Finite spectral triples as towerless intervals

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We test the reading of the finite factor of almost-commutative spectral triples as the remnant of a fifth dimension with its Kaluza-Klein tower removed. The Connes-Lott two-point space is exactly the minimal spectral truncation of a chiral interval of length l, with mass $\pi/(2l)$, and with a bulk mass the interval carries a light pair that truncates to Connes-Lott at any length. As $l \to \infty$ at fixed bulk mass, the pair becomes chiral wall modes and a one-dimensional Weyl law appears above the gap. As $l \to 0$, a light sector of fixed mass survives along a tuned bulk mass or along a scaled boundary coupling; the latter gives the Dirac Yukawa block $\Phi_Q \hat{Y} \Phi_U$. The counts 9 and 11 are KO-degrees of the two odd factorizations of a KO-6 triple. In the reduced Standard Model representation, charged paired sectors admit both, one or three unpaired neutrinos admit only KO-degree 7, and the full chiral representation admits neither. We propose reading these as labels of limits, with the physical point in the even interior; that identification is not constructed here. For the heat cutoff, the Taylor mass of the W obeys a weighted sum rule whose small-mass form is the relation $8M_W^2 = \sum N_f m_f^2$ of the asymptotic spectral action: a top alone above a fixed cutoff gives $M_W \sim \Lambda^2/m_t \to 0$, and $m_t/M_W \to \sqrt{8/3}$ needs $m_t/\Lambda \to 0$. At leading order, in the cases computed, matching the finite spectral action to the tower requires a tower-scale fermion. In the spectral propinquity, the finite limits of connected commutative metric triples are points carrying a finite Dirac block, while compressions at fixed truncation generate finite algebras whose structure is set by the symmetries of the spectral window. A two-sheet parent reproduces the algebra, bimodule and real structure of the Chamseddine-Connes classification, with stated inputs. A nonzero wall coupling on a chosen channel between the sheets gives a rank-one block whose commutation algebra is $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with the fibres, the grading and the constant matrix multipliers as inputs.

I. INTRODUCTION

Between 1979 and 1987 the Kaluza-Klein programme sought the Standard Model in the geometry of extra dimensions. The Higgs field appeared as the internal components of a higher-dimensional gauge field [1–4], and Witten's search for a realistic Kaluza-Klein theory found that $SU(3) \times SU(2) \times U(1)$ can arise from isometries of the internal space while the fermion quantum numbers are difficult to achieve [5]. In 1983 he made the diffi-

culty precise [6]: for a ground state $M^4 \times B$ with B compact and all gauge fields coming from isometries of B, a theorem of Atiyah and Hirzebruch makes the index of the internal Dirac operator vanish as a character of the isometry group, so the four-dimensional fermion quantum numbers are vectorlike. The result is true in that scope, and the same text names its escapes: elementary gauge fields in the higher dimension, and a unified theory, perhaps a supergravity or superstring theory, that fixes the gauge group. From 1984 attention moved to supersymmetry and superstrings, and the heterotic string took both escapes, with elementary gauge fields whose group the theory fixes. The reviews of 1986 and 1987 record that move and close the programme [7, 8]; we read their open-problem sections only through abstracts and later retrospectives [9, 10], the second of which counts Witten's chirality result as one of three reasons the coset programme lost ground, with the large negative cosmological constant and the string models.

Noncommutative geometry entered particle physics with a Kaluza-Klein reading from the start. In 1988 Madore generalized Kaluza-Klein theory by replacing the functions on the internal space with a matrix algebra, for which the internal Laplacian has a finite spectrum [11]. Connes and Lott wrote the Standard Model as a gauge theory on a spacetime "of Kaluza-Klein type, where the fiber is formed of a two-point set", with the Higgs field as the connection along the discrete direction [12]. In its simplest form the fibre has algebra $\mathbb{C} \oplus \mathbb{C}$ and Dirac operator $m\sigma_1$, and its two points lie at Connes distance $1/|m|$. Coquereaux, Esposito-Farèse and Vaillant called their gauging of discrete directions "in a sense, a discrete analog" of the Kaluza-Klein approach [13], and in 1994 Viet and Wali discretized the fifth dimension [14]. In the texts and reference lists we read, none of these founding papers cites the chirality obstruction; the advantage they claim for the discrete direction is finiteness, and chirality is supplied by the grading of the finite Hilbert space.

The two-point space is the oldest layer of the present theory, which adds a real structure, a KO-dimension and a first-order condition [15–17]; the finite space of the noncommutative Standard Model has KO-dimension 6, so that its product with four-dimensional spacetime has total KO-dimension $10 \equiv 2 \pmod 8$ [15, 16, 18]. We read the two-point space as the minimal spectral truncation of a chiral interval, whose walls carry the chirality that the obstruction excludes from a closed internal manifold, so

that the obstruction and the discrete direction are two statements about the same limit. Within the searches recorded in the Supplemental Material, part H (the history of the field, Section 1.4), this link has not been made in print before.

KO-dimension and metric dimension differ: the finite space has metric dimension 0 and KO-dimension 6. So the numbers 9, 10 and 11 below are KO-degrees of factorizations, while the emergence or loss of metric dimension is measured by Weyl laws and requires a singular limit. We make the discrete-dimension reading precise and test it against a proposal suggested by the Kaluza-Klein picture: with massless weak bosons a dimension should open (10 to 11), with infinitely heavy weak bosons and top a dimension should close (10 to 9), and the physical point, with its finite Fermi constant and measured weak angle, lies between these limits. Below we give exact results for an interval model. The maps from KO signs to interval geometries, and from the length l to the measured parameters, are part of the proposal and are not constructed here.

Our model of the fifth dimension is the chiral interval $I_l = [0, l]$, with boundary conditions that place the two chiralities at opposite ends. Table I summarizes what the picture gives. The full proofs are in the Supplemental Material; the main text gives their structure.

II. SETTING AND CONVENTIONS

Let $H = L^2(I_l; \mathbb{C}^2)$ with $\psi = (L, R)$, grading $\Gamma = \mathrm{diag}(1, -1)$, and for a real bulk mass M

$$(D_M\psi)_L = R' + MR, \qquad (D_M\psi)_R = -L' + ML. \quad (1)$$

We use three boundary types: (a) $R(0) = R(l) = 0$; (b) $L(0) = L(l) = 0$; (c) $R(0) = 0$, $L(l) = 0$. Each defines a self-adjoint operator with compact resolvent anticommuting with Γ.

A real structure is an antiunitary J with $J^2 = \epsilon$, $JD = \epsilon' DJ$, $J\Gamma = \epsilon'' \Gamma J$; KO-dimension 6 means $(\epsilon, \epsilon', \epsilon'') = (+, +, -)$.

A spectral truncation of (A, H, D) to a spectral subspace PH is (PAP, PH, PDP). The map $a \mapsto PaP$ is the compression. For functions it is the classical Toeplitz compression, and its use for truncating spectral triples is due to [19, 20]. The image PAP is an operator system, in general not an algebra.

TABLE I. Results, by question.

