Exercise Set 1: ’t Hooft Anomaly Matching

The Dualised Standard Model — Part III Exercises

Lent Term 2027

Conventions for these exercises

Quantity	**Convention**
Dynkin index	$T(\square) = \frac{1}{2}$ per Weyl fermion
Anomaly coefficient	$A(\square) = 1$, $A(\overline{\square}) = -1$
Baryon number	$B[Q] = 1/N_c$, $B[\widetilde{Q}] = -1/N_c$
Anomaly matching	$\mathcal{A}_{\mathrm{UV}} = \mathcal{A}_{\mathrm{IR}}$ for global symmetries

References: Lecture 1, §§??–??. Anomaly coefficient definition: $A(R)\, d^{abc} = \mathrm{Tr}_R[T^a\{T^b, T^c\}]$.

Problem 1: Gauge anomaly cancellation in the Standard Model

⋆ (15 min)

Background. Gauge anomaly cancellation is a *consistency requirement*: if $\sum_{\text{L-handed Weyl}} \mathrm{Tr}[T^a\{T^b, T^c\}] \neq 0$ for the gauge group generators, the theory is quantum mechanically inconsistent. In this problem, you will verify that the Standard Model fermion content satisfies this requirement.

Setup. One generation of SM fermions under $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$:

Field	$\mathrm{SU}(3)_c$	$\mathrm{SU}(2)_L$	Y	Chirality
$Q_L = (u_L, d_L)$	$\mathbf{3}$	$\mathbf{2}$	$+1/6$	L
u_R	$\mathbf{3}$	$\mathbf{1}$	$+2/3$	R
d_R	$\mathbf{3}$	$\mathbf{1}$	$-1/3$	R
$L_L = (\nu_L, e_L)$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	L
e_R	$\mathbf{1}$	$\mathbf{1}$	-1	R

Convention: All anomaly computations use left-handed Weyl fermions. Right-handed Weyl fermions ψ_R are converted to left-handed via $\psi_R \to \overline{\psi}_L$ with conjugate representation and *negated* $\mathrm{U}(1)$ charges.

Questions.

(a) Verify that the $\mathrm{SU}(3)_c^2\,\mathrm{U}(1)_Y$ anomaly vanishes.

(b) Verify that the $\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_Y$ anomaly vanishes.

(c) Verify that the $\mathrm{U}(1)_Y^3$ anomaly vanishes.

Hint: Convert all right-handed fields to left-handed conjugates before summing. For $\mathrm{SU}(N)^2\,\mathrm{U}(1)$, the anomaly is $\sum_i T(R_i)\,Y_i$ summed over left-handed Weyl fermions only.

Solution to Problem 1. **First, convert to all left-handed Weyl fermions.** The right-handed fields become left-handed conjugates with negated hypercharge:

L-handed Weyl field	$\mathrm{SU}(3)_c$	$\mathrm{SU}(2)_L$	Y
$Q_L = (u_L,\, d_L)$	$\mathbf{3}$	$\mathbf{2}$	$+1/6$
u_L^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	$-2/3$
d_L^c	$\overline{\mathbf{3}}$	$\mathbf{1}$	$+1/3$
$L_L = (\nu_L,\, e_L)$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
e_L^c	$\mathbf{1}$	$\mathbf{1}$	$+1$

(a) $\mathrm{SU}(3)_c^2\,\mathrm{U}(1)_Y$**:**

The anomaly coefficient is $\mathcal{A} = \sum_i T(R_i^{(\mathrm{SU}(3))})\,Y_i$, where the sum runs over all left-handed Weyl fermions that carry $\mathrm{SU}(3)_c$ charge:

$$\begin{aligned}\mathcal{A}[\mathrm{SU}(3)_c^2\,\mathrm{U}(1)_Y] &= \underbrace{T(\Box)\times 2\times(+\tfrac{1}{6})}_{Q_L\text{: 2 doublet components}} + \underbrace{T(\Box)\times 1\times(-\tfrac{2}{3})}_{u_L^c} + \underbrace{T(\Box)\times 1\times(+\tfrac{1}{3})}_{d_L^c} \\ &= \tfrac{1}{2}\big(\tfrac{2}{6}-\tfrac{2}{3}+\tfrac{1}{3}\big) = \tfrac{1}{2}\big(\tfrac{1}{3}-\tfrac{2}{3}+\tfrac{1}{3}\big) = \tfrac{1}{2}\times 0 = 0. \quad \checkmark \end{aligned} \tag{1}$$

