Exercise Set 3: Seiberg Duality in Practice

The Dualised Standard Model — Part III Exercises

Lent Term 2027

Conventions for these exercises

Quantity	**Convention**
Dynkin index	$T(\Box) = T(\overline{\Box}) = \frac{1}{2}$
Cubic anomaly	$A(\Box) = 1,\ \ A(\overline{\Box}) = -1$
R-charge (superfield)	$R[Q] = (N_f - N_c)/N_f;\ \ R[W] = 2$
R-charge (fermion)	$R_{\text{ferm}} = R_{\text{sf}} - 1$ (chiral); $R_\lambda = +1$ (gaugino)
Baryon charges	$B[Q] = +1,\ \ B[\widetilde{Q}] = -1;\quad B[q] = N_c/(N_f - N_c)$
N_f counting	Dirac pairs $(Q_i, \widetilde{Q}_i)$
Magnetic theory	$\mathrm{SU}(N_f - N_c)$ with $(q, \widetilde{q}, M)$; $\quad W_{\text{mag}} = \frac{1}{\mu} M\, q\, \widetilde{q}$

References: Lecture 3 (§4, anomaly matching); Lecture 4 (§5, APS deformation).

Solutions can be cross-checked against `scripts/verify_anomaly_matching.py`.

Problem 1: Anomaly Matching for $\mathrm{SU}(3)$ with $N_f = 6$ ⋆⋆ (30 min)

Background. In Lecture 3, we verified all six independent 't Hooft anomaly coefficients as polynomial identities in N_c and N_f. In this problem, you will carry out the explicit numerical computation for the benchmark case $N_c = 3$, $N_f = 6$.

Setup. The electric theory is $\mathrm{SU}(3)$ with $6 \times (Q, \widetilde{Q})$ and $W = 0$. The magnetic dual is $\mathrm{SU}(6-3) = \mathrm{SU}(3)$ with $6 \times (q, \widetilde{q})$, gauge-singlet mesons $M^i{}_j$ (6×6 matrix), and $W_{\text{mag}} = (1/\mu)\, M\, q\, \widetilde{q}$.

Questions.

(a) **Quantum numbers.** Write down the complete quantum number tables for both theories, including both superfield and fermion R-charges. Use the values:

$$R_{\text{sf}}[Q] = (6-3)/6 = 1/2, \qquad R_{\text{ferm}}[Q] = -1/2,$$
$$R_{\text{sf}}[q] = 3/6 = 1/2, \qquad R_{\text{ferm}}[q] = -1/2,$$
$$R_{\text{sf}}[M] = 2(6-3)/6 = 1, \qquad R_{\text{ferm}}[M] = 0,$$
$$B[q] = 3/(6-3) = 1.$$

(b) **Compute all six anomaly coefficients** on both sides:

(i) $\mathrm{SU}(6)_L^3$
(ii) $\mathrm{SU}(6)_R^3$
(iii) $\mathrm{SU}(6)_L^2\,\mathrm{U}(1)_B$
(iv) $\mathrm{SU}(6)_L^2\,\mathrm{U}(1)_R$
(v) $\mathrm{U}(1)_B^2\,\mathrm{U}(1)_R$
(vi) $\mathrm{Tr}[R_{\mathrm{ferm}}]$ (gravitational anomaly)

(c) Verify that all six coefficients match.

Hint: For the mixed anomaly $\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_X$, the formula is $\mathcal{A} = \sum_f (\text{gauge mult}) \times T(R_L^{(f)}) \times X_f$, summed over all fermions charged under $\mathrm{SU}(N_f)_L$. Remember that only fermion (not superfield) R-charges appear in anomaly triangles.