Two-point space	The minimal spectral truncation of the chiral interval, with $\|m\| = \pi/(2l)$ (Theorem A).
10 to 11	In the type-(c) family at fixed bulk mass, a dimension opens with the tower restored as $l \to \infty$: chiral wall modes, one-dimensional Weyl law (Theorem B).
10 to 9	A light sector of fixed mass survives collapse along a tuned bulk mass (type (c)) or a scaled boundary coupling. The truncation stays Connes-Lott, and the whole triple converges to a point carrying a Dirac block (Theorems B, C, Proposition G).
Yukawa block	The Dirac Yukawa block of D_F is obtained in the boundary-Yukawa limit, $\Phi_Q \hat{Y} \Phi_U$. The algebra and Majorana block are not derived there (Theorem C).
9 and 11	KO-degrees of two factorizations of one even triple. J, charge conjugation and particle number do not select. In the reduced representation, charged paired sectors admit both, and one or three unpaired neutrinos admit only 11; the full chiral representation admits neither (Theorem D).
Heavy W	For $m \ll \Lambda$: needs a heavy Dirac fermion, with $m_t/M_W \to \sqrt{8/3}$ as $\Lambda \to \infty$ faster than m_t (Proposition E). For the heat cutoff and Taylor mass, the relation is the small-mass form of a weighted sum: a top alone above a fixed cutoff drops out and $M_W \to 0$ (Proposition E.1).
W against the tower	The fixed truncation is not gauge covariant. At leading order, with stated trace and gauge boundary conditions, matching on a common profile space needs a sum rule; in the case computed, a tower-scale fermion (Proposition F).
Chirality	In the full chiral representation, neither odd factorization exists. The label 9 uses vectorlike sectors. In the type-(c) limit chirality sits on walls. As a proposal: the odd labels belong to limits, and the physical point to the even interior (Section VI).
Finite algebra	Compressed functions generate it at fixed truncation, with structure set by the symmetries of the window (complex generation, or a real form from real functions). A two-sheet parent gives the classified bimodule, with two sheets, restricted algebras and the grading as inputs (Proposition H). A wall coupling gives the rank-one block of $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with the fibres, grading and channel as inputs (Proposition I).

III. THE TWO-POINT SPACE IS A TRUNCATED INTERVAL

Theorem A. *Take $M = 0$ and type (c). The spectrum of D_0 is $\pm(r - \frac{1}{2})\pi/l$, $r \geq 1$. The lowest spectral subspace is spanned by $u_L = \sqrt{2/l}\,(\cos(\pi x/2l), 0)$ and $u_R = \sqrt{2/l}\,(0, \sin(\pi x/2l))$; on it $PD_0P = m\sigma_1$ with $m = \pi/(2l)$, $P\Gamma P = \sigma_3$, and the compression of $C(I_l)$ has range the diagonal algebra $\mathbb{C} \oplus \mathbb{C}$. Hence the truncation is the Connes-Lott two-point space with $|m| = \pi/(2l)$, and the Connes distance between the two states of the compressed operator system is $2l/\pi$.*

Structure of the proof. The spectrum follows by separation. Pointwise orthogonality of the L and R components makes PfP diagonal, with entries $\int f|u_L|^2$ and $\int f|u_R|^2$, and $f = 1$, $f = x$ span the diagonal algebra (Supplemental Material, part 05). □

The two states pull back to the smeared measures $|u_L|^2 dx$ and $|u_R|^2 dx$ on I_l, which differ from endpoint evaluations. With the inherited real structure, the full diagonal algebra fails the first-order condition: for $e = \mathrm{diag}(1, 0)$ one has $e^\circ = e$ and $[[m\sigma_1, e], e^\circ] = m\sigma_1$ (Supplemental Material, part 08).

The discrete version is a staggered chain on

$$H_N = \mathrm{span}\{L_1, \ldots, L_{N-1}, R_2, \ldots, R_N\} \tag{2}$$

with $B_N = h_N(I - S)$, whose spectrum $2h_N \sin((2r - 1)\pi/(4N - 2))$ converges to that of D_0 and whose sites lie at distance $|i-j|/h_N$; the two-point space is its $N = 2$ case, as in dimensional deconstruction [21].

Proposition A.1 (bare limits). *At fixed truncation, $m \to 0$ produces no heat dimension, and $m \to \infty$ leaves no light fermions. In the product with any finite KO-6 triple,*

$$D_F^2 = D_0^2 \otimes 1 + 1 \otimes D_I^2 \geq \frac{\pi^2}{4l^2}, \tag{3}$$

so the literal collapse $l \to 0$ of the massless interval has an empty light sector at every fixed cutoff.

IV. BULK MASS: THE LIGHT PAIR AND BOTH LIMITS

Theorem B. *1.* Chirality from boundary conditions. *For every M and l, $\ker D_M$ is $\mathbb{C}(e^{Mx}, 0)$ for type*

(a), $\mathbb{C}(0, e^{-Mx})$ *for type (b) and* 0 *for type (c), with indices* $+1, -1, 0$*. For a finite sum of representations* V_ρ *of a compact group with types* $\tau(\rho)$*, the equivariant index is* $\sum_{\tau=a}[V_\rho] - \sum_{\tau=b}[V_\rho]$*, which realizes* $[V_L] - [V_R]$*.*

2. Light pair. *For type (c),* $M = -\mu < 0$ *and* $\mu l > 1$*, there is exactly one positive eigenvalue below* μ*,*

$$\lambda_0 = \mu \operatorname{sech}(\kappa_0 l), \qquad \tanh(\kappa_0 l) = \frac{\kappa_0}{\mu}, \tag{4}$$

with $\lambda_0 = 2\mu e^{-\mu l}(1 + o(1))$*, and the others are* $(\mu^2 + k_n^2)^{1/2}$ *with* $k_n \in (n\pi/l, (n + \frac{1}{2})\pi/l)$*. The profiles satisfy* $R_0(x) = L_0(l - x) = \sinh(\kappa_0 x)$ *up to normalization.*

3. Truncation. *The light pair truncates to Connes-Lott with* $D = \lambda_0 \sigma_1$*, and the Connes distance between its states is* $1/\lambda_0$*.*

4. Decompactification *(*$l \to \infty$*, fixed* μ*).* $\lambda_0 \to 0$*; the normalized profiles converge, uniformly on compacta and in* L^2*, to* $\sqrt{2\mu}\, e^{-\mu y}$ *at the two walls, exact chiral zero modes of the half-line operators; and* $\#\{n : \lambda_n \le E\} = (l/\pi)\sqrt{E^2 - \mu^2} + O(1)$ *for* $E > \mu$*.*

5. Collapse at fixed light mass. *For* $m_f > 0$ *and* $l < 1/m_f$ *there is a unique* $\mu(l)$ *with* $\lambda_0 = m_f$*. As* $l \to 0$*, with* $c = \mu(l)l$ *and* $L = \log(2/(m_f l))$*,*

$$c = -W_{-1}\left(-\frac{m_f l}{2}\right) + o(1) = L + \log L + o(1), \tag{5}$$

a statement about this localization model. The next eigenvalue exceeds $(\mu^2 + \pi^2/l^2)^{1/2}$*, and the truncation stays Connes-Lott with mass* m_f *at each* l*. At fixed* μ *instead, every eigenvalue eventually has* $|\lambda| > \mu$*, and the smallest is* $(\pi/2l)(1 + o(1))$*.*

Structure of the proof. Item 1: $D\psi = 0$ forces $L = \alpha e^{Mx}$, $R = \beta e^{-Mx}$, and exponentials do not vanish, so each boundary condition removes one coefficient. Eliminating R reduces type (c) to $-L'' + M^2 L = \lambda^2 L$ with $L(l) = 0$, $L'(0) = ML(0)$; root counting for $\tanh t = t/c$ and $\tan t = t/c$ gives item 2. Item 3 is the computation $Du_L = \lambda_0 u_R$ with orthogonal chiral supports. For item 4, $\|L_0\|^2 \sim e^{2s}/(8\kappa_0)$ with $\kappa_0 \to \mu$ gives the profiles, and the location of k_n gives the counting. Item 5 uses

that $F(c) = c^2\mathrm{sech}^2 s(c)$, with $s = c\tanh s$, decreases strictly from 1 to 0 on $(1,\infty)$ (Supplemental Material, part 09). □

So in this one-field family the dimension emerges with the tower restored (item 4), and the collapse keeps a finite light sector along the curve $\mu(l)$ (item 5); Theorem C gives another route. The excited levels combine a quantized internal momentum with the bulk mass in quadrature, $\lambda_n^2 = \mu^2 + k_n^2$, while the light pair has imaginary momentum $k = i\kappa_0$, with $\lambda_0^2 = \mu^2 - \kappa_0^2$.