(b) $\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_Y$**:**

The anomaly coefficient is $\mathcal{A} = \sum_i T(R_i^{(\mathrm{SU}(2))})\,Y_i$, summing over left-handed Weyl fermions carrying $\mathrm{SU}(2)_L$ charge:

$$\begin{aligned}\mathcal{A}[\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_Y] &= \underbrace{T(\Box_{\mathrm{SU}(2)})\times 3\times(+\tfrac{1}{6})}_{Q_L\text{: 3 colours}} + \underbrace{T(\Box_{\mathrm{SU}(2)})\times 1\times(-\tfrac{1}{2})}_{L_L} \\ &= \tfrac{1}{2}\big(\tfrac{3}{6}-\tfrac{1}{2}\big) = \tfrac{1}{2}\big(\tfrac{1}{2}-\tfrac{1}{2}\big) = 0. \quad \checkmark \end{aligned} \tag{2}$$

(c) $\mathrm{U}(1)_Y^3$**:**

The anomaly coefficient is $\mathcal{A} = \sum_i Y_i^3$, summing over all left-handed Weyl fermions with their multiplicities:

$$
\begin{aligned}
\mathcal{A}[\mathrm{U}(1)_Y^3] = &\underbrace{3\times 2\times(+\tfrac{1}{6})^3}_{Q_L:\,\text{3 colours}\times\text{2 doublet}} + \underbrace{3\times 1\times(-\tfrac{2}{3})^3}_{u_L^c:\,\text{3 colours}} + \underbrace{3\times 1\times(+\tfrac{1}{3})^3}_{d_L^c:\,\text{3 colours}} \\
&+ \underbrace{1\times 2\times(-\tfrac{1}{2})^3}_{L_L:\,\text{2 doublet}} + \underbrace{1\times 1\times(+1)^3}_{e_L^c} \\
&= 6\times\tfrac{1}{216} + 3\times(-\tfrac{8}{27}) + 3\times\tfrac{1}{27} + 2\times(-\tfrac{1}{8}) + 1 \\
&= \tfrac{6}{216} - \tfrac{24}{27} + \tfrac{3}{27} - \tfrac{2}{8} + 1 \\
&= \tfrac{1}{36} - \tfrac{21}{27} - \tfrac{1}{4} + 1. \qquad (3)
\end{aligned}
$$

Converting to a common denominator of 108:

$$
= \tfrac{3}{108} - \tfrac{84}{108} - \tfrac{27}{108} + \tfrac{108}{108} = \tfrac{3-84-27+108}{108} = \tfrac{0}{108} = 0. \quad \checkmark \qquad (4)
$$

Key point: All three anomalies vanish. This is a *consistency check* on the SM fermion content—if any anomaly were non-zero, the Standard Model would be quantum mechanically inconsistent. Note that this is *not* anomaly matching; it is anomaly *cancellation*. □

Problem 2: 't Hooft anomaly matching for SU(3) QCD with $N_f = 2$

$\star\star$ **(30 min)**

Background. 't Hooft anomaly matching is a constraint on the IR spectrum: the anomaly coefficients of exact global symmetries must be equal when computed using UV or IR degrees of freedom (Lecture 1, Theorem **??**).

Setup. Consider SU(3) QCD with $N_f = 2$ massless Dirac quarks (u, d) in the fundamental representation.

- **UV theory:** 2 Dirac quarks, each decomposing into a left-handed Weyl fermion q_L^i $(i = 1, 2)$ in the $(\Box, \Box_L)$ of $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L$ and a right-handed Weyl fermion q_R^i in the $(\Box, \Box_R)$ of $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_R$. Equivalently, in all-left-handed notation: q_L^i in $(\Box, \Box_L, \mathbf{1}_R)$ and $\tilde{q}_L^i$ in $(\overline{\Box}, \mathbf{1}_L, \Box_R)$.
- **Exact global symmetry:** $G_{\text{global}} = \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B$ (the axial $\mathrm{U}(1)_A$ is broken by the ABJ anomaly and is *not* matched).
- **Baryon number:** $B[q_L] = 1/3$, $B[\tilde{q}_L] = -1/3$.
- **IR hypothesis:** Confinement with chiral symmetry breaking $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \to \mathrm{SU}(2)_V$. The massless IR spectrum consists of 3 Goldstone pions π^a $(a = 1, 2, 3$, adjoint of $\mathrm{SU}(2)_V$, $B = 0)$ and we need to check whether composite baryons contribute to anomaly matching.

