

$$g(x) = \cos(3\pi x/l),$$

equation (18) gives $Q(g) = 0$, while $g(0) = 1$ and $g(l) = -1$. Thus neither endpoint evaluation factors through Q.

The pullbacks of the truncated endpoint states are instead the probability measures

$$d\mu_L = \frac{2}{l}\cos^2\frac{\pi x}{2l}\,dx, \qquad d\mu_R = \frac{2}{l}\sin^2\frac{\pi x}{2l}\,dx.$$

Their distance using the **original** continuum seminorm is

$$d_{D_l}(\mu_L,\mu_R) = \int_0^l \left|\int_0^x (d\mu_L - d\mu_R)\right| dx = \frac{4l}{\pi^2}. \tag{21}$$

The equality follows by integration by parts and is attained by $f = -x$. Equations (10), (20), and (21) concern three different state/metric comparisons.

Status: PROVED

Part 4. The two limits

Amendment (Terra 6.1, 3 Nov 2026; report). The statements of this part are correct for the stated bare chain, with one precision: the condition N/l -> infinity gives uniform convergence of the dispersion on bounded energy windows and of normalized heat traces and counting densities, and does not by itself give convergence of the unnormalized difference of heat traces (for N = floor(l^(5/4)) it diverges). Escapes for the empty light sector: a tuned bulk mass or a scaled boundary coupling.

Statement. At fixed finite rank, $m \to 0$ produces infinitely separated finite geometry, with heat dimension zero. A resolved interval requires a growing tower. As $m \to \infty$, the metric collapses, but the light Hilbert space is zero.

Hypotheses. Identify

$$m = m_1 = \frac{\pi}{2l},$$

use normalization (1), and hold physical energies or heat times fixed unless stated otherwise.

Proof.

For fixed N, $D_N = mT_N$ with T_N independent of m. Therefore, as $m \to 0$,

$$D_N \to 0, \qquad K_N(t) \to 2(N-1), \qquad d_N(i,j) \to \infty \quad (i \neq j).$$

For the minimal truncation,

$$K_2(t) = 2e^{-tm^2}.$$

Every finite N has ultraviolet heat dimension zero. Making its finitely many modes light does not change that conclusion.

For the full interval, a fixed energy E is in the resolved one-dimensional regime when

$$\frac{1}{l} \ll E.$$

For its finite-chain approximation, one additionally needs

$$\boxed{\frac{1}{l} \ll E \ll \frac{N}{l}.} \tag{22}$$

Thus, as $l \to \infty$, an exact continuum limit throughout a fixed positive energy window follows if $N/l \to \infty$. Equivalently, as $m \to 0$, one needs $Nm \to \infty$.

The continuum counting function and heat density make the emergence explicit:

$$\#\{|\lambda| \le E\} = 2\left\lfloor \frac{lE}{\pi} + \frac{1}{2} \right\rfloor, \qquad \frac{K_l(t)}{l} \longrightarrow \frac{1}{\sqrt{\pi t}} \quad (l \to \infty). \tag{23}$$

Based at an endpoint, the metric limit is a half-line; the opposite endpoint recedes to infinity.

The order of limits matters even for the normalized heat trace:

$$\lim_{l\to\infty}\lim_{N\to\infty} \frac{K_N(t)}{l} = \frac{1}{\sqrt{\pi t}}, \qquad \lim_{N\to\infty}\lim_{l\to\infty} \frac{K_N(t)}{l} = 0. \tag{24}$$

Likewise, taking $t \to 0$ at fixed N gives dimension zero, whereas taking $N \to \infty$ first gives the interval's dimension one.

As $m \to \infty$, all distances tend to zero. At fixed N, the algebra elements with uniformly bounded commutators reduce to constants. But every eigenvalue has magnitude at least m, so

$$\|(D_N - i)^{-1}\| \le \frac{1}{\sqrt{m^2+1}} \longrightarrow 0. \tag{25}$$

The same holds for D_l, after identifying the varying intervals by dilation. Its heat trace tends to zero at every fixed $t > 0$.

Thus the metric limit is a point, while the embedded resolvent limit has

$$H_{\text{light}} = \ker T_N = 0.$$

There is no finite operator limit on the original nonzero Hilbert space, and no surviving fermion pair at fixed energy.

Finally, the two-state spectral truncation exists for

$$m_1 \le E < 3m_1.$$

A genuine spectral cutoff $E \ll 1/l$ retains no modes. The ratio between the first two positive levels is always 3, so the minimal pair has no parametrically large separation from the next level.

Status: PROVED

Verdict on hypothesis H (theorem T1)

Supported for T1's geometric assertion. Connes–Lott is exactly the minimal nonzero grading-preserving spectral compression of the opposite-chirality interval above, with

$$l = \frac{\pi}{2|m|}.$$

The chain recovers its spectrum, heat window, and metric.

The precise replacement for the stronger interpretation of H is:

> The two sheets are truncated endpoint evaluation states, whose pullbacks are smeared over the interval. Refinement restores the interval. Decompactification produces a one-dimensional heat regime only when the number of retained modes grows sufficiently rapidly. Collapse identifies the metric points but removes all fermionic modes of this particular interval at fixed energy.

Accordingly, a nonempty two-state sector at $E \ll 1/l$, or surviving light fermions after collapse, is disproved by this model. The construction supplies no KO-6 or Standard Model identification.

Independent numerical check (Claude, 2 Nov 2026)

For l = 1 and N = 2, 3, 5, 9 we checked the following. The spectrum of D_N agrees with +/- 2 h_N sin((2r-1) theta_N) to 1e-14. The lowest positive eigenvalue equals pi/(2l) for every N. For random real site functions, ||[D_N, pi_N(a)]|| equals h_N max abs(a_{i+1} - a_i), which is the identity behind d_N(i,j) = abs(i-j)/h_N. The endpoint distance d_N(1,N) = 0.6366, 0.7869, 0.8844, 0.9398 tends to l. For the minimal truncation, Q(cos(3 pi x/l)) = 0, and the smeared-state distance 0.405290 matches 4l/pi^2. The truncated distance is 2l/pi = 0.6366, identical to d_2(1,2).

Literature (../LITERATURE.md, section A). The Kaluza-Klein reading of the two-point space is in Connes-Lott (sections I and III: "two copies of Z separated by a distance L"); the two points as points of a fifth dimension are in Viet-Wali 1996; the two-point model as an approximation of a manifold, to be related to dimensional reduction, is in Dimakis-Muller-Hoissen 1994 (section III, Example 4, and section VII). Connes-van Suijlekom (section 1) note that compressions P A P are operator systems, of which the chiral interval is an exception. The truncation theorem with |m| = pi/(2l) was not found in print.

Part 06

The proposed uniqueness is false: **the same KO-6 triple can admit both reductions**. Moreover, under the stated definition, the strict heavy-limit triple is a doubling of KO-7, although it is not a doubling of KO-5. The distinction between the full real Hilbert space and its physical particle subspace is essential.

1. Lemma — Product convention and necessary correction of the "Clifford point."

Under ordinary spectral-triple conventions, a graded triple on `C^2` is not itself an odd KO-1 or KO-7 triple. The appropriate interpretation is an odd point

$$P_k = (\mathbb{C}, \mathbb{C}, 0, K), \qquad k = 1 \text{ or } 7,$$

together with the auxiliary `C^2` introduced by the odd-times-odd product. For real algebras, use the point algebra `R`, so that the product algebra remains `A`.

I use the following products, with the standard antilinear tensor product of real structures.

For even × even,

$$D = D_1 \otimes 1 + \gamma_1 \otimes D_2, \qquad \gamma = \gamma_1 \otimes \gamma_2,$$
$$J = J_1 \otimes (J_2 \gamma_2^b), \qquad b = (1 - \epsilon_1'')/2.$$

For even × odd, use the same formula for `D`, no grading, and

$$J = (J_1\gamma_1^a) \otimes J_2, \qquad (-1)^a = \epsilon_1''\epsilon_2'.$$

Odd × even is obtained by transporting the even-first construction through the tensor flip.

For odd × odd,

$$H = H_1 \otimes H_2 \otimes \mathbb{C}^2, \quad D = D_1 \otimes 1 \otimes \sigma_1 + 1 \otimes D_2 \otimes \sigma_2, \quad \gamma = 1 \otimes 1 \otimes \sigma_3,$$

$$J = J_1 \otimes J_2 \otimes (BK),$$

where

$(\epsilon_1', \epsilon_2')$	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
B	σ_1	1	σ_3	$-i\sigma_2$.

This agrees, up to harmless phases, with Dąbrowski–Dossena, equation (25), Table 6.

Proof. In the odd-odd case,

$$B\sigma_1 = \epsilon_1'\sigma_1 B, \qquad B\sigma_2 = -\epsilon_2'\sigma_2 B.$$

The second minus sign compensates for complex conjugation of `sigma_2`. Consequently `JD=DJ`; computing `J^2` and `J gamma` gives the signs of the sum degree. The grading twists in the other two formulas follow by the same calculation. Order zero and first order pass to these products.

Thus the intended doubling has Hilbert space `H_0 ⊗ C^2`, with the displayed grading on the auxiliary factor. All subsequent statements use this corrected interpretation and the zero-Dirac point. ■

2. Theorem — Exact doubling criterion in KO-dimension six.

Let `(A,H,D,J,gamma)` be a finite KO-6 triple. For `s=±1`, the following are equivalent:

1. It is a product `T_n × P_k`, where

$$(s, n, k) = (+1, 7, 7) \quad \text{or} \quad (-1, 5, 1).$$

2. There is a complex-linear unitary `U` satisfying

$$U^2 = 1, \quad U\gamma = -\gamma U, \quad [U, a] = [U, a^o] = [U, D] = 0, \quad UJ = sJU,$$

   where `a^o=Ja^\*J^{-1}`.

Write `H_L=H_{\gamma=+1}`, `H_R=H_{\gamma=-1}`, and

$$D = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}.$$

Equivalently, there is an `A`-bimodule unitary `V:H_L→H_R` such that

$$V^*M = (V^*M)^*, \qquad (V^*C)^2 = s, \quad C = J|_{H_L} : H_L \longrightarrow H_R.$$

Here `C` and `V^\*C` are antilinear.

Proof. Given `U`, identify

$$H_L \oplus H_L \longrightarrow H, \qquad (x, y) \longmapsto x + Uy.$$

Put

$$H_0 = H_L, \qquad d = (UD)|_{H_L}, \qquad j = (UJ)|_{H_L}.$$

The assumptions give

$$d = d^*, \qquad j^2 = s, \qquad jd = s\,dj.$$

Hence `(j^2,jd/dj)` is `(+,+)` for KO-7 and `(-,-)` for KO-5. In these coordinates,

$$a = \begin{pmatrix} a_0 & 0 \\ 0 & a_0 \end{pmatrix}, \quad D = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix}, \quad \gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$J(x,y) = (sjy, jx).$$

These are precisely the products in Result 1: `B=σ_1` for `(7,7)` and `B=-iσ_2` for `(5,1)`.

The descended opposite action is

$$ja_0^* j^{-1} = a^o|_{H_L}.$$

Since `[d,a_0]=U[D,a]|_{H_L}`, order zero and first order descend. Conversely, the product has `U=1⊗σ_1`, with all the required properties.

Finally,

$$U = \begin{pmatrix} 0 & V^* \\ V & 0 \end{pmatrix}.$$

Its commutation with `D` is equivalent to self-adjointness of `V^\*M`, while `UJ=sJU` is equivalent to `(V^\*C)^2=s`.

If one prefers an anticommuting convention, replacing `U` by `i gamma U` exchanges `[U,D]=0` with `{U,D}=0` and preserves `s`. ■

In particular, ordinary equality of left and right module multiplicities is only part of the criterion; compatibility with `D` and `J` must also be checked.

3. Counterexample — J does not select a unique reduction; electrodynamics admits both.

Consider the electrodynamics triple in the basis `(e_L,e_R,bar e_L,bar e_R)`. After a phase change compatible with `A,J,gamma`, its matrices are

$$\rho(a,b) = \mathrm{diag}(a,a,b,b), \qquad \gamma = \mathrm{diag}(1,-1,-1,1),$$

$$J = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} K, \qquad D = m \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \quad m \geq 0.$$

These are the input data of van den Dungen–van Suijlekom, §4.1, equations (4.1)–(4.2).

Define

$$U_+ = \mathrm{diag}(\sigma_1, \sigma_1), \qquad U_- = \mathrm{diag}(\sigma_1, -\sigma_1).$$

Proof. Both involutions reverse the grading and commute with the algebra, opposite algebra, and `D`. Direct multiplication gives

$$U_+ J = JU_+, \qquad U_- J = -JU_-.$$

Thus the same triple, with the same `J`, admits both factorizations.

On `H_0=span(e_L,bar e_R)`, the two descended triples have

$$d_+ = mI_2, \qquad j_+ = \sigma_1 K$$

and

$$d_- = \mathrm{diag}(m,-m), \qquad j_- = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} K.$$

They have KO-degrees seven and five, respectively. This also holds at `m=0`. ■

4. Lemma — Exact neutral-sector criterion, including Majorana masses.

Suppose a KO-6 sector has

$$H = H_+ \oplus H_-, \qquad \dim H_+ = \dim H_- = r,$$

and both algebra actions are the same scalar character throughout this sector. Then:

- a KO-7 reduction always exists;
- a KO-5 reduction exists exactly when **every singular value of `M`, including zero, has even multiplicity**.

Proof. Choose an orthonormal basis of `H_+` and its `J`-image as the basis of `H_-`. Then

$$J = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} K.$$

The equation `JD=DJ` says `M^T=M`. Takagi factorization, implemented by a basis change preserving this form of `J`, reduces `M` to

$$\Sigma = \mathrm{diag}(\mu_1, \dots, \mu_r), \qquad \mu_i \geq 0.$$

Taking `V=I` satisfies Result 2 with `s=+1`.

For `s=-1`, the descended antiunitary `j` squares to `-1` and commutes with `d^2=D^2|_{H_+}`. Every eigenspace of `d^2` must therefore have even complex dimension. These dimensions are exactly the singular-value multiplicities.

Conversely, pair equal entries of `Sigma` and choose `V=σ_2` on each pair. Then `V` is a skew-symmetric Hermitian unitary commuting with `Sigma`; it satisfies the criterion with `s=-1`. ■

A single neutral Weyl state and its conjugate therefore **are** a KO-7 doubling, even without a Dirac partner. They cannot be a KO-5 doubling. The same statement holds for a single Majorana pair, whether its mass vanishes or not.

5. Proposition — The strict one-generation heavy limit is a KO-7 doubling, and KO-5 is obstructed.

Here is an explicit convention for the Standard Model representation. Set

$$F = \mathbb{C} \oplus \mathbb{C}^3, \qquad H_p = (\mathbb{C}^2 \otimes F)_L \oplus (\mathbb{C}^2 \otimes F)_R, \qquad H = H_p \oplus H_{\bar{p}}.$$

For `a=(lambda,q,m)`, put

$$Q_\lambda = \mathrm{diag}(\lambda, \bar{\lambda}), \qquad B_{\lambda,m} = \mathrm{diag}(\lambda, m),$$
$$\rho_p(a) = (q \otimes I_4) \oplus (Q_\lambda \otimes I_4),$$
$$\rho_{\bar{p}}(a) = (I_2 \otimes B_{\lambda,m}) \oplus (I_2 \otimes B_{\lambda,m}).$$

Use

$$J(p, \bar{p}) = (K\bar{p}, Kp), \qquad \gamma = \mathrm{diag}(I_8, -I_8, -I_8, I_8).$$

On particles, the Dirac block is

$$S_p = \begin{pmatrix} 0 & Y^* \\ Y & 0 \end{pmatrix},$$

where `Y` has entries `y_nu,y_e,y_u I_3,y_d I_3` on the corresponding species. The antiparticle block is `bar S_p`; the additional Majorana block connects `nu_R` with `J nu_R`. This realizes the bimodule and mass-block structure in CCM, Proposition 2.4 and Theorem 2.21.

Assume precisely the strict projection described in canonical §3: remove `nu_R,J nu_R` and the complete up-type Dirac sector, and restrict to

$$B = \{(\lambda, Q_\lambda, m)\} \simeq \mathbb{C} \oplus M_3(\mathbb{C}).$$

The surviving space consists of `d_L,d_R,e_L,e_R,nu_L` and their conjugates, with `D_light=PSP`.

Proof. The surviving down-quark and electron masses pair identical `B`-bimodules. After making their masses nonnegative, each sector has both involutions from Result 3, with the identity on color.

On

$$N = \mathrm{span}(\nu_L, J\nu_L),$$

both algebra actions are the scalar character `lambda`, and `PSP|_N=0`. Thus

$$\gamma_N = \sigma_3, \qquad J_N = \sigma_1 K, \qquad U_N = \sigma_1$$

gives a KO-7 reduction. Combining this with `U_+` on the charged sectors gives a global KO-7 reduction.

A KO-5 reduction is impossible: the full space has dimension eighteen, so its descended space would have dimension nine, which cannot carry an antiunitary squaring to `-1`. More specifically, the neutral bimodule is inequivalent to the charged bimodules, so any permitted `U` preserves it; Result 4 already obstructs KO-5 there. ■

Thus the anticipated claim "the neutral sector prevents any doubling" is false under the definition in Question 1.

6. Proposition — Modifications and the distinction from physical vectorlikeness.

For the reduced algebra `B` above, the following conclusions hold.

Surviving neutral content	Odd reductions of the full light triple
No neutral sector	Both KO-5 and KO-7
`nu_L,nu_R` and conjugates, pure Dirac mass	Both KO-5 and KO-7
Only `nu_L,J nu_L`	KO-7 only
A single isolated Majorana pair	KO-7 only
General neutral Dirac–Majorana block	Always KO-7; KO-5 exactly under Result 4

Proof. The charged sectors admit both reductions by Result 3. Pure Dirac neutrinos have the same four-state construction, so adding or retaining `nu_R` restores both possibilities, including when its Dirac mass is zero.

For one generation with Dirac mass `m` and right Majorana mass `R`, the neutral symmetric matrix can be written

$$M_N = \begin{pmatrix} 0 & m \\ m & R \end{pmatrix}.$$

Phase changes make `m,R` nonnegative. Its singular values are

$$\mu_\pm = \frac{\sqrt{|R|^2 + 4|m|^2} \pm |R|}{2}.$$

They coincide exactly when `R=0`. Hence any nonzero right Majorana mass obstructs KO-5 in this one-generation neutral sector, but never KO-7. Treating that sector separately leaves a charged KO-5 reduction; it does not turn the complete direct sum into a KO-5 reduction. ■

There is an additional physically meaningful condition absent from Question 1. Let `Q` be `+1` on particles and `-1` on antiparticles.

No-go theorem under the extra hypothesis `[U,Q]=0`. An unpaired `nu_L` prevents doubling.

Proof. Such a `U` must map particle states of positive grading to particle states of negative grading within the same bimodule. No particle `nu_R` exists. The neutral involution used in Result 5 instead maps `nu_L` to its conjugate and violates `[U,Q]=0`. ■

This is why physical chirality and doubling of the complete real triple are different tests.

7. Proposition — Three generations and arbitrary flavor mixing.

Assume the reduced algebra acts trivially on generation indices.

If all three right-neutrino Majorana pairs are removed and only the top Dirac sector is made heavy, the light space has

$$96 - 6 - 12 = 78$$

complex dimensions. If all three up-type sectors are removed, it has dimension fifty-four. In either case, with three surviving left neutrinos and `D_light=PSP`, the light triple admits KO-7 and excludes KO-5.

Dropping the remaining neutral sector, or retaining three right neutrinos with purely Dirac neutrino masses, permits both reductions.

Proof. Each surviving charged species has equal left/right flavor multiplicities. Singular-value decomposition of its mass matrix uses bimodule unitaries; conjugate transformations on antiparticles preserve `J`. Consequently every charged sector reduces to copies of Result 3, irrespective of mass rank or flavor mixing.

The three unpaired neutral states give `r=3`, `M_N=0` in Result 4, excluding KO-5 and admitting KO-7. Equivalently, the proposed descended spaces have odd dimensions thirty-nine and twenty-seven.

With three right neutrinos retained, the general neutral matrix is

$$M_N = \begin{pmatrix} 0 & Y_\nu^T \\ Y_\nu & R \end{pmatrix}, \qquad R^T = R.$$

For `R=0`, every singular value of `Y_nu` appears twice, so both reductions exist. For general `R`, KO-7 always exists, while KO-5 exists exactly when every singular value of this six-by-six matrix has even multiplicity. Thus nonzero Majorana masses are not an unconditional obstruction in several generations; the exact obstruction is the stated multiplicity condition. ■

8. No-go theorem — The massless Standard Model triple is not a doubling.

Let the algebra remain the full

$$A = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$$

with the representation of Result 5, for `g≥1` generations. Then no involution satisfying Result 2 exists, for any admissible `D`. In particular this applies at `v=0`, when only the Majorana block remains.

Proof. Consider the central projection

$$e_{\mathbb{H}} = (0, 1_{\mathbb{H}}, 0).$$

Its represented range lies entirely in the particle left-handed sector:

$$\dim \rho(e_{\mathbb{H}})H_+ = 8g, \qquad \dim \rho(e_{\mathbb{H}})H_- = 0.$$

An algebra-linear grading-reversing unitary would identify these spaces, which is impossible.

In physical representation-ring notation,

$$V_L = (3,2)_{1/6} \oplus (1,2)_{-1/2},$$

$$V_R = (3,1)_{2/3} \oplus (3,1)_{-1/3} \oplus (1,1)_{-1} \oplus (1,1)_0.$$

Thus `I_G=[V_L]-[V_R]≠0`. For the complete real graded Hilbert space, the relevant gauge character is

$$[H_+] - [H_-] = I_G - \overline{I_G},$$

not merely `I_G`. It too is nonzero: the coefficient of `(3,2)\_{1/6}` is `g`. Either this full-character calculation or the central-projection argument proves the obstruction. ■

Consequence for the 10 -> 9 claim

9. Proposition — What dimension reduction is actually established.

The heavy projection itself preserves KO-degree six, as canonical §3 explicitly states. Hence its spacetime product still has degree `4+6=10 mod 8`.

A subsequent choice of a KO-5 reduction removes a degree-one Clifford factor and gives the formal representative `4+5=9`. Such a reduction exists for the charged truncation or the pure-Dirac completion, but **both also admit KO-7 reductions**. Their fixed `J` does not select nine.

For the strict light spectrum with one or three unpaired left neutrinos, **KO-5 is obstructed; only KO-7 is available among the proposed reductions**. This gives the formal representative eleven. If particle preservation is additionally required, that strict light spectrum admits neither reduction.

Proof. These are Results 2–7. For completeness, the spacetime real structures are

$$J_{4+5} = (J_M \gamma_M) \otimes j_5, \qquad J_{4+7} = J_M \otimes j_7.$$

Thus even the odd reductions require different product conventions. No finite-dimensional operation here changes heat dimension: a finite internal factor remains spectrally zero-dimensional. Consequently the claimed loss of one dimension is, at most, a chosen Clifford-factor removal—not a proved metric collapse, and not a consequence of vectorlike charged matter alone. ■ —

Independent numerical check (Claude, 2 Nov 2026)

We used the electrodynamics data of Result 3, with m = 0.7, J = [[0, I_2],[I_2, 0]] K and gamma = diag(1,-1,-1,1). The triple has J^2 = 1, JD = DJ and J gamma = -gamma J, so its KO-dimension is 6. Both U_+ = diag(sigma_1, sigma_1) and U_- = diag(sigma_1, -sigma_1) satisfy U^2 = 1 and U gamma = -gamma U. Both commute with random algebra elements, their opposites and D; and U_+ J = J U_+, U_- J = -J U_-. On H_0 = span(e_L, bar e_R), U_+ descends to j^2 = +1 with jd = dj (KO-7) and U_- to j^2 = -1 with jd = -dj (KO-5). For No-go 8 we also rechecked by hand the dimension count dim rho(e_H) H_+ = 8g, dim rho(e_H) H_- = 0 in the representation of Result 5.

Terminology. Here "doubling" means the factorization of an even triple as an odd triple times a Clifford point (the doubling of an odd triple in Dabrowski-Dossena's product construction). It is unrelated to the "fermion doubling" of the noncommutative Standard Model (Lizzi-Mangano-Miele-Sparano). Prior work on products and on J versus J gamma: ../LITERATURE.md, section D; the factorization criterion and the 9/11 results were not found in print.

Referee note

Terra 6.1 (3 Nov 2026; report) checked Result 3, Lemma 4, Proposition 5, Result 7, No-go theorem 8 and the no-go under [U, Q] = 0: all correct. One amendment to Result 7: "arbitrary flavor mixing" means arbitrary matrices within the displayed species blocks, with the heavy projection chosen onto the specified top Dirac sector; mixing between inequivalent species is outside the displayed Dirac operator. Escapes: for KO-5, neutral-sector pairing or a symmetry enforcing even multiplicities; for No-go 8, a change of algebra or representation, the even description, and boundary or defect realizations of chirality; for the no-go under [U, Q] = 0, a matching particle bimodule of opposite grading.

Part 07

Does the chiral interval select 9 or 11?

Questions 1-3 of prompts/07. The Astra job stopped on the codex usage limit before producing output, so these are Claude Opus 5.5's proofs, checked by hand. Astra job 08 confirmed Lemmas 1-3 under the pointwise reality hypothesis of Lemma 1 and answered Question 4: see lemma 08 and the amendments below.

Let I_l = [0, l], H = L^2(I_l, C^2) with psi = (L, R), D = E d/dx with E = [[0, 1], [-1, 0]], Dom D = {psi in H^1 : R(0) = 0, L(l) = 0}, Gamma = diag(1, -1) (job 05). D and Gamma are real.

Lemma 1 (real structures on the chiral interval). Let J be antiunitary with J f J^{-1} = conj(f) for f in C^infty(I_l), J Dom D = Dom D, J^2 = eps, JD = eps' DJ, J Gamma = eps'' Gamma J. Then J = W K with W = diag(alpha, beta) constant, abs(alpha) = abs(beta) = 1, and

```
eps = +1,   eps'' = +1,   alpha = eps' beta,   eps' in {+1, -1}.
```

Both values of eps' occur, and they are exchanged by J -> J Gamma. The chiral boundary conditions exclude every real structure anticommuting with Gamma; the bare line allows such ones, e.g. W = E with signs (-1, +1, -1).

Proof. Antilinearity together with J f = conj(f) J gives J = W K for a unitary matrix function W(x). Since E is real, D J psi = E W' conj(psi) + E W conj(psi'), while J D psi = W E conj(psi'). Hence eps' D J = J D forces E W' = 0, so W is constant, and W E = eps' E W. Preservation of R(0) = 0 for arbitrary L(0) forces W_21 = 0, and preservation of L(l) = 0 forces W_12 = 0. So W is diagonal and unitary. Then W conj(W) = 1 gives eps = +1, and W commutes with Gamma, so eps'' = +1. Finally diag(alpha, beta) E = [[0, alpha], [-beta, 0]] and E diag(alpha, beta) = [[0, beta], [-alpha, 0]], so alpha = eps' beta. Replacing J by J Gamma keeps eps = eps'' = +1 and flips eps'. ■

With Connes' even convention (eps' = +1) the chiral interval therefore has KO-dimension 0 mod 8, although its metric dimension is 1.

Lemma 2 (compression). Every J of Lemma 1 commutes with P = 1_{[0,E]}(abs(D)). On the truncation span(u_L, u_R) of job 05 it induces J_2 = diag(alpha, beta) K, with D_2 = m_1 sigma_1, Gamma_2 = sigma_3 and the signs (+1, eps', +1).

Proof. JD = eps' DJ gives J D^2 = D^2 J, so J commutes with abs(D) and with its spectral projections. The basis functions u_L = sqrt(2/l)(cos(pi x/2l), 0) and u_R = sqrt(2/l)(0, sin(pi x/2l)) are real, so J u_L = alpha u_L and J u_R = beta u_R. ■

Lemma 3 (the reflection fixes s = eps'). Let (sigma psi)(x) = a (R(l - x), L(l - x)). Then sigma preserves Dom D, anticommutes with Gamma and commutes with constants; [sigma, D] = 0 holds exactly when the phases on the two components agree, and sigma^2 = 1 forces a = +1 or -1. On the truncation, sigma = a sigma_1. For every J of Lemma 1,

```
sigma J = s J sigma   with   s = beta/alpha = eps'.
```

Proof. The swap with x -> l - x exchanges the conditions R(0) = 0 and L(l) = 0. With general phases a, b on the two components, (D sigma psi)_L = -b L'(l - x) and (sigma D psi)_L = -a L'(l - x), so commutation forces a = b. One checks sigma u_L = u_R and sigma u_R = u_L. Then sigma J psi = a (beta conj R(l - x), alpha conj L(l - x)) and J sigma psi = a (alpha conj R(l - x), beta conj L(l - x)), which gives s = beta/alpha = eps'. ■

Consequence. As the interval collapses, only constants keep bounded commutators, and the reflection becomes a canonical L <-> R involution. Its sign against J is s = eps', so the interval fixes s exactly as far as the real structure on it is fixed, and no further. The 9 versus 11 ambiguity of job 06 is the same as the standard ambiguity J versus J gamma for even real spectral triples (Dabrowski-Dossena). This matches job 06, Proposition 9: the two odd reductions use J_M gamma_M (for 4 + 5) and J_M (for 4 + 7) on spacetime. Under Connes' convention eps' = +1 the reflection gives s = +1, i.e. KO-7, formally 11.

Conjecture (Question 4; false as stated, see amendment 4). For F = F_0 x (collapsing chiral interval) with the product conventions of job 06, Lemma 1, the induced U = 1 (x) sigma satisfies UJ = sJU with s = eps' (interval) times a sign determined by the product grading twist. In particular, which of 9 and 11 is selected is fixed by choosing which spacetime real structure (J_M or J_M gamma_5) represents physical charge conjugation. A KO-0 interval does not change the KO-degree of a product, so the "Clifford point" of job 06 must come from the odd x odd product structure rather than from the interval alone; the open point is how the two descriptions match. Attempted by: Claude Opus 5.5 (Questions 1-3 above). The gpt-6-astra job was cut off by the usage limit before attempting it.

Amendments from job 08

1. Lemma 1 needs its hypothesis J f J^-1 = conj(f). Without it, J_ref = sigma_1 R_l K (R_l the reflection x -> l - x) preserves the chiral boundary conditions and has the KO-6 signs (Counterexample 08.4), so the chiral boundary conditions alone do not force KO-dimension 0.
2. Lemma 2 holds for the compressed operators. With the full compressed diagonal algebra and the inherited J_2, the two-point data fail the first-order condition: [[D_2, e], e°] = D_2 for e = diag(1, 0) (Counterexample 08.4).
3. The sign s = eps' in the Consequence above is the sign of the reflection against the interval's own J. In the standard internal-first KO-6 product the required grading twist changes it: the bare reflection

1 (x) sigma always has s = -1, formally 4 + 5 = 9, for either eps' (Theorem 08.5).
4. Question 4 (Conjecture 07.4) is false as stated: a fixed spacetime J_M and particle-number conservation admit both signs (Proposition 08.9, Counterexample 08.11, Theorem 08.12).

Part 08

Product real structures and the choice between 9 and 11

Prompt: prompts/08 (Task A: check Lemmas 07.1-07.3; Task B: Conjecture 07.4). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5: by hand (results 1-3, 4 second part, 8, 12) and numerically in Python (results 5, 6, 7, 11 and 8; section at the end).

Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): all 12 numbered statements are correct. Two points of scope: the case t = -1 in Result 1 uses the extended convention 0_-, since the standard KO-0 signs have t = +1; and Theorem 7 concerns the scalar-algebra truncation, since the full compressed two-sheet algebra with its inherited real structure fails first order (Counterexample 4).

Outcome. Lemmas 07.1-07.3 hold under the pointwise reality hypothesis. Conjecture 07.4 is false as stated: Counterexample 11 fixes the physical D_F, J_F, Gamma_F and particle number Q and still admits admissible involutions of both signs. What holds instead: in the standard internal-first KO-6 product, the bare interval reflection always has s = -1 (Theorem 5, formally 4+5 = 9); the product is exactly T_5(m) x P_1 with d_5 = D_0 + m Gamma_0 (Theorem 7); particle preservation pairs the two signs whenever [Q, D] = 0 (Theorem 12); and the literal collapse of the massless interval leaves no light sector (No-go theorem 8).

We use the sign convention

$$J^2 = \epsilon, \qquad JD = \epsilon' DJ, \qquad J\Gamma = \epsilon'' \Gamma J.$$

Unless stated otherwise, KO-6 means the standard signs $(+,+,-)$. We require the order-zero and first-order conditions when calling data a real spectral triple; we impose no orientability or Poincare-duality assumption.

1. Lemma (07.1: classification under pointwise reality).

Let

$$H_I = L^2([0,l],\mathbb{C}^2), \quad D_I = E\partial_x, \quad E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \Gamma_I = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with

$$\operatorname{Dom} D_I = \{(L,R) \in H^1 : R(0) = 0,\ L(l) = 0\}.$$

Suppose J_I is antiunitary, preserves this domain, and satisfies

$$J_I f J_I^{-1} = \overline{f}, \qquad J_I D_I = t D_I J_I, \quad t \in \{+1,-1\}.$$

Then

$$J_I = \operatorname{diag}(\alpha,\beta)K, \qquad |\alpha| = |\beta| = 1, \qquad \alpha = t\beta.$$

Consequently,

$$(\epsilon_I, \epsilon_I', \epsilon_I'') = (+,t,+).$$

Proof. The linear unitary $W = J_I K$ commutes with every scalar multiplication operator. Hence W is multiplication by a measurable unitary matrix $W(x)$.

Domain preservation, applied to compactly supported smooth sections, gives $W \in H^1_{\mathrm{loc}}$. Comparing the coefficients of derivatives in

$$WE\overline{\psi'} = tE(W'\overline{\psi} + W\overline{\psi'})$$

gives

$$WE = tEW, \qquad EW' = 0.$$

Thus W is constant. Preservation of the boundary conditions gives $W_{21} = W_{12} = 0$, because the remaining endpoint traces are arbitrary.

Writing $W = \mathrm{diag}(\alpha, \beta)$, the equation $WE = tEW$ becomes $\alpha = t\beta$. Finally,

$$W\overline{W} = 1, \qquad W\Gamma_I = \Gamma_I W.$$

These imply the asserted signs. Conversely, every displayed J_I satisfies all the hypotheses. □

Thus Lemma 07.1 is correct under its stated pointwise reality hypothesis. Up to phase, the two possibilities are K and $\Gamma_I K$. They are 0_+ and 0_- in the extended convention; Connes' even convention selects $t = +1$. See Dąbrowski–Dossena, arXiv v1, Table 1, p. 5.

2. Lemma (07.2: spectral compression).

Every J_I in Result 1 commutes with the spectral projections of $|D_I|$. On the minimal nonzero grading-preserving spectral subspace,

$$D_2 = m\sigma_1, \qquad \Gamma_2 = \sigma_3, \qquad J_2 = \mathrm{diag}(\alpha, \beta)K, \qquad m = \frac{\pi}{2l}.$$

Proof. The relation $J_I D_I = tD_I J_I$ implies that J_I preserves every eigenspace of D_I^2, hence every spectral subspace of $|D_I|$.

Solving $D_I\psi = \lambda\psi$ with the boundary conditions gives $\lambda = \pm(r - \frac{1}{2})\pi/l$. The lowest pair has the real orthonormal chiral basis

$$u_L = \sqrt{\frac{2}{l}}\begin{pmatrix}\cos(mx)\\ 0\end{pmatrix}, \qquad u_R = \sqrt{\frac{2}{l}}\begin{pmatrix}0\\ \sin(mx)\end{pmatrix}.$$

Direct differentiation gives $D_I u_L = mu_R$, $D_I u_R = mu_L$. Also $J_I u_L = \alpha u_L$, $J_I u_R = \beta u_R$. □

This proves Lemma 07.2 as a statement about the compressed operators. Result 4 below addresses the first-order condition.

3. Lemma (07.3: reflection sign).

Let $R_l\psi(x) = \psi(l - x)$. A component-exchanging reflection

$$\sigma = \begin{pmatrix}0 & a\\ b & 0\end{pmatrix} R_l, \qquad |a| = |b| = 1,$$

commutes with D_I exactly when $a = b$. Requiring additionally $\sigma^2 = 1$ gives $a = b = \pm 1$. For these reflections,

$$\sigma\Gamma_I = -\Gamma_I\sigma, \qquad \sigma J_I = tJ_I\sigma.$$

On the minimal truncation, $\sigma = \pm\sigma_1$.

Proof. Reflection exchanges the two endpoint conditions. Since $\partial_x R_l = -R_l\partial_x$, commutation with D_I requires

$$-E\begin{pmatrix}0 & a\\ b & 0\end{pmatrix} = \begin{pmatrix}0 & a\\ b & 0\end{pmatrix} E,$$

which is equivalent to $a = b$. Its square is then a^2, so $a = \pm 1$.

For real a, multiplication of the matrices representing σ and J_I gives

$$\sigma J_I = tJ_I\sigma$$

using $\alpha = t\beta$. The identities $\cos(m(l - x)) = \sin(mx)$ and $\sin(m(l - x)) = \cos(mx)$ give the compressed reflection. □

Thus all three lemmas agree with the direct calculation. Notice, however, that

$$\sigma f\sigma^{-1} = f(l - \cdot).$$

The reflection commutes with constants, but generally does not commute with the interval algebra.

4. Counterexample (two stronger interpretations of the interval lemmas fail).

First, chiral boundary conditions alone do not force KO-dimension zero. Without pointwise reality, the antiunitary

$$J_{\mathrm{ref}} = \sigma K, \qquad \sigma = \sigma_1 R_l,$$

has the standard KO-6 signs.

Second, the minimal two-point compression with its full diagonal algebra and inherited J_2 is not a real spectral triple satisfying the first-order condition when $m > 0$.

Proof. For the first assertion,

$$J_{\mathrm{ref}}^2 = 1, \qquad J_{\mathrm{ref}} D_I = D_I J_{\mathrm{ref}}, \qquad J_{\mathrm{ref}} \Gamma_I = -\Gamma_I J_{\mathrm{ref}}.$$

It implements

$$J_{\mathrm{ref}} f J_{\mathrm{ref}}^{-1} = \overline{f}(l - \cdot).$$

The opposite action is still scalar multiplication, so order zero and first order hold. This example violates precisely the pointwise reality hypothesis of Result 1.

For the second assertion, the compressed algebra is

$$A_2 = \{\mathrm{diag}(a, b)\}.$$

Indeed, compression sends 1 to 1 and $\cos(\pi x/l)$ to $\frac{1}{2}\sigma_3$. Let $e = \mathrm{diag}(1, 0)$. The inherited diagonal J_2 gives $e^o = e$, whereas

$$[[D_2, e], e^o] = D_2 \neq 0.$$

Moreover, $\sigma_1 e \sigma_1 = 1 - e$, so the reflection is not an A_2-module map. □

For the finite products below, we therefore explicitly restrict the interval algebra to scalar constants. This restriction makes the compressed interval a real triple and makes its reflection algebra-linear. It is an additional assumption about the proposed collapsed model.

5. Theorem (B1–B2: the internal-first KO-6 product).

Let

$$F_0 = (A_0, H_0, D_0, J_0, \Gamma_0)$$

be a finite real KO-6 triple:

$$J_0^2 = 1, \qquad J_0 D_0 = D_0 J_0, \qquad J_0 \Gamma_0 = -\Gamma_0 J_0.$$

Take either the interval or its scalar-algebra truncation, and use the internal-first product

$$D_F = D_0 \otimes 1 + \Gamma_0 \otimes D_I, \qquad \Gamma_F = \Gamma_0 \otimes \Gamma_I.$$

Among the grading-twisted tensor real structures

$$J_{a,b} = (J_0 \Gamma_0^a) \otimes (J_I \Gamma_I^b), \qquad a, b \in \{0, 1\},$$

the standard KO-6 conditions force

$$a = 0, \qquad t(-1)^b = -1.$$

For the induced reflection $U = 1 \otimes \sigma$,

$$\boxed{U J_F = -J_F U.}$$

Proof. Direct calculation gives

$$J_{a,b}^2 = (-1)^a, \qquad J_{a,b} \Gamma_F = -\Gamma_F J_{a,b}.$$

The two Dirac summands satisfy

$$J_{a,b}(D_0 \otimes 1) = (-1)^a (D_0 \otimes 1) J_{a,b},$$

$$J_{a,b}(\Gamma_0 \otimes D_I) = -t(-1)^b (\Gamma_0 \otimes D_I) J_{a,b}.$$

Thus $J_F^2 = 1$ requires $a = 0$, and $J_F D_F = D_F J_F$ requires $t(-1)^b = -1$.

Since $\sigma J_I = t J_I \sigma$ and $\sigma \Gamma_I = -\Gamma_I \sigma$,

$$U J_{a,b} = t(-1)^b J_{a,b} U.$$

The required product sign is therefore -1. □

Explicitly:

Interval $t = \epsilon'_I$	Required b	Product reflection sign s
$+1$	1	-1
-1	0	-1

For $t = +1$, this is exactly job 06's formula

$$J_F = J_0 \otimes (J_I \Gamma_I).$$

Changing J_I to $J_I \Gamma_I$ while adjusting the required twist leaves the resulting J_F unchanged, up to phase. The general formula $s = t(-1)^b$ is proved, but t and b are not independent choices when the product must have standard KO-6 signs. The relevant product constructions are Dąbrowski–Dossena, §4.1, equation (18) and Tables 2–3.

A KO-0 factor leaves the KO-degree unchanged. Hence F_0 must already have degree six in this ordinary even-even construction. An odd internal triple multiplied by an even interval remains odd under the standard product rules.

6. Proposition (B1–B2: the grading and the lift of reflection).

The preceding construction has three necessary distinctions.

1. The interval grading alone cannot be the total KO-6 grading for a tensor real structure of the displayed form.
2. Twisting the reflection by the internal grading can change s, but generally destroys commutation with D_F.
3. Transporting all data between equivalent product conventions preserves s.

Proof. If one sets $\Gamma_F = 1 \otimes \Gamma_I$, then every $J_{a,b}$ commutes with Γ_F, contradicting the required KO-6 relation. The product grading must be $\Gamma_0 \otimes \Gamma_I$.

For

$$U_c = \Gamma_0^c \otimes \sigma, \qquad c \in \{0, 1\},$$

we obtain

$$U_c J_F = (-1)^c \, t(-1)^b J_F U_c,$$

and

$$[U_c, D_F] = [\Gamma_0^c, D_0] \otimes \sigma.$$

Thus U_0 always commutes with D_F. For $c = 1$,

$$[U_1, D_F] = 0 \quad \Longleftrightarrow \quad D_0 = 0,$$

because $\{\Gamma_0, D_0\} = 0$. When $D_0 = 0$, both lifts are admissible and have opposite signs.

To compare product conventions, let

$$W = P^0_+ \otimes 1 + P^0_- \otimes \Gamma_I, \qquad P^0_\pm = \frac{1 \pm \Gamma_0}{2}.$$

Then

$$W D_F W = D_0 \otimes \Gamma_I + 1 \otimes D_I,$$

while

$$W(1 \otimes \sigma)W = \Gamma_0 \otimes \sigma.$$

Unitary transport preserves the relation $UJ = sJU$. Keeping the bare symbol $1 \otimes \sigma$ after changing convention can therefore mean choosing a different involution. □

The interval can still carry the physical L/R label. For example, if $\Gamma_0 = Q_0$ distinguishes particles from antiparticles, then

$$\Gamma_F = Q_0 \otimes \Gamma_I$$

has the usual reversal of finite grading between a particle and its conjugate.

Multiplying the total real structure by Γ_F also flips s, but changes its signs to $(-,-,-)$. It therefore leaves the standard KO-6 convention required by job 06.

7. Theorem (B1: exact reconciliation with the odd Clifford point).

For the scalar-algebra minimal truncation, choose the harmless overall phase so that

$$D_F = D_0 \otimes 1 + m\Gamma_0 \otimes \sigma_1, \quad \Gamma_F = \Gamma_0 \otimes \sigma_3, \quad J_F = J_0 \otimes (\sigma_3 K).$$

With $U = 1 \otimes \sigma_1$, this triple is exactly the odd-odd product

$$\boxed{F \cong T_5(m) \times P_1,}$$

where

$$T_5(m) = (A_0, H_0,\ d_5 = D_0 + m\Gamma_0,\ j_5 = J_0\Gamma_0),$$

and

$$P_1 = (\mathbb{C}, \mathbb{C}, 0, K).$$

For real algebras, use the scalar point algebra $\mathbb{R}$.

Proof. The operator d_5 is self-adjoint, and

$$j_5^2 = -1, \qquad j_5 d_5 = -d_5 j_5.$$

Its opposite action is unchanged, and $[d_5, a] = [D_0, a]$, so order zero and first order descend.

Let $e_\pm$ be the eigenvectors of σ_3, and define

$$R : H_0 \longrightarrow H_{F,+}, \qquad Rx = P^0_+ x \otimes e_+ + P^0_- x \otimes e_-.$$

This is unitary onto the positive total-grading subspace. Direct calculation gives

$$R^* U D_F R = D_0 + m\Gamma_0, \qquad R^* U J_F R = J_0 \Gamma_0.$$

Under the unitary identification

$$(x, y) \longmapsto Rx + URy,$$

the operators become

$$D_F = \begin{pmatrix} 0 & d_5 \\ d_5 & 0 \end{pmatrix}, \quad \Gamma_F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_F(x, y) = (-j_5 y, j_5 x).$$

These are precisely the odd-odd product matrices with auxiliary matrix $B = -i\sigma_2$. □

More generally, any admissible involution in a standard KO-6 triple gives

$$d = (UD)|_{H_+}, \qquad j = (UJ)|_{H_+},$$

with

$$j^2 = s, \qquad jd = s\,dj.$$

Thus

$$s = -1: \quad F \cong T_5 \times P_1, \qquad s = +1: \quad F \cong T_7 \times P_7.$$

The auxiliary matrices are respectively $-i\sigma_2$ and σ_1, in agreement with Dąbrowski–Dossena, equation (25), Table 6, p. 12.

The even KO-0 interval is therefore not identified with an odd point. Its two-dimensional space becomes the auxiliary doubling space after reorganizing the complete product, while its mass enters d_5. The point itself has one-dimensional Hilbert space and zero Dirac operator. Because its Dirac operator is zero, the distinction between P_1 and P_7 includes a choice of KO label.

8. No-go theorem (literal collapse produces no light finite sector).

Assume the heavy-limit model uses exactly the product Dirac operator of Result 5, with $m = \pi/(2l)$. At every fixed energy cutoff Λ, its light subspace is zero for sufficiently small l.

Proof. Anticommutation of D_0 and Γ_0 gives

$$D_F^2 = D_0^2 \otimes 1 + 1 \otimes D_I^2.$$

The chiral interval and its minimal truncation satisfy

$$D_I^2 \geq \frac{\pi^2}{4l^2}.$$

Hence

$$|D_F| \geq \frac{\pi}{2l}, \qquad \mathbf{1}_{[0,\Lambda]}(|D_F|) = 0 \quad \text{if } l < \frac{\pi}{2\Lambda}.$$

The same conclusion holds for the alternative product convention by unitary equivalence. □

Consequently, the exact product supplies an algebraic model at finite l, but cannot yield a nonempty light vectorlike sector by literal collapse at fixed energy. Retaining such a sector requires additional dynamics or a modified limiting prescription.

9. Proposition (B3: spacetime grading twists do not select the finite involution).

Let the four-dimensional Euclidean spacetime triple satisfy

$$J_M^2 = -1, \qquad J_M D_M = D_M J_M, \qquad J_M \gamma_M = \gamma_M J_M.$$

For an odd reduction (d, j_s) with

$$j_s^2 = s, \qquad j_s d = s\,d j_s,$$

the job-06 even-odd convention requires

$$J_{M\times T_s} = (J_M \gamma_M^{a_s}) \otimes j_s, \qquad (-1)^{a_s} = s.$$

Therefore

$$J_{4+5} = (J_M \gamma_M) \otimes j_5, \qquad J_{4+7} = J_M \otimes j_7.$$

These formulas do not imply that choosing physical spacetime charge conjugation selects s.

Proof. For

$$D_{M\times T_s} = D_M \otimes 1 + \gamma_M \otimes d,$$

the proposed real structure has commutation signs $(-1)^{a_s}$ and s against the two summands. They must agree. Its square is $-s$, giving KO-1 for $s = -1$ and KO-3 for $s = +1$.

We can also retain the removed point and compare the complete products. Write

$$F = T_s \times P_k, \qquad J_F = j_s \otimes (B_s K),$$

where $B_- = -i\sigma_2$, $B_+ = \sigma_1$. The unitary

$$V = P_+^M \otimes 1 \otimes 1 + P_-^M \otimes 1 \otimes \sigma_1$$

identifies $(M \times T_s) \times P_k$ with $M \times F$. In particular,

$$V((J_M \gamma_M^{a_s}) \otimes j_s \otimes B_s K)V^* = J_M \otimes J_F.$$

Here the extra factor $\gamma_M^{a_s}$ cancels because $\sigma_1 B_s = s B_s \sigma_1$. □

Thus both reconstructions can use the same original J_M. The different spacetime factors in the reduced products are required by the product rules.

10. Proposition (B3: Euclidean and Lorentzian charge conjugation).

In the conventions used by Chamseddine–Connes–Marcolli and van den Dungen–van Suijlekom, the Euclidean spacetime charge-conjugation operator is J_M, with

$$(J_M^2, \epsilon'_M, \epsilon''_M) = (-, +, +).$$

The operator $J_M\gamma_5$ is its other even real-structure convention. Lorentzian charge conjugation requires changing signature and is not obtained merely by choosing between these two Euclidean operators.

Proof and source identification. CCM state, "The operator J is the charge conjugation operator," in §3.1, and use $J = J_M \otimes J_F$ in §3.2. Their calculation in §3.5.2 explicitly uses $J_M D_M = D_M J_M$. See CCM, arXiv v1, §§3.1–3.2, p. 20, and §3.5.2, p. 27.

Van den Dungen–van Suijlekom likewise identify J_M as "the charge conjugation operator on the spinors" in Example 2.1 and use $J = J_M \otimes J_F$ in Definition 2.3. Their fermionic action uses this J_M. See the published paper, pp. 435–436 and Proposition 4.3, p. 452.

For an explicit signature comparison, take Lorentzian signature $(+---)$, chiral gamma matrices, and

$$\gamma_E^0 = \gamma_L^0, \qquad \gamma_E^j = -i\gamma_L^j.$$

Up to overall phases, standard conjugations are

$$C_E = i\gamma_E^0\gamma_E^2 K, \qquad C_L = -i\gamma_L^2 K.$$

These are the conventions in D'Andrea–Kurkov–Lizzi, arXiv v2, equations (3.4), (3.13)–(3.15).

The Clifford relations give

$$C_E^2 = -1, \quad [C_E, \gamma_5] = 0, \quad [C_E, D_E] = 0, \qquad D_E = -i\gamma_E^\mu \nabla_\mu,$$

whereas

$$C_L^2 = 1, \qquad C_L\gamma_5 = -\gamma_5 C_L.$$

In the displayed common matrix basis,

$$C_L = -i\gamma_E^0 C_E.$$

Both C_E and $C_E\gamma_5$ square to -1 and preserve Euclidean chirality; neither is already C_L.

Finally, for real m, Lorentzian charge conjugation carries

$$(i\gamma_L^\mu(\partial_\mu + ieA_\mu) - m)\psi = 0$$

to the same equation with charge $-e$. Composing C_L with γ_5 instead reverses the scalar mass sign. Thus the usual massive Dirac-field identification distinguishes charge conjugation once the signature and Dirac-operator conventions are fixed. □

This identification determines the starting spacetime convention. By Result 9, it does not determine the finite involution.

11. Counterexample (fixed physical real structure permits both 9 and 11).

There is a single scalar-interval product with fixed D_F, J_F, Γ_F, Q admitting particle-preserving involutions of both signs.

Proof. Let τ_i act on an internal particle/antiparticle space and σ_i on the interval truncation. Set

$$\rho(a,b) = \mathrm{diag}(a,b) \otimes 1, \qquad D_F = m\tau_3 \otimes \sigma_1,$$

$$\Gamma_F = \tau_3 \otimes \sigma_3, \qquad J_F = (\tau_1 \otimes \sigma_3)K, \qquad Q = \tau_3 \otimes 1, \quad m > 0.$$

This is Result 5 with

$$D_0 = 0, \quad \Gamma_0 = \tau_3, \quad J_0 = \tau_1 K.$$

It has standard KO-6 signs. Its Dirac operator commutes with the algebra, so the first-order condition holds.

Define

$$U_- = 1 \otimes \sigma_1, \qquad U_+ = \tau_3 \otimes \sigma_1 = QU_-.$$

Both are grading-reversing unitary involutions commuting with D_F, the two algebra actions, and Q. Nevertheless,

$$U_- J_F = -J_F U_-, \qquad U_+ J_F = J_F U_+.$$

The unitary

$$W = \mathrm{diag}(1, \sigma_3)$$

on the particle/antiparticle blocks transforms the data into

$$D_F' = m\, 1 \otimes \sigma_1, \qquad J_F' = (\tau_1 \otimes 1)K,$$

while preserving Γ_F, Q, ρ. The transformed involutions are

$$U_-' = \mathrm{diag}(\sigma_1, -\sigma_1), \qquad U_+' = \mathrm{diag}(\sigma_1, \sigma_1).$$

Thus this is also the electrodynamics counterexample in job 06, expressed as an interval product. Tensoring with a fixed spacetime J_M leaves both possibilities available. □

This disproves the assertion in Conjecture 07.4 that physical spacetime charge conjugation alone selects the reduction. Selecting specifically the bare reflection $1 \otimes \sigma$ is an additional restriction and gives $s = -1$ in the internal-first convention.

12. Theorem (B4: particle preservation and sign pairing).

Let Q be a self-adjoint involution equal to $+1$ on particles and -1 on antiparticles.

For an interval product with $Q = Q_0 \otimes 1$, the bare reflection automatically satisfies

$$[1 \otimes \sigma, Q] = 0.$$

More generally, assume a standard KO-6 triple satisfies

$$[Q, D] = [Q, \Gamma] = [Q, a] = [Q, a^o] = 0, \qquad QJ = -JQ.$$

If an admissible doubling involution U preserves particle number, then QU is another particle-preserving doubling involution with the opposite sign against J.

Proof. The tensor-product assertion is immediate. For the general assertion, $[Q, U] = 0$ gives

$$(QU)^2 = 1.$$

The remaining commutation and grading properties follow from those of Q and U. If $UJ = sJU$, then

$$(QU)J = sQJU = -sJ(QU).$$

Also $[QU, Q] = 0$. □

Thus particle preservation does not select a sign in a particle-number-conserving sector. The additional assumption $[Q, D] = 0$ is essential to the sign-pairing argument; Majorana terms can violate it.

Conversely, particle-preserving doubling requires the particle spaces of opposite grading to be isomorphic as bimodules: U restricts to such an isomorphism. An unpaired left-handed neutrino with no matching right-handed particle in its bimodule therefore obstructs particle-preserving doubling. The formal interval product builds in the missing partner and cannot represent that strict light spectrum without changing its particle content.

Consequence for 9 versus 11

Lemmas 07.1–07.3 hold under pointwise reality, but their proposed physical consequence does not follow. For the internal-first standard KO-6 product, the required grading twist gives the bare interval reflection $s = -1$, hence a KO-5 reduction and formal $4+5 = 9$. Changing the interval's even real-structure convention consistently does not change this result. A fixed spacetime charge conjugation J_M, and even particle-number preservation, can coexist with both reductions; choosing the lift of L/R reflection remains additional data. Finally, literal collapse of this chiral interval leaves no light fermions, so neither 9 nor 11 has been established as the dimension of a surviving heavy-W/Z limit.

Independent numerical check (Claude, 3 Nov 2026)

Python (numpy), antiunitaries represented as J = M K, with J X J^-1 = M conj(X) M^-1, J^2 = M conj(M), and U J = s J U iff U M = s M conj(U). F_0: H_0 = C^2 (+) C^2 with Gamma_0 = diag(g, -g), D_0 = diag(d, conj(d)) (d off-diagonal, random complex entry), J_0 = block swap times K; KO-6 signs verified. Interval truncation D_I = m sigma_1 (m = 0.7), Gamma_I = sigma_3, J_I = diag(t, 1) K.

- **Theorem 5.** Of the eight choices (t, a, b), exactly (t, a, b) = (+1, 0, 1) and (-1, 0, 0) give the KO-6 signs (+, +, -); both give s = -1 for U = 1 (x) sigma_1, and U commutes with D_F.
- **Theorem 7.** With R x = P_+ x (x) e_+ + P_- x (x) e_-: R* U D_F R = D_0 + m Gamma_0 and R* U J_F R = J_0 Gamma_0, with j_5^2 = -1 and j_5 d_5 = -d_5 j_5 (KO-5 signs).
- **Proposition 6.** Gamma_0 (x) sigma_1 commutes with D_F for D_0 = 0 and fails to commute for D_0 != 0.
- **Counterexample 11.** KO-6 signs, order zero and first order hold for random diagonal algebra elements; U_- and U_+ are both admissible (involutive, grading-reversing, commuting with D_F, Q, the algebra and its opposite), with s = -1 and s = +1.
- **No-go theorem 8.** D_F^2 = D_0^2 (x) 1 + 1 (x) D_I^2 holds exactly.
- **Counterexample 4 (first order), by hand.** For e = diag(1, 0) and the inherited diagonal J_2, e° = e and [[m sigma_1, e], e] = m sigma_1.

Terminology. Here "doubling" means the factorization of an even triple as an odd triple times a Clifford point (the doubling of an odd triple in Dabrowski-Dossena's product construction). It is unrelated to the "fermion doubling" of the noncommutative Standard Model (Lizzi-Mangano-Miele-Sparano). Prior work on products and on J versus J gamma: ../LITERATURE.md, section D; the factorization criterion and the 9/11 results were not found in print.

Part 09

The chiral interval with a bulk mass: chirality, light pairs and both limits

Theorem T3 of ../STRATEGY.md, core part. Proved by Claude Opus 5.5 without Astra or Fable; checked numerically against an independent first-order discretization. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026): of 11 statements, 4 correct, 3 needed amendments, 2 had gaps (closed by the referee) and

2 were wrong as first stated (Corollary 9.10 and Lemma 9.11). All corrections are incorporated below; the referee report is appended.

Setting

Let I = [0, l], H = L^2(I; C^2) with psi = (L, R), and Gamma = diag(1, -1). For M in R let A = -d/dx + M, so that A^dagger = d/dx + M, and

```
D_M psi = (A^dagger R, A L),   i.e.   (D psi)_L = R' + M R,   (D psi)_R = -L' + M L.
```

At M = 0 this is the operator of job 05. We consider three boundary conditions:

- (a) R(0) = R(l) = 0;
- (b) L(0) = L(l) = 0;
- (c) R(0) = 0 and L(l) = 0 (the chiral interval of job 05).

Lemma 9.1. For each of (a), (b) and (c), D_M is self-adjoint on {psi in H^1 : boundary condition} and has compact resolvent.

Proof. For psi, phi in H^1 the mass terms cancel (M is real) and integration by parts gives <D psi, phi> - <psi, D phi> = [conj(R) phi_L - conj(L) phi_R]_0^l. Each condition is a Lagrangian subspace of the four-dimensional space of boundary values: two independent conditions on which the form vanishes. Distributional differentiation shows Dom D_M^* is contained in H^1, so the boundary form forces the same conditions and D_M is self-adjoint. The estimate norm(psi') <= norm(D_M psi) + abs(M) norm(psi) makes the graph norm control the H^1 norm, so the resolvent maps L^2 boundedly into H^1, which embeds compactly in L^2. ■

Chirality lives in the boundary conditions

Theorem 9.2. For every M in R and l > 0:

- (a) ker D = C (e^{Mx}, 0), so ind = dim ker_{Gamma=+1} - dim ker_{Gamma=-1} = +1;
- (b) ker D = C (0, e^{-Mx}), so ind = -1;
- (c) ker D = 0, so ind = 0.

Proof. D psi = 0 iff A L = 0 and A^dagger R = 0, i.e. L = alpha e^{Mx} and R = beta e^{-Mx}. Exponentials never vanish, so a condition at either end forces the corresponding coefficient to 0. ■

Corollary 9.3 (equivariant index). Let a compact group G act on H_G = (+)_rho V_rho (x) L^2(I; C^2), a sum of finitely many finite-dimensional representations V_rho, with boundary type tau(rho) in {a, b, c} and mass M_rho on each summand. (With infinitely many summands the operator need not be Fredholm: the referee report gives a counterexample.) Then the G-index of D in R(G) is

```
sum over tau(rho) = a of [V_rho]   minus   sum over tau(rho) = b of [V_rho],
```

independent of every M_rho and of l. Assigning (a) to the summands of V_L and (b) to those of V_R realizes I_G = [V_L] - [V_R] of research_note 7. Vectorlike pairs of type (c) contribute 0. ■

Spectrum of the mixed condition with a bulk mass

Theorem 9.4. Take condition (c). The spectrum is symmetric under lambda -> -lambda, all eigenvalues are simple, and 0 is not an eigenvalue. Write c = mu l.

1. M = -mu with mu > 0 and c > 1: there is exactly one positive eigenvalue below mu,

   ```
   lambda_0 = mu sech(kappa_0 l),   where kappa_0 in (0, mu) is the unique positive root of
    ↪  tanh(kappa l) = kappa/mu,
   ```

   and the remaining positive eigenvalues are lambda_n = sqrt(mu^2 + k_n^2), n >= 1, with k_n the unique root of tan(k l) = k/mu in (n pi/l, (n + 1/2) pi/l).

2. M = -mu and c < 1: no eigenvalue in (0, mu]. The positive eigenvalues are sqrt(mu^2 + k_n^2), n >= 0, with k_0 in (0, pi/(2l)) and k_n in (n pi/l, (n + 1/2) pi/l) roots of tan(k l) = k/mu. At c = 1 the lowest eigenvalue is exactly mu.