V. BOUNDARY YUKAWA MATRICES AND TOWERLESS TEXTURES

Let $Q \in L^2(I_l;\mathbb{C}^2)\otimes\mathbb{C}^{N_Q}$ and $U \in L^2(I_l;\mathbb{C}^2)\otimes\mathbb{C}^{N_U}$ carry diagonal bulk masses, and for a matrix X impose $Q_R(0) = 0$, $U_L(0) = 0$, $Q_R(l) = XU_R(l)$, $U_L(l) = -X^\dagger Q_L(l)$. This defines a self-adjoint D_X anticommuting with Γ. Write $c_i^Q = -m_i^Q l$, $c_j^U = m_j^U l$ (positive c localizes a field away from the Yukawa wall), $\phi(c)^2 = 2c/(e^{2c}-1)$, $\Phi_Q = \mathrm{diag}\,\phi(c_i^Q)$ and $\Phi_U = \mathrm{diag}\,\phi(c_j^U)$. Let F_Q, F_U be the diagonal matrices of values at $x = l$ of the normalized zero modes, so that $F = \Phi/\sqrt{l}$, and let Λ_Q, Λ_U be the smallest lowest positive eigenvalues of the single fields with one Dirichlet condition at each end (type (c)).

Theorem C. *1.* Index and symmetry breaking. $\ker D_X \cong \ker X^\dagger \oplus \ker X$ *in the two chiralities, so* $\mathrm{ind}\, D_X = N_Q - N_U$ *for all* X. *If a compact group acts, the domain is invariant under the stabilizer* H_X *of* X, *and* $\mathrm{ind}_{H_X} D_X = [\rho_Q] - [\rho_U]$ *restricted to* H_X. *All zero modes lift iff* X *is invertible, which requires* $\rho_Q \cong \rho_U$ *as* H_X*-representations; for* $SU(3)\times U(1)_{\mathrm{em}}$ *and leptons without* ν_R, *a massless* ν_L *is forced for every* X.

2. Interpolation. *For one massless flavor the positive spectrum is* θ/l *and* $(n\pi \pm \theta)/l$, $\theta = \arctan|X|$*: the light pair moves from the chiral zero modes of types (a)⊕(b) at* $X = 0$ *to the Connes-Lott mass* $\pi/(2l)$ *of two copies of type (c) as* $|X| \to \infty$.

3. Light spectrum. *With* A, B *the diagonal matrices of* $\omega\coth(\omega l) \pm m$, $\omega^2 = m^2 - \lambda^2$, *the light eigenvalues are the fixed points* $\lambda = \sigma_k(A(\lambda)^{1/2} X B(\lambda)^{1/2})$,

and with $Y_0 = F_Q X F_U$,

$$\sigma_k(Y_0)\Big(1 - \frac{\lambda_k^2}{\Lambda_Q^2} - \frac{\lambda_k^2}{\Lambda_U^2}\Big) \le \lambda_k \le \sigma_k(Y_0), \qquad (6)$$

which holds for λ_k *below* $\min(\Lambda_Q, \Lambda_U)$.

4. Towerless limit. *As* $l \to 0$ *with the c's and* $\hat{Y} = X/l$ *fixed, the spectrum in* $[-E, E]$ *converges to* 0, *with multiplicity* $N_Q + N_U - 2\,\mathrm{rank}\,\hat{Y}$, *and to the* $\pm\sigma_k(M_F)$ *with* $\sigma_k(M_F) \le E$, *where*

$$M_F = \Phi_Q \hat{Y} \Phi_U, \qquad D_F = \begin{pmatrix} 0 & M_F \\ M_F^\dagger & 0 \end{pmatrix} \qquad (7)$$

is the Yukawa block of the Chamseddine-Connes finite Dirac operator. All other eigenvalues diverge.

5. Hierarchies. *For graded* $F = \mathrm{diag}(f_1 \le \cdots \le f_N)$ *and* G *likewise, let* $T_p = \{N-p+1, \ldots, N\}$, $f_{T_p} = \prod_{i \in T_p} f_i$ *(likewise* g_{T_p}*),* $\Delta_p = |\det X_{T_p T_p}| \neq 0$, D_p *the largest* $p \times p$ *minor of* X *in absolute value,* $\epsilon_p^F = f_{N-p}/f_{N-p+1}$ *(likewise* ϵ_p^G*), and* $K_p = \binom{N}{p} - 1$. *Then* $\sigma_1 \cdots \sigma_p(FXG)$ *lies between* $f_{T_p} g_{T_p} \Delta_p$ *and*

$$f_{T_p} g_{T_p} \big[\Delta_p + D_p\big((1 + K_p \epsilon_p^F)(1 + K_p \epsilon_p^G) - 1\big)\big]. \quad (8)$$

If the entries of $\hat{Y}$ *are of order one and its trailing minors are bounded below, the masses follow*

$$\phi(c_k^Q)\phi(c_k^U) \approx 2\sqrt{c_k^Q c_k^U}\, e^{-c_k^Q - c_k^U} : \qquad (9)$$

order-one differences of localization give exponential hierarchies [22–24].

Structure of the proof. Solving the constant-coefficient equations and imposing the wall conditions gives the secular equation $\det(\lambda^2 - A(\lambda) X B(\lambda) X^\dagger) = 0$. The entries of A, B are decreasing and concave in λ^2, so each lies above its chord; this gives the bounds of item 3. With $\hat{Y}$ fixed, $\Lambda \sim 1/l$ gives item 4. Item 5 applies the singular values of the p-th compound matrix, dominated by its $T_p T_p$ entry (Supplemental Material, part 12). □

Item 2 gives a second route to collapse at fixed light mass, with no bulk mass: for one massless flavor and $X(l) = \tan(ml)$, the light eigenvalue is exactly m for

small l, and the higher levels diverge. Item 4 is its multiflavor form.

The mixing pattern $|V_{ij}| \sim \phi(c_i^Q)/\phi(c_j^Q)$ holds in numerical samples (Python) and is known in the extra-dimensional flavor literature [25, 26]. For $M_s = FX_sG_s$ with nonzero trailing minors, there is an upper bound with explicit constants, independent of the right-hand profiles: $|V_{ij}| \leq 3K_uK_df_i/f_j$ for $i < j$, with K_s built from the ratios of minors (Supplemental Material, part 12, from the referee report). There is no universal lower bound, since identical sectors give $V = 1$.

VI. REAL STRUCTURES: THE CHOICE BETWEEN 9 AND 11

We call the factorization of a KO-6 triple as an odd triple times a point a doubling; this usage is unrelated to fermion doubling.

Theorem D. *1.* Factorization criterion. *A KO-6 triple T factors as $T_n \times P_k$ with $(s, n, k) = (+1, 7, 7)$ or $(-1, 5, 1)$ iff there is a unitary U with $U^2 = 1$, $U\Gamma = -\Gamma U$, $[U, a] = [U, a^\circ] = [U, D] = 0$ and $UJ = sJU$.*

2. The real structure does not select s. *The almost-commutative electrodynamics triple admits involutions of both signs. In a neutral sector with scalar algebra character, the KO-7 factorization always exists, and the KO-5 factorization exists exactly when every singular value of the Majorana block, including zero, has even multiplicity. So the strict light spectrum of the Standard Model representation, with one or three unpaired left-handed neutrinos and no Majorana term, admits only $s = +1$; two unpaired neutral pairs with $D = 0$ admit both signs; and while the full algebra keeps $\mathbb{H}$, neither factorization exists.*

3. The interval reflection. *On the chiral interval with pointwise reality ($JfJ^{-1} = \bar{f}$), the real structures are $\mathrm{diag}(\alpha, \beta)K$ with $\alpha = \epsilon'\beta$, of KO-dimension 0 for $\epsilon' = +1$ and an extended sign convention for $\epsilon' = -1$, and the reflection $x \mapsto l - x$ exchanging L and R has sign ϵ' against J. In the internal-first KO-6 product $F_0 \times I_l$, the KO-6 conditions force the grading twist $J_F = J_0 \otimes J_I\Gamma_I^b$ with $\epsilon'(-1)^b = -1$,*

and the bare reflection $1 \otimes \sigma$ has $s = -1$ for either interval convention.