Questions.

(a) Compute the UV anomaly coefficients: $\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B$, $\mathrm{SU}(2)_R^2\,\mathrm{U}(1)_B$, $\mathrm{U}(1)_B^3$, $\mathrm{U}(1)_B$-gravitational. Also explain why the purely cubic anomaly $\mathrm{SU}(2)_L^3$ vanishes identically for *any* SU(2) representation (recall that $d^{abc} = 0$ for SU(2), since all representations are pseudo-real).

(b) The IR theory has 3 massless Goldstone pions. Pions are bosons and do not contribute to fermion triangle anomalies. What massless fermion must exist in the IR for anomaly matching to work?

(c) Assuming the massless baryon is a nucleon doublet (p, n) in the fundamental of $\mathrm{SU}(2)_V$ (embedded as $(\Box_L, \Box_R)$ of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$) with $B = 1$, compute the IR anomaly coefficients and verify $\mathcal{A}_{\text{UV}} = \mathcal{A}_{\text{IR}}$ for each.

Hint for (b): The pions have $B = 0$ and are bosons. The mixed anomaly $\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B$ is non-zero in the UV. What fermion can match it?

Solution to Problem 2. **(a) UV anomaly coefficients.**

The UV theory has $N_c = 3$ colours and $N_f = 2$ flavours. In the all-left-handed convention:

- $q_L^{i\alpha}$ $(i = 1, 2;\ \alpha = 1, 2, 3)$: 6 Weyl fermions in $(\Box_3, \Box_2, \mathbf{1})$ with $B = 1/3$.
- $\tilde{q}_L^{i\alpha}$: 6 Weyl fermions in $(\overline{\Box}_3, \mathbf{1}, \Box_2)$ with $B = -1/3$.

$\underline{\mathrm{SU}(2)_L^3 \text{ and } \mathrm{SU}(2)_R^3}$: These vanish identically. For SU(2), the totally symmetric tensor $d^{abc} = \mathrm{Tr}[T^a\{T^b, T^c\}] = 0$ because all SU(2) representations are pseudo-real ($R \cong \overline{R}$), so $A(R)+A(\overline{R}) = 0$ while $A(R) = A(\overline{R})$, forcing $A(R) = 0$. Therefore:

$$\mathcal{A}_{\mathrm{UV}}[\mathrm{SU}(2)_L^3] = \mathcal{A}_{\mathrm{UV}}[\mathrm{SU}(2)_R^3] = 0. \tag{5}$$

This is a general property of SU(2)—the non-trivial anomaly matching constraints come from the *mixed* anomalies below, not the purely cubic ones.

$\underline{\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B}$: The mixed anomaly involves $T(R_{\mathrm{SU}(2)})$ and the $\mathrm{U}(1)_B$ charge. Only q_L contributes:

$$\mathcal{A}_{\mathrm{UV}}[\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B] = N_c \times T(\Box_{\mathrm{SU}(2)}) \times B[q_L] = 3 \times \tfrac{1}{2} \times \tfrac{1}{3} = \tfrac{1}{2}. \tag{6}$$

$\underline{\mathrm{U}(1)_B^3}$: Both q_L and $\tilde{q}_L$ contribute:

$$\mathcal{A}_{\mathrm{UV}}[\mathrm{U}(1)_B^3] = N_c N_f \times B[q_L]^3 + N_c N_f \times B[\tilde{q}_L]^3 = 6 \times \tfrac{1}{27} + 6 \times (-\tfrac{1}{27}) = 0. \tag{7}$$

$\underline{\mathrm{U}(1)_B\text{-gravitational}}$: Proportional to $\sum_i B_i$:

$$\mathcal{A}_{\mathrm{UV}}[\mathrm{U}(1)_B\text{-grav}] = N_c N_f \times B[q_L] + N_c N_f \times B[\tilde{q}_L] = 6 \times \tfrac{1}{3} + 6 \times (-\tfrac{1}{3}) = 0. \tag{8}$$

Summary of UV anomalies:

Anomaly	$\mathcal{A}_{\mathrm{UV}}$
$\mathrm{SU}(2)_L^3$	0 (identically)
$\mathrm{SU}(2)_R^3$	0 (identically)
$\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B$	1/2
$\mathrm{U}(1)_B^3$	0
$\mathrm{U}(1)_B$-grav	0

(b) The pions do not match the anomalies.