3. M = +mu > 0: every positive eigenvalue is sqrt(mu^2 + k_n^2) with k_n in ((n - 1/2) pi/l, n pi/l), n >= 1, a root of tan(k l) = -k/mu, so all exceed sqrt(mu^2 + (pi/2l)^2).

For M = 0 the spectrum is +/-(n + 1/2) pi/l, n >= 0 (job 05).

Proof. If D psi = lambda psi with lambda > 0, then R = A L / lambda and A^dagger A L = lambda^2 L, where A^dagger A = -d^2/dx^2 + M^2. The conditions become L(l) = 0 and L'(0) = M L(0). Conversely, every solution of this Sturm-Liouville problem with lambda^2 > 0 yields the eigenvector (L, A L/lambda), whose R-component vanishes at 0. The Sturm-Liouville problem has simple eigenvalues, and the condition L(l) = 0 forces each eigenfunction into one of three forms:

- E = M^2 + k^2: L = sin(k(l - x)), giving tan(k l) = -k/M;
- E = M^2 - kappa^2 with 0 < kappa < abs(M): L = sinh(kappa(l - x)), giving tanh(kappa l) = -kappa/M;
- E = M^2: L = l - x, giving M l = -1.

Counting roots for M = -mu, with t = k l or t = kappa l:

- g(t) = tanh t - t/c is concave on t > 0, with g(0) = 0, g'(0) = 1 - 1/c and g -> -infinity. It has a positive root iff c > 1, and that root is unique.
- h(t) = tan t - t/c is negative on (n pi - pi/2, n pi). On (n pi, n pi + pi/2) we have h' = sec^2 t - 1/c, which changes sign at most once from - to +. Since h(n pi) <= 0 and h tends to +infinity, there is exactly one root there for n >= 1. For n = 0 there is one iff c < 1.

For M = +mu, the function tan t + t/c is strictly increasing on (n pi - pi/2, n pi), from -infinity to n pi/c > 0, so it has exactly one root there. The hyperbolic equation and M l = -1 have no solutions. Finally, s = kappa_0 l satisfies s = c tanh s, which gives lambda_0^2 = mu^2 - kappa_0^2 = mu^2 sech^2 s. ■

Remark 9.4a (the boundary identity; from the Farsi-Latrémolière assessment of 3 Nov 2026, section 4.1). Write D_M = D_0 + M sigma_1 with D_0 the massless operator. For psi = (L, R) in the domain of condition (c),

```
||D_M psi||^2 = ||D_0 psi||^2 + M^2 ||psi||^2 + M (|R(l)|^2 + |L(0)|^2),
```

since 2 Re <D_0 psi, sigma_1 psi> = [|R|^2]_0^l - [|L|^2]_0^l and the condition kills R(0) and L(l). So the anticommutator {D_0, sigma_1} vanishes in the interior and its quadratic form is the boundary term: this is exactly the hypothesis that Theorem 3.4 of Farsi-Latrémolière needs and that the interval fails, by the term that creates the wall modes (results/R1_counterexample_farsi_latremoliere.md). For M = +mu the term is nonnegative and the identity gives ||D_M psi||^2 >= ((pi/(2l))^2 + mu^2) ||psi||^2, a one-line proof of the bound in item 3; for M = -mu it is negative, which is where item 1 finds its eigenvalue below mu.

Lemma 9.5 (size and shape of the light pair). For c > 1,

```
mu sech(mu l) <= lambda_0 <= 2 mu e^{-kappa_0 l},   so   lambda_0 = 2 mu e^{-mu l} (1 + o(1))   as
↪ mu l -> infinity.
```

The pair is spanned by L_0 = sinh(kappa_0 (l - x)), concentrated at x = 0, and R_0 = A L_0 / lambda_0 = (kappa_0 cosh(kappa_0(l - x)) - mu sinh(kappa_0(l - x)))/lambda_0, which vanishes at 0 and is concentrated at x = l. Moreover ||R_0|| = ||L_0||.

Proof. The lower bound holds because sech is decreasing and kappa_0 < mu. For the upper bound, lambda_0^2 = (mu - kappa_0)(mu + kappa_0) <= 2 mu (mu - kappa_0) = 2 mu^2 (1 - tanh(kappa_0 l)) <= 4 mu^2 e^{-2 kappa_0 l}. The equality of norms follows from ||A L_0||^2 = <A^dagger A L_0, L_0> = lambda_0^2 ||L_0||^2. For the asymptotic, put s = kappa_0 l, so that c = s/tanh s and 0 < c - s = c(1 - tanh s) <= 2c e^{-2s} -> 0; hence lambda_0 = mu sech s = 2 mu e^{-c} e^{c-s}/(1 + e^{-2s}) = 2 mu e^{-c}(1 + o(1)). Finally, mu = kappa_0 coth(kappa_0 l) and lambda_0 = kappa_0/sinh(kappa_0 l) give

the exact identity R_0(x) = sinh(kappa_0 x) = L_0(l - x): the two profiles are mirror images, decreasing away from their walls, with localization on the scale 1/kappa_0 as s -> infinity (and approaching multiples of l - x and x as c -> 1). ■

Theorem 9.6 (the light pair is Connes-Lott with m = lambda_0). Let P_0 project onto the eigenvalues +/- lambda_0 (c > 1), and use the basis u_L = (L_0, 0)/||L_0|| and u_R = (0, R_0)/||R_0||. Then:

- P_0 D P_0 = lambda_0 sigma_1 and P_0 Gamma P_0 = sigma_3;
- the compression of C(I) is Q(f) = diag(int f abs(u_L)^2, int f abs(u_R)^2), whose range is C + C;
- the truncated end states are at Connes distance 1/lambda_0.

Proof. We have D u_L = A L_0/||L_0|| = lambda_0 u_R. Pointwise orthogonality of the L and R components makes Q(f) diagonal. Q(1) = I and Q(x) = diag(_L, _R) with _L < _R, so the range is the full diagonal algebra. Finally, |||[lambda_0 sigma_1, diag(a, b)]|| = lambda_0 abs(a - b). ■

The two-sheet distance is therefore 1/(fermion mass), as in the Standard Model finite geometry; it depends on l only through lambda_0. At M = 0 (job 05) it was 2l/pi.

The two limits

Lemma 9.7. For c > 1 let s(c) be the positive root of tanh s = s/c, and set F(c) = (lambda_0 l)^2 = c^2 sech^2 s(c). Then F is strictly decreasing on (1, infinity), F(1+) = 1 and F(infinity) = 0.

Proof. Differentiating tanh s = s/c gives s'(1/c - sech^2 s) = s/c^2. Here 1/c - sech^2 s > 0, since 1/c = tanh(s)/s and sinh 2s > 2s. Using c^2 - s^2 = c^2 sech^2 s, the condition F'(c) = 2c - 2 s s' < 0 is equivalent to c^2 sech^2 s < c^3 sech^2 s, i.e. to c > 1. The limits follow from s -> 0 as c -> 1 and s/c -> 1 as c -> infinity. ■

Theorem 9.8 (decompactification, l -> infinity at fixed mu, M = -mu). We have lambda_0 <= 2 mu e^{-kappa_0 l} -> 0 with kappa_0 -> mu. The normalized profiles of L_0 near x = 0 and of R_0 near x = l converge, uniformly on compact sets of y and in L^2([0, infinity)) after extension by zero, to sqrt(2 mu) e^{-mu y}, where y is the distance to the respective wall: these are exact chiral zero modes of the two half-line operators (proof: norm(L_0)^2 = sinh(2s)/(4 kappa_0) - 1/2 ~ e^{2s}/(8 kappa_0), kappa_0 -> mu, L_0(y) = R_0(l - y), and the same formula gives a uniform exponential bound). Since k_n lies in (n pi/l, (n + 1/2) pi/l), the rest of the spectrum obeys

```
#{n : lambda_n <= E} = (l/pi) sqrt(E^2 - mu^2) + O(1)   for E > mu,
```

a one-dimensional Weyl law (for the positive spectrum; both signs double the leading term). One dimension emerges above the bulk gap mu, and the chirality sits on the two walls. ■

Theorem 9.9 (collapse at fixed light mass). Fix m_f > 0. For each l in (0, 1/m_f) there is a unique mu(l) > 1/l with lambda_0 = m_f. As l -> 0:

- c(l) = mu(l) l -> infinity, with c = log(2c/(m_f l)) + o(1); equivalently c = -W_{-1}(-m_f l/2) + o(1) (Lambert W, lower branch), so c = L + log L + o(1) with L = log(2/(m_f l)): the leading scaling log(1/(m_f l))/l carries a log log correction, and it is specific to this localization model;
- the next eigenvalue exceeds sqrt(mu(l)^2 + (pi/l)^2) -> infinity;
- the light pair stays Connes-Lott with mass m_f (Theorem 9.6), while the interval collapses to a point in Gromov-Hausdorff distance. This is a statement about the compression at each l. As whole triples in the spectral propinquity, the family converges to (C, C^2, m_f sigma_1), a point carrying a massive two-dimensional Dirac block, at distance at least min(sqrt(2)/2, (1/m_f - l)/2) from the Connes-Lott triple (Corollary 15.5 of lemma 15). Both statements hold: the metric compares the whole algebra, and the compression is a different object.

Proof. Existence and uniqueness follow from Lemma 9.7 applied to F(c) = (m_f l)^2 in (0, 1). As l -> 0 we have F -> 0, so c -> infinity, and Lemma 9.5 gives m_f = 2 (c/l) e^{-c}(1 + o(1)). The bound on the next eigenvalue is Theorem 9.4.1. ■

Corollary 9.10 (collapse at fixed bulk mass). If instead mu is fixed and l -> 0, then eventually c < 1, every eigenvalue has absolute value greater than mu, and the smallest absolute value is (pi/(2l))(1 + o(1)) (the lowest root t_0 = k_0 l of tan t/t = 1/(mu l) tends to pi/2). No light fermion survives. This extends Proposition 05-4 to M != 0.

Gauge fields on the interval (scale lemma for T4)

Lemma 9.11. For -d^2/dx^2 on [0, l]:

- Neumann-Neumann conditions give eigenvalues (n pi/l)^2, n >= 0, including a massless mode;
- Neumann-Dirichlet (in either order) gives ((n - 1/2) pi/l)^2, n >= 1;
- Dirichlet-Dirichlet gives (n pi/l)^2, n >= 1.

A gauge field with a Dirichlet condition at one end and a Neumann condition at the other therefore has lightest mass pi/(2l), which equals the Connes-Lott mass of the massless chiral interval (job 05); with Dirichlet conditions at both ends the lightest mass is pi/l. ■

Remark: fermion mass as internal momentum

In this family, masses are internal momenta, with an explicit rule for their quantization. At M = 0 the Connes-Lott mass is the lowest allowed momentum (r - 1/2) pi/l, quantized by the chiral boundary conditions. With a bulk mass, lambda_n^2 = mu^2 + k_n^2 adds the bulk mass and the quantized momentum in quadrature. The light pair's mass, mu sech(kappa_0 l), is exponentially sensitive to mu l. Order-one differences of mu_i l between generations therefore give hierarchical masses of about 2 mu_i e^{-mu_i l}. This is the localization mechanism for Yukawa hierarchies (Arkani-Hamed-Schmaltz 2000; Gherghetta-Pomarol 2000; Grossman-Neubert 2000). In noncommutative-geometry terms, the finite operator D_F of the two-sheet space is the towerless remnant of internal momentum, and its texture becomes localization and overlap data.

Numerical check (Claude)

The secular equations of Theorem 9.4 were compared with the eigenvalues of an independent first-order discretization: the chain of job 05 with mass term (M/2)(R_{i+1} + R_i), N = 4000.

mu (l = 1)	secular	chain
1e-4	1.57073, 4.71237, 7.85397	1.57054, 4.71178, 7.85299
0.5	1.26828, 4.63129, 7.80591	1.26809, 4.63070, 7.80493
3	0.30255, 5.06274, 8.05194	0.30244, 5.06227, 8.05103
6	0.02975, 7.04564, 9.33873	0.02972, 7.04536, 9.33799

At mu = 6, 2 mu e^{-mu l} = 0.029745, against lambda_0 = 0.029747. Lemma 9.7 was confirmed on (1, 40] using the cancellation-free form c^2 sech^2 s. For the collapse at m_f = 1:

l	mu l	mu	lambda_0	next eigenvalue
0.5	2.234	4.47	1.000000	9.56
0.1	4.501	45.0	1.000000	59.2
0.01	7.284	728	1.000000	812
0.001	9.893	9893	1.000000	10487

Referee report (gpt-6.1-terra, effort max, 3 Nov 2026)

Prompt: prompts/terra_check_09.md. Reproduced as received; the corrections above implement it. Claude checked the two errors, the identity R_0(x) = L_0(l - x) and the asymptotic of 9.5 by hand.

We take $\ell > 0$ throughout. All checks below are analytic; we used no numerical computation.

Lemma 9.1 Verdict: Correct with amendment. The compact-resolvent argument needs the graph-norm estimate

$$\|\psi'\|_{L^2} \le \|D_M\psi\|_{L^2} + |M|\,\|\psi\|_{L^2}.$$

The boundary form in the proof is correct. Each proposed boundary subspace is maximal isotropic. To apply this to the adjoint, distributional differentiation shows that every vector in $\operatorname{Dom} D_M^*$ lies in H^1; the boundary form then forces it to satisfy the same boundary conditions. Thus D_M is self-adjoint.

The displayed estimate makes the graph norm control the H^1-norm. Consequently, the resolvent maps L^2 boundedly into H^1, and compactness of $H^1 \hookrightarrow L^2$ gives compact resolvent.

Theorem 9.2 Verdict: Correct.

The zero-mode equations give

$$L(x) = \alpha e^{Mx}, \qquad R(x) = \beta e^{-Mx}.$$

The boundary conditions force $\beta = 0$ in case (a), $\alpha = 0$ in case (b), and both coefficients to vanish in case (c). The grading preserves each domain and anticommutes with D_M, so the stated chiral indices follow.

Corollary 9.3 Verdict: Correct with amendment. Assume that the displayed direct sum has finitely many finite-dimensional G-representations. Under that hypothesis, Theorem 9.2 gives the stated element of $R(G)$.

Without such a hypothesis, the direct-sum operator need not be Fredholm, even if every summand has boundary type (c). For a counterexample, take $\ell = 1$, $G = S^1$, the distinct one-dimensional representations $V_j(z) = z^j$, $j \ge 2$, and $M_j = -j$. Choose a smooth χ equal to 1 on $[0, 1/2]$ and equal to 0 at 1, and put

$$f_j(x) = \chi(x)e^{-jx}.$$

Then $(f_j, 0)$ satisfies condition (c), while

$$A_j f_j = -\chi'(x)e^{-jx}, \qquad \frac{\|A_j f_j\|}{\|f_j\|} = O\big(\sqrt{j}\, e^{-j/2}\big) \longrightarrow 0.$$

After normalization, these vectors lie in distinct orthogonal representation summands. The direct sum has no kernel, but it is not bounded below and hence is not Fredholm. Its ordinary $R(G)$-valued Fredholm index is therefore undefined.

Theorem 9.4 Verdict: Correct with amendment. Specify $\mu > 0$ in the negative-mass cases and $n \ge 1$ in case 3. For an exhaustive list, add $M = 0$, where

$$\operatorname{spec} D_0 = \left\{ \pm \frac{(n + \frac{1}{2})\pi}{\ell} : n \ge 0 \right\}.$$

At $c = 1$, the threshold eigenvalue is μ, and the remaining positive roots have $n \ge 1$ in the intervals stated in case 1.

For $\lambda \ne 0$, eliminating R gives

$$-L'' + M^2 L = \lambda^2 L, \qquad L(\ell) = 0, \quad L'(0) = ML(0).$$

Conversely, every solution gives $R = AL/\lambda$. The squared operator is nonnegative, and Theorem 9.2 excludes $\lambda = 0$. Separated boundary conditions make each squared eigenvalue simple; $\Gamma D_M = -D_M\Gamma$ gives the symmetric spectrum.

For $M = -\mu$, put $c = \mu\ell$. In the subthreshold case, the equation is

$$\frac{\tanh t}{t} = \frac{1}{c}, \qquad t = \kappa\ell > 0.$$

The left side decreases strictly from 1 to 0, so there is exactly one positive root if $c > 1$, and none if $c \le 1$. It satisfies $\kappa < \mu$.

For the oscillatory roots, let $h(t) = \tan t - t/c$. It is negative on each positive interval $(n\pi - \pi/2, n\pi)$. On $(n\pi, n\pi + \pi/2)$, its derivative increases strictly. For $n \ge 1$, it starts negative in value and tends to $+\infty$, so it has exactly one root. On $(0, \pi/2)$, it has exactly one positive root when $c < 1$, and none when $c \ge 1$. At $c = 1$, the separate linear solution $L = \ell - x$ gives $\lambda = \mu$.

For $M = +\mu$, $\tan t + t/c$ increases strictly from $-\infty$ to a positive value on each $((n - \frac{1}{2})\pi, n\pi)$, $n \ge 1$, giving exactly one root there. It has no positive root on the other half-periods, and there is no subthreshold or threshold solution. These counts establish all three stated cases.

Lemma 9.5 Verdict: Gap, closed here. The displayed proof does not justify the relative asymptotic or make "concentrated" precise.

Write $s = \kappa_0 \ell$ and $c = \mu\ell$. The root equation gives $c = s/\tanh s$. As $c \to \infty$, $s \to \infty$ and $s/c \to 1$; moreover,

$$0 < c - s = c(1 - \tanh s) \le 2ce^{-2s} \longrightarrow 0.$$

Thus

$$\lambda_0 = \mu \operatorname{sech} s = 2\mu e^{-c} \frac{e^{c-s}}{1 + e^{-2s}} = 2\mu e^{-c}(1 + o(1)).$$

The two stated inequalities also follow directly from $s < c$ and $\operatorname{sech} s \le 2e^{-s}$.

The root equation simplifies the right profile exactly:

$$R_0(x) = \sinh(\kappa_0 x) = L_0(\ell - x).$$

This proves the equality of norms and shows that the two profiles decrease away from their respective walls. Exponential localization on the scale $1/\kappa_0$ holds when $s \to \infty$. It is not uniform throughout $c > 1$: at fixed ℓ, as $c \downarrow 1$, the normalized profiles approach multiples of $\ell - x$ and x.

Theorem 9.6 Verdict: Correct.

The two chiral basis vectors span the $\pm\lambda_0$ eigenspaces together, and the first two matrix identities follow from

$$D_M u_L = \lambda_0 u_R, \qquad D_M u_R = \lambda_0 u_L.$$

By the reflection identity in Lemma 9.5, the normalized left density exceeds the right density on the left half of the interval, with the reverse inequality on the right half. Hence

$$\langle x \rangle_L < \ell/2 < \langle x \rangle_R.$$

Thus $Q(1)$ and $Q(x)$ span the diagonal algebra, and the commutator norm is $\lambda_0 |a - b|$. The distance between its two pure states is $1/\lambda_0$.

The pullbacks of these states through Q are the two smeared probability measures, not literal endpoint evaluations. Also, the unnumbered sentence immediately after the proof needs correction: the distance is **not determined by ℓ alone**, but it does depend on ℓ. In fact, Lemma 9.7 gives $1/\lambda_0 > \ell$ for $c > 1$.

Lemma 9.7 Verdict: Correct.

Set $q = \operatorname{sech}^2 s$. Since

$$\frac{1}{c} = \frac{\tanh s}{s} > q,$$

implicit differentiation is valid and gives

$$s' = \frac{s}{c(1 - cq)}, \qquad F'(c) = \frac{2cq(1 - c)}{1 - cq} < 0.$$

As $c \downarrow 1$, the inverse relation $c = s/\tanh s$ gives $s \to 0$, so $F \to 1$. As $c \to \infty$, $s \to \infty$ and $c/s \to 1$, so $c^2 \operatorname{sech}^2 s \to 0$.

Theorem 9.8 Verdict: Gap, closed here. The statement asserts convergence of the normalized profiles without proving it or specifying a mode of convergence. The convergence holds uniformly on compact sets of wall distance and in $L^2([0,\infty))$, after extension by zero.

Let $B_\ell = \|L_0\| = \|R_0\|$. With $s = \kappa_0 \ell$,

$$B_\ell^2 = \frac{\sinh(2s)}{4\kappa_0} - \frac{\ell}{2} \sim \frac{e^{2s}}{8\kappa_0}.$$

At fixed μ, $s/c \to 1$, so $\kappa_0 \to \mu$. For fixed $y \geq 0$,

$$\frac{L_0(y)}{B_\ell} = \frac{R_0(\ell - y)}{B_\ell} \longrightarrow \sqrt{2\mu}\, e^{-\mu y}.$$

The same formula supplies a uniform exponential bound, proving the stated L^2 convergence. The limits are exact zero modes of the two separately oriented half-line operators, with left and right chirality respectively. They are not zero modes at finite ℓ.

For $E > \mu$, set $K = \sqrt{E^2 - \mu^2}$. The root bounds

$$n\pi < k_n \ell < (n + \tfrac{1}{2})\pi$$

give

$$\#\{n \geq 1 : k_n \leq K\} = \frac{\ell K}{\pi} + O(1).$$

This is the displayed one-sided counting law. Counting both signs of the Dirac spectrum doubles its leading term.

Theorem 9.9 Verdict: Correct.

Lemma 9.7 makes F a continuous strictly decreasing map from $(1,\infty)$ onto $(0,1)$. Since $m_f \ell \in (0,1)$, it gives the unique $c = \mu\ell > 1$. As $\ell \to 0$, $c \to \infty$, and Lemma 9.5 yields

$$m_f \ell = 2ce^{-c}(1 + o(1)), \qquad c = \log \frac{2c}{m_f \ell} + o(1).$$

The first root above the light pair has $k_1 > \pi/\ell$, proving the stated lower bound and divergence.

The original interval metric is $|x - y|$: the mass term commutes with scalar functions, so it does not change the Dirac commutator seminorm. Therefore

$$d_{\mathrm{GH}}([0,\ell], \{\text{point}\}) = \ell/2 \longrightarrow 0.$$

The compressed two-state geometry uses a different seminorm and retains distance $1/m_f$.

Corollary 9.10 Verdict: Error. The assertion that "every eigenvalue exceeds μ" is false: the spectrum is symmetric and contains negative eigenvalues.

The corrected statement is that, for fixed bulk mass and sufficiently small ℓ, **every eigenvalue has absolute value greater than** μ, and the smallest absolute eigenvalue is

$$\min_{\lambda \in \operatorname{spec} D_M} |\lambda| = \frac{\pi}{2\ell}(1 + o(1)).$$

For $M = -\mu < 0$, the lowest root $t_0 = k_0 \ell \in (0, \pi/2)$ satisfies

$$\frac{\tan t_0}{t_0} = \frac{1}{\mu\ell} \longrightarrow \infty,$$

so $t_0 \to \pi/2$, proving the formula. For $M = 0$ it is exact; the positive-mass case follows similarly from the root just above $\pi/2$. Thus the conclusion that no light fermion survives is correct.

Lemma 9.11 Verdict: Error. The three eigenvalue lists are correct, but the gauge-field conclusion lacks a boundary condition at the other end.

For a Dirichlet condition at **both** ends, which in particular includes a Dirichlet condition at one end, the lightest mass is π/ℓ, not $\pi/(2\ell)$. The conclusion is correct for a Dirichlet condition at one end **and a Neumann condition at the other**, for the stated operator $-d^2/dx^2$.

Even if "at one end" is read as exactly one Dirichlet condition, another boundary condition need not be Neumann: $u(0) = 0$, $u'(\ell) = u(\ell)/\ell$ admits the zero mode $u(x) = x$.

Amendments and errors

- **9.1:** Add the graph-norm estimate for compact resolvent.
- **9.3:** Require a finite direct sum, or impose other hypotheses ensuring an equivariant Fredholm operator with index in $R(G)$.
- **9.4:** Specify the root indices and $\mu > 0$, and include $M = 0$ for completeness.
- **9.5:** Supply the relative-asymptotic argument and state localization precisely; the gap is closed above.
- **9.6:** Correct the adjacent unnumbered claim that the distance is unrelated to ℓ.
- **9.8:** Prove and specify the profile convergence; the gap is closed above.
- **9.10:** Replace "every eigenvalue exceeds μ" by the corresponding statement about absolute values.
- **9.11:** Require Neumann at the opposite endpoint for the claimed lightest mass.

Overall: 11 statements checked; 4 correct, 3 correct with amendment, 2 gaps closed above, and 2 errors.

Literature (../LITERATURE.md, section B). Opposite-chirality boundary modes with exponentially small mixing are in Jackiw-Rebbi 1976, Kaplan 1992 and Shamir 1993 (section 3); collapse of spectral triples to a point is treated by Farsi-Latremoliere (J. London Math. Soc., 2026). The tuned collapse mu(l) of Theorem 9.9 was not found in print.

Part 10

Toeplitz compression of the twistor coordinates on CP^3

Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026): the three identities are correct; the displayed fourth moment had an index error (pairing two barred indices), corrected below. This answers Part A of parked job 04 (research_note 4 and canonical 4, which labelled the result "Checked calculation", although no script computes it). Proved by Claude Opus 5.5 without Astra or Fable; Monte Carlo check with 2 x 10^6 samples.

Let Gamma_1, ..., Gamma_5 be Hermitian generators of Cl(5) on C^4, so that Gamma_a Gamma_b + Gamma_b Gamma_a = 2 delta_ab. For [w] in CP^3 put x_a([w]) = w* Gamma_a w / w* w, and define the level-one Berezin-Toeplitz compression

```
Q(f) = 4 int_{CP^3} f([w]) P_w dmu([w]),
```

where P_w is the rank-one projector onto w and mu is the Fubini-Study probability measure, normalized so that Q(1) = I.

Proposition 10.1.

```
Q(x_a) = Gamma_a / 5,   Q(x_a)^2 = I/25,   Q(x_a^2) = I/5.
```

In particular Q is not multiplicative.

Proof. Lift to w uniform on S^7 in C^4. For n = 4, writing w = g/abs(g) with g a standard complex Gaussian (abs(g) and g/abs(g) are independent), Wick's theorem gives

```
E[w_j w_k conj(w_i) conj(w_l)] = (delta_ij delta_kl + delta_jl delta_ki) / (n(n+1)),
```

and the sixth moment is the sum over the six matchings of unbarred with barred indices, divided by n(n+1)(n+2) = 120.

First, Q(x_a)_kl = 4 sum_ij (Gamma_a)_ij E[conj(w_i) w_j w_k conj(w_l)] = (4/20)(tr(Gamma_a) delta_kl + (Gamma_a)_kl) = Gamma_a/5, because tr Gamma_a = 0.

Second, Q(x_a^2)_kl = (4/120) sum over matchings of {j, q, k} with {i, p, l} of the corresponding contraction of (Gamma_a)_ij (Gamma_a)_pq. Matchings that contain tr Gamma_a vanish. The remaining three give tr(Gamma_a^2) delta_kl = 4 delta_kl, (Gamma_a^2)_kl = delta_kl and (Gamma_a^2)_kl = delta_kl. The total is 6 delta_kl, so Q(x_a^2) = (24/120) I = I/5.

The middle identity follows from Gamma_a^2 = I. ■

Numerical check. With Gamma_a built as Pauli tensor products (Clifford relations verified), the Monte Carlo deviations are max abs(Q(x_a) - Gamma_a/5) <= 0.0009 and max abs(Q(x_a^2) - I/5) <= 0.0006, at a sampling error of about 10^-3.

Part 11

Heavy W in the spectral-action family forces a heavy fermion

Theorem T5 of ../STRATEGY.md. Proved by Claude Opus 5.5 without Astra or Fable. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026): Proposition 11.1 correct; Corollaries 11.2 and 11.3 and the regime amendment overstated their conclusions as first written; corrected below, report appended. The result is conditional on the Chamseddine-Connes-Marcolli normalization at the matching (unification) scale.

Hypothesis (CCM relation). In the spectral action of the almost-commutative Standard Model, the gauge normalization g^2 f_0 / (2 pi^2) = 1/4 and the Higgs normalization by a = Tr(Y_nu* Y_nu + Y_e* Y_e + 3(Y_u* Y_u + Y_d* Y_d)) give, at the matching scale,

```
8 M_W^2 = sum over generations of (m_nu^2 + m_e^2 + 3 m_u^2 + 3 m_d^2),
```

where the m are Dirac masses (Chamseddine-Connes-Marcolli, Adv. Theor. Math. Phys. 11 (2007), section 5). In outline (H the unrescaled Higgs field of CCM; the normalization is the three-generation one): the Dirac masses satisfy sum N_f m_f^2 = a abs()^2, while M_W^2 = (g^2/4)(a f_0/pi^2) abs()^2 with f_0 = pi^2/(2 g^2). Majorana masses do not enter a.

Proposition 11.1. Assume the CCM relation at fixed gauge coupling, with N_g generations, and let m_max be the largest Dirac mass. Then

```
m_max / (2 sqrt 2) <= M_W <= sqrt(N_g) m_max.
```

In particular M_W -> infinity if and only if m_max -> infinity.

Proof. Every term has weight at least 1, so 8 M_W^2 >= m_max^2. Each generation has weights 1 + 1 + 3 + 3 = 8, so 8 M_W^2 <= 8 N_g m_max^2. ■

Corollary 11.2 (no-go for the fixed-mass prescription). The heavy-W prescription of canonical 14.1 and research_state 9 (v -> infinity with y_f v fixed for every fermion and the gauge couplings fixed, so that the Yukawa couplings tend to zero) is incompatible with the CCM relation: the right-hand side stays bounded while M_W -> infinity. In the spectral-action family a heavy-W/Z limit forces m_max -> infinity, with m_max >= M_W/sqrt(N_g); some species is then unbounded along a subsequence, though no single fixed species need diverge along the whole family (the referee report gives a family with m_e^2 + m_nu^2 = 8 t^2 in which each mass returns to zero infinitely often). ■

Corollary 11.3 (only the top heavy). Assume that the up-type Dirac mass of one generation (three colors) diverges and every other Dirac mass stays bounded. Then

```
m_t^2 = (8/3) M_W^2 + O(1),   so   m_t / M_W -> sqrt(8/3), approximately 1.633.
```

Top dominance is a hypothesis here: the relation alone also allows, for example, y_e = 2g with every other Yukawa zero, which gives m_e = 2 sqrt(2) M_W and m_t = 0. Under the hypothesis, the surviving SU(3) x U(1)_em theory is vectorlike and anomaly-free, and job 03 gives the residual kinetic weights. ■

Remark (interval comparison). In the massless chiral interval of job 05, the fermion truncation mass and the Neumann-Dirichlet gauge mass both equal pi/(2l) (Lemma 9.11). Under top dominance (Corollary 11.3), the CCM relation instead fixes m_t / M_W = sqrt(8/3) at matching. The interval model and the spectral action therefore assign the top and the W to the same scale with different order-one ratios. Matching them is part of T4.

Amendment (regime of validity; review of 3 Nov 2026; corrected after the referee report). The CCM relation is a relation in the truncated asymptotic (heat-kernel) action at the matching scale Lambda; small Dirac masses relative to Lambda alone do not make it exact for the full finite-cutoff action, which also needs control of momenta and of the Majorana sector. With fixed Yukawa couplings, M_W and every mass with nonzero Yukawa diverge together as v -> infinity, with ratios fixed by all the Yukawas through the matching constraint (for example y_t = y_nu = g gives m_t/M_W = sqrt 2); Corollary 11.3's ratio needs top dominance. At fixed Lambda a heavy-W limit leaves every fixed small-mass hierarchy (m_max <= eps Lambda implies M_W <= sqrt(N_g) eps Lambda), so it needs Lambda -> infinity jointly or a calculation of the finite-cutoff action. The weights f(m^2/Lambda^2) of lemma 03, Part A, hold under its commuting finite-operator hypothesis and do not apply directly to the broken gauge and Higgs kinetic terms (Part C's w_L(x, 0) = 1 - 3x/4 + O(x^2) differs from f); near the cutoff the coefficients need their own calculation (STRATEGY T5b).

Literature (../LITERATURE.md, section E). The mass relation and the top ratio are in CCM 2007: equation (1.5), Lemma 4.6 equation (4.16) and section 5.4 equation (5.27). Corollary 11.3 restates that result; Proposition 11.1 and Corollary 11.2 apply it to the heavy-W prescription of the corpus.

Referee report on lemmas 10 and 11 (gpt-6.1-terra, effort max, 3 Nov 2026)

Prompt: prompts/terra_check_10_11.md. Reproduced as received; the corrections above (and in lemma 10) implement it. Claude checked the counterexamples (the rotating e/nu family; y_e = 2g; y_t = y_nu = g) and the corrected fourth moment by hand.

We count Proposition 10.1, Proposition 11.1, the two corollaries, and the regime-of-validity amendment as five statements. The checks below are analytic; we did not rerun the Monte Carlo calculation.

Proposition 10.1 **Verdict: Correct with amendment.** All three identities and the normalization of Q are correct. The displayed fourth-moment formula in the proof has an index error, and its subsequent contraction is consequently incorrect as written.

For w uniformly distributed on $S^7 \subset \mathbb{C}^4$, the moments are

$$\mathbb{E}\left[\prod_{r=1}^{p} w_{j_r}\overline{w_{i_r}}\right] = \frac{\sum_{\sigma\in S_p}\prod_{r=1}^{p}\delta_{j_r i_{\sigma(r)}}}{4\cdot 5\cdots(3+p)}.$$

Thus the fourth moment required in the proof is

$$\mathbb{E}[w_j w_k \overline{w_i}\,\overline{w_l}] = \frac{\delta_{ij}\delta_{kl}+\delta_{jl}\delta_{ki}}{20}.$$

The file instead has $\delta_{il}\delta_{kj}$ as its second term. For example, with $i = l = 1$ and $j = k = 2$, its formula gives $1/20$, whereas the expectation is zero by phase invariance. Contracting the erroneous formula with A_{ij} gives A_{lk}, not the claimed A_{kl}.

The sixth-moment statement is correct: its denominator is $4\cdot 5\cdot 6 = 120$, and its numerator is the sum over all six matchings. Also,

$$Q(1) = 4\mathbb{E}[ww^*] = I.$$

For any 4×4 matrix A, the corrected moments give

$$Q(w^* A w) = \frac{\mathrm{tr}(A) I + A}{5}$$

and

$$Q((w^* A w)^2) = \frac{((\mathrm{tr}\, A)^2 + \mathrm{tr}(A^2)) I + 2\,\mathrm{tr}(A) A + 2A^2}{30}.$$

For $A = \Gamma_a$, conjugation by a different Clifford generator shows $\mathrm{tr}\,\Gamma_a = 0$, while $\Gamma_a^2 = I$ and $\mathrm{tr}(\Gamma_a^2) = 4$. Consequently,

$$Q(x_a) = \frac{\Gamma_a}{5}, \qquad Q(x_a)^2 = \frac{I}{25}, \qquad Q(x_a^2) = \frac{I}{5}.$$

Proposition 11.1 **Verdict: Correct.** For a fixed, finite N_g, the proof follows from the assumed relation. If S denotes its weighted mass sum, then

$$m_{\max}^2 \le S \le 8 N_g m_{\max}^2.$$

Substituting $S = 8M_W^2$ proves both bounds and the equivalence of the two divergence statements.

The CCM relation is quoted correctly for the three-generation model. In Chamseddine–Connes–Marcolli, arXiv:hep-th/0610241v1, equations (1.5), (4.10), (4.15)–(4.17), and (5.25)–(5.26), the weights are $1, 1, 3, 3$; g is the weak coupling at matching; and the neutrino terms are **Dirac Yukawa mass parameters**, not the eigenmasses of the full seesaw matrix. The Majorana matrix does not enter the trace a, although it enters other terms in the action. CCM states that the mass relation is valid "at unification scale."

The outline preceding the proposition should identify its $\langle H \rangle$ as the **unrescaled** spectral Higgs field. CCM's canonically normalized field is $\mathbf{H} = (\sqrt{a f_0}/\pi)\varphi$. Equivalently, using the standard normalized Yukawas and v,

$$m_f = \frac{y_f v}{\sqrt{2}}, \qquad M_W = \frac{g v}{2}, \qquad \sum_f N_f y_f^2 = 4g^2,$$

which gives the quoted mass relation.

There is also a scope distinction for $N_g \neq 3$. Proposition 11.1 remains true under its explicitly assumed relation, but CCM derives the coefficient 8 for three generations. If we replicate the same gauge representations N_g times and normalize their gauge kinetic term in the same way, the gauge trace scales by $N_g/3$, giving instead

$$\sum_f N_f m_f^2 = \frac{8 N_g}{3} M_W^2.$$

Thus the N_g-generation version with coefficient 8 is an additional assumption beyond the cited normalization.

Corollary 11.2 **Verdict: Error.** The fixed-mass no-go and the displayed bound are correct. The subsequent assertion that a particular Dirac mass must *tend to infinity* is stronger than the relation proves for a general family.

If all finitely many Dirac masses are bounded, Proposition 11.1 bounds M_W. Moreover, it gives exactly

$$m_{\max} \geq \frac{M_W}{\sqrt{N_g}}.$$

Thus $M_W \to \infty$ forces $m_{\max} \to \infty$, and at least one fixed species is unbounded along a subsequence. It need not force one fixed species to tend to infinity along the whole family.

For a counterexample with fixed gauge coupling, let $t \to \infty$, put $M_W = t$, and take all Dirac masses except one charged-lepton mass and one neutrino Dirac mass to be zero. Set

$$\theta(t) = \frac{\pi}{4}(1 + \sin t), \quad m_e = 2\sqrt{2}\, t \cos\theta(t), \quad m_\nu = 2\sqrt{2}\, t \sin\theta(t).$$

Then $m_e^2 + m_\nu^2 = 8t^2$, so the assumed CCM relation holds. Nevertheless, each of m_e and m_ν vanishes on an unbounded sequence. The corresponding Yukawa couplings vary; the gauge coupling can remain fixed.

The description of the fixed-mass prescription should also say **gauge couplings fixed**: nonzero fixed fermion masses with $v \to \infty$ require their Yukawa couplings to tend to zero.

Corollary 11.3 **Verdict: Error.** The displayed asymptotic formula is correct under the stated "only the top heavy" hypothesis. The claim that a heavy top follows from the spectral action, without that choice or another hypothesis selecting it, is false.

Let R be the weighted sum of all the other Dirac masses squared. Under the hypothesis, $R = O(1)$, and

$$8M_W^2 = 3m_t^2 + R \quad \Longrightarrow \quad m_t^2 = \frac{8}{3}M_W^2 - \frac{1}{3}R.$$

This proves the stated $O(1)$ formula and $m_t/M_W \to \sqrt{8/3}$.

For a counterexample to the unconditional heavy-top claim, take $y_e = 2g$ and every other normalized Yukawa coupling, including y_t, equal to zero. The matching constraint is satisfied, $m_e = 2\sqrt{2}\, M_W$, and $m_t = 0$. The relation therefore permits a heavy charged lepton with no heavy top. CCM's equation (5.27) likewise obtains the top ratio by neglecting the other Yukawa couplings; its equation (5.29) considers a comparable neutrino Yukawa instead. The interval-comparison remark needs the same top-dominance qualification.

The assertion about the surviving $SU(3) \times U(1)_{\mathrm{em}}$ representation is correct for the stated Standard Model field content: each surviving charged Dirac fermion has identical left- and right-handed representations of the unbroken group. Their anomaly contributions cancel pairwise, and neutral Majorana states do not contribute to these gauge anomalies.

Amendment on the regime of validity **Verdict: Error.** The warning about a fixed cutoff is justified, but the amendment overstates both the fixed-Yukawa conclusion and the form of the threshold weights.

First, fixed Yukawa couplings do not by themselves imply a heavy top or the ratio in Corollary 11.3. A fixed nonzero y_t gives

$$\frac{m_t}{M_W} = \frac{\sqrt{2}\, y_t}{g},$$

but its value depends on the other Yukawas through the matching constraint. For example, $y_t = y_\nu = g$, with all others zero, satisfies that constraint and gives $m_t/M_W = \sqrt{2}$; both masses diverge as $v \to \infty$. If $y_t = 0$, the top does not diverge. Corollary 11.3 applies only when the other Dirac masses remain bounded.

Second, the displayed CCM relation is a relation in the **truncated asymptotic action**. Small Dirac masses relative to Λ alone do not establish it as an exact relation for the full finite-cutoff action: the expansion also needs control of the other relevant scales, including momenta and the Majorana sector, or a suitable uniform remainder estimate. CCM's own discussion places a Majorana scale of order Λ in one scenario; see its equations (5.16)–(5.21).

At fixed Λ, the amendment correctly observes that a heavy-W limit cannot remain within a fixed small-mass hierarchy. Indeed, if $m_{\max} \le \varepsilon\Lambda$, Proposition 11.1 gives $M_W \le \sqrt{N_g}\varepsilon\Lambda$. A limit within that hierarchy requires a joint family with $\Lambda \to \infty$, together with the necessary control of the remaining scales.

Finally, the simple weights $f(m^2/\Lambda^2)$ in lemma 03, Part A have a commuting finite-operator hypothesis. They cannot be assigned generally to broken gauge and Higgs kinetic terms. In particular, that same file's Part C gives, for a broken weak generator with one heavy and one massless partner and the squared-operator cutoff $f(x) = e^{-x}$,

$$w_L(x,0) = 3\int_0^1 u^2 e^{-xu}\,du = 1 - \frac{3}{4}x + O(x^2),$$

which differs from $f(x) = 1 - x + O(x^2)$. Near the cutoff, the appropriate coefficients and any replacement for the CCM relation require a calculation under specified hypotheses.

Amendments and errors found

- Correct the fourth-moment indices in Proposition 10.1's proof; the three proposition identities are unchanged.
- Specify the unrescaled Higgs field in the CCM outline and retain the three-generation scope of the cited normalization.
- In Corollary 11.2, assert divergence of $m_{\max}$, or add a hypothesis ensuring that one fixed species tends to infinity.
- In Corollary 11.3 and the interval remark, make top dominance a hypothesis; it is not forced by the mass relation.
- In the regime amendment, qualify fixed-Yukawa claims, distinguish the truncated action from the finite-cutoff action, and restrict the simple threshold weights to their applicable commuting setting.

Overall: 5 statements checked; 1 correct, 1 amended, 0 gaps, 3 errors.

Part 12

Boundary Yukawa matrices on the interval: from chiral to vectorlike boundary conditions, and towerless textures

Theorems T3 (remaining part) and T8 of ../STRATEGY.md. Proved by Claude Opus 5.5 without Astra or Fable. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026): of 9 statements, 4 correct, 2 needed amendments and 3 were wrong as first stated (Theorem 12.4 item 2, Theorem 12.7, Corollary 12.9); all corrections are incorporated below and the referee report is appended. The secular equation was checked against direct numerical integration of the ODE; the bounds of Theorem 12.6 and Lemma 12.8 were checked on random samples (section "Numerical check").

Setting

Let I = [0, 1]. A doublet-type field Q in L^2(I; C^2) (x) C^{N_Q} and a singlet-type field U in L^2(I; C^2) (x) C^{N_U} have components (L, R), with grading Gamma = diag(1, -1) on each. As in lemma 09,

the bulk operator is (D psi)_L = R' + M R, (D psi)_R = -L' + M L, with diagonal bulk masses M_Q = diag(m_i^Q) and M_U = diag(m_j^U). For a complex N_Q x N_U matrix X, let D_X be this operator on the domain of H^1 functions with

```
Q_R(0) = 0,   U_L(0) = 0,   Q_R(1) = X U_R(1),   U_L(1) = -X^dagger Q_L(1).
```

At X = 0, Q has boundary type (a) and U type (b) of lemma 09. The matrix X is a boundary Yukawa coupling: it couples the two fields only at the wall x = l. We write c_i^Q = -m_i^Q l and c_j^U = m_j^U l, so that c > 0 means that the zero mode of the field at X = 0 is localized near x = 0, away from the Yukawa wall.

Lemma 12.1. D_X is self-adjoint with compact resolvent, and Gamma D_X = -D_X Gamma.

Proof. The boundary form of lemma 9.1, summed over both fields, is [conj(Q_R) phi_{Q,L} - conj(Q_L) phi_{Q,R} + conj(U_R) phi_{U,L} - conj(U_L) phi_{U,R}] evaluated between 0 and l. At x = l the conditions leave Q_L and U_R free, which is half of the boundary values, and on them the form becomes conj(U_R) X^dagger phi_{Q,L} - conj(Q_L) X phi_{U,R} - conj(U_R) X^dagger phi_{Q,L} + conj(Q_L) X phi_{U,R} = 0. At x = 0 the conditions are those of types (a) and (b). Hence the conditions define a Lagrangian subspace, and the argument of lemma 9.1 applies. Gamma flips the signs of Q_R and U_R together, so it preserves each condition. ■

Kernel, index and symmetry breaking

Theorem 12.2. For every X,

```
ker D_X = {(e^{M_Q x} a, 0; 0, 0) : X^dagger e^{M_Q l} a = 0}  (+)  {(0, 0; 0, e^{-M_U x} b) : X
↪  e^{-M_U l} b = 0},
```

so dim ker_{Gamma=+1} = N_Q - rank X, dim ker_{Gamma=-1} = N_U - rank X, and ind D_X = N_Q - N_U.

Proof. As in Theorem 9.2, D psi = 0 forces L = e^{Mx} a and R = e^{-Mx} b on each field. The conditions at 0 remove Q_R and U_L, and the conditions at l reduce to X U_R(l) = 0 and X^dagger Q_L(l) = 0. ■

Let a compact group G act unitarily on C^{N_Q} and C^{N_U} by rho_Q and rho_U, commuting with the bulk masses. The domain of D_X is G-invariant exactly for g in the stabilizer H_X = {g : rho_Q(g) X = X rho_U(g)}.

Corollary 12.3 (equivariant index and a no-go). In R(H_X),

```
ind_{H_X} D_X = [ker X^dagger] - [ker X] = [rho_Q] - [rho_U]   (restricted to H_X),
```

because X maps (ker X)^perp isomorphically and H_X-equivariantly onto ran X = (ker X^dagger)^perp. At X = 0 we have H_X = G, and the index is the chiral class [V_L] - [V_R] of Corollary 9.3. D_X has no zero modes iff X is invertible. This requires rho_Q and rho_U to be isomorphic as H_X-representations, i.e. the residual theory must be vectorlike. For the Standard Model with H = SU(3) x U(1)_em, the quark sector satisfies this. A lepton sector without nu_R does not: a massless left-handed neutrino is forced for every X preserving electromagnetism (H contained in H_X), in agreement with the unpaired nu_L of job 06. ■

Exact secular equation

For a field of mass m and lambda > 0 let omega = (m^2 - lambda^2)^{1/2}, and read omega coth(omega l) as k cot(k l) when omega = ik and as 1/l when omega = 0. Define

```
alpha_i(lambda) = omega coth(omega l) + m_i^Q,     beta_j(lambda) = omega coth(omega l) - m_j^U,
```

A = diag(alpha_i) and B = diag(beta_j). Let W be the set of lambda > 0 with lambda^2 < m^2 + pi^2/l^2 for every bulk mass m of the system.

Theorem 12.4. For lambda in W, lambda is an eigenvalue of D_X iff det(lambda^2 - A(lambda) X B(lambda) X^dagger) = 0, and its multiplicity is dim ker(lambda^2 - A X B X^dagger). Moreover:

1. alpha_i(0) = 2m/(1 - e^{-2ml}) with m = m_i^Q, which equals f_{Q_i}^2, the value at x = l of the normalized zero mode e^{mx}; likewise beta_j(0) = 2m/(e^{2ml} - 1) = f_{U_j}^2 for e^{-mx}. In terms of c, f^2 = phi(c)^2/l with phi(c)^2 = 2c/(e^{2c} - 1) and phi(0) = 1.
2. On its own window 0 < lambda^2 < m^2 + pi^2/l^2, each alpha_i (and each beta_j) is a strictly decreasing, concave function of lambda^2, positive at 0 and tending to -infinity at the edge of that window, so it vanishes at exactly one point: the lowest positive eigenvalue Lambda_i^Q of Q_i alone with R(0) = 0 and L(l) = 0, respectively Lambda_j^U of U_j alone with L(0) = 0 and R(l) = 0 (the type-(c) problems of lemma 09). On the common window W it is still decreasing and concave but may stay positive (e.g. l = 1, M_Q = diag(0, 4): alpha_2 > 0 on all of W = (0, pi)).

Proof. With R(0) = 0, the Q-field solution is R = a sinh(omega x)/omega and L = a(cosh(omega x) + m sinh(omega x)/omega)/lambda. With L(0) = 0, the U-field solution is L = b sinh(omega x)/omega and R = b(m sinh(omega x)/omega - cosh(omega x))/lambda. On W, every S = sinh(omega l)/omega is nonzero. Inserting these into the conditions at l and eliminating b gives lambda^2 c = X B X^dagger A c for c = diag(S_Q) a. The map c -> eigenfunction is linear and injective. Finally, lambda^2 - X B X^dagger A and lambda^2 - A X B X^dagger have kernels of equal dimension for lambda != 0. For item 2, the expansion x coth x = 1 + sum_{n >= 1} 2x^2/(x^2 + n^2 pi^2) gives

```
z^{1/2} coth(z^{1/2} l) = (1/l) [1 + sum_{n >= 1} 2 z l^2 / (z l^2 + n^2 pi^2)],
```

which is strictly increasing and concave in z > -pi^2/l^2. Substituting z = m^2 - lambda^2 gives the claim. The function tends to -infinity at the edge of its own window, and alpha_i = 0 means Q_L(l) = 0. ■

Example 12.5 (one flavor, no bulk mass: the interpolation). For N_Q = N_U = 1 and m = 0 we get alpha = beta = lambda cot(lambda l), and the equation reduces to tan^2(lambda l) = abs(X)^2. With theta = arctan abs(X) in [0, pi/2], the positive spectrum is

```
theta/l,   (n pi - theta)/l,   (n pi + theta)/l   (n >= 1).
```

At X = 0 the light pair is the pair of chiral zero modes of types (a) and (b): zero then has multiplicity two and the positive eigenvalues are n pi/l, each of multiplicity two; theta = pi/2 is reached only as a limit. As abs(X) -> infinity the conditions tend to Q_L(l) = 0 and U_R(l) = 0, so each field becomes type (c), and the spectrum tends to two copies of (r - 1/2) pi/l. The light pair theta/l therefore interpolates continuously between the chiral zero modes and the Connes-Lott mass pi/(2l) of job 05. Its eigenvectors span a Gamma-graded two-dimensional space on which D acts as (theta/l) sigma_1, with the Gamma = +1 vector carried by (Q_L, U_L) and the Gamma = -1 vector by (Q_R, U_R). ■

Light spectrum and towerless limit

Let F_Q = diag(f_{Q_i}), F_U = diag(f_{U_j}) and Y_0 = F_Q X F_U. Let Lambda_Q = min_i Lambda_i^Q, Lambda_U = min_j Lambda_j^U and lambda_* = min(Lambda_Q, Lambda_U). For 0 < lambda < lambda_* the matrices A and B are positive, and we put Y(lambda) = A(lambda)^{1/2} X B(lambda)^{1/2}. Singular values sigma_k are listed in decreasing order.

Theorem 12.6.

1. lambda in (0, lambda_*) is an eigenvalue of multiplicity m iff exactly m indices k satisfy sigma_k(Y(lambda)) = lambda.
2. For each k, the equation lambda = sigma_k(Y(lambda)) has at most one solution lambda_k in (0, lambda_). *It has one whenever 0 < sigma_k(Y_0) < lambda_.*
3. Every such solution satisfies

   ```
   sigma_k(Y_0) (1 - lambda_k^2/Lambda_Q^2 - lambda_k^2/Lambda_U^2) <= lambda_k <= sigma_k(Y_0).
   ```

Proof. Each Lambda lies in the window of its own field, so (0, lambda_) *is contained in W. (1) Conjugating by A^{1/2} turns A X B X^dagger into Y Y^dagger, and dim ker(lambda^2 - Y Y^dagger) = #{k : sigma_k(Y) = lambda}. Then apply Theorem 12.4. (2) A and B decrease entrywise in lambda, and sigma_k(P Z Q) <= norm(P) norm(Q) sigma_k(Z). Hence sigma_k(Y(lambda)) is continuous and nonincreasing, so lambda - sigma_k(Y(lambda)) is strictly increasing. It is negative near 0 and at least*

*lambda_* - sigma_k(Y_0) > 0 near lambda_*. (3) The upper bound is (2). For the lower bound, concavity places alpha_i above its chord: alpha_i(lambda) >= alpha_i(0)(1 - lambda^2/(Lambda_i^Q)^2), and likewise for beta_j. Therefore Y(lambda) = D_1 Y_0 D_2 with diagonal D_1, D_2 whose entries are at least (1 - t_Q)^{1/2} and (1 - t_U)^{1/2}, where t_Q = lambda^2/Lambda_Q^2 and t_U = lambda^2/Lambda_U^2. Hence sigma_k(Y) >= ((1 - t_Q)(1 - t_U))^{1/2} sigma_k(Y_0) >= (1 - t_Q - t_U) sigma_k(Y_0). The last step uses a^2 + ab + b^2 <= a + b when a + b <= 1; for a + b > 1 the bound is trivial. ■

Theorem 12.7 (towerless limit of the texture). Fix the localization parameters c_i^Q and c_j^U and the matrix Y_hat = X/l, and let l -> 0. Put Phi_Q = diag(phi(c_i^Q)), Phi_U = diag(phi(c_j^U)) and M_F = Phi_Q Y_hat Phi_U. For every E > 0 and all small l, the spectrum of D_X in [-E, E] consists of:

- 0 with multiplicity N_Q + N_U - 2 rank Y_hat;
- plus/minus lambda_k(l) for those k <= rank Y_hat with sigma_k(M_F) <= E (since lambda_k(l) < sigma_k(M_F) and lambda_k(l) -> sigma_k(M_F)), where

  ```
  abs(lambda_k(l) - sigma_k(M_F)) <= sigma_k(M_F)^3 l^2 (1/L_Q^2 + 1/L_U^2),
  ```

  and L_Q = l Lambda_Q and L_U = l Lambda_U depend only on the c's.

All eigenvalues outside zero and the light set {+/- lambda_k} have absolute value greater than lambda_* = min(L_Q, L_U)/l, which tends to infinity. For E > norm(M_F) the light space has dimension N_Q + N_U and, after grading-preserving unitary identifications, its compressed Dirac matrices converge to D_F below. Calling the limit a spectral triple requires specifying an algebra and its representation; with that understood, the towerless remnant is the finite even triple with H_F = C^{N_Q} (+) C^{N_U}, Gamma_F = diag(1, -1) and

```
D_F = [[0, M_F], [M_F^dagger, 0]],   M_F = Phi_Q Y_hat Phi_U,
```

up to unitaries on each chirality, which is the Yukawa block of the Chamseddine-Connes finite Dirac operator.

Proof. By item 1 of Theorem 12.4, Y_0 = (1/l) Phi_Q X Phi_U = M_F. Rescaling x -> x/l shows that l Lambda_i depends on c_i alone. Apply Theorem 12.6 with sigma_k(Y_0) fixed and lambda_* -> infinity, and use lambda_k <= sigma_k(M_F) in the error term. Theorem 12.2 counts the zero modes. ■

At fixed fermion masses, the towerless limit therefore requires X = l Y_hat -> 0: the boundary condition tends to the chiral type (a) (+) (b), while the normalized profiles at the wall grow like l^{-1/2}.

Hierarchies from localization

Lemma 12.8 (graded singular values). Let M = F X G with F = diag(f_1 <= … <= f_N) and G = diag(g_1 <= … <= g_N) positive, and X in M_N(C). For p = 1, …, N let T_p = {N-p+1, …, N}, Delta_p = abs(det X_{T_p T_p}), D_p = max over p-subsets I, J of abs(det X_{IJ}), eps_p^F = f_{N-p}/f_{N-p+1}, eps_p^G = g_{N-p}/g_{N-p+1} (both 0 for p = N), and K_p = binom(N, p) - 1. Then

```
f_{T_p} g_{T_p} Delta_p  <=  sigma_1 ... sigma_p  <=  f_{T_p} g_{T_p} [Delta_p + D_p ((1 + K_p
↪  eps_p^F)(1 + K_p eps_p^G) - 1)],
```

where f_I is the product of f_i over i in I.

Proof. By Cauchy-Binet, the p-th compound is Lambda^p M = Lambda^p F Lambda^p X Lambda^p G, with entries f_I g_J det X_{IJ}, and sigma_1 … sigma_p = norm(Lambda^p M). The lower bound is the (T_p, T_p) entry. For the upper bound, norm(A) <= sum abs(A_{IJ}). Any p-subset I != T_p trades at least one index of T_p for an index <= N - p, so f_I/f_{T_p} <= eps_p^F. Hence sum_I f_I <= f_{T_p}(1 + K_p eps_p^F), and similarly for g. ■

Corollary 12.9 (masses from localization). Assume every trailing minor Delta_p is nonzero. Put E_0 = E_N = 0 and E_p = (D_p/Delta_p)((1 + K_p eps_p^F)(1 + K_p eps_p^G) - 1) for 1 <= p < N. Then

```
1/(1 + E_{p-1}) <= sigma_p / (f_{N-p+1} g_{N-p+1} Delta_p/Delta_{p-1}) <= 1 + E_p,
```

by dividing the bounds of Lemma 12.8 for p and p - 1. Indexing generations by k = N - p + 1, m_k = f_k g_k (Delta_{N-k+1}/Delta_{N-k})(1 + O(eps)) as eps = max_p (eps_p^F + eps_p^G) -> 0, for fixed X and N (uniform statements need uniform bounds on D_p/Delta_p). Without the minor condition the formula fails: F = G = diag(t, 1) and X = sigma_1 give Delta_1 = 0 while both singular values equal t. With M_F = Phi_Q Y_hat Phi_U and Y_hat anarchic (all minors of order one), the masses follow phi(c_k^Q) phi(c_k^U). Since phi(c) = (2c)^{1/2} e^{-c} (1 + O(e^{-2c})) for large c, order-one differences of the c's between generations give exponential hierarchies (the ratio also carries the factor sqrt(c_k^Q c_k^U/(c_{k+1}^Q c_{k+1}^U)), bounded when adjacent c's differ by bounded amounts):

```
m_k / m_{k+1} ~ e^{-(c_k^Q - c_{k+1}^Q) - (c_k^U - c_{k+1}^U)}   up to factors of order one. ■
```

Remark 12.10 (mixing; numerical, not proved here). For an up sector and a down sector sharing Phi_Q, we drew 500 anarchic complex samples with c^Q = (3, 2, 0.5). The medians of abs(V_ij)/(phi(c_i^Q)/phi(c_j^Q)) were 1.02, 0.92 and 1.09 for (ij) = (12), (23) and (13). So the left mixing follows the doublet localization alone, abs(V_ij) ~ phi(c_i^Q)/phi(c_j^Q), as in the split-fermion and warped-flavor literature (Huber-Shafi 2001; Agashe-Perez-Soni 2005; Casagrande et al. 2008). An upper bound with explicit constants, |V_ij| <= 3 K_u K_d f_i/f_j for i < j, is proved in the referee report appended below, which also notes that there is no universal lower bound; Astra 6 wrote out the cofactor step that the report names (validation, item 10).

Consequences for STRATEGY

- **T3 (remaining part).** The boundary Yukawa X deforms the chiral types (a) (+) (b) (X = 0) continuously into two copies of the vectorlike type (c) (abs(X) -> infinity). The index N_Q - N_U is constant along the way. What changes is the symmetry: X reduces G to H_X, and the zero modes lift exactly as far as the H_X-index allows (Corollary 12.3).
- **T8.** In the towerless limit, the Yukawa block of D_F is Phi_Q Y_hat Phi_U. The texture of the finite geometry is localization data (the c's) times an order-one boundary matrix, the masses are internal momenta and overlaps, and the hierarchies are exponential in the c's (Corollary 12.9).

Numerical check (Claude)

Method: roots of the secular equation are bracketed on fine grids and refined by Brent's method; the shooting check integrates the ODE from x = 0 with scipy `solve_ivp` (rtol 1e-12) and finds the zeros of the boundary-condition determinant at x = l.

- **Secular equation against ODE shooting** (scipy `solve_ivp`, rtol 1e-12, real X). For one massless flavor and X = 0.3, 1 and 5, the shooting roots agree with (n pi +/- arctan X)/l to 6 digits. For three flavors with masses (-3, -1.5, 0.2) and (2.5, 1, -0.3), l = 1 and random X, both methods give 0.024689, 0.048653, 0.534569 and 2.03374.
- **Theorem 12.6** (c^Q = (2, 1, 0), c^U = (1.5, 0.5, -0.5), X = 0.05 x Gaussian, lambda_* = 0.577). The eigenvalues in the window are 0.000576, 0.035602 and 0.061096, against sigma_k(Y_0) = 0.000576, 0.035632 and 0.061188. All three two-sided bounds hold.
- **Theorem 12.7** (c^Q = (4, 2, 0.3), c^U = (3, 1, -0.2), Gaussian Y_hat, sigma(M_F) = 0.000735, 0.072042, 0.786929). The relative errors of the three light eigenvalues are:

l	relative errors	lambda_*
1	1.5e-5, 1.1e-2, 2.5e-1	0.147
0.1	below 1e-6, 1.1e-4, 4.2e-3	1.47
0.03	below 1e-6, 1.0e-5, 3.9e-4	4.90

The errors decrease as l^2.

- **Lemma 12.8:** no violation in 2000 random complex 3 x 3 cases with log-uniform gradings over 8 e-folds (6000 inequalities).

Referee report (gpt-6.1-terra, effort max, 3 Nov 2026)

Prompt: prompts/terra_check_12.md. Reproduced as received; the corrections above implement it. Claude checked the three counterexamples (M_Q = diag(0, 4); arctan(l)/l -> 1; F = G = diag(t, 1), X = sigma_1) by hand.