4. Exact reconciliation. *For the truncated interval with scalar algebra, $F_0 \times I_l \cong T_5(m) \times P_1$ with $d_5 = D_0 + m\Gamma_0$ and $j_5 = J_0\Gamma_0$: the interval mass enters the odd Dirac operator, and the interval's two states become the doubling space. With the full compressed two-sheet algebra, the inherited data fail first order and the reflection does not commute with that algebra.*

5. No physical selection. *Spacetime grading twists (J_M versus $J_M\gamma_5$) do not select the finite involution. A product with fixed D_F, J_F, Γ_F and particle number Q admits particle-preserving involutions of both signs. More generally, if Q is a self-adjoint involution commuting with D, Γ, the algebra and its opposite, with $QJ = -JQ$, then particle-preserving involutions come in pairs U, QU of opposite sign.*

Structure of the proof. Given U, put $d = UD$ and $j = UJ$ on $\{\Gamma = +1\}$; then $j^2 = s$, $jd = sdj$, and $(x, y) \mapsto x + Uy$ exhibits the product. For item 2, Takagi factorization reduces the Majorana block to singular values; $j^2 = -1$ commuting with d^2 forces even multiplicities, and pairing equal entries with σ_2 gives the converse. Items 3 to 5 are direct computations with the stated operators, with $(QU)J = -sJ(QU)$ for the pairing (Supplemental Material, part 06, Supplemental Material, part 07 and Supplemental Material, part 08). □

The values 9 and 11 are therefore the KO-degrees, shifted by 4, of the two odd factorizations of one even triple. Replacing J by $J\Gamma$ only changes signs inside the even classification [27], and identifying either factorization with the dimension of a parent geometry would need a change of parity in K-homology that these results do not supply.

The two labels differ by sector. Each charged paired sector of the reduced representation is invariant under the algebra, its opposite, D and J, and admits both signs, so the formal label 9 is available for the charged fermions sector by sector. The obstruction to $s = -1$ for the whole triple sits in the neutral sector: with one or three neutrinos and no ν_R, only $s = +1$ occurs there, and that neutral involution does not preserve particle number, so with particle preservation imposed on this spectrum neither reduction is available. With ν_R and pure

Dirac masses both signs occur. A Majorana mass breaks $[Q, D] = 0$ and the pairing of item 5, and $s = -1$ exists exactly when every singular value of the neutral block has even multiplicity, so a generic Majorana block leaves only $s = +1$.

Chirality at the odd labels. In odd dimension the Clifford volume element is central, so an irreducible spinor has no chirality operator. Witten's argument for an odd number of extra dimensions ([5], pp. 421–423) runs as follows: the reversal $\gamma_i \mapsto -\gamma_i$ of the internal gamma matrices fixes $[\gamma_i, \gamma_j]$ and flips the internal helicity, so on an orientable internal space, with the gauge group acting by coordinate transformations, four-dimensional fermions of opposite helicity carry the same gauge charges, and the low-energy content is vectorlike. That paper treats eleven-dimensional supergravity on $M^4 \times M^{pqr}$; the ways out it considers are a non-orientable internal space and torsion. The finite statement here is parallel: in the Standard Model representation of $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, neither odd factorization exists (item 2). Such arguments depend on their hypotheses: chiral modes can sit on boundaries or defects, and chiral couplings can arise for massive gauge fields linked to non-Killing vector fields [28].

We propose reading the objection as a constraint on limits. The label 9 uses the charged paired sectors, which are vectorlike under the reduced $SU(3) \times U(1)_{\mathrm{em}}$. A neutral Weyl fermion has a real gauge representation and can be written in Majorana form, though it stays a chiral spacetime fermion; for an odd number of them, the obstruction to $s = -1$ is the parity forced by $j^2 = -1$. In the type-(c) family as $l \to \infty$ at fixed μ, the pair becomes chiral zero modes of the two half-lines, and a region receding from both walls sees only the bulk operator, with threshold μ (Theorem B, item 4). This shares a feature with Hořava-Witten theory [29], where chirality is carried by the boundaries of an odd bulk. The content differs: there, an E_8 vector multiplet sits on each boundary, with one ten-dimensional chirality for a common orientation, while here two modes of one bulk Dirac operator have opposite Γ-chirality. The same-chirality analogue is type (a) at both ends, with opposite localization masses, and unbroken $E_8 \times E_8$ allows no bilinear connector between the boundary multiplets (Supplemental Material, part 20). Two parts of the proposal are not constructed here. The assignment of the label 11 to this limit is open, since the interval product of item 4 also has a KO-5 fac-

torization. And we take the physical point, with its finite Fermi constant and measured weak angle, to lie at finite l in the even interior, where the KO-6 structure accommodates chirality, without relating l quantitatively to those parameters.

VII. SPECTRAL ACTION: HEAVY FERMIONS AND THE GAUGE SECTOR

Proposition E. *Assume the Chamseddine-Connes-Marcolli relation $8M_W^2 = \sum N_f m_f^2$ over Dirac masses at the matching scale, with N_g generations, and let $m_{\max}$ be the largest Dirac mass. Then*

$$\frac{m_{\max}}{2\sqrt{2}} \le M_W \le \sqrt{N_g}\, m_{\max}. \tag{10}$$

Hence a heavy-W limit at fixed fermion masses is incompatible with the relation, and if only the up-type mass of one generation diverges, $m_t/M_W \to \sqrt{8/3}$, the top-mass relation of [16].

Proof. The weights over one generation sum to 8, and the largest mass has weight at least 1 (Supplemental Material, part 11). □

The relation belongs to the truncated heat-kernel expansion, with its coefficient $\mathrm{Tr}\, Y^\dagger Y$ polynomial in the masses and higher terms in m^2/Λ^2 dropped. It is therefore a statement for $m \ll \Lambda$. The case of fixed cutoff is the following.

Proposition E.1. *Let $S = \mathrm{Tr}\, f(D_A^2/\Lambda^2)$ on $\mathbb{R}^4$, as a relative trace or action density, with a constant vacuum V_0 with eigenvalues m_i, and let $a_\mu(q)T$ be a transverse vector. For $f(z) = e^{-tz\Lambda^2}$ the quadratic action is exact, $\int_q |a|^2 C(q^2)$, with $d_{ij} = m_i - m_j$ and*

$$C(q^2) = \frac{1}{8\pi^2} \sum_{ij} |T_{ij}|^2 \Big[\big(\tfrac{2}{t} + d_{ij}^2 + q^2\big) I_{ij} - \tfrac{1}{t}\big(e^{-tm_i^2} + e^{-tm_j^2}\big) \Big], \tag{11}$$

where $I_{ij} = \int_0^1 e^{-t[(1-u)m_i^2 + um_j^2 + u(1-u)q^2]} du$. Its small-mass expansion gives the mass formula of Proposition F and the CCM relation. In the three-generation Standard Model data with only one up-type Dirac mass m, $f(z) = e^{-z}$ and $x = m^2/\Lambda^2$:

1. *the Taylor mass is*

$$\frac{M_W^2}{\Lambda^2} = \frac{3[1-(1+x)e^{-x}]}{x[3+w(x)]}, \tag{12}$$

with $w(x) = 3[2-(x^2+2x+2)e^{-x}]/x^3$*, so that* $M_W^2 \sim \frac{3}{8}m^2$ *for* $x \to 0$ *and* $M_W^2 \sim \Lambda^4/m^2$ *for* $x \to \infty$*; a unique positive timelike pole has the same limits, and both obey* $M_W^2 \leq \Lambda^2/e$*;*