Pions are Goldstone bosons and have no fermion triangle diagrams. The mixed anomaly $\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B = 1/2$ in the UV, so there must be massless *fermions* in the IR to produce a non-zero anomaly coefficient.

The lightest baryons in 2-flavour QCD are the nucleons (p, n), which form a doublet under the unbroken $\mathrm{SU}(2)_V$. If the nucleons remain massless (or light compared to Λ_{QCD}—which is the hypothesis for the chiral limit), they provide the required fermionic degrees of freedom.

(c) IR anomaly matching with a nucleon doublet.

The massless nucleon doublet (p, n) transforms as follows under the unbroken $\mathrm{SU}(2)_V \subset \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$:

- As a left-handed Weyl fermion, it is in the $(\Box_L, \Box_R)$ of $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ (it transforms as a fundamental under both, which is the correct embedding of the $\mathrm{SU}(2)_V$ fundamental into $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$).
- Baryon number: $B[N] = 1$ (the nucleon is a 3-quark state).

More precisely, the nucleon doublet consists of a left-handed Weyl component N_L in $(\Box_L, \mathbf{1}_R)$ and a right-handed Weyl component N_R in $(\mathbf{1}_L, \Box_R)$. In the all-left-handed convention, N_R becomes $\widetilde{N}_L$ in $(\mathbf{1}_L, \overline{\Box}_R)$.

However, this does not produce the right matching. The correct approach is to recognise that the nucleon is a *Dirac* fermion: the Dirac nucleon doublet contains N_L (left-handed, $\square$ of $\mathrm{SU}(2)_L$, $\mathbf{1}$ of $\mathrm{SU}(2)_R$, $B=1$) and N_R (right-handed, $\mathbf{1}$ of $\mathrm{SU}(2)_L$, $\square$ of $\mathrm{SU}(2)_R$, $B=1$). In the all-left-handed convention, $N_R \to \tilde{N}_L$ with $\overline{\square}_R$ and $B=-1$ (as conjugation negates the U(1) charge for the right-handed component when converting to a left-handed antiparticle).

Wait—this is actually subtle. Let us be careful. For a Dirac nucleon with baryon number $B=1$:

- N_L: left-handed, $\square_L$, $\mathbf{1}_R$, $B=+1$.
- N_R: right-handed, $\mathbf{1}_L$, $\square_R$, $B=+1$.

Convert N_R to a left-handed field: $N_R \to \tilde{N}_L^c$ (left-handed, $\mathbf{1}_L$, $\overline{\square}_R$, $B=-1$).

Now compute the IR anomalies:

As noted in part (a), $\mathrm{SU}(2)_L^3$ and $\mathrm{SU}(2)_R^3$ are identically zero ($d^{abc}=0$ for SU(2)), so the non-trivial constraints come from the mixed anomalies.

<u>$\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B$ (the key non-trivial check):</u>

UV (from part (a)): $\mathcal{A}_{\mathrm{UV}}[\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B]=\frac{1}{2}$.

IR (N_L contributes with $B=1$):

$$\mathcal{A}_{\mathrm{IR}}[\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B]=1\times T(\square_{\mathrm{SU}(2)})\times 1=\tfrac{1}{2}. \tag{9}$$

Match: $\frac{1}{2}=\frac{1}{2}$. ✓

<u>$\mathrm{SU}(2)_R^2\,\mathrm{U}(1)_B$:</u>

UV:

$$\mathcal{A}_{\mathrm{UV}}[\mathrm{SU}(2)_R^2\,\mathrm{U}(1)_B]=3\times T(\square_{\mathrm{SU}(2)})\times(-\tfrac{1}{3})=-\tfrac{1}{2}. \tag{10}$$

(Using $B[\tilde{q}_L]=-1/3$ for the fields transforming under $\mathrm{SU}(2)_R$.)