Solution to Problem 1. **(a) Quantum number tables.**

Electric theory (SU(3) gauge, $N_f = 6$):

Field	$\mathrm{SU}(3)_{\mathrm{gauge}}$	$\mathrm{SU}(6)_L$	$\mathrm{SU}(6)_R$	$\mathrm{U}(1)_B$	R_{sf}	R_{ferm}
Q	$\square$	$\square$	$\mathbf{1}$	$+1$	$1/2$	$-1/2$
$\widetilde{Q}$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	-1	$1/2$	$-1/2$
λ	adj	$\mathbf{1}$	$\mathbf{1}$	0	$+1$	$+1$

Magnetic theory ($\mathrm{SU}(3)_{\mathrm{mag}}$ gauge, $N_f = 6$):

Field	$\mathrm{SU}(3)_{\mathrm{mag}}$	$\mathrm{SU}(6)_L$	$\mathrm{SU}(6)_R$	$\mathrm{U}(1)_B$	R_{sf}	R_{ferm}
q	$\square$	$\overline{\square}$	$\mathbf{1}$	$+1$	$1/2$	$-1/2$
$\widetilde{q}$	$\overline{\square}$	$\mathbf{1}$	$\square$	-1	$1/2$	$-1/2$
M	$\mathbf{1}$	$\square$	$\overline{\square}$	0	1	0
λ_{mag}	adj	$\mathbf{1}$	$\mathbf{1}$	0	$+1$	$+1$

Note: $B[q] = N_c/(N_f - N_c) = 3/3 = 1$, which coincidentally equals $B[Q]$ for this particular case.

(b) Anomaly coefficient computations.

(i) $\mathrm{SU}(6)_L^3$**.**
Electric: Only Q transforms under $\mathrm{SU}(6)_L$ (as $\square$), with 3 gauge colours:

$$\mathcal{A}_1^{\mathrm{el}} = 3 \times A(\square) = 3 \times 1 = \boxed{3}.$$

Magnetic: Two fields contribute:

- q: 3 mag. colours, $\overline{\square}$ under $\mathrm{SU}(6)_L$: $3 \times A(\overline{\square}) = 3 \times (-1) = -3$.
- M: gauge singlet, $\square$ under $\mathrm{SU}(6)_L$ with 6 copies (from the $\mathrm{SU}(6)_R$ antifund index): $6 \times A(\square) = 6 \times 1 = 6$.

$$\mathcal{A}_1^{\text{mag}} = -3 + 6 = \boxed{3}. \quad \checkmark$$

(ii) $\mathrm{SU}(6)_R^3$**.**
By left–right symmetry ($Q \leftrightarrow \widetilde{Q}$, $q \leftrightarrow \widetilde{q}$, $M^i{}_j \to M^j{}_i$):
Electric: $\widetilde{Q}$ in $\overline{\Box}$ of $\mathrm{SU}(6)_R$, 3 colours: $3 \times A(\overline{\Box}) = -3$.
Magnetic: $\widetilde{q}$ in $\Box$ of $\mathrm{SU}(6)_R$, 3 mag colours: $3 \times A(\Box) = 3$. M in $\overline{\Box}$ of $\mathrm{SU}(6)_R$ with 6 copies: $6 \times A(\overline{\Box}) = -6$. Total: $3 - 6 = -3$.

$$\mathcal{A}_2^{\text{el}} = -3 = \mathcal{A}_2^{\text{mag}}. \quad \checkmark$$

(iii) $\mathrm{SU}(6)_L^2\,\mathrm{U}(1)_B$**.**
Formula: $\mathcal{A} = \sum_f (\text{gauge mult}) \times T(R_L) \times B_f$.
Electric: Q: 3 colours, $\Box$ of $\mathrm{SU}(6)_L$, $B = +1$:

$$\mathcal{A}_3^{\text{el}} = 3 \times \tfrac{1}{2} \times 1 = \boxed{\tfrac{3}{2}}.$$

Magnetic:

- q: 3 colours, $\overline{\Box}$ of $\mathrm{SU}(6)_L$, $B = 1$: $3 \times \frac{1}{2} \times 1 = \frac{3}{2}$.
- M: $B = 0$, no contribution.

$$\mathcal{A}_3^{\text{mag}} = \boxed{\tfrac{3}{2}}. \quad \checkmark$$

(iv) $\mathrm{SU}(6)_L^2\,\mathrm{U}(1)_R$**.**
Formula: $\mathcal{A} = \sum_f (\text{gauge mult}) \times T(R_L) \times R_{\text{ferm},f}$.
Electric: Q: 3 colours, $\Box$ of $\mathrm{SU}(6)_L$, $R_{\text{ferm}} = -1/2$:

$$\mathcal{A}_4^{\text{el}} = 3 \times \tfrac{1}{2} \times (-\tfrac{1}{2}) = \boxed{-\tfrac{3}{4}}.$$