2. $M_Z^2 \sim \frac{2}{3}xe^{-x}\Lambda^2$ *for* $x \to \infty$*, so that* M_W^2/M_Z^2 *does not stay at* $5/8$*, while the neutral-current angle stays at* $\cos^2\theta_{\rm NC} = 5/8$ *and the ratio of the photon and gluon coefficients stays at* $8/3$ *for all* x*;*

3. *the top is separated: the finite trace in the vacuum density is* $84 + 12e^{-x}$*, and among single charged species only an up-type quark can be removed while keeping the two values in item 2 (for a down-type quark they become* $59/92$ *and* $46/15$*, for a charged lepton* $17/28$ *and* $7/3$*);*

4. *the broken vectors keep a positive kinetic form, so that they do not decouple by becoming heavy. The light space consists of the five other quarks and the leptons, on which* $SU(3) \times U(1)_{\rm em}$ *acts with* b_L *paired with* b_R*; its electroweak* $SU(2)$ *does not act linearly on it.*

Structure of the proof. The Duhamel expansion of the heat trace to second order has the vertices $(2p+q)\cdot a$, $\gamma^{\mu\nu}q_\mu a_\nu$ and $d_{ij}\gamma^\mu\gamma_5 a_\mu$, and a quadratic vertex. The heavy Dirac block has eigenvalues $\pm m$ and connects to massless partners with $d^2 = m^2$, while the other doublets contribute a massless kinetic term, which gives the limits (Supplemental Material, part 18). □

For the heat cutoff and the Taylor mass $M_W^2 = C(0)/C'(0)$, and for doublets with one massive member and one massless, item 1 is the CCM relation with a form factor. With $\rho(x) = 2[1-(1+x)e^{-x}]/x^2$,

$$8\kappa M_W^2 = \sum_f N_f m_f^2\, \rho(m_f^2/\Lambda^2), \qquad \kappa = \frac{P + \sum_i d_i w(x_i)}{6}, \tag{13}$$

where P is the particle-trace weight of complete massless doublets, $d_i = N_i/2$ that of each massive doublet, and $P + \sum d_i = 6$. For the top alone, $\kappa = (3+w)/4$. If all $x_i \to 0$, then $\rho, \kappa \to 1$, which is the CCM relation.

In the top-alone family, m_t/M_W differs from $\sqrt{8/3}$ for every $x > 0$, and approaches it only as $x \to 0$, so a limit with the relation intact needs $m_t/\Lambda \to 0$; the departure is 5% at $m \approx 0.45\Lambda$, and M_W has a unique maximum of 0.524Λ at $m \approx 1.42\Lambda$ (Python, for the location). A massive diagonal contribution to the trace is exponentially suppressed above the cutoff, and a mixed loop with a massless partner has $\rho \sim 2/x^2$. With no other mass in the vacuum, the W mass then has no source. This regime lies beyond the range of the expansion, and whether the classical action is meaningful for $m > \Lambda$ is a separate question.

The Taylor statement holds for nonnegative cutoffs, including those flat near 0. With a complete massless doublet retained, the Taylor mass tends to 0 for any finite set of heavy doublets. With every doublet of equal mass and no light fermions, the poles satisfy $2x < M_W^2/\Lambda^2 < 2x+2$ and $\frac{16}{5}x < M_Z^2/\Lambda^2 < \frac{16}{5}x + 2$, so they diverge at fixed cutoff with $M_W^2/M_Z^2 \to 5/8$; there the kernel vanishes at fixed momentum and the Taylor kinetic term has the wrong sign for $x > 2$. In the top-alone family, a limit with $m_t/M_W \to \sqrt{8/3}$ and all scales diverging exists jointly with $\Lambda \to \infty$ and $m/\Lambda \to 0$. For nonnegative f of unbounded support, no finite $c > 0$ is stationary along $V = cV_*$, so these are masses about a prescribed background. All of this is classical and bosonic. So in the families treated, a heavy W and Z at fixed cutoff is found only without a massless complete doublet.

Proposition F. *Fluctuate the product of spacetime with the chiral interval by a five-dimensional gauge field (A_μ, A_5) and compress to the light pair. Then:*

1. *the sheet connections are the weighted averages $A_{L,R} = \int w_{L,R} A_\mu$ and the link is $\phi = m + iB$ with $B = \int_0^l l^{-1} \sin(\pi x/l) A_5 \, dx$, so the image is $\{\phi + \phi^* = 2m\}$ and not every Connes-Lott fluctuation arises;*

2. *the fixed compression is not gauge covariant: $A_5 = a\cos(2\pi x/l)$ has trivial Wilson line and changes the compressed spectrum;*

3. *at leading order, in the stated trace and vacuum, the spectral action gives the vector mass matrix $\mathcal{M}^2 = \frac{3}{2} Q^{-1/2} K Q^{-1/2}$. For the minimal two-sheet model its eigenvalues are 0 and $6m^2$: a massless diagonal vector and a vector of mass $\sqrt{6}\, m$. With the*

CCM three-generation trace and one massive colour triplet, $M_W^2 = \frac{3}{8}m^2$. Neumann-Dirichlet conditions, which are separate input, give the tower mass $m = \pi/(2l)$.

Structure of the proof. Items 1 and 2 are direct integrals with the profiles of Theorem A. Item 3 is the a_4 coefficient $\eta\,\mathrm{Tr}F^2 + 3\eta\,\mathrm{Tr}(D_\mu V)^2$ with $\eta = f_0/(24\pi^2)$ (Supplemental Material, part 13). □

So in these cases the finite spectral action and the interval tower disagree on the vector mass, and the value depends on the whole fermion trace. With the gauge-covariant projection onto the light pair of the fluctuated operator, the covariance defect disappears, and on a common profile space equality of the two masses becomes an explicit trace-and-overlap sum rule (Supplemental Material, part 14). In the one-fermion Abelian mixed-boundary reduction, it fails at zero bulk mass, where $\mathcal{M}^2_{\mathrm{spec}} = \frac{3}{5}\mathcal{M}^2_{\mathrm{KK}}$, and holds at some tuned bulk mass $0 < M_* < 3\pi/(4l)$, at which the matched fermion has a mass of the order of the tower scale. At leading spectral-action order, if all fermion masses retained in the finite trace are uniformly bounded, then $\mathcal{M}^2_{\mathrm{spec}} \leq 6\|V_0\|^2$, so no matching is possible in the collapse. Equality of one mass does not establish equality of the full gauge or scalar actions. In these results the gauge sector ties the W to a fermion at the tower scale, in line with Proposition E.

VIII. WHOLE TRIPLES AND TRUNCATIONS

The results above concern compressions at each l. For the limit of the whole triple we use the spectral propinquity Λ_{spec} of Latrémolière [30] (doi:10.1016/j.aim.2022.108393), a metric on metric spectral triples up to unitary equivalence, that is, on triples whose Lipschitz seminorm $L(a) = \|[D, a]\|$ has kernel $\mathbb{R}1$ and induces the weak* topology on states. A tunnel between two such triples is a quantum compact metric space, carrying a Hilbert module with a D-norm, that maps onto both; its extent is the Hausdorff distance between its state space and theirs, and the propinquity Λ is the infimum of extents. Then Λ_{spec} is the infimum of $\sqrt{2}/2$ and of those ε for which some tunnel has both extent and separation of the families e^{itD_j}, $0 \leq t \leq 1/\varepsilon$, below ε. The proofs below use

only this definition, written out in Supplemental Material, part 15, Conventions, and facts proved there, chiefly $\Lambda_{\mathrm{spec}}(T_1, T_2) \geq \min(\sqrt{2}/2, |\operatorname{diam} T_1 - \operatorname{diam} T_2|/2)$ (consequence (c)) and a comparison of norms along a tunnel (Lemma 15.1).