IR ($\tilde{N}_L^c$ has $\overline{\square}_R$ and $B=-1$): the Dynkin index $T(\overline{\square})=T(\square)=1/2$, so

$$\mathcal{A}_{\mathrm{IR}}[\mathrm{SU}(2)_R^2\,\mathrm{U}(1)_B]=1\times T(\square_{\mathrm{SU}(2)})\times(-1)=-\tfrac{1}{2}. \tag{11}$$

Match: $-\frac{1}{2}=-\frac{1}{2}$. ✓

<u>$\mathrm{U}(1)_B^3$:</u>

UV: 0 (computed above).

IR: N_L has $B=+1$, $\tilde{N}_L^c$ has $B=-1$:

$$\mathcal{A}_{\mathrm{IR}}[\mathrm{U}(1)_B^3]=2\times(+1)^3+2\times(-1)^3=2-2=0. \tag{12}$$

(The factor of 2 accounts for the two components of each SU(2) doublet.)

Match: $0=0$. ✓

<u>$\mathrm{U}(1)_B$-gravitational:</u>

UV: 0.

IR:

$$\mathcal{A}_{\mathrm{IR}}[\mathrm{U}(1)_B\text{-grav}]=2\times(+1)+2\times(-1)=0. \tag{13}$$

Match: $0=0$. ✓

Conclusion: All anomaly matching conditions are satisfied by the IR spectrum consisting of 3 massless Goldstone pions (which do not contribute to anomalies) plus a massless nucleon Dirac doublet. The $\mathrm{SU}(2)^3$ purely cubic anomaly vanishes identically for SU(2) (no d-symbol), so the non-trivial checks are the mixed anomalies $\mathrm{SU}(2)^2_{L,R}\,\mathrm{U}(1)_B$.

Remark: In the real world, the nucleon mass $m_N \approx 940$ MeV is not zero. Anomaly matching applies in the *chiral limit* $m_q \to 0$. The matching tells us that if the quarks were exactly massless, the nucleon would need to be massless too (or else the anomaly matching would require massless baryonic fermions in another representation). $\square$

Problem 3: General anomaly matching for $\mathrm{SU}(N_c)$ with N_f fundamentals ★★ (30 min)

Background. We now compute the UV anomaly coefficients for the most general case that underpins all subsequent anomaly matching in these lectures.

Setup. $\mathrm{SU}(N_c)$ gauge theory with N_f massless Dirac flavours in the fundamental representation. In the all-left-handed convention:

- Q_i^α ($i = 1, \dots, N_f$; $\alpha = 1, \dots, N_c$): $N_c N_f$ left-handed Weyl fermions in $(\Box_{\mathrm{SU}(N_c)}, \Box_{\mathrm{SU}(N_f)_L}, \mathbf{1}_{\mathrm{SU}(N_f)_R})$ with baryon number $B[Q] = 1/N_c$.
- $\widetilde{Q}^{i\alpha}$: $N_c N_f$ left-handed Weyl fermions in $(\overline{\Box}_{\mathrm{SU}(N_c)}, \mathbf{1}_{\mathrm{SU}(N_f)_L}, \Box_{\mathrm{SU}(N_f)_R})$ with baryon number $B[\widetilde{Q}] = -1/N_c$.

Exact global symmetry for anomaly matching: $G_{\text{exact}} = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B$. (The axial $\mathrm{U}(1)_A$ is anomalous under $\mathrm{SU}(N_c)$ and is *not* matched.)

Questions.

(a) Compute all 6 independent UV anomaly coefficients:

  (i) $\mathrm{SU}(N_f)_L^3$
  (ii) $\mathrm{SU}(N_f)_R^3$
  (iii) $\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_B$
  (iv) $\mathrm{SU}(N_f)_R^2\,\mathrm{U}(1)_B$
  (v) $\mathrm{U}(1)_B^3$
  (vi) $\mathrm{U}(1)_B$-gravitational

(b) State which of these must be matched by any proposed IR spectrum.

(c) Explain why $\mathrm{U}(1)_A$ is not included in the matching conditions.

Hint: For $\mathrm{SU}(N_f)_L^3$, the Q_i transform as the fundamental of $\mathrm{SU}(N_f)_L$, and there are N_c copies (one per colour).