Magnetic:

- q: 3 mag colours, $\overline{\Box}$ of $\mathrm{SU}(6)_L$, $R_{\text{ferm}} = -1/2$: $3 \times \frac{1}{2} \times (-\frac{1}{2}) = -\frac{3}{4}$.
- M: 6 copies, $\Box$ of $\mathrm{SU}(6)_L$, $R_{\text{ferm}} = 0$: $6 \times \frac{1}{2} \times 0 = 0$.

$$\mathcal{A}_4^{\text{mag}} = -\tfrac{3}{4} + 0 = \boxed{-\tfrac{3}{4}}. \quad \checkmark$$

(v) $\mathrm{U}(1)_B^2\,\mathrm{U}(1)_R$**.**
Formula: $\mathcal{A} = \sum_f n_f \times B_f^2 \times R_{\text{ferm},f}$, where n_f is the total multiplicity (gauge $\times$ flavour).
Electric:

- Q: $3 \times 6 = 18$ fermions, $B = 1$, $R_{\text{ferm}} = -1/2$: $18 \times 1 \times (-1/2) = -9$.
- $\widetilde{Q}$: 18 fermions, $B = -1$, $R_{\text{ferm}} = -1/2$: $18 \times 1 \times (-1/2) = -9$.
- Gaugino: $B = 0$, no contribution.

$$\mathcal{A}_5^{\mathrm{el}} = -9 + (-9) = \boxed{-18}.$$

Magnetic:

- q: $3 \times 6 = 18$ fermions, $B = 1$, $R_{\mathrm{ferm}} = -1/2$: $18 \times 1 \times (-1/2) = -9$.
- $\widetilde{q}$: 18 fermions, $B = -1$, $R_{\mathrm{ferm}} = -1/2$: $18 \times 1 \times (-1/2) = -9$.
- M: $B = 0$, no contribution.
- Gaugino: $B = 0$, no contribution.

$$\mathcal{A}_5^{\mathrm{mag}} = -9 + (-9) = \boxed{-18}. \quad \checkmark$$

(vi) $\mathrm{Tr}[R_{\mathrm{ferm}}]$ **(gravitational).**

Electric:

- Q: 18 fermions, $R_{\mathrm{ferm}} = -1/2$: $18 \times (-1/2) = -9$.
- $\widetilde{Q}$: 18 fermions, $R_{\mathrm{ferm}} = -1/2$: -9.
- Gaugino: $3^2 - 1 = 8$ fermions, $R_{\mathrm{ferm}} = +1$: $+8$.

$$\mathcal{A}_6^{\mathrm{el}} = -9 - 9 + 8 = \boxed{-10}.$$

Magnetic:

- q: $3 \times 6 = 18$ fermions, $R_{\mathrm{ferm}} = -1/2$: -9.
- $\widetilde{q}$: 18 fermions, $R_{\mathrm{ferm}} = -1/2$: -9.
- M: $6^2 = 36$ fermions, $R_{\mathrm{ferm}} = 0$: 0.
- Gaugino: $3^2 - 1 = 8$ fermions, $R_{\mathrm{ferm}} = +1$: $+8$.

$$\mathcal{A}_6^{\mathrm{mag}} = -9 - 9 + 0 + 8 = \boxed{-10}. \quad \checkmark$$

(c) Summary table.