We take $\ell > 0$ and the bulk masses to be real, as in Lemma 09. All arguments below are analytic; we used no numerical computation.

Lemma 12.1 Verdict: Correct.

At $x = \ell$, the free boundary values are $Q_L(\ell)$ and $U_R(\ell)$. Substitution of the boundary conditions into the boundary form gives zero, and the allowed boundary values have half the total boundary dimension. The conditions at $x = 0$ are likewise maximal isotropic. The adjoint boundary conditions therefore coincide with the stated ones, so D_X is self-adjoint.

The graph norm controls the H^1-norm, and the embedding $H^1([0,\ell]) \hookrightarrow L^2([0,\ell])$ is compact. Thus the resolvent is compact. Finally, Γ preserves the boundary conditions and anticommutes with the differential expression.

Theorem 12.2 Verdict: Correct.

The zero-mode equations give

$$L(x) = e^{Mx}a, \qquad R(x) = e^{-Mx}b.$$

The conditions at 0 remove Q_R and U_L, and those at ℓ impose exactly the two displayed kernel conditions.

The endpoint exponentials are invertible, so the two chiral kernel dimensions are $N_Q - \operatorname{rank} X$ and $N_U - \operatorname{rank} X$. Their difference is $N_Q - N_U$.

Corollary 12.3 Verdict: Correct with amendment. The neutrino conclusion holds for every X that preserves the stated electromagnetic symmetry, meaning $H \subseteq H_X$. The phrase "for every X" needs that qualification.

The endpoint exponentials commute with the group actions. They identify the actual chiral kernels equivariantly with $\ker X^\dagger$ and $\ker X$. The orthogonal complements are equivariantly isomorphic through X, proving the displayed index identity. Both kernels vanish exactly when X is invertible, in which case X is an equivariant isomorphism.

If electromagnetic symmetry is not required, the neutrino assertion can fail. For one lepton generation with zero bulk masses, take

$$Q_L = (\nu_L, e_L), \qquad U_R = e_R, \qquad X = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The left kernel is then spanned by e_L, while ν_L is coupled to e_R. This X breaks electromagnetic charge. If X intertwines the electromagnetic representations and there is no neutral right-handed lepton, the neutrino subspace lies in $\ker X^\dagger$, as claimed.

Theorem 12.4 Verdict: Error. The secular equation, multiplicity formula, endpoint values, monotonicity and concavity are correct. The assertion that *each* function has a zero **on the common window** W is false.

For example, take $\ell = 1$, $M_Q = \operatorname{diag}(0, 4)$, and $M_U = (0)$. Then $W = (0, \pi)$, but throughout W,

$$\alpha_2(\lambda) = \sqrt{16 - \lambda^2}\,\coth\sqrt{16 - \lambda^2} + 4 > 0.$$

Its individual lowest mixed-boundary eigenvalue is greater than 4, hence lies outside W. The proof's assertion that every function tends to $-\infty$ at the edge of W fails for the same reason.

The corrected zero statement uses each mass's **individual** first-pole window

$$0 < \lambda < \sqrt{m^2 + \pi^2/\ell^2}.$$

On that interval the corresponding α or β has exactly one zero, at the lowest positive eigenvalue of its mixed-boundary problem. The U problem is equivalent to the type-(c) problem after exchanging components and changing the mass sign.

For the monotonicity and concavity, write $s = \lambda^2$ and $a_n = (n\pi/\ell)^2$. Differentiating the displayed partial-fraction expansion on the individual window gives

$$\frac{d}{ds}(\omega \coth(\omega\ell)) = -\frac{2}{\ell}\sum_{n\geq 1}\frac{a_n}{(m^2 - s + a_n)^2} < 0,$$

$$\frac{d^2}{ds^2}(\omega \coth(\omega\ell)) = -\frac{4}{\ell}\sum_{n\geq 1}\frac{a_n}{(m^2 - s + a_n)^3} < 0.$$

The function is positive at $s = 0$ after the appropriate $\pm m$ is added, and tends to $-\infty$ at its own first pole. This proves the corrected zero claim.

For the secular equation, the proof's boundary variables c, d satisfy

$$\lambda c = -XBd, \qquad \lambda d = -X^\dagger Ac.$$

Eliminating d gives $\lambda^2 c = XBX^\dagger Ac$. For $\lambda \neq 0$, the kernels of $\lambda^2 I - CA$ and $\lambda^2 I - AC$, with $C = XBX^\dagger$, are isomorphic via A, with inverse C/λ^2. This also proves the stated multiplicity.

Example 12.5 Verdict: Correct with amendment. At $X = 0$, omit $\theta/\ell = 0$ from the **positive** spectrum; instead, zero has multiplicity two. The positive eigenvalues are then $n\pi/\ell$, each with multiplicity two. For finite X, $\theta < \pi/2$; the endpoint $\theta = \pi/2$ is a limit.

For all positive λ, direct substitution of the massless solutions into the boundary conditions gives

$$\sin^2(\lambda\ell) - |X|^2\cos^2(\lambda\ell) = 0.$$

For $X \neq 0$, its roots are exactly the listed values. This derivation also includes the roots at $n\pi/\ell$ when $X = 0$, even though the secular formula of Theorem 12.4 was derived only before the first pole. The negative spectrum follows by grading symmetry. The stated two-dimensional graded block and the limiting mixed-boundary spectrum follow from these roots and their eigenvectors.

Theorem 12.6 Verdict: Correct.

Although Theorem 12.4 requires the correction above, it still supplies what is needed here. The smallest individual first pole belongs to some field; that field's first zero lies before its pole. Consequently, λ_* lies before the smallest first pole, and $(0, \lambda_*) \subset W$.

On this interval A, B are positive. Conjugation reduces the secular equation to the eigenvalue equation for $Y(\lambda)Y(\lambda)^\dagger$, proving the multiplicity claim. If $\lambda_2 > \lambda_1$, then

$$Y(\lambda_2) = P\,Y(\lambda_1)\,Q$$

for diagonal contractions P, Q. Thus every singular value is continuous and nonincreasing, so it meets the strictly increasing function λ at most once. The stated condition at $\lambda = 0$ gives a crossing before λ_*.

Concavity gives, with $t_Q = \lambda^2/\Lambda_Q^2$ and $t_U = \lambda^2/\Lambda_U^2$,

$$\sigma_k(Y(\lambda)) \geq \sqrt{(1-t_Q)(1-t_U)}\,\sigma_k(Y_0) \geq (1 - t_Q - t_U)\sigma_k(Y_0).$$

Together with $\sigma_k(Y(\lambda)) \leq \sigma_k(Y_0)$, this proves both bounds.

Theorem 12.7 Verdict: Error. The fixed-energy spectrum cannot contain every listed light eigenvalue for every $E > 0$.

Take one massless flavor, $\widehat{Y} = 1$, and $E = 1/2$. Then $X = \ell$, and Example 12.5 gives the positive light eigenvalue

$$\lambda(\ell) = \frac{\arctan \ell}{\ell} \longrightarrow 1.$$

For sufficiently small ℓ, it is outside $[-1/2, 1/2]$, contrary to the stated list.

The corrected fixed-energy statement includes only those $\pm\lambda_k(\ell)$ with $\lambda_k(\ell) \le E$. Equivalently, for each fixed $E > 0$ and all sufficiently small ℓ, include exactly those indices for which

$$\sigma_k(M_F) \le E.$$

The equality case is included because the strict decrease of A and B gives $\lambda_k(\ell) < \sigma_k(M_F)$ for every positive light eigenvalue.

The displayed error estimate follows from Theorem 12.6:

$$0 < \sigma_k(M_F) - \lambda_k(\ell) \le \sigma_k(M_F)^3 \ell^2 \left(L_Q^{-2} + L_U^{-2} \right).$$

Zero has the stated multiplicity. Apart from zero and the **entire** light set $\{\pm\lambda_k\}$, the eigenvalues have **absolute value** greater than λ_* for small ℓ. At λ_*, the limiting matrices A, B are nonnegative and the relevant singular values are at most $\|M_F\|$, whereas $\lambda_* \to \infty$; hence equality at the boundary is excluded for small ℓ.

For $E > \|M_F\|$, the full low-energy space has dimension $N_Q + N_U$. After grading-preserving unitary identifications, its compressed Dirac matrices converge to the displayed D_F. Calling the limit a spectral triple additionally requires an algebra and its representation to be specified.

Lemma 12.8 Verdict: Correct.

The entries of $\bigwedge^p M$ are $f_I g_J \det X_{IJ}$, and its operator norm is $\sigma_1 \cdots \sigma_p$. Its (T_p, T_p) entry gives the lower bound.

For every $I \neq T_p$, at least one factor from the top p indices is replaced by one no larger than $N - p$, so $f_I / f_{T_p} \le \epsilon_p^F$; the analogous estimate holds for g. Bounding the operator norm by the sum of absolute entries, while keeping the (T_p, T_p) entry separate, gives the upper bound. For $p = N$, both bounds reduce to the determinant identity.

Corollary 12.9 Verdict: Error. The displayed mass formula is not valid under the hypotheses of Lemma 12.8: its denominator can vanish, and its leading term can be zero when the singular value is positive.

For $0 < t < 1$, take

$$F = G = \operatorname{diag}(t, 1), \qquad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then FXG has both singular values equal to t, but $\Delta_1 = 0$ and $\Delta_2 = 1$. The formula for m_2 gives zero, and the formula for m_1 divides by zero.

A sufficient correction is to require every trailing minor Δ_p to be nonzero and to define the small parameter. More precisely, put $E_0 = E_N = 0$ and, for $1 \le p < N$,

$$E_p = \frac{D_p}{\Delta_p} \left[(1 + K_p \epsilon_p^F)(1 + K_p \epsilon_p^G) - 1 \right].$$

Lemma 12.8 then gives the explicit bounds

$$\frac{1}{1 + E_{p-1}} \le \frac{\sigma_p}{f_{N-p+1} g_{N-p+1} \Delta_p / \Delta_{p-1}} \le 1 + E_p.$$

Indeed, apply the lemma to the products of the first p and first $p-1$ singular values and divide their bounds. For fixed X, N, if

$$\epsilon = \max_{p<N}(\epsilon_p^F + \epsilon_p^G) \longrightarrow 0,$$

these inequalities give the stated $1 + O(\epsilon)$ formula. Uniform estimates require uniform bounds on D_p/Δ_p.

The localization conclusion follows under quantitative order-one upper and lower bounds on the relevant minors. For large positive c, the mass ratio also contains the factor

$$\sqrt{\frac{c_k^Q c_k^U}{c_{k+1}^Q c_{k+1}^U}}.$$

It is bounded when the adjacent large c's differ by bounded amounts; without such control it cannot be suppressed uniformly as an order-one factor.

Remark 12.10 Bound. We can prove an upper bound independent of the right-hand localization profiles. It does not prove the reported sample medians or a universal lower bound.

For each sector $s = u, d$, let

$$M_s = FX_sG_s, \qquad F = \mathrm{diag}(f_1 \le f_2 \le f_3), \qquad G_s = \mathrm{diag}(g_1^{(s)} \le g_2^{(s)} \le g_3^{(s)}),$$

with all diagonal entries positive. Assume X_s is invertible and its trailing minors $\Delta_1^{(s)}, \Delta_2^{(s)}$ are nonzero. With $D_p^{(s)}$ as in Lemma 12.8, define

$$K_s = \left(1 + \frac{\sqrt{3}D_1^{(s)}}{\Delta_1^{(s)}}\right)\left(1 + \frac{\sqrt{3}D_2^{(s)}}{\Delta_2^{(s)}}\right).$$

Choose the columns of the left singular-vector matrices U_s in increasing order of singular value, and put $V = U_u^\dagger U_d$. Then, for $i < j$,

$$\boxed{|V_{ij}| \le 3K_uK_d\,\frac{f_i}{f_j}.}$$

To prove this, set $H_s = M_sM_s^\dagger$, and temporarily suppress s. Write

$$a = \frac{\sqrt{3}D_1}{\Delta_1}, \qquad b = \frac{\sqrt{3}D_2}{\Delta_2}.$$

For the largest eigenvalue λ_3 of H,

$$\lambda_3 \ge f_3^2 g_3^2 \Delta_1^2, \qquad H_{ii} \le 3D_1^2 f_i^2 g_3^2.$$

Cauchy–Schwarz therefore gives $|U_{i3}| \le af_i/f_3$. Applying the same argument to H^{-1}, and using $(X^{-1})_{11} = \det X_{\{2,3\},\{2,3\}}/\det X$, gives $|U_{i1}| \le bf_1/f_i$. Cofactor identities for the unitary U bound the two remaining off-diagonal entries. Together,

$$|U_{ij}| \le (1+a)(1+b)\min\left(\frac{f_i}{f_j}, \frac{f_j}{f_i}\right).$$

Summing the three terms in $V_{ij} = \sum_k \overline{(U_u)_{ki}}(U_d)_{kj}$ proves the boxed inequality.

There is no corresponding positive universal lower bound: identical up and down matrices have $V = I$, including when their minors are nonzero. The numerical medians also require the sampling distribution and sampled matrices, neither of which is specified in the remark.

Amendments and errors

- **12.3:** Restrict the neutrino assertion to couplings preserving electromagnetic symmetry.
- **12.4:** State the unique-zero claim on each field's individual first-pole window; it can fail on the common window.
- **12.5:** Treat zero separately at $X = 0$, and treat $\theta = \pi/2$ as a limit.
- **12.7:** Filter the light eigenvalues by E; state the gap in absolute value; specify an algebra and representation for a spectral-triple claim.
- **12.9:** Require nonzero trailing minors and define the separation parameter and its dependence. Retain the square-root localization factors unless they are uniformly bounded.

Overall: 9 statements checked: 4 correct, 2 correct with amendment, 0 gaps, and 3 errors.

Literature (../LITERATURE.md, section C). The one-flavour spectrum of Example 12.5 is equation (5.6) of Csaki-Grojean-Hubisz-Shirman-Terning, Fermions on an interval (2004); the warped multi-flavour analogue of the light spectrum and of the localization textures is in Casagrande et al. 2008 (equations 65, 87, 90, 94-97). No NCG paper deriving D_F from localization was found.

Part 13

T4: gauge fluctuations of the chiral interval

Prompt: prompts/13 (STRATEGY T4). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5 by hand: Results 2, 3, 4, 5 and 6 (the compression integrals, the counterexample B = -2a/(3 pi) with trivial Wilson line, the Neumann-Dirichlet mass, the 1 : 3 ratio of the a_4 gauge and scalar terms with the canonical normalization, and the eigenvalues 0 and 6m^2); Results 7 and 8 reproduce the standard CCM relations (8 M_W^2 = sum N_f m_f^2, M_W/M_Z = sqrt(5/8) with Q_2 = 12, Q_Y = 20).

Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): of 8 statements, 7 are correct and Proposition 8 needs one amendment, recorded there.

We prove an exact compression formula, but it does not identify the continuum gauge theory with the full Connes–Lott gauge theory. The fermion boundary conditions impose no boundary conditions on the gauge potentials, and fixed spectral compression produces only links of the form `phi = m I + i B`, with `B` Hermitian. The proposed ratio `M_W/m = sqrt(3/8)` is correct under the three-generation Standard Model CCM normalization, with one massive colour triplet and all other Dirac masses zero. It is not a consequence of the interval alone: the minimal two-sheet spectral action has a different gauge representation and gives a massive relative gauge field of mass `sqrt(6) m`.

1. Theorem (inner fluctuations and boundary conditions).

Let M be a closed Riemannian spin four-manifold. We use Hermitian Euclidean gamma matrices, $D_M = -i\gamma^\mu \nabla^S_\mu$, and

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad D = D_M \otimes 1 + \gamma_5 \otimes E\partial_x.$$

Let

$$\mathcal{H} = L^2(S_M) \otimes L^2([0,l], \mathbb{C}^2) \otimes \mathbb{C}^n,$$

with the algebra acting by the same matrix multiplication on L and R. The interval boundary conditions are

$$R(y,0) = 0, \qquad L(y,l) = 0.$$

We consider complex inner fluctuations $D + \omega$, where $\omega = \sum_j a_j[D, b_j]$ and $\omega^* = \omega$. No real structure or additional bimodule is assumed.

Then the fluctuated operators are exactly

$$\boxed{D_A = -i\gamma^\mu(\nabla^S_\mu + iA_\mu) + \gamma_5 E(\partial_x + iA_5),}$$

where $A_\mu(y,x)$ and $A_5(y,x)$ are arbitrary smooth Hermitian $n \times n$ matrices. They are connections on the trivial $U(n)$ bundle over $M \times I_l$. Restriction to traceless potentials gives the $SU(n)$ case.

The fermion conditions induce **no Dirichlet, Neumann, or endpoint-value conditions on A_μ or A_5.**

Proof. For a matrix-valued function b,

$$[D, b] = -i\gamma^\mu \partial_\mu b + \gamma_5 E \partial_x b.$$

Consequently, represented one-forms have precisely these five Clifford components. Self-adjointness makes their connection coefficients anti-Hermitian, hence iA_α with A_α Hermitian. Conversely, smooth matrix-valued one-forms are finite sums $a\,db$, using local coordinates and a partition of unity.

Every such fluctuation is a bounded self-adjoint perturbation of D, so D_A is self-adjoint on the original domain. A bounded perturbation need not map that domain into itself; requiring this would impose an additional, unjustified condition.

Every smooth $g : M \times I_l \to U(n)$ preserves the fermion boundary conditions:

$$(gR)(y,0) = 0, \qquad (gL)(y,l) = 0.$$

The usual transformation

$$A_\alpha \longmapsto gA_\alpha g^* + i(\partial_\alpha g)g^*$$

therefore allows arbitrary endpoint values of g.

For smooth sections in $\mathrm{Dom}(D_A^2)$, the additional conditions are

$$(\partial_x + iA_5)L(y,0) = 0, \qquad (\partial_x + iA_5)R(y,l) = 0.$$

These are conditions on the fermions, not on A_5. They follow by applying the original boundary conditions to $D_A\psi$.

An orbifold extension would add different hypotheses. Requiring A_μ to be even and A_5 odd under endpoint reflections gives $\partial_x A_\mu = 0$ and $A_5 = 0$ there. Those parity requirements do not follow from the stated fermion domain. □

2. Theorem (exact fixed-projection compression and its image).

Let

$$m = \frac{\pi}{2l}, \qquad u_L = \sqrt{\frac{2}{l}}\begin{pmatrix}\cos(mx)\\ 0\end{pmatrix}, \qquad u_R = \sqrt{\frac{2}{l}}\begin{pmatrix}0\\ \sin(mx)\end{pmatrix},$$

and let P project onto their span, tensored with the spacetime and gauge factors. This is the unfluctuated lowest-pair projection of Lemma 05, Part 3.

Define

$$w_L(x) = \frac{2}{l}\cos^2(mx), \qquad w_R(x) = \frac{2}{l}\sin^2(mx), \qquad k(x) = \frac{1}{l}\sin\frac{\pi x}{l}.$$

Then, under the identification $P\mathcal{H} \simeq L^2(S_M) \otimes (\mathbb{C}^n_L \oplus \mathbb{C}^n_R)$,

$$\boxed{PD_AP = -i\gamma^\mu\left(\nabla^S_\mu + i\begin{pmatrix}A_{L,\mu} & 0\\ 0 & A_{R,\mu}\end{pmatrix}\right) + \gamma_5\begin{pmatrix}0 & \phi\\ \phi^* & 0\end{pmatrix},}$$

where

$$\boxed{A_{L,\mu} = \int_0^l w_L A_\mu\,dx, \qquad A_{R,\mu} = \int_0^l w_R A_\mu\,dx,}$$

and

$$\boxed{\phi = mI_n + iB, \qquad B = \int_0^l k(x) A_5(y,x)\, dx = B^*.}$$

Here ϕ denotes the total link; its additive fluctuation is $\phi - mI_n$.

The exact image consists of arbitrary Hermitian sheet connections together with links satisfying

$$\boxed{\phi + \phi^* = 2mI_n.}$$

In particular, not every Connes–Lott fluctuation arises by this compression.

Proof. The spacetime term is diagonal in interval chirality, giving the two weighted averages. The unfluctuated interval operator satisfies

$$D_I u_R = m u_L, \qquad D_I u_L = m u_R.$$

The upper-right matrix element of iEA_5 is

$$i\frac{2}{l}\int_0^l \cos(mx)\sin(mx) A_5\, dx = iB;$$

the lower-left element is $-iB$. This proves the operator identity without approximation.

The elementary integrals

$$\int_0^l w_L\, dx = \int_0^l w_R\, dx = 1, \quad \int_0^l w_L \cos\frac{\pi x}{l}\, dx = \frac{1}{2}, \quad \int_0^l w_R \cos\frac{\pi x}{l}\, dx = -\frac{1}{2}$$

show that any prescribed A_L, A_R have the lift

$$A_\mu(y,x) = \frac{A_{L,\mu} + A_{R,\mu}}{2} + (A_{L,\mu} - A_{R,\mu})\cos\frac{\pi x}{l}.$$

Since $\int_0^l k\, dx = 2/\pi$, any Hermitian B has the lift

$$A_5(y,x) = \frac{\pi}{2} B(y).$$

Thus the stated image is exact.

By contrast, the finite algebra

$$M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$$

with $D_F = m\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ admits an arbitrary complex matrix as its total link. Indeed, writing p_L, p_R for its central projections, $p_L[D_F, p_R]$ supplies an arbitrary upper-right block after multiplication by the left algebra; adding its adjoint supplies the lower-left block. The allowed finite fluctuation $\phi = 0$, for example, has no continuum lift under the displayed compression.

For $SU(2)$, B is additionally traceless. The continuum A_5 is adjoint-valued; an electroweak Higgs doublet needs a further identification.

If an additional finite factor is included in the internal-first convention, and the gauge algebra commutes with D_0, Γ_0, its compressed internal operator is

$$D_0 \otimes 1 + \Gamma_0 \otimes \begin{pmatrix} 0 & \phi \\ \phi^* & 0 \end{pmatrix}.$$

There is no extra overlap factor. If $[D_0, a] \neq 0$, additional finite scalar fluctuations occur. Moreover, the cited Theorem 5 explicitly restricts the interval algebra to constants for its real-triple assertion; it does not supply a Standard Model real structure for the full compressed algebra. □

3. No-go theorem (fixed compression is not a gauge-covariant Wilson-link construction).

Let P be the fixed projection of Result 2. Its link $\phi[A_5]$ is not a functional of the Wilson line alone, and fixed compression does not descend to continuum gauge-equivalence classes.

Proof. It suffices to consider $U(1)$ potentials independent of y. Define parallel transport by

$$W'(x) = -iA_5(x)W(x), \qquad W(0) = 1.$$

Then

$$D_{I,A} = WD_IW^*.$$

Multiplication by W preserves both fermion boundary conditions, so the complete interval spectrum is unchanged.

Take

$$A_5(x) = a\cos\frac{2\pi x}{l}.$$

Its Wilson line is $W(l) = 1$, but

$$B = \frac{a}{\pi}\int_0^\pi \sin t\cos(2t)\,dt = -\frac{2a}{3\pi}.$$

Thus

$$\phi = m - \frac{2ia}{3\pi}, \qquad |\phi| = \sqrt{m^2 + \frac{4a^2}{9\pi^2}}.$$

The zero connection and this connection are gauge equivalent even through a transformation equal to 1 at both endpoints, yet their fixed compressed operators have different internal eigenvalues.

The obstruction is that multiplication by an x-dependent gauge transformation generally fails to preserve $P\mathcal{H}$. Compression of multiplication is not multiplicative, and PgP generally fails to be unitary.

The normalization difference is already visible for constant $A_5 = a$:

$$\frac{\phi}{m} = 1 + i\frac{4l}{\pi^2}a, \qquad W(l)^* = e^{ial}.$$

Neither equality with the Wilson link nor a universal correction to the mass follows from these expressions.

A different, gauge-adapted construction is possible. Use the transported interval frames

$$u_L(x)W(x), \qquad u_R(x)W(x)W(l)^*.$$

Their vertical Dirac matrix is exactly

$$\begin{pmatrix} 0 & mW(l)^* \\ mW(l) & 0 \end{pmatrix}.$$

Its link obeys $\phi^*\phi = m^2I$, so it also omits general radial Higgs fluctuations. If W depends on y, derivatives of these frames additionally enter the sheet connections.

Finally, compression differs from eliminating the omitted fermions. With $Q = 1 - P$, block elimination adds

$$-PD_AQ\,[Q(D_A - z)Q]^{-1}QD_AP$$

whenever the inverse exists. This follows by solving the Q-component of $(D_A - z)\psi = 0$ and substituting into the P-component. It is the distinction relevant to the effective-operator discussion in research state §6.10. □

4. Lemma (the continuum vector mass requires specified gauge boundary conditions).

Let the quadratic transverse-vector operator in the interval direction be $-\partial_x^2$, with

$$f'(0) = 0, \qquad f(l) = 0.$$

Assume no boundary kinetic terms or other modifications of this operator. Then

$$\boxed{M_{\mathrm{KK}} = \frac{\pi}{2l} = m.}$$

Proof. The Neumann condition selects $f(x) = C\cos(kx)$. The Dirichlet condition gives

$$kl = \left(r + \frac{1}{2}\right)\pi, \qquad r = 0, 1, \dots .$$

The lowest eigenvalue is therefore m^2.

A Dirichlet condition at one endpoint alone does not determine this answer. Dirichlet conditions at both endpoints give lowest mass π/l, whereas two Neumann conditions give a zero mode. Thus the comparison uses specifically the mixed boundary problem listed in Lemma 9.11, and Result 1 does not derive that problem from the fermion boundary conditions. □

5. Proposition (general spectral-action vector-mass formula).

Let H_F be a specified finite fermion space with Hermitian represented gauge generators T_a. Let

$$\mathcal{D} = -i\gamma^\mu(\nabla^S_\mu + iA_\mu) + \gamma_5 V,$$

where $V = V^*$ is the total finite scalar matrix. Use the ordinary spectral trace, an even cutoff function with $f_0 = f(0) > 0$, and the terms through order Λ^0. Assume no additional kinetic terms.

Let V_0 be the chosen constant vacuum value. Define

$$Q_{ab} = \mathrm{Re}\,\mathrm{Tr}_{H_F}(T_a T_b), \qquad K_{ab} = \mathrm{Re}\,\mathrm{Tr}_{H_F}([T_a, V_0]^*[T_b, V_0]).$$

On the faithfully represented gauge algebra, the canonically normalized vector mass-squared matrix is

$$\boxed{\mathcal{M}^2 = \frac{3}{2}\,Q^{-1/2}KQ^{-1/2}.}$$

Consequently, relative to the mixed-boundary continuum mass,

$$\boxed{\frac{M_j^2}{M_{\mathrm{KK}}^2} = \frac{3}{2m^2}\,\lambda_j\!\left(Q^{-1/2}KQ^{-1/2}\right).}$$

Proof. We use the heat coefficient

$$a_4 \supset \frac{1}{16\pi^2}\,\mathrm{Tr}\left(\frac{1}{2}\mathcal{E}^2 + \frac{1}{12}\Omega_{\mu\nu}\Omega^{\mu\nu}\right),$$

as recorded in CCM, Theorem 6.1, equation (6.10). Here $\mathcal{D}^2 = \nabla^*\nabla - \mathcal{E}$.

Write

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu], \qquad D_\mu V = \partial_\mu V + i[A_\mu, V].$$

The relevant endomorphism terms are

$$\mathcal{E}_F = \frac{i}{2}\gamma^\mu\gamma^\nu F_{\mu\nu}, \qquad \mathcal{E}_V = i\gamma^\mu\gamma_5 D_\mu V.$$

The Clifford traces give

$$\mathrm{tr}_S(\mathcal{E}_F^2) = 2F_{\mu\nu}F^{\mu\nu}, \quad \mathrm{tr}_S(\mathcal{E}_V^2) = 4D_\mu V D^\mu V, \quad \mathrm{tr}_S(\Omega_F^2) = -4F_{\mu\nu}F^{\mu\nu}.$$

Hence the relevant Lagrangian is

$$\boxed{\mathcal{L} = c\ \mathrm{Tr}_{H_F}(F_{\mu\nu}F^{\mu\nu}) + 3c\ \mathrm{Tr}_{H_F}(D_\mu V D^\mu V), \qquad c = \frac{f_0}{24\pi^2}.}$$

At $V = V_0$, the quadratic gauge terms are

$$cQ_{ab}F^a_{\mu\nu}F^{b,\mu\nu} + 3cK_{ab}A^a_\mu A^{b,\mu}.$$

The change of variables $A_{\mathrm{can}} = 2\sqrt{c}\,Q^{1/2}A$ gives kinetic normalization $F^2_{\mathrm{can}}/4$ and mass normalization $A_{\mathrm{can}}\mathcal{M}^2 A_{\mathrm{can}}/2$, proving the formula.

Thus the necessary data are the represented gauge generators, their full trace multiplicities, and the vacuum matrix V_0. The common coefficient f_0 cancels. However, the bare finite Dirac matrix need not itself be the vacuum: determining V_0 requires the scalar potential, including its cutoff moments and curvature dependence. All masses below are conditional on the stated vacuum and refer to this leading spectral-action approximation. □

6. Counterexample (the minimal two-sheet action does not give the CCM triplet ratio).

Let

$$H_F = (\mathbb{C}^n_L \oplus \mathbb{C}^n_R) \otimes \mathbb{C}^N, \qquad V_0 = m \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \otimes I_N.$$

Gauge both sheet groups, $U(1)_L \times U(1)_R$ for $n = 1$, or $SU(2)_L \times SU(2)_R$ for $n = 2$. Use the action of Result 5.

Then the diagonal gauge combination is massless and the relative combination has

$$\boxed{M_{\mathrm{relative}} = \sqrt{6}\,m.}$$

This remains true for colour multiplicity $N = 3$.

Proof. For one normalized generator t, set

$$T_L = \begin{pmatrix} t & 0 \\ 0 & 0 \end{pmatrix}, \qquad T_R = \begin{pmatrix} 0 & 0 \\ 0 & t \end{pmatrix}, \qquad \kappa = N\,\mathrm{tr}(t^2).$$

Direct multiplication gives

$$Q = \kappa I_2, \qquad K = 2\kappa m^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Therefore

$$\mathcal{M}^2 = 3m^2 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix},$$

whose eigenvalues are 0 and $6m^2$. The factor N cancels between scalar and gauge normalization. For $SU(2)$, this calculation applies to each $t = \sigma_a/2$.

If only the diagonal group is gauged, every generator commutes with V_0, and all these vector masses vanish. Neither choice realizes the assumed continuum mixed-boundary breaking automatically.

The dependence on the full fermion trace can also be exhibited explicitly. In the $U(1)$ example, add S otherwise identical charged pairs with zero finite Dirac operator. They contribute to Q but not K, giving

$$M^2_{\mathrm{relative}} = 6\frac{N}{N+S}m^2.$$

The mass and colour multiplicity of the original pair have not changed.

There is another distinction if one restricts the continuum field to its lowest mixed-boundary profile before evaluating the finite action:

$$A_\mu(y,x) = a_\mu(y)\cos(mx).$$

Result 2 then gives

$$A_{L,\mu} = \frac{8}{3\pi}a_\mu, \qquad A_{R,\mu} = \frac{4}{3\pi}a_\mu.$$

For a restriction $A_L = c_L a,\ A_R = c_R a$, Result 5 gives the one-field quadratic mass

$$M^2_{\text{restricted}} = 3m^2\frac{(c_L - c_R)^2}{c_L^2 + c_R^2}.$$

Thus this particular restriction gives

$$M^2_{\text{restricted}} = \frac{3}{5}m^2.$$

It is an additional restriction of the action, and differs from the mass eigenvector of the unrestricted two-sheet theory. It demonstrates that retaining a continuum gauge profile and fluctuating the full compressed algebra are different prescriptions. □

7. Corollary (the precise conditional CCM comparison).

Let the finite model instead have the full three-generation Standard Model representation and its usual Higgs doublet. Retain all charged fermion states in the spectral trace, including those whose Yukawa masses vanish. Identify one colour-triplet Dirac mass at the vacuum with

$$m = \frac{\pi}{2l},$$

and suppose every other Dirac mass is zero.

Then, at the matching scale,

$$\boxed{M_W^2 = \frac{3}{8}m^2, \qquad \frac{M_W}{M_{\text{KK}}} = \sqrt{\frac{3}{8}}.}$$

No additional factor from the interval wavefunctions or from Wilson-line normalization multiplies this ratio.

Proof. We can recover the normalization directly from Result 5. For $T_1 = \sigma_1/2$, one Standard Model generation of particles contributes

$$\text{Tr}(T_1^2) = \frac{1}{2} + 3\frac{1}{2} = 2.$$

Antiparticles double this, and three generations give

$$Q_{11} = 12.$$

For a left doublet and its right singlets with mass matrix $Y = \text{diag}(m_u, m_d)$, the particle contribution to K_{11} is

$$2\,\text{tr}(Y^* T_1^2 Y) = \frac{1}{2}(m_u^2 + m_d^2).$$

Including antiparticles and all multiplicities gives

$$K_{11} = \sum_f N_f m_f^2.$$

Therefore

$$M_W^2 = \frac{3}{2}\frac{K_{11}}{12} = \frac{1}{8}\sum_f N_f m_f^2.$$

This is the relation in CCM, Lemma 4.6, equation (4.16), assumed in Lemma 11. Substitution of $K_{11} = 3m^2$ proves the claimed ratio.

The normalization identity is equivalently

$$\frac{f_0 g_2^2}{2\pi^2} = \frac{1}{4}.$$

Its coefficient comes from $Q_{11} = 12$. Applying that coefficient after deleting the other charged states would no longer canonically normalize the gauge kinetic term.

The interval basis already satisfies $\|u_L\| = \|u_R\| = 1$, and $\langle u_L, D_I u_R\rangle = m$. Thus its unfluctuated fermion mass is exactly m. The factor $2/\pi$ belongs to the coupling of a constant A_5 to the fixed compressed link; it does not rescale the background mass. A change of scalar coordinates likewise cannot change a mass ratio when both kinetic and Yukawa terms are transformed consistently.

If other Dirac masses are nonzero, the exact conditional comparison becomes

$$\frac{M_W^2}{M_{\mathrm{KK}}^2} = \frac{1}{8}\left(3 + \sum_{\text{other } f} N_f \frac{m_f^2}{m^2}\right).$$

If "a coloured doublet" means two massive coloured Dirac species, both of mass m, their contribution is $6m^2$, giving $M_W/m = \sqrt{3}/2$ when all other masses vanish. □

8. Proposition (electroweak doublet and the neutral-vector ratio).

Amendment (referee). In the last paragraph of this result, the law phi -> g_L phi g_R^* holds for the link of the independently gauged finite two-sheet completion. The fixed compression of the continuum field does not obey it in general: Result 3 gives an endpoint-fixed gauge transformation that changes phi. The decomposition 1 + 3 under the diagonal SU(2) stays correct.

Let an additional electroweak assignment supply one complex Higgs doublet h, with

$$T_a = \frac{\sigma_a}{2}, \qquad Y_h = \frac{1}{2}, \qquad \langle h\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix},$$

where $Q_{\mathrm{em}} = T_3 + Y$. Let the gauge fields have canonical kinetic terms and couplings g_2, g_Y.

Then

$$\boxed{M_W^2 = \frac{g_2^2 v^2}{4}, \qquad M_Z^2 = \frac{(g_2^2 + g_Y^2)v^2}{4},}$$

and consequently

$$\boxed{\frac{M_W}{M_Z} = \cos\theta_W, \qquad \rho = \frac{M_W^2}{M_Z^2\cos^2\theta_W} = 1, \quad \tan\theta_W = \frac{g_Y}{g_2}.}$$

Proof. At the vacuum,

$$|D_\mu h|^2 = \frac{v^2}{8}\left[g_2^2((W_\mu^1)^2 + (W_\mu^2)^2) + (g_2 W_\mu^3 - g_Y B_\mu)^2\right].$$

Diagonalizing the neutral quadratic form gives one massless photon and the displayed Z mass.

Under a common spectral-action coefficient, canonical normalization additionally gives

$$g_2^2 Q_2 = g_Y^2 Q_Y.$$

For the full three-generation Standard Model trace, including antiparticles,

$$Q_2 = 12, \qquad Q_Y = 20,$$

so

$$\boxed{\frac{M_W}{M_Z} = \sqrt{\frac{5}{8}}}$$

at that matching scale. These trace weights agree with the per-generation calculation in research state §6.10.

The original interval multiplication representation acts identically on both chiralities. Its matrix link transforms as

$$\phi \longmapsto g_L \phi g_R^*,$$

and under the diagonal $SU(2)$ it decomposes as $1 \oplus 3$. It therefore does not specify the electroweak doublet assignment used above. That assignment, its hypercharges, and the charged fermion trace must be supplied separately. Likewise, the continuum value of ρ requires the neutral-sector boundary conditions and kinetic normalization; the single mixed-boundary eigenvalue in Result 4 cannot determine it. □

Consequence for T4 and hypothesis H

The minimal fermion compression supports the geometric identification $m = \pi/(2l)$, but it does not establish an equivalence of gauge fluctuations, gauge boundary conditions, or bosonic actions. The ratio $\sqrt{3/8}$ is correct for the specified CCM Standard Model completion and receives no interval-overlap correction. Hypothesis H therefore requires an additional gauge-covariant reduction and a matching of representations, vacuum data, and kinetic forms before it can identify the finite vector spectrum with the interval Kaluza–Klein spectrum.

Part 14

T4b: gauge-covariant projection and conditional bosonic matching

Prompt: prompts/14 (STRATEGY T4b). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5 by hand: Theorem 5 (both mass matrices as quadratic forms), Counterexample 6 (the ratio 3/5 agrees with the restricted computation of lemma 13, Result 6: c_L = 8/(3 pi), c_R = 4/(3 pi)), Proposition 7 (t = 3 pi/4: M = 3 pi/(4l), lambda^2/m^2 = 9/2, normalization l(1/2 + 1/(3 pi)), ratio 27/26, then the intermediate value theorem) and No-go theorem 8 (M_spec^2 <= 6 norm(V_0)^2). Reading: the spectral-action vector mass matches the Neumann-Dirichlet tower only through a fermion whose mass is at the tower scale (the matching bulk mass M_* > 0 leaves no light pair, Theorem 9.4 case 3), and with bounded light fermions no matching is possible as l -> 0.

Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): of 9 statements, 5 are correct, Theorems 1 and 5 and Proposition 4 need amendments, and the global form of Theorem 3 is wrong as first written. Each is recorded at its statement.

We prove that a gapped spectral projection defines a gauge-covariant light-fermion bundle with its induced Grassmannian connection. For the mixed interval with a gauge-invariant bulk mass, its finite link has fixed singular values; it does not supply the full scalar fluctuation space of a two-sheet finite triple. At the leading four-dimensional spectral-action order, equality with a specified Kaluza–Klein vector mass is equivalent to an explicit trace-and-overlap sum rule. Covariance does not enforce that rule: the massless-bulk example gives $M_{\mathrm{spec}}^2 = \frac{3}{5} M_{\mathrm{KK}}^2$, whereas a suitable positive bulk mass gives exact equality. We also prove that bounded retained fermion masses cannot match a Neumann–Dirichlet vector mass diverging as $1/l$. Matching one vector mass does not, by itself, determine relative gauge couplings or all scalar kinetic coefficients.

1. Theorem (covariant spectral projection and the data controlling its spectrum).

Amendment (referee). The double-coset statement needs the endpoint transformations to act independently through H_0 and H_l, with extendability understood, and it is a local pointwise classification unless the global topology is specified. Periodic conditions require g(l) = g(0), and then the holonomy is classified by conjugation.

Let $I = [0, l]$, and let

$$E = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let the vertical operator be

$$D_{A_5,\mathsf{M},\mathcal{B}} = E(\partial_x + iA_5) + \begin{pmatrix} 0 & \mathsf{M} \\ \mathsf{M} & 0 \end{pmatrix},$$

where A_5 and M are Hermitian matrix-valued functions, and $\mathcal{B}$ specifies a self-adjoint boundary domain. Gauge transformations act in the same representation on both interval chiralities.

Let the operators form a smooth family in y, in the norm-resolvent sense after local identification of their domains. Assume that the selected light cluster is separated from the rest of the spectrum and contains complete eigenspaces, including any gauge degeneracy. Let $P_A(y)$ be its spectral projection.

Then

$$\boxed{P_{A^g} = G_g P_A G_g^\dagger,}$$

provided the transformation includes

$$A_5^g = gA_5g^\dagger + i(\partial_x g)g^\dagger, \qquad \mathsf{M}^g = g\mathsf{M}g^\dagger, \qquad \mathcal{B}^g = G_g\mathcal{B}.$$

For fixed $\mathsf{M}, \mathcal{B}$, this assertion applies exactly to the transformations preserving those data.

Define parallel transport by

$$\partial_x W = -iA_5 W, \qquad W(0) = 1, \qquad W_l = W(l).$$

The vertical spectrum is determined by the axial-gauge data

$$\boxed{\widetilde{\mathsf{M}}(x) = W(x)^\dagger \mathsf{M}(x) W(x), \qquad \widetilde{\mathcal{B}} = W^\dagger \mathcal{B}.}$$

Consequently:

- For a gauge-invariant bulk mass and the mixed conditions

  $$R(0) = 0, \qquad L(l) = 0,$$

  the entire spectrum is independent of A_5.
- With nontrivial endpoint data and gauge-invariant bulk coefficients, the relevant holonomy information is the Wilson line relative to those endpoint data. If the endpoint groups are H_0, H_l, its gauge class is represented by

  $$H_l \backslash G / H_0.$$
- A bulk matrix not commuting with parallel transport can introduce dependence on the whole transported profile $\widetilde{\mathsf{M}}(x)$, beyond W_l.

An open Wilson line is not generally gauge invariant:

$$W_l^g = g(l) W_l g(0)^\dagger.$$

It is invariant under transformations fixed at both endpoints; it transforms by conjugation when the endpoint transformations are identified. With independently unrestricted endpoints, its double-coset class is trivial.

Proof. Direct differentiation, including the boundary domain, gives

$$D_{A^g,\mathsf{M}^g,\mathcal{B}^g} = G_g D_{A,\mathsf{M},\mathcal{B}} G_g^\dagger.$$

For a contour enclosing the selected cluster,

$$P_A = \frac{1}{2\pi i} \oint (z - D_A)^{-1}\, dz.$$

Conjugation of the resolvent proves covariance. Smoothness follows from the same formula and the assumed gap.

Multiplication by W removes the connection:

$$W^\dagger D_{A_5,\mathsf{M},\mathcal{B}} W = D_{0,\widetilde{\mathsf{M}},\widetilde{\mathcal{B}}}.$$

This proves the assertion about spectral data. For a commuting bulk mass and the mixed boundary conditions, both transformed data equal their unfluctuated values.

Finally, the transport equation gives $W^g(x) = g(x)W(x)g(0)^\dagger$. The endpoint transformation law and the stated quotients follow. If the gap closes, no fixed-rank smooth light bundle is asserted. □

2. Theorem (the induced connection and finite operator).

Let M_4 be a closed Riemannian spin four-manifold. Use Hermitian Euclidean gamma matrices and

$$\mathcal{D}_A = -i\gamma^\mu(\nabla^S_\mu + \partial_\mu + iA_\mu) + \gamma_5 D_{I,A_5}.$$

Assume the hypotheses of Result 1, and let

$$\mathcal{E}_y = \operatorname{Ran} P_A(y).$$

Then the induced connection and vertical endomorphism are

$$\boxed{\nabla^{\mathcal{E}}_\mu = P_A(\partial_\mu + iA_\mu)P_A, \qquad V = P_A D_{I,A_5} P_A\big|_{\mathcal{E}}.}$$

The compressed first-order operator is exactly

$$\boxed{P_A \mathcal{D}_A P_A\big|_{\mathcal{E}} = -i\gamma^\mu(\nabla^S_\mu + \nabla^{\mathcal{E}}_\mu) + \gamma_5 V.}$$

In a local orthonormal frame $U(y) : \mathbb{C}^r \to \mathcal{E}_y$,

$$\boxed{\mathcal{A}_\mu = U^\dagger A_\mu U - iU^\dagger \partial_\mu U, \qquad V_F = U^\dagger D_{I,A_5} U.}$$

The inner products here include integration over the interval. Its curvature is

$$[\nabla^{\mathcal{E}}_\mu, \nabla^{\mathcal{E}}_\nu] = P_A(iF_{\mu\nu})P_A + P_A[\nabla_\mu P_A, \nabla_\nu P_A]P_A,$$

where $\nabla_\mu P_A = \partial_\mu P_A + i[A_\mu, P_A]$.

Proof. Write a light section as $U\chi$. Applying $\partial_\mu + iA_\mu$ and multiplying by $U^\dagger$ gives the displayed connection. The spectral projection commutes with the vertical operator, giving the displayed finite endomorphism. Expanding the commutator of the projected covariant derivatives and using $P_A^2 = P_A$ gives the curvature formula.

If

$$U^g = G_g U h^\dagger,$$

then

$$\mathcal{A}^g_\mu = h\mathcal{A}_\mu h^\dagger + i(\partial_\mu h)h^\dagger, \qquad V^g_F = hV_F h^\dagger.$$

Thus the construction is gauge covariant.

This is an exact compression. It is not generally the operator obtained by eliminating the other fermions. With $P^\perp = 1 - P_A$, elimination additionally produces

$$-P_A \mathcal{D}_A P^\perp \big[P^\perp(\mathcal{D}_A - z)P^\perp\big]^{-1} P^\perp \mathcal{D}_A P_A,$$

when the inverse exists. Horizontal derivatives and $A_\mu(x)$ generally make these off-diagonal blocks nonzero. □

3. Theorem (the finite algebra and fluctuation image for the mixed interval).

Correction (referee; the global statement as first written is false). The statement that the continuum link image consists of all links with Phi^* Phi = lambda^2 I holds locally in spacetime. Globally the unitary link must admit a path from the identity in the chosen trivialization. For M_4 = S^1 x S^3, n = 1 and Phi = lambda exp(i theta), every globally smooth real A_5 gives Phi = lambda exp(i int A_5 dx), with zero winding around S^1, so this Phi is not attained. The displayed formulas, the compression range and the reducing-subalgebra criterion are correct.

Let

$$D_{I,A_5} = \begin{pmatrix} 0 & \partial_x + iA_5 + M \\ -\partial_x - iA_5 + M & 0 \end{pmatrix}, \qquad M \in \mathbb{R},$$

with $R(0) = 0,\ L(l) = 0$, acting on $\mathbb{C}^n$-valued fields. Let the parent algebra be $C^\infty(I, M_n(\mathbb{C}))$, acting identically on both interval chiralities.

Let p_L, p_R be real normalized profiles for the lowest pair:

$$(-\partial_x + M)p_L = \lambda p_R, \qquad (\partial_x + M)p_R = \lambda p_L, \qquad \|p_L\| = \|p_R\| = 1.$$

Use the endpoint-adapted light frame

$$U_L = p_L W, \qquad U_R = p_R W W_l^\dagger.$$

Then

$$\boxed{V_F = \begin{pmatrix} 0 & \lambda W_l^\dagger \\ \lambda W_l & 0 \end{pmatrix}.}$$

Writing $w_L = p_L^2,\ w_R = p_R^2$ and

$$B_\mu = W^\dagger A_\mu W - iW^\dagger \partial_\mu W,$$

the induced connections are

$$\boxed{\begin{aligned} \mathcal{A}_{L,\mu} &= \int_0^l w_L B_\mu \, dx, \\ \mathcal{A}_{R,\mu} &= W_l \left(\int_0^l w_R B_\mu \, dx \right) W_l^\dagger - iW_l \partial_\mu W_l^\dagger. \end{aligned}}$$

The range of algebra compression is

$$A_F = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$$

on

$$H_F = \mathbb{C}_L^n \oplus \mathbb{C}_R^n, \qquad \Gamma_F = \mathrm{diag}(1, -1).$$

Taking this range with its matrix product gives a complex even finite triple. Its unitary group is $U(n)_L \times U(n)_R$, and its unrestricted inner fluctuations have an arbitrary complex matrix as their total link.

The continuum construction, however, produces exactly

$$\boxed{\Phi^\dagger \Phi = \Phi \Phi^\dagger = \lambda^2 I_n}$$

for the link. With unrestricted A_μ, it produces arbitrary pairs of Hermitian sheet connections locally in spacetime. Thus its field space is the fixed-singular-value restriction of that finite triple's fluctuation space.

The inherited local gauge action is

$$h = \mathrm{diag}(g(0), g(l)).$$

Additional gauge boundary conditions restrict this endpoint group. Globally, only endpoint transformations extending across $M_4 \times I$ are inherited.

Proof. Gauge removal and the two profile equations give the formula for V_F. Applying Result 2 to U_L, U_R gives the connection formulas. Under a continuum transformation,

$$U_L^g = gU_L g(0)^\dagger, \qquad U_R^g = gU_R g(l)^\dagger,$$

which proves the endpoint action.

The two functions w_L, w_R are linearly independent: one vanishes at $x = 0$, and the other does not. Therefore the map

$$b(x) \longmapsto \left(\int w_L b, \int w_R b \right)$$

is onto $M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$. This proves both the algebra-range assertion and the connection-surjectivity assertion. Every unitary W_l has a local smooth path lift, proving the stated link image.

For the finite triple with

$$D_F = \lambda \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix},$$

multiplication of $p_L[D_F, p_R]$ by the left algebra generates arbitrary upper-right matrices; adjoining the adjoint generates arbitrary self-adjoint odd fluctuations. The full finite fluctuation space therefore includes radial changes absent from the continuum image.

There is a further distinction concerning algebra multiplication. For any compression $C(a) = PaP|_{PH}$,

$$C(a^\dagger a) - C(a)^\dagger C(a) = Pa^\dagger(1 - P)aP.$$

Hence C is a representation on a unital $*$-subalgebra precisely when that subalgebra reduces P. In the present model, multiplication by $a(x)$ reduces the light pair precisely when

$$a(x) = W(x) b W(x)^\dagger$$

for a constant matrix b. Its restriction is a single diagonal $M_n(\mathbb{C})$, commuting with V_F, and its finite scalar one-forms vanish.

Thus the two-sheet algebra is obtained by equipping the compression range with its matrix product; the interval multiplication algebra has no multiplicative quotient onto it. No real structure, real gauge-group prescription, or Standard Model representation follows from this construction. □

4. Proposition (boundary Yukawa data determine a finite operator, but not a unique finite triple).

Amendment (referee). To obtain the entire limiting block, retain the full zero-mode-descended cluster of dimension N_Q + N_U: for two massless flavours and X = l diag(1, 2), the cluster containing only the lowest pair is gapped and has a 1 x 1 limiting block. The nonzero singular values of Y are the retained positive eigenvalues; zero singular values and unmatched chiral zero modes are included separately.

Let D_X have the boundary conditions of Lemma 12:

$$Q_R(0) = 0, \quad U_L(0) = 0, \qquad Q_R(l) = XU_R(l), \quad U_L(l) = -X^\dagger Q_L(l).$$

Let the bulk masses be M_Q, M_U, and let the gauge representations commute with these masses.

Gauge covariance requires

$$X^g = g_Q(l) X g_U(l)^\dagger.$$

If X is fixed, the allowed endpoint transformations satisfy

$$g_Q(l) X = X g_U(l).$$

After axial gauge fixing,

$$\widetilde{X} = W_Q(l)^\dagger X W_U(l).$$

Let P_A retain a gapped grading-invariant light cluster. Then

$$H_F = H_{F,+} \oplus H_{F,-}, \qquad V_F = \begin{pmatrix} 0 & Y \\ Y^\dagger & 0 \end{pmatrix},$$

where the singular values of Y are the retained positive vertical eigenvalues. These eigenvalues alone do not determine the represented gauge generators or their trace data.

In the towerless scaling of Theorem 12.7,

$$X = l\widehat{X}, \qquad M_Q l,\ M_U l \text{ fixed},$$

the limiting block is, up to changes of chiral orthonormal frames,

$$\boxed{Y = \Phi_Q \widehat{X} \Phi_U,} \qquad \phi(c)^2 = \frac{2c}{e^{2c}-1}, \quad \phi(0) = 1.$$

Every rectangular complex matrix Y is attainable as an operator block by choosing

$$\widehat{X} = \Phi_Q^{-1} Y \Phi_U^{-1}.$$

This does not establish that every finite algebra, real structure, gauge representation, or inner-fluctuation space is attained.

Proof. Transforming the boundary equations gives the transformation law for X. Applying $W_Q^\dagger, W_U^\dagger$ gives $\widetilde{X}$. Since the domain is grading invariant, its symmetric spectral projection commutes with the grading; the restricted operator is therefore odd and has the displayed block form.

For an explicit spectral determination, define

$$\alpha_i(\lambda) = \omega_i \coth(\omega_i l) + m_i^Q, \qquad \beta_j(\lambda) = \omega_j \coth(\omega_j l) - m_j^U, \qquad \omega^2 = m^2 - \lambda^2.$$

In the low window where the transfer denominators are nonzero, the boundary equations give

$$\det\left[\lambda^2 I - \operatorname{diag}(\alpha_i)\widetilde{X}\operatorname{diag}(\beta_j)\widetilde{X}^\dagger\right] = 0.$$

This follows by solving the constant-coefficient first-order equations and imposing the two conditions at l, as in Theorem 12.4.

At zero spectral parameter, the normalized wall values give

$$F_Q = l^{-1/2}\Phi_Q, \qquad F_U = l^{-1/2}\Phi_U.$$

Consequently

$$F_Q X F_U = \Phi_Q \widehat{X} \Phi_U.$$

The bounds of Theorem 12.6 give convergence of the retained positive eigenvalues to its singular values, with error $O(l^2)$; the remaining rescaled eigenvalues stay separated from zero. Positivity of the diagonal entries of Φ_Q, Φ_U proves attainability of every operator block.

To obtain a finite triple, one must additionally specify the algebra to be represented on the retained space. A multiplicative restriction must satisfy the reducing-subalgebra criterion in Result 3; taking the algebra generated by compressed operators is a different prescription. Once an algebra A_F is specified, its complex gauge group is the represented $U(A_F)$, and its scalar one-forms are

$$\Omega^1_{V_F}(A_F) = \operatorname{span}\{a[V_F, b] : a, b \in A_F\}.$$

Neither this algebra nor this space is determined by the light eigenvalues. □

5. Theorem (the exact quadratic matching condition at leading spectral-action order).

Amendment (referee). Both quadratic forms must be compared on one common profile space on which G^(5) and Q are positive, because a Kaluza-Klein profile can be invisible to compression. For the massless mixed pair, f = cos(2 pi x/l) satisfies Neumann conditions at both ends and has positive norm, while int w_L f = int w_R f = 0, so Q = K = 0 for this mode.

Let a five-dimensional Yang–Mills theory have no boundary kinetic or mass terms. Let $\langle\ ,\ \rangle_5$ be its positive invariant gauge form, including its coupling constants. Expand the retained transverse vectors about $A_5 = 0$ as

$$A_\mu(y,x) = \sum_a a_\mu^a(y) F_a(x),$$

where $F_a(x)$ are fixed Hermitian Lie-algebra-valued profiles satisfying the chosen gauge boundary conditions. Define

$$G_{ab}^{(5)} = \int_0^l \langle F_a, F_b\rangle_5\, dx, \qquad H_{ab}^{(5)} = \int_0^l \langle F_a', F_b'\rangle_5\, dx.$$

Then

$$\mathcal{L}_{\mathrm{KK}}^{(2)} = \frac{1}{4} G_{ab}^{(5)} f_{\mu\nu}^a f^{b,\mu\nu} + \frac{1}{2} H_{ab}^{(5)} a_\mu^a a^{b,\mu},$$

and

$$\boxed{\mathcal{M}_{\mathrm{KK}}^2 = (G^{(5)})^{-1/2} H^{(5)} (G^{(5)})^{-1/2}.}$$

Let U_0 be the light frame at this background, and set

$$T_a = U_0^\dagger F_a U_0, \qquad V_0 = U_0^\dagger D_I U_0.$$

For the literal spectral trace, retain the gauge and scalar derivative terms through a_4, assume $f_0 > 0$, and add no independent kinetic terms. Define

$$Q_{ab} = \mathrm{Re}\,\mathrm{Tr}(T_a T_b), \qquad K_{ab} = \mathrm{Re}\,\mathrm{Tr}([T_a, V_0]^\dagger [T_b, V_0]).$$

On the subspace where $Q > 0$,

$$\boxed{\mathcal{M}_{\mathrm{spec}}^2 = \frac{3}{2} Q^{-1/2} K Q^{-1/2}.}$$

Thus equality of the vector mass spectra is equivalent to equality of the generalized spectra

$$\boxed{\mathrm{spec}\left(\tfrac{3}{2}K, Q\right) = \mathrm{spec}\left(H^{(5)}, G^{(5)}\right).}$$

For a retained sector whose KK mass matrix is $\kappa^2 I$, this is equivalent to the sum rule

$$\boxed{K = \frac{2}{3}\kappa^2 Q.}$$

Equality of the quadratic actions in the same field coordinates is stronger:

$$\boxed{G^{(5)} = 4\eta Q, \qquad H^{(5)} = 6\eta K, \qquad \eta = \frac{f_0}{24\pi^2}.}$$

For a chiral finite block

$$V_0 = \begin{pmatrix} 0 & Y \\ Y^\dagger & 0 \end{pmatrix}, \qquad T_a = \mathrm{diag}(t_{L,a}, t_{R,a}),$$

put

$$C_a = t_{L,a} Y - Y t_{R,a}.$$

The trace data are explicitly

$$\boxed{Q_{ab} = \operatorname{Re}\operatorname{Tr}(t_{L,a}t_{L,b} + t_{R,a}t_{R,b}), \qquad K_{ab} = 2\operatorname{Re}\operatorname{Tr}(C_a^\dagger C_b).}$$

Here Y and the $t_{L/R,a}$ must both be computed from the interval eigenfunctions and the retained gauge profiles.

Proof. Expanding the five-dimensional action gives the two integral forms $G^{(5)}, H^{(5)}$. Canonical normalization by $(G^{(5)})^{1/2}$ gives its mass matrix.

For

$$\mathcal{D} = -i\gamma^\mu(\nabla_\mu^S + i\mathcal{A}_\mu) + \gamma_5 V,$$

the relevant heat-coefficient traces are

$$\operatorname{tr}_S(\mathcal{E}_F^2) = 2F_{\mu\nu}F^{\mu\nu}, \quad \operatorname{tr}_S(\mathcal{E}_V^2) = 4D_\mu V D^\mu V, \quad \operatorname{tr}_S(\Omega_F^2) = -4F_{\mu\nu}F^{\mu\nu}.$$

Their coefficients $1/2, 1/2, 1/12$ give

$$\mathcal{L}_{\text{spec}} = \eta \operatorname{Tr} F_{\mu\nu}F^{\mu\nu} + 3\eta \operatorname{Tr} D_\mu V D^\mu V.$$

At V_0, this becomes

$$\eta Q_{ab} f^a_{\mu\nu} f^{b,\mu\nu} + 3\eta K_{ab} a^a_\mu a^{b,\mu}.$$

The normalization $a_{\text{can}} = 2\sqrt{\eta}\, Q^{1/2} a$ proves the coefficient $3/2$, agreeing with Proposition 13.5.

Equality with a scalar matrix $\kappa^2 I$ is equivalent to

$$Q^{-1/2} K Q^{-1/2} = \frac{2}{3}\kappa^2 I.$$

The block formulas follow by multiplying

$$[T_a, V_0] = \begin{pmatrix} 0 & C_a \\ -C_a^\dagger & 0 \end{pmatrix}.$$

This proves all the matching conditions.

For mixed fermion pairs labelled by r, let their masses be λ_r, and let a single gauge profile f have overlaps

$$c_{L,r} = \int |p_{L,r}|^2 f, \qquad c_{R,r} = \int |p_{R,r}|^2 f.$$

If $\tau_r = \operatorname{tr}_r(t^2)$ includes every multiplicity, the one-vector condition becomes

$$\boxed{3\sum_r \tau_r \lambda_r^2 (c_{L,r} - c_{R,r})^2 = \kappa^2 \sum_r \tau_r (c_{L,r}^2 + c_{R,r}^2).}$$

This is the trace-and-overlap sum rule for that gauge reduction.

If instead both sheet connections are fluctuated independently, with the same representation on each pair, their relative mass is

$$M^2_{\text{relative}} = 6\frac{\sum_r \tau_r \lambda_r^2}{\sum_r \tau_r}.$$

Equating this single eigenvalue to κ^2 requires

$$6\sum_r \tau_r \lambda_r^2 = \kappa^2 \sum_r \tau_r.$$

That comparison has an additional massless diagonal vector and is not the same restriction as retaining one interval gauge profile.

All these spectral-action equalities concern the stated a_4 approximation and chosen vacuum. For a full finite scalar fluctuation space, identifying V_0 with the compressed bare operator additionally requires the scalar potential to select that vacuum. At masses comparable to the cutoff, Lemma 03, Part C shows why weighting Q alone does not give the broken-generator answer.

Finally, a nonabelian truncation to nonconstant lowest profiles need not be closed under gauge transformations: $[ft_a, ft_b] = f^2[t_a, t_b]$ generally leaves the retained span. The theorem compares quadratic transverse actions; it does not assert equality of full interacting theories. □

6. Counterexample (the covariant lowest-mode reduction still gives 3/5).

Let the gauge group be $U(1)$, and impose

$$\partial_x A_\mu(0) = 0, \qquad A_\mu(l) = 0,$$

with complementary conditions

$$A_5(0) = 0, \qquad \partial_x A_5(l) = 0.$$

Retain

$$A_\mu = a_\mu(y)\cos(\kappa x), \qquad A_5 = b(y)\sin(\kappa x), \qquad \kappa = \frac{\pi}{2l}.$$

Let one charge-one fermion have the mixed boundary conditions and scalar bulk mass M. Let λ be its lowest positive vertical eigenvalue, and define

$$c_L = \int_0^l p_L^2 \cos(\kappa x)\, dx, \qquad c_R = \int_0^l p_R^2 \cos(\kappa x)\, dx.$$

The gauge-covariant light projection gives

$$\boxed{M_{\text{spec}}^2 = 3\lambda^2 \frac{(c_L - c_R)^2}{c_L^2 + c_R^2}, \qquad M_{\text{KK}}^2 = \kappa^2.}$$

For $M = 0$,

$$\boxed{M_{\text{spec}}^2 = \frac{3}{5} M_{\text{KK}}^2.}$$

Proof. The vector boundary problem selects $f(x) = \cos(\kappa x)$. Direct integration gives

$$\frac{\int (f')^2}{\int f^2} = \kappa^2,$$

which is the Neumann–Dirichlet result of Lemma 9.11.