Solution to Problem 3. **(a) UV anomaly coefficients.**

(i) $\mathrm{SU}(N_f)_L^3$: Only Q_i^α transforms under $\mathrm{SU}(N_f)_L$. For each fixed colour α, the N_f fields Q_i^α form the fundamental of $\mathrm{SU}(N_f)_L$ with anomaly coefficient $A(\Box) = 1$. Summing over N_c colours:

$$\boxed{\mathcal{A}_{\mathrm{UV}}[\mathrm{SU}(N_f)_L^3] = N_c.} \tag{14}$$

(ii) $\mathrm{SU}(N_f)_R^3$: By $L \leftrightarrow R$ symmetry (the $\widetilde{Q}^i$ play the role of Q_i for $\mathrm{SU}(N_f)_R$):

$$\boxed{\mathcal{A}_{\mathrm{UV}}[\mathrm{SU}(N_f)_R^3] = N_c.} \tag{15}$$

<u>(iii) $\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_B$</u>: This is a mixed anomaly: two $\mathrm{SU}(N_f)_L$ generators and one $\mathrm{U}(1)_B$ generator. The Q_i^α contribute with Dynkin index $T(\Box_{\mathrm{SU}(N_f)}) = 1/2$ and baryon number $B = 1/N_c$:

$$\boxed{\mathcal{A}_{\mathrm{UV}}[\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_B] = N_c \times T(\Box_{\mathrm{SU}(N_f)}) \times \frac{1}{N_c} = N_c \times \frac{1}{2} \times \frac{1}{N_c} = \frac{1}{2}.} \tag{16}$$

<u>(iv) $\mathrm{SU}(N_f)_R^2\,\mathrm{U}(1)_B$</u>: The $\widetilde{Q}^i$ have $T(\Box_{\mathrm{SU}(N_f)}) = 1/2$ and $B = -1/N_c$:

$$\boxed{\mathcal{A}_{\mathrm{UV}}[\mathrm{SU}(N_f)_R^2\,\mathrm{U}(1)_B] = N_c \times \frac{1}{2} \times \left(-\frac{1}{N_c}\right) = -\frac{1}{2}.} \tag{17}$$

<u>(v) $\mathrm{U}(1)_B^3$</u>: Summing B^3 over all left-handed Weyl fermions:

$$\mathcal{A}_{\mathrm{UV}}[\mathrm{U}(1)_B^3] = N_c N_f \times \left(\frac{1}{N_c}\right)^3 + N_c N_f \times \left(-\frac{1}{N_c}\right)^3 = \frac{N_f}{N_c^2} - \frac{N_f}{N_c^2} = 0. \tag{18}$$

$$\boxed{\mathcal{A}_{\mathrm{UV}}[\mathrm{U}(1)_B^3] = 0.} \tag{19}$$

<u>(vi) $\mathrm{U}(1)_B$-gravitational</u>: The gravitational anomaly coefficient is $\sum_i B_i$:

$$\mathcal{A}_{\mathrm{UV}}[\mathrm{U}(1)_B\text{-grav}] = N_c N_f \times \frac{1}{N_c} + N_c N_f \times \left(-\frac{1}{N_c}\right) = N_f - N_f = 0. \tag{20}$$

$$\boxed{\mathcal{A}_{\mathrm{UV}}[\mathrm{U}(1)_B\text{-grav}] = 0.} \tag{21}$$

Summary:

Anomaly triple	$\mathcal{A}_{\mathrm{UV}}$	**Depends on**
$\mathrm{SU}(N_f)_L^3$	N_c	$A(\Box) = 1$, colour multiplicity
$\mathrm{SU}(N_f)_R^3$	N_c	$L \leftrightarrow R$ symmetry
$\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_B$	$+1/2$	$T(\Box) = 1/2$, $B = 1/N_c$
$\mathrm{SU}(N_f)_R^2\,\mathrm{U}(1)_B$	$-1/2$	$T(\Box) = 1/2$, $B = -1/N_c$
$\mathrm{U}(1)_B^3$	0	Q and $\widetilde{Q}$ cancel
$\mathrm{U}(1)_B$-grav	0	Q and $\widetilde{Q}$ cancel

These are exactly the UV anomaly coefficients tabulated in Lecture 1, §**??**. They will be the primary tool for verifying Seiberg duality in Exercise Set 3.

(b) *All six* anomaly coefficients must be matched by any proposed IR spectrum. If even one coefficient fails to match, the proposed spectrum is ruled out.