Coefficient	**Electric**	**Magnetic**	**Match?**
$\mathrm{SU}(6)_L^3$	3	3	✓
$\mathrm{SU}(6)_R^3$	-3	-3	✓
$\mathrm{SU}(6)_L^2\,\mathrm{U}(1)_B$	$3/2$	$3/2$	✓
$\mathrm{SU}(6)_L^2\,\mathrm{U}(1)_R$	$-3/4$	$-3/4$	✓
$\mathrm{U}(1)_B^2\,\mathrm{U}(1)_R$	-18	-18	✓
$\mathrm{Tr}[R]$ (gravitational)	-10	-10	✓

All six independent anomaly coefficients match. These values are consistent with the output of `scripts/verify_anomaly_matching.py` for $N_c = 3$, $N_f = 6$. □

Problem 2: Mass Deformation from $N_f = 6$ to $N_f = 5$ ⋆⋆⋆ (20 min)

Background. A powerful consistency check of Seiberg duality is that giving a mass to one flavour on both sides of the duality produces the correct dual pair with $N_f - 1$ flavours. This was presented in Lecture 3, §5, for general (N_c, N_f).

Questions.

Start with SU(3) SQCD with $N_f = 6$ flavours. Add a mass term for the 6th flavour: $\Delta W = m\, Q_6\, \widetilde{Q}^6$.

(a) What is the resulting **electric** theory after integrating out the massive quarks?

(b) What is the **magnetic** dual of this deformed theory? (Use the general Seiberg duality rule.)

(c) On the magnetic side of the *original* ($N_c = 3, N_f = 6$) theory, the mass deformation maps to $\Delta W = m\, M^6{}_6$ (via $M^6{}_6 = Q_6\, \widetilde{Q}^6$). Show that the F-term equation for $M^6{}_6$ forces $\langle q_6\, \widetilde{q}^6 \rangle \neq 0$, and explain how this *Higgses* the magnetic gauge group from SU(3) to SU(2).

(d) As a check, verify the anomaly coefficient $\mathrm{SU}(5)_L^3$ for the deformed dual pair $(\mathrm{SU}(3), N_f = 5) \leftrightarrow (\mathrm{SU}(2), N_f = 5)$.

Hint for (c): The total magnetic superpotential is $W = (1/\mu)\, M^i{}_j\, q_i\, \widetilde{q}^j + m\, M^6{}_6$. Setting $\partial W / \partial M^6{}_6 = 0$ gives $q_6\, \widetilde{q}^6 = -\mu\, m$.

Solution to Problem 2. **(a)** The mass term $m\, Q_6\, \widetilde{Q}^6$ makes the 6th flavour heavy. Below the scale m, integrating out Q_6 and $\widetilde{Q}^6$ leaves:

$$\boxed{\text{Electric:}\quad \mathrm{SU}(3) \text{ with } N_f = 5 \text{ flavours, } W = 0\,.}$$

(b) Applying the Seiberg duality rule to SU(3) with $N_f = 5$:

$$\text{Magnetic gauge group:}\quad \mathrm{SU}(N_f - N_c) = \mathrm{SU}(5-3) = \mathrm{SU}(2)\,,$$

with 5 magnetic quark pairs $(q_i, \widetilde{q}^i)$, a 5×5 meson matrix M', and $W_{\mathrm{mag}} = (1/\mu')\, M' q \widetilde{q}$.

$$\boxed{\text{Magnetic:}\quad \mathrm{SU}(2) \text{ with } N_f = 5 \text{ flavours } + M' + W_{\mathrm{mag}}\,.}$$

(c) The total superpotential of the magnetic SU(3) theory with the mass deformation is:

$$W = \frac{1}{\mu}\, M^i{}_j\, q_i\, \widetilde{q}^j + m\, M^6{}_6\,.$$

The F-term equation for $M^6{}_6$ is:

$$\frac{\partial W}{\partial M^6{}_6} = 0 \qquad \Longrightarrow \qquad \frac{1}{\mu}\, q_6\, \widetilde{q}^6 + m = 0 \qquad \Longrightarrow \qquad \langle q_6\, \widetilde{q}^6 \rangle = -\mu\, m \neq 0\,.$$

The nonzero VEV of q_6 (in the fundamental of $\mathrm{SU}(3)_{\mathrm{mag}}$) breaks the magnetic gauge group by the Higgs mechanism:

$$\mathrm{SU}(3)_{\mathrm{mag}} \xrightarrow{\langle q_6 \rangle \neq 0} \mathrm{SU}(2)_{\mathrm{mag}}\,.$$

After integrating out the massive fields (q_6, $\widetilde{q}^6$, $M^6{}_j$, $M^i{}_6$), the surviving magnetic theory is precisely SU(2) with 5 flavours plus the 5×5 meson M', matching the prediction of part (b). ✓

(d) Check $\mathrm{SU}(5)_L^3$ for $(\mathrm{SU}(3), N_f = 5) \leftrightarrow (\mathrm{SU}(2), N_f = 5)$.