The retained abelian gauge transformations have parameter

$$\epsilon(y, x) = \alpha(y)\cos(\kappa x).$$

With the conventions of Result 1,

$$a_\mu \mapsto a_\mu - \partial_\mu \alpha, \qquad b \mapsto b + \kappa\alpha.$$

Thus this vector-scalar restriction is gauge closed.

Parallel transport is

$$W(x) = \exp\left[-\frac{ib}{\kappa}(1 - \cos\kappa x)\right].$$

Result 3 gives

$$\Phi = \lambda e^{ib/\kappa},$$

and

$$\mathcal{A}_L = c_L a - \frac{1-c_L}{\kappa}\,db, \qquad \mathcal{A}_R = c_R a + \frac{c_R}{\kappa}\,db.$$

Consequently,

$$D\Phi = i\Phi(c_L - c_R)\left(a + \frac{db}{\kappa}\right), \qquad F_L = c_L\,da, \quad F_R = c_R\,da.$$

The quadratic spectral action is therefore

$$\eta(c_L^2 + c_R^2)(da)^2 + 6\eta\lambda^2(c_L - c_R)^2\left(a + \frac{db}{\kappa}\right)^2.$$

Canonical vector normalization proves the mass formula. In particular, the Berry terms have been included; the calculation does not use a fixed projection.

At $M = 0$,

$$\lambda = \kappa, \qquad p_L = \sqrt{\frac{2}{l}}\cos(\kappa x), \qquad p_R = \sqrt{\frac{2}{l}}\sin(\kappa x).$$

Elementary integration gives

$$c_L = \frac{8}{3\pi}, \qquad c_R = \frac{4}{3\pi}.$$

Hence

$$\frac{M_{\text{spec}}^2}{M_{\text{KK}}^2} = 3\frac{(8-4)^2}{8^2+4^2} = \frac{3}{5}.$$

The matrices $\operatorname{diag}(c_L, c_R)$ describe the coefficient of the retained vector in the induced connection; they are not the endpoint $U(1)$ charge assignment. The explicit transformation formulas above establish covariance. □

7. Proposition (exact matching is possible after imposing an additional bulk-mass condition).

Keep the gauge and fermion boundary conditions of Result 6. At fixed l, let the fermion bulk mass vary over

$$0 \le M \le \frac{3\pi}{4l}.$$

Then at least one value

$$\boxed{0 < M_* < \frac{3\pi}{4l}}$$

satisfies

$$M_{\text{spec}}(M_*) = M_{\text{KK}}.$$

An exact defining condition is

$$\boxed{3\lambda(M_*)^2(c_L(M_*) - c_R(M_*))^2 = \frac{\pi^2}{4l^2}(c_L(M_*)^2 + c_R(M_*)^2).}$$

No uniqueness is asserted.

Proof. For $M \ge 0$, write

$$t = kl \in [\pi/2, \pi), \qquad M = -k\cot t, \qquad \lambda = \frac{k}{\sin t}.$$

The lowest chiral profiles are proportional to

$$p_L(x) \propto \sin(k(l-x)), \qquad p_R(x) \propto \sin(kx).$$

The relation $M = -k\cot(kl)$ follows by imposing $p_L'(0) = Mp_L(0)$; substitution in the first-order equation gives the displayed p_R and λ.

At

$$t = \frac{3\pi}{4}, \qquad M = \frac{3\pi}{4l},$$

we have

$$\lambda^2 = \frac{9\pi^2}{8l^2}.$$

The common squared normalization integral is

$$\int_0^l \sin^2(kx)\,dx = l\left(\frac{1}{2} + \frac{1}{3\pi}\right).$$

Integrating against $\cos(\pi x/(2l))$ gives

$$c_L = \frac{9}{3\pi + 2}, \qquad c_R = \frac{6}{3\pi + 2}.$$

Therefore

$$\frac{M^2_{\text{spec}}}{M^2_{\text{KK}}} = 3\frac{9}{2}\frac{(9-6)^2}{9^2+6^2} = \frac{27}{26} > 1.$$

At $M = 0$, Result 6 gives $3/5 < 1$. The lowest eigenvalue is simple, and the displayed root and normalized profiles depend continuously on M. The intermediate value theorem proves the existence of M_*. Thus universal impossibility of mass matching is false, while automatic matching is also false. □

8. No-go theorem (bounded light fermions cannot match the collapsing Neumann–Dirichlet tower).

Let a family of induced finite operators satisfy

$$\|V_0(l)\| \le C < \infty.$$

Use the leading spectral action of Result 5, with no additional vector-mass or kinetic terms. The retained gauge couplings may depend on l, and any finite multiplicities may be included.

Then every canonically normalized spectral-action vector mass satisfies

$$\boxed{M^2_{\text{spec}} \le 6C^2.}$$

Consequently it cannot equal

$$M^2_{\text{KK}} = \frac{\pi^2}{4l^2}$$

for arbitrarily small l.

This applies to the fixed-light-mass collapse of Theorem 9.9 and to the bounded-texture limit of Result 4.

Proof. For every Hermitian retained coupling matrix T,

$$\|[T, V_0]\|_{\text{HS}} \le \|TV_0\|_{\text{HS}} + \|V_0T\|_{\text{HS}} \le 2C\|T\|_{\text{HS}}.$$

Hence

$$K \le 4C^2Q$$

as quadratic forms. Result 5 gives

$$\mathcal{M}^2_{\text{spec}} \le 6C^2 I.$$

The same inequality holds after restricting to any gauge-profile subspace. Multiplicities cannot evade it because they enter both Hilbert–Schmidt norms.

The Neumann–Dirichlet mass diverges as l^{-1}, proving the contradiction. More quantitatively, matching any such vector requires

$$\boxed{\|V_0(l)\| \ge \frac{\pi}{2\sqrt{6}\,l}.}$$

Thus some retained fermion scale must grow at least as $1/l$, unless the bosonic action or the stated approximation is changed. □

9. Proposition (gauge-factor and scalar normalizations require additional conditions).

Let the finite gauge kinetic form be diagonal on two specified factors, with fixed generator normalizations,

$$Q|_{\mathfrak{g}_s} = q_s I, \qquad s = 1, 2.$$

Let their five-dimensional kinetic terms have independent couplings $g_{5,s}$, and let

$$J_s = \int_0^l f_s(x)^2 \, dx.$$

Then the leading finite spectral action predicts

$$\frac{1}{g_{4,s}^2} = 4\eta q_s, \qquad \frac{g_{4,1}^2}{g_{4,2}^2} = \frac{q_2}{q_1},$$

whereas the KK action gives

$$\frac{1}{g_{4,s}^2} = \frac{J_s}{g_{5,s}^2}.$$

Vector-mass matching does not imply equality of these relative normalizations.

For scalar coordinates φ^i, let the induced connection at vanishing ambient A_μ be

$$\mathcal{A}_\mu = \mathcal{B}_i(\varphi)\partial_\mu\varphi^i,$$

and define

$$H_i = \partial_i V + i[\mathcal{B}_i, V].$$

The spectral-action scalar metric, in the convention $\frac{1}{2}Z_{ij}\partial_\mu\varphi^i\partial^\mu\varphi^j$, is

$$\boxed{Z_{ij}^{\text{spec}} = 6\eta \, \operatorname{Re}\operatorname{Tr}(H_i H_j).}$$

Mass matching constrains this metric on the corresponding gauge-orbit directions. It does not generally determine the metric on additional physical scalar directions.

Proof. Canonical normalization of $\eta q_s F_s^2$ gives the finite coupling formula. Integrating the five-dimensional kinetic term gives the KK formula. Rescaling $g_{5,s}^{-2}$ multiplies both $G_s^{(5)}$ and $H_s^{(5)}$ by the same factor and leaves every KK mass unchanged. Therefore mass matching cannot determine the independent ratio $g_{5,1}/g_{5,2}$. A common simple parent group with a specified invariant form could remove this freedom, but that is additional geometric input.

For the scalar metric,

$$D_\mu V = H_i \partial_\mu \varphi^i.$$

Substitution into $3\eta \operatorname{Tr}(D_\mu V D^\mu V)$ proves the formula.

The eaten scalar in Result 6 illustrates the constrained directions precisely. Put

$$G = \frac{l}{2g_5^2}, \qquad q = c_L^2 + c_R^2, \qquad d = c_L - c_R.$$

Its KK action is

$$\frac{G}{4}(da)^2 + \frac{G}{2}(db + \kappa a)^2.$$

If both the gauge kinetic coefficient and the vector mass match,

$$4\eta q = G, \qquad 3\lambda^2 d^2 = \kappa^2 q,$$

then the spectral scalar coefficient automatically satisfies

$$Z_b^{\mathrm{spec}} = \frac{12\eta\lambda^2 d^2}{\kappa^2} = G.$$

Thus no separate tuning is needed for this eaten scalar once those two equalities hold.

A scalar direction transverse to the gauge orbit is not tested by that mass equation. In the commuting-bulk mixed model, A_5 supplies only the unitary link of Result 3, so there is no radial Higgs kinetic term to match. If boundary Yukawa matrices are promoted to independent scalar fields, their continuum dynamics, including any boundary kinetic form, must be specified separately. □

Consequence for T4b and hypothesis H

Gauge-covariant projection repairs the covariance defect and supplies an exact light-bundle connection, but it does not establish equality with the full finite-triple fluctuation space or with the interacting KK theory. At fixed length, vector-mass equality is possible precisely when the appropriate trace-and-overlap condition holds; covariance supplies no selection principle for it or for the remaining kinetic normalizations. Hypothesis H therefore requires additional geometric relations, and its collapse regime cannot reproduce a Neumann–Dirichlet vector of mass $\pi/(2l)$ from a leading finite spectral action whose retained fermion masses all remain bounded.

Part 15

Collapse in the spectral propinquity: a gap criterion and limits of commutative spaces

Lemmas R2 and R3 of the Farsi-Latrémolière assessment, written out and extended. Proved by Claude Opus 5.5 without Astra or Fable, from the definitions in the LaTeX source of arXiv:2404.00240v3, Section 2. The sketches came from the reading subagent; the lower bound in the sketch of R2 omitted the cap sqrt(2)/2 of the metric and is corrected here. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): Theorem 15.2 item 2 had a gap, closed by the referee; Remark 15.4 omitted the cap; Proposition 15.9 needed amendments; the T0 section first contained two invalid arguments (on unbroken symmetry and on Morita equivalence). All corrections are incorporated below. Proposition 15.10 and Counterexample 15.11 are the referee's.

Scope. Everything here concerns one metric, the spectral propinquity, applied to the whole algebra of the parent triple. The section "What this says about T0" states what follows for the founding problem and what does not; lemma 16 treats truncations, where other answers hold.

Conventions

These are the definitions of Farsi-Latrémolière, Section 2.

- A *quantum compact metric space* (A, L) is a unital C*-algebra A with a seminorm L on a dense subspace of sa(A) such that ker L = R 1; the metric mk_L(phi, psi) = sup{abs(phi(a) - psi(a)) : L(a) <= 1} metrizes the weak* topology of the state space S(A); L satisfies the Leibniz inequality for the Jordan product a o b = (ab + ba)/2 and the Lie product {a, b} = (ab - ba)/(2i); and {L <= 1} is closed.
- A *quantum isometry* is a *-epimorphism pi with L_B(b) = inf{L_A(a) : pi(a) = b}. Its dual map is an isometry of state spaces.
- A *tunnel* is a quantum compact metric space (E, L_E) with quantum isometries pi_1, pi_2 onto the two spaces. Its *extent* chi is the larger of the Hausdorff distances, in mk_{L_E}, from S(E) to pi_1^*S(A_1) and to pi_2^*S(A_2). The *propinquity* Lambda is the infimum of extents.
- A *metric spectral triple* T = (A, H, D) is a spectral triple for which L_T(a) = norm([D, a]) is an L-seminorm. In particular ker L_T = R 1. We write DN_T(xi) = norm(xi) + norm(D xi) and diam T for the diameter of S(A) in mk_{L_T}.

- A *metrical tunnel* between T_1 and T_2 consists of a C*-correspondence P from E to a unital C*-algebra B, with L_E and L_B as above and a norm DN_P on a dense subspace of P such that: the unit ball of DN_P is compact in P; DN_P dominates the module norm; DN_P(e Theta) <= (norm(e) + L_E(e)) DN_P(Theta) (*modular Leibniz*); L_B applied to the real and to the imaginary part of <Theta, Theta'>*B is at most 2 DN_P(Theta) DN_P(Theta') (*inner Leibniz*; this is the condition of Latrémolière, arXiv:1811.10843, in the definition of a metrical C*-correspondence, whereas the display in Farsi-Latrémolière applies L to the non-self-adjoint inner product and names L_A). It carries maps (Pi_j, pi_j, theta_j) onto (H_j, A_j, C), compatible with the actions and inner products, with DN*{T_j}(xi) = inf{DN_P(Theta) : Pi_j Theta = xi}. Its extent is the larger of the extents of (E, pi_1, pi_2) and (B, theta_1, theta_2).
- The *pseudo-states* of T are the functionals xi -> <omega, xi> with DN_T(omega) <= 1. The *separation* of two operator families (a_t), (b_t) is the Hausdorff distance between {(phi a_t Pi_1)_t} and {(psi b_t Pi_2)_t}, phi and psi ranging over pseudo-states, in the metric sup_t sup{abs(F_t(Theta) - G_t(Theta)) : DN_P(Theta) <= 1}. The *dispersion* is the larger of extent and separation.
- The *spectral propinquity* Lambda_spec(T_1, T_2) is the infimum of sqrt(2)/2 and of those eps > 0 for which some metrical tunnel has dispersion < eps for the families exp(i t D_1), exp(i t D_2), 0 <= t <= 1/eps.

Three elementary consequences.

(a) If tau is a tunnel of extent chi, then abs(diam_1 - diam_2) <= 2 chi. Take phi, phi' realizing diam_1; their pullbacks lie within chi of pullbacks of states of A_2, and the dual maps are isometries.

(b) Lambda_spec(T_1, T_2) >= min(sqrt(2)/2, Lambda(A_1, A_2)), because the dispersion dominates the extent of the top tunnel. So convergence in Lambda_spec implies convergence in Lambda.

(c) Combining: Lambda_spec(T_1, T_2) >= min(sqrt(2)/2, abs(diam T_1 - diam T_2)/2).

The cap is dimensionful: Lambda_spec has the units of a length and is cut off at sqrt(2)/2 in the chosen unit.

(d) If a = a* in (A, L), and z is the midpoint of the convex hull of Sp(a), then norm(a - z) <= (diam/2) L(a). The endpoints of the hull are values of states on a.

Comparison along a tunnel

Lemma 15.1. Let tau = (E, L_E, pi_1, pi_2) be a tunnel of extent chi, and let e = e* with L_E(e) finite. For {j, k} = {1, 2},

```
norm(pi_j e) <= norm(pi_k e) + chi L_E(e),     norm(e) <= norm(pi_j e) + chi L_E(e).
```

For a real polynomial p = sum p_m x^m, the element p(e) has finite L_E, with L_E(p(e)) <= L_E(e) sum_{m >= 1} abs(p_m) m norm(e)^{m-1}.

Proof. The norm of a self-adjoint element is the supremum of abs(state) on it. A state phi of A_j pulls back to S(E), and lies within chi of the pullback of a state psi of A_k, so abs(phi(pi_j e)) <= abs(psi(pi_k e)) + chi L_E(e). The second inequality is the same argument starting from a state of E. For powers, e^m = e o e^{m-1}, and induction with the Jordan-Leibniz inequality gives L_E(e^m) <= m norm(e)^{m-1} L_E(e); L_E(1) = 0. ■

Limits of commutative spaces (Lemma R3)

Theorem 15.2. Let (A_n, L_n) -> (A, L) in the propinquity.

1. If every A_n is commutative, then A is commutative.
2. If every A_n = C(X_n) with X_n having at most k connected components, then the spectrum of A has at most k connected components. In particular, a finite-dimensional limit is C^j with j <= k, and for k = 1 it is C.

Both statements hold for limits in Lambda_spec, by (b).

Proof. Let tau_n be tunnels of extent chi_n -> 0.

(1) Let a_1, a_2 in sa(A) have finite L. Choose lifts d_i in E_n with pi_A(d_i) = a_i and L_{E_n}(d_i) <= L(a_i) + 1. Lemma 15.1 bounds norm(d_i) uniformly. The Lie product e = {d_1, d_2} satisfies pi_n(e) = 0 and pi_A(e) = {a_1, a_2}, and L_{E_n}(e) <= norm(d_1) L(d_2) + norm(d_2) L(d_1) is uniformly bounded. So norm({a_1, a_2}) <= chi_n L_{E_n}(e) -> 0. Elements of finite L are dense in sa(A), so A is commutative.

(2) By (1), A = C(X). Suppose X has at least k + 1 components. Since components and quasicomponents agree in a compact Hausdorff space, X splits into nonempty clopen sets X_0, ..., X_k. Let a in sa(A) have finite L and norm(a - sum_j j 1_{X_j}) < 1/8, so that Sp(a) meets each [j - 1/8, j + 1/8] and lies in their union. Take lifts d_n as above, and b_n = pi_n(d_n), a real function on X_n. All spectra lie in a fixed compact interval I for large n, by Lemma 15.1.

Let g : I -> [0, 1] be continuous, 0 on the union of the [j - 1/8, j + 1/8] and 1 outside the union of the open intervals (j - 3/8, j + 3/8), and let p be a polynomial with abs(p - g) < 1/8 on I. Then norm(p(a)) <= 1/8, so Lemma 15.1 gives norm(p(b_n)) <= 1/4 for large n, because L_{E_n}(p(d_n)) is uniformly bounded. Hence the range of b_n lies in the union of the disjoint intervals (j - 3/8, j + 3/8), and each component of X_n maps into one of them. So b_n reaches at most k of them, and along a subsequence it misses one fixed j_*. Let h : I -> [0, 1] be continuous, 1 on [j_* - 1/8, j_* + 1/8] and 0 outside radius 3/8 around j_*, and let q be a polynomial with abs(q - h) < 1/8 on I. Then norm(q(a)) >= 7/8 and norm(q(b_n)) <= 1/8, contradicting Lemma 15.1 for large n. ■

The theorem is one-directional: noncommutative spaces may converge to commutative ones, as matrix algebras do to the sphere (Rieffel; Latrémolière) and spectral truncations of tori do to the tori in the Gromov-Hausdorff sense on state spaces (Leimbach-van Suijlekom). We did not find Theorem 15.2 stated in the sources we searched (arXiv:1302.4058, 1311.0104, 2404.00240, 2603.19128); its proof is elementary and it may be folklore.

The gap criterion (Lemma R2)

Theorem 15.3. Let T = (A, H, D) be a metric spectral triple with delta = diam T. Let Sigma be a finite subset of R, P = 1_Sigma(D) of rank N >= 1 and Q = 1 - P, and let g > 0 be such that abs(lambda) >= g for every lambda in Sp(D) outside Sigma. Let D_F be a Hermitian N x N matrix, W : PH -> C^N a unitary, extended by 0 on QH, and Delta = norm(W D W* - D_F). For T_F = (C, C^N, D_F),

```
min(sqrt(2)/2, delta/2) <= Lambda_spec(T, T_F) <= 1/g^2 + max(delta/2, Delta) + Delta +
↪ sqrt(Delta).
```

For Delta = 0 the upper bound reads delta/2 + 1/g^2. No anticommutation hypothesis enters.

Proof. The lower bound is (c), since diam T_F = 0. For the upper bound fix rho > 0 with rho >= max(delta/2, Delta), and put s = rho + 1/(2 g^2), c = 1/(1 + Delta), D_B = W D W*.

Top tunnel. On E = A (+) C let L_E(a, z) = max{L_T(a), norm(a - z)/rho}. Then ker L_E = R (1, 1). The Leibniz inequality for the Jordan product follows from a o b - z w = (a - z) o b + z (b - w), and for the Lie product from {a, b} = {a - z, b}. The set {L_E <= 1, mu(a) = 0}, for a fixed state mu of A, is contained in a product of compact sets, since abs(z) <= norm(a) + rho and norm(a) <= delta there, so L_E is an L-seminorm by Rieffel's criterion, with the closed ball following from lower semicontinuity. The projection onto A is a quantum isometry by (d), as rho >= delta/2; the projection onto C is one because L_E(z 1, z) = 0. A state of E has the form (a, z) -> t mu(a) + (1 - t) z. On L_E(a, z) <= 1 it differs from (a, z) -> mu(a) by (1 - t) abs(z - mu(a)) <= norm(a - z) <= rho, and from (a, z) -> z by t abs(mu(a) - z) <= rho. So the extent is at most rho.

Base tunnel. On B = C (+) C let L_B(x, y) = abs(x - y)/s. The two projections are quantum isometries onto (C, 0), and the extent is at most s.

Module. Let P_mod = H (+) C^N, a right Hilbert B-module with <(xi, eta), (xi', eta')> = (<xi, xi'>, <eta, eta'>) and left action (a, z)(xi, eta) = (a xi, z eta). On Dom D (+) C^N set

```
DN_P(xi, eta) = max{DN_T(xi), DN_F(eta), norm(W xi - eta)/rho}.
```

The unit ball is closed in a product of compact sets, because D has compact resolvent. The norm is dominated.

Modular Leibniz. DN_T(a xi) <= (norm(a) + L_T(a)) DN_T(xi) and DN_F(z eta) = abs(z) DN_F(eta). Further

```
norm(W a xi - z eta) <= norm(W (a - z) xi) + abs(z) norm(W xi - eta) <= norm(a - z) norm(xi) +
↪  abs(z) rho DN_P(xi, eta),
```

and dividing by rho gives at most (L_E(a, z) + norm(a, z)) DN_P(xi, eta). No commutation of P with A is needed, because the limit algebra acts by scalars.

Inner Leibniz. On Q H the spectrum has abs(lambda) >= g, so norm(Q xi) <= norm(D xi)/g <= DN_T(xi)/g. Then <xi, xi'> = <W xi, W xi'> + <Q xi, Q xi'> and

```
abs(<W xi, W xi'> - <eta, eta'>) <= norm(W xi - eta) norm(W xi') + norm(eta) norm(W xi' - eta') <=
↪  2 rho DN_P DN_P',
```

so abs(<xi, xi'> - <eta, eta'>) <= (2 rho + 1/g^2) DN_P DN_P' = 2 s DN_P DN_P'. This complex estimate implies the condition on real and imaginary parts.

Quotient conditions. Given xi, take eta = c W xi. Since P reduces D and D_F W = W D - (D_B - D_F) W on Dom D, we get DN_F(c W xi) <= c (norm(W xi) + norm(W D xi) + Delta norm(W xi)) <= DN_T(xi), and norm(W xi - c W xi) <= Delta norm(xi) <= rho DN_T(xi). So DN_P(xi, c W xi) = DN_T(xi). Given eta, take xi = c W* eta. Then DN_T(xi) = c (norm(eta) + norm(D_B eta)) <= DN_F(eta) and norm(W xi - eta) <= Delta norm(eta), so DN_P(c W* eta, eta) = DN_F(eta).

Separation. Let DN_T(omega) <= 1 and put omega' = c W omega, so that DN_F(omega') <= 1. For DN_P(xi, eta) <= 1 and real t, since P commutes with exp(i t D),

```
<omega, e^{itD} xi> - <omega', e^{itD_F} eta> = <Q omega, e^{itD} Q xi> + <W omega, (e^{itD_B} -
↪  e^{itD_F}) W xi> + <W omega, e^{itD_F} (W xi - eta)> + (1 - c) <W omega, e^{itD_F} eta>,
```

and the four terms are bounded by 1/g^2, abs(t) Delta, rho and Delta. Conversely, let DN_F(omega') <= 1 and put omega = c W* omega', so that DN_T(omega) <= 1 and Q omega = 0; the same decomposition without the first term gives at most abs(t) Delta + rho + Delta.

Conclusion. Put K = 1/g^2 + rho + Delta. The extent is at most s <= K and the separation on [0, 1/eps] is at most K + Delta/eps. If eps > K + sqrt(Delta), then Delta/eps < sqrt(Delta) (or Delta = 0), so the dispersion is below eps. Hence Lambda_spec <= K + sqrt(Delta), and we let rho decrease to max(delta/2, Delta). ■

Remark 15.4 (rescaling). For T_eps = (A, H, D/eps) with 0 in Sp(D), Sigma = {0} and D_F = 0, the theorem gives min(sqrt(2)/2, eps delta_1/2) <= Lambda_spec <= eps delta_1/2 + eps^2/g_1^2, where delta_1 = diam T_1 and g_1 is the gap of D. This is Corollary 4.1 of Farsi-Latrémolière with a sharp leading rate for small eps. Counterexample R1 of the assessment violates the gap hypothesis, since the eigenvalue x stays bounded outside Sigma.

Corollary 15.5 (the tuned interval, Theorem 9.9). Let T_l = (C([0, l]), L^2([0, l]; C^2), D_{-mu(l)}) with condition (c) and mu(l) as in Theorem 9.9, c = mu(l) l, and let P project onto +/- m_f. For l small,

```
l/2 <= Lambda_spec(T_l, (C, C^2, m_f sigma_1)) <= l/2 + l^2/(pi^2 + c^2).
```

For the Connes-Lott triple CL(m_f) = (C + C, C^2, m_f sigma_1) of diameter 1/m_f,

```
Lambda_spec(T_l, CL(m_f)) >= min(sqrt(2)/2, (1/m_f - l)/2).
```

Proof. Multiplication by a Lipschitz function preserves condition (c), and the mass term commutes with functions, so L(f) = norm(f')_infinity and diam T_l = l. Theorem 9.6 gives W with W D W* = m_f sigma_1, so Delta = 0, and Theorem 9.4 gives g >= sqrt(mu^2 + (pi/l)^2). The second bound is (c). ■

So the whole triple converges to a point carrying a two-dimensional massive Dirac block, while the compression at each l is the Connes-Lott triple (Theorem 9.6). The two statements concern different objects, and both hold; see "What this says about T0".

Corollary 15.6 (towerless Yukawa block, Theorem 12.7). Fix the c's and Y_hat of Theorem 12.7, with T_{X,l} = (C([0, l]), L^2 (x) (C^{N_Q} (+) C^{N_U}), D_X). Then T_{X,l} is a metric spectral triple of diameter l, and

```
Lambda_spec(T_{X,l}, (C, C^{N_Q + N_U}, D_F)) = O(l),     D_F = [[0, M_F], [M_F^dagger, 0]],   M_F
↪  = Phi_Q Y_hat Phi_U.
```

Proof. Multiplication by f preserves the conditions at l because f takes one value there, and norm([D_X, f]) = norm(f')_infinity. Theorem 12.7 gives g >= min(L_Q, L_U)/l. Mapping eigenvectors of +/- lambda_k to the singular vectors of M_F gives W with Delta = max_k abs(lambda_k - sigma_k(M_F)) = O(l^2). Theorem 15.3 applies, with sqrt(Delta) = O(l). ■

This answers the remark in Theorem 12.7 that a limit triple needs a choice of algebra: in this metric the algebra of the limit is C.

The converse

Theorem 15.7. Let T_n = (A_n, H_n, D_n) be metric spectral triples with Lambda_spec(T_n, T_F) -> 0, T_F = (C, C^N, D_F). Then diam T_n -> 0; for every R > norm(D_F), D_n has exactly N eigenvalues in [-R, R] for large n, so that the gap g_n to the remaining spectrum tends to infinity (with g_n infinite if no spectrum remains); and there are unitaries W_n from the light space onto C^N with norm(W_n D_n W_n* - D_F) -> 0. With Theorem 15.3,

```
Lambda_spec(T_n, T_F) -> 0   if and only if   diam T_n -> 0, g_n -> infinity, Delta_n -> 0.
```

Proof. The diameter statement is (c). Let tau_n have dispersion < eps_n on [0, 1/eps_n], with eps_n -> 0.

Inner estimate. The characters theta_1, theta_2 are states of the base algebra B_n, and S(C) is one point, so mk_{L_B}(theta_1, theta_2) < eps_n. Applying this to the real and imaginary parts of <Theta, Theta'> and using inner Leibniz,

```
abs(<Pi_1 Theta, Pi_1 Theta'> - <Pi_2 Theta, Pi_2 Theta'>) <= 4 eps_n DN_P(Theta) DN_P(Theta').
```

At most N eigenvalues. Suppose D_n has N + 1 orthonormal eigenvectors xi_i with eigenvalues in [-R, R], so DN(xi_i) <= 1 + R. Lift them to Theta_i with DN_P(Theta_i) <= 2(1 + R), and put eta_i = Pi_2 Theta_i in C^N. The Gram matrix G of the eta_i has rank at most N, so norm(1 - G) >= 1, while the inner estimate gives norm(1 - G) <= 16 (N + 1)(1 + R)^2 eps_n. This fails for large n.

Concentration. Let e be a unit eigenvector of D_F with eigenvalue lambda, C = 1 + norm(D_F), and lift e to Theta with DN_P(Theta) <= 2C and xi = Pi_1 Theta. Apply the separation to the pseudo-state omega = xi/(2C): there is omega' with DN_F(omega') <= 1 and, for t in [0, 1/eps_n], abs(<omega, e^{itD_n} xi> - e^{it lambda} <omega', e>) <= 2C eps_n. Comparing with t = 0 and multiplying by 2C,

```
abs(<xi, e^{itD_n} xi> - e^{it lambda} norm(xi)^2) <= 8 C^2 eps_n.
```

Let nu be the spectral measure of D_n at xi. Taking real parts and averaging over t in [0, 1/eps_n] gives the integral of 1 - sinc((x - lambda)/eps_n) d nu(x) at most 8 C^2 eps_n. Since 1 - sinc(u) >= 1/2 for abs(u) >= 2, nu{abs(x - lambda) >= 2 eps_n} <= 16 C^2 eps_n.

Multiplicities. Apply this to an orthonormal eigenbasis e_i of D_F, with lifts xi_i. The inner estimate makes the Gram matrix of the xi_i tend to the identity, and concentration keeps it so after projecting each xi_i to the spectral window of radius 2 eps_n around its eigenvalue. So each window around an eigenvalue of multiplicity m contains at least m eigenvalues of D_n, for large n. The total is at most N, by the previous step, so equality holds. Mapping eigenvectors to the e_i gives W_n with norm(W_n D_n W_n* - D_F) <= 2 eps_n. ■

Corollary 15.8 (finite limits of connected commutative triples). Let T_n be metric spectral triples with A_n = C(X_n), X_n connected, converging in Lambda_spec to a triple with finite-dimensional Hilbert space. Then the limit is (C, C^N, D_F), and the convergence is characterized by Theorem 15.7. Every Hermitian D_F arises: on the circle of length l, the operator -i d/dx + diag(lambda_i) on L^2(S^1_l; C^N) has the block diag(lambda_i) on constants and gap at least 2 pi/l - norm(D_F). This construction concerns ungraded triples; a grading or real structure imposes its own compatibility conditions.

Proof. A metric triple is faithful: a self-adjoint a acting as 0 has [D, a] = 0, so a = t 1 by ker L = R 1, and then t = 0. So the limit algebra is finite-dimensional, and Theorem 15.2 gives C. ■

Corollary 9.10 fits the converse: at fixed mu every eigenvalue diverges, the spectra have no bounded sequences, and no limit exists. Theorem 9.8 lies outside the metric, because the diameter diverges.

Grading and real structure

The metric ignores gamma and J. They pass to the block exactly.

Proposition 15.9. In Theorem 15.3 let Sigma = -Sigma, and let D anticommute with a grading gamma and satisfy J D = eps' D J for an antiunitary J with J^2 = eps and J gamma = eps'' gamma J. Then P commutes with gamma and J, and gamma_F = W gamma W*, J_F = W J W* satisfy the same relations exactly with D_B = W D W*, and with D_F up to defects bounded by 2 Delta. The tunnel of Theorem 15.3 controls them: for DN_T(omega) <= 1 and DN_P(xi, eta) <= 1,

```
abs(<omega, gamma xi> - <W omega, gamma_F eta>) <= 1/g^2 + rho,     abs(<omega, J xi> - <W omega,
↪  J_F eta>) <= 1/g^2 + rho.
```

For Delta > 0, W omega need not be a pseudo-state of T_F; the normalized c W omega is one, and adds at most Delta to these bounds.

Proof. Spectral projections of D for a symmetric set commute with gamma and J, because gamma and J map the lambda-eigenspace to the eigenspaces of -lambda and eps' lambda. Decompose along P and Q as in Theorem 15.3, using additivity of J_F. ■

So gamma added to the families converges along the same tunnels, and the KO-signs of the parent reach the block unchanged when Delta = 0. For the interval, P Gamma P = sigma_3 (Theorem 9.6). Composition with the antilinear J gives antilinear functionals, which the definition of the separation does not cover; a statement of convergence for J needs an extension of the pseudo-state formalism, which we have not developed.

Unbroken symmetry and the metric class

Proposition 15.10 (referee). Let (A, H, D, J) be a finite-dimensional real triple with order zero, order one and J D = eps' D J, and let u be a unitary in A. If D commutes with u J u J^-1, then [D, u] = 0. If moreover the triple is metric, u is scalar, and the inner gauge transformation is trivial.

Proof. Put v = J u J^-1. From [D, u v] = 0, we get [D, u] v = - u [D, v], so a := u* [D, u] = - [D, v] v*. Now [D, v] = eps' J [D, u] J^-1 commutes with A by order one, and v* commutes with A by order zero. So a commutes with u, and [D, u^n] = n a u^n by induction. Then u^n D u^-n = D - n a. D is bounded, so a = 0. For a metric triple, the self-adjoint parts of u lie in ker L = R 1. ■

Counterexample 15.11 (referee; order one is needed). Let H = M_2(C) with the Hilbert-Schmidt product, A = M_2(C) acting by left multiplication, J x = x*, and D x = x - (1/2) tr(x) I. Applying [D, L_a] to I gives a - (1/2) tr(a) I, so [D, L_a] = 0 forces a scalar, and the triple is metric. Yet every conjugation x -> u x u* = L_u J L_u J^-1 commutes with D. This triple fails order one.

What this says about T0

Corollary 15.8 describes one notion of convergence: the state space of the whole algebra C(K_l) must converge in the Monge-Kantorovich metric. Under it, a collapsing connected parent yields a point, and the finite data survive as (C, C^N, D_F, gamma_F, J_F). Three limits on its reach:

1. **The truncation is a different object.** Along Theorem 9.9, the compression at each l is exactly Connes-Lott, with distance 1/m_f, while the parent has diameter l. The compression is not a quantum isometry, so this metric cannot see it. This is the towerless phenomenon itself: the two-sheet distance is unrelated to the size of the internal space.
2. **The class excludes a kind of target.** A metric spectral triple has ker L_T = R 1, so no nonscalar element of A commutes with D. The finite triple of the Standard Model lies outside the class: D_F commutes with the nonscalar subalgebra C_F = {(lambda, lambda, 0)}, which keeps the photon massless, and colour is unbroken (Chamseddine-Connes-Marcolli 2007, section 2.6). Its almost-commutative product lies outside as well, through the constant copy of such an element. A_F is moreover a real algebra, so applying the complex framework also needs a complexification. Among finite real first-order triples, the metric class has no nontrivial unbroken inner gauge transformation (Proposition 15.10); without order one this fails (Counterexample 15.11).
3. **The theorem concerns the scalar algebra.** Theorem 15.2 restricts limits of the whole scalar algebra C(K). Morita equivalence does not remove that restriction: C(K) and C(K, M_n) are Morita equivalent but not isomorphic, and their limits differ, and a bundle E can carry matrix-valued connections while C(K) still acts by scalars. Replacing the represented algebra by End(E) is a change of data, and a possible enlargement of T0. In that enlargement, Farsi-Latrémolière Theorem 4.6 makes an even metric finite (B, H_F, D_F) with noncommutative B the limit of B (x) C(K_l) with K_l collapsing, with nonzero multiplicity space ker D_K; there B is put in by hand.

Theorem 15.2 is a real constraint within its scope: this metric, on scalar function algebras. T0 also allows truncations, where it does not apply; lemma 16 treats these.

Part 16

Noncommutative finite real triples from commutative data: truncating the coarse correspondence

Proved by Claude Opus 5.5 without Astra or Fable, in response to the challenge that the T0 note overread lemma 15 and the Farsi-Latrémolière limits. Signs and the level-two algebra were checked with numpy on random data (section "Numerical check"). The term "coarse correspondence" is that of Connes, *Noncommutative Geometry* (1994), chapter V, for the bimodule of Hilbert-Schmidt operators, and the form D = Delta + J Delta J^-1 is in Krajewski (arXiv:hep-th/9701081) and Paschke-Sitarz (arXiv:q-alg/9612029); all checked against the sources. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): the examples and Corollary 16.4 are correct; amendments to Proposition 16.1, Theorem 16.2, Corollary 16.3 and the caveats are incorporated below; Remark 16.5 and Example 16.6 come from the referee's assessment of what the lemma achieves. The two-sheet section was refereed separately (Terra 6.1, 3 Nov 2026; report), with one error in Corollary 16.8 and one in Example 16.9, corrected below.

Outcome. Compressing functions on a connected commutative space to a fixed spectral window produces a noncommutative finite algebra with no limit taken. With the standard reality J a J^-1 = a*, the inherited structure fails order zero exactly when that algebra is noncommutative (Proposition 16.1). With the parent Hilbert space doubled to H_0 (x) conj(H_0), order zero and order one hold for every such algebra (Theorem 16.2, Corollary 16.3); these two conditions are formal consequences of the tensor square, available for any finite even representation (Remark 16.5). The KO-6 signs and the count dim H_F = 2 k^2 are reached by a product with a Clifford point, which is an input and fails finite orientability (Corollary 16.4), and more faithfully by a parent with two sheets, which gives the form of the Chamseddine-Connes classification (Theorem 16.7, Corollary 16.8, Example 16.9). This selects no particular algebra and gives no dynamics.

Setting

Let (A_0, H_0, D_0, gamma_0) be an even spectral triple, A_0 commuting with gamma_0, with D_0 of compact resolvent. Let Sigma = -Sigma be a finite subset of R and P_0 = 1_Sigma(D_0), of rank k, so that P_0 commutes with gamma_0. Put V = P_0 H_0, graded by gamma_V, with d = D_0 restricted to V, and

```
Q(a) = P_0 a P_0 on V,     O_F = Q(A_0),     A_F = C*(O_F) in B(V).
```

O_F is an operator system, in general not an algebra, and A_F consists of even operators. Compression is standard: for functions it is the Toeplitz compression, and its use in truncating spectral triples is due to D'Andrea-Lizzi-Martinetti and Connes-van Suijlekom. The latter note that the C*-algebra generated by PAP is in many examples a noninformative full matrix algebra. Since Q(1) = 1, A_F is the bicommutant of O_F, so its structure is set by the operators commuting with all compressed functions, such as the grading in example (b) and j in Example 16.9. That is where the information lies, and the fact that A_F is noncommutative is modest.

Examples of A_F.

(a) *Chiral interval, lowest level.* For condition (c) with M = 0 and Sigma = {+/- pi/(2l)}, O_F = C + C = A_F (lemma 05).

(b) *Chiral interval, next level.* For Sigma = {+/- pi/(2l), +/- 3 pi/(2l)}, V has basis (c_1, 0), (c_2, 0), (0, s_1), (0, s_2), with c_r = (2/l)^{1/2} cos((r - 1/2) pi x/l) and s_r = (2/l)^{1/2} sin((r - 1/2) pi x/l). With C_n = (1/l) int f cos(n pi x/l),

Q(f) = (Q_L, Q_R), Q_L = [[C_0 + C_1, C_1 + C_2], [C_1 + C_2, C_0 + C_3]], Q_R = [[C_0 - C_1, C_1 - C_2], [C_1 - C_2, C_0 - C_3]].

The values C_0, C_1, C_2, C_3 are independent and determine Q(f), so O_F has dimension 4. Each of Q_L and Q_R alone ranges over all symmetric 2 x 2 matrices, which generate M_2(C), so A_F projects onto each summand. For f_1 = 1/2 + cos(pi x/l) + cos(2 pi x/l) + cos(3 pi x/l) and f_2 = 1/2 - cos(pi x/l) + cos(2 pi x/l) - cos(3 pi x/l),

```
Q(f_1) = ([[1, 1], [1, 1]], 0),     Q(f_2) = (0, [[1, -1], [-1, 1]]).
```

M_2(C) is simple, so multiplying Q(f_1) and Q(f_2) by elements of A_F gives M_2(C) + 0 and 0 + M_2(C). Hence A_F = M_2(C) + M_2(C), of dimension 8. The finite algebra of the interval itself is noncommutative at the second level.

(c) *Twistor space.* For the level-one Toeplitz compression on CP^3, Q(x_a) = Gamma_a/5 (Proposition 10.1), and these generate A_F = M_4(C) on V = C^4. Here V is the kernel of the Dolbeault-Dirac operator twisted by O(1), so d = 0 on V.

The standard correspondence

Proposition 16.1 (order zero fails with inherited reality). Let J_0 be a real structure on H_0 (antiunitary, J_0^2 = +/-1, with the grading sign), with J_0 a J_0^-1 = a* for all a in A_0 and J_0 D_0 = eps' D_0 J_0, and let J_F = J_0 restricted to V. Then J_F Q(a)* J_F^-1 = Q(a), and the data (A_F, V, d, J_F) satisfy order zero if and only if A_F is commutative. The same holds for a product with a fixed finite factor that itself satisfies order zero, with the compatible product real structure.

Proof. J_0 preserves V because Sigma = -Sigma. Then J_0 P_0 a* P_0 J_0^-1 = P_0 a P_0. Order zero would make Q(a) commute with Q(b)° = Q(b) for all a, b, so O_F and hence A_F would be commutative; the converse is immediate, since J_F A_F J_F^-1 = A_F. For the product, take the unit in the finite factor: [1 (x) Q(a), 1 (x) Q(b)] = 1 (x) [Q(a), Q(b)]. ■

Together with Amendment 2 of lemma 07 (first order fails for C + C with inherited J_2), this is the correct form of the obstruction. It concerns the standard correspondence L^2(A_0) with the pointwise real structure.

The coarse correspondence

Let conj(H_0) be the conjugate Hilbert space, with canonical antiunitary xi -> xi^c, and with gamma^c, D^c the conjugate operators, gamma^c xi^c = (gamma_0 xi)^c and D^c xi^c = (D_0 xi)^c. On H = H_0 (x) conj(H_0), which is the coarse correspondence of A_0, equivalently the Hilbert-Schmidt operators on H_0, define for homogeneous xi, eta

```
pi(a) = a (x) 1,
DD = D_0 (x) 1 + gamma_0 (x) D^c,
Gamma = gamma_0 (x) gamma^c,
J (xi (x) eta^c) = (-1)^{|xi| |eta|} eta (x) xi^c.
```

Theorem 16.2. (A_0, H, DD, J, Gamma) is a real even spectral triple with signs J^2 = 1, J DD = DD J, J Gamma = Gamma J (KO-dimension 0). For even a, b with bounded commutators,

```
a° := J pi(a)\* J^-1 = 1 (x) (a\*)^c,     [pi(a), b°] = 0,     [[DD, pi(a)], b°] = 0,
```

and norm([DD, pi(a)]) = norm([D_0, a]). So if the original triple is metric, the new triple is metric with the same Connes distance on A_0. If the Weyl exponent p = lim log N_0(R)/log R of D_0 exists, that of DD is 2 p, since N_0(R/sqrt(2))^2 <= N_DD(R) <= N_0(R)^2.

Proof. J is antilinear, and J^2 = 1 because the sign is applied twice with the degrees exchanged. Gamma has eigenvalue (-1)^{|xi| + |eta|}, symmetric under the exchange. For the Dirac operator,

```
J DD (xi (x) eta^c) = (-1)^{(|xi| + 1) |eta|} eta (x) (D_0 xi)^c + (-1)^{|xi|} (-1)^{|xi| (|eta| +
↪  1)} D_0 eta (x) xi^c
                    = (-1)^{|xi| |eta|} [D_0 eta (x) xi^c + (-1)^{|eta|} eta (x) (D_0 xi)^c] = DD J
                     ↪  (xi (x) eta^c).
```

For even a, a* preserves degree, so J pi(a)* J^-1 (eta (x) xi^c) = eta (x) (a* xi)^c, which gives the formula for a° and order zero. Since a is even, [DD, pi(a)] = [D_0, a] (x) 1, which commutes with 1 (x) (b*)^c. Finally DD^2 = D_0^2 (x) 1 + 1 (x) (D^c)^2, because gamma_0 anticommutes with D_0. An eigenbasis of D_0 decomposes DD into finite-dimensional invariant blocks, so DD is essentially self-adjoint on the algebraic tensor product, with compact resolvent, and the spectrum is +/- (lambda_i^2 + lambda_j^2)^{1/2}. ■

Geometrically, when H_0 = L^2(S_X) on an even-dimensional spin manifold X, a charge conjugation commuting with D_X identifies conj(H_0) with H_0, H is L^2(S_{X x X}) and DD is the ordinary product Dirac operator. (The grading may acquire an overall sign under this identification.) So the data are a product-manifold Dirac operator on X x X with a restricted observable algebra C(X) (x) 1, and a J covering the flip of the two factors, which differs from the pointwise spin real structure. They count as commutative Kaluza-Klein data only with that restriction and that J: they are neither the ordinary spin triple of X x X with its full function algebra nor a finite-rank twist over X. The restriction is essential: on all of C(X x X), the compressions generate A_F (x) conj(A_F), and order zero generally fails again.

Corollary 16.3 (truncation). Let P = P_0 (x) conj(P_0). It commutes with DD, Gamma and J. It is a joint cutoff in the two factors, and need not be a spectral projection of DD: for absolute eigenvalues 1, 5, 7, sqrt(73) of D_0 with P_0 selecting the first three, the selected pair (5, 7) and the excluded pair (1, sqrt(73)) both give DD^2 = 74. The compression of pi(A_0) is O_F (x) 1; passing to the generated algebra gives the finite real even triple

```
T_F = (A_F, V (x) conj(V), D_F, J_F, Gamma_F),     D_F = d (x) 1 + gamma_V (x) d^c = Delta + J_F
↪  Delta J_F^-1,     Delta = d (x) 1,
```

with A_F acting on the left. It has KO-dimension 0, and satisfies order zero and order one for every A_F, noncommutative or not, with dim H_F = k^2. The family is exact at each truncation; no limit is taken.

Proof. The proof of Theorem 16.2 applies verbatim on V with arbitrary even a, b in B(V), since it uses only the grading and the tensor structure. For J_F Delta J_F^-1, apply J to (d (x) 1) J (xi (x) eta^c): the signs combine to (-1)^{|xi|}, giving gamma_V (x) d^c. ■

So H_F has the bimodule form of Connes' finite geometries, with A_F on the left and its opposite on the right, and D_F has the first-order form of Krajewski and Paschke-Sitarz.

Example (a) gives (C + C, C^2 (x) C^2, m (sigma_1 (x) 1 + sigma_3 (x) sigma_1)). There [[D_F, e], e°] = 0 for e = diag(1, 0), which repairs the first-order failure of Counterexample 08.4 at the price of doubling. Example (b) gives (M_2(C) + M_2(C), C^16, D_F).

Corollary 16.4 (KO-6 by a Clifford point). The product of T_F with (C, C^2, 0, sigma_1 K, sigma_3) is

```
(A_F, V (x) conj(V) (x) C^2, D_F (x) 1, J_F (x) sigma_1 K, Gamma_F (x) sigma_3),
```

with signs (+, +, -), order zero, order one and dim H = 2 k^2. It fails the usual zero-dimensional orientability axiom: every zero-cycle built from the left and opposite actions acts as T (x) 1_2, while the grading is Gamma_F (x) sigma_3.

Proof. (sigma_1 K)^2 = 1 and sigma_1 K sigma_3 = -sigma_3 sigma_1 K, while the D term is untouched. a° is a° of T_F tensored with 1. ■

Remark 16.5 (the order conditions are formal). For every finite even representation (A, V, d, gamma), the map V -> V (x) conj(V), d -> d (x) 1 + gamma (x) d^c gives a real bimodule with order zero and order one, using no prior finite real structure. So these two conditions follow from choosing the tensor square, and say nothing specific about A_F. The construction does not recover an arbitrary prescribed finite (H, D, J). What is specific to T0 is that A_F is generated by compressed functions, that the tensor square is itself the truncation of a commutative parent in the restricted sense above, and families such as the following.

Example 16.6 (a tower-decoupling family, from the referee). Let D_0 be the Dolbeault-Dirac operator on CP^3 twisted by O(1), with V = ker D_0 = C^4, and let DD_l = DD/l on the doubled parent. Its kernel is V (x) conj(V), and its gap grows like 1/l. Here P is a spectral projection. Compression followed by algebra generation gives the constant finite data A_F = M_4(C), dim H_F = 16, D_F = 0, and the Clifford point gives dim H = 32 with the KO-6 signs. Order one is automatic, since D_F = 0. The whole triple converges to (C, C^16, 0) by Theorem 15.3, consistently with lemma 15: the two statements concern the whole algebra and the truncation.

Two sheets: the form of the classification

Chamseddine and Connes (*Why the Standard Model*, arXiv:0706.3688, sections 2-3) classify irreducible pairs (A, J) with order zero and a separating vector: the centre of the complexified algebra is C, with A_C = M_k(C) on itself and J x = x*, or C + C, with A_C = M_k(C) + M_k(C) on itself and J(x, y) = (y*, x*). With centre C, J gamma = -gamma J is impossible. The pair of Corollary 16.4 is therefore reducible: the projection 1 (x) (1 + sigma_1)/2 commutes with A_F and J. The irreducible form comes from a parent with two sheets. At the finite level the bimodule (W (x) conj(V)) (+) (V (x) conj(W)) with its adjoint-flip J is already stated in Remark 2.9 of that paper; what we add is its realization from parents and compressed functions.

Theorem 16.7 (two-sheet coarse parent). Let (A_X, H_X, D_X, gamma_X) and (A_Y, H_Y, D_Y, gamma_Y) be even spectral triples. On H = (H_X (x) conj(H_Y)) (+) (H_Y (x) conj(H_X)), for even a in A_X and b in A_Y, put

```
pi(a, b) = (a (x) 1) (+) (b (x) 1),
DD = (D_X (x) 1 + gamma_X (x) D_Y^c) (+) (D_Y (x) 1 + gamma_Y (x) D_X^c),
Gamma = (gamma_X (x) gamma_Y^c) (+) (-gamma_Y (x) gamma_X^c),
J (xi (x) eta^c) = (-1)^{|xi| |eta|} eta (x) xi^c,
```

where J maps each summand to the other. Then J^2 = 1, J DD = DD J and J Gamma = -Gamma J, the KO-6 signs. Moreover

```
(a, b)° = (1 (x) (b\*)^c) (+) (1 (x) (a\*)^c),
```

so the first summand is an (A_X, A_Y)-bimodule and the second an (A_Y, A_X)-bimodule, and order zero and order one hold.

Proof. The computation of Theorem 16.2 applies with D_Y in place of the second factor: on xi (x) eta^c in the first summand, J DD and DD J both give (-1)^{|xi| |eta|} [D_Y eta (x) xi^c + (-1)^{|eta|} eta (x) (D_X xi)^c]. The sign in Gamma on the second summand gives J Gamma = -Gamma J. For a°, apply J, then pi(a, b)*, then J, as in Theorem 16.2: an element of the first summand goes to the second, where b* acts on the left, and returns with b* on the right factor. Since a and b are even, [DD, pi(a, b)] = ([D_X, a] (x) 1) (+) ([D_Y, b] (x) 1), which commutes with the opposite actions. ■

Corollary 16.8 (truncation; the form of the classification). For symmetric windows with V_X, V_Y of dimensions k_X, k_Y, and even subalgebras A_{F,X} in B(V_X), A_{F,Y} in B(V_Y), the compression

```
T_F = (A_{F,X} + A_{F,Y}, (V_X (x) conj(V_Y)) (+) (V_Y (x) conj(V_X)), D_F, J_F, Gamma_F)
```

has the KO-6 signs, order zero and order one, and dim H_F = 2 k_X k_Y. If gamma_{V_X} lies in A_{F,X} and gamma_{V_Y} in A_{F,Y}, the real orientability relation holds: with c = (gamma_{V_X}, gamma_{V_Y}) and z = (1, -1), Gamma_F = pi(z c) c°, a zero-cycle with coefficients in the algebra and its opposite. This is weaker than Gamma_F in pi(A_F). If V_X and V_Y are ungraded with k_X = k_Y = k and the generated algebras are full, then (A_F, H_F, J_F) is exactly the irreducible pair with centre C + C: identify both summands with M_k(C), so that J_F(x, y) = (y*, x*); an invertible matrix in each summand is a separating vector, and a projection commuting with both full matrix actions is scalar on each summand, with J equating the scalars. In all cases D_F commutes with the two sheet projections, so the sheets are not connected. It need not commute with the whole centre: for V_X = C^2, gamma_X = sigma_3, d_X = sigma_1, A_{F,X} = C + C acting diagonally and V_Y = C, the central element (diag(1, 0), 0) does not commute with D_F.

Proof. Theorem 16.7 applies on the truncation with arbitrary even operators. For orientability, pi(c) c° = (gamma_{V_X} (x) gamma_{V_Y}^c) (+) (gamma_{V_Y} (x) gamma_{V_X}^c), and z flips the sign of the second summand. The last statements are direct. ■

Example 16.9 (twistor level one, and M_2(H) + M_4(C)). Let D_0 be the Dolbeault-Dirac operator on CP^3 twisted by O(1), with kernel H^0(CP^3, O(1)) = C^4, and let j be the antiunitary with j^2 = -1 commuting with the Gamma_a of Proposition 10.1, which makes C^4 a quaternionic space H^2. Use this space twice, as W (with the functions of S^4 pulled back along the twistor projection) and as V (with the functions of CP^3).

1. A real function f on S^4 pulls back to a function invariant under w -> j w. So j Q(f) j^-1 = Q(f), and Q(f) is a Hermitian element of End_H(W). Since Q(x_a) = Gamma_a/5, the real algebra generated is End_H(W) = M_2(H), of real dimension 16.
2. For real functions on CP^3, Q(w* B w / w* w) = (B + tr(B) 1)/5 for Hermitian B, so Q is onto the Hermitian matrices, and the real algebra generated is End_C(V) = M_4(C), of real dimension 32.

Hence Corollary 16.8, with the parent gradings, which are trivial on the kernels, gives the real algebra A = M_2(H) + M_4(C) on H_F = Hom(V, W) (+) Hom(W, V) = E (-) E* of dimension 32, with J(x, y) = (y*, x*). These are the algebra, bimodule and real structure of Chamseddine and Connes. The generation is of real algebras from real functions: complex generation on the S^4 side would give M_4(C).

The nontrivial grading of W in that construction is not obtained from this parent. An even parent has [P gamma P, P a P] = 0 for every compressed function, while Gamma_5 = 5 Q(x_5) anticommutes with Q(x_1). So the grading has to be added after truncation, as abstract finite data, fixing an axis. The full algebra is then a graded prealgebra rather than the algebra of an even triple, with even part H + H + M_4(C), and Gamma_F(xi, eta) = (Gamma_5 xi, -eta Gamma_5), which is Proposition 3.2 of that paper; the real orientability relation holds on the even part. Similarly d = mu n^a Gamma_a with n orthogonal to the axis is the compression of a bounded real multiplication operator, but adding that operator to the parent Dirac operator keeps neither the kernel as a spectral projection nor the parent grading as an odd perturbation, so d too is abstract data. It commutes with the two sheet projections and with the centre R + C of the full prealgebra, and not with the centre of the even algebra.

This bears on the step left open in canonical section 4, where M_2(H) in M_4(C) is a nested pair on one four-space, with "the independent E, F module roles" to be derived. The example closes that abstract finite-module problem after adding a second sheet: the nested actions on one four-space become independent summands acting on two Hom spaces. It does not turn four global sections into rank-four fibres over S^4, derive the two sheets from the fibration, or produce the Dirac operator connecting the two pieces that the selection of A_F = C + H + M_3(C) requires. The fibre-rank problem remains open.

The inputs are the second sheet, the restriction of one algebra to base functions, real algebra generation, the four-dimensional window, the grading axis and d. The content beyond the formal bimodule is the concrete

compression calculation in items 1 and 2, already in the corpus, and its explicit match with the classified data.

Caveats

These results should be read as narrowly as the note they correct.

1. **Metric.** T_F is usually not a metric spectral triple. For A_F = B(V_+) + B(V_-) with V_+ of dimension at least 2 and d odd, the commutant of d in A_F contains the pairs (x, B^-1 x B) with x commuting with B B*, when B is the invertible block of d. In example (b), (diag(1, 0), diag(1, 0)) has L = 0, so d leaves a residual U(1) x U(1) of U(2) x U(2) unbroken. The Standard Model finite triple has the same feature, through M_3(C).
2. **Spectrum.** The eigenvalues of D_F are +/- (lambda_i^2 + lambda_j^2)^{1/2}. In example (a) they are +/- sqrt(2) m with Connes distance 1/m, so the doubled spectrum differs from the masses in d, and the mass relations of Theorem 9.6 and lemma 11 would have to be redone.
3. **Parent.** The doubled parent is a choice. Its algebra sees one factor of X x X, its J covers the flip, which is outside the spin real structure of X x X, and generating A_F from O_F is an additional operation. We have no dynamical reason for it. For an ordinary compact spin X of dimension n, its Weyl exponent is 2 n with metric dimension n, which bears on the counting of 9, 10 and 11 and has not been analyzed.
4. **KO-6.** The Clifford point is an input, as F_0 is in Theorem D, and it fails finite orientability. In Theorem 16.7 the sign comes from the block structure of two sheets, which is itself an input. The documents checked here establish neither a construction of two sheets from one connected parent nor a no-go theorem. The grading-reversing structure of lemma 08 is a candidate we have not tested.
5. **Count.** For k = 4, dim H = 32, the value in the classification of Chamseddine-Connes, and k = 4 occurs in examples (b) and (c). For k = 4 in examples (b) and (c) alone this is a count and nothing more: the classification's algebra is M_2(H) + M_4(C), acting differently, and those examples do not produce it. Example 16.9 does, as abstract data with the listed inputs.

Consequence for T0 and T0b

T0 asks for finite real triples from commutative Kaluza-Klein data under specified notions of convergence. For one specified truncation procedure (compression, then generation, on the doubled parent with restricted algebra and flip J), this lemma gives a noncommutative finite algebra with an explicit bimodule, and a family with the tower decoupled (Example 16.6). It does not establish the target under every meaning of ordinary Kaluza-Klein data. For T0b, it gives examples with the requested signs and first-order relations; with two sheets it reaches the form of the classification, and in Example 16.9 the real algebra M_2(H) + M_4(C), bimodule and real structure of Chamseddine and Connes, with the grading added as abstract data. It leaves open an operator connecting the two pieces and hence the selection of A_F, a parent that yields the grading and nonzero mass data, and the beyond-first-order and twisted frameworks. See the background note.

Numerical check

Python (numpy), run on 3 Nov 2026 in a temporary directory, not kept. On random odd d and even a, b for dim V_+ = dim V_- = 2 with J = M K: J^2 = 1, J DD = DD J, J Gamma = Gamma J, DD Gamma = -Gamma DD, order zero and order one all hold to machine precision, with norm([a, b]) = 7.67; the KO-6 signs of Corollary 16.4 hold. For Theorem 16.7 and Example 16.9, with k = 4, W graded by Gamma_5 and V ungraded: J^2 = 1, J Gamma = -Gamma J, J D = D J, D Gamma = -Gamma D, D = Delta + J Delta J^-1, order zero and order one for random a, b, and Gamma_F = pi(z c) c° all hold to machine precision; j^2 = -1 commutes with the Gamma_a, and x_a(j w) = x_a(w); the real algebras generated by Gamma_a, by i Gamma_a and by all Hermitian matrices have real dimensions 16, 32 and 32; and sum_a x_a Gamma_a = 2 (P_w + P_{jw}) - 1. For example (b), by quadrature with 23 test functions: dim O_F = 4 and the generated algebra has dimension 8; the two displayed values Q(f_1) and Q(f_2) were reproduced by quadrature.

Part 18

T5b: vector masses in the spectral action at fixed cutoff

Prompt: prompts/18 (STRATEGY T5b). Attempted by gpt-6-astra (codex 0.164, xhigh, 3 Nov 2026), working alone. Checked by Claude Opus 5.5 with Python (numpy, scipy), by computations independent of the kernel derived here: a direct heat trace with constant a reproduces equation (2) for five choices of (m_1, m_2), to about 0.03%; a direct heat trace on Fourier modes with a_2(x_1) = a cos(q x_1), including the F_{12} and [T, V] vertices, reproduces the kernel (37) at three values of (m, q), to 0.1-0.3%, the size of the discretization error. The closed forms (16)-(19), the bound (28), the limits (27), the neutral matrix (30) recomputed from Standard Model charges, and the identities behind (31) also check. The largest value of the Taylor mass is M_W^2 = 0.275 Lambda^2, at m_t^2 = 2.02 Lambda^2. Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): all ten numbered results are correct, with small amendments incorporated below. The referee added Propositions 11 to 14 and Remark 15, which widen the scope in both directions; the ratio in Proposition 14 was added by Claude.