(c) Why $\mathrm{U}(1)_A$ is excluded. The axial symmetry $\mathrm{U}(1)_A$ (under which $Q \to e^{i\alpha}Q$ and $\widetilde{Q} \to e^{i\alpha}\widetilde{Q}$) is *anomalous* with respect to the gauge group $\mathrm{SU}(N_c)$. The ABJ anomaly explicitly breaks $\mathrm{U}(1)_A$ to the discrete subgroup $\mathbb{Z}_{2N_f}$. Since $\mathrm{U}(1)_A$ is not an exact symmetry of the quantum theory, the 't Hooft matching argument (which relies on gauging the global symmetry via spectator fields) does not apply to it. The $\mathrm{U}(1)_A$ anomaly is a genuine symmetry *breaking*, not a matching constraint. □

Problem 4: Anomaly matching with antisymmetric matter $\star\star\star\star$ (45 min)

Background. This problem extends anomaly matching to representations beyond the fundamental. The antisymmetric representation $\wedge^2$ appears in theories with SU(5) unification and will feature prominently in the dualised Standard Model (Lectures 6–7).

Setup. Consider SU(N) gauge theory with the following matter content (all Weyl fermions, left-handed):

- One Weyl fermion Ψ in the antisymmetric representation $\wedge^2$ (dimension $N(N-1)/2$).
- $(N-4)$ Weyl fermions χ_j ($j = 1, \dots, N-4$) in the antifundamental $\overline{\Box}$ (dimension N).

Questions.

(a) **Gauge anomaly cancellation.** Using $A(\wedge^2) = N-4$ and $A(\overline{\Box}) = -1$, verify that the gauge anomaly vanishes:

$$A(\wedge^2) + (N-4) \times A(\overline{\Box}) = (N-4) + (N-4) \times (-1) = 0.$$

(b) **Global symmetries.** Identify the global symmetry group. The $(N-4)$ antifundamentals χ_j transform as the fundamental of a global SU($N-4$) flavour symmetry. What U(1) global symmetries exist?

(c) **UV anomaly coefficient.** Compute the SU$(N-4)^3$ anomaly in the UV. Only the χ_j transform under SU($N-4$).

(d) **Special case:** SU(5)**.** For $N = 5$: the theory has one antisymmetric (**10**) and one antifundamental ($\overline{\mathbf{5}}$). This is a *chiral* gauge theory (the fermion content is not vector-like).

(i) Verify gauge anomaly cancellation for SU(5): $A(\wedge^2) = 5 - 4 = 1$ and $A(\overline{\Box}) = -1$.

(ii) Explain why this theory has no gauge-invariant mass term (unlike vector-like theories with $Q + \widetilde{Q}$).

(iii) Discuss why anomaly matching is particularly constraining for chiral theories.

Hint for (b): Assign U(1) charges by requiring that no gauge-invariant operator is charged under the supposed U(1). You may also use the mixed SU$(N)^2\,$U(1) anomaly cancellation condition to fix the U(1) charges.

Solution to Problem 4. **(a) Gauge anomaly cancellation.**

The gauge anomaly coefficient is:

$$\begin{aligned}\mathcal{A}_{\text{gauge}} &= A(\wedge^2) + (N-4) \times A(\overline{\Box}) \\ &= (N-4) + (N-4) \times (-1) \\ &= (N-4) - (N-4) = 0. \quad \checkmark\end{aligned} \tag{22}$$

This is the requirement for quantum consistency: the antisymmetric representation and $(N-4)$ antifundamentals produce equal and opposite cubic anomaly coefficients.

(b) Global symmetries.

The $(N-4)$ antifundamentals χ_j rotate into each other under a global $\mathrm{SU}(N-4)$ flavour symmetry. The antisymmetric Ψ is a singlet under this $\mathrm{SU}(N-4)$.

There is also a non-anomalous U(1) symmetry. To find it, we assign charges q_Ψ and q_χ to Ψ and χ respectively, and require:

(i) The $\mathrm{SU}(N)^2\,\mathrm{U}(1)$ mixed anomaly vanishes (for the global U(1) to be non-anomalous with respect to the gauge group):

$$T(\wedge^2)\times q_\Psi + (N-4)\times T(\overline{\Box})\times q_\chi = 0.$$

Using $T(\wedge^2) = (N-2)/2$ and $T(\overline{\Box}) = 1/2$:

$$\frac{N-2}{2}\,q_\Psi + (N-4)\times\frac{1}{2}\,q_\chi = 0,$$

giving $q_\Psi/q_\chi = -(N-4)/(N-2)$.