Electric: Q in $\square$ of $\mathrm{SU}(5)_L$, 3 colours: $\mathcal{A}^{\mathrm{el}} = 3 \times 1 = 3$.

Magnetic: Magnetic gauge group is SU(2).

- q: 2 mag colours, $\overline{\square}$ of $\mathrm{SU}(5)_L$: $2 \times (-1) = -2$.
- M': 5 copies, $\square$ of $\mathrm{SU}(5)_L$: $5 \times 1 = 5$.

$\mathcal{A}^{\mathrm{mag}} = -2 + 5 = 3$.

$$\boxed{\mathcal{A}^{\mathrm{el}} = 3 = \mathcal{A}^{\mathrm{mag}}}. \quad ✓$$

□

Problem 3: R-charges and the Superpotential ⋆⋆ (15 min)

Background. The non-anomalous R-symmetry plays a central role in Seiberg duality. At the superconformal fixed point, the R-charge determines the scaling dimension of chiral primary operators via $\Delta = \frac{3}{2}|R|$.

Questions.

(a) **Derive $R[Q]$.** For $\mathrm{SU}(N_c)$ SQCD with N_f massless flavours, the anomaly-free R-symmetry satisfies $\mathrm{Tr}[\mathrm{SU}(N_c)^2\,\mathrm{U}(1)_R] = 0$. Using the fermion R-charges and the fact that the gaugino has $R_{\text{ferm}} = +1$, derive:

$$R[Q] = \frac{N_f - N_c}{N_f}\,.$$

(b) **Magnetic R-charges.** Using the Seiberg duality map and the requirement $R[W_{\text{mag}}] = 2$, determine $R[q]$, $R[\widetilde{q}]$, and $R[M]$ as functions of N_c and N_f.

(c) **Verify $R[W_{\text{mag}}] = 2$.** Show that

$$R[M] + R[q] + R[\widetilde{q}] = 2\,.$$

(d) **Conformal dimension and the conformal window.** At the IR fixed point, the conformal dimension of a chiral primary scalar operator $\mathcal{O}$ is $\Delta(\mathcal{O}) = \frac{3}{2}R[\mathcal{O}]$. Compute the dimension of the meson $M = Q\widetilde{Q}$ in the electric theory. For which value of N_f does $\Delta[M] = 1$ (free field)? What is the physical significance of this value?

Hint for (a): The anomaly coefficient $\mathrm{Tr}[\mathrm{SU}(N_c)^2\,\mathrm{U}(1)_R]$ receives contributions from all fermions charged under the gauge group: Q-fermions, $\widetilde{Q}$-fermions, and the gaugino. Each contributes $T(R) \times R_{\text{ferm}}$.

Solution to Problem 3. **(a)** The anomaly $\mathrm{Tr}[\mathrm{SU}(N_c)^2\,\mathrm{U}(1)_R] = 0$ requires the sum over all gauge-charged fermions:

$$\begin{aligned} 0 &= N_f \cdot T(\square) \cdot R_{\text{ferm}}[Q] + N_f \cdot T(\overline{\square}) \cdot R_{\text{ferm}}[\widetilde{Q}] + T(\text{adj}) \cdot R_{\text{ferm}}[\lambda] \\ &= N_f \cdot \tfrac{1}{2} \cdot (R[Q] - 1) + N_f \cdot \tfrac{1}{2} \cdot (R[Q] - 1) + N_c \cdot 1 \\ &= N_f\,(R[Q] - 1) + N_c\,. \end{aligned} \tag{1}$$

Solving for $R[Q]$:

$$N_f\,R[Q] = N_f - N_c \qquad \Longrightarrow \qquad \boxed{R[Q] = \frac{N_f - N_c}{N_f}}\,.$$