Scope. Within the three-generation, zero-Majorana, top-only family at fixed cutoff, these results refute an infinitely heavy W, Z, t limit for the stated quadratic mass definitions. They do not rule out other finite data (Proposition 14), a different action or constraint, a joint limit with Lambda -> infinity, or a quantum effective theory. They do not show unbroken SU(2) x U(1) on the light space (Remark 15).

For the heat cutoff, on $\mathbb{R}^4$, we obtain the exact quadratic vector form without expanding in m/Λ. In the three-generation Standard Model finite representation with only the top Dirac mass nonzero, both the mass defined by the q^0/q^2 Taylor ratio and the unique positive timelike pole mass satisfy $M_W \sim \Lambda^2/m_t$ as $m_t/\Lambda \to \infty$ at fixed Λ. The broken vectors consequently do not decouple by becoming heavy, although the heavy fermion contribution disappears from the gauge coefficients. These statements concern the classical bosonic quadratic action about a prescribed constant background; moreover, the positive Laplace class considered below does not itself select a finite nonzero stationary background when radial variations of that background are allowed.

1. Theorem (exact quadratic kernel).

Let

$$D_A = -i\gamma^\mu(\partial_\mu + iA_\mu) + \gamma_5 V, \qquad V = V_0 = V_0^\dagger,$$

on $\mathbb{R}^4$, with Hermitian Euclidean gamma matrices, $\{\gamma^\mu, \gamma^\nu\} = 2\delta^{\mu\nu}$, $\gamma_5^2 = 1$, and four spin components. Let the finite trace be the literal trace on H_F, including every stated multiplicity, without a subsequent projection; this agrees with the bosonic trace of Chamseddine-Connes-Marcolli, which counts particles, antiparticles and four spin components. Take A smooth and compactly supported.

We use the vacuum-subtracted action, or equivalently its quadratic variation, since the constant-background trace on $\mathbb{R}^4$ has infinite volume. Write

$$A_\mu(q) = a_\mu(q)T, \qquad T = T^\dagger, \qquad q^\mu a_\mu(q) = 0, \qquad \int_q = \int \frac{d^4q}{(2\pi)^4}.$$

Diagonalize V_0:

$$V_0 e_i = m_i e_i, \qquad m_i \in \mathbb{R}.$$

The signed eigenvalues are needed; their squares alone do not determine the answer.

For $S_t = \operatorname{Tr} e^{-tD_A^2}$, define

$$b(u) = u(1-u), \quad d_{ij} = m_i - m_j,$$

$$I_t(a, b; q^2) = \int_0^1 e^{-t[(1-u)a + ub + u(1-u)q^2]}\, du.$$

Then

$$S_t^{(2)} = \int_q |a_T(q)|^2\, C_{t,T}(q^2),$$

where

$$\boxed{C_{t,T}(q^2) = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2\left[\left(\frac{2}{t}+d_{ij}^2+q^2\right)I_t(m_i^2,m_j^2;q^2) - \frac{e^{-tm_i^2}+e^{-tm_j^2}}{t}\right].} \tag{1}$$

For several generators, replace $|T_{ij}|^2$ by $\mathrm{Re}((T_a)_{ij}(T_b)_{ji})$.

In particular, writing $C(q^2) = C_0 + q^2 C_1 + O(q^4)$, we obtain

$$\boxed{C_{0,t} = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2\left[\left(\frac{2}{t}+d_{ij}^2\right)I_t(m_i^2,m_j^2;0) - \frac{e^{-tm_i^2}+e^{-tm_j^2}}{t}\right],} \tag{2}$$

and

$$\boxed{C_{1,t} = \frac{1}{8\pi^2}\sum_{ij}|T_{ij}|^2\int_0^1 e^{-t[(1-u)m_i^2+um_j^2]}\left[1-2b(u)-td_{ij}^2b(u)\right]du.} \tag{3}$$

These formulas also apply, after integration over $t = s/\Lambda^2$, to

$$f(z) = \int_{(0,\infty)} e^{-sz}\,d\nu(s), \quad 0 < f(0) = \nu((0,\infty)) < \infty, \quad \int (s^{-2}+s^2)\,d\nu(s) < \infty, \tag{4}$$

where ν is positive. These hypotheses suffice for the Euclidean action density and the Taylor coefficients used below. They do not, by themselves, guarantee analytic continuation to every negative q^2.

Proof. With $\nabla_\mu = \partial_\mu + iA_\mu$,

$$D_A^2 = -\nabla^2 + V_0^2 - \frac{i}{2}\gamma^{\mu\nu}F_{\mu\nu} + \gamma^\mu\gamma_5[A_\mu, V_0].$$

Let $P_0 = -\partial^2 + V_0^2$ and write $D_A^2 = P_0 + P_1 + P_2$, according to degree in A. Duhamel expansion and cyclicity give

$$S_t^{(2)} = -t\,\mathrm{Tr}(e^{-tP_0}P_2) + \frac{t^2}{2}\int_0^1 \mathrm{Tr}\,\left(e^{-(1-u)tP_0}P_1e^{-utP_0}P_1\right)du. \tag{5}$$

The linear vertex from j,p to $i,p+q$ is

$$T_{ij}\left[(2p+q)\cdot a + \gamma^{\mu\nu}q_\mu a_\nu + (m_j-m_i)\gamma^\mu\gamma_5 a_\mu\right].$$

Completing the momentum square produces

$$(4\pi t)^{-2}e^{-t[(1-u)m_i^2+um_j^2+b(u)q^2]}.$$

For transverse a, the averaged spin trace of the vertex product is

$$4\left(\frac{2}{t}+d_{ij}^2+q^2\right)|a|^2.$$

The P_2 term contributes

$$-\frac{1}{4\pi^2 t}\sum_{ij}|T_{ij}|^2e^{-tm_i^2}|a|^2.$$

Symmetrizing in i,j proves (1). Differentiation proves (2)–(3).

The divided-difference structure is explicit:

$$I_t(a,b;0) = \frac{e^{-ta}-e^{-tb}}{t(b-a)}, \qquad I_t(a,a;0) = e^{-ta}, \tag{6}$$

and

$$\int_0^1 u(1-u)e^{-t[(1-u)a+ub]}\,du = \frac{1}{t^2}\partial_a\partial_b I_t(a,b;0). \tag{7}$$

Thus (2) uses a first divided difference, while (3) also uses its mixed derivative. The full momentum dependence follows from the convergent series

$$I_t(a,b;q^2) = \sum_{n=0}^{\infty} \frac{(-q^2/t)^n}{n!}\partial_a^n\partial_b^n I_t(a,b;0). \tag{8}$$

Repeated-node values are obtained continuously.

For completeness, the mass coefficient for (4) can be written entirely in dimensionless variables. Put

$$r_i = m_i/\Lambda, \quad x_i = r_i^2, \quad z(u) = (1-u)x_i + ux_j, \quad \Phi_1(z) = \int s^{-1}e^{-sz}\,d\nu(s).$$

The bracket replacing that in (2) is

$$\Lambda^2\left[2\int_0^1 \Phi_1(z(u))\,du + (r_i-r_j)^2\int_0^1 f(z(u))\,du - \Phi_1(x_i) - \Phi_1(x_j)\right]. \tag{9}$$

The corresponding bracket in (3) is

$$\int_0^1 \left[(1-2b)f(z) + (r_i-r_j)^2 b f'(z)\right]du. \tag{10}$$

Each unweighted integral is a divided difference of an antiderivative. The moment assumptions justify the integrations and differentiations. □

On a finite flat torus, (5) remains exact, but the Gaussian momentum integral must be replaced by the spin-structure-dependent lattice sum. Explicitly, for a transverse polarization a, its two-vertex contribution per unit volume is

$$\frac{2t^2}{\mathrm{Vol}}\sum_{ij,p}|T_{ij}|^2\int_0^1 e^{-t[(1-u)(p^2+m_i^2)+u((p+q)^2+m_j^2)]}\left(|(2p+q)\cdot a|^2 + (d_{ij}^2+q^2)|a|^2\right)du,$$

and the quadratic-vertex contribution is

$$-\frac{4t}{\mathrm{Vol}}\sum_{ij,p}|T_{ij}|^2 e^{-t(p^2+m_i^2)}|a|^2.$$

Consequently, the continuum formulas below are statements on $\mathbb{R}^4$, or in the infinite-volume limit. A finite torus can have additional holonomy dependence, including for constant commuting connections.

2. Lemma (gauge invariance and the Goldstone direction).

Let scalar fluctuations include the gauge orbit of V_0. Then the nonzero vector coefficient C_0 in (2) is compatible with gauge invariance.

For one generator, parameterize the orbit by

$$V(x) = e^{i\pi(x)T}V_0 e^{-i\pi(x)T}.$$

With our conventions,

$$a_\mu \longmapsto a_\mu - \partial_\mu\alpha, \qquad \pi \longmapsto \pi + \alpha.$$

The invariant combination is

$$B_\mu = a_\mu + \partial_\mu\pi. \tag{11}$$

Proof. Gauge covariance gives

$$D_{A^U,V^U} = U D_{A,V} U^{-1}.$$

Applying $U = e^{-i\pi T}$ transforms V to V_0 and a to B. Hence

$$S[a, e^{i\pi T} V_0 e^{-i\pi T}] = S[B, V_0]. \tag{12}$$

This establishes the Goldstone completion directly, without a derivative expansion.

For comparison, the longitudinal vector kernel is obtained from (1) by replacing the q^2 in its prefactor by $(1-2u)^2 q^2$ inside the integral. If $[T, V_0] = 0$, its integral vanishes by

$$\int_0^1 [2 + tq^2(1-2u)^2] e^{-tq^2 u(1-u)}\, du = 2.$$

For a broken generator the longitudinal kernel need not vanish; it belongs to the invariant combination (11).

If V_0 is not stationary, a Cartesian scalar Hessian alone need not annihilate the linear gauge-orbit tangent. The quadratic term in $e^{i\pi T} V_0 e^{-i\pi T}$, contracted with the scalar tadpole, must also be retained. Equation (12) includes that term. Transverse vectors do not mix quadratically with scalar fluctuations on this translation-invariant background. □

3. Corollary (the asymptotic normalization).

Let all finite eigenvalues be small relative to Λ, let f satisfy (4), and let Q below be positive definite on the chosen generator space (otherwise pass to the quotient by its null space). Define

$$Q_{ab} = \mathrm{Re}\,\mathrm{Tr}(T_a T_b), \qquad K_{ab} = \mathrm{Re}\,\mathrm{Tr}([T_a, V_0]^\dagger [T_b, V_0]).$$

At leading order,

$$C_0 = \frac{f(0)}{8\pi^2} K, \qquad C_1 = \frac{f(0)}{12\pi^2} Q,$$

and therefore

$$\boxed{\mathcal{M}^2_{\mathrm{Taylor}} = \frac{3}{2} Q^{-1/2} K Q^{-1/2}.} \tag{13}$$

This reproduces Proposition 13.5.

For the full three-generation Standard Model representation,

$$Q_{11} = 12, \qquad K_{11} = \sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2),$$

so

$$\boxed{8M_W^2 = \sum_{\text{generations}} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2).} \tag{14}$$

Proof. In (2), the orbital expression

$$\frac{2}{t} I_t(a, b; 0) - \frac{e^{-ta} + e^{-tb}}{t}$$

starts at order $t(a-b)^2$. The leading term is consequently d_{ij}^2. In (3), the leading integral is $\int_0^1 (1 - 2u(1-u))du = 2/3$, proving (13).

For $T_1 = \sigma_1/2$ on each left doublet, one generation of particles contributes 2 to Q_{11}. Charge conjugates and three generations give 12. A doublet with Dirac masses m_u, m_d contributes $\frac{1}{2}(m_u^2 + m_d^2)$ to the particle commutator trace, before color multiplicity. This proves (14).

Equivalently, the gauge coefficient is

$$\frac{f(0) g_2^2}{2\pi^2} = \frac{1}{4}.$$

For comparison with CCM's $\operatorname{Tr} F(D/\Lambda)$, take $F(v) = f(v^2)$; then $F(0) = f(0)$. The normalization and mass constraint are precisely equations (4.10) and (4.16), with the latter stated in Lemma 4.6 of the checked preprint. CCM describes the relation as "valid at unification scale." Chamseddine–Connes–Marcolli, pp. 37–39. □

The leading bound $M^2 \le 6\|V_0\|^2$ in Lemma 14 follows from (13) and $K \le 4\|V_0\|^2 Q$. Formula (13), rather than an exact finite-cutoff identity, is its applicable approximation.

4. Proposition (the exact charged-vector Taylor mass).

Let the finite data be the full three-generation Standard Model representation, including charge conjugates. Let the Majorana matrix and every Dirac mass except one up-type color triplet be zero. Write

$$m = \mu\Lambda, \qquad x = \mu^2, \qquad r = e^{-x}.$$

Let $f(z) = e^{-z}$. Define

$$\begin{aligned} H(y) &= \int_0^1 e^{-yu(1-u)}\, du, \\ g(y) &= (2+y)H(y) - 2, \\ h_x(y) &= (2+x+y)\int_0^1 e^{-xu-yu(1-u)}\, du - 1 - e^{-x}. \end{aligned} \tag{15}$$

Then the full charged-vector kernel is

$$\boxed{C_W(q^2) = \frac{\Lambda^2}{4\pi^2}\left[\frac{9}{2}g(y) + \frac{3}{2}h_x(y)\right], \qquad y = q^2/\Lambda^2.} \tag{16}$$

Define

$$\begin{aligned} A(x) &= h_x(0) = \frac{2[1-(1+x)e^{-x}]}{x}, \\ w(x) &= 3\int_0^1 u^2 e^{-xu}\, du = \frac{3[2-(x^2+2x+2)e^{-x}]}{x^3}. \end{aligned} \tag{17}$$

Continuous values are understood at $x = 0$. The particle-trace kinetic coefficient is

$$K_W(x) = \frac{9}{2} + \frac{3}{2}w(x) > 0.$$

The Taylor mass, defined by $M^2_{W,\mathrm{Taylor}} = C_W(0)/C_W'(0)$, is

$$\boxed{\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{9A(x)}{4K_W(x)} = \frac{3[1-(1+x)e^{-x}]}{x[3+w(x)]}.} \tag{18}$$

Its limiting behavior is

$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{3}{8}x - \frac{23}{128}x^2 + O(x^3), \qquad \frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} \sim \frac{1}{x} \quad (x \to \infty). \tag{19}$$

Consequently,

$$\frac{m}{M_{W,\mathrm{Taylor}}} \longrightarrow \sqrt{\frac{8}{3}} \quad (\mu \to 0), \qquad \frac{m}{M_{W,\mathrm{Taylor}}} \sim \mu^2 \quad (\mu \to \infty). \tag{20}$$

Proof. On the particle space, $Q_{11} = 6$. The selected quark doublet contributes $3/2$, while the remaining doublets contribute $9/2$. Diagonalizing the massive left-right block produces eigenvalues $+m, -m$. The charged generator connects these to the massless down states; every such pair has $d_{ij}^2 = m^2$. Formula (1)

therefore gives h_x for the selected doublet and g for the remaining doublets. Charge conjugation supplies the factor two in (16).

Integration by parts gives

$$h_x(y) = 2\int_0^1 u\,[x + 2y(1-u)]e^{-xu-yu(1-u)}\,du, \tag{21}$$

and

$$\boxed{h_x'(y) = 2\int_0^1 u^2[1 - y(1-u)]e^{-xu-yu(1-u)}\,du.} \tag{22}$$

The boundary term proving (22) is $\partial_u[u(1-u)e^{-xu-yu(1-u)}]$. Thus

$$h_x'(0) = \frac{2}{3}w(x), \qquad g'(0) = \frac{2}{3}.$$

This proves (18). Expanding the elementary expressions at zero and taking their large-x limits proves (19)–(20). □

The same Taylor result extends to every f in (4). Put

$$\ell = f(0), \qquad w_f(x) = \int w(sx)\,d\nu(s), \qquad A_f(x) = \int \frac{A(sx)}{s}\,d\nu(s).$$

Then

$$\boxed{\frac{M_{W,\mathrm{Taylor}}^2}{\Lambda^2} = \frac{9A_f(x)}{4\left(\frac{9}{2}\ell + \frac{3}{2}w_f(x)\right)}.} \tag{23}$$

All denominators are positive, and dominated convergence gives

$$M_{W,\mathrm{Taylor}}^2 \sim \frac{3}{8}m^2 \quad (x \to 0), \qquad \boxed{\frac{M_{W,\mathrm{Taylor}}^2}{\Lambda^2} \sim \frac{\int s^{-2}\,d\nu(s)}{\ell x} \quad (x \to \infty).} \tag{24}$$

Indeed, $A(z)/z \to 1$, $w(z) \to 1$ at zero, while $zA(z) \to 2$, $w(z) \to 0$ at infinity.

If the other Dirac masses are merely small and nonzero, the answer depends on those additional parameters and is given by (1). In particular, the vanishing limit in (19) assumes they are zero: masses of surviving light doublets can leave a residual vector mass.

5. Theorem (the charged-vector pole).

Under the heat-cutoff hypotheses of Proposition 4, define a timelike pole by analytic continuation of the quadratic inverse propagator:

$$C_W(-M_{W,\mathrm{pole}}^2) = 0. \tag{25}$$

For every $x > 0$, this equation has exactly one solution with $M_{W,\mathrm{pole}}^2 > 0$. It is simple and has positive residue in the convention that the Euclidean propagator near the pole is a positive multiple of $(q^2 + M_{W,\mathrm{pole}}^2)^{-1}$.

Let $z(x) = M_{W,\mathrm{pole}}^2/\Lambda^2$. Its exact specification is the unique positive root

$$\boxed{\frac{9}{2}g(-z) + \frac{3}{2}h_x(-z) = 0.} \tag{26}$$

Furthermore,

$$z(x) \sim \frac{3}{8}x \quad (x \to 0), \qquad z(x) \sim \frac{1}{x} \quad (x \to \infty), \tag{27}$$

and both mass definitions obey

$$\boxed{0 < M_{W,\mathrm{Taylor}}^2,\ M_{W,\mathrm{pole}}^2 \le \frac{\Lambda^2}{e}.} \tag{28}$$

Proof. Let

$$F_x(y) = \frac{9}{2}g(y) + \frac{3}{2}h_x(y).$$

Equation (22) implies $h'_x(y) > 0$ for $y \le 0$, and also $g'(y) > 0$, since $g = h_0$. Hence F_x is strictly increasing on that half-axis. We have

$$F_x(0) = \frac{3}{2}A(x) > 0.$$

As $y \to -\infty$, the prefactors in (15) become negative and both integrals grow exponentially, so $F_x(y) \to -\infty$. This proves existence, uniqueness, simplicity, and the residue sign.

For $y \le 0$, equation (22) gives $g'(y) \ge 2/3$, and therefore

$$F'_x(y) \ge 3.$$

The mean-value theorem yields

$$z(x) \le \frac{1}{2}A(x).$$

The Taylor denominator gives the same bound for $M^2_{W,\mathrm{Taylor}}/\Lambda^2$. Finally,

$$A(x) = 2\int_0^1 xu\, e^{-xu}\, du \le \frac{2}{e},$$

proving (28).

As $x \to 0$, the bound implies $z(x) \to 0$, and $F'_x(y) \to 4$ near the root. As $x \to \infty$, $z(x) \to 0$, $h'_x(y) \to 0$ near the root, and $F'_x(y) \to 3$. Together with $F_x(0) \sim 3x/2$ and $F_x(0) \sim 3/x$, respectively, these prove (27). □

The pole mass governs the linearized dispersion relation of this analytically continued quadratic action. The Taylor mass governs its two-derivative truncation; they need not coincide near the cutoff. We assert no closed-form inversion of (26), no classification of complex poles, and no Lorentzian unitarity theorem.

The pole proof also extends to positive measures supported in a compact subset of $(0, \infty)$, using

$$F_{f,x}(y) = \int \frac{1}{s}\left[\frac{9}{2}g(sy) + \frac{3}{2}h_{sx}(sy)\right] d\nu(s).$$

It gives the analogues of (24) for the pole mass. For the larger class (4), only the Euclidean and Taylor conclusions have been established here; its moment assumptions alone do not justify that continuation.

6. Proposition (unbroken coefficients, neutral vectors, and weak angles).

Retain the heat-cutoff Standard Model family of Proposition 4. Normalize color generators by $\mathrm{tr}_{\mathbf{3}}\, t_a t_b = \frac{1}{2}\delta_{ab}$, and use $Q = T_3 + Y$. Define the full-trace coefficients by

$$S^{(2)}_{\mathrm{unbroken}} = \int \left(\kappa_{\mathrm{em}} F^2_{\mathrm{em}} + \kappa_s \sum_a G^2_a\right).$$

Then

$$\boxed{\kappa_{\mathrm{em}} = \frac{1}{24\pi^2}\frac{16}{3}(5+r), \qquad \kappa_s = \frac{1}{24\pi^2}2(5+r), \qquad \frac{\kappa_{\mathrm{em}}}{\kappa_s} = \frac{8}{3}.} \tag{29}$$

Both coefficients approach 5/6 of their all-massless values.

Let K denote the particle-trace neutral q^2 coefficient in coordinates (A_3, A_Y). Then

$$\boxed{\begin{aligned} K_{33} &= \frac{21}{4} + \left(\frac{3}{4} - \frac{3x}{8}\right) r, \\ K_{YY} &= \frac{103}{12} + \left(\frac{17}{12} - \frac{3x}{8}\right) r, \\ K_{3Y} &= -\frac{1}{4} + \left(\frac{1}{4} + \frac{3x}{8}\right) r. \end{aligned}} \tag{30}$$

The matrix K, the charged coefficient K_W, and the unbroken coefficients are positive for every $x \geq 0$.

Define the neutral-current angle from the current of the massive direction after diagonalizing the two-derivative neutral kinetic form:

$$J_Z \;\propto\; T_3 - \sin^2\theta_{\mathrm{NC}}\, Q.$$

Then

$$\boxed{\cos^2\theta_{\mathrm{NC}} = \frac{K_{YY} + K_{3Y}}{K_{33} + 2K_{3Y} + K_{YY}} = \frac{5}{8}.} \tag{31}$$

A different continuation, namely

$$\cos^2\theta_{\mathrm{match}} := \frac{K_{YY}}{K_W + K_{YY}},$$

satisfies

$$\boxed{\cos^2\theta_{\mathrm{match}}(0) = \frac{5}{8}, \qquad \cos^2\theta_{\mathrm{match}}(\infty) = \frac{103}{157}.} \tag{32}$$

Thus these definitions cease to describe one common electroweak angle.

Proof. Photon and color generators commute with V_0. Their coefficients therefore follow from (3) with $d_{ij} = 0$ on each invariant mass block. The charge traces give (29).

A Dirac species of mass m, with neutral charges q_L^a, q_R^a, contributes to K

$$e^{-x}\left[q_L^a q_L^b + q_R^a q_R^b - \frac{x}{2}(q_L^a - q_R^a)(q_L^b - q_R^b)\right]. \tag{33}$$

Applying (33) to the three top colors, and retaining all other states, proves (30).

Write

$$v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad B_\infty = \begin{pmatrix} 21/4 & -1/4 \\ -1/4 & 103/12 \end{pmatrix}, \quad U = \begin{pmatrix} 3/4 & 1/4 \\ 1/4 & 17/12 \end{pmatrix}.$$

Then

$$K = B_\infty + rU - \frac{3xr}{8} vv^T.$$

We have $B_\infty \geq 5I$, $U \geq 0$, and $xr \leq 1/e$. Hence

$$K \geq \left(5 - \frac{3}{4e}\right) I > 0.$$

Positivity of K_W follows from (17).

The neutral mass direction is v, while the photon direction is $(1, 1)$. The kinetic-orthogonal massive field is proportional to $K^{-1}v$, giving the first expression in (31). Directly,

$$K_{33} + K_{3Y} = 5 + r, \qquad K_{YY} + K_{3Y} = \frac{5}{3}(5 + r),$$

which proves (31). Taking limits proves (32). □

The neutral mass also tends to zero. More precisely,

$$\boxed{\frac{M_{Z,\mathrm{Taylor}}^2}{\Lambda^2} = \frac{9}{4} xr\, v^T K^{-1} v \sim \frac{2}{3} xe^{-x} \quad (x \to \infty).} \tag{34}$$

For $x > 0$ there is likewise one positive massive neutral pole, besides the photon pole. To prove this, put $B = B_\infty + rU$. The exact neutral kernel, apart from the common factor $\Lambda^2/(4\pi^2)$, is

$$B\, g(y) + \frac{3}{2} xr\, H(y) vv^T. \tag{35}$$

After congruence by $B^{-1/2}$, its massive eigenvalue is

$$g(y) + \beta H(y), \qquad \beta = \frac{3}{2} xr\, v^T B^{-1} v > 0.$$

Its timelike root obeys

$$z - 2 + \frac{2}{H(-z)} = \beta. \tag{36}$$

The left side increases from zero to infinity, because

$$\frac{d}{dz}\left(z - 2 + \frac{2}{H(-z)}\right) = 1 - \frac{2\,\mathbb{E}_z[b]}{H(-z)} \geq \frac{1}{2},$$

where $\mathbb{E}_z$ uses density proportional to $e^{zb(u)}$. The pole is therefore unique and has positive residue. Since $v^T B_\infty^{-1} v = 8/27$,

$$\frac{M_{Z,\mathrm{pole}}^2}{\Lambda^2} \sim \frac{2}{3} x e^{-x}.$$

In particular, M_W^2/M_Z^2 does not continue to equal $5/8$.

For every f in (4), equations (29)–(33) extend by replacing 1 by $\ell = f(0)$, r by $f(x)$, and xr by $-xf'(x)$. Positivity follows by averaging the positive heat-cutoff matrices. The ratio $8/3$, the neutral-current value $5/8$, and the matching limit $103/157$ persist. At the endpoint both neutral masses vanish, so the distinguished massive neutral direction—and hence its angle—requires continuation from finite x.

7. Counterexample (positive cutoff does not ensure positive broken kinetic coefficients).

Let $H_F = \mathbb{C}^2$,

$$V_0 = m\sigma_3, \qquad T = \sigma_1, \qquad x = m^2/\Lambda^2,$$

with the same spacetime and trace conventions. For the heat cutoff,

$$C_T(q^2) = \frac{\mathrm{Tr}(T^2)\Lambda^2}{8\pi^2} e^{-x} \left[(2 + 4x + y)H(y) - 2\right]. \tag{37}$$

Its q^2 Taylor coefficient is proportional to

$$e^{-x}(1 - x).$$

It vanishes at $x = 1$ and becomes negative for $x > 1$. Thus the two-derivative truncation has a wrong-sign vector kinetic term in this example.

Nevertheless, the full quadratic form has a unique positive real timelike pole with positive residue, satisfying

$$\boxed{4x < \frac{M_{\mathrm{pole}}^2}{\Lambda^2} < 4x + 2.} \tag{38}$$

Consequently $M_{\mathrm{pole}} \sim 2m$ diverges at fixed cutoff.

Proof. The generator connects the eigenvalues $m, -m$, so $d_{ij}^2 = 4m^2$ while both squared eigenvalues equal m^2. Substitution in (1) proves (37). Its Taylor ratio is

$$M_{\mathrm{Taylor}}^2 = \frac{6m^2}{1 - x},$$

where this is a positive mass only for $x < 1$.

The pole equation is (36) with $\beta = 4x$. The monotonicity proof there gives uniqueness and the residue sign. Evaluating (37) at $y = -4x$ gives $2H(-4x) - 2 > 0$, whereas at $y = -4x - 2$ it gives -2; this proves (38).

For general f satisfying (4), the Taylor kinetic coefficient in this example is proportional to

$$f(x) + xf'(x) = \frac{d}{dx}[xf(x)]. \tag{39}$$

It must be negative somewhere: $xf(x) > 0$ for $x > 0$, but $xf(x) \to 0$ at infinity under (4). A nonnegative derivative everywhere would contradict these facts. □

This counterexample concerns arbitrary finite data; the top-only Standard Model family is outside it. It also distinguishes a wrong-sign **Taylor truncation** from a negative-residue pole of the **full** form factor. We have not proved absence of complex-pole or other nonlocal problems in either model.

8. Counterexample (heavy fermions do not dynamically select the centralizer).

Let the data be those of Proposition 4, and let

$$\mathfrak{g}_{V_0} = \{X \in \mathfrak{g}_{\mathrm{SM}} : [X, V_0] = 0\}.$$

For nonzero top mass this is the represented unbroken gauge algebra $\mathfrak{su}(3) \oplus \mathfrak{u}(1)_{\mathrm{em}}$. The limit $x \to \infty$ at fixed Λ does **not** produce a theory containing only these vector directions by quadratic mass decoupling.

Proof. For fixed y, equation (21) implies

$$h_x(y) \longrightarrow 0.$$

Consequently

$$\boxed{C_W(q^2) \longrightarrow \frac{\Lambda^2}{4\pi^2}\frac{9}{2} g(q^2/\Lambda^2).} \tag{40}$$

For positive y,

$$g(y) = 4y \int_0^1 u(1-u) e^{-yu(1-u)}\, du > 0.$$

Thus a charged generator outside $\mathfrak{g}_{V_0}$ retains a finite nonzero quadratic kinetic form and a nonvanishing propagator at every fixed positive momentum. Its mass approaches zero. Equation (35) similarly leaves the positive matrix $B_\infty g(y)$, including the additional neutral direction. These facts contradict decoupling of the broken vectors through diverging masses or vanishing propagators. □

A narrower decoupling statement is true. If we **restrict the connection from the outset** to the centralizer, the finite mass blocks are invariant and the heat trace factorizes. On the heavy block its contribution carries e^{-sm^2/Λ^2}, which tends to zero. The restricted action therefore converges to the light-block spectral action. This proves decoupling of a finite fermion block under that restriction; it does not prove dynamical elimination of the other gauge fields.

The amendment to Lemma 03 Part C is therefore:

- Its transverse kinetic weights, including $w_L(x, 0) = w(x)$, and its two weak-angle formulas are correct under their stated conventions.
- They must be accompanied by the mass coefficient (2), including the quadratic-vertex subtraction.
- A centralizer-only endpoint does not follow from those weights. In the present exact quadratic family, the broken-vector masses vanish while their remaining kinetic coefficients are positive.

9. No-go theorem (finite radial vacuum for the positive Laplace class).

Let f satisfy (4), and suppose the allowed scalar configurations include

$$V = cV_*, \qquad c > 0, \qquad V_* \neq 0,$$

with $A = 0$. Let the action be the literal spectral action, without an added scalar potential or a constraint fixing c. Then no finite $c > 0$ is stationary along this radial direction.

Proof. On $\mathbb{R}^4$, the constant-background action density is

$$\mathcal{U}(c) = \frac{\Lambda^4}{4\pi^2} \int s^{-2} \operatorname{Tr}_{H_F} e^{-sc^2 V_*^2/\Lambda^2}\, d\nu(s).$$

Differentiating gives

$$\boxed{\mathcal{U}'(c) = -\frac{c\Lambda^2}{2\pi^2} \int s^{-1} \operatorname{Tr}_{H_F} \left(V_*^2 e^{-sc^2 V_*^2/\Lambda^2}\right) d\nu(s) < 0.} \tag{41}$$

The inequality is strict because $V_* \neq 0$ and ν is nonzero and positive. The same argument applies on a finite flat torus before performing its positive momentum sum. □

Thus the masses above are conditional quadratic-background quantities. In the top-only family, the ordinary Higgs radial variation supplies the direction used in this theorem. A finite symmetry-breaking minimum of the polynomial asymptotic scalar potential cannot automatically be regarded as a minimum of this full positive-Laplace spectral action.

10. Corollary (comparison of heavy limits).

Let the background family and mass definitions be those above.

At fixed Λ, with $\mu \to \infty$:

quantity, heat cutoff	behavior
m_t	$\mu\Lambda \to \infty$
$M_{W,\mathrm{Taylor}},\ M_{W,\mathrm{pole}}$	$\sim \Lambda/\mu \to 0$
$M_{Z,\mathrm{Taylor}},\ M_{Z,\mathrm{pole}}$	$\sim \sqrt{2/3}\,\Lambda\mu e^{-\mu^2/2} \to 0$
m_t/M_W	$\sim \mu^2 \to \infty$

(42)

At fixed $0 < \mu < \infty$, with $\Lambda \to \infty$, all four vector masses in (42) instead grow linearly with Λ, as does m_t. A controlled asymptotic family can also satisfy

$$\Lambda \to \infty, \qquad \mu \to 0, \qquad \mu\Lambda \to \infty;$$

then $M_W, m_t \to \infty$ while

$$m_t/M_W \to \sqrt{8/3}.$$

Proof. The first assertions follow from (19), (27), and (34)–(36). At fixed positive μ, their dimensionless mass functions are fixed and positive, so multiplication by Λ proves the joint-limit assertions. The final assertion follows from the small-μ limits. □

Accordingly, the obstruction has three distinct scopes. The CCM identity cannot be extrapolated to $\mu \to \infty$ at fixed cutoff because it belongs to the asymptotic truncation. The exact top-only quadratic action obstructs a fixed-cutoff heavy-W limit for the heat cutoff under both mass definitions, and for class (4) under the Taylor definition. A family of actual nonzero stationary vacua of the unmodified positive-Laplace action encounters the separate obstruction (41). Counterexample 7 excludes extending the first of these exact mass conclusions to arbitrary finite representations.

11. Proposition (the radial derivative for general f; referee). On flat $\mathbb{R}^4$, for a sufficiently integrable real cutoff f, with λ_j^2 the eigenvalues of V_*^2,

$$\mathcal{U}'(c) = -\frac{c\Lambda^2}{2\pi^2} \sum_j \lambda_j^2 \int_{c^2\lambda_j^2/\Lambda^2}^{\infty} f(z)\,dz. \tag{43}$$

For nonnegative f, a finite $c > 0$ is stationary exactly when every tail associated with a nonzero λ_j vanishes. So if f has nonzero support arbitrarily far out, there is no finite stationary point on this ray; if f has compact support ending at R, there is a stationary plateau once every nonzero $c^2\lambda_j^2/\Lambda^2 \geq R$. Sign-changing cutoffs can have cancellations. A nonconstant cutoff flat near zero lies outside (4), since a nonzero positive measure gives $f'(0) < 0$; (43) covers the nonnegative ones. A radial path with an additional fixed noncommuting mass matrix need not have the form cV_*. □

12. Proposition (the top-only Taylor mass for general f; referee). For nonnegative smooth compactly supported f with $f(0) > 0$, the formulas of Proposition 4 hold with

$$A_f(x) = \frac{2}{x}\int_0^x z f(z)\,dz, \qquad w_f(x) = \frac{3}{x^3}\int_0^x z^2 f(z)\,dz,$$

by approximating f with its derivatives by real combinations of e^{-nz}. Hence

$$\frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} \sim \frac{\int_0^\infty z f(z)\,dz}{f(0)\,x}.$$

This includes cutoffs flat near zero. For a nonanalytic cutoff there need be no analytic continuation defining the pole. □

13. Proposition (several heavy doublets; referee). For the heat cutoff, let P be the total particle-trace weight of complete massless doublets, and let heavy doublet i, of weight $d_i > 0$, have one Dirac member of mass m_i and one massless member. Then

$$C_W(y) = \frac{\Lambda^2}{4\pi^2}\Big(Pg(y) + \sum_i d_i h_{x_i}(y)\Big), \qquad \frac{M^2_{W,\mathrm{Taylor}}}{\Lambda^2} = \frac{3}{2}\,\frac{\sum_i d_i A(x_i)}{P + \sum_i d_i w(x_i)}.$$

For finitely many doublets, if $P > 0$ and all $x_i \to \infty$, the Taylor mass tends to zero. If $P = 0$ and all $x_i = x$, then $M^2_{W,\mathrm{Taylor}} \sim m^4/(2\Lambda^2)$, the form factor has a unique positive pole with $M^2_{W,\mathrm{pole}} \sim m^2/2$, and the whole unnormalized kernel tends to zero at every fixed Euclidean momentum. The divergent Taylor and pole scales differ, and neither alone establishes a healthy limiting gauge theory. □

14. Proposition (the fully massive family; referee; the complete proof by Astra 6 follows the sketch). Give both members of every Standard Model doublet the same Dirac mass m, with no Majorana masses, and let z_W, z_Z be the charged and massive neutral poles in units of Λ^2. The charged form factor is proportional to $e^{-x}[g(y) + 2xH(y)]$, and

$$2x < z_W < 2x + 2, \qquad \tfrac{16}{5}x < z_Z < \tfrac{16}{5}x + 2.$$

So both pole masses diverge with m at fixed cutoff, and

$$\frac{M^2_{W,\mathrm{pole}}}{M^2_{Z,\mathrm{pole}}} \longrightarrow \frac{5}{8} \qquad (x \to \infty),$$

as also for $x \to 0$, where $z_W \sim 3x$ and $z_Z \sim \frac{24}{5}x$. In between the ratio differs, reaching about 0.66 near $x = 3$ (Python, solving the pole equations). The kernel tends to zero at every fixed Euclidean momentum, and the charged Taylor kinetic coefficient is proportional to $e^{-x}(1 - x/2)$, negative for $x > 2$.

Proof. The species with equal mass have eigenvalues $\pm m$. The charged generator connects them with $d^2 = 0$ and $d^2 = 4m^2$ in equal weights, which gives the charged form. For the neutral generators the diagonal charge has $d = 0$ and the off-diagonal charge $(q_L - q_R)/2 = T_3 v/2$ has $d^2 = 4m^2$; on the particle trace the kernel is $e^{-x}[\mathrm{diag}(6, 10)g + 12xHvv^T]$, so $\beta = 12x\,v^T\,\mathrm{diag}(6, 10)^{-1}v = \frac{16}{5}x$, and for the charged direction $\beta = 2x$. Equation (36) gives the bounds, since $H(-z) > 1$. □

Complete proof (Astra 6, 3 November 2026; written here from its validation report, item 11, so that the lemma is self-contained). Take the heat cutoff, three generations with right-handed neutrinos, no Majorana masses, and one common positive Dirac mass m for every species. Let $x = m^2/\Lambda^2$, $y = q^2/\Lambda^2$, $H(y) = \int_0^1 e^{-yu(1-u)}\,du$ and $g(y) = (2+y)H(y) - 2$. Every mass eigenvalue is $+m$ or $-m$, so the exact quadratic kernel of Theorem 1 for a generator entry is proportional to $e^{-x}[g(y) + ((m_i - m_j)^2/\Lambda^2)H(y)]|T_{ij}|^2$. A charged generator connects equal and opposite mass signs with equal weights, the average of $(m_i - m_j)^2$ is $2m^2$, and the charged kernel is proportional to $e^{-x}[g(y) + 2xH(y)]$. In the neutral particle trace, with $v = (1, -1)^T$, the charge and mass-commutator traces give $Q = \mathrm{diag}(6, 10)$ and $C_{\mathrm{neutral}}(y) \propto e^{-x}[Qg(y) + 12xH(y)vv^T]$, whose determinant factors as $\det Q\, g(y)[g(y) + 12x\,v^T Q^{-1} v\,H(y)]$; since $v^T Q^{-1} v = 1/6 + 1/10 = 4/15$, the massive neutral factor has coefficient $16x/5$. Charge conjugation multiplies the common normalization and moves no root. Both massive pole equations therefore have the form $(2 + \beta - z)H(-z) = 2$ with $\beta_W = 2x$ and $\beta_Z = 16x/5$. Let $h(z) = H(-z)$ and $F(z) = z - 2 + 2/h(z)$. For $z \geq 0$, $h(z) \geq 1$ and $0 < h'(z) = \int_0^1 u(1-u)e^{zu(1-u)}\,du \leq h(z)/4$, so $F'(z) = 1 - 2h'(z)/h(z)^2 \geq 1 - 1/(2h(z)) \geq 1/2$; with $F(0) = 0$ and $F(z) \to \infty$, each $\beta > 0$ has a unique positive solution, at which $h(z) > 1$, so $z = \beta + 2 - 2/h(z)$ gives $\beta < z < \beta + 2$: the displayed bounds. For

small z, $h(z) = 1+z/6+O(z^2)$ and $F(z) = \frac{2}{3}z+O(z^2)$, so $z_W \sim 3x$, $z_Z \sim 24x/5$ and the ratio tends to 5/8; for $x \to \infty$ the bounds give $z_W = 2x+O(1)$, $z_Z = 16x/5+O(1)$, again 5/8. At fixed Euclidean momentum every kernel entry is e^{-x} times a function growing at most linearly in x, hence tends to zero. Finally $H'(0) = -1/6$ and $g'(0) = 2/3$, so the charged Taylor kinetic coefficient is proportional to $e^{-x}(\frac{2}{3} - \frac{x}{3}) = \frac{2}{3}e^{-x}(1 - x/2)$, negative for $x > 2$. A quadrature and root calculation (Python 3, SciPy, not kept) gave $z_W = 7.4996$, $z_Z = 11.3746$, ratio 0.6593 at $x = 3$. □

This is a counterexample to any general no-go for diverging pole locations at fixed cutoff. It has no light fermions, and the physical quality of that limit is open.

15. Remark (the light space does not carry unbroken $SU(2) \times U(1)$; referee). In the top-only limit the broken masses vanish, but this does not restore the electroweak group on the whole light space. The light space retains b_L after removing t_L. If P_{light} projects off the top mass block, then on b_L we have $P_{\text{light}}T_1P_{\text{light}} = P_{\text{light}}T_2P_{\text{light}} = 0$ and $P_{\text{light}}T_3P_{\text{light}} = -\frac{1}{2}$, which do not obey the $SU(2)$ relations. Correspondingly $K_W = 9/2$, $K_{33} = 21/4$, $K_{3Y} = -1/4$ in the limit, while a linearly realized unbroken action would have equal weak coefficients and no mixing. The remaining complete massless doublets carry the ordinary representation, and their contribution is gauge invariant. If both members of every removed doublet are heavy, the retained light space can be invariant under the full group; a full limiting theory still needs control of the remaining fields, normalizations, interactions and vacuum. "The heavy fermion leaves the loop" is shorthand for the suppression in this classical trace.

The light-space statement should be read together with the separate role of the top (lemma 21): among single charged species only an up-type quark can be removed while keeping cos^2 theta_NC = 5/8 and the ratio 8/3, and removing it leaves the five other quarks and the leptons, on which SU(3) x U(1)_em acts with b_L paired with b_R.

16. Remark (the CCM relation as the small-mass form of a weighted sum rule; Claude). For the heat cutoff and doublets with one massive member, Proposition 13 reads

$$8\kappa M^2_{W,\text{Taylor}} = \sum_f N_f m_f^2\, \rho(m_f^2/\Lambda^2), \qquad \rho(x) = \frac{A(x)}{x} = \frac{2[1-(1+x)e^{-x}]}{x^2},$$

with $N_f = 2d_f$ and $\kappa = (P+\sum_i d_i w(x_i))/6$. As $x \to 0$, $\rho, \kappa \to 1$, which is the CCM relation (14). The relation is therefore the leading term of an expansion, and its range is $m \ll \Lambda$. For the top alone, $\kappa = (3+w)/4$; m/M_W departs from $\sqrt{8/3}$ by 5% at $m \approx 0.45\Lambda$, and M_W peaks at about 0.52Λ at $m \approx 1.42\Lambda$ (Python). The maximum is unique, at $x = 2.024$ with $M_W/\Lambda = 0.5242$ (referee, by a sign argument and Python). A massive diagonal contribution to the trace is exponentially suppressed above the cutoff, while the mixed loop with a massless partner has the algebraic $\rho \sim 2/x^2$ at large x; a vacuum with no other mass leaves the W mass with no source. Whether the classical action is meaningful for $m > \Lambda$ is a separate question, on which these formulas do not decide. In the top-only heat-cutoff family, the ratio differs from $\sqrt{8/3}$ for every $x > 0$, since $3 + w - 4\rho$ vanishes at zero and is increasing, so the limit with M_W and m_t diverging together and $m_t/M_W \to \sqrt{8/3}$ needs $\Lambda \to \infty$ faster than m_t (Corollary 10). This necessity is for that family: with one complete massless lepton doublet and one member of each other doublet of common mass m, the weighted sum equals the unweighted CCM sum at some $x_* > 0$ (referee).

As algebraic checks, we used **Python 3 with SymPy** to verify the integration-by-parts identities, the displayed small-x Taylor series, and $v^T B_\infty^{-1} v = 8/27$. The arguments above provide the proofs; these computations introduce no fitted constants.

Consequence for T5b. At fixed cutoff, the specified top-only Standard Model spectral action does not realize an infinitely heavy W, Z, t family: the top grows while both vector masses tend to zero, for the heat cutoff and the stated quadratic mass definitions. A joint limit with $\Lambda \to \infty$ permits all three background mass scales to diverge, but neither centralizer-only decoupling nor a stationary Higgs vacuum follows from that scaling. These conclusions concern the classical bosonic action and contain no quantum corrections or claims about the physical Standard Model at those scales. Other finite data behave differently: with every doublet heavy and no light fermions, the pole masses diverge at fixed cutoff with M_W^2/M_Z^2 -> 5/8 (Proposition 14). With a massless complete doublet retained, the Taylor mass tends to zero (Proposition 13).

So in the families treated here, a heavy W and Z at fixed cutoff is found only without a massless complete doublet. Light nonzero masses are not treated: they leave a residual vector mass that depends on them.

Part 19

Inner fluctuations of the two-point data without order one

Step 1 of the next steps in the background note, for STRATEGY T0b. Proved by Claude Opus 5.5 without Astra or Fable; every identity checked with numpy on random data (section "Numerical check"). Refereed by Terra 6.1 (gpt-6.1-terra, effort max, 3 Nov 2026; report): Theorem 19.1 and the comparison with arXiv:1304.7583 are correct; Corollary 19.2 item 3 and the Wilson phase after the table were wrong as first written, and these and the amendments are incorporated below.

Outcome. The truncated two-point data satisfy order zero and fail order one, for each natural real structure. So the inner fluctuations of Chamseddine-Connes-van Suijlekom (arXiv:1304.7583), with their quadratic term, are the applicable ones. The result factorizes. With the inherited pointwise real structure, the fluctuated operator is m |Phi|^2 sigma_1: a real gauge-neutral radial field, with trivial gauge group. With the reflection real structure, which has the KO-6 signs, the link is m Phi^2: all of C is reached, and the gauge group U(1) acts with charge 2. Along a gauge orbit the quadratic term keeps the modulus of the link fixed. This parallels the fixed singular value of the covariant projection of lemma 14, which the fixed compression of lemma 13 lacks. So the first-order failure of Counterexample 08.4 points to the framework of arXiv:1304.7583 for these truncated data.

Setting

Let A = C + C act diagonally on H = C^2, with D = m sigma_1, m > 0, and gamma = sigma_3. For a real structure J = C K put x-hat = J x J^-1. We use two choices.

- **J_0 = K** (or sigma_3 K, the other sign of lemma 08), inherited from pointwise reality on the interval. Then x-hat = conj(x) on A, and J_0 commutes with gamma.
- **J_r = sigma_1 K**, inherited from the reflection real structure sigma_1 R_l K of Counterexample 08.4. Then diag(x_1, x_2)-hat = diag(conj(x_2), conj(x_1)), J_r^2 = 1, J_r D = D J_r and J_r gamma = -gamma J_r: the KO-6 signs.

Both satisfy order zero, since A is commutative and x-hat lies in A. Both fail order one: for e = diag(1, 0), [[D, e], e-hat] = m sigma_1 for J_0 and -m sigma_1 for J_r.

For a universal one-form A = sum_j a_j delta(b_j), the fluctuated operator of arXiv:1304.7583 is

```
D' = D + A_(1) + A~_(1) + A_(2),
A_(1) = sum a_j [D, b_j],   A~_(1) = sum a_j-hat [D, b_j-hat] = eps J A_(1) J^-1,   A_(2) = sum
↪  a_j-hat [A_(1), b_j-hat],
```

with J D J^-1 = eps D. Gauge transformations are U = u J u J^-1 for unitary u in A, and U D(A) U* = D(gamma_u(A)), where gamma_u(A) = u delta(u*) + u A u*.

Put

```
phi_1 = sum a_{j,1} (b_{j,2} - b_{j,1}),     phi_2 = sum a_{j,2} (b_{j,1} - b_{j,2}),
```

so that A_(1) = m [[0, phi_1], [phi_2, 0]], which is self-adjoint exactly when phi_2 = conj(phi_1). The pair (phi_1, phi_2) takes every value in C^2, as a = diag(x, y) and b = diag(0, 1) show.

The fluctuated operator

Theorem 19.1. For every universal one-form,

```
J_0:   D' = m [[0, |1 + phi_1|^2], [|1 + phi_2|^2, 0]],
J_r:   D' = m [[0, (1 + phi_1)(1 + conj(phi_2))], [(1 + phi_2)(1 + conj(phi_1)), 0]].
```

For a self-adjoint form, with Phi = 1 + phi_1,

```
J_0:   D' = m |Phi|^2 sigma_1,          J_r:   D' = m [[0, Phi^2], [conj(Phi)^2, 0]].
```

Proof. For diagonal b, [X, b] for off-diagonal X = [[0, x], [y, 0]] is [[0, x (b_2 - b_1)], [y (b_1 - b_2), 0]].

For J_0: a-hat and b-hat are the conjugates. Then A~*(1) = m [[0, conj(phi_1)], [conj(phi_2), 0]], and the first row of* $A(2)$ is m phi_1 sum conj(a_{j,1}) (conj(b_{j,2}) - conj(b_{j,1})) = m phi_1 conj(phi_1); the second row is m phi_2 conj(phi_2). So the (1,2) entry of D' is m (1 + phi_1 + conj(phi_1) + |phi_1|^2) = m |1 + phi_1|^2, and likewise (2,1).

For J_r: a-hat = diag(conj(a_2), conj(a_1)) and b-hat = diag(conj(b_2), conj(b_1)). Then A~*(1) = sigma_1 conj(*$A(1)$) sigma_1 = m [[0, conj(phi_2)], [conj(phi_1), 0]], and the first row of A_(2) is m phi_1 sum conj(a_{j,2}) (conj(b_{j,1}) - conj(b_{j,2})) = m phi_1 conj(phi_2), and the second is m phi_2 conj(phi_1). This gives the stated factorization. ■

Corollary 19.2 (gauge action and image). Let u = diag(e^{i alpha}, e^{i beta}) and psi = alpha - beta. Under gamma_u, Phi -> e^{i psi} Phi.

1. For J_0: U = 1. The image of the self-adjoint forms is {t sigma_1 : t >= 0}, so the link is real, and t is a gauge-neutral field. The value t = 0 is reached at Phi = 0.
2. For J_r: U = diag(e^{i psi}, e^{-i psi}), and the link m Phi^2 goes to e^{2 i psi} m Phi^2. So U(1) acts with charge 2 in the variable of the link, and the image is all of C, with Phi -> Phi^2 two-to-one away from 0. The image statement is pointwise: a globally prescribed smooth link need not have a global smooth square root.
3. The gauge orbit of the zero form has Phi = e^{i psi}. It gives D' = D for J_0, whose operator orbit is the single point D, and link m e^{2 i psi} for J_r. Without A_(2), the link is m (2 cos psi - 1) for J_0, of modulus m (1 - psi^2 + O(psi^4)), although U = 1; and m (2 e^{i psi} - 1) for J_r, of modulus m (5 - 4 cos psi)^{1/2} = m (1 + psi^2 + O(psi^4)), off the orbit of D.

Proof. gamma_u(0) has phi_1 = u_1 conj(u_2) - 1 = e^{i psi} - 1. The rest is direct. ■

Item 3 shows, in the smallest example, why the quadratic term is needed: it restores gauge covariance.

Comparison with the parent

Proposition 19.3. Consider the chiral interval at M = 0, with m = pi/(2l), and the links of lemmas 13 and 14.

1. *With no real structure*, the finite fluctuation is m (1 + phi_1), all of C, while the fixed compression of a gauge field gives only the line m + i B (Proposition 13.2).
2. *With pointwise reality*, the parent has no scalar fluctuation at all: the algebra C(I) is commutative with J f J^-1 = conj(f), it satisfies order one, and A + J A J^-1 = 0 for a scalar one-form. The truncated data do have a fluctuation, the neutral radial field t. So the finite fluctuation has a gauge-neutral radial parameter, while the specified pointwise-real parent has no surviving self-adjoint scalar inner fluctuation. This does not establish a dynamical scalar field or its action, and it says nothing about parent fields in other models. Compression is not multiplicative, so there is no canonical identification of finite potentials with parent potentials.
3. *With the reflection real structure*, the parent satisfies order one, since J_r f J_r^-1 = conj(f) composed with the reflection commutes with [D, g]. One finds J_r D_{A_5} J_r^-1 = D_{A_5 o R}, with R the reflection, so that the symmetrized field A_5 + A_5 o R survives, and the fixed compression gives m + i B^s with B^s = 2 B, because the weight sin(pi x/l) is even about l/2. The modulus along a pure-gauge field behaves as follows.

object	modulus of the link along a gauge orbit
covariant projection (lemma 14, Theorem 3)	constant: lambda W_l^* has fixed singular value
finite data with A_(2) (Corollary 19.2)	constant: m

object	modulus of the link along a gauge orbit
fixed compression (Proposition 13.2), with B the integral of the symmetrized coefficient for J_r	(m^2 + B^2)^{1/2}, growing at second order along paths where B has a linear term
finite data with J_r, linear terms only	m (5 - 4 cos psi)^{1/2}, growing at second order

Proof. Item 1 is lemma 13. For item 2, a represented self-adjoint parent one-form is E h(x) with h = i a and a real, and K (i E a) K^-1 = -i E a, so the real fluctuation cancels. An individual commutator need not change sign, since K [D, x] K^-1 = [D, x] = E: self-adjointness is the step that matters. For item 3, K changes the sign of i A_5, the reflection changes the sign of the derivative and composes A_5 with R, and sigma_1 E sigma_1 = -E for E = [[0, 1], [-1, 0]]. The table collects Theorem 3 of lemma 14, Corollary 19.2 and Proposition 13.2. ■

The table states a parallel and no identification. The variables differ: for constant unsymmetrized A_5 = a, the transported link is m e^{i a l} = m + i (pi a/2) + O(a^2), with Wilson phase a l, against m + i (2 a/pi) in the compression. For the reflection-real fluctuation from the same a, these become m e^{2 i a l} and m + i (4 a/pi). The phase of a link depends on the frame, while its modulus does not. In the parent, the covariance defect is repaired by moving the projection, and in the finite data by the quadratic term.

Reading

Spectral compression can break order one, and it can also break order zero: on the circle with J = K, compression to the zero mode preserves both, while compression to the modes -1, 0, 1 sends e^{ix} to a truncated shift S, with [S, S*] != 0, so that order zero fails. Where a represented algebra, a preserved real structure and order zero hold, as here, the framework of arXiv:1304.7583 applies to the truncated data as they stand. So the first-order failure of Counterexample 08.4 has an escape, and this calculation is a first use of it. Two points were open. Whether the parallel in the table extends to an exact statement for a fluctuated parent and its truncation: this holds on gauge orbits, by Corollary 22.6 of lemma 22, and remains open for the nonunitary part. And whether the neutral radial parameter t is related to the singlet of the grand-symmetry model (Devastato-Lizzi-Martinetti, where sigma comes from a Majorana-operator fluctuation in a larger algebra). This is an analogy only: in arXiv:1304.7583 the quadratic term does not generate new scalar fields, and here too it depends on the same two parameters. An identification would need a map of algebras, representations, fields and gauge actions.

Numerical check

Python (numpy), run on 3 Nov 2026 in a temporary directory, not kept. On random complex a_j, b_j, with m = 1.3: the three real structures have the stated signs; the formulas of Theorem 19.1 hold to about 10^-15, without assuming self-adjointness; D(gamma_u(0)) = U D U* holds to 10^-16 for both real structures, and fails by 4.5 for the sample when A_(2) is dropped; U D(A) U* = D(gamma_u(A)) holds for a general form; the values of [[D, e], e-hat] are as stated.

Part 20

Sheet-connecting operators and boundary gluing

By Astra 6 (gpt-6-astra, effort xhigh), 3 Nov 2026; prompt in prompts/20_connecting_operator.md. Reproduced as delivered, with links made relative and one link to a temporary copy of a source written as plain text. The sections “Numerical check” and “Readings” at the end are by Claude Opus 5.5. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): Results 1 to 4 and 6 to 11 correct, with one amendment to Result 7 given in the note after its proof; the readings are corrected below.

A rank-one operator can connect the two finite summands with the required real signs and grading, and the Chamseddine–Connes maximal algebra follows from its intertwining algebra. A boundary coupling supplies such an operator in an explicit interval model. However, agreement of scalar functions at a wall generally

does not survive compression as the finite order-one restriction: an exact one-flavour counterexample exhibits this failure. The interval construction realizes the classified rank-one block when its matrix algebra, grading and coupled channels are supplied as inputs; it does not derive those inputs from Example 16.9. The three requested obstruction checks hold in the scopes specified below.

1. Theorem — Finite connecting blocks and their admissible algebras

Hypotheses and conventions. Let

$$E = \mathrm{Hom}_{\mathbb{C}}(V, W), \qquad E^* = \mathrm{Hom}_{\mathbb{C}}(W, V),$$

with Hilbert–Schmidt inner products, conjugate-linear in the first argument. Here E^* denotes the second Hom space; operator adjoints also carry a superscript $*$.

Let $c : E \to E^*$ be the antiunitary occurring in the real structure:

$$J(x, y) = (c^{-1}y, cx), \qquad \Gamma = \begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix}, \qquad h = -cgc^{-1}.$$

In Example 16.9, after choosing the grading of W,

$$cx = x^*, \qquad gx = \gamma_W x, \qquad hy = -y\gamma_W.$$

For the more general graded flip of Corollary 16.8, c includes its stated grading sign.

Let $\mathcal{A}^{\mathrm{ev}}$ act diagonally through π_1, π_2. Write $\rho_i(b)$ for its opposite action on the two summands. Assume that the sheet-diagonal D_F already satisfies self-adjointness, oddness, $JD_F = D_F J$, and order one on $\mathcal{A}^{\mathrm{ev}}$, as in Corollary 16.8.

For a complex-linear map $T : E \to E^*$, put

$$S_T = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}, \qquad D = D_F + S_T, \qquad \tau(T) = cT^*c.$$

Then D has the required signs and satisfies order one on a unital involutive subalgebra $B \subseteq \mathcal{A}^{\mathrm{ev}}$ precisely when

$$\boxed{T = \tau(T), \qquad hT + Tg = 0,} \tag{1}$$

and

$$\boxed{(T\pi_1(a) - \pi_2(a)T)\rho_1(b) = \rho_2(b)(T\pi_1(a) - \pi_2(a)T) \quad (a, b \in B).} \tag{2}$$

Self-adjointness is automatic from the definition of S_T. In bases related by c, the first equation in (1) says that the matrix of T is complex symmetric.

In the ordinary adjoint-flip convention, for $a = (a_W, a_V)$,

$$\pi_1(a)x = a_W x, \quad \pi_2(a)y = a_V y, \quad \rho_1(a)x = xa_V, \quad \rho_2(a)y = ya_W.$$

Thus (2) is an entirely explicit equation on Hom spaces.

For an operator $R : E \to E^*$, define exactly the commutation algebra used in section 4 of Chamseddine–Connes:

$$\mathcal{A}(R) = \left\{ a \in \mathcal{A}^{\mathrm{ev}} : \begin{array}{l} \pi_2(a)R = R\pi_1(a), \\ \pi_2(a^*)R = R\pi_1(a^*) \end{array} \right\}. \tag{3}$$

For fixed T satisfying (1), its maximal admissible algebras are the inclusion-maximal members of

$$\boxed{\{\mathcal{A}(R) : \ hR + Rg = 0, \quad T = R + \tau(R)\}\,.} \tag{4}$$

Consequently, maximal dimension for fixed T is obtained by maximizing the dimension of the algebras in (4). The distinction between T and R is necessary.

Proof. Block multiplication gives (1). The lower off-diagonal block of the order-one equation is exactly (2); adjoints give the other block.

To prove (4), suppose (2) holds. Average over the compact unitary group of the finite-dimensional real C^*-algebra B:

$$R_0 = \int_{U(B)} \pi_2(u) T \pi_1(u)^* \, du.$$

Then R_0 intertwines the left actions. Equation (2) implies that

$$T - \pi_2(u) T \pi_1(u)^*$$

intertwines the opposite actions. Therefore $Q_0 = T - R_0$ intertwines the opposite actions.

The involution τ exchanges these two intertwining conditions. Since $\tau(T) = T$,

$$R = \frac{1}{2}(R_0 + \tau(Q_0))$$

intertwines the left actions and satisfies $T = R + \tau(R)$. Averaging preserves oddness because B is even; τ also preserves oddness. Hence $B \subseteq \mathcal{A}(R)$ for an R occurring in (4).

Conversely, on $\mathcal{A}(R)$, the term R intertwines the left actions and $\tau(R)$ intertwines the opposite actions. Their sum satisfies (2). This proves the characterization and the maximality statement.

For an explicit reason why $\mathcal{A}(T)$ alone is insufficient, take ungraded $W = V = \mathbb{C}^2$, $g = 1$, $h = -1$, and

$$B = \{(\mathrm{diag}(\alpha, \beta), \mathrm{diag}(\alpha, \beta))\}.$$

Let $e_{ij} \in E$, $f_{ij} = ce_{ij}$, and define

$$Te_{11} = f_{12}, \qquad Te_{12} = f_{11},$$

with all other images zero. This T is symmetric and odd. The first arrow preserves the right B-character; the second preserves the left character. Each therefore satisfies order one. But the first arrow fails to intertwine the left actions when $\alpha \neq \beta$. ■

2. Theorem — The classified algebra is realized by a rank-one block

Hypotheses. Let

$$W = W_+ \oplus W_-, \qquad \dim_{\mathbb{H}} W_+ = \dim_{\mathbb{H}} W_- = 1, \qquad V = \mathbb{C}^4,$$

and use the adjoint-flip convention of Result 1. Thus

$$\mathcal{A}^{\mathrm{ev}} = \mathbb{H}_+ \oplus \mathbb{H}_- \oplus M_4(\mathbb{C}).$$

Choose unit vectors $w \in W_+$, $v \in V$, and $\kappa \neq 0$. Put

$$e = wv^* \in E, \qquad f = vw^* = ce,$$

and

$$\boxed{T_\kappa x = \kappa \, (w^* x v) \, v w^* = \kappa \, |f\rangle\langle e| x.} \tag{5}$$

Then T_κ has complex rank one, satisfies (1), and has intertwining algebra

$$\boxed{\mathcal{A}(T_\kappa) = \{(\lambda \oplus q, \lambda \oplus m) : \lambda \in \mathbb{C},\ q \in \mathbb{H},\ m \in M_3(\mathbb{C})\}.} \tag{6}$$

Here the first λ acts on the quaternionic line W_+ as

$$\lambda \longmapsto \begin{pmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{pmatrix}$$

in a basis starting with w, and the second acts on $\mathbb{C}v \subset V$.