A convenient normalisation is:

$$q_\Psi = -(N-4), \qquad q_\chi = (N-2). \tag{23}$$

The full global symmetry is:

$$G_{\text{global}} = \mathrm{SU}(N-4)\times\mathrm{U}(1). \tag{24}$$

(c) $\mathrm{SU}(N-4)^3$ anomaly in the UV.

Only the χ_j transform under $\mathrm{SU}(N-4)$. Each χ_j is in the antifundamental of $\mathrm{SU}(N)$, which has N colour components. For each colour $\alpha = 1,\dots,N$, the fields χ_j^α $(j = 1,\dots,N-4)$ form the fundamental of $\mathrm{SU}(N-4)$ (note: the flavour index runs over j, and each χ_j is in the $\overline{\Box}$ of $\mathrm{SU}(N)$, so there are N colour copies).

$$\boxed{\mathcal{A}_{\text{UV}}[\mathrm{SU}(N-4)^3] = N\times A(\Box_{\mathrm{SU}(N-4)}) = N.} \tag{25}$$

This is the anomaly coefficient that any proposed IR spectrum must match.

(d) Special case: SU(5).

For $N = 5$:

- Antisymmetric $\wedge^2$ of SU(5): the **10** representation (dimension $5\times 4/2 = 10$).
- $(N-4) = 1$ antifundamental: $\overline{\mathbf{5}}$.
- Global symmetry: $\mathrm{SU}(1)\times\mathrm{U}(1) = \mathrm{U}(1)$ (trivial flavour group since there is only one antifundamental).

(i) Gauge anomaly cancellation:

$$A(\wedge^2_{\mathrm{SU}(5)}) + 1\times A(\overline{\Box}_{\mathrm{SU}(5)}) = (5-4) + (-1) = 1-1 = 0. \quad \checkmark \tag{26}$$

(ii) This is a *chiral* gauge theory. In a vector-like theory (such as QCD with $Q+\widetilde{Q}$), the fermions come in conjugate pairs and one can always write a gauge-invariant mass term $m\,Q\widetilde{Q}$.

Here, the **10** and $\overline{\mathbf{5}}$ are *not* conjugate representations of SU(5)—the **10** is the antisymmetric tensor $\Psi^{[\alpha\beta]}$ and the $\overline{\mathbf{5}}$ is χ_α. The only gauge-invariant bilinear one could form is $\epsilon_{\alpha\beta\gamma\delta\epsilon}\Psi^{[\alpha\beta]}\Psi^{[\gamma\delta]}\chi^\epsilon$, but this is a cubic (Yukawa-type) coupling, not a mass term. There is *no* gauge-invariant fermion bilinear $\bar{\psi}\psi$, so no mass term is possible.

(iii) Anomaly matching is particularly constraining for chiral theories because:

1. **No decoupling.** In a vector-like theory, fermions can acquire a mass and decouple without affecting anomaly matching (their contribution vanishes from both UV and IR simultaneously). In a chiral theory, fermions cannot be given an arbitrary mass, so the anomaly coefficients are "protected" and provide strong constraints on the IR spectrum.

2. **No continuous deformation.** A chiral theory cannot be continuously deformed to a weakly-coupled theory (by adding a mass and taking it large), so one must rely on non-perturbative arguments. Anomaly matching is one of the few exact tools available.

3. **Topological robustness.** The anomaly coefficient is an integer (or rational, depending on normalisation) that cannot change under continuous deformation of the theory. For chiral theories, this integer is generically non-zero and thus provides a "topological" constraint on the IR physics.

Preview: The matter content $\mathbf{10} + \overline{\mathbf{5}}$ of SU(5) is exactly the matter content of one generation of SM fermions in the Georgi–Glashow SU(5) GUT. The anomaly cancellation we verified above is the GUT-level statement of the SM anomaly cancellation from Problem 1. This type of chiral matter content will appear in the dualised SM construction (Lectures 6–7). □

End of Exercise Set 1. These anomaly coefficients will be used in Exercise Set 3 to verify Seiberg duality.

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