(b) The meson field M maps to the composite $Q\widetilde{Q}$, so its R-charge is:

$$R[M] = R[Q] + R[\widetilde{Q}] = 2\,\frac{N_f - N_c}{N_f}\,.$$

From the requirement $R[W_{\text{mag}}] = R[M] + R[q] + R[\widetilde{q}] = 2$ and the symmetry $R[q] = R[\widetilde{q}]$:

$$R[q] = R[\widetilde{q}] = \frac{2 - R[M]}{2} = \frac{2 - 2(N_f - N_c)/N_f}{2} = \frac{N_c}{N_f}\,.$$

$$\boxed{R[q] = R[\tilde{q}] = \frac{N_c}{N_f}\,, \qquad R[M] = \frac{2(N_f - N_c)}{N_f}\,.}$$

(c) Verification:

$$R[M] + R[q] + R[\tilde{q}] = \frac{2(N_f - N_c)}{N_f} + \frac{N_c}{N_f} + \frac{N_c}{N_f} = \frac{2N_f - 2N_c + 2N_c}{N_f} = \frac{2N_f}{N_f} = 2\,. \quad \checkmark$$

(d) The conformal dimension of the meson is:

$$\Delta[M] = \tfrac{3}{2}\,R[M] = \tfrac{3}{2} \times \frac{2(N_f - N_c)}{N_f} = \frac{3(N_f - N_c)}{N_f}\,. \tag{2}$$

Setting $\Delta[M] = 1$ (the dimension of a free scalar field):

$$\frac{3(N_f - N_c)}{N_f} = 1\,, \tag{3}$$

$$3N_f - 3N_c = N_f\,, \tag{4}$$

$$2N_f = 3N_c\,, \tag{5}$$

$$N_f = \frac{3N_c}{2}\,. \tag{6}$$

$$\boxed{\Delta[M] = 1 \quad \text{at} \quad N_f = \frac{3N_c}{2}\,.}$$

Physical significance: $N_f = 3N_c/2$ is the *lower boundary of the SUSY conformal window.* At this value, the meson becomes a free field and decouples from the interacting sector. Below $N_f = 3N_c/2$, the conformal dimension would fall below the unitarity bound $\Delta \geq 1$, signalling that the conformal description breaks down. The magnetic theory enters the free magnetic phase: it is IR-free, and the correct description is in terms of the weakly-coupled magnetic quarks and mesons.

This connects directly to Exercise Set 2, Problem 3(c), where the conformal window was determined from the unitarity bound. □

Problem 4: s-Confinement at $N_f = N_c + 1$ ⋆⋆⋆ (25 min)

Background. The case $N_f = N_c + 1$ is a limiting case of Seiberg duality where the magnetic gauge group becomes trivial. The theory *s-confines*: it confines without chiral symmetry breaking, and the low-energy description consists entirely of gauge-singlet composite fields. This was first analysed by Seiberg [1] and systematically by Intriligator and Seiberg [2].

Questions.

(a) In the Seiberg dual of $\mathrm{SU}(N_c)$ SQCD with $N_f = N_c + 1$ flavours, what is the magnetic gauge group?

(b) Since the magnetic gauge group is trivial, the magnetic "quarks" q_i and $\widetilde{q}^j$ are gauge-singlet fields. These are identified with the electric baryons:

$$B_i \longleftrightarrow q_i\,, \qquad \widetilde{B}^j \longleftrightarrow \widetilde{q}^j\,.$$

Together with the mesons $M^i{}_j$, these form the complete set of light degrees of freedom. Write the most general superpotential consistent with the global symmetries $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$.

(c) Show that the exact confining superpotential takes the form:

$$W = \frac{1}{\Lambda^{2N_c-1}}\Big(B_i\, M^i{}_j\, \widetilde{B}^j - \det M\Big)\,. \tag{7}$$

Verify the **dimensional consistency**: show that each term has dimension $[\text{mass}]^3$.

(d) For $N_c = 3$, $N_f = 4$, verify the anomaly coefficient $\mathrm{SU}(4)_L^3$ for the confined description (mesons M, baryons B_i, $\widetilde{B}^j$) against the fundamental description (Q, $\widetilde{Q}$, λ).