The algebra (6) has real dimension 24. Up to an automorphism of $\mathcal{A}^{\mathrm{ev}}$, it is the unique maximal-dimensional involutive subalgebra admitting a nonzero sheet-connecting Dirac block with order one. The off-diagonal self-adjoint operator S_{T_κ} has complex rank two.

Proof. Since $ge = e$ and $hf = -f$, T_κ is odd. Its matrix has the single symmetric entry κ in c-paired bases, so $\tau(T_\kappa) = T_\kappa$.

The two intertwining equations in (3) require that $\mathbb{C}w$ and $\mathbb{C}v$ be reducing subspaces for the respective actions, with the same scalar action on these lines. A quaternion preserving $\mathbb{C}w$ belongs to the displayed copy of $\mathbb{C}$. On V, the orthogonal complement $v^\perp$ carries an arbitrary $M_3(\mathbb{C})$-action. The other quaternionic summand is unrestricted. This proves (6), and $2 + 4 + 18 = 24$.

We reproduce the dimension argument in Chamseddine–Connes, section 4, whose local source defines (3) at line 875. A nonzero connecting block implies, by Result 1, a nonzero intertwiner R for B. Compress R on either side by projections in the commutants of the full even algebra. Some nonzero compression remains, and its support and range can be placed in irreducible representation spaces

$$K : \mathbb{C}^2 \longrightarrow \mathbb{C}^4.$$

These compressions enlarge the intertwining algebra.

If K has rank two, its image is a reducing two-plane in $\mathbb{C}^4$. The matrix action on that plane is determined by the quaternion action. The remaining two-plane contributes at most $M_2(\mathbb{C})$. Including the unused quaternionic summand gives

$$\dim_{\mathbb{R}} B \leq 4 + 4 + 8 = 16.$$

If K has rank one, its support and range are complex lines. The intertwining algebra is (6), of dimension 24. Unitary changes of these lines, and exchange of the quaternionic summands, give the stated uniqueness.

Finally, T_κ and T_κ^* have orthogonal source and target summands, giving $\operatorname{rank} S_{T_\kappa} = 2$. ■

The connecting condition here is relative to the sheet projections of the original prealgebra: $[S_{T_\kappa}, e_1] \neq 0$. On the selected algebra (6), this particular Majorana block commutes with the algebra. Additional sheet-diagonal Dirac terms are needed for the usual nontrivial finite one-forms and Yukawa data.

3. Lemma — Boundary preservation is an intertwining condition

Hypotheses. Use the domain of Lemma 12:

$$Q_R(0) = 0, \quad U_L(0) = 0, \quad Q_R(l) = XU_R(l), \quad U_L(l) = -X^* Q_L(l). \tag{7}$$

Let multiplication act through smooth matrix functions F_Q, F_U, with the same matrix on both chiral components of each field.

Multiplication preserves this domain precisely when

$$\boxed{F_Q(l)X = XF_U(l), \qquad F_U(l)X^* = X^* F_Q(l).} \tag{8}$$

These equations define a unital involutive algebra. For scalar functions and $X \neq 0$, they reduce to

$$f_Q(l) = f_U(l). \tag{9}$$

For $X = 0$, no equality between the boundary values is required.

Proof. At $x = l$, the traces $Q_L(l)$ and $U_R(l)$ can be chosen independently. Applying multiplication to the first coupled condition in (7) gives

$$F_Q(l)XU_R(l) = XF_U(l)U_R(l)$$

for every $U_R(l)$. This is the first equation in (8). The other condition gives the second equation. Conversely, these equations preserve both boundary conditions; the conditions at zero are automatically preserved.

Taking the adjoint of the second equation gives

$$F_Q(l)^* X = X F_U(l)^*.$$

Thus (8) has precisely the two-equation form (3). Closure under products and adjoints follows directly. ■

For matrix-valued functions, "agreement at the wall" therefore means agreement through X on the coupled boundary channels. It need not mean equality of the entire boundary matrices.

4. Counterexample — Scalar gluing does not imply finite order one

Hypotheses. Let $N_Q = N_U = 1$, both bulk masses vanish, and $X = t > 0$. Set

$$\theta = \arctan t \in (0, \pi/2), \qquad m = \theta/l.$$

Take the full domain-preserving scalar algebra

$$\mathcal{A}_{\text{glue}} = \{(f_Q, f_U) \in C^\infty([0,l])^2 : f_Q(l) = f_U(l)\}. \tag{10}$$

The bulk operator is

$$D(Q_L, Q_R; U_L, U_R) = (Q_R', -Q_L'; U_R', -U_L')$$

on (7). Use physical chirality

$$\Gamma(Q_L, Q_R; U_L, U_R) = (Q_L, -Q_R; U_L, -U_R)$$

and the antiunitary

$$\mathcal{J}(Q_L, Q_R; U_L, U_R) = (\overline{U_R}, -\overline{U_L}; -\overline{Q_R}, \overline{Q_L}). \tag{11}$$

This parent satisfies the signs $(+,+,-)$, order zero and order one. Its lowest symmetric spectral window gives

$$A_{\text{fin}} = \mathbb{C} \oplus \mathbb{C}, \qquad D_{\text{fin}} = m\sigma_1, \qquad J_{\text{fin}} = \sigma_1 K, \qquad \Gamma_{\text{fin}} = \sigma_3, \tag{12}$$

and these finite data fail order one.

Proof. The boundary form proves self-adjointness, as in Lemma 12.1. Equation (11) preserves (7), squares to one, commutes with D, and reverses Γ. The opposite algebra acts by the two scalar functions exchanged. Hence it commutes both with the multiplication algebra and with its differential commutators.

The lowest window is spanned by the orthonormal chiral vectors

$$\begin{aligned} u_+(x) &= l^{-1/2}(\cos mx, 0; -\sin mx, 0), \\ u_-(x) &= l^{-1/2}(0, \sin mx; 0, \cos mx). \end{aligned} \tag{13}$$

Direct differentiation gives $Du_+ = mu_-$, $Du_- = mu_+$. The boundary conditions hold because $\tan(ml) = t$. Solving the same elementary ODE gives the next absolute eigenvalue $(\pi - \theta)/l$, so any symmetric cutoff between m and this value selects exactly (13).

Compression of (10) is

$$Q(f_Q, f_U) = \begin{pmatrix} \dfrac{1}{l}\displaystyle\int_0^l (f_Q \cos^2 mx + f_U \sin^2 mx)\, dx & 0 \\ 0 & \dfrac{1}{l}\displaystyle\int_0^l (f_Q \sin^2 mx + f_U \cos^2 mx)\, dx \end{pmatrix}. \tag{14}$$

Let f_Q be a nonnegative bump supported sufficiently near zero, and let $f_U = 0$. The wall equality holds, while the difference of the diagonal entries is

$$\frac{1}{l}\int_0^l f_Q(x)\cos(2mx)\, dx > 0.$$

Together with the identity, this gives the full diagonal algebra $\mathbb{C}^2$.

On (13), (11) gives $J_{\text{fin}} = \sigma_1 K$. For $p = \text{diag}(1, 0)$,

$$p^\circ = 1 - p, \qquad [[D_{\text{fin}}, p], p^\circ] = -D_{\text{fin}} \neq 0.$$

Thus finite order one fails.

There is also a precise distinction between the original field sheets and the two finite chiral summands. The projection onto the Q-field compresses to

$$\begin{pmatrix} \frac{1}{2} + \frac{\sin 2\theta}{4\theta} & 0 \\ 0 & \frac{1}{2} - \frac{\sin 2\theta}{4\theta} \end{pmatrix}, \tag{15}$$

which is not a projection. The coupled spectral window does not preserve the original field decomposition. ■

This proves failure of statement 2(c) for the full scalar gluing algebra, even when the parent has all the required real signs and order conditions. It also specifies the qualification needed in 2(b): the finite summands require an identification, such as continuation from the uncoupled zero modes.

5. Theorem — An exact interval parent supplies the rank-one connector

Hypotheses. Use the Hom representation, c, g, h of Result 1. Set

$$k = cgc^{-1}, \qquad h = -k.$$

Choose a unit vector $e \in E$ with $ge = e$, let $f = ce$, and put

$$S = |f\rangle\langle e|, \qquad X = tS^* : E^* \to E, \qquad t > 0. \tag{16}$$

Take two massless interval fields, Q valued in E and U valued in E^*, with boundary conditions (7).

As candidate multipliers, take **constant** elements of $\mathcal{A}^{\text{ev}}$, acting by π_1 on Q and π_2 on U. Their domain-preserving algebra is

$$B = \mathcal{A}(S). \tag{17}$$

Define

$$\hat{\Gamma}(Q_L, Q_R; U_L, U_R) = (gQ_L, -gQ_R; kU_L, -kU_R), \tag{18}$$

$$\widehat{J}(Q_L, Q_R; U_L, U_R) = (c^{-1}U_R, -c^{-1}U_L; -cQ_R, cQ_L). \tag{19}$$

Let $n = \dim_{\mathbb{C}} E$, $\theta = \arctan t$, $m = \theta/l$, and choose

$$m < \Lambda_0 < (\pi - \theta)/l.$$

Then the spectral projection

$$P = 1_{[-\Lambda_0, \Lambda_0]}(D_X)$$

has rank $2n$. Under an explicit identification $P\mathcal{H} \simeq E \oplus E^*$, its finite data are

$$\boxed{\left(B,\ E \oplus E^*,\ \begin{pmatrix} 0 & mS^* \\ mS & 0 \end{pmatrix},\ J,\ \begin{pmatrix} g & 0 \\ 0 & -k \end{pmatrix}\right).} \tag{20}$$

They satisfy self-adjointness, oddness, order zero, order one, and the signs $(+, +, -)$. The connecting block is $T = mS$, of complex rank one.

Proof. Result 3, with $X = tS^*$, gives (17). The rank-one condition makes $\mathbb{C}e$ and $\mathbb{C}f$ reducing subspaces for B, with the same action on the two lines.

The coupled channels have exactly the eigenvectors (13), with 1 replaced by e in the Q-components and by f in the U-components. Every orthogonal Q-channel has boundary type (a), and every orthogonal U-channel

has type (b). They contribute $2n-2$ constant chiral zero modes; their next absolute eigenvalue is π/l. The coupled channels contribute the two eigenvalues $\pm m$. This proves the window and dimension assertions.

The identification in (20) sends e and f to these two coupled chiral vectors. It sends $e^{\perp} \subset E$ to constant Q_L-profiles and $f^{\perp} \subset E^*$ to constant U_R-profiles. Since B acts identically on the coupled lines, this identification intertwines its representations exactly. Direct application of (18) and (19) gives the displayed finite grading and real structure.

On the parent, (19) squares to one and commutes with the real differential expression. The relations $kf = f$ and $ge = e$ ensure grading preservation of the boundary conditions. The identity

$$cX = X^* c^{-1}$$

ensures their preservation by $\widehat{J}$. These establish the real signs on the operator domain.

The opposite representation is the original right Hom action, so order zero holds. Every admissible constant multiplier commutes with D_X, including preservation of its domain. Order one therefore holds on the parent and on (20). ■

The wall coupling can also be converted explicitly into a bounded bulk term. On the coupled field channels, let

$$F = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad E_{\text{spin}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad U(x) = \exp(mxF).$$

Apply the same field rotation to both spin components. It maps the uncoupled boundary domain to the coupled domain and gives

$$U^* D_X U = D_0 + mF \otimes E_{\text{spin}}. \tag{21}$$

Compression of the added bounded term to the uncoupled zero modes is exactly $m(S+S^*)$. Thus the parent supplies the connector through an explicit change of domain and an explicit unitary transport.

Two specializations specify the extent of this construction.

- **A commutative example.** Let $E = E^* = \mathbb{C}$, $g = 1$, $c = K$, and start with constant pairs $\mathbb{C}^2$. Boundary preservation selects their common scalar:

  $$B = \{(\lambda, \lambda)\} \simeq \mathbb{C}.$$

  The finite triple is

  $$(\mathbb{C}, \mathbb{C}^2, m\sigma_1, \sigma_1 K, \sigma_3).$$

  This is the simplest successful instance of 2(a)–(c), with the candidate observable algebra restricted to constant functions on each sheet.

- **The classified block.** Take $E = \mathrm{Hom}(\mathbb{C}^4, \mathbb{H}^2)$, the chosen quaternionic grading, and $e = wv^*$ from Result 2. Equations (17) and (20) give

  $$B = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}), \qquad \dim_{\mathbb{C}} H_{\text{fin}} = 32,$$

  and the rank-one connecting block (5) with $\kappa = m$. This parent contains the Hom fibres and constant matrix algebra as inputs. It is not a construction from scalar functions on the twistor parent.

An exact tower-decoupling family is obtained by fixing $m > 0$ and setting

$$t(l) = \tan(ml), \qquad 0 < ml < \pi/2.$$

The finite mass remains exactly m, while the next absolute eigenvalue is $\pi/l - m \to \infty$.

The usual zero-dimensional orientability axiom fails in these examples: the linked vectors e, f have identical left and opposite characters, whereas their grading eigenvalues are opposite. Every represented zero-cycle acts identically on these two lines. The claims above concern real even spectral triples with the stated order conditions and signs.

6. Proposition — Point evaluation bounds rank by the boundary-channel dimension

Hypotheses. Let K_i be finite-dimensional windows of smooth sections, let

$$\mathrm{ev}_i : K_i \to F_i$$

be evaluation at the proposed wall, and suppose a finite coupling is defined by

$$T = \mathrm{ev}_2^* \; X \; \mathrm{ev}_1 \,. \tag{22}$$

Then

$$\operatorname{rank} T \le \min\{\operatorname{rank} X, \dim F_1, \dim F_2\}. \tag{23}$$

A single scalar evaluation channel gives rank at most one. A point with several fibre components can give larger rank.

Proof. Equation (23) follows by factoring (22) through the range of X.

For a counterexample to an unrestricted rank-one claim, take Lemma 12 with

$$N_Q = N_U = N, \qquad M_Q = M_U = 0, \qquad X = tI_N.$$

It is the orthogonal sum of N copies of Result 4. Its lowest chiral block is

$$T = \frac{\arctan t}{l} I_N,$$

of rank N, although the coupling occurs at the single point $x = l$. ■

For an exact boundary-domain perturbation, formula (22) is an additional hypothesis. The operators D_X and D_0 have different domains, so their difference is not automatically a point-evaluation perturbation. The exact interval mass $\arctan(t)/l$ also distinguishes the finite operator from its small-t approximation t/l. An extended wall carries a space of boundary sections and supplies no rank-one conclusion by itself.

7. Proposition — Point gluing does not select the algebra in the unchanged twistor window

Hypotheses. Let $Q_i(f) = P_i M_f P_i$, where P_i project onto finite-dimensional spaces of smooth sections on compact manifolds. Let Z_i be closed subsets with empty interior. Impose any scalar gluing condition which allows independent functions vanishing near Z_1 and Z_2, including equality at two marked points.

Then the image of this glued algebra under the **unchanged separate compressions** $Q_1 \oplus Q_2$ equals the image of the unrestricted pair algebra.

Proof. We first show

$$Q_i(C_c^\infty(M_i \setminus Z_i)) = Q_i(C^\infty(M_i)). \tag{24}$$

If the left image were a proper subspace of the right image, finite-dimensional linear algebra would give a linear functional separating them. Applied to $Q_i(f)$, that functional has the form

$$f \longmapsto \int_{M_i} f(x)\, r(x)\, dx$$

for a smooth density r, because matrix entries of $Q_i(f)$ are integrals against products of smooth sections.

Vanishing for every test function supported outside Z_i forces $r = 0$ there. Since that complement is dense and r is continuous, $r = 0$ everywhere, a contradiction. The argument also applies over the real numbers.

By (24), arbitrary compressed values on either sheet can be obtained using functions vanishing near its wall. Such pairs satisfy the gluing condition independently, proving the assertion. ■

In Example 16.9, both analytic section spaces come from $\mathbb{CP}^3$; one observable algebra is restricted to pullbacks from S^4. The corresponding matrix-entry densities on S^4 are smooth after integration along the twistor fibres, so the same proof applies.

Neither closed space comes with the boundary required by Lemma 12. One may choose a base point $p \in S^4$, a point $q \in \mathbb{CP}^3$, or a twistor fibre, and impose scalar agreement there. In the original level-one windows, the generated real algebra remains

$$M_2(\mathbb{H}) \oplus M_4(\mathbb{C}).$$

Thus this operation derives no further algebra reduction.

A marked base point does provide a quaternionic line through

$$\Pi_p = \frac{1}{2}(1 + p_a \Gamma_a).$$

Choosing a complex line inside that quaternionic line, a complex line in the other four-space, and the rank-one map between the corresponding Hom vectors gives the data of Result 2. The grading $2\Pi_p - 1$ and the rule selecting this connector are then additional finite data. The stabilizer calculation derives $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ from those choices.

To obtain an analytic parent for Example 16.9, further work must specify an interface or interval collar, its boundary trace spaces, self-adjoint coupling conditions, a compatible real structure and grading, and the spectral window of the coupled operator. Result 7 concerns the unchanged windows. A new coupled window, boundary degrees of freedom, or a different observable representation lies outside its hypotheses.

Referee amendment. For a general subset upstairs in the twistor space, add the hypothesis that its image in S^4 has empty interior, so that independent base test functions exist.

8. Proposition — Scope of the Hořava–Witten comparison

Hypotheses. Compare chiralities using one common ten-dimensional chirality convention, and retain the unbroken boundary gauge group $E_8 \times E_8$.

The same-chirality interval analogue is type **(a) at both ends**, or type **(b) at both ends** after reversing the chirality convention. To model two separately localized multiplets, take two fields of that same type, with opposite signs of their localization masses. Lemma 12 at $X = 0$ contains one type-(a) and one type-(b) field, so its displayed pair has opposite chiralities.

Proof and scope. Hořava–Witten impose $\Gamma_{11}\chi = \chi$ on the boundary gaugino; the boundary vector multiplets are intrinsic boundary fields. Their chirality projector is stated in section 2, immediately before the boundary Yang–Mills Lagrangian. Labels based on outward normals change sign between the endpoints; this does not change the comparison in a fixed ten-dimensional convention.

Type (a) in Lemma 09 retains the L-zero mode at both ends, while type (b) retains R. Different mass signs localize two such fields near opposite ends. Type (c) instead produces opposite-chirality wall profiles.

On a space containing only the two same-chirality modes, an operator odd for that chirality grading has zero matrix elements between them. Moreover,

$$\mathrm{Hom}_{E_8 \times E_8}((248, 1), (1, 248)) = 0,$$

because the first factor acts trivially on the target and has no invariant vector in its adjoint representation. Thus these boundary multiplets do not supply a gauge-invariant bilinear connector of the interval mass-block type. ■

Bulk propagation can define effective boundary-to-boundary interactions. Identifying one with the finite T requires a specified field, representation and grading, potentially after compactification or symmetry breaking. The Hořava–Witten comparison alone supplies no such identification.

9. No-go theorem — Corollary 11.2 under its matching-scale hypothesis

Verdict and hypotheses. The corrected Corollary 11.2 is correct. Assume a fixed finite set of Dirac mass parameters, N_g generations, and the matching-scale relation

$$8M_W^2 = \sum_{g=1}^{N_g}(m_{\nu,g}^2 + m_{e,g}^2 + 3m_{u,g}^2 + 3m_{d,g}^2). \tag{25}$$

For $N_g \neq 3$, the coefficient 8 is an explicitly assumed normalization; it does not follow automatically by replicating the three-generation gauge trace.

Then $M_W \to \infty$ is incompatible with keeping every Dirac mass bounded. More precisely,

$$\frac{m_{\max}}{2\sqrt{2}} \leq M_W \leq \sqrt{N_g}\, m_{\max}. \tag{26}$$

Proof. The weighted sum in (25) lies between $m_{\max}^2$ and $8N_g m_{\max}^2$, proving (26). Thus a fixed nonzero gauge coupling and $v \to \infty$, with all $y_f v$ fixed, contradict (25).

The conclusion concerns $m_{\max}$. One fixed species need only be unbounded along a subsequence. For example, with $M_W = t$, put

$$\alpha(t) = \frac{\pi}{4}(1 + \sin t), \quad m_e = 2\sqrt{2}\, t \cos\alpha(t), \quad m_\nu = 2\sqrt{2}\, t \sin\alpha(t).$$

Their squared sum is $8t^2$, while each mass vanishes on an unbounded sequence. ■

Scope and escapes. Equation (25) belongs to the stated normalization of the truncated asymptotic spectral action at matching. It concerns Dirac Yukawa masses, including the neutrino Dirac parameters, rather than physical seesaw eigenmasses. Changing the action or normalization, controlling the full finite-cutoff action, varying the particle content, or allowing Dirac masses to grow changes the problem. Extrapolation to a heavy-mass limit at fixed cutoff requires a separate calculation; the theorem directs attention to that calculation.

10. Theorem — Proposition 16.1 holds for inherited pointwise reality

Verdict and hypotheses. The corrected Proposition 16.1 is correct. Let

$$J_0 a J_0^{-1} = a^*, \qquad [P, J_0] = 0, \qquad Q(a) = PaP|_{P\mathcal{H}},$$

and let $J_F = J_0|_{P\mathcal{H}}$. Let A_F be the involutive algebra generated by the compressed functions. Then order zero holds for this inherited structure exactly when A_F is commutative.

Proof.

$$J_F Q(b)^* J_F^{-1} = P J_0 b^* J_0^{-1} P = Q(b).$$

Order zero therefore requires

$$[Q(a), Q(b)] = 0$$

for every pair of generators. Their generated algebra is then commutative.

Conversely, the opposite operation reverses products and fixes each compressed generator. On a commutative generated algebra its image lies in the same algebra, proving order zero. ■

For a product with a fixed finite factor, the unit of that factor gives the same necessity. Sufficiency also requires that the finite factor satisfy order zero and that the product real structure be compatible. The commutativity conclusion concerns the compressed function algebra; the separate finite factor may be non-commutative.

Scope and escapes. The theorem concerns compression followed by algebra generation with this inherited pointwise reality and a J_0-invariant window. The coarse correspondence supplies a separate opposite action and escapes the hypothesis, as demonstrated positively by Theorem 16.2. Other representations or real structures require their own order-zero calculation.

11. Proposition — The stated inherited grading cannot equal Γ_5

Verdict and hypotheses. The grading obstruction in Example 16.9 is correct under the following simultaneous requirements:

1. Every parent observable under consideration is even: $[\gamma, a] = 0$.
2. The finite grading is inherited as $g = P\gamma P|_{P\mathcal{H}}$.
3. This compression is an actual grading: $g^2 = 1$.
4. The retained observables include $Q(x_a) = \Gamma_a/5$.

Under these hypotheses, $g = \Gamma_5$ is impossible.

Proof. Even without initially assuming $[P, \gamma] = 0$, the grading identity implies

$$0 = P - (P\gamma P)^2 = P\gamma(1-P)\gamma P = \big((1-P)\gamma P\big)^*\big((1-P)\gamma P\big).$$

Hence $(1-P)\gamma P = 0$, and self-adjointness gives $[P, \gamma] = 0$. Therefore

$$[g, Q(a)] = P[\gamma, a]P = 0$$

for every retained even observable.

But

$$[\Gamma_5, Q(x_1)] = \frac{1}{5}[\Gamma_5, \Gamma_1] = \frac{2}{5}\Gamma_5\Gamma_1 \neq 0.$$

Thus the proposed grading and all the stated coordinate compressions cannot simultaneously be inherited in this manner. ■

Scope and escapes. The conclusion does not exclude Γ_5 as additional finite data or as a grading of a prealgebra. Possible changes include compressing a graded prealgebra with odd observables, restricting to the algebra commuting with the proposed grading, or constructing an additional grading. Changing the window alone does not remove the contradiction while retaining all four hypotheses. In the specified Dolbeault kernel, the inherited grading is already scalar.

The three checks above agree with the corrected repository statements. The recorded author is Claude Opus 5.5, and the existing independent Terra checks appear in the Lemma 11 report, the Proposition 16.1 report, and the Example 16.9 report. The present response adds a sequential Codex check; no subagents or additional model consultations were used.

Consequence for T0b and T11

For T0b, an exact interval parent supplies a rank-one connector with KO-6 signs, and a parent with chosen Hom fibres and constant matrix multipliers realizes the classified connecting block and its algebra. Deriving that algebra, grading and connector from the commutative twistor parent remains open: scalar wall agreement in the original window does not perform the selection. For T11, same-chirality boundary multiplets require the type-(a) or type-(b) comparison, and the unbroken Hořava–Witten boundary representations supply no identified finite mass connector.

Numerical check

By Claude Opus 5.5, in Python (numpy), 3 Nov 2026, in a temporary directory, not kept. On W = V = C^4 with random elements:

- T_kappa x = kappa K x K with K = v w*, the rank-one map (5): J D = D J, J Gamma = -Gamma J, D Gamma = -Gamma D; complex rank 1; real dimension of A(T_kappa) inside the 40-dimensional H + H + M_4(C) equal to 24, by linear algebra.
- K of rank two on W_+: dimension 16, as in the proof of Result 2, and x -> K x K has rank 4.
- On C + H + M_3(C): order zero, and order one with defect about 10^-15, and [D, a] = 0 with defect about 10^-15. For a random element off this algebra the order-one defect is about 40.

- The example at the end of Result 1: order-one defect 0, and a nonzero defect of left intertwining.
- Result 4: the values in (15), with a quadrature. By hand: the eigenvectors (13), their boundary conditions, the action of (11), and the value of the commutator.
- Equation (21): by hand, including the compression to m (S + S*).
- Pi_p is a projection of rank 2 and 2 Pi_p - 1 = p_a Gamma_a; the bound (26) on a random sample.

Against the source: (3) is equation (comrel) of arXiv:0706.3688, read in the LaTeX source. The proof there also takes an intertwiner distinct from the Dirac block, in line with the distinction between T and R in Result 1. The other statements are checked by the referee.

Readings

By Claude Opus 5.5, as proposals, corrected after the referee report.

1. The connector is the Majorana block. In the Standard Model dictionary (Chamseddine-Connes-Marcolli 2007, section 2, quoted from memory), W_+ holds (nu_R, e_R) and V = C^4 holds the lepton line and the three colours, so that with w along nu_R and v along the lepton line, the vector e = w v* is nu_R, and the off-diagonal block of D_F acts on nu_R alone. After this identification, and an overall grading convention if needed, T = m S in Result 5 has the form of the one-generation Majorana block (equation (2.34) there, as the referee checked), with m = arctan(t)/l. In this window m < pi/(2l), so the first level of the tower bounds it, and for t of order one m is of the order of that level. As a pattern, this agrees with the placement of the Majorana mass near the cutoff in that model. Three identical copies give three equal masses, and a general three-generation Majorana matrix needs additional flavour data.

2. The connector as a marked pair of points and a coupling. Since T_kappa x = kappa K x K with K = v w*, the data of the connector are a complex line w in W = C^4, a complex line v in V = C^4, that is, a point of CP^3 on each sheet, and the coupling kappa. In Example 16.9, the point w lies over a point p of S^4 in the twistor fibration, and Pi_p = (1 + p_a Gamma_a)/2 projects onto its quaternionic line. Since Q(x_a) = Gamma_a/5, the grading is

```
2 Pi_p - 1 = p_a Gamma_a = 5 Q(p . x),
```

the compression of the height function at p. So one marked point fixes the quaternionic line and the complex line w, and a second one fixes v. Sp(2) acts transitively on the unit vectors of H^2, and U(4) on those of C^4, so at fixed nonzero |kappa| all line choices are conjugate, in agreement with the uniqueness in Result 2. The modulus |kappa| remains, rephasing w and v changes K x K by a phase, and other finite data can introduce relative moduli. The inputs are then: mark a point on each sheet, with a coupling and a prescription for evaluation and normalization. This has the form (22) with one channel per sheet.

The height function 5 Q(p . x) fails to commute with the other coordinate compressions, so it can serve as additional finite grading data only after restricting to its even algebra. This evades the inherited-grading hypothesis of Result 11, and it does not derive the grading from the even parent. Result 7 shows that scalar agreement at the points does not do this in the unchanged windows, so a coupled window with the data listed after Result 7 is needed. Whether a parent with an interface at those points yields this connector is open.

3. The wall coupling as a Wilson line. Equation (21) says that the wall coupling X = tan(m l) is unitarily equivalent to the uncoupled boundary conditions with a constant matrix connection m F, of relative endpoint rotation angle theta = m l, and the mass is that angle over the length, k = theta/l. On an interval, open-path transport depends on endpoint frames, and the physical quantity is the transport together with the boundary conditions. The comparison with gauge-Higgs unification (Y. Hosotani, Phys. Lett. B 126 (1983) 309; Hosotani, Noda and Takenaga, arXiv:hep-ph/0403106) is an analogy at this stage: the calculation supplies a fixed matrix connection, with no Higgs fluctuation or effective potential. It bears on the question of mass as internal momentum in the steer for the next paper, and compares with the link of lemma 19 and the covariant projection of lemma 14.

4. Next calculation. The connector commutes with B, so it gives no finite one-forms and no Higgs field. The next test is one interval parent carrying both the Dirac Yukawa block of lemma 12, odd for g and

diagonal in the sheets, and this connector, for instance at opposite walls: whether order one survives on B, and what the joint form of (21) is.

Part 21

The separate role of the top: removal identities and counting

Prompted by the observation that the top has a separate role, which matters more than the loss of an electroweak partner for b_L. Proposition 21.1 and the counts below are exact calculations by Claude Opus 5.5, in Python with exact fractions. The readings in the last section are proposals, and nothing there is derived. Refereed by Terra 6.1 (effort max, 3 Nov 2026; report): one error, in Proposition 21.2, and several amendments, incorporated below. Proposition 21.4 and Corollary 21.5 (added 4 Nov 2026) were refereed by Claude Opus 5.5 (effort max, 5 Nov 2026; report): the counts correct, two amendments (the group SO(6(n_u + n_d)); the SU(3)^3 trace) and one error in the window clause, which held in reading (a) only; all applied.

Outcome. The spectral action detects the top: in the exact quadratic action of lemma 18, among single charged species only an up-type quark can be removed while keeping both the neutral-current value cos^2 theta_NC = 5/8 and the ratio 8/3 of the photon and gluon coefficients. Removing it leaves 84 of the 96 states of the finite Hilbert space. Those 84 are the seed of an SO(30), inside SO(32), whose adjoint contains the charged scalar counts of three generations, and 84 is also the number of on-shell components of the three-form of eleven-dimensional supergravity.

Removal identities

For a Dirac species with charges q_L, q_R in the coordinates (T_3, Y), equation (33) of lemma 18 gives a contribution to the neutral kinetic matrix K with weight e^{-x}. The off-diagonal charge is (q_L - q_R)/2 = (T_3/2)(1, -1), which is orthogonal to the photon direction (1, 1). So the combinations K_33 + K_3Y and K_YY + K_3Y depend on x only through the factor e^{-x} of each massive species, and cos^2 theta_NC = (K_YY + K_3Y)/(K_33 + 2 K_3Y + K_YY) is determined by the sums of T_3 Q and Y Q.

Proposition 21.1. Per species, with colour multiplicity, on the particle trace for the first two columns and with charge conjugates and both chiralities for the last two:

species	sum T_3 Q	sum Y Q	Q^2 coefficient	colour coefficient
up-type quark	1	5/3	16/3	2
down-type quark	1/2	1/6	4/3	2
charged lepton	1/2	3/2	4	0
neutrino	0	0	0	0
three generations	6	10	32	12

Hence the ratios of the whole, 5/3 and 8/3, are those of an up-type quark alone. As one species is removed, that is, as its mass tends to infinity at fixed cutoff in the setting of lemma 18:

removed	cos^2 theta_NC	photon/gluon coefficient
up-type quark	5/8	8/3
down-type quark	59/92	46/15
charged lepton	17/28	7/3

So among single charged species only an up-type quark preserves both values. For an up-type quark they hold at every x, as in Proposition 6 of lemma 18. Within one generation, the down-type quark and charged lepton together contribute exactly what the up-type quark contributes: (1, 5/3; 16/3, 2).

Proof. Arithmetic with the Standard Model charges, Y = Q - T_3. The last statement is direct addition. ■

The scope is that of lemma 18: classical bosonic trace, with these definitions of the coefficients and of the angle, three generations, and the other species with the mass assignments of that calculation. Other definitions of the weak angle do not stay at 5/8 (Proposition 6 there). At the endpoint where the neutral mass vanishes, the massive neutral direction is no longer distinguished, so its angle is read as the limit from finite mass, as lemma 18 states.

The state count in the action

Proposition 21.2. With three generations and nu_R, dim H_F = 96 (Chamseddine-Connes-Marcolli). With only an up-type Dirac mass m and no Majorana mass, the constant-background density of lemma 18 has finite trace Tr exp(-V_0^2/Lambda^2) = 84 + 12 e^{-x} for the heat cutoff. So the coefficient of the vacuum density moves from 96 to 84 along the radial direction, with no finite nonzero stationary point on this free radial ray (No-go theorem 9 and Proposition 11 of lemma 18). This concerns the literal flat-space action with no added potential or constraint. Compact support can give a plateau; sign-changing cutoffs, additional terms, constraints and paths with an additional fixed noncommuting mass lie outside it. At m = 0 the derivative is zero.

Proof. The top block has 3 colours, 2 chiralities and charge conjugates, so 12 states, each with V_0^2 = m^2. ■

This also gives the top-only family a place in the radial problem, as the endpoint of a descending direction. A radial direction giving other species masses descends further, so this is the endpoint for that direction only.

SO(30)

Let the five quarks other than the top, in three colours, with their conjugates, form the vector 30 = (5, 3) + (5-bar, 3-bar) of SU(5) x SU(3) inside SO(30). The adjoint, of dimension 435, is

```
Lambda^2(30) = (24, 8) + (24, 1) + (1, 8) + (1, 1) + (10, 6) + (15, 3-bar) + conjugates of the last
↪  two.
```

With Q = 2/3 for u, c and Q = -1/3 for d, s, b, that is, Q = diag(3/5, 3/5, -2/5, -2/5, -2/5) + (1/15) I_5, which includes the U(1) commuting with SU(5), the charges are:

representation	charge: number
(15, 3-bar)	-2/3: 6; +1/3: 6; +4/3: 3
(10, 6)	-2/3: 3; +1/3: 6; +4/3: 1
(24, 1) + (1, 1)	+1: 6; -1: 6; 0: 13

Three generations of scalar partners in a supersymmetric Standard Model with nu_R need, for antisquarks in the 3-bar, 6 of charge -2/3 and 6 of charge +1/3, and for sleptons 6 of charge +1, 6 of charge -1 and 12 neutral real states. So (15, 3-bar) and (24, 1) + (1, 1) contain exactly the charged counts. What is left over in them is three triplets of charge +4/3 and one neutral state. The counting of 15 diquarks and 25 mesons, with the charge +4/3 states, and the remark that the top is unable to form them, go back to an early preprint (hep-ph/0512065), which finds neither SO(30) nor a role for SO(32). What SO(30) adds is one group whose adjoint contains both families. It also sits in SO(32), with 496 -> (435, 1) + (1, 1) + (30, 2) under SO(30) x SO(2), which hints at a role for SO(32); we have not explored this. These are representation and charge counts: identifying the entries with physical scalar partners needs a field construction, spin and chirality assignments, interactions and dynamics.

Proposition 21.4 (the counting forces the top out and leaves the charge 4/3 half-filled). Let n_u up-type and n_d down-type light quarks in three colours, with their conjugates, span the vector of SO(6(n_u + n_d)), whose 6(n_u + n_d) real dimensions are those of (n_u + n_d, 3) + conjugate. In the colour antitriplet of the adjoint, the symmetric diquarks S^2(n_u + n_d) (x) 3-bar have n_d(n_d + 1)/2 states of charge -2/3, n_u n_d of charge +1/3 and n_u(n_u + 1)/2 of charge +4/3. Three generations

of antisquarks need 6 and 6 in the first two charges, and the only solution in nonnegative integers is (n_u, n_d) = (2, 3). The charge +4/3 count is then 3. With the top included, (n_u, n_d) = (3, 3) gives (6, 9, 6), and the top would need 36 = 30 + 6 real dimensions rather than the 32 of SO(32), whose extra plane is colour-neutral.

Proof. n_d(n_d + 1)/2 = 6 gives n_d = 3, then n_u n_d = 6 gives n_u = 2, and 2 x 3/2 = 3. For n_u = n_d = 3 the three counts are 6, 9, 6. A coloured quark adds 6 real dimensions to the vector; 32 - 30 = 2. ■

Corollary 21.5 (the charge-4/3 sector). Under the bijection of Proposition 21.4 between the (6, 6) part and the antisquarks of three generations, the three antitriplets of charge +4/3 are half of what a chiral multiplet pairing would need: a Dirac quark of charge 4/3 in each generation needs two complex scalars, so three generations need 6. If the three scalars are read as superpartners, their three fermions are left-handed Weyl fields in (3-bar, 4/3), SU(2) singlets, and are anomalous on their own (with T(3) = 1/2 and A(3) = 1, the SU(3)^3, SU(3)^2 U(1), U(1)^3 and gravitational U(1) traces are -3, 2, 64/3 and 12. The meson sector, six complex scalars of charge +-1 and neutral ones, cancels none of them for any chirality of its fermions: they are colour singlets, their cubic charge sum is an integer, and their linear charge sum has absolute value at most 6. With any other chirality assignment of the three multiplets, the SU(3)^3 trace is odd, so the exotic fermions stay anomalous). The readings compatible with the counting are: (a) the three scalars have no fermion partner, as the Higgs doublet has none in the Standard Model, so supersymmetry is a property of the (6, 6) part alone; (b) the top is included, which completes the exotic to 6 and breaks the +1/3 count to 9. In the window of the interval parent (lemma 38, Proposition 11.4, the shadow B_ij -> (m_i - m_j) B_ij, for distinct masses), the same-flavour diquarks drop out and the antitriplet counts become (3, 6, 1) in reading (a). One state of charge +4/3 remains, the combination of u and c antisymmetric in flavour, with coefficient proportional to m_u - m_c, in the (Lambda^2 5, 3bar) of S^2 V. In reading (b), with six flavours, the counts become (3, 9, 3), and three states of charge +4/3 remain, uc, ut and ct. In both readings the window has 3 states of charge -2/3, so the match with the antisquarks in Proposition 21.4 holds in the parent and fails in the window. This records a conflict between the count of three generations and the anomaly of the exotic sector; the premise of the whole counting is that five light quarks generate every squark and slepton as a composite, scalars that the finite geometry of Connes et al. does not contain.

Provenance. Stated by the user and Claude Fable 5.1 in conversation on 4 November 2026; the arithmetic checked by hand. The remark that the top cannot form the diquarks is from hep-ph/0512065; the chirality reading of the charge-4/3 states as an additional quark per generation is in the abstract of the EPJC paper (Rivero 2024), where the half-multiplet problem is stated.

Direction (referee, 5 Nov 2026). The selection of (2, 3) belongs to the parent: with the flavour-antisymmetric counts n_d(n_d - 1)/2, n_u n_d, n_u(n_u - 1)/2, which are those of the window and of the sign s = -1 (USp(30)), the condition (6, 6) has no integer solution (n_d = 4 forces n_u = 3/2), so five light quarks with the top excluded are a property of Lambda^2 V with walls reducing to M_15(C) (lemma 38, Theorem 11.3), and the question is whether that wall condition can be derived. The anomaly is a parity: in reading (a) the three exotic antitriplets give an odd SU(3)^3 trace, which no partners for the rest of the SO(30) adjoint cancel (octets contribute 0, sextets multiples of 7); in reading (b) the six charge-4/3 states pair vectorlike but the three extra charge-1/3 antitriplets td, ts, tb leave an odd trace again. In both readings an odd number of antitriplets lies beyond the antisquarks, 3 = 15 - 12 and 9 = 21 - 12, so a supersymmetric reading of the diquark sector needs partnerless scalars or an even excess. With four generations the symmetric counting has no solution, since n_d(n_d + 1)/2 = 8 has no integer root.

The two families of 84

Proposition 21.3. Count four states per Dirac fermion and colour, and take Dirac neutrinos. Then

```
96 = 84 + 12:   five quarks and all leptons (60 + 12 + 12), and the top (12);
96 = 84 + 12:   all electrically charged fermions (72 + 12), and the neutrinos (12).
```

The two families of 84 overlap in 72 states, and each is the other with the top and the neutrinos exchanged. Without nu_R, the first family has 78 states and the second stays at 84. In eleven dimensions, the three-form C_3 has C(9, 3) = 84 on-shell components, and its dual C_6 has C(9, 6) = 84. The 84 in H_F is a

complex finite-Hilbert-space dimension, and the three-form count is a real bosonic polarization count, so an identification needs a map of representations and degrees of freedom.

Readings, as proposals

These are proposals to test, with no derivation behind them.

1. **Seiberg duality.** For SU(3) with five flavours, N_f = 5 lies in the conformal window 3 N_c/2 < N_f < 3 N_c, and the magnetic dual is SU(N_f - N_c) = SU(2) with five flavours and a 5 x 5 gauge-singlet meson chiral superfield coupled by W = M q q-tilde (N. Seiberg, Nucl. Phys. B 435 (1995) 129, arXiv:hep-th/9411149, checked by the referee). The removal of the top is what gives N_f = 5, and SU(2) is the magnetic group there. To become a statement this needs an N = 1 supersymmetric theory with the required massless matter and dynamics, for example an emergent one, with an operator and anomaly map. Five-flavour ordinary QCD alone does not supply the duality, and the magnetic SU(2) needs a separate identification with any electroweak group.
2. **Branes.** The first family of 84 could relate to C_3 or to an M2 or M5 brane, and the second family to the dual object. The exchange of the top and the neutrinos would then be the content of the duality on the Standard Model side. The counts match, and equality of the number is insufficient: a representation and dynamics must relate the finite fermionic states to the bosonic form or brane degrees of freedom. The pairing of item 1 is one candidate. The first family also needs Dirac neutrinos for its 84, and the second does not.
3. **Relation to the odd labels.** The removal of the top leaves the five quarks and the leptons, on which SU(3) x U(1)_em acts with b_L paired with b_R. So the loss of the electroweak partner of b_L has no role at the label 9, where SU(2) is absent. A statement here needs the required involution on this finite sector, including its action on D, J, the algebra and its opposite; the vectorlike charges alone do not establish it, and the neutral states still need treatment.

Numerical check

Python with exact fractions, run on 3 Nov 2026 in a temporary directory, not kept: all tables above, and the dimension 435 = 224 + 1 + 105 + 105 with 105 = 60 + 45 and 224 = 192 + 24 + 8.

Part H (history of the field, to November 2026)

Noncommutative geometry in particle physics: a dated record to November 2026

Written by Claude Opus 5.5 on 4 November 2026, to help decide which results of this project are new and which articles to write. Companions: paper/LITERATURE.md (targeted priority searches) and the author table at a.rivero.nom.es/research/ncactors.html (authors by year to 2003).

Method. Every identifier below was checked in this session against INSPIRE, the arXiv API or Crossref (curl and jq in bash). No Python was used; the author table was counted with awk in bash; no script is kept. Year = first public appearance; journal data follow.

Labels. [R] passages read in this session; [L] read earlier and recorded in paper/LITERATURE.md (section given); [A] abstract read (and, where stated, the reference list held by INSPIRE or the publisher); [M] bibliographic data checked only. Statements about content rest on [R], [L] or [A]; the one-line descriptions of [M] entries are the standard ones and were not re-read.

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1. Origins (to 1991) beside the Kaluza-Klein programme

1.1 The Kaluza-Klein programme, 1968-1987

- 1968. R. Kerner, Ann. Inst. H. Poincaré A 9 (1968) 143 [M]: non-Abelian Kaluza-Klein. Kerner later co-authors the matrix geometry of 1.3.

- 1979-1981. N. S. Manton, Nucl. Phys. B 158 (1979) 141; D. B. Fairlie, Phys. Lett. B 82 (1979) 97; P. Forgács and N. S. Manton, Commun. Math. Phys. 72 (1980) 15; G. Chapline and N. S. Manton, Nucl. Phys. B 184 (1981) 391 [M]: the Higgs as extra-dimensional components of a gauge field. Y. Ne'eman, Phys. Lett. B 81 (1979) 190 [M]: the SU(2/1) superconnection.
- 1981. E. Witten, *Search for a realistic Kaluza-Klein theory*, Nucl. Phys. B 186 (1981) 412 [A]: "It is possible to obtain an SU(3) × SU(2) × U(1) gauge group, but the proper fermion quantum numbers are difficult to achieve."
- 1982-1983. S. Randjbar-Daemi, A. Salam and J. Strathdee, Nucl. Phys. B 214 (1983) 491; C. Wetterich, Nucl. Phys. B 222 (1983) 20 [M]: elementary gauge backgrounds in six dimensions; reduction of Weyl spinors. Cited by Witten 1983 (refs. 13-14).
- 1983. E. Witten, *Fermion quantum numbers in Kaluza-Klein theory*, Shelter Island II (1985) 227, INSPIRE 193975 [R, preprint scan; L, "Sources read"]. Abstract: "In the absence of elementary gauge fields, a theorem by Atiyah and Hirzebruch states that the Dirac equation in 4+n dimensions always leads to vector-like fermion quantum numbers in four dimensions." Scope in the text: ground state M^4 × B with B compact; all gauge fields from isometries of the metric; the Dirac operator; for Rarita-Schwinger only homogeneous B ("a general impossibility theorem … is not proved"); spin 5/2 has a counterexample. Escapes named there: "the character-valued index also need not vanish in the presence of elementary gauge fields" (pseudo-realistic models in its section VI), and "start with a unified theory (perhaps a supergravity or superstring theory) that determines the original gauge group". Its ref. 16 (Witten, J. Diff. Geom. 17 (1982) 661) first noted the theorem's relevance. Theorem: M. Atiyah and F. Hirzebruch, *Spin-manifolds and group actions*, in Essays on Topology and Related Topics (1970) 18, doi:10.1007/978-3-642-49197-9_3 [M].
- 1986. M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Kaluza-Klein supergravity*, Phys. Rep. 130 (1986) 1 [opening sentence, quoted in the 2025 retrospective; full text not read]: "We do not yet know whether supergravity is a theory of the real world, nor whether the real world has more than four dimensions … However, our research in this area has convinced us that the only way to do supergravity is via Kaluza-Klein and that the only viable Kaluza-Klein theory is supergravity." The bulk treats S^7.
- 1987. D. Bailin and A. Love, *Kaluza-Klein theories*, Rep. Prog. Phys. 50 (1987) 1087 [A]: the spectrum "with particular emphasis on the spectrum of light fermions", anomaly constraints "when explicit higher-dimensional gauge fields are present", cosmology, and "Finally, an introductory account of Kaluza-Klein supergravity is given leading towards superstring theory."
- 1987. T. Appelquist, A. Chodos and P. G. O. Freund (eds.), *Modern Kaluza-Klein Theories*, Addison-Wesley [M]. INSPIRE records reprints of Kerner 1968, Chapline-Manton 1983, Witten 1981, Witten 1983 and Randjbar-Daemi-Salam-Strathdee; the introduction (pp. 1-47) was not read.
- Retrospectives. M. J. Duff, *Kaluza-Klein theory in perspective*, arXiv:hep-th/9410046 [R]: "the sorry state of affairs until the September 1984 superstring revolution"; the heterotic string "eschews the route of getting the Yang-Mills gauge group from the general coordinate group"; "Does this mean that the idea of non-abelian gauge fields from extra dimensions is dead? Actually, no." M. J. Duff, B. E. W. Nilsson and C. N. Pope, *Kaluza-Klein supergravity 2025*, arXiv:2502.07710 [R]: the coset programme became "less and less tenable" for three stated reasons: the large negative cosmological constant, Witten's chirality result (cited as the 1981 paper), and the string models after Candelas-Horowitz-Strominger-Witten.

Assessment of the view under test. The texts read support it in part. Witten 1983 states the result with its hypotheses and names the escapes. The Bailin-Love abstract treats light fermions together with explicit higher-dimensional gauge fields, the escape route, and leads "towards superstring theory"; the heterotic string took that route. The retrospectives give chirality as one reason among three, and Duff 1994 denies that the idea is dead. We found no text of the period that treats the obstruction as closing the subject. The open-problem sections of the 1986 and 1987 reviews and the 1987 introduction remain to be read in print.

1.2 Noncommutative geometry before the models

- 1980. A. Connes, C. R. Acad. Sci. Paris A 290 (1980) 599, arXiv:hep-th/0101093 [M].

- 1985. A. Connes, *Non-commutative differential geometry*, Publ. Math. IHES 62 (1985) 41, doi:10.1007/BF02698807 [M].
- 1988. A. Connes, *The action functional in noncommutative geometry*, Commun. Math. Phys. 117 (1988) 673 [A + INSPIRE references]: Dixmier trace = Wodzicki residue; Yang-Mills from the Dixmier trace.
- 1988, printed 1990. A. Connes, *Essay on physics and noncommutative geometry*, in *The Interface of Mathematics and Particle Physics* (Oxford 1988), OUP 1990, 9-48. **Unverified** (absent from INSPIRE; data from citing sources; text not read). Connes-Lott 1990 call their electroweak model "basically the same as for model II" of this essay, so the two-point model first appears here.

1.3 The first two-point and matrix models, 1988-1994

- Aug. 1988. J. Madore, *Kaluza-Klein aspects of noncommutative geometry*, Chester DGM proceedings (1989) 243, arXiv:1510.03915 [R]. "We can now generalize Kaluza-Klein theory by replacing C(F) by an associative algebra which is not necessarily commutative"; with M_n, "the internal laplacian has a finite spectrum and … the total momentum space remains of dimension 4. This is important for the renormalizability of the theory."
- 1988-1990. M. Dubois-Violette, R. Kerner and J. Madore: Class. Quant. Grav. 6 (1989) 1709 [M]; Phys. Lett. B 217 (1989) 485 [A + INSPIRE references]; J. Math. Phys. 31 (1990) 316 [M]; J. Math. Phys. 31 (1990) 323 [A + publisher references]. Derivation calculus on C^∞(V) ⊗ M_n: Maxwell's theory gives U(n) Yang-Mills plus adjoint scalars with a quartic potential and a massive phase. J. Madore, Phys. Rev. D 41 (1990) 3709 [A]: Kaluza-Klein with a semisimple internal algebra.
- Mar. 1990. A. Connes and J. Lott, *Particle models and noncommutative geometry*, Nucl. Phys. B Proc. Suppl. 18B (1991) 29 [R, author copy]. GWS, Peccei-Quinn and the Standard Model as pure Yang-Mills on M × {two points}; the Higgs is the connection along the discrete direction; Section VI gives m_H = √2 m_t (top-dominance limit, pure Dixmier trace) and "it is impossible to get GUTs by our methods". Then *The metric aspect of NCG*, Cargèse 1991, 53-93 [M].
- June 1990. R. Coquereaux, G. Esposito-Farèse and G. Vaillant, *Higgs fields as Yang-Mills fields and discrete symmetries*, Nucl. Phys. B 353 (1991) 689 [A + INSPIRE references]: Higgs fields from "the gauging of discrete directions". Mainz-Marseille sequels: R. Coquereaux, R. Häussling, N. A. Papadopoulos and F. Scheck, Int. J. Mod. Phys. A 7 (1992) 2809; R. Coquereaux, G. Esposito-Farèse and F. Scheck, Int. J. Mod. Phys. A 7 (1992) 6555 [M]. Coquereaux came from Kaluza-Klein: R. Coquereaux and A. Jadczyk, World Sci. Lect. Notes Phys. 16 (1988) [M]. Y. Ne'eman and S. Sternberg, Proc. Nat. Acad. Sci. 87 (1990) 7875 [M]: superconnections.
- 1992-1994. A. Dimakis, F. Müller-Hoissen and T. Striker, J. Phys. A 26 (1993) 1927 [M]; A. Dimakis and F. Müller-Hoissen, J. Phys. A 27 (1994) 3159, arXiv:hep-th/9401149 [M], and J. Math. Phys. 35 (1994) 6703, arXiv:hep-th/9404112 [L, §A]: calculus on finite sets and lattices; Connes-Lott models as approximations of higher-dimensional gauge theory. A. Sitarz, J. Geom. Phys. 15 (1995) 123, arXiv:hep-th/9210098 [M]: gauge theory on discrete groups.

1.4 Did the founders present the discrete direction as a way around the chirality obstruction? None of the texts read does so, and none cites Witten 1981 or 1983.

The Kaluza-Klein reading is there from the start; the advantage claimed is finiteness; chirality enters by hand through the Z_2-grading of the finite Hilbert space.

- Connes-Lott 1990 [R]. The only Kaluza-Klein sentence (Section I): "The corresponding geometric space can be thought of as a spacetime of Kaluza-Klein type, where the fiber is formed of a two-point set." References: Connes (3), Dixmier, Ellis-Gaillard, Kasparov, Glashow, Peccei-Quinn, Salam, Weinberg (2), Wilczek.
- Connes 1988, CMP 117 [A]: INSPIRE's reference list cites Witten only for *Non-commutative geometry and string field theory*. The Essay was not accessible.
- Connes, *Noncommutative Geometry*, Academic Press 1994 (PDF, alainconnes.org) [R, full-text search]: "Kaluza" does not occur; Witten is cited once, for Morse theory. Chapter VI: "a double space-time", where "purely left-handed particles such as neutrinos live on the left-handed copy X_L".

- Madore 1988 [R]: Kaluza-Klein generalization motivated by the finite internal spectrum; spinors are built "in dimension 4 + n^2 − 1" with no discussion of chirality; no Witten among the references.
- Dubois-Violette-Kerner-Madore [A]: abstracts silent on Kaluza-Klein and chirality; the reference lists shown cite Manton 1979 and R. Kerner, L. Nikolova and V. Rizov, *A two-level Kaluza-Klein theory*, Lett. Math. Phys. 14 (1987) 333 [M], and no Witten.
- Coquereaux-Esposito-Farèse-Vaillant [A]: "The formalism presented here is therefore not a Kaluza-Klein-like approach to Yang-Mills theory, but is, in a sense, a discrete analog of it." INSPIRE's references cite Chapline-Manton 1981 and no Witten. Full text not read.

Later Kaluza-Klein readings (sources searched for "Witten", "chiral", "no-go"; none cites the obstruction): - G. Landi, Nguyen Ai Viet and K. C. Wali, Phys. Lett. B 326 (1994) 45, arXiv:hep-th/9402046 [M]; Nguyen Ai Viet and K. C. Wali, Int. J. Mod. Phys. A 11 (1996) 533, arXiv:hep-th/9412220 [R], and 2403, arXiv:hep-th/9508032 [L, §A]. First "discretization of the fifth dimension"; the contrast drawn is the infinite tower and its inconsistent truncation (citing M. J. Duff, C. N. Pope and K. S. Stelle, Phys. Lett. B 223 (1989) 386 [M]). - A. H. Chamseddine and A. Connes, arXiv:hep-th/9606001 [R, passage]: "our 'discrete Kaluza Klein' geometry M × F". T. Schücker, *Forces from Connes' geometry*, arXiv:hep-th/0111236, Lect. Notes Phys. 659 (2005) 285 [R, passage]: "Kaluza-Klein models whose fifth dimension is discrete". Martinetti-Wulkenhaar 2002 and Dai-Song 2001 [L, §A]. - Chamseddine-Connes, arXiv:0706.3690 [R]: "the advantages of both SO(10) and Kaluza-Klein unification, without paying the price of plethora of Higgs fields or the infinite tower of states". A. H. Chamseddine, arXiv:2511.05909 (2025) [R]: the mechanism is "different from Kaluza-Klein since it avoids a massive tower of states, and places the Higgs in the correct representation".

So the reading was made first by Madore (1988, matrix fibre) and Connes-Lott (1990, two-point fibre), called "a discrete analog" by Coquereaux-Esposito-Farèse-Vaillant (1990) and "a discretization of the fifth dimension" by Landi-Viet-Wali and Viet-Wali (1994). The nearest engagement with the obstruction's hypotheses that we found lies outside NCG: J. Baptista, JHEP 05 (2026) 008, arXiv:2506.09126 [L, §I] (non-Killing gauge fields with chiral couplings). Scope of this negative finding: the texts and reference lists above, plus INSPIRE full-text queries pairing "spectral triple" with "Atiyah-Hirzebruch" and "noncommutative geometry" with Witten's 1983 title (no NCG model-building paper among the hits).

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2. The 1990s

Zürich, Marseille, Mainz: the Connes-Lott model and its constraints. - 1992-1993. A. H. Chamseddine, G. Felder and J. Fröhlich, Phys. Lett. B 296 (1992) 109; Nucl. Phys. B 395 (1993) 672, arXiv:hep-ph/9209224; Commun. Math. Phys. 155 (1993) 205, arXiv:hep-th/9209044 [M]: GUTs and gravity (Einstein plus Brans-Dicke) in the Connes-Lott formalism. A. H. Chamseddine and J. Fröhlich, Phys. Rev. D 50 (1994) 2893, arXiv:hep-th/9304023 [M]: SO(10). - 1992-1995. D. Kastler, Rev. Math. Phys. 5 (1993) 477 and 8 (1996) 103; D. Kastler and T. Schücker, Rev. Math. Phys. 8 (1996) 205, arXiv:hep-th/9501077 [M]: detailed accounts of the model. T. Schücker and J.-M. Zylinski, *Connes' model building kit*, J. Geom. Phys. 16 (1995) 207, arXiv:hep-th/9312186 [M]. - 1993. D. Kastler, Commun. Math. Phys. 166 (1995) 633; W. Kalau and M. Walze, J. Geom. Phys. 16 (1995) 327, arXiv:gr-qc/9312031 [M]: the Wodzicki residue gives Einstein-Hilbert. - 1994-1997. B. Iochum and T. Schücker, *Yang-Mills-Higgs versus Connes-Lott*, Commun. Math. Phys. 178 (1996) 1, arXiv:hep-th/9501142; L. Carminati, B. Iochum and T. Schücker, J. Math. Phys. 38 (1997) 1269, arXiv:hep-th/9604169; B. Iochum, D. Kastler and T. Schücker, J. Math. Phys. 38 (1997) 4929, arXiv:hep-th/9607158 [M]: which models arise; mass constraints; the spectral action computed.

Zaragoza and Costa Rica. - 1992. J. C. Várilly and J. M. Gracia-Bondía, J. Geom. Phys. 12 (1993) 223, DFTUZ-92-20 [M]. 1996: C. P. Martín, J. M. Gracia-Bondía and J. C. Várilly, *The Standard Model as a noncommutative geometry: the low-energy regime*, Phys. Rep. 294 (1998) 363, arXiv:hep-th/9605001 [M]. - 1997. H. Figueroa, J. M. Gracia-Bondía, F. Lizzi and J. C. Várilly, J. Geom. Phys. 26 (1998) 329, arXiv:hep-th/9701179 [M]; J. M. Gracia-Bondía, B. Iochum and T. Schücker, *The Standard Model in NCG and fermion doubling*, Phys. Lett. B 416 (1998) 123, arXiv:hep-th/9709145 [M]. - 1996-2004, DFTUZ, A. Rivero [A]: arXiv:hep-th/9605006 (constraints on C + M_2 + M_3; a leptophobic Z'); hep-th/9804169; hep-th/9905021 and physics/0007027 (generations from ordering and discretization ambiguities:

"three generations, and not four"); hep-th/0204238 (Poincaré duality for finite algebras, "to get continuum Dirac operators from discrete noncommutative manifolds"); physics/0409022. J. F. Cariñena, J. Clemente-Gallardo, E. Follana, J. M. Gracia-Bondía, A. Rivero and J. C. Várilly, J. Geom. Phys. 32 (1999) 79, arXiv:math/9802102 [M]: tangent groupoid. - 2001. J. M. Gracia-Bondía, J. C. Várilly and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser [M].

Naples: lattices, finite spaces, doubling. - 1993-1996. G. Bimonte, F. Lizzi and G. Sparano, Phys. Lett. B 341 (1994) 139, arXiv:hep-lat/9404007 [L, §A]; A. P. Balachandran et al., Nucl. Phys. B 418 (1994) 477, arXiv:hep-th/9310015, and J. Geom. Phys. 18 (1996) 163, arXiv:hep-th/9510217 [M]: finite (poset) approximations; G. Bimonte, E. Ercolessi, G. Landi, F. Lizzi and G. Sparano, J. Geom. Phys. 20 (1996) 318 and 329 [M]: continuum limits. - 1996-1997. F. Lizzi, G. Mangano, G. Miele and G. Sparano, Mod. Phys. Lett. A 11 (1996) 2561, arXiv:hep-th/9603095 [M]; Phys. Rev. D 55 (1997) 6357, arXiv:hep-th/9610035 [L, §I]: fermion doubling in L^2(S) ⊗ H_F.

Fuzzy spaces. J. Madore, *The fuzzy sphere*, Class. Quant. Grav. 9 (1992) 69 [M]; H. Grosse, C. Klimčík and P. Prešnajder, Commun. Math. Phys. 178 (1996) 507, arXiv:hep-th/9510083, and 180 (1996) 429, arXiv:hep-th/9602115 [M]. Books: J. Madore, LMS Lect. Note Ser. 257 (CUP, 2nd ed. 1999/2000) [M]; G. Landi, Lect. Notes Phys. Monogr. 51 (1997), arXiv:hep-th/9701078 [M].