Proposition G. *Let $T = (A, H, D)$ be a metric spectral triple of diameter δ. Let $\Sigma \subset \mathbb{R}$ be finite with $P = 1_\Sigma(D)$ of rank N, let $|\lambda| \geq g$ on the rest of the spectrum, let $W : PH \to \mathbb{C}^N$ be unitary and $\Delta = \|WDW^* - D_F\|$. Then*

$$\begin{aligned} \min\big(\tfrac{\sqrt{2}}{2}, \tfrac{\delta}{2}\big) &\leq \Lambda_{\mathrm{spec}}\big(T, (\mathbb{C}, \mathbb{C}^N, D_F)\big) \\ &\leq g^{-2} + \max\big(\tfrac{\delta}{2}, \Delta\big) + \Delta + \sqrt{\Delta}, \end{aligned} \tag{14}$$

with no anticommutation hypothesis. Conversely, a sequence T_n converges to $(\mathbb{C}, \mathbb{C}^N, D_F)$ iff $\delta_n \to 0$ and there exist rank-N spectral projections P_n and unitaries W_n with gaps $g_n \to \infty$ and $\Delta_n \to 0$. Moreover:

1. *the tuned interval of Theorem B satisfies*

$$\frac{l}{2} \leq \Lambda_{\mathrm{spec}}\big(T_l, (\mathbb{C}, \mathbb{C}^2, m_f\sigma_1)\big) \leq \frac{l}{2} + \frac{l^2}{\pi^2 + c^2}, \tag{15}$$

   and stays at distance at least $\min(\sqrt{2}/2, (1/m_f - l)/2)$ from the Connes-Lott triple;

2. *the Yukawa family of Theorem C converges at rate $O(l)$ to $(\mathbb{C}, \mathbb{C}^{N_Q+N_U}, D_F)$;*

3. *propinquity limits of commutative spaces are commutative, and the number of connected components does not increase, so the finite limits of connected commutative triples are exactly the triples $(\mathbb{C}, \mathbb{C}^N, D_F)$;*

4. *for a symmetric block, the grading and real structure descend to the block with their signs.*

Structure of the proof. The tunnel on $A \oplus \mathbb{C}$ has seminorm $\max\{L(a), \|a - z\|/\rho\}$, and on $H \oplus \mathbb{C}^N$ the D-norm is $\max\{\mathrm{DN}(\xi), \mathrm{DN}_F(\eta), \|W\xi - \eta\|/\rho\}$; the Leibniz inequalities hold because the limit algebra acts by scalars, and $\|(1 - P)\xi\| \leq \|D\xi\|/g$ bounds the rest. The converse uses the inner Leibniz inequality for the rank and the separation for concentration of spectral measures. Item 3 applies a comparison of norms along a tunnel to Lie products and to polynomials in one element (Supplemental Material, part 15). □

So one family has two limiting descriptions, and both hold: the whole triple converges to a point carrying a Dirac block, while the compression at each l is Connes-Lott with distance $1/m_f$. The metric compares whole algebras, and the compression is not a quantum isometry. Item 3 is a statement about this metric on scalar function algebras. The class of metric triples also excludes the Standard Model finite triple, where D_F commutes with the nonscalar subalgebra $\{(\lambda, \lambda, 0)\}$ [16, Sec. 2.6].

Proposition H. *Let P_0 be a finite-rank spectral projection, for a symmetric set, of an even triple $(A_0, H_0, D_0, \gamma_0)$, with $V = P_0 H_0$, $d = D_0|_V$ and $Q(a) = P_0 a P_0$. Let A_F be the C*-algebra generated by $Q(A_0)$.*

1. *Since $Q(1) = 1$, A_F is the bicommutant of $Q(A_0)$ in $\mathrm{End}_{\mathbb{C}}(V)$, so its structure is set by the operators commuting with all compressed functions. For the chiral interval with the two lowest levels, $Q(C(I_l))$ is a four-dimensional operator system; the grading commutes with it, and $A_F = M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$. On $\mathbb{CP}^3$ at level one, $A_F = M_4(\mathbb{C})$.*

2. *With a real structure satisfying $J_0 a J_0^{-1} = a^*$, the inherited data satisfy order zero iff A_F is commutative.*

3. *On the coarse correspondence $H_0 \otimes \overline{H_0}$, with A_0 acting on the left, $\mathcal{D} = D_0 \otimes 1 + \gamma_0 \otimes \overline{D_0}$ and J the graded flip, one has a real even triple of signs $(+,+,+)$, with the same Connes distance on A_0, and its truncation $(A_F, V \otimes \overline{V}, d \otimes 1 + \gamma_V \otimes \bar{d}, J_F, \Gamma_F)$ satisfies order zero and order one. A product with a Clifford point gives the KO-6 signs and $\dim H = 2k^2$, with $k = \dim V$.*

4. *KO-6 is reached in the irreducible form by a parent with two sheets, on $(H_X \otimes \overline{H_Y}) \oplus (H_Y \otimes \overline{H_X})$ with J exchanging the summands. For equal dimensions and full matrix algebras, the truncation is the pair $M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$, $J(x, y) = (y^*, x^*)$, of the classification [17]. At twistor level one, real functions on S^4 and on $\mathbb{CP}^3$ generate the real algebras $M_2(\mathbb{H})$ and $M_4(\mathbb{C})$, giving that algebra with its bimodule of dimension 32.*

Structure of the proof. Item 1: the algebra is unital and finite-dimensional. On the interval, each chiral

block of $Q(f)$ ranges over all symmetric 2×2 matrices, which generate $M_2(\mathbb{C})$, and explicit test functions give an element supported on each block alone, $(\left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right), 0)$ and $(0, \left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right))$; simplicity of $M_2(\mathbb{C})$ then gives both summands. Item 2: $J_F Q(a)^* J_F^{-1} = Q(a)$. Items 3 and 4: the graded flip satisfies $J\mathcal{D} = \mathcal{D}J$ by direct computation, a° acts on the other tensor factor, and $[\mathcal{D}, \pi(a)] = [D_0, a] \otimes 1$. The twistor case uses $Q(x_a) = \Gamma_a/5$, the quaternionic structure j commuting with Γ_a, and $Q(w^* B w/w^* w) = (B + \mathrm{tr}(B)1)/5$ (Supplemental Material, part 10 and Supplemental Material, part 16). □

Connes and van Suijlekom observe that the C*-algebra generated by PAP is, in many examples, a noninformative full matrix algebra [20]. That is the case here apart from the symmetries of the window. The grading commutes with compressed functions and gives the block structure on the interval. The antiunitary j, with $j^2 = -1$, commutes with the compressions of real functions on S^4, which generate the real algebra $M_2(\mathbb{H})$, while the complex algebra generated by the same functions is $M_4(\mathbb{C})$. So item 1 is modest, and the information sits in those symmetries.

The order conditions in items 3 and 4 are formal: the same tensor square works for any finite even representation, and the two-sheet bimodule is already in [17, Remark 2.9]. What is specific here is that the tensor square arises from a commutative parent, namely the product Dirac operator on $X \times X$, with algebra restricted to $C(X) \otimes 1$ and J covering the flip. The Clifford point is an input and fails finite orientability. In item 4 the second sheet, the base-function restriction and the window are inputs. The nontrivial grading of the classification is added after truncation as abstract data, since an even parent cannot give a grading anticommuting with compressed functions. In this parent the two sheets are not connected, so the selection of $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ is not reached there; Proposition I supplies a connector in another parent. The triples are usually not metric, and the eigenvalues of the doubled operator add in quadrature.

