Exercise Set 4: $\mathcal{N} = 2$ to $\mathcal{N} = 1$ Deformation

The Dualised Standard Model — Part III Exercises

Lent Term 2027

Conventions for these exercises

Quantity	**Convention**
Dynkin index	$T(\square) = T(\overline{\square}) = \frac{1}{2}$; $\quad T(\text{adj}) = N_c$
N_f counting ($\mathcal{N} = 2$)	N_f hypermultiplets; each $= (Q_i, \widetilde{Q}_i)$ in $\mathcal{N} = 1$ language
Adjoint chiral	Φ_{adj} from $\mathcal{N} = 2$ vector; **not** counted in N_f
$\mathcal{N} = 2$ superpotential	$W_{\mathcal{N}=2} = \sqrt{2}\, Q_i\, \Phi\, \widetilde{Q}^i$ (required by extended SUSY)
$\mathcal{N} = 2$ R-charges	$R[Q] = R[\widetilde{Q}] = 1$; $\quad R[\Phi] = 0$; $\quad R[W] = 2$
FI parameter	ξ, $\;[\xi] = [\text{mass}]^2$ (real parameter)
APS deformation	$W_{\text{def}} = \mu\, \text{Tr}\Phi^2$; $\quad [\mu] = [\text{mass}]$
$\mathcal{N} = 1$ R-charge	$R[Q] = (N_f - N_c)/N_f$; $\quad R[W] = 2$ (after Φ decouples)

References: Lecture 4 (§§1–5); Seiberg–Witten [1]; Argyres–Plesser–Seiberg [2].

Problem 1: $\mathcal{N} = 2$ Multiplet Decomposition and Beta Function ⋆⋆ (20 min)

Background. In Lecture 4, §1, we decomposed $\mathcal{N} = 2$ multiplets into $\mathcal{N} = 1$ language. In this problem you will carry out the decomposition explicitly, count degrees of freedom, and verify the $\mathcal{N} = 2$ beta function using the $\mathcal{N} = 1$ formula.

Setup. Consider $\mathcal{N} = 2$ SU(N_c) SQCD with N_f hypermultiplets.

Questions.

(a) Write out the complete $\mathcal{N} = 1$ field content of the $\mathcal{N} = 2$ theory. For each $\mathcal{N} = 1$ multiplet, state the field type (vector or chiral), the representation under SU(N_c), and the component fields (bosons + fermions). Organise your answer in a table.

(b) Count the on-shell degrees of freedom per hypermultiplet. Show that one $\mathcal{N} = 2$ hypermultiplet contains $4 + 4$ real bosonic and fermionic degrees of freedom respectively.

(c) Compute the $\mathcal{N} = 1$ one-loop beta function coefficient b_0 for this field content using the formula

$$b_0 = 3\, T(\text{adj}) - \sum_{\text{chirals}} T(R_i)\,.$$

Show that $b_0 = 2N_c - N_f$, matching the $\mathcal{N} = 2$ result from Lecture 4, Eq. (2).

(d) Evaluate your result for two cases:

(i) SU(2) with $N_f = 4$ hypermultiplets. Is the theory asymptotically free?
(ii) SU(3) with $N_f = 5$ hypermultiplets. Is the theory asymptotically free?

(e) The $\mathcal{N} = 2$ superpotential is $W_{\mathcal{N}=2} = \sqrt{2}\, Q_i\, \Phi\, \widetilde{Q}^i$.

(i) Verify that $[W] = [\text{mass}]^3$.
(ii) Using the $\mathcal{N} = 2$ R-charges $R[Q] = R[\widetilde{Q}] = 1$ and $R[\Phi] = 0$, verify $R[W] = 2$.

Hints:

- For (a): remember that the $\mathcal{N} = 2$ vector multiplet decomposes into one $\mathcal{N} = 1$ vector *and* one $\mathcal{N} = 1$ chiral in the adjoint. The adjoint chiral is part of the gauge sector, not matter.
- For (c): the chiral multiplets contributing to the beta function are Φ (adjoint) plus the $2N_f$ fundamentals/antifundamentals from the hypermultiplets.

Solution to Problem 1. **(a) $\mathcal{N} = 1$ decomposition.**

$\mathcal{N} = 2$ **origin**	$\mathcal{N} = 1$ **type**	**Field**	SU(N_c) **rep.**	**Components**
Vector mult.	Vector	$V = (A_\mu, \lambda)$	adj	gauge boson + gaugino
(gauge sector)	Chiral	$\Phi = (\phi, \psi, F)$	adj	complex scalar + Weyl fermion
Hypermultiplet	Chiral	$Q_i = (q_i, \psi_{Q_i}, F_{Q_i})$	$\square$	complex scalar + Weyl fermion
($\times N_f$)	Chiral	$\widetilde{Q}_i = (\tilde{q}_i, \psi_{\widetilde{Q}_i}, F_{\widetilde{Q}_i})$	$\overline{\square}$	complex scalar + Weyl fermion

Key point: The adjoint chiral Φ is part of the $\mathcal{N} = 2$ gauge sector and is *not* counted in N_f.

(b) Degree-of-freedom counting per hypermultiplet.

One hypermultiplet contains two $\mathcal{N} = 1$ chiral multiplets (Q_i and $\widetilde{Q}_i$). Each chiral multiplet has:

- **Bosons:** one complex scalar = 2 real on-shell DOF.
- **Fermions:** one Weyl fermion = 2 real on-shell DOF.

Therefore one hypermultiplet has:

$$\text{Bosonic DOF} = 2 + 2 = 4\,, \qquad \text{Fermionic DOF} = 2 + 2 = 4\,.$$

$$\boxed{\text{One hypermultiplet:}\quad 4 + 4 \text{ real on-shell DOF.}}$$

This is the standard $\mathcal{N} = 2$ matter multiplet with equal numbers of bosonic and fermionic degrees of freedom, as required by supersymmetry.

(c) Beta function computation.

The $\mathcal{N} = 1$ one-loop beta function coefficient is:

$$b_0 = 3\,T(\text{adj}) - \sum_{\text{chirals}} T(R_i)\,.$$

Gauge contribution: $3\,T(\text{adj}) = 3N_c$.

Chiral multiplet contributions:

- Φ in adj: $\quad T(\text{adj}) = N_c$.
- N_f copies of Q_i in $\square$: $\quad N_f \times T(\square) = N_f \times \frac{1}{2} = \frac{N_f}{2}$.
- N_f copies of $\widetilde{Q}_i$ in $\overline{\square}$: $\quad N_f \times T(\overline{\square}) = N_f \times \frac{1}{2} = \frac{N_f}{2}$.

Total chiral contribution:

$$\sum_{\text{chirals}} T(R_i) = N_c + \frac{N_f}{2} + \frac{N_f}{2} = N_c + N_f\,.$$

Therefore:

$$b_0 = 3N_c - (N_c + N_f) = 2N_c - N_f\,. \tag{1}$$

$$\boxed{b_0^{\mathcal{N}=1} = 2N_c - N_f = b_0^{\mathcal{N}=2}\,.} \quad \checkmark$$

The $\mathcal{N} = 1$ computation reproduces the $\mathcal{N} = 2$ result exactly, as it must—the two descriptions are equivalent.

(d) Specific cases.

(i) SU(2) *with* $N_f = 4$*:*

$$b_0 = 2(2) - 4 = 0\,.$$

$b_0 = 0$: the theory is **exactly conformal** at one loop. It is not asymptotically free; the coupling does not run. This is the boundary of the $\mathcal{N} = 2$ asymptotic freedom window $N_f < 2N_c$.

(ii) SU(3) *with* $N_f = 5$*:*

$$b_0 = 2(3) - 5 = 1 > 0\,.$$

$b_0 > 0$: the theory is **asymptotically free**.

(e) Superpotential checks.

(i) Dimensional analysis: In natural units ($\hbar = c = 1$), chiral superfield scalars have $[\phi] = [\text{mass}]$. Therefore:

$$[Q] = [\text{mass}], \qquad [\Phi] = [\text{mass}], \qquad [\widetilde{Q}] = [\text{mass}].$$

$$[W_{\mathcal{N}=2}] = [\sqrt{2}\,Q\,\Phi\,\widetilde{Q}] = [\text{mass}] \times [\text{mass}] \times [\text{mass}] = [\text{mass}]^3\,. \quad \checkmark$$

(ii) R-charge check: Using the $\mathcal{N} = 2$ R-charges $R[Q] = R[\widetilde{Q}] = 1$ and $R[\Phi] = 0$:

$$R[W_{\mathcal{N}=2}] = R[Q] + R[\Phi] + R[\widetilde{Q}] = 1 + 0 + 1 = 2\,. \quad \checkmark$$

Remark 0.1 ($\mathcal{N} = 2$ vs. $\mathcal{N} = 1$ R-charges). The $\mathcal{N} = 2$ R-charges $R[Q] = 1$, $R[\Phi] = 0$ differ from the $\mathcal{N} = 1$ values $R[Q] = (N_f - N_c)/N_f$. The two R-symmetries coincide only at the $\mathcal{N} = 2$ conformal point $N_f = 2N_c$ (where $(N_f - N_c)/N_f = 1/2 \neq 1$). The $\mathcal{N} = 2$ R-symmetry is a specific linear combination of the $\mathcal{N} = 1$ R-symmetry and the extra $\mathrm{SU}(2)_R$ symmetry of the extended algebra.

$\square$

Problem 2: Fayet–Iliopoulos D-term and Symmetry Breaking ⋆⋆ (25 min)

Background. In Lecture 4, §4, we introduced the Fayet–Iliopoulos D-term for U(1) gauge factors. The FI parameter ξ plays a crucial role in the $\mathcal{N} = 2$ story: on the Coulomb branch, the effective FI parameters at singular points can trigger monopole condensation. This exercise develops the mechanics of the FI D-term in a simple abelian setting.

Setup. Consider a U(1) gauge theory with gauge coupling g and one chiral superfield Φ of charge $q = +1$, with no superpotential ($W