Hints:

(a) Apply $\mathrm{SU}(N_f - N_c) = \mathrm{SU}((N_c+1) - N_c) = \mathrm{SU}(1)$.

(c) The mesons M have dimension [mass] (see CONVENTIONS.md). The baryons B are products of N_c fundamental fields, each of dimension [mass], so $[B] = [\text{mass}]^{N_c}$. The dynamical scale Λ has $[\Lambda] = [\text{mass}]$.

(d) The baryon B_i transforms as $\overline{\square}$ of $\mathrm{SU}(N_f)_L$ (antisymmetric product of N_c fundamentals with $N_f = N_c + 1$ gives the antifundamental of $\mathrm{SU}(N_c+1)_L$).

Solution to Problem 4. **(a)** The magnetic gauge group is:

$$\mathrm{SU}(N_f - N_c) = \mathrm{SU}((N_c+1) - N_c) = \mathrm{SU}(1)\,.$$

But $\mathrm{SU}(1)$ is the trivial group (containing only the identity). Therefore:

$$\boxed{\text{There is \textbf{no} magnetic gauge group.}}$$

The magnetic theory has no gauge dynamics: all "magnetic" fields are gauge singlets.

(b) The confined degrees of freedom and their quantum numbers under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$:

Field	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$	$\mathrm{U}(1)_B$	$R_{\rm sf}$
$M^i{}_j$	$\square$	$\overline{\square}$	0	$2(N_f - N_c)/N_f = 2/(N_c+1)$
B_i	$\overline{\square}$	$\mathbf{1}$	$+N_c$	$N_c(N_f - N_c)/N_f = N_c/(N_c+1)$
$\widetilde{B}^j$	$\mathbf{1}$	$\square$	$-N_c$	$N_c/(N_c+1)$

(Here $B_i = \epsilon^{a_1\cdots a_{N_c}} Q^{i_1}_{a_1} \cdots Q^{i_{N_c}}_{a_{N_c}}$ transforms as the N_c-th antisymmetric product of $\square$ under $\mathrm{SU}(N_f)_L$; for $N_f = N_c + 1$, this is the $\overline{\square}$ of $\mathrm{SU}(N_c+1)_L$.)

The most general superpotential consistent with these symmetries is:

$$W = \frac{\alpha}{\Lambda^{2N_c-1}} B_i\, M^i{}_j\, \widetilde{B}^j + \frac{\beta}{\Lambda^{2N_c-1}} \det M\,, \tag{8}$$

where α and β are dimensionless constants. These are the only two independent invariants that can be formed:

- $B_i\, M^i{}_j\, \widetilde{B}^j$ contracts all flavour indices and has the correct R-charge, baryon charge, and flavour transformation properties.
- $\det M$ is an $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ invariant with $B = 0$ and $R = N_f \times 2/(N_c+1) = 2$ (using $N_f = N_c + 1$).

(c) The exact result from [1, 2] is $\alpha = 1$, $\beta = -1$:

$$\boxed{W = \frac{1}{\Lambda^{2N_c-1}} \Big(B_i\, M^i{}_j\, \widetilde{B}^j - \det M \Big)\,.}$$

(The relative sign and normalisation are fixed by requiring smooth limits as various masses are taken to infinity, matching the ADS superpotential for $N_f < N_c$, and by instanton calculations.)

Dimensional check. We need $[W] = [\text{mass}]^3$:

- $[M] = [\text{mass}]$ (elementary meson field).
- $[B] = [\text{mass}]^{N_c}$ (product of N_c fundamentals, each of dimension [mass]).
- $[\Lambda] = [\text{mass}]$.

For the first term:

$$\frac{[B \cdot M \cdot \widetilde{B}]}{[\Lambda^{2N_c-1}]} = \frac{[\text{mass}]^{N_c} \cdot [\text{mass}] \cdot [\text{mass}]^{N_c}}{[\text{mass}]^{2N_c-1}} = \frac{[\text{mass}]^{2N_c+1}}{[\text{mass}]^{2N_c-1}} = [\text{mass}]^2\,.$$

Wait — this gives $[\text{mass}]^2$, not $[\text{mass}]^3$. We need to be more careful about the dimension of B.