Proposition I. *Let* $E = \mathrm{Hom}(V, W)$ *and* $E^* = \mathrm{Hom}(W, V)$*, with* $J(x, y) = (y^*, x^*)$*, grading* $gx = \gamma_W x$*,* $hy = -y\gamma_W$*, and the even algebra* $\mathcal{A}^{\mathrm{ev}}$ *acting by* π_1, π_2*, with opposite actions* ρ_1, ρ_2*. Write* c *for the adjoint map* $E \to E^*$*.*

1. *A block $T : E \to E^*$ added to a sheet-diagonal D_F that already has these signs and order one keeps the signs iff $T = cT^*c$ and $hT + Tg = 0$. It satisfies order one on an involutive $B \subseteq \mathcal{A}^{\mathrm{ev}}$ iff $T\pi_1(a) - \pi_2(a)T$ intertwines $\rho_1(b)$ and $\rho_2(b)$ for all $a, b \in B$. The maximal such B are the maximal $\mathcal{A}(R) = \{a : \pi_2(a)R = R\pi_1(a),\ \pi_2(a^*)R = R\pi_1(a^*)\}$, over odd R with $T = R + cR^*c$.*

2. *Let $W = W_+ \oplus W_- = \mathbb{H} \oplus \mathbb{H}$, $V = \mathbb{C}^4$, and let $w \in W_+$, $v \in V$ be unit vectors. Then, for $\kappa \neq 0$, $T_\kappa x = \kappa K x K$, with $K = vw^*$, has complex rank one, and $\mathcal{A}(T_\kappa) = \{(\lambda \oplus q, \lambda \oplus m)\} \cong \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, the algebra of the classification [17].*

3. *Let Q, U be massless interval fields valued in E, E^*, with the boundary conditions of Theorem C and $X = tS^*$, where $t > 0$ and $S = |ce\rangle\langle e|$ for a unit vector e with $ge = e$. The constant multipliers preserving the domain are exactly $B = \mathcal{A}(S)$. With $\theta = \arctan t$ and $m = \theta/l$, a symmetric cutoff in $(m, (\pi - \theta)/l)$ gives the finite data $(B, E \oplus E^*, m(S + S^*), J, \Gamma)$, of signs $(+, +, -)$, with order zero and order one. For $t(l) = \tan(ml)$, the mass stays m and the next level diverges.*

4. *The transport $\exp(mxF)$, with F rotating the coupled channels of Q and U, carries D_X to the uncoupled operator plus a constant term mF times the Clifford generator, so after a unitary change of domain the wall coupling is a constant matrix connection with relative endpoint rotation angle $\theta = ml$.*

5. *For scalar multipliers and $X \neq 0$, preservation of the domain is the gluing $f_Q(l) = f_U(l)$, and this does not pass to the truncation: in the one-flavor window the compressions generate $\mathbb{C} \oplus \mathbb{C}$ with $D = m\sigma_1$ and $J = \sigma_1 K$, which fails order one. For separate compressions in unchanged windows, including the level-one twistor windows, scalar gluing on closed sets with empty interior, which allows independent functions off those sets, leaves the image unchanged.*

Structure of the proof. Item 1: block multiplication, then averaging over the unitary group of B, which separates the part of T intertwining the left actions from the part intertwining the opposite ones. Item 2: the intertwining equations make $\mathbb{C}w$ and $\mathbb{C}v$ reducing with one common

scalar, of real dimension $2+4+18=24$; maximality and uniqueness up to automorphism follow the argument of [17], where a block of rank two gives dimension 16. Item 3: the coupled channels have the eigenvectors $\cos(mx)$ and $\sin(mx)$ with explicit chiral supports, the other channels are chiral zero modes, and $cX = X^*c^{-1}$ preserves the domain. Item 5: for the window statement, a functional on the compressed image is integration against a smooth density, which vanishes if it vanishes off a closed set with empty interior (Supplemental Material, part 20). □

Items 2 and 3 realize the classified connecting block in an exact parent, on $E = \mathrm{Hom}(\mathbb{C}^4, \mathbb{H}^2)$ with $\dim H_F = 32$. The Hom fibres, the constant matrix multipliers, the grading and the coupled channel are inputs, and deriving them from the commutative twistor parent remains open. On B the block commutes with the algebra, so it supplies no finite one-forms, and the Yukawa block needs further sheet-diagonal terms, which one wall coupling can carry together with it [31]. With w along ν_R and v along the lepton line, T has the form of the one-generation Majorana block [16], and the bound $m < \pi/(2l)$ ties that mass to the first level of the tower; a general three-generation Majorana matrix needs further flavour data.

We propose reading the connector as a marked pair of points with a coupling. Since $K = vw^*$, its data are a point of $\mathbb{CP}^3$ on each sheet and κ. At fixed $|\kappa|$ the line choices are conjugate, since $Sp(2)$ and $U(4)$ act transitively, and $|\kappa|$ remains a modulus. On the twistor window, w lies over a point p of S^4, and the compressed height function $5Q(p \cdot x) = p_a \Gamma_a$ can serve as added finite grading after restricting to its even algebra. This evades the inheritance hypothesis of Proposition H and does not derive the grading, and a parent with an interface at those points has not been constructed.

IX. DISCUSSION

What the towerless picture gives. The finite Dirac operator of an almost-commutative geometry behaves as the remnant of internal momentum on an interval: its excited levels combine quantized momenta with bulk masses, its light masses are set by localization and boundary couplings, its texture is localization data together with the matrix $\hat{Y}$, and its chirality is carried by boundary conditions, with an ordinary index that

is stable under boundary couplings and an equivariant index read on the stabilizer of the coupling. In the families treated, the emergence limit needs the tower, and the collapse limit keeps a finite light sector along a tuned bulk mass or a scaled coupling. In the reduced representation the labels 9 and 11 attach to sectors: charged paired sectors carry both, and an odd number of unpaired neutrinos carries only 11.

What remains open. The results are spectral and kinematic. The length, the bulk masses and the boundary data are free, and chirality is an input through boundary conditions, as in orbifold constructions. The results reach the Connes-Lott layer and the Yukawa block as compressions. Beyond that layer, Proposition H gives finite algebras at fixed truncation, with order zero and order one on a restricted doubled parent. Proposition I supplies an operator connecting the sheets in a parent with chosen fibres. The derivation of those fibres, the selection of a particular finite algebra, the KO-6 sign and the grading from a commutative parent, and any dynamical origin remain open. Proposition G decides the whole-triple limit in one metric. Proposition E.1 concerns one classical action at quadratic order about a prescribed background. It shows that, for a top alone, the heavy limit of the 10 to 9 proposal keeps the CCM ratio only as $m_t/\Lambda \to 0$, hence jointly with $\Lambda \to \infty$; at fixed cutoff, with a light sector retained, the vectors end up light in the families treated. The link between the KO labels and these limits, and between l and measured parameters, remains a proposal. A companion note [31] carries Theorem C to the whole finite Dirac operator, as the relative position of two gradings at the walls, with the compression as an exact average, for a matrix-valued parent. There the sum rule of Proposition E has the coefficient $(3-d)/24$, with d the spectral dimension of the interval at the cutoff, so the relation $8M_W^2 = \sum N_f m_f^2$ is its value at $d = 0$.

No-go results and their escapes. Each negative statement above holds in a stated family, at a stated order, and we state it together with its hypotheses and known escapes, as the Coleman-Mandula theorem is stated together with supersymmetry. The list below collects them. We read them as guides to where to go next; none of them closes the question it bears on, and other escapes are possible.

- No net chiral spectrum from an odd bulk: smooth closed product, standard reduction, exact isometries. *Escapes:* Boundaries and defects [29]; mas-

sive non-Killing gauge fields [28]; as our proposal, an even interior with odd limits.

- No odd factorization: full chiral Standard Model representation, involution of Theorem D. *Escapes:* Reduced algebra and representation, with vector-like sectors (Theorem D).

- Only KO-7: scalar-character neutral sector, a singular value of odd multiplicity, zero included. *Escapes:* Even multiplicity; ν_R with Dirac pairing (Theorem D).

- Empty light sector in collapse: massless type-(c) interval, fixed spectral cutoff (Proposition A.1). *Escapes:* Type-(a) zero modes; tuned bulk mass (Theorem B); scaled boundary coupling (Theorem C).