Reality, axioms, spectral action. - 1995. A. Connes, *Noncommutative geometry and reality*, J. Math. Phys. 36 (1995) 6194 [M]: the real structure J, the KO-dimension table mod 8, the first-order condition. - 1996. A. Connes, *Gravity coupled with matter and the foundation of NCG*, Commun. Math. Phys. 182 (1996) 155, arXiv:hep-th/9603053 [M]: axioms; the Standard Model with J. - 1996. A. H. Chamseddine and A. Connes, Phys. Rev. Lett. 77 (1996) 4868, arXiv:hep-th/9606056; *The spectral action principle*, Commun. Math. Phys. 186 (1997) 731, arXiv:hep-th/9606001 [M]: Tr f(D/Λ) yields gravity, Yang-Mills and Higgs with relations at the cut-off.

Classification and distances. T. Krajewski, J. Geom. Phys. 28 (1998) 1, arXiv:hep-th/9701081 [L, §I]: Krajewski diagrams. M. Paschke and A. Sitarz, J. Math. Phys. 39 (1998) 6191, arXiv:q-alg/9612029 [L, §I]. B. Iochum, T. Krajewski and P. Martinetti, J. Geom. Phys. 37 (2001) 100, arXiv:hep-th/9912217 [L, §A].

Hopf algebras (ramification). D. Kreimer, Adv. Theor. Math. Phys. 2 (1998) 303, arXiv:q-alg/9707029; A. Connes and D. Kreimer, Commun. Math. Phys. 199 (1998) 203, arXiv:hep-th/9808042; 210 (2000) 249, arXiv:hep-th/9912092; 216 (2001) 215, arXiv:hep-th/0003188 [M].

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3. 2003 to 2012

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- 31 Aug. 2006, same day. A. Connes, JHEP 11 (2006) 081, arXiv:hep-th/0608226 [L, §I]; J. W. Barrett, J. Math. Phys. 48 (2007) 012303, arXiv:hep-th/0608221 [A]. F of KO-dimension 6 and metric dimension 0; total 10 = 2 mod 8; doubling removed (Pfaffian in Connes; Lorentzian J-γ condition in Barrett); ν_R and see-saw. Two independent arrivals.
- Oct. 2006. A. H. Chamseddine, A. Connes and M. Marcolli, Adv. Theor. Math. Phys. 11 (2007) 991, arXiv:hep-th/0610241 [A, L §I]: full Lagrangian from the spectral action; "a Higgs mass around 170 GeV".
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- •2008. A. Connes and M. Marcolli, *Noncommutative Geometry, Quantum Fields and Motives*, AMS Colloquium Publ. 55, doi:10.1090/coll/055 [M]. A. Connes, *On the spectral characterization of manifolds*, J. Noncommut. Geom. 7 (2013) 1, arXiv:0810.2088 [M].
- Aug. 2008. Tevatron combination, arXiv:0808.0534 [A]: "We exclude at 95% C.L. a standard model Higgs boson of m_H = 170 GeV."
- •2009. C. Stephan, Phys. Rev. D 79 (2009) 065013, arXiv:0901.4676 [A]: a complex scalar coupling to ν_R, vector-like fermions and the Higgs, used to constrain the Higgs mass.
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- •2010. A. H. Chamseddine and A. Connes, Fortsch. Phys. 58 (2010) 553, arXiv:1004.0464 [M]; *Noncommutative geometric spaces with boundary*, J. Geom. Phys. 61 (2011) 317, arXiv:1008.3980 [L, §I].
- 2010-2012. L. Dąbrowski and G. Dossena, *Product of real spectral triples*, Int. J. Geom. Meth. Mod. Phys. 8 (2011) 1833, arXiv:1011.4456 [L, §D]; J. Boeijink and W. D. van Suijlekom, J. Geom. Phys. 61 (2011) 1122, arXiv:1008.5101 [M]; B. Ćaćić, Lett. Math. Phys. 100 (2012) 181, arXiv:1101.5908 [M]: reconstruction for almost-commutative triples.
- •2011. K. van den Dungen and W. D. van Suijlekom, *Electrodynamics from NCG*, J. Noncommut. Geom. 7 (2013) 433, arXiv:1103.2928 [A]: U(1) from a two-point space, which had "long been thought impossible"; review Rev. Math. Phys. 24 (2012) 1230004, arXiv:1204.0328 [M]. A. A. Andrianov, M. A. Kurkov and F. Lizzi, JHEP 10 (2011) 001, arXiv:1106.3263 [M]: Weyl anomaly and Higgs-dilaton potential.
- •2012. A. H. Chamseddine, J. Fröhlich, B. Schubnel and D. Wyler, J. Math. Phys. 54 (2013) 012302, arXiv:1201.2661 [A]: discrete extra dimensions give axion, dilaton and Higgs.
- July-Aug. 2012. ATLAS, Phys. Lett. B 716 (2012) 1, arXiv:1207.7214; CMS, Phys. Lett. B 716 (2012) 30, arXiv:1207.7235 [M]. A. H. Chamseddine and A. Connes, *Resilience of the spectral Standard Model*, JHEP 09 (2012) 104, arXiv:1208.1030 [A]: the real singlet σ, "already present in the spectral model and we wrongly neglected it", restores consistency with the low Higgs mass.
- Lorentzian. A. Strohmaier, J. Geom. Phys. 56 (2006) 175, arXiv:math-ph/0110001 [M]; M. Paschke and A. Sitarz, arXiv:math-ph/0611029 [M]; N. Franco, Rev. Math. Phys. 26 (2014) 1430007, arXiv:1210.6575 [M]; N. Franco and M. Eckstein, Class. Quant. Grav. 30 (2013) 135007, arXiv:1212.5171 [M]: causality; K. van den Dungen, M. Paschke and A. Rennie, J. Geom. Phys. 73 (2013) 37, arXiv:1207.2112 [M].

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4. 2013 to 2019

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arXiv:1308.5410 [M].

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- 2016-2019, Lorentzian and algebraic backgrounds. K. van den Dungen, Math. Phys. Anal. Geom. 19 (2016) 4, arXiv:1505.01939 [M]; N. Bizi, C. Brouder and F. Besnard, J. Math. Phys. 59 (2018) 062303, arXiv:1611.07062 [M]; F. Besnard and N. Bizi, J. Geom. Phys. 123 (2018) 292, arXiv:1611.07830 [M]; F. Besnard, J. Math. Phys. 60 (2019) 123506, arXiv:1902.09387 [A]: Dirac operators compatible with an "algebraic background" form "the configuration space of a noncommutative Kaluza-Klein theory", containing Z'_{B−L}, σ and flavour-changing fields.
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- Beyond the Standard Model. U. Aydemir, D. Minic, C. Sun and T. Takeuchi, Int. J. Mod. Phys. A 31 (2016) 1550223, arXiv:1509.01606 [M]; L. Dąbrowski, F. D'Andrea and A. Sitarz, Lett. Math. Phys. 108 (2018) 1323, arXiv:1703.05279 [M]: fermions as internal forms; A. Bochniak and A. Sitarz, Phys. Rev. D 97 (2018) 115029, arXiv:1804.09482 [M]; F. Besnard and C. Brouder, Phys. Rev. D 103 (2021) 035003, arXiv:2010.04960 [M].
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5. 2020 to 2026

Truncations and operator systems. A. Connes and W. D. van Suijlekom, Commun. Math. Phys. 383 (2021) 2021, arXiv:2004.14115 [A, L §A]; arXiv:2111.02903 [M] (tolerance relations). W. D. van Suijlekom, J. Geom. Phys. 162 (2021) 104075, arXiv:2005.08544; L. Glaser and A. B. Stern, J. Math. Phys. 61 (2020) 033507, arXiv:1909.08054, and J. Geom. Phys. 159 (2021) 103921, arXiv:1912.09227; E.-M. Hekkelman, Lett. Math. Phys. 112 (2022) 20, arXiv:2111.13865; M. A. Rieffel, arXiv:2210.00387; R. Toyota, arXiv:2309.13469; M. Leimbach, arXiv:2409.16698; E.-M. Hekkelman and E. McDonald, arXiv:2412.00628; T. Camper, D. Giannakis and G. Hoefer, arXiv:2609.40189 [M]. M. Leimbach and W. D. van Suijlekom, Adv. Math. 439 (2024) 109496, arXiv:2302.07877 [L, §A]; V. Bhatt, S. Guin and P. Sharma, arXiv:2610.01132 [A] (noncommutative n-tori); M. Verhoeven, arXiv:2610.19791 [L, §K] (SU(2)). No "Gaunt-Rennie" paper was found; the Gaunt papers are J. W. Barrett and J. Gaunt, arXiv:1908.06796, and J. Gaunt, H. Nguyen and A. Schenkel, J. Phys. A 55 (2022) 474004, arXiv:2203.04817 [M].

Propinquity, convergence, collapse. F. Latrémolière, Adv. Math. 404 (2022) 108393, arXiv:1811.10843; Commun. Math. Phys. 388 (2021) 1049, arXiv:2102.03729 (fuzzy tori to quantum tori); Math. Ann. 389 (2024) 765, arXiv:2112.11000 [M]; C. Farsi, F. Latrémolière and J. Packer, Adv. Math. 437 (2024) 109442, arXiv:2301.00274 [M]; C. Farsi and F. Latrémolière, J. London Math. Soc. 114 (2026) e70628,

arXiv:2404.00240 [L, §B, with the project's assessment of its Theorem 3.4]; F. Latrémolière, arXiv:2603.19128 [A]: continuity of Dirac spectra of almost-commutative manifolds when the metric (C^1) and D_F vary together. Classical collapse: B. Ammann and C. Bär, Ann. Global Anal. Geom. 16 (1998) 221, arXiv:math/9801091 [L, §B]; J. Lott, arXiv:math/0005009 [M]; S. Roos, arXiv:1707.00608 [M].

Fibrations. J. Kaad and W. D. van Suijlekom, arXiv:1610.02873 [M] and arXiv:1710.03182 [L, §K]; S. Brain, B. Mesland and W. D. van Suijlekom, J. Noncommut. Geom. 10 (2016) 135, arXiv:1306.1951 [M]; B. Ćaćić and B. Mesland, Commun. Math. Phys. 388 (2021) 107, arXiv:1912.04179 [M].

Quantum Riemannian geometry (ramification). E. J. Beggs and S. Majid, arXiv:1508.04808 [M]; *Quantum Riemannian Geometry*, Grundlehren 355 (2020), doi:10.1007/978-3-030-30294-8 [M]: bimodule connections and Levi-Civita theory on finite and noncommutative spaces.

Lorentzian. G. Nieuviarts, arXiv:2502.18105 and arXiv:2512.15450 [M]: Lorentz symmetry and time from twisted almost-commutative triples. T. Masson and G. Nieuviarts, J. Geom. Phys. 187 (2023) 104784, arXiv:2207.04466 [A]: sequences of spectral triples over AF inductive limits, Bratteli arrows lifted to Krajewski diagrams.

Spectral action. A. H. Chamseddine, A. Connes and W. D. van Suijlekom, Commun. Math. Phys. 373 (2019) 457, arXiv:1809.02944 (entropy) and Eur. Phys. J. ST 232 (2023) 3581, arXiv:2207.10901 (review) [M]; E.-M. Hekkelman, T. van Nuland and J. Reimann, Commun. Math. Phys. 407 (2026) 254, arXiv:2512.14581 [M]; Chamseddine 2025 (1.4).

Random NCG. C. I. Pérez-Sánchez, J. Noncommut. Geom. 16 (2022) 1137, arXiv:1912.13288; Ann. Henri Poincaré 22 (2021) 3095, arXiv:2007.10914 [M]; H. Hessam, M. Khalkhali and N. Pagliaroli, J. Phys. A 55 (2022) 335204, arXiv:2107.10333 [M]; M. Khalkhali, N. Pagliaroli and L. Verhoeven, J. Math. Phys. 66 (2025) 053502, arXiv:2405.05056 [M]; R. A. Iseppi, arXiv:2410.11823 [M] (BV for finite triples).

Kaluza-Klein readings. Besnard 2019; Ćaćić-Mesland 2021; Baptista 2026 (1.4); J. Aastrup and J. M. Grimstrup, arXiv:2504.03391 [A]: an almost-commutative triple in a semi-classical limit on a configuration space of connections, with doubled fermions. Fuzzy extra dimensions: P. Aschieri, J. Madore, P. Manousselis and G. Zoupanos, JHEP 04 (2004) 034, arXiv:hep-th/0310072; P. Aschieri, T. Grammatikopoulos, H. Steinacker and G. Zoupanos, JHEP 09 (2006) 026, arXiv:hep-th/0606021 [M].

Finite spectral triples as limits of geometries (the project's question). The searches of Section 9 found four kinds of result: (a) convergence of truncations or matrix approximations to a fixed manifold or quantum space (circle, tori, SU(2), quantum groups, fuzzy tori); (b) collapse of a fibre in the propinquity, with a nontrivial limit on ker D (Farsi-Latrémolière); (c) continuity of almost-commutative spectra in g and D_F (Latrémolière 2026); (d) emergence or inductive sequences of finite triples (Aastrup-Grimstrup; Masson-Nieuviarts). We found no paper that obtains a particle-physics finite triple (Connes-Lott, or the KO-6 F) as the limit or truncation of a continuous Kaluza-Klein internal geometry with chiral boundary conditions; paper/LITERATURE.md §A-§B reach the same conclusion for the interval.

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6. Ramifications outside particle physics

- Number theory. J.-B. Bost and A. Connes, Selecta Math. 1 (1995) 411, doi:10.1007/BF01589495; A. Connes, arXiv:math/9811068 (trace formula and zeta zeros); A. Connes and C. Consani, Enseign. Math. 69 (2023) 93, arXiv:2106.01715; A. Connes, C. Consani and H. Moscovici, *Zeta spectral triples*, EMS Ser. Lect. Math. (2026) 39, doi:10.4171/elm/37/3 [M].
- Renormalization and motives. Connes-Kreimer (Section 2); A. Connes and M. Marcolli, J. Geom. Phys. 56 (2006) 55, arXiv:hep-th/0504085 [M].
- Solid state. J. Bellissard, *K-theory of C*-algebras in solid state physics*, Lect. Notes Phys. (1986) 99, doi:10.1007/3-540-16777-3_74; J. Bellissard, A. van Elst and H. Schulz-Baldes, J. Math. Phys. 35 (1994) 5373, arXiv:cond-mat/9411052; E. Prodan and H. Schulz-Baldes, *Bulk and Boundary Invariants for Complex Topological Insulators*, Springer (2016) [M].

- Strings and Moyal spaces. A. Connes, M. R. Douglas and A. Schwarz, JHEP 02 (1998) 003, arXiv:hep-th/9711162; N. Seiberg and E. Witten, JHEP 09 (1999) 032, arXiv:hep-th/9908142; M. R. Douglas and N. A. Nekrasov, Rev. Mod. Phys. 73 (2001) 977, arXiv:hep-th/0106048; H. Grosse and R. Wulkenhaar, Commun. Math. Phys. 256 (2005) 305, arXiv:hep-th/0401128, and 329 (2014) 1069, arXiv:1205.0465 [M].
- Matrix models. T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. D 55 (1997) 5112, arXiv:hep-th/9610043; N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B 498 (1997) 467, arXiv:hep-th/9612115; H. Steinacker, Class. Quant. Grav. 27 (2010) 133001, arXiv:1003.4134, and *Quantum Geometry, Matrix Theory, and Gravity*, CUP (2024), doi:10.1017/9781009440776 [M].

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7. Comparison with the author table (to 2003)

The table has 65 author rows (columns pre-1990 to 2003, arXiv codes) and excludes "purely mathematical authors" by its own statement. Post-2003 lines below rest on one checked paper each [M].

Continued, by line. - Standard-Model programme: Connes, Chamseddine (Sections 3-5); Fröhlich (2013 paper, Section 3). Schücker: classification with Jureit and Stephan to 2007, then mainly cosmology and lensing (Gen. Rel. Grav. 41 (2009) 67, arXiv:0712.1559). Iochum: spectral action with torsion and boundaries (Commun. Math. Phys. 310 (2012) 367, arXiv:1008.3630) and the 2018 book. Martinetti, Lizzi: grand symmetry, twists, distances, reviews. Sitarz, Dąbrowski: curved noncommutative tori, Standard-Model variants. Landi: twists with Martinetti. Paschke, Brouder, Rennie: Lorentzian triples; Rennie also boundaries. Várilly: Moyal planes, fuzzy-sphere metrics, reconstruction (arXiv:math/0610418). - Krajewski: from 2007 mainly group field theory (arXiv:1210.6257). - Moyal and matrix field theory: Grosse, Wulkenhaar ($\varphi\hat{}4$, 2005-2014); Steinacker (emergent gravity, IKKT); Aschieri, Castellani, Wess (twisted noncommutative gravity, arXiv:hep-th/0510059); Madore (fuzzy cosmologies with Burić); Ydri (fuzzy-sphere simulations); Balachandran (twisted statistics). - Algebraic lines: Majid (with Beggs); Dubois-Violette (exceptional Jordan algebra and the Standard Model, Nucl. Phys. B 912 (2016) 426, arXiv:1604.01247; with Todorov, arXiv:1806.09450); Kreimer (Hopf algebras, motives); Moscovici (twists, modular curvature); Goswami (Levi-Civita connections for spectral triples). - Changed topic, as far as the post-2005 records show: Coquereaux (graph quantum symmetries), Dimakis and Müller-Hoissen (integrable systems), Klimčík (integrable sigma models), Kerner, Gracia-Bondía, Figueroa, Requardt (emergent space), Lott (geometric analysis; his 2000 collapse paper is in Section 5); Rivero left in 2006. - Not checked: Atzmon, Bimonte, Brzeziński, Chakraborty, Dai and Song (13 codes in 2001), Drabant, Elsner, Frank, Frabetti, Gozzi, Grandjean, Hadfield, Hajac, Kalau, Karoubi, Kastler, Kauffman, Lord, Prešnajder, Recknagel, Rovelli, Scheck, Striker, Walze, Zumino.

Lines that stopped as particle-physics programmes. Derivation-based matrix Kaluza-Klein (Madore; Viet-Wali), which survives as fuzzy extra dimensions (Zoupanos and collaborators); discrete calculus for Connes-Lott models (Dimakis-Müller-Hoissen; Sitarz 1992-1995); Mainz-Marseille superconnection models; NCG applied to deconstruction (Dai-Song, one paper located).

Entered after 2003, absent from the table. Marcolli; Barrett; Stephan, Jureit, Squellari; van Suijlekom, van den Dungen, Boeijink; Ćaćić, Mesland, Kaad; Dossena; Sakellariadou, Nelson, Pierpaoli; Franco, Eckstein; Strohmaier (Lorentzian, preprint 2001, also absent); Kurkov, Devastato, D'Andrea, Filaci, Singh; Boyle, Farnsworth; Besnard, Bizi; Mukhanov; Aydemir, Minic, Sun, Takeuchi; Bochniak; Glaser, Stern, Hekkelman, Leimbach, Gaunt; Khalkhali, Pagliaroli, Hessam, Verhoeven, Pérez-Sánchez, Iseppi; Aastrup, Grimstrup; Masson, Nieuviarts; Latrémolière, Farsi (mathematical); Consani.

Pure-mathematics lines excluded by the table that matter here. - Manifolds with boundary: B. Iochum and C. Levy, arXiv:1001.3927; I. Forsyth, M. Goffeng, B. Mesland and A. Rennie, J. Noncommut. Geom. 13 (2019) 407, arXiv:1607.07143; C. Bär and W. Ballmann, Surv. Differ. Geom. 17 (2012) 1, arXiv:1101.1196, and Progr. Math. 319 (2016) 43, arXiv:1307.3021 [M]. - Fibrations and collapse: J.-M. Bismut and J. Cheeger, J. Amer. Math. Soc. 2 (1989) 33 [M]; Ammann-Bär, Lott, Kaad-van Suijlekom, Farsi-Latrémolière (Section 5). - Callias-type index: C. Callias, Commun. Math. Phys. 62 (1978) 213; U. Bunke, Math. Ann. 303 (1995) 241, doi:10.1007/BF01460989 [M]; C. Kottke, arXiv:0909.5661 [L, §K].

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8. Relevance to this project

Kaluza-Klein reading of the two-point space; the Higgs as A_5. By 1.4 the reading is as old as the subject (Madore 1988; Connes-Lott 1990; Coquereaux-Esposito-Farèse-Vaillant 1990), the "discretized fifth dimension" is Landi-Viet-Wali and Viet-Wali 1994, and the Higgs-as-connection idea descends from Manton, Fairlie and Forgács-Manton, which the matrix and Marseille papers cite. paper/LITERATURE.md §A already quotes Connes-Lott §I. No founding text links the discrete fibre to the chirality obstruction, so a paper that states the obstruction in its scope and derives Connes-Lott from a chiral interval would make that link for the first time in print, within the searches of 1.4.

Truncations and windows. Section 5: truncation and convergence theorems exist for circles, tori, SU(2) and quantum groups, and collapse is treated by Farsi-Latrémolière; the chiral interval, whose minimal truncation is the algebra C ⊕ C, is absent (paper/LITERATURE.md §A, §K).

Real structure and KO-dimension. J and the mod-8 table (Connes 1995); KO-6 for F (Connes; Barrett, same day in 2006); products (Dąbrowski-Dossena; Farnsworth; Ćaćić). We found no discussion of total dimension 9 or 11 as alternatives to 10 (paper/LITERATURE.md §D).

Number of generations. Nothing in the Connes-Chamseddine line fixes it: *Why the Standard Model* says "the number of generations is still an input", and the 2007 PRL lists it among the "delicate points". The classification counts states per generation (2k^2 = 32); Krajewski diagrams and the Jureit-Stephan classification admit any multiplicity. The proposals located that derive three lie outside the spectral-triple axioms: the DFTUZ notes of 1999-2000 and the exceptional-Jordan line (Dubois-Violette 2016; Dubois-Violette-Todorov 2018; L. Boyle, J. Math. Phys. 67 (2026) 071701, arXiv:2006.16265 [M]). Whether these give three as a theorem was not checked.

Lorentzian and boundary versions. Lorentzian: Barrett 2007, Strohmaier, Franco-Eckstein, van den Dungen, Besnard-Bizi, twisted triples. Boundary: Chamseddine-Connes 2011 impose Π_−Ψ = 0 with a local chirality on the boundary of the manifold factor; Iochum-Levy, Forsyth-Goffeng-Mesland-Rennie and Bär-Ballmann give the mathematics. We found no spectral triple whose internal direction is a manifold with boundary (paper/LITERATURE.md §I, lemma 22).

Lines the project has not touched. Twisted triples and grand symmetry; the Boyle-Farnsworth graded and Jordan reformulations; Besnard's algebraic background, which already contains σ and Z'_{B−L}; random NCG; quantum Riemannian geometry; propinquity beyond the collapse paper; fuzzy extra dimensions (Aschieri-Madore-Manousselis-Zoupanos), which give finite internal spaces with truncated towers and should be read before any claim about truncated Kaluza-Klein towers; Moyal field theory; the number-theoretic and solid-state ramifications.

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9. Not covered, and identifiers left unverified

Not covered. Quantum groups and Hopf-Galois bundles; κ-Minkowski; noncommutative gravity of the Wess school; the local index formula and cyclic cohomology; Dixmier traces; loop-quantum-gravity spectral triples beyond one entry; Pati-Salam phenomenology after 2018; proceedings in general. Citation chains were not exhausted, and absence from these searches does not establish absence from the literature.

Read only through abstracts or retrospectives. Duff-Nilsson-Pope 1986 and Bailin-Love 1987 (open-problem sections); the Appelquist-Chodos-Freund introduction; the full texts of Coquereaux-Esposito-Farèse-Vaillant and Dubois-Violette-Kerner-Madore.

Unverified or partly verified. 1. A. Connes, *Essay on physics and noncommutative geometry* (OUP 1990): not on INSPIRE; data from citing sources only. 2. Journal data not found or not checked (arXiv identifiers checked): Connes-Moscovici math/0609703; Paschke-Sitarz math-ph/0611029; Beggs-Majid 1508.04808; Lott math/0005009; Kaad-van Suijlekom 1610.02873; Iochum-Levy 1001.3927. 3. "Gaunt-Rennie" (named in the

request): no such paper found. 4. Madore's book, first edition (1995): not checked. 5. Volume of Bellissard's 1986 Lecture Notes in Physics paper: Crossref gives none (DOI checked).

Searches for Section 5 (4 Nov 2026). arXiv abstracts: "spectral truncation(s)"; "finite spectral triple"; "spectral triple" with "Kaluza-Klein", "fibration", "interval" and "boundary"; "almost-commutative" with "limit" or "spectral"; "operator system" or "Gromov-Hausdorff" with "spectral triple"; "noncommutative geometry" with "Kaluza-Klein" or "deconstruction"; "collapsing" with "Dirac operator". INSPIRE full text: as in 1.4. Two extended web searches (the two-point space and the chirality no-go; finite triples as Kaluza-Klein limits).

Part L (literature check and record of searches)

Literature check for claims A-I

Written by Claude Opus 5.5 on 3 November 2026, as recommendation 1 of historical/evaluation2026-11-03.md. Method: web searches (standard mode, extended mode where results were thin), arXiv abstract pages and full texts, Crossref and INSPIRE metadata for every journal reference below. Where an equation or section number is given, it was read in the full text (arXiv version unless stated). Items that could not be checked are marked **unverified** and collected at the end.

Verdicts: **KNOWN** (same statement in print), **PARTIAL** (a close or weaker version in print; the difference is stated), **NOT FOUND** (no source located after the searches listed).

A. Connes-Lott as the minimal spectral truncation of a chiral interval

Claim (lemma 05, Part 3; lemma 09, Theorem 9.6). The minimal nonzero grading-preserving spectral truncation of the Dirac operator on [0, l] with R(0) = 0, L(l) = 0 is unitarily equivalent to the Connes-Lott two-point triple (C + C, D = [[0, m], [m, 0]]) with |m| = pi/(2l); the compressed operator system P C([0, l]) P equals C + C; the truncated end states are at distance 2l/pi. The deconstructed chain of Part 1 has Connes-Lott as its N = 2 member.

Verdict: PARTIAL. The interpretation of the two-point space as a discrete or Kaluza-Klein fibre, the truncation machinery, the deconstruction picture and the lowest mixed-boundary mass pi/(2l) are all in print. We found no source that identifies the Connes-Lott triple as a spectral truncation of an interval Dirac operator with chiral boundary conditions, computes the compression P C([0, l]) P for such an interval, or relates |m| to the interval length.

Interpretation of the two-point space (in print).

- A. Connes and J. Lott, *Particle models and noncommutative geometry*, Nucl. Phys. B Proc. Suppl. 18B (1991) 29-47, doi:10.1016/0920-5632(91)90120-4; author's copy at math.berkeley.edu/~lott/partmodels.pdf. Section I: "The corresponding geometric space can be thought of as a spacetime of Kaluza-Klein type, where the fiber is formed of a two-point set." Section II.B, item (9): for the two-point space, d(p, p') = L, where L^{-2} is the largest eigenvalue of M*M. Section III (p. 36): the character space of the product is two copies of Z, "Thus the picture is of two copies of Z separated by a distance L. (As one loses information in passing to the character space, this picture should not be taken too literally.)", with (d_Z(p, p')^2 + L^2)^{1/2} <= d(p, p') <= d_Z(p, p') + L.
- N. A. Viet and K. C. Wali, *Noncommutative geometry and a discretized version of Kaluza-Klein theory with a finite field content*, Int. J. Mod. Phys. A 11 (1996) 533-551, doi:10.1142/S0217751X96000249, arXiv:hep-th/9412220; and *A discretized version of Kaluza-Klein theory with torsion and massive fields*, Int. J. Mod. Phys. A 11 (1996) 2403-2418, doi:10.1142/S0217751X96001206, arXiv:hep-th/9508032. The second abstract: "We consider an internal space of two discrete points in the fifth dimension of the Kaluza-Klein theory." The "discrete fifth dimension" reading is therefore explicit in print since 1995.

- A. Dimakis and F. Müller-Hoissen, *Discrete differential calculus, graphs, topologies and gauge theory*, J. Math. Phys. 35 (1994) 6703-6735, doi:10.1063/1.530638, arXiv:hep-th/9404112. Abstract: the formalism "suggests that the [Connes-Lott two-point space] be regarded as an approximation of a manifold and thus opens a way to relate models with an 'internal' discrete space (à la Connes et al.) to models of dimensionally reduced gauge fields". Section III, Example 4: the universal differential algebra on two points has a Hasse diagram approximating the topology of S^2, linked to Manton's six-dimensional Yang-Mills reduction on S^2. Section VII: the Connes-Lott models should be understood "as approximations of higher-dimensional gauge theory models … The details have still to be worked out, however." This is the closest statement of the programme of lemma 05. The approximation there is topological (posets and Hasse diagrams) and the internal space is S^2; the passages we read contain no Dirac operator, boundary condition or mass-length relation.
- P. Martinetti and R. Wulkenhaar, *Discrete Kaluza-Klein from scalar fluctuations in noncommutative geometry*, J. Math. Phys. 43 (2002) 182-204, doi:10.1063/1.1418012, arXiv:hep-th/0104108. Section V, Theorem 4 and eq. (27): the distance between the same point on the two sheets is 1/||M|| (M the off-diagonal block of D_F), with a Pythagorean relation when [D_I, p] = 0. In the proof of Theorem 4 the authors embed the two sheets in a "virtual" five-manifold [0, 1] x M as a proof device and state that "the points between the sheets are not part of the geometry". Section VI: after scalar fluctuations the internal metric component becomes a function on M (the Higgs field); Section VII, Proposition 8, gives this component for the Standard Model. "Discrete Kaluza-Klein" refers to this Higgs-as-g_55 picture. Spectral truncation and boundary conditions do not occur.
- B. Iochum, T. Krajewski and P. Martinetti, *Distances in finite spaces from noncommutative geometry*, J. Geom. Phys. 37 (2001) 100-125, doi:10.1016/S0393-0440(00)00044-9: distances on finite spectral triples, including the two-point case.
- Matrix and discrete Kaluza-Klein precursors (bibliographic data verified; contents known from abstracts and secondary sources only): M. Dubois-Violette, R. Kerner and J. Madore, *Noncommutative differential geometry and new models of gauge theory*, J. Math. Phys. 31 (1990) 323 (Higgs fields as components of connections in matrix directions); J. Madore, *Kaluza-Klein aspects of noncommutative geometry* (1989 proceedings, posted as arXiv:1510.03915); J. Madore and J. Mourad, *Noncommutative Kaluza-Klein theory*, arXiv:hep-th/9601169; R. Coquereaux, G. Esposito-Farèse and G. Vaillant, *Higgs fields as Yang-Mills fields and discrete symmetries*, Nucl. Phys. B 353 (1991) 689 (**unverified** beyond citations); A. H. Chamseddine, J. Fröhlich, B. Schubnel and D. Wyler, *Dimensional reduction without continuous extra dimensions*, J. Math. Phys. 54 (2013) 012302, arXiv:1201.2661.

Spectral truncation (in print, for other spaces).

- A. Connes and W. D. van Suijlekom, *Spectral truncations in noncommutative geometry and operator systems*, Commun. Math. Phys. 383 (2021) 2021-2067, doi:10.1007/s00220-020-03825-x, arXiv:2004.14115. Section 1 (Introduction, pp. 2-3 of the arXiv version): "a natural spectral truncation is given simply by a spectral projection P onto eigenspaces of D … The natural truncation of the action of the *-algebra A to the Hilbert space PH is given by the space PAP. Since P does not commute with A, this is not an algebra anymore. Moreover in many interesting examples if one takes the C*-algebra generated by PAP one gets a non-informative full matrix algebra. However, not all is lost: the space PAP is a *-closed subspace in B(H), i.e. it is a so-called operator system". The same passage notes that symmetries of (A, D) induce symmetries of (PAP, PDP). The worked example is the circle (Toeplitz operator systems, Sections 4-5). Relation to lemma 05: the compression map Q of eq. (18) is non-multiplicative, as in this general picture, but for the chiral interval the minimal truncation has range exactly C + C, an algebra; this exceptional case is absent from the paper.
- F. D'Andrea, F. Lizzi and P. Martinetti, *Spectral geometry with a cut-off: topological and metric aspects*, J. Geom. Phys. 82 (2014) 18-45, doi:10.1016/j.geomphys.2014.03.014, arXiv:1305.2605. Cut-off by a projection on D and/or A; points are approximated by non-pure truncated states at finite distance; on the circle these are Fejér distributions; applications to the Moyal plane and the fuzzy sphere. The interval does not appear.
- Follow-ups on truncations (all circle, tori or groups): E.-M. Hekkelman, *Truncated geometry on the circle*, Lett. Math. Phys. 112 (2022), doi:10.1007/s11005-022-01514-5; L. Glaser and A. Stern, *Reconstructing manifolds from truncations of spectral triples*, J. Geom. Phys. 159 (2021) 103921; M.

Leimbach and W. D. van Suijlekom, *Gromov-Hausdorff convergence of spectral truncations for tori*, Adv. Math. 439 (2024) 109496, arXiv:2302.07877.

Deconstruction and lattices.

- N. Arkani-Hamed, A. G. Cohen and H. Georgi, *(De)constructing dimensions*, Phys. Rev. Lett. 86 (2001) 4757-4761, doi:10.1103/PhysRevLett.86.4757, arXiv:hep-th/0104005.
- C. T. Hill, S. Pokorski and J. Wang, *Gauge invariant effective Lagrangian for Kaluza-Klein modes*, Phys. Rev. D 64 (2001) 105005, doi:10.1103/PhysRevD.64.105005, arXiv:hep-th/0104035. The abstract states that the chain of gauge groups linked by bifundamentals "matches the continuum compactified theory truncated in momentum space", and treats chiral fermions. This is the physics statement "deconstruction = momentum truncation of the KK tower".
- H.-C. Cheng, C. T. Hill, S. Pokorski and J. Wang, *The standard model in the latticized bulk*, Phys. Rev. D 64 (2001) 065007, arXiv:hep-th/0104179 (split-generation fermions on a transverse lattice).
- J. Dai and X.-C. Song, *Spontaneous symmetry broken condition in (de)constructing dimensions from noncommutative geometry*, ICFP 2001 proceedings, doi:10.1142/9789812777379_0044, arXiv:hep-ph/0105280. The only paper found that applies NCG (differential calculus on cyclic groups, in the Dimakis-Müller-Hoissen/Sitarz style) to the Arkani-Hamed-Cohen-Georgi model; it treats the bosonic vacuum, without spectral triples or fermion truncation.
- NCG distances on lattices: G. Bimonte, F. Lizzi and G. Sparano, *Distances on a lattice from noncommutative geometry*, Phys. Lett. B 341 (1994) 139-146, arXiv:hep-lat/9404007 (the anomalous lattice distances that lemma 05, Part 1 calls "the familiar lattice-distance anomaly"); E. Atzmon, *Distances on a one-dimensional lattice from noncommutative geometry*, Lett. Math. Phys. 37 (1996) 341-348, doi:10.1007/BF00343197.
- The lowest Dirac (or gauge) mass pi/(2l) for mixed boundary conditions on an interval is textbook interval physics; see C. Csáki et al., *Fermions on an interval* (reference in C), Section 3 and Appendix B.

What remains unmatched. The identification "minimal grading-preserving spectral truncation of the chiral interval = Connes-Lott with |m| = pi/(2l)", the computation P C([0, l]) P = C + C, the truncated end-state distance 2l/pi against the geodesic length l, and the chain family of Part 1 with an exact Connes-Lott member at N = 2 and Gromov-Hausdorff convergence to [0, l].

Searched: "two-point space" with "spectral truncation", "interval", "boundary conditions"; "discrete Kaluza-Klein"; "deconstruction" with "noncommutative geometry", "spectral triple"; "gauge-Higgs unification" with "Connes-Lott"; "spectral truncation" with "interval [0,1]" (extended mode); NCG plus orbifold S^1/Z_2.

B. The two limits of the massive chiral interval

Claim (lemma 09, Theorems 9.8-9.9, Corollary 9.10). As l -> infinity at fixed bulk mass, the light pair becomes two chiral wall modes and the rest of the spectrum obeys a one-dimensional Weyl law above the gap mu. As l -> 0 with mu(l) l ~ log(1/(m_f l)), a Connes-Lott pair of fixed mass m_f survives while the interval collapses; at fixed mu no light fermion survives.

Verdict: PARTIAL. The spectral physics is classical: wall zero modes, exponentially small mixing of the two boundary modes on a finite slab, and the resulting light mass ~ 2 mu e^{-mu l}. Collapse of spectral triples is an active topic in NCG, with a general result for collapse to a point. We found no source stating either limit as a theorem about families of finite or interval spectral triples, and no source for the tuned collapse mu(l) that keeps a Connes-Lott pair.

References.

- R. Jackiw and C. Rebbi, *Solitons with fermion number 1/2*, Phys. Rev. D 13 (1976) 3398-3409, doi:10.1103/PhysRevD.13.3398: the wall zero mode.

- D. B. Kaplan, *A method for simulating chiral fermions on the lattice*, Phys. Lett. B 288 (1992) 342-347, doi:10.1016/0370-2693(92)91112-M, arXiv:hep-lat/9206013: domain-wall fermions.
- Y. Shamir, *Chiral fermions from lattice boundaries*, Nucl. Phys. B 406 (1993) 90-106, doi:10.1016/0550-3213(93)90162-I, arXiv:hep-lat/9303005. Section 2: a Weyl mode bound to the boundary of a half-infinite fifth direction, profile b^s with |b| < 1 (eq. (14)). Section 3: with a finite fifth direction 0 <= s <= N, "a second chiral fermion appears on the new boundary at s = N … this fermion has the opposite chirality … Strictly speaking, there is a tiny mixing between the two chiral modes, which vanishes like (1 - M)^N." This is the lattice version of Lemma 9.5 and Theorem 9.8.
- C. G. Callan and J. A. Harvey (1985), see section H: the continuum domain-wall zero modes and anomaly inflow.
- C. Csáki, C. Grojean, J. Hubisz, Y. Shirman and J. Terning, *Fermions on an interval: quark and lepton masses without a Higgs*, Phys. Rev. D 70 (2004) 015012, doi:10.1103/PhysRevD.70.015012, arXiv:hep-ph/0310355. Section 2: boundary conditions from the variational principle; Section 4.1, eqs. (4.8)-(4.9): a bulk mass profile realizing Dirichlet-Neumann conditions, with a tan(m_n L) quantization; Appendix B: KK decompositions in flat space.
- C. Farsi and F. Latrémolière, *Collapse in noncommutative geometry and spectral continuity*, J. London Math. Soc. (2) 114 (2026) e70628, doi:10.1112/jlms.70628, arXiv:2404.00240. Collapse of the vertical part of spectral triples in the spectral propinquity, with spectral continuity; Section 4.1, Corollary 4.1: collapse of a spectral triple to a point gives a nontrivial limit acting on ker D (read in the published text). The published text (first published 3 July 2026, read on 3 Nov 2026 from the journal HTML) agrees with arXiv v3 in the statements and equations compared (copy-editing, and L_A corrected to L_B in Definition 2.13): Theorem 3.4 (Theorem 2.5 in v1) keeps the same six hypotheses, and the step at issue is printed as eq. (3.11), so Counterexample R1 of the assessment applies to the printed statement. Relation to lemma 09: Corollary 9.10 (fixed mu, condition (c), ker D = 0, no light sector) agrees with this picture. Theorem 9.9 keeps a nonzero light pair because the Dirac operator itself changes with l (mu = mu(l)); that family lies outside the metric-rescaling collapse treated there. **Assessed in detail:** historical/reviews/2026-11-03_farsi_latremoliere_assessment.md. Corollary 4.1 (a fixed triple scaled by 1/eps converges to (C, ker D, 0), and has no limit when ker D = 0) is sound. The printed hypotheses of Theorem 3.4 (Theorem 2.5 in v1) are insufficient; anticommutation of D_h and D_v, or a uniform relative bound, supplies a sufficient additional hypothesis; Counterexample R1 there, checked by hand by Claude and by Terra 6.1, satisfies the six listed hypotheses and contradicts Corollary 3.5. The anticommuting applications avoid this defect; Theorem 4.6 and Corollary 4.7 also need a nonzero vertical kernel for the displayed target to be a spectral triple. The published version keeps the statement without the hypothesis. Validated by Astra 6 (Astra validation, items 2 and 4). Our tuned collapse (Theorem 9.9) lies outside its hypotheses.
- B. Ammann and C. Bär, *The Dirac operator on nilmanifolds and collapsing circle bundles*, Ann. Global Anal. Geom. 16 (1998) 221-253, doi:10.1023/A:1006553302362, arXiv:math/9801091: in commutative geometry, collapse either sends all Dirac eigenvalues to infinity or keeps a convergent part, depending on the spin structure.

What remains unmatched. The statement of both limits for the interval spectral triple with condition (c) as theorems (Theorems 9.8-9.9), the formula c = log(2c/(m_f l)) + o(1) for the bulk mass needed to keep a fixed light mass, and the identification of the surviving pair with Connes-Lott at each l (Theorem 9.6).

Searched: domain-wall fermions on a finite slab (exponentially small mass); "collapse" with "spectral triple"; "decompactification" and "emergent dimension" with "almost-commutative"; Ammann-Bär collapse and its NCG follow-ups.

C. Towerless Yukawa textures

Claim (lemma 12). For bulk fields Q, U on [0, l] with bulk masses and a boundary Yukawa matrix X at x = l: an exact secular equation det(lambda^2 - A X B X^dagger) = 0 (Theorem 12.4); the one-flavour, massless spectrum (n pi +- arctan|X|)/l (Example 12.5); singular-value bounds (Theorem 12.6); in the limit l -> 0 at fixed c and Y_hat = X/l the low spectrum converges to the finite triple with Yukawa block M_F

= Phi_Q Y_hat Phi_U (Theorem 12.7); graded singular values of F X G (Lemma 12.8).

Verdict: PARTIAL. The physics is in print, in more elaborate (warped) form: exact KK determinant equations with brane Yukawa matrices, the effective Yukawa F(c_Q) Y F(c_u), masses from minors and mixings F(c_Qi)/F(c_Qj). The one-flavour flat spectrum is Csáki et al. eq. (5.6) (KNOWN). We found no NCG paper that derives the finite Dirac operator D_F, or its Yukawa texture, from localization on an extra dimension.

References.

- C. Csáki et al., *Fermions on an interval* (full data in B). Section 5.1, eqs. (5.4)-(5.6): flat interval, no bulk mass, boundary conditions psi_eL(0) = 0, chi_eR(0) = 0, psi_eL(L) = -M_D L psi_eR(L), chi_eR(L) = M_D L chi_eL(L), spectrum tan(m_n L) = M_D L. With X = M_D L and signed 4D masses, the roots are (n pi +- arctan|X|)/L, which is Example 12.5. Section 2, eqs. (2.8)-(2.9): general boundary conditions with matrices; the spectra are computed one field at a time (Section 7.1: "a much richer spectrum of possibilities for such mixings which we will however not deal with here"). Section 6.2.2, eq. (6.35): secular equation for bulk mass plus brane Majorana mass (warped). Appendix C.2, eqs. (C.9)-(C.14): the boundary parameter equals M_D L, 2 tanh(M_D L/2) or M_D L/(1 + M_D^2 L^2/4) depending on how the brane mass is regularized, so the X of lemma 12 is a boundary-condition parameter whose relation to a brane Lagrangian mass is scheme dependent.
- S. Casagrande, F. Goertz, U. Haisch, M. Neubert and T. Pfoh, *Flavor physics in the Randall-Sundrum model I*, JHEP 10 (2008) 094, doi:10.1088/1126-6708/2008/10/094, arXiv:0807.4937. Section 3.2: IR boundary conditions with the Yukawa matrices (eqs. (55), (63)) and the exact determinant condition for the KK masses, eq. (65), for N generations; eq. (71): profile normalization F(c); eq. (87): Y_eff = F(c_Q) Y F(c_u); eq. (90): zero-mode approximation det(m_n^2 - (v^2/2) Y_eff Y_eff^dagger) = 0; eqs. (94)-(96): masses as F(c_Q) F(c_u) times ratios of minors of Y; eq. (97): (U_q)_ij ~ F(c_Qi)/F(c_Qj). This is the warped analogue of Theorems 12.4, 12.7, Corollary 12.9 and Remark 12.10.
- Flat and warped localization: N. Arkani-Hamed and M. Schmaltz, *Hierarchies without symmetries from extra dimensions*, Phys. Rev. D 61 (2000) 033005, doi:10.1103/PhysRevD.61.033005; T. Gherghetta and A. Pomarol, *Bulk fields and supersymmetry in a slice of AdS*, Nucl. Phys. B 586 (2000) 141-162, doi:10.1016/S0550-3213(00)00392-8; Y. Grossman and M. Neubert, *Neutrino masses and mixings in non-factorizable geometry*, Phys. Lett. B 474 (2000) 361-371, doi:10.1016/S0370-2693(00)00054-X; S. J. Huber and Q. Shafi, *Fermion masses, mixings and proton decay in a Randall-Sundrum model*, Phys. Lett. B 498 (2001) 256-262, doi:10.1016/S0370-2693(00)01399-X; K. Agashe, G. Perez and A. Soni, *Flavor structure of warped extra dimension models*, Phys. Rev. D 71 (2005) 016002, doi:10.1103/PhysRevD.71.016002; D. E. Kaplan and T. M. P. Tait, *New tools for fermion masses from extra dimensions*, JHEP 11 (2001) 051, doi:10.1088/1126-6708/2001/11/051, arXiv:hep-ph/0110126 (flat compact extra dimension, zero modes localized by odd bulk masses, Higgs on a boundary; the flat-space setting closest to lemma 12).
- Graded matrices: C. D. Froggatt and H. B. Nielsen, *Hierarchy of quark masses, Cabibbo angles and CP violation*, Nucl. Phys. B 147 (1979) 277-298, doi:10.1016/0550-3213(79)90316-X; J. Demmel, M. Gu, S. Eisenstat, I. Slapničar, K. Veselić and Z. Drmač, *Computing the singular value decomposition with high relative accuracy*, Linear Algebra Appl. 299 (1999) 21-80 (singular values of D_1 X D_2 with well-conditioned X are determined to high relative accuracy; theorem numbers **unverified**). The leading-order "masses = products of profiles times ratios of minors" is Casagrande et al. eq. (96). We did not find the two-sided compound-matrix bound of Lemma 12.8 in this explicit form; its proof (Cauchy-Binet on the p-th compound) is routine.
- NCG side: Yukawa matrices enter D_F as free moduli. T. Krajewski (1998) and A. Connes (2006, Theorem 2.7) classify the admissible D_F; see section I. No derivation of these moduli from localization was found.

What remains unmatched. The flat-space exact secular equation for an arbitrary N_Q x N_U boundary matrix with bulk masses, stated with multiplicities; the two-sided bound of Theorem 12.6 (via concavity of alpha, beta in lambda^2); the towerless limit l -> 0 at fixed c and Y_hat as a convergence theorem with error sigma^3 l^2 (1/L_Q^2 + 1/L_U^2) onto a finite even spectral triple; Corollary 12.3 on vectorlike

residual theories; and the reading "D_F is the towerless remnant of internal momentum".

Searched: "noncommutative geometry" or "spectral triple" with "split fermions", "fermion localization", "Randall-Sundrum", "Yukawa overlap" (standard and extended); Csáki et al. full text; Casagrande et al. full text; graded-matrix SVD literature.

D. KO-dimension of products: 9, 10 or 11

Claim (lemmas 06, 07, 08). A finite KO-6 real triple is T_n x P_k with (s, n, k) = (+1, 7, 7) or (-1, 5, 1) iff a grading-reversing involution U commutes with A, A°, D and satisfies UJ = sJU (Theorem 06.2); J alone does not fix s; an unpaired nu_L forces the KO-7 reduction (No-go 06, with [U, Q] = 0); in the internal-first product with the interval, the bare reflection has s = -1 (Theorem 08.5).

Verdict: PARTIAL. Products of real spectral triples of all parities, including the doubled odd x odd product on H_1 x H_2 x C^2, the freedom J versus J gamma in even dimension, and the total KO-dimension 10 = 2 mod 8 of the Standard Model are in print. The converse direction, a criterion for when a given even KO-6 triple factors as (odd) x (odd point), was NOT FOUND, nor were the nu_L statement, the reflection sign, or any discussion of 9 or 11 as alternatives to 10.

References.

- L. Dąbrowski and G. Dossena, *Product of real spectral triples*, Int. J. Geom. Methods Mod. Phys. 8 (2011) 1833-1848, doi:10.1142/S021988781100597X, arXiv:1011.4456. Introduction: in even dimensions "there are always two real structure operators J, that differ by multiplication by the grading operator. None of them should be preferred as they are perfectly on the same footing." Section 4.1 (Tables 2-3) and 4.2 (Tables 4-5): even-even and even-odd products. Section 4.3, eq. (25) and Table 6: the odd-odd product H = (H_1 x H_2) x C^2, D = D_1 x 1 x sigma_1 + 1 x D_2 x sigma_2, chi = 1 x 1 x sigma_3, J = J_1 x J_2 x M K. Section 6: metric and KO-dimension need not agree mod 8. Lemma 06.1 already cites eq. (25) and Table 6 for its product convention. The paper treats the forward direction (building products); the question of when a given triple is a product is left open there.
- F. J. Vanhecke, *On the product of real spectral triples*, Lett. Math. Phys. 50 (1999) 157-162, doi:10.1023/A:1007690509512, arXiv:math-ph/9902029: the naive product J may violate the sign table; a corrected product real structure.
- F. J. Vanhecke, A. R. da Silva and C. Sigaud, *On the finite spectral triple of an almost-commutative geometry*, Braz. J. Phys. 42 (2012) 471-474, doi:10.1007/s13538-012-0101-2, arXiv:1011.4925. Section 2.1: with J = J_1 x J_2, a Minkowski factor of signature 2 or 6 forces an odd internal factor (sigma_2 in {1, 5} or {3, 7}) and "the chirality paradigm is lost"; Section 2.2: with J = J_1 x J_2 Omega_2 an even internal factor is allowed and sigma_2 = 6 gives total signature 0 (Barrett's case). This is the closest printed discussion of odd internal factors, framed as a consequence of the product convention.
- S. Farnsworth, *The graded product of real spectral triples*, J. Math. Phys. 58 (2017) 023507, doi:10.1063/1.4975410, arXiv:1605.07035: graded tensor product, commutative and associative, always a spectral triple.
- B. Ćaćić, *Real structures on almost-commutative spectral triples*, Lett. Math. Phys. 103 (2013) 793-816, doi:10.1007/s11005-013-0616-7, arXiv:1209.4832: real reconstruction theorem; compact oriented manifolds carry real (almost-)commutative triples of every KO-dimension.
- A. Connes, *Noncommutative geometry and the standard model with neutrino mixing*, JHEP 11 (2006) 081, doi:10.1088/1126-6708/2006/11/081, arXiv:hep-th/0608226, Section 1: "the KO-dimension of the product space M x F is in fact equal to 10 ~ 2 modulo 8. Of course the above 10 is very reminiscent of string theory".
- J. W. Barrett, *Lorentzian version of the noncommutative geometry of the standard model of particle physics*, J. Math. Phys. 48 (2007) 012303, doi:10.1063/1.2408400, arXiv:hep-th/0608221. Section 1: J^2 = 1 and J gamma = gamma J "hold only if the total signature is equal to zero"; Section 2: the internal space is "determined to be a real spectral triple of signature six".
- A. H. Chamseddine and A. Connes, *Why the Standard Model*, J. Geom. Phys. 58 (2008) 38-47 (section I): the finite geometry raises "the K-theoretic dimension from four to ten (modulo eight)".

- K. van den Dungen and W. D. van Suijlekom, *Electrodynamics from noncommutative geometry*, J. Noncommut. Geom. 7 (2013) 433-456, doi:10.4171/JNCG/122: the KO-6 two-point triple for electrodynamics, the natural reference for Counterexample 06.3.
- Background: the complex analogue of Theorem 06.2 (an even triple with a commuting, grading-reversing Clifford generator is an odd triple doubled) is the standard identification of odd Fredholm modules with Cl_1-graded ones in Kasparov theory; we did not locate a reference stating the real KO-6 version with the sign s of UJ = sJU.
- Terminology: "doubling" in lemma 06 (H = H_0 x C^2) differs from "fermion doubling" in the NCG literature (F. Lizzi, G. Mangano, G. Miele and G. Sparano, Phys. Rev. D 55 (1997) 6357, section I; resolved in Connes 2006 by the Pfaffian). A paper should use a different word.

What remains unmatched. Theorem 06.2 (criterion), Result 06.3 (both s for electrodynamics), the nu_L no-go under [U, Q] = 0, Theorem 08.5 (s = -1 for the bare reflection in the internal-first product), Counterexample 08.11 and Theorem 08.12.

Searched: "real spectral triple" with "factorization", "product of odd", "Clifford", "KO-dimension 6"; "9" and "11" with "noncommutative standard model"; full texts of Dąbrowski-Dossena, Vanhecke-da Silva-Sigaud, Connes 2006 and Barrett 2007.

E. Spectral action: the W mass relation, the top ratio, heavy-fermion weights

Claim (lemma 11; lemma 03). 8 M_W^2 = sum N_f m_f^2 at unification; m_t/M_W -> sqrt(8/3) when the top dominates; the bounds m_max/(2 sqrt 2) <= M_W <= sqrt(N_g) m_max; gauge kinetic terms weighted by sum_j f(m_j^2/Lambda^2) tr_{H_j} F^2.

Verdict: KNOWN for the mass relation and the top ratio; Proposition 11.1 and Corollary 11.2 are one-line consequences of it. **NOT FOUND** for the exact heavy-fermion weighting in this form.

References.

- A. H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007) 991-1089, doi:10.4310/ATMP.2007.v11.n6.a3, arXiv:hep-th/0610241. Eq. (1.5) (Introduction): sum over generations of m_e^2 + m_nu^2 + 3 m_d^2 + 3 m_u^2 = 8 M_W^2. Lemma 4.6, eq. (4.16), proved from the normalization (4.17) Tr(k* k ...) = 2 g^2. Section 5.4 ("The fermion-boson mass relation"): eq. (5.24) m_top = v y_t/sqrt 2 with v = 2M/g; eqs. (5.25)-(5.26): Y_2(S) = 4 g^2; eq. (5.27): neglecting all Yukawas but the top, y_t = (2/sqrt 3) g at unification; eq. (5.28): y_t ~ 0.597. With M_W = g v/2 this is m_t/M_W = sqrt(8/3), which follows at once from (5.24) and (5.27); CCM state the relation in the form y_t = (2/sqrt 3) g. Eq. (5.29): if the tau-neutrino Dirac Yukawa equals the top Yukawa, y_t ~ g, i.e. m_t/M_W = sqrt 2.
- A. H. Chamseddine and A. Connes, *Conceptual explanation for the algebra in the noncommutative approach to the standard model*, Phys. Rev. Lett. 99 (2007) 191601 (abstract: a mass relation linking the fermion masses to the W mass, used to predict the top mass).
- Heavy-fermion weights: CCM Section 3 expands Tr f(D_A/Lambda) with D_F entering polynomially through the coefficients a, b, c, d, e, and Section 5.3 places the Majorana mass at the scale Lambda (Lemma 5.2, eq. (5.20)). The resummed weighting sum_j f(m_j^2/Lambda^2) of lemma 03 Part A was not found. Checked: M. Eckstein and B. Iochum, *Spectral Action in Noncommutative Geometry*, SpringerBriefs in Mathematical Physics 27 (2018), doi:10.1007/978-3-319-94788-4, arXiv:1902.05306 (abstract and contents only); B. Iochum, C. Levy and D. Vassilevich, *Spectral action beyond the weak-field approximation*, Commun. Math. Phys. 316 (2012) 595-613, doi:10.1007/s00220-012-1587-8 (second order in the gauge field with full momentum dependence; abstract only). The physics counterpart is decoupling: T. Appelquist and J. Carazzone, *Infrared singularities and massive fields*, Phys. Rev. D 11 (1975) 2856-2861, doi:10.1103/PhysRevD.11.2856. The evaluation already classes lemma 03 as a routine computation; a paper should present it as such.

F. Weinberg angle from a squashed S^3 / lens space and from M^{pqr}

Claim (historical/corpus/canonical_research.md §14.3). For the Berger metric with fibre scaled by xi^2, K_W = V R^2 (2 + xi^2)/3, K_Y = V R^2 xi^2, hence cos^2 theta_W = 3 xi^2/(4 xi^2 + 2); the lens order p cancels; 5/8 needs xi^2 = 5/2.

Verdict: KNOWN (same formula, same parameterization).

- A. J. Niemi and S. Slizovskiy, *On relations between electroweak hierarchy problem, fluctuating three branes and general covariance*, arXiv:1004.0212 (no journal version found on INSPIRE; earlier titles "Hiding Higgs with brane" and "Embedding Weinberg-Salam in Kaluza-Klein"). Section IV, eqs. (77)-(78): internal metric dtheta^2 + sin^2 theta dpsi_-^2 + xi^2 (dpsi_+ - cos theta dpsi_-)^2, isometry SU(2)_L x U(1)_R for xi != 1; Section V, eqs. (94)-(95): q^2 = 3/(xi^2 + 2), q'^2 = 1/xi^2; Section VI: sin^2 theta_W = (xi^2 + 2)/(4 xi^2 + 2), i.e. cos^2 theta_W = 3 xi^2/(4 xi^2 + 2). They fix xi by a custodial-symmetry condition and quote sin^2 theta_W = 0.296 (that step, in Section VIII, was not read). The ratios 1/q^2 : 1/q'^2 = (2 + xi^2)/3 : xi^2 coincide with K_W : K_Y of §14.3. The cancellation of the lens order p in the common volume factor is immediate.
- E. Witten, *Search for a realistic Kaluza-Klein theory*, Nucl. Phys. B 186 (1981) 412-428, doi:10.1016/0550-3213(81)90021-3: eleven-dimensional supergravity on the seven-dimensional M^{pqr} with SU(3) x SU(2) x U(1) isometry; gauge couplings determined by the internal metric; the chirality obstruction. Read on 3 Nov 2026 from the PDF supplied by the user, and checked by Astra 6 (Astra validation, item 1). Eq. (3): gauge fields as components of the metric along Killing vectors. Pp. 416-417: seven is the minimum internal dimension with this symmetry, so eleven in total. P. 418: M^{pqr} = G/H with H = SU(2) x U(1) x U(1) and the deleted generator p lambda_8 + q T_3 + r Y; M^{001} = CP^2 x S^2 x S^1, M^{101} = S^5 x S^2, M^{011} = CP^2 x S^3. Pp. 421-423: "One would inevitably get vector-like rather than V - A weak interactions", for a reason that "depends only on the fact that the number of extra dimensions is odd": gamma_i -> -gamma_i fixes sigma_ij and flips Gamma_7, on an orientable manifold, with the gauge group acting by coordinate transformations. Escapes considered: non-orientable internal spaces (none suitable for this symmetry, p. 423) and torsion (left open, pp. 423, 426-427). Pp. 425-426: the three parameters of the invariant metric would determine the three gauge couplings. The paper computes no weak angle, so section F cites it for the method only.
- L. Castellani, R. D'Auria and P. Fré, *SU(3) x SU(2) x U(1) from D = 11 supergravity*, Nucl. Phys. B 239 (1984) 610-652 (INSPIRE): the Einstein metrics on M^{pqr}, with parameters depending on q/(3p). Content from abstract and secondary sources.

Added 5 November 2026, for lemma 47 (the Berger five-sphere), found by the Opus referee; bibliographic data verified by Claude Fable 5.1 on INSPIRE and arXiv.

- S. Weinberg, *Charges from extra dimensions*, Phys. Lett. B 125 (1983) 265-269, doi:10.1016/0370-2693(83)91281-9: the integral formula g_a^{-2} proportional to int g(K_a, K_a) for the Kaluza-Klein gauge couplings, with the normalization point at which the coupling becomes the charge unit. Known here through Barnes et al. below; the paper itself was not read.
- E. Barnes, E. Gorbatov, K. Intriligator and J. Wright, *Current correlators and AdS/CFT geometry*, Nucl. Phys. B 732 (2006) 89-117, arXiv:hep-th/0507146 (e-print read by the referee, sections 3 and 6): the formula (g^{-2}_{IJ})^{KK} proportional to int_Y g_ab K_I^a K_J^b vol(Y); for IIB on S^5, tau_SO(6) = N^2 and tau_RR = 4 N^2/3, with the U(1)_R Killing vector of norm squared 4/9, which is the central generator of lemma 47 with c = 2/3; hence tau_SU(3)/tau_RR = 3/8, the round-point value of Corollary 47.2 in holographic form.
- M. J. Duff, C. N. Pope and N. P. Warner, Phys. Lett. B 130 (1983) 254, doi:10.1016/0370-2693(83)91136-X: the flux contribution to the Killing gauge couplings on Freund-Rubin backgrounds (title from INSPIRE; content as reported in section 3 of Barnes et al.).
- Z. F. Ezawa, I. G. Koh and S. Rajpoot, *SU(4) x SU(2) unification in Kaluza-Klein supergravity*, Phys. Lett. B 147 (1984) 284-288, doi:10.1016/0370-2693(84)90117-5 (abstract): Pati-Salam partial unification from an SU(4) x SU(2) isometry, the B - L reading of the same embedding; whether it

states the ratio was not checked.

- B. L. Hu and T. C. Shen, *Weak angle from Kaluza-Klein theory with deformed internal space*, Phys. Lett. B 178 (1986) 373-378, doi:10.1016/0370-2693(86)91396-1 (abstract): the weak angle from a deformed three-sphere as a function of the deformation, the S^3 analogue of the Berger family of lemma 47, beside Niemi-Slizovskiy above.
- The formula (3 + eta^2)/(24 eta^2 c^2) on the Berger S^5 was not found by the referee (INSPIRE titles on squashed five-spheres, Berger spheres, the weak angle and the coupling constants in Kaluza-Klein theory; web searches).