Corrected dimensional analysis. In SQCD, the elementary chiral superfield Q has classical dimension $[Q] = [\text{mass}]$ (the scalar component of a chiral superfield has mass dimension 1 in $d = 4$). The baryon operator is:

$$B = \epsilon^{a_1\cdots a_{N_c}} Q^{i_1}_{a_1} \cdots Q^{i_{N_c}}_{a_{N_c}}\,, \qquad [B] = [\text{mass}]^{N_c}\,.$$

The meson $M^i{}_j = Q^i \widetilde{Q}_j$ has $[M] = [\text{mass}]^2$ in the *electric* theory (it is a composite of two dimension-1 fields).

In the *magnetic* (confined) description, M is an elementary field with $[M] = [\text{mass}]^2$ inherited from the duality map. (Alternatively, at the conformal fixed point, the dimension is $\Delta[M] = (3/2)R[M]$; but for the superpotential counting, we use the UV dimension.)

Now:

$$\frac{[B \cdot M \cdot \widetilde{B}]}{[\Lambda^{2N_c-1}]} = \frac{[\text{mass}]^{N_c} \cdot [\text{mass}]^2 \cdot [\text{mass}]^{N_c}}{[\text{mass}]^{2N_c-1}} = \frac{[\text{mass}]^{2N_c+2}}{[\text{mass}]^{2N_c-1}} = [\text{mass}]^3 . \quad \checkmark$$

For $\det M$ with $N_f = N_c + 1$:

$$\frac{[\det M]}{[\Lambda^{2N_c-1}]} = \frac{[\text{mass}]^{2N_f}}{[\text{mass}]^{2N_c-1}} = \frac{[\text{mass}]^{2(N_c+1)}}{[\text{mass}]^{2N_c-1}} = [\text{mass}]^{2N_c+2-2N_c+1} = [\text{mass}]^3 . \quad \checkmark$$

Both terms have the correct dimension.

(d) Anomaly check for $\mathrm{SU}(4)_L^3$ with $N_c = 3$, $N_f = 4$.

Electric (fundamental) theory:

$$Q \text{ in } \square \text{ of } \mathrm{SU}(4)_L , \quad 3 \text{ colours:} \quad \mathcal{A}^{\text{el}} = 3 \times A(\square) = 3 \times 1 = 3 .$$

Confined theory:

- $M^i{}_j$: transforms as $\square$ under $\mathrm{SU}(4)_L$, with 4 copies (from $\mathrm{SU}(4)_R$ antifund index): $4 \times A(\square) = 4 \times 1 = 4$.
- B_i: transforms as $\overline{\square}$ of $\mathrm{SU}(4)_L$ (the 3-index antisymmetric product of $\square$ of $\mathrm{SU}(4)$ is $\overline{\square}$ of $\mathrm{SU}(4)$, since $\binom{4}{3} = 4$ and it transforms as the dual): $1 \times A(\overline{\square}) = 1 \times (-1) = -1$.
- $\widetilde{B}^j$: singlet under $\mathrm{SU}(4)_L$, no contribution.

$$\mathcal{A}^{\text{conf}} = 4 + (-1) + 0 = 3 .$$

$$\boxed{\mathcal{A}^{\text{el}} = 3 = \mathcal{A}^{\text{conf}}} . \quad \checkmark$$

Remark 0.1 (The s-confinement programme). The result $\mathrm{SU}(N_c)$ with $N_f = N_c + 1$ is the simplest case of *s-confinement* (confinement without chiral symmetry breaking). The systematic classification of all s-confining theories was carried out by Csaki, Schmaltz, and Skiba [3]. The existence of an exact confining superpotential (7) is one of the most striking predictions of supersymmetric dynamics.

□

End of Exercise Set 3. These exercises provide hands-on practice with the core machinery of Seiberg duality: anomaly matching, mass deformations, R-charge analysis, and s-confinement. The tools developed here will be essential when we apply duality to the Standard Model in Lectures 5–7.

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