- Fixed compression not gauge covariant: local gauge transformations, fixed light projection (Proposition F). *Escapes:* A projection invariant under the transformation; covariant projection.

- First order fails: two-point compression, inherited J (Theorem A). *Escapes:* Fluctuations without order one [32]; doubled parent (Proposition H).

- Order zero fails: noncommutative A_F, J_0-invariant window, inherited pointwise reality (Proposition H). *Escapes:* Doubled or two-sheet parent; another opposite action.

- No matching with bounded fermions: leading action, positive kinetic form, stated trace, common profiles (Proposition F). *Escapes:* Tower-scale fermion, which removes the bound, while full matching needs more; other order or action.

- $M_W \to 0$ at fixed cutoff: massless complete doublet, stated heavy family, Taylor mass, $f(0) > 0$, classical quadratic action (Proposition E.1). *Escapes:* Joint limit $\Lambda \to \infty$; every doublet heavy; a doublet of fixed nonzero mass; $f(0) = 0$; quantum effective action.

- No finite nonzero radial vacuum: ray $V = cV_*$, flat background, nonnegative cutoff of unbounded support, no added term (Proposition E.1). *Escapes:* Compact support (a plateau); sign-changing cutoffs; a potential or constraint; a compact radial direction on the interval, with its tower [31].

- Proposed Γ_5 not inherited: even observables, $g \triangleq P\gamma P$, $g^2 = 1$, coordinates retained (Proposition H). *Escapes:* Graded prealgebra; added finite data with a restricted even algebra.

- Scalar gluing selects no algebra: closed sets with empty interior, independent functions off them, unchanged separate windows (Proposition I). *Escapes:* Coupled window; boundary degrees of freedom; restricted multipliers; changed representation.

- Finite limits are points with a Dirac block: connected scalar algebras, this propinquity class (Proposition G). *Escapes:* Truncation; matrix-valued parent algebras; other notions of convergence.

Relation to the literature. Theorem A: the Kaluza-Klein reading of the two-point space is in Connes-Lott [12], the two points as points of a fifth dimension in [14], the two-point model as an approximation of a manifold in [33], and a discrete Kaluza-Klein reading in [34]; compression and spectral truncation are standard [19, 20]. We did not find their application to the chiral interval, with $|m| = \pi/(2l)$ and $PC(I_l)P = \mathbb{C} \oplus \mathbb{C}$. Theorem B: wall modes with exponentially small mixing are classical [35–37], and collapse of spectral triples to a point is treated in [38] (doi:10.1112/jlms.70628), Corollary 4.1 (Corollary 3.1 in arXiv:2404.00240v1): a fixed metric triple (A, H, D) with $0 \in \mathrm{Sp}(D)$ gives $\Lambda_{\mathrm{spec}}\big((A, H, D/\varepsilon), (\mathbb{C}, \ker D, 0)\big) \to 0$ as $\varepsilon \to 0$. This is the case $\Sigma = \{0\}$, $D_F = 0$ of Proposition G (Supplemental Material, part 15, Remark 15.4). In Theorem B, $\mu(l)l \to \infty$, so the family is no rescaling of a fixed triple, and its limit keeps the block $m_f\sigma_1$; we did not find it elsewhere. Theorem C: the one-flavor spectrum is in [39], Eq. (5.6), and warped multiflavor textures are in [40]; we did not find a derivation of D_F from localization in noncommutative geometry. Theorem D: products of real triples and the choice J versus $J\Gamma$ are in [27, 41]; we did not find the factorization criterion or the results on 9 and 11. Proposition E restates [16], Eqs. (4.16) and (5.27). The top is also separate in that it is unable to form mesons or diquarks [42]. Proposition G: the tunnels follow the proof of [38], Theorem 3.4 (Theorem 2.5 in arXiv:2404.00240v1; Theorem 3.4 in v3 and in print, with the same six hypotheses). That theorem asserts that if $D = D_h + D_v$, with 0 isolated in $\mathrm{Sp}(D_v)$, satisfies

six conditions on commutators, on $\ker D_v$ and on a positive projection of A onto a subalgebra B, then the triples with $D_h + D_v/\varepsilon$ converge to $(B, \ker D_v, pD_hp)$, with p the projection on $\ker D_v$. The scalar family $A = B = \mathbb{C}$ on ℓ^2, with $D_v e_n = ne_n$ and $D_h e_n = -n^2 e_n$, meets the six conditions; at $\varepsilon = 1/(n + x/n)$, for fixed $x > 0$, the eigenvalues 0 and x persist, so the converse in Proposition G excludes the limit $(\mathbb{C}, \mathbb{C}e_0, 0)$ (Counterexample R1, ours; Supplemental Material, part R1, Section 3.1). The step that fails bounds $\|D_v\xi\|/\varepsilon$ by $\|D_\varepsilon\xi\|$ (Eq. (3.11) in print); anticommutation of D_h and D_v, which all its applications satisfy, or a uniform relative bound repairs it, and Proposition G uses the gap of D instead. Proposition H: the first-order form $D = \Delta + J\Delta J^{-1}$ appears in the classification of finite triples [43, 44], and the Hilbert-Schmidt bimodule is the coarse correspondence [45]. Inner fluctuations without order one [32, 46] apply to the two-point data: they factorize as $m|\Phi|^2\sigma_1$ for pointwise J, with a trivial gauge group, and as a link $m\Phi^2$ of charge 2 for the reflection J, whose quadratic term keeps the modulus fixed along a gauge orbit (Supplemental Material, part 19). Proposition I: the algebra $\mathcal{A}(T)$ and the maximality argument are in [17], Section 4; the parent is what is added here. As mathematics, most ingredients are known; what may be new is their combination in one limiting construction. Statements that something was not found refer to searches that are recorded, with queries, in the Supplemental Material (part L); they do not establish absence.

Open problems. (0) A family of commutative Kaluza-Klein data $(C^\infty(K_l), L^2(S_{K_l} \otimes E_l), D_l, J_l, \gamma_l)$ and notions of spectral, operator-algebraic, K-theoretic or truncation convergence under which it yields a finite real triple $(A_F, H_F, D_F, J_F, \gamma_F)$ with the structures of the current framework. (i) Which symmetries of a spectral window can select the finite algebra, and a physical origin for the doubled parent. (ii) The two sheets, the fibres and the grading from a commutative parent; and an interface realizing the connector of Proposition I there; one wall coupling carries both the connector and the Yukawa block for a matrix-valued parent [31]. (iii) Metric-compatible truncations for curved internal spaces. (iv) A lower bound for the mixing on generic data. (v) The bosonic matching beyond the sum rule, and heavy limits beyond the families treated.

PROVENANCE AND SUPPLEMENTAL MATERIAL

This paper has no listed authors. The results were produced by language models, and the part numbers below are those of the Supplemental Material, which contains the full proofs, the record of literature searches and the referee reports. Numerical checks name their language.

Result	Part	Proved by	Checked by
A, A.1	05, 08	Astra 6	Claude (Python); Terra 6.1
B	09	Claude	Claude (Python); Terra 6.1 (two errors corrected)
C	12	Claude	Claude (Python); Terra 6.1 (three errors corrected)
D	06, 07, 08	Astra 6; Claude	Claude (by hand, Python); Astra 6; Terra 6.1 (one summary corrected)
E	11	Claude	Claude; Terra 6.1 (corrections applied)
E.1	18, 21	Astra 6; Terra 6.1; Claude	Claude (Python, direct heat traces); Terra 6.1 (one error in part 21 corrected)
F	13, 14	Astra 6	Claude (by hand); Terra 6.1 (one error, four amendments)
G	15	Claude	Terra 6.1 (corrections applied)
H	10, 16, 19	Claude	Python; Terra 6.1 (four errors corrected)
I	20	Astra 6	Claude (Python, by hand); Terra 6.1
R1	R1	Claude	Claude (by hand); Terra 6.1; Astra 6 (printed text)

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