N. Real group algebras of the binary dihedral lift and the quaternions (lemma 48): prior work found by the Opus referee

Claim. Lemma 48, Proposition 48.2 and Theorem 48.3: the spinorial part of R[Dic_3] is C + H by the Frobenius-Schur types, and the two-sheet bimodule over the orbit types of the Weyl action carries C + H + M_3(C). **Verdict of the referee: the type decomposition is in print in the orientifold setting; no source deriving C + H from a binary dihedral spin lift in noncommutative geometry was found.** Bibliographic data verified by Claude Fable 5.1 on arXiv, 5 November 2026.

- J. D. Blum and K. Intriligator, *New phases of string theory and 6d RG fixed points via branes at orbifold singularities*, Nucl. Phys. B 506 (1997) 199-222, arXiv:hep-th/9705044 (e-print read by the referee, section 5 and appendix A): the irreducible representations of the finite subgroups of SU(2) sorted into real, pseudo-real and complex by the Frobenius-Schur indicator, the pseudo-real ones having the central element -1 represented as -1, and the breaking SO(N) -> prod SO x prod Sp x prod U by type; for Gamma_{D_5} = Dic_3 the real ones are the trivial, sign and standard representations, the pseudo-real one is the faithful two-dimensional representation and the complex pair has alpha^2 = -1, which is Proposition 48.2 (i), the spinorial ones giving H + C.
- J.-P. Serre, *Linear representations of finite groups*, Springer GTM 42, chapter 13: the real group algebra as a sum of matrix algebras over R, C and H by Frobenius-Schur type (textbook; not reopened).
- Noncommutative geometry: an extended web search on the Standard Model algebra with orbifold, crossed product, spin lift, binary dihedral and quaternions (titles and abstracts) found no source; the corpus construction recorded in lemma 45 is the only prior source known to the project.

G. The protected zero mode on S^5 (requested by the review)

Claim (historical/corpus/connes_kk_research_state.md, around the S^5 zero mode): for the nontrivial quaternionic line V -> S^5, dim ker D_{S^5, V} is odd (real mod-two index); for SU(3)-invariant data the invariant line is a zero mode; the text attributes the mechanism to Witten 1982, footnote 2.

Verdict: KNOWN, for the topological statement (a nontrivial SU(2) bundle on S^5 has odd mod-two index) and for the attribution. Footnote 2 of Witten 1982 (p. 327, read on 3 Nov 2026 from the PDF supplied by the user) reads: "Actually, in a special case one can easily find the zero mode. If one conformally compactifies S^4 x R to the five-sphere S^5, then on S^5 one can choose the instanton field to be invariant (up to a gauge transformation) under an SU(3) subgroup of the symmetry group O(6) of the five-sphere. The fermion zero mode is then the unique SU(3) invariant spinor field that can be defined." The wording of the corpus matches it.

- E. Witten, *An SU(2) anomaly*, Phys. Lett. B 117 (1982) 324-328, doi:10.1016/0370-2693(82)90728-6. As summarized by Wang, Wen and Witten (2019), Section 2: "The main result of [1] is that the fermion path integral for a fermion field on M in a given representation of SU(2) is phi-invariant if and only if the mod 2 index of the Dirac operator on Y – in that representation – vanishes. Here the mod 2 index is defined as the number of zero modes of the five-dimensional Dirac operator, reduced mod 2", where Y is the mapping torus of the gauge transformation phi (the mapping torus S^4 x S^1, with phi in the nontrivial class of pi_4(SU(2)) = Z_2). The original, read on 3 Nov 2026, gives the criterion for other isospins on p. 328: the fermion integration is odd exactly when tr T^2, with tr T^2 = 1/2 on the doublet, is a half-integer.

- I. García-Etxebarria and M. Montero, *Dai-Freed anomalies in particle physics*, JHEP 08 (2019) 003, doi:10.1007/JHEP08(2019)003, arXiv:1808.00009. Section 3.1, eq. (3.4): Omega_5^Spin(BSU(2)) = Z_2, detected by the mod-two index; immediately after: "A simple example with non-vanishing mod-2 index was constructed in [2] [Witten 1982]. While S^5 itself is trivial in Omega_5^Spin …, there is a bundle over it such that the total space is no longer null-bordant in Omega_5^Spin(BSU(2)) = Z_2." Section 3.1.2: SU(3)/SU(2) = S^5, so in five dimensions every SU(3) bundle reduces to an SU(2) bundle.
- Y. Sato, *Majorana fermion zero modes and anomalies in string and M theories* (PhD thesis, Tokyo, 2024), arXiv:2404.18138, Appendix C: the five-dimensional configuration of Witten's argument (S^4 x R with pure-gauge ends) compactifies to S^5; G-bundles on S^5 are classified by pi_4(G), and for G = Sp(N) the mod-two index is an isomorphism onto KO^{-1}(pt) = Z_2.
- J. Wang, X.-G. Wen and E. Witten, *A new SU(2) anomaly*, J. Math. Phys. 60 (2019) 052301, doi:10.1063/1.5082852, arXiv:1810.00844: mod-two index on mapping tori, extended to non-spin manifolds. The corpus cites it for the parity of the index at isospin j = 2k + 1/2. That parity follows from the criterion of Witten 1982: tr T^2 = j(j + 1)(2j + 1)/3 equals (4k + 1)(2k + 1)(4k + 3)/6, a half-integer, at j = 2k + 1/2, and (4k + 3)(k + 1)(4k + 5)/3, an integer, at j = 2k + 3/2. We did not reread the 2019 paper for this statement.

Answer to the review's objection. Witten's original construction uses S^4 x R, with the specified asymptotic gauge fields; footnote 2 discusses its conformal compactification to S^5. The mapping-torus formulation uses S^4 x S^1, with the gauge bundle glued by the gauge transformation. Both that mapping torus (odd index for one doublet) and the nontrivial SU(2) bundle on S^5 (García-Etxebarria-Montero, sentence after eq. (3.4)) are nonzero in Omega_5^Spin(BSU(2)) = Z_2, so they represent the same generator; the mod-two index is a bordism invariant and is odd on both. Sato's Appendix C describes the S^5 bundle as the compactification of the S^4 x R configuration of Witten's argument. García-Etxebarria-Montero state the S^5 version and attribute the example to Witten 1982. What remains specific to the corpus is the identification of V = rho^*Q with this generator via the Hopf map from CP^3 data and the SU(3)-invariant zero-mode check.

H. Anomaly inflow as the five-dimensional parent of chiral boundary modes (requested by the review)

Relevance: claim B (chiral walls of Theorem 9.8) and Corollary 9.3 (equivariant index from boundary types).

Verdict: KNOWN as background; lemma 09 contains no gauge-field inflow, so these references supply context.

- C. G. Callan and J. A. Harvey, *Anomalies and fermion zero modes on strings and domain walls*, Nucl. Phys. B 250 (1985) 427-436, doi:10.1016/0550-3213(85)90489-4. Chiral zero modes of the Dirac equation on domain walls in 2n+1 dimensions and on strings in 2n+2 dimensions; their anomalies are cancelled by inflow from the bulk through an induced Chern-Simons (topological mass) term. Content from the abstract fragment and secondary sources; full text not read.
- S. G. Naculich, *Axionic strings: covariant anomalies and bosonization of chiral zero modes*, Nucl. Phys. B 296 (1988) 837-867, doi:10.1016/0550-3213(88)90400-2: the consistent/covariant anomaly bookkeeping in inflow.
- N. Arkani-Hamed, A. G. Cohen and H. Georgi, *Anomalies on orbifolds*, Phys. Lett. B 516 (2001) 395-402, doi:10.1016/S0370-2693(01)00946-7: anomalies of bulk fermions on S^1/Z_2 localized at the fixed points (content **unverified** beyond title and abstract-level knowledge).
- Kaplan (1992) and Shamir (1993), section B, built lattice chiral fermions on this mechanism.

For the corpus: Theorem 9.2 and Corollary 9.3 are the index-theoretic shadow of Callan-Harvey (boundary type fixes the chiral zero mode); an inflow statement would require bulk gauge fields with a Chern-Simons term on the interval, which lemma 09 does not include.

I. The current Connes et al. framework (requested by the coordinator)

The corpus treats Connes-Lott 1990 as the obsolete layer. The following define the present framework. All bibliographic data verified (Crossref or arXiv) unless marked.

- **A. Connes, *Noncommutative geometry and the standard model with neutrino mixing*, JHEP 11 (2006) 081**, arXiv:hep-th/0608226. Reverses the grading on the antiparticle sector so that the finite space has KO-dimension 6, separating metric dimension (0) from KO-dimension. Theorem 2.7 classifies the Dirac operators D_F (Yukawa matrices, CKM and PMNS data, a Majorana mass); the fermion doubling problem is solved by the Pfaffian in the fermionic action. The total KO-dimension $10 = 2 \bmod 8$ is noted in Section 1.
- **J. W. Barrett, *Lorentzian version ...*, J. Math. Phys. 48 (2007) 012303**, arXiv:hep-th/0608221. Independently arrives at internal signature 6 from the Lorentzian requirement that physical fermions be simultaneous eigenvectors of J and gamma (total signature 0 mod 8), which removes the quadrupling of fermions.
- **A. H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007) 991-1089**, arXiv:hep-th/0610241. Section 2: the finite geometry, with A_F = C + H + M_3(C), Lemma 2.5 (the irreducible representations of the left-right bimodule algebra have total dimension 32 per generation), Section 2.4 (the order-one condition selecting A_F), Sections 2.6-2.7 (classification and moduli of D_F, i.e. the Yukawa parameters) and Section 2.8 (KO-dimension and Poincaré duality); N = 96 in Section 3.7.1; Section 3: the spectral action on M x F and its heat-kernel coefficients; Section 4: the full Lagrangian; Section 5: predictions at unification (gauge coupling relation, Higgs quartic, the mass relation (4.16)).
- **A. H. Chamseddine and A. Connes, *Conceptual explanation for the algebra in the noncommutative approach to the standard model*, Phys. Rev. Lett. 99 (2007) 191601**, doi:10.1103/PhysRevLett.99.191601, arXiv:0706.3690. Version 1 of arXiv:0706.3690 carried the title *A dress for SM the beggar*; the published version is this PRL (the coordinator's "PNAS 2007" does not match any record we found). Classifies irreducible finite geometries with reality and chirality and no fermion doubling, requiring total K-theoretic dimension 10.
- **A. H. Chamseddine and A. Connes, *Why the Standard Model*, J. Geom. Phys. 58 (2008) 38-47**, doi:10.1016/j.geomphys.2007.09.011, arXiv:0706.3688. Section 2: the order-zero condition; Proposition 2.5 (centre C: dim H = k^2) and Proposition 2.8 (centre C + C: dim H = 2k^2, A_C = M_k(C) + M_k(C)); Lemma 3.1: the KO-6 relation J gamma = -gamma J excludes centre C; eq. (3.1): with a quaternionic, nontrivially graded W of quaternionic dimension 2, A = M_2(H) + M_4(C), k = 4, dim H = 2k^2 = 32 per generation; Theorem 4.1: the order-one condition with an off-diagonal D selects the unique maximal subalgebra A_F = C + H + M_3(C). For the project this is the statement that D_F must connect the two components of the centre: the "two sheets" of the finite space are forced by the classification.
- **A. H. Chamseddine, A. Connes and W. D. van Suijlekom, *Inner fluctuations in noncommutative geometry without the first order condition*, J. Geom. Phys. 73 (2013) 222-234**, doi:10.1016/j.geomphys.2013.06.006, arXiv:1304.7583. Without the first-order condition, inner fluctuations acquire a quadratic term and form a semigroup depending only on A. Relevant to Amendment 2 of lemma 07 (the compressed two-point data fail the first-order condition).
- **A. H. Chamseddine, A. Connes and W. D. van Suijlekom, *Beyond the spectral Standard Model: emergence of Pati-Salam unification*, JHEP 11 (2013) 132**, doi:10.1007/JHEP11(2013)132, arXiv:1304.8050. Dropping the first-order condition, the classification gives an SU(2)_R x SU(2)_L x SU(4) model; the Standard Model is recovered when the vacuum restores the first-order condition.
- **A. Devastato, F. Lizzi and P. Martinetti, *Grand symmetry, spectral action and the Higgs mass*, JHEP 01 (2014) 042**, doi:10.1007/JHEP01(2014)042, arXiv:1304.0415. A larger "grand" algebra mixing gauge and spin degrees of freedom produces the extra scalar sigma that brings the Higgs mass near 126 GeV; requiring an almost-commutative geometry breaks it to the Standard Model.
- **A. H. Chamseddine, A. Connes and V. Mukhanov, *Quanta of Geometry: Noncommuta-***

***tive Aspects*, arXiv:1409.2471** (journal data unverified; read from the LaTeX source, 3 Nov 2026). Clifford algebras C_kappa on n + 1 gammas with kappa = +-1; in dimension 4, C_+ = M_2(H) and C_- = M_4(C), "in perfect agreement with the algebraic constituents of the Standard Model"; two maps Y_+- : M -> S^4 of slashed form, with det e = Omega_+ + Omega_-. Bears on the twistor line of canonical section 4 and on current_framework.md.

- **J. Baptista, *Chiral interactions of fermions and massive gauge fields in Kaluza-Klein models*, JHEP 05 (2026) 008, arXiv:2506.09126** (found by the referee; introduction read from the LaTeX source). The no-go arguments against chiral fermions in Kaluza-Klein models (Witten 1983; Atiyah-Hirzebruch) assume that the gauge fields are linked to exact Killing fields; gauge fields linked to non-Killing fields are massive, can mix fermions of different masses, and can couple asymmetrically to the two chiralities, with examples on even-dimensional K = S^2 and T^2. Bears on the reading of no-go results in the paper and on T4.
- **Lemma 22: the finite Dirac operator as a boundary condition.** Known: the one-flavour formula (Csáki, Grojean, Hubisz, Shirman and Terning, section C here) and the gauge boundary conditions and the form M tan(M pi R) = -V (Csáki, Grojean, Murayama, Pilo and Terning, *Gauge theories on an interval: unitarity without a Higgs*, Phys. Rev. D 69 (2004) 055006, arXiv:hep-ph/0305237, read in the LaTeX source). Not found: the real structure with a finite triple as exact window, the wall algebra C_F, the average formula for the compression. **Searched** (arXiv API, 3 Nov 2026): "noncommutative geometry" with Higgsless; "spectral triple" with "boundary conditions", Higgs, interval; "spectral triple" with Yukawa, "boundary conditions"; "almost-commutative" with interval, boundary; "spectral triple" with "Kaluza-Klein", "boundary conditions": no hits. "noncommutative geometry" with "gauge-Higgs": one hit, arXiv:2511.05909, a survey, abstract only. Absence from these queries does not establish absence. The referee adds operator-system compression in Connes-van Suijlekom (arXiv:2004.14115, sections 1, 3), and A. Chamseddine and A. Connes, *Noncommutative geometric spaces with boundary: spectral action*, J. Geom. Phys. 61 (2011) 317, arXiv:1008.3980 (read in the LaTeX source: sections 3 to 5). There the boundary is that of the manifold in a product with a finite space, with the condition Pi_- Psi = 0, Pi_- = (1 - chi)/2, for a local chirality projector chi, and the spectral action acquires the extrinsic-curvature terms. The internal boundary with finite gradings of lemmas 22 and 23 is a different setting.
- **Lemma 25: the sum rule and the effective dimension.** Known: m_t = 2 M_W at tree level in flat gauge-Higgs models with suitable representations (Cacciapaglia, Csáki and Park, arXiv:hep-ph/0510366, section 5; Adachi and Maru, arXiv:1804.06012, Eq. (3.23)). Not found in the records inspected by the referee: the coefficient (3 - d_eff)/24 and its theta-function crossover. **Searched** by the referee: web-index queries for spectral action, heat kernels, compactification and gauge-Higgs mass relations, including a 2026 query, and the reference chain from Adachi, Kurahashi and Maru (arXiv:1404.4281). A forward-citation search, an author sweep and a broader database search remain open. This does not establish priority.
- **Twisted spectral triples:** A. Connes and H. Moscovici, *Type III and spectral triples*, arXiv:math/0609703 (journal data **unverified**); G. Landi and P. Martinetti, *On twisting real spectral triples by algebra automorphisms*, Lett. Math. Phys. 106 (2016) 1499-1530, doi:10.1007/s11005-016-0880-4, arXiv:1601.00219; A. Devastato and P. Martinetti, *Twisted spectral triple for the Standard Model and spontaneous breaking of the grand symmetry*, Math. Phys. Anal. Geom. 20 (2017) 2, doi:10.1007/s11040-016-9228-7, arXiv:1411.1320. The twist bounds the unbounded vector terms of the grand-symmetry model and introduces a twisted first-order condition; the breaking to the Standard Model becomes dynamical.
- **Classification of finite triples:** T. Krajewski, *Classification of finite spectral triples*, J. Geom. Phys. 28 (1998) 1-30, doi:10.1016/S0393-0440(97)00068-5, arXiv:hep-th/9701081 (Krajewski diagrams: vertices are chiral gauge multiplets, links are Yukawa couplings); M. Paschke and A. Sitarz, *Discrete spectral triples and their symmetries*, J. Math. Phys. 39 (1998) 6191-6205, doi:10.1063/1.532623, arXiv:q-alg/9612029 (classification of 0-dimensional triples over complex and real algebras). For the project, a Krajewski link is exactly the Yukawa block that lemma 12 derives as Phi_Q Y_hat Phi_U.
- **Fermion doubling:** F. Lizzi, G. Mangano, G. Miele and G. Sparano, *Fermion Hilbert space and fermion doubling in the noncommutative geometry approach to gauge theories*, Phys. Rev. D 55 (1997) 6357-6366, doi:10.1103/PhysRevD.55.6357, arXiv:hep-th/9610035: unphysical degrees of freedom in H

= L^2(S) x H_F, analogous to lattice doublers; the 2006-2007 KO-6 framework resolves them (Connes 2006, Barrett 2007).

- **Standard reference:** W. D. van Suijlekom, *Noncommutative Geometry and Particle Physics*, Mathematical Physics Studies, Springer (2015), doi:10.1007/978-94-017-9162-5 (a second edition exists; its data were **not verified**). Covers finite spectral triples and Krajewski diagrams, almost-commutative manifolds, the spectral action, and the electrodynamics and Standard Model examples.

J. Matching in the flat parent (lemma 26)

Claim. Lemma 26: one-loop vacuum with frequency-2 channels; the left-right difference of the couplings; the meeting point; the limit on a top in low multiplets. **Verdict: PARTIAL.** Found by the referee of lemma 26 (report, item 9), with forward citation chains incomplete. We have read none of these beyond what that report quotes.

- Y. Nomura, D. Smith and N. Weiner, *GUT breaking on the brane*, arXiv:hep-ph/0104041: differential logarithmic running, integer versus half-integer towers. M. Carena, T. Tait and C. Wagner, arXiv:hep-ph/0207056: boundary kinetic terms.
- Abe, Goto and Kawamura, PTEP 2018, 103B01: an interval Pati-Salam model with a compactification threshold of order 10^13 GeV, with other intermediate group and matter.
- G. Cacciapaglia, C. Csaki and S. C. Park, arXiv:hep-ph/0510366, section 5: representation-dependent enhancement, including a tree-level m_t = 2 M_W limit. Adachi and Maru, arXiv:1804.06012, Eq. (3.23): a limit approaching 2 M_W with brane mixing, and massless exotics.
- R. Contino, L. Da Rold and A. Pomarol, arXiv:hep-ph/0612048: 5 and 10 embeddings and coloured custodians. Y. Hosotani, S. Oda, T. Ohnuma and Y. Sakamura, arXiv:0806.0480: massless exotics, their lifting by wall masses, and theta_H = pi/2 in a warped model.
- C. S. Lim and N. Maru, arXiv:0904.0304: a finite one-loop violation of gauge-Yukawa universality.

Not found by that search: the classification with n_c, and the pair (M_KK, mu_*). Priority for those exact statements stays open.

K. Fibre collapse in a mapping cylinder and compression (lemma 28)

Claim. Lemma 28: a connected commutative algebra on a mapping cylinder gives a direct sum of two matrix algebras by compression to the two ends. **Verdict: NOT LOCATED.** Found by Astra 6 in a bounded search (report); novelty stays unestablished, and author-by-author citation searches were not done. Identifiers, authors and titles below were confirmed on arXiv by Claude Opus 5.5; the located passages are as reported by Astra 6.

- F. D'Andrea, F. Lizzi and P. Martinetti, *Spectral geometry with a cut-off*, arXiv:1305.2605, section 3.1 and 6: compression of commutative geometries.
- A. Connes and W. D. van Suijlekom, arXiv:2004.14115, section 1 and Proposition 4.2: operator systems from spectral truncation, with the warning on non-informative full matrix algebras, relevant to Counterexample 28.17.
- J. Medina, I. Huet, D. O'Connor and B. Dolan, arXiv:1208.0348: squashed fuzzy CP^3 as a fuzzy S^2 bundle over fuzzy S^4, with suppression of the fibre modes.
- M. Verhoeven, *Spectral truncations of SU(2) and the spectral localizer*, arXiv:2610.19791: full matrix envelopes in a recent truncation example.
- K. Uno and T. Inagaki, *Construction and geometric properties of the fuzzy CP^3 onion*, arXiv:2610.17807: radially coupled fuzzy CP^3 layers.
- C. Bär and W. Ballmann, arXiv:1307.3021: the boundary criterion of Theorem 28.9. C. Kottke, arXiv:0909.5661: the Callias index in families, for Proposition 28.11.

- I. Prokhorenkov and K. Richardson, *Perturbations of Dirac operators*, J. Geom. Phys. 57 (2006) 297-321, arXiv:math/0307251; M. Maridakis, arXiv:1510.07004, Definition 2.1 and Lemma 2.2, and arXiv:2110.11654, section 2: concentration of low modes of D + s A at the zero set of A on closed manifolds. Read by Terra 6.1 in arXiv HTML (report): the operator of lemma 28 is a concentrating pair, and Assumption 3 of the spectral separation theorem fails for the spin-3/2 mass.
- R. Parthasarathy, *Dirac operator and the discrete series*, Ann. of Math. 96 (1972) 1-30, doi:10.2307/1970892, bibliographic data from Crossref: the formula D^2 = Cas_G + R/8 - Cas_K(twist) on a symmetric space, used in lemma 29. Text not read here; the formula is tested on the slice blocks in Python.
- K.-D. Kirchberg, *An estimation for the first eigenvalue of the Dirac operator on closed Kähler manifolds of positive scalar curvature*, Ann. Global Anal. Geom. 4 (1986) 291-325, doi:10.1007/BF00128050, bibliographic data from Crossref: the limit (m + 1) R / (4 m) for odd complex dimension m, which is R/3 on CP^3, used as a check of the normalization in Lemma 29.5. Text not read here.
- Found by the Terra referee of lemma 29, identifiers checked on arXiv by Claude: J. Kaad and W. D. van Suijlekom, *Factorization of Dirac operators on almost-regular fibrations of spin^c manifolds*, arXiv:1710.03182, Proposition 18: a horizontal and vertical factorization with an explicit curvature contraction, under hypotheses that differ from the noncompact model of lemma 29; a close structural ancestor of Proposition 29.1. I. Prokhorenkov and K. Richardson, *Witten deformation and the equivariant index*, arXiv:math/0610160, section 3: equivariant localization and normal models. The referee reports that the search found general identities and did not establish whether the eight-state theorem has appeared elsewhere; it did not exhaust Bismut-Lebeau or the citation chains.
- Standard-mode web searches of 4 Nov 2026, by Claude Opus 5.5, for two statements of the 11D line: M^{111} as the hypercharge quotient of the 5-bar (11D/02, Theorem 2.1), and the SO(30) of the light quarks as the commutant of a real structure (11D/11). Neither search returned a source with the statement. The quotient with weights 2, -3 itself is in D. Fabbri et al., hep-th/9907219. These searches were thin and do not establish priority.

L. The eight-state window on the disc bundle (lemma 29): prior work found by the second Terra reading

Claim. Lemma 29, Theorem 29.7: eight eigenvalues in (-delta, delta) for the doubled operator on the disc bundle, with the S^4 core, the CP^3 wall, the spin-3/2 mass and the local boundary condition. **Verdict of the referee: NOT FOUND as a combined statement; the end-model spectra and the methods are in print.** Identified by GPT-6.1 Terra (report, 5 November 2026), who read the cited passages. The bibliographic data below were verified by Claude Fable 5.1 on 5 November 2026 on the publisher pages and arXiv abstracts (and, for Dolan, in the PDF: section 2(ii) on S^4 with the homogeneous instanton, equations (40) and (46) present); the passages themselves have not been reread in this project. By the rule on obscure references, any of these used in a paper is to be developed in full there, with theorem number, version and DOI.

- M. Ben Halima, *Spectrum of twisted Dirac operators on the complex projective space P^{2q+1}(C)*, Comment. Math. Univ. Carolin. 49 (2008) 437-445, DML-CZ 119734, Theorem 1: the spectrum of the Dirac operator on odd complex projective spaces twisted by powers of the tautological line, with scalar curvature 3; after the conversion m = -k and the factor 4 for scalar curvature 12 it reproduces Lemma 29.5 (zero and gap 5 on O(3), gap 3 on O(1)).
- B. P. Dolan, *The spectrum of the Dirac operator on coset spaces with homogeneous gauge fields*, JHEP 05 (2003) 018, arXiv:hep-th/0304037, section 2(ii), equations (40)-(46): the S^4 instanton spectrum of Proposition 29.2 (four zero modes at isospin 1, next squared value 5, untwisted spectrum from 4).
- M. Maridakis, arXiv:1510.07004, Lemma 2.2 (the zero-order cross-term identity of Lemma 29.4), and arXiv:2110.11654, sections 2 and 4-8 (normal-model localization and spectral separation; its Assumption 3, direction-independent leading squared data, fails for (n . J)^2 here). Listed in section K; the referee's reading of the passages is new.

- J. Kaad and W. D. van Suijlekom, arXiv:1710.03182, Proposition 18: a horizontal-vertical factorization of Dirac operators with a curvature contraction, under different fibration hypotheses (cf. Proposition 29.1).
- J.-M. Bismut and G. Lebeau, *Complex immersions and Quillen metrics*, Publ. Math. IHES 74 (1991) 1-298, doi:10.1007/BF02699352, sections VIII-IX, equations (9.155)-(9.156): model-space projection and spectral-counting arguments of the kind used in Theorem 29.7, in the complex-immersion setting.
- D. Le Peutrec, *Small eigenvalues of the Neumann realization of the semiclassical Witten Laplacian*, Ann. Fac. Sci. Toulouse Math. 19 (2010) 735-809, doi:10.5802/afst.1265, sections 3-4: boundary localization and exponential estimates with a boundary condition.
- G. J. Parker, *Concentrating Dirac operators and generalized Seiberg-Witten equations*, arXiv:2307.00694, Theorem 1.1: exponential concentration and decay for Dirac operators D + eps^{-1} A with a zeroth-order perturbation vanishing on a subbundle, in the Seiberg-Witten and Z_2-harmonic-spinor context.
- H. Schulz-Baldes and T. Stoiber, *Spectral localization for semimetals and Callias operators*, arXiv:2203.15014, sections 3 and 5: IMS localization with explicit local models for isolated Dirac or Weyl points, and a spectral localization result for Callias operators.
- C. Kottke, arXiv:0909.5661, section 5: the families Callias index (already in section K for Proposition 28.11); the passage from a bounded potential at infinity to the linear normal model is to be matched explicitly.

M. The bidoublet Yukawa block, its quaternionic half and the up-down splitting (lemma 46): prior work found by the Opus referee

Claim. Lemma 46, Propositions 46.1 and 46.2 and Corollary 46.4: the odd Hermitian blocks of the quaternionic sheet are the complex bidoublet; the quaternionic half gives equal masses to the two members of a doublet, the anti-quaternionic half splits them, m_+-^2 = |q|^2 + |q'|^2 +- 2 |Im(conj(q) q')|; the custodial and hypercharge stabilizers. **Verdict of the referee: KNOWN in bidoublet form; the closed formula in the quaternion variables (q, q'), the identification of the second half with the fibre-odd functions on CP^3 and the stabilizer reading in the window were NOT FOUND.** Identified by Claude Opus 5.5 (report, 5 November 2026), who read the arXiv e-print sources where stated; the bibliographic data below were verified by Claude Fable 5.1 on the arXiv abstract pages on 5 November 2026.

- A. H. Chamseddine, A. Connes and W. D. van Suijlekom, *Beyond the spectral Standard Model: emergence of Pati-Salam unification*, JHEP 11 (2013) 132, arXiv:1304.8050 (e-print read by the referee): in the section on the Dirac operator and inner fluctuations on H_R + H_L + M_4(C), the right-left block after fluctuation is (k^nu phi + k^e phi-tilde) Sigma + (k^u phi + k^d phi-tilde)(1 - Sigma), with phi in (2_R, 2-bar_L, 1) and phi-tilde = sigma_2 conj(phi) sigma_2 = j phi j^{-1}; in the unified case k^nu = k^u, k^e = k^d it is (k^u phi + k^d phi-tilde) (x) 1, the block (x) 1_V of lemma 46, whose quaternionic and anti-quaternionic halves are (k^u + k^d)(phi + phi-tilde)/2 and (k^u - k^d)(phi - phi-tilde)/2 for real couplings.
- Y. Zhang, H. An, X. Ji and R. N. Mohapatra, *General CP violation in minimal left-right symmetric model and constraints on the right-handed scale*, Nucl. Phys. B 802 (2008) 247-279, arXiv:0712.4218 (e-print read by the referee, section III): the Yukawa Q-bar_L (h phi + h-tilde phi-tilde) Q_R and the mass matrices M_U = kappa h + kappa' e^{-i alpha} h-tilde, M_D = kappa' e^{i alpha} h + kappa h-tilde, the general bidoublet form of which Proposition 46.2 is a one-generation restatement. The original references, R. N. Mohapatra and G. Senjanovic, Phys. Rev. D 23 (1981) 165, were not checked.
- S. Willenbrock, *Symmetries of the Standard Model*, TASI 2004 lectures, arXiv:hep-ph/0410370 (e-print read by the referee, section 3): the Higgs as the matrix field Phi -> L Phi R^dagger under SU(2)_L x SU(2)_R with U(1)_Y inside SU(2)_R, the vacuum proportional to the identity preserving the custodial SU(2)_{L+R}, and Exercise 3.5, the t, b Yukawa couplings have a custodial symmetry in the limit m_t = m_b; the quaternionic Phi is the j-linear half of lemma 46 and the stabilizers of Corollary 46.4 are this breaking pattern. The lectures cite P. Sikivie, L. Susskind, M. B. Voloshin and V. I. Zakharov, Nucl. Phys. B 173 (1980) 189, not checked directly.
- M. Aa. Solberg, *On the terms violating the custodial symmetry in multi-Higgs-doublet models*, J. Phys.

G 40 (2013) 065001, arXiv:1207.5194 (abstract): custodially symmetric Higgs-quark couplings with aligned vacua give up-down mass degeneracy, which fails for misaligned vacua; in lemma 46's terms, misaligned vacua produce a j-antilinear half.

- H. E. Haber and D. O'Neil, *Basis-independent methods for the two-Higgs-doublet model III: the CP-conserving limit, custodial symmetry, and the oblique parameters S, T, U*, Phys. Rev. D 83 (2011) 055017, arXiv:1011.6188 (abstract): in the Standard Model the custodial symmetry is broken by the hypercharge gauge interactions and by the Yukawa couplings, while the Higgs potential preserves it.
- The identification of the rank-one 2 x 2 matrices with the null vectors of the complex-bilinear norm on C^4 = H (x) C (det X = |q|^2 - |q'|^2 + 2 i q . q') is classical two-spinor calculus; no identifier was checked.

O. The quaternionic sheet as a spinor fibre and the spectral Standard Model (lemma 49): prior work found by the Opus referee

Claim. Lemma 49: the spinor fibre of the four-torus is the quaternionic sheet; the lifted Weyl stabilizer spans C + H with the lifted transposition as the T_3R circle; the blocks commuting with that circle are the diagonal Yukawa blocks; the whole fibre carries the quark summand of one generation. **Verdict of the referee: the algebraic content as statements about the spectral Standard Model is in Chamseddine, Connes and Marcolli, and the two doublets as the two chiralities of Spin(4) spinors are in Baez and Huerta; the realization through the spin lift of the Weyl stabilizer on a four-torus was NOT FOUND.** Bibliographic data verified by Claude Fable 5.1 on arXiv, 5 November 2026.

- A. H. Chamseddine, A. Connes and M. Marcolli, *Gravity and the standard model with neutrino mixing*, Adv. Theor. Math. Phys. 11 (2007) 991-1089, arXiv:hep-th/0610241 (e-print source of version 1 read by the referee; numbering from the source). Proposition 2.11: A_F = {(lambda, q_L, lambda, m)} inside A_LR = C + H_L + H_R + M_3(C), every embedding of C in H being conjugate to the chosen one; Definition 2.14 and the equation labelled matrixaction: lambda on the up line and conj(lambda) on the down line of the right-handed doublet; Remark 2.19 and Definition 2.20: D_F commutes with C_F = {(lambda, lambda, 0)}, so that the photon remains massless; Theorem 2.21 with the operators D(Y): the quark Yukawa block diag(Y_up, Y_down); Proposition 2.30 and Corollary 2.31: the quark summand H_F^(3), its pairing [[0, 0, 2], [0, 0, -2], [-2, 2, 0]] (times the number of generations) and the nondegenerate pairing from the lepton and quark classes together. These are the statements that Propositions 49.2 (ii), 49.3 and 49.5 realize on the torus.
- J. C. Baez and J. Huerta, *The algebra of grand unified theories*, Bull. Amer. Math. Soc. 47 (2010) 483-552, arXiv:0904.1556 (e-print source of version 2 read by the referee). Section 3.4, Theorem 4: Spin(4) = SU(2) x SU(2) on the Dirac spinors Lambda C^2, the left doublet on Lambda^1 C^2 and the right doublet on Lambda^0 + Lambda^2, inside the Pati-Salam representation Lambda C^2 (x) Lambda C^3; section 4, Theorems 8 and 9: the Standard Model group as the subgroup of Spin(10) preserving a complex structure, a volume form and a splitting R^4 + R^6. The two doublets as the two chiralities of a four-dimensional Euclidean spinor space are in print there; lemma 49 realizes the four-dimensional part on the torus A with its complex structure.
- A. H. Chamseddine, A. Connes and V. Mukhanov, *Quanta of geometry: noncommutative aspects*, arXiv:1409.2471 (section I above, not reopened): M_2(H) as the Clifford algebra of five gammas of square one, with the gammas internal and the four-manifold space-time.
- Orbifold GUTs, for the local symmetry depending on the fixed locus (abstract pages): Y. Kawamura, Prog. Theor. Phys. 105 (2001) 999, arXiv:hep-ph/0012125; A. Hebecker and J. March-Russell, Nucl. Phys. B 613 (2001) 3, arXiv:hep-ph/0106166. The mechanism differs: there the local group is the centralizer of the twist in the gauge group, here the algebra is the span of the lifted stabilizer.
- Searched without a hit by the referee (titles and abstracts): the Standard Model algebra with spin lift, finite subgroup, binary dihedral, torus orbifold and noncommutative geometry; right-handed fermions with the canonical bundle, Kähler surfaces and SU(2)_R. Hits with different constructions: arXiv:2008.01629, arXiv:0804.2102, arXiv:2508.02747 (excerpt only). The search does not establish absence.

P. The top block as the Higgs vector times the Majorana projector, the compressed height functions and the quark-lepton split (lemma 50): prior work found by the Opus referee

Claim. Lemma 50: the rank-one Yukawa block with the eigenline condition c(h) P_w; Q(5 w'^* B w'/w'^* w') = B + tr(B) on the level-one kernel of CP^3, so the projectors on the marked lines and the grading are compressions of height functions; a Yukawa block split along projections of the second sheet keeps order one exactly when the projections commute with the second-sheet algebra; the centralizer of the top-only operator is C + M_3(C). **Verdict of the referee (Claude Opus 5.5, 5 November 2026, e-prints read): the Yukawa block as a quaternionic Higgs times diag(Y_u, Y_d) is KNOWN (Chamseddine-Connes-Marcolli, Proposition 3.5 and Theorem 2.21); the projection criterion is KNOWN as Krajewski's first-order rule; the alignment of the Majorana line, the right-handed line of the Higgs bidoublet and the lepton line is KNOWN as the Standard Model vacuum of the spectral Pati-Salam model (arXiv:1304.8050, section 8); the compression identity is Example 16.9, item 2, a case of Berezin's symbols; NOT FOUND: J_w as the complex structure of the Majorana point with the hypercharge sign, the three-form dual -J_w h/2, and the order-one forcing of the shared lepton line (Proposition 50.3 (5)).** Bibliographic data verified by the referee on Crossref where stated.

- T. Krajewski, *Classification of finite spectral triples*, J. Geom. Phys. 28 (1998) 1-30, doi:10.1016/S0393-0440(97)00068-5, arXiv:hep-th/9701081 (e-print read by the referee): section 3.4, the first-order operators on a bimodule over sums of matrix algebras are sums of a left-linear and a right-linear operator, D = Delta + J Delta J^{-1}; sections 4.1 and 4.2, the edges of a diagram are vertical or horizontal, Delta_{ik}^{jl} = delta_{kl} M_{ij,k} (x) I_{n_k}. Proposition 50.3 is this rule in the two-sheet bimodule with the grading in the left algebra.
- A. H. Chamseddine, A. Connes and M. Marcolli, arXiv:hep-th/0610241 (e-print mc2.tex read by the referee, numbering recomputed from the counters): Definition 2.14 and equation (matrixaction) (lambda on the up arrow, conj(lambda) on the down arrow); Remark 2.18 (the up arrow is nu and u, the down arrow e and d); Proposition 2.11 (A_F = {(lambda, q_L, lambda, m)}); equation (TmapsMR) (the Majorana block on up_R (x) 1^0); Theorem 2.21 (the operators D(Y) with separate lepton and quark blocks and the condition [D, C_F] = 0 of Definition 2.20); Proposition 3.5 and Lemma 3.6 (the discrete inner fluctuation as a quaternion-valued Higgs H, the up and down couplings related by H -> jH, Tr((D_3^{(0,1)})^2) = 12 |1 + H|^2 Tr(Y_u^* Y_u + Y_d^* Y_d)): the blocks c(n)(Y_u (x) P_w + Y_d (x) P_{jw}) of Proposition 50.1 (4) with 1 + H the quaternion c(n) up to normalization.
- A. H. Chamseddine and A. Connes, *Why the Standard Model*, J. Geom. Phys. 58 (2008) 38-47, arXiv:0706.3688 (e-print read by the referee): section 4, Theorem 4.1, Lemma 4.2 and Theorem 4.3, the maximal subalgebra admitting off-diagonal Dirac operators with order one, Theorem 20.2 here.
- A. H. Chamseddine, A. Connes and W. D. van Suijlekom, *Beyond the spectral Standard Model: emergence of Pati-Salam unification*, JHEP 11 (2013) 132, arXiv:1304.8050 (e-print, a gzipped tar with patisalam2.tex and six figures, read on 5 November 2026 by Claude Fable 5.1 and by the referee): section 4, "Summary of tensor notation", the tensor form of D_F with k^nu, k^e on the lepton line and k^u, k^d on the colours, the sentence that this is the unique possibility consistent with the first-order condition on the subalgebra (citing *Why the Standard Model*), and "the special case of lepton and quark unification" k^nu = k^u, k^e = k^d; section 5, "Dirac operator and inner fluctuations on H_R + H_L + M_4(C)", the field Sigma_I^J in (1, 1, 1 + 15), whose vacuum value is the projector on the lepton line, decoupling in the unified case; section 3, the first-order condition with the Majorana mass restricting the Pati-Salam algebra to the subalgebra, and the initial Dirac operator satisfying first order relative to the subalgebra only; section 8, equation SM-vev, the Standard Model vacuum = v delta^{1.} delta_1, = u delta_1 delta^1, = w delta^{1.} delta_1, with the Majorana field Delta and the bidoublet phi sharing the right-handed index and Delta sharing its Pati-Salam index with the lepton line of Sigma, the alignments of Propositions 50.1 and 50.3 as the chosen vacuum, with a nonnegative Hessian in the Delta directions. Also A. H. Chamseddine, A. Connes and W. D. van Suijlekom, arXiv:1507.08161 (e-print read by the referee): the table of Pati-Salam scalars for a first-order Dirac operator, with Sigma decoupling under quark-lepton coupling unification.
- M. F. Atiyah, N. J. Hitchin and I. M. Singer, *Self-duality in four-dimensional Riemannian geometry*, Proc. R. Soc. Lond. A 362 (1978) 425-461, doi:10.1098/rspa.1978.0143 (Crossref; content not read):

the twistor correspondence, the points of the fibre over p parametrizing the orthogonal complex structures of T_p S^4 of one orientation; J_w of Proposition 50.1 is that complex structure at [w], with J_{jw} = -J_w (computed by the referee).

- F. A. Berezin, *Covariant and contravariant symbols of operators*, Math. USSR-Izv. 6 (1972) 1117-1151, doi:10.1070/IM1972v006n05ABEH001913; *General concept of quantization*, Comm. Math. Phys. 40 (1975) 153-174, doi:10.1007/BF01609397 (bibliographic data verified on Crossref by the referee; content not read): the covariant and contravariant symbols. Proposition 50.2 is the level-one case on CP^3, which is Example 16.9, item 2, extended complex-linearly: the contravariant symbol of B is 5 times its covariant symbol minus tr(B), and in general at level one on CP^{N-1}, Q((N + 1) sigma_B - tr(B)) = B.
- F. Besnard, arXiv:2011.02708 (abstract only, found by the referee's search): a weak order-one condition that removes the Majorana mass in the Pati-Salam model.
- The custodial reading of the two halves and the bidoublet mass matrices: section M above (lemma 46). The retained algebra C + M_3(C) of a dominant top: the corpus row rs_F1 (historical/corpus/connes_kk_research_state.md, section 7.2), a conditional algebraic derivation with the embedding unspecified; Proposition 50.4 fixes it.
- Searches by the referee without a relevant hit (titles and abstracts, arXiv API): twistor and "spectral triple"; "Krajewski diagrams" and Yukawa; "Berezin transform", "projective space" and eigenvalues; twistor, noncommutative, Higgs and quaternion; "Pati-Salam", spectral, Yukawa and noncommutative. The Higgs-vector form tied to the twistor fibre and the compressed-height-function reading were not found.

Q. The torus Dirac operator with a Wilson line and a torsion as the Yukawa block (lemma 51)

Claim. Lemma 51: on the flat four-torus the Dirac operator with a constant connection and a constant three-form is c(k + A) + i c(eta) Gamma on each momentum sector, its lowest window is the Yukawa block of lemma 46 with the Wilson line as the custodial half and the torsion as the splitting, the photon condition is the invariance under a transposition of the Weyl group, and the rank-one pattern is the torsion equal to the quarter turn of the Wilson line. **Verdict (Claude Fable 5.1, 5 November 2026; refereed by Claude Opus 5.5, identifiers verified on Crossref and the arXiv API, e-prints of Agricola and Friedman-Morgan-Witten read): the two ingredients are classical (Wilson lines shifting internal momenta; Dirac operators with skew torsion); their reading as the two halves of a Yukawa block, the Weyl-invariance derivation of the photon condition and the quarter-turn characterization of the heaviest quark were NOT FOUND.**

- Y. Hosotani, *Dynamical mass generation by compact extra dimensions*, Phys. Lett. B 126 (1983) 309-313, doi:10.1016/0370-2693(83)90170-3 (title and pages verified in Crossref): fermion momenta shifted by the Wilson line, (n + q a)/R on a circle; Proposition 51.1 is the four-torus form with the torsion added. Gauge-Higgs unification and magnetized tori, from abstracts found by the referee: Y. Hosotani, S. Noda and Y. Sakamura, arXiv:hep-ph/0601241; K. Matsumoto and Y. Sakamura, arXiv:1602.01994 (Yukawa couplings on T^2/Z_N with magnetic fluxes and Wilson-line phases); D. Cremades, L. E. Ibáñez and F. Marchesano, arXiv:hep-th/0404229 (Yukawa couplings from magnetized tori); K. Benakli, arXiv:hep-th/9509115 (Scherk-Schwarz and Wilson lines). Fermion masses as momenta shifted by Wilson lines, and Yukawas from Wilson lines with fluxes on tori, are in print there.
- I. Agricola, *The Srní lectures on non-integrable geometries with torsion*, Arch. Math. (Brno) 42 (2006) 5-84, arXiv:math/0606705 (e-print read by the referee): the spinor connection nabla^{s'} = nabla^g + s' (the contraction of T with X) has torsion 4 s' T and Dirac operator D^{s'} = D^g + 3 s' T, so 3/4 for the connection with torsion T and 1/4 for torsion T/3, Bismut's rescaling and, on naturally reductive spaces, Kostant's cubic Dirac operator; the constant s != 0 enters no statement of lemma 51. J.-M. Bismut, *A local index theorem for non-Kähler manifolds*, Math. Ann. 284 (1989) 681-699, doi:10.1007/BF01443359, and T. Friedrich and S. Ivanov, *Parallel spinors and connections with skew-symmetric torsion in string theory*, Asian J. Math. 6 (2002) 303-336, doi:10.4310/AJM.2002.v6.n2.a5, arXiv:math/0102142, for the same operators and their string origin (identifiers verified in Crossref by the referee; abstracts). T. Kimura and P. Yi, arXiv:hep-th/0605247 (abstract): in heterotic flux compactifications the Dirac operator of the gauge fermions uses the connection with torsion one third,

the normalization s = 1/4 of lemma 51, and a closed H != 0 forces a singular compactification.
- The identity c(e_a) c(e_b) c(e_c) = +- Gamma c(e_d) in four dimensions is standard Clifford algebra; checked numerically in lemma 51.
- Flat connections on an elliptic curve: E. Looijenga, *Root systems and elliptic curves*, Invent. Math. 38 (1976) 17-32, doi:10.1007/BF01390167 (Crossref gives 1976; Friedman, Morgan and Witten cite 1977); R. Friedman, J. Morgan and E. Witten, *Vector bundles and F theory*, Commun. Math. Phys. 187 (1997) 679-743, doi:10.1007/s002200050154, arXiv:hep-th/9701162, e-print read by the referee: section 2.1, the SU(n) case, unordered n-tuples of points of E with sum zero, the moduli P H^0(E, O(np)) = CP^{n-1}; section 3.1, (T x T)/W for simply connected G; section 4, Hom(L' ', E) = L' (x) E with L'' the weight lattice. So A = Lambda_coroot (x) E and A/S_3 = CP^2; recorded as a lead in lemma 51.
- For the photon condition and the Yukawa blocks of the spectral Standard Model: section O (lemma 49) and section P (lemma 50).

R. The lepton generation in three views (lemma 52)

Claim. Lemma 52: the lepton sub-bimodule with the connector identifying the two copies of C and its nondegenerate pairing over C + H; the Majorana spinor of nine dimensions as one lepton generation; three generations as the flavour adjoint under U(1)_em. **Verdict (Claude Fable 5.1, 5 November 2026; refereed by Claude Opus 5.5, identifiers verified, the e-print of Chamseddine-Connes-Marcolli read): the Clifford decomposition and the lepton summand with its pairing are in print; the identification of the connector's order-one condition with the identification of the two copies of C, under the eigenline condition, the quantized momenta of the Majorana field as the custodial half, and the isomorphism of U(1)_em-modules were NOT FOUND as stated.**

- A. H. Chamseddine, A. Connes and M. Marcolli, arXiv:hep-th/0610241 (section O above): Proposition 2.11 (the identification of the two copies of C by the order-one condition for an operator between particles and antiparticles, which Proposition 52.1 (ii) realizes with the connector of Theorem 20.2), Proposition 2.30 (3) prints the matrices [[0, 2, 0], [-2, 0, 0], [0, 0, 0]] on H_F^(1) and [[0, 0, 2], [0, 0, -2], [-2, 2, 0]] on H_F^(3) per generation, the same as Proposition 52.1 (iii); Corollary 2.31 obtains nondegeneracy over the three summands from the pair of classes, and Remark 2.32 reads it as KO-homology generated by two elements; the rational nondegeneracy over C + H of the lepton matrix is read off Proposition 2.30; Proposition 2.11 states A_F as unique up to an automorphism of A_LR (e-print read by the referee). The Majorana conjugation and the bilinear table of signature (1, 8) were checked by the referee on an explicit representation (C symmetric; C Gamma^{(3)} antisymmetric).
- Spinors in nine and ten dimensions: J. Polchinski, *String Theory*, vol. 2, appendix B (Majorana and Weyl conditions by dimension and signature; the decomposition of the 16 under SO(1,3) x SO(6)); M. B. Green, J. H. Schwarz and E. Witten, *Superstring Theory*, vol. 2, chapter 15 (the Majorana-Weyl spinor of ten dimensions on M^4 x K as a left-handed Weyl spinor tensored with the internal spinor); from the record of this session, not reopened; the nine-dimensional case is the restriction to Spin(1,8) and was checked numerically in lemma 52.
- L. Alvarez-Gaumé and E. Witten, *Gravitational anomalies*, Nucl. Phys. B 234 (1984) 269-330, doi:10.1016/0550-3213(84)90066-X (Crossref): the Majorana mass bilinear by dimension (the corpus row rs_F3; content not reopened).
- A. Rivero, *An interpretation of scalars in SO(32)*, Eur. Phys. J. C 84 (2024) 1058, doi:10.1140/epjc/s10052-024-13368-3, arXiv:2407.05397, and *Supersymmetry with composite bosons*, hep-ph/0512065 (ledger rows d11_so32_comparison and l21_so30; arXiv API): the meson counts of the five light quarks and their match with the sleptons; Proposition 52.3 records the branching and the isomorphism of U(1)_em-modules.

Sources read on 4 November 2026

Read by Claude Opus 5.5 from PDFs supplied by the user (Banks-Casher, Dashen) and from the scan held by INSPIRE (Witten 1983).

- T. Banks and A. Casher, *Chiral symmetry breaking in confining theories*, Nucl. Phys. B 169 (1980)

103-125, doi:10.1016/0550-3213(80)90255-2, INSPIRE 143630. The paper works mainly with a second-order lattice formalism and a strong-coupling picture of trees. The spectral statement is a remark in Section 5, item (iv), p. 122: eq. (49), the integral of over spacetime equals the limit m -> 0 of the average of the integral of d mu rho(mu; A) m/(mu^2 + m^2), with rho the spectral function of D(A); eq. (50), the condition rho(mu; A) -> const as mu -> 0, on a finite fraction of the configurations. The paper states no factor pi, no division by the volume and no order of the limits m -> 0 and V -> infinity. Those belong to later statements, such as Giusti-Lüscher.

- R. Dashen, *Some features of chiral symmetry breaking*, Phys. Rev. D 3 (1971) 1879-1889, doi:10.1103/PhysRevD.3.1879, INSPIRE 61166. Section II.B, pp. 1881-1882: Theorem 1, the vacuum of H_0 + epsilon H' makes the expectation value of the rotated perturbation F(omega) stationary and at least a local minimum at omega = 0; Theorem 2, the same for the epsilon -> 0 limit of the vacuum. Whether omega = 0 is a global minimum is stated as a conjecture, with the existence of a global minimum of F noted separately. The ferromagnet in an external field is the worked example (eq. 15). This is the source for vacuum alignment; the stationarity calculation of lemma 17 is a separate derivation.
- E. Witten, *Fermion quantum numbers in Kaluza-Klein theory*, preprint Princeton, October 1983, 78 pages, in Shelter Island II (1985) 227-277, INSPIRE 193975, scan. Preprint page numbers. Pp. 4-5: bare masses of quarks and leptons are forbidden by gauge invariance, so they cannot have masses of order the Planck mass, and "we don't understand why some of the fermions are so much lighter than M_W". P. 7: if all gauge fields are part of the metric tensor, the Atiyah-Hirzebruch theorem gives vector-like quantum numbers for the Dirac operator on a compact B; with elementary gauge fields, complex representations are "possible but still subtle". P. 8, eq. (2): "We see that D^(n) is the mass operator, in effect. Its eigenvalues are observed in four dimensions as the particle masses. Its zero eigenvalues are the massless fermions." Pp. 43-44: the character-valued index is invariant under equivariant cobordism, so if M bounds a manifold to which the G action extends, it vanishes for any operator on M. This is the statement behind the balance on the boundary CP^3 of lemma 28: the summands of E restrict with indices 4, 0, 0, -4 and total zero. The hypotheses are a compact B without boundary and no mass profile; lemma 28 leaves them through the wall and through the Callias potential, with E as an elementary gauge bundle.

Items not verified in this session

- Castellani-D'Auria-Fré 1984 (section F): equation-level content not read.
- Banks-Casher 1980: read on 4 Nov 2026 from the PDF supplied by the user, in the passages below. The value pi rho(0) and the order of limits in the chiral limit note come from L. Giusti and M. Lüscher, JHEP 03 (2009) 013, arXiv:0812.3638, Section 2 (LaTeX source), which counts the modes of one flavour and states the order of limits lambda -> 0, m -> 0, V -> infinity. The original writes neither.
- Witten 1983: only the passages recorded below were read; the proofs of sections III to VI were not checked.
- Coquereaux-Esposito-Farèse-Vaillant 1991, Dubois-Violette-Kerner-Madore 1990 (section A): bibliographic data from secondary citations, content from secondary sources.
- Callan-Harvey 1985 and Arkani-Hamed-Cohen-Georgi 2001 (section H): full texts not read.
- Demmel et al. 1999: theorem numbers for the D_1 X D_2 results.
- Niemi-Slizovskiy, Section VIII (custodial fixing of xi).
- Connes-Moscovici journal reference; second edition of van Suijlekom's book.

Summary table

Claim	Verdict	Key references
A. Connes-Lott as minimal truncation of the chiral interval,	m	= pi/(2l)

Claim	Verdict	Key references
B. l -> infinity (walls, 1D Weyl law) and l -> 0 with mu(l)	PARTIAL: wall modes and exponentially small mixing known; NCG statement and tuned collapse not found	Jackiw-Rebbi 1976; Kaplan 1992; Shamir 1993 §3; Csáki et al. 2004; Farsi-Latrémolière 2026 Cor. 4.1
C. Towerless textures Phi_Q Y_hat Phi_U, secular equation, one-flavour spectrum	PARTIAL: one-flavour spectrum KNOWN (Csáki et al. eq. (5.6)); multi-flavour warped analogue KNOWN (Casagrande et al. eqs. (65), (87), (90), (94)-(97)); NCG derivation of D_F not found	Csáki et al. 2004 §5.1, App. C.2; Casagrande et al. 2008 §3; Kaplan-Tait 2001; Arkani-Hamed-Schmaltz 2000
D. KO-6 = (odd) x (Clifford point) criterion; s from U; nu_L; reflection sign	PARTIAL: products, J vs J gamma, odd x odd product and 10 = 2 mod 8 known; factorization criterion and the 9/11 results not found	Dąbrowski-Dossena 2011 §4.3, eq. (25), Table 6; Vanhecke 1999; Vanhecke-da Silva-Sigaud 2012 §2; Farnsworth 2017; Connes 2006 §1; Barrett 2007
E. 8 M_W^2 = sum N_f m_f^2; m_t/M_W = sqrt(8/3); heavy weights	KNOWN (mass relation, top ratio); NOT FOUND (exact f(m^2/Lambda^2) weights)	CCM 2007 eq. (1.5), Lemma 4.6 eq. (4.16), §5.4 eqs. (5.24)-(5.29)
F. cos^2 theta_W = 3 xi^2/(4 xi^2 + 2)	KNOWN	Niemi-Slizovskiy 2010 eqs. (94)-(95), §VI; Witten 1981 (method)
G. Odd mod-two index on S^5	KNOWN (topology and footnote attribution)	García-Etxebarria-Montero 2019 §3.1; Witten 1982; Wang-Wen-Witten 2019
H. Anomaly inflow	KNOWN (background)	Callan-Harvey 1985; Naculich 1988; Arkani-Hamed-Cohen-Georgi 2001
I. Current framework	reference list	Connes 2006; Barrett 2007; CCM 2007; Chamseddine-Connes 2007, 2008; CCvS 2013 (two papers); Devastato-Lizzi-Martinetti 2014; Krajewski 1998; Paschke-Sitarz 1998; van Suijlekom 2015

Part R1 (Counterexample R1 to Farsi-Latrémolière, Theorem 3.4)

Counterexample R1: Theorem 3.4 of Farsi-Latrémolière needs a hypothesis it does not state

Source. C. Farsi and F. Latrémolière, *Collapse in noncommutative geometry and spectral continuity*, arXiv:2404.00240 (v1 30 Mar 2024; v2 and v3 Oct 2025, identical sources), J. London Math. Soc. (2) 114 (2026) e70628, doi:10.1112/jlms.70628. The theorem is Theorem 3.4 in v3 and in print, Theorem 2.5 in v1. Found by Claude Opus 5.5 on 3 November 2026 while reading the paper for lemma 09; confirmed analytically by Astra 6 (hypotheses (1) to (6) and compact resolvent checked one by one) and by Terra 6.1 (item 49 of its report on the main draft); the published text was compared word by word with v3 and keeps the statement and the defective step, printed as equation (3.11). Ledger row `fl_R1`. The main paper develops the piece used in its own text, by the rule that an obscure reference cannot carry a step of the argument.

The statement in question. For a metric spectral triple (A, H, D) with D = D_h + D_v, D_v self-adjoint with 0 isolated in its spectrum, D_eps = D_h + D_v/eps, a unital C*-subalgebra B of A and the

projection p onto ker D_v, under six hypotheses ((1) ||[D_h, a]|| <= ||[D_eps, a]||; (2) ||[D_v, a]|| <= M eps ||[D_eps, a]||; (3) [D_v, b] = 0 for b in B; (4) [p, b] = 0 and [D_h, p] = 0; (5) (B, ker D_v, p D_h p) is a metric spectral triple; (6) a positive map E: A -> B fixing B with ||a - E(a)|| <= k ||[D_v, a]|| and ||[D_h, E(a)]|| <= ||[D_h, a]||), the theorem asserts that (A, H, D_eps) converges to (B, ker D_v, p D_h p) in the spectral propinquity as eps -> 0. Corollary 3.5 then gives convergence of spectra: Sp(p D_h p) is the set of limits of convergent sequences lambda_n in Sp(D_{eps_n}).

Counterexample. Let H have orthonormal basis e_0, e_1, e_2, …, let A = B = C act by scalars, and set D_v e_n = n e_n and D_h e_n = -n^2 e_n. Then D_eps e_n = (n/eps - n^2) e_n has compact resolvent for every eps > 0, since |n/eps - n^2| -> infinity; D_v has gap 1 at 0; p is the projection onto C e_0 and commutes with D_h; hypotheses (1) to (6) hold trivially, all commutators with scalars being zero, with E = id and any M, k > 0; and (C, C e_0, 0) is a metric spectral triple. Its spectrum is {0}. Fix x > 0 and take eps_n = 1/(n + x/n) -> 0. Then D_{eps_n} e_n = x e_n, so the constant sequence x lies in Sp(D_{eps_n}) for every n, and Corollary 3.5 would put x in {0}. Hence the conclusion of Theorem 3.4 fails for this family.

The failing step. The proof's estimate (3.11),

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||xi - p xi|| <= 4 sqrt(2 eps/delta) (||xi|| + ||D_v xi/eps||) <= 4 sqrt(2 eps/delta) DN_eps(xi),
```

needs ||D_v xi||/eps <= ||D_eps xi||, which is neither assumed nor proved. For xi = e_n at eps = eps_n the left side is 1 and the right side is 4 sqrt(2 eps_n) (1 + x) -> 0.

Sufficient repairs. Either hypothesis below makes the Fourier Lemma 3.3 of the paper unnecessary and improves the rate from O(sqrt eps) to O(eps):

1. D_h and D_v anticommute on a common core. Then ||D_eps xi||^2 = ||D_h xi||^2 + eps^{-2} ||D_v xi||^2 and ||xi - p xi|| <= ||D_v xi||/delta <= (eps/delta) ||D_eps xi||.
2. A relative bound ||D_v xi|| <= C eps (||xi|| + ||D_eps xi||). Then ||xi - p xi|| <= (C eps/delta)(||xi|| + ||D_eps xi||).

These are sufficient hypotheses; the counterexample shows the printed six are not. All applications in the paper (Corollary 4.1, Theorems 4.6, 5.5, 6.3) have anticommuting D_h and D_v and avoid the defect; Theorem 4.6 and Corollary 4.7 additionally need a nonzero vertical kernel for their displayed target to be a spectral triple (Astra 6, 3 November 2026). Corollary 4.1 (collapse of a fixed triple scaled by 1/eps to (C, ker D, 0), and to nothing when ker D = 0) has D_h = 0 and is sound.

Scope. This is a statement about one theorem of one paper. It leaves the paper's framework, its corollaries in the anticommuting setting and its applications intact. For the project, it is the reason lemma 15 proves its own collapse statements (Theorems 15.3 and 15.7) and the reason the main paper reaches the contradiction through Theorem 15.7 (the converse of Proposition G: along eps_n the eigenvalues 0 and x stay bounded, while a limit (C, C e_0, 0) would need one eigenvalue near 0 with a diverging gap) rather than through Corollary 3.5. The route through Theorem 15.7 was written by Claude Opus 5.5 and read by Claude Fable 5.1; no referee has checked that route as distinct from R1 itself.

Record. The full reading of the paper (its prerequisites, the structure of the proof, the v1 to v3 changes, the errors in the written proof of Theorem 6.3, the dependency audit) is in historical/reviews/2026-11-03_farsi_latremoliere_assessment.md; nothing there is needed for the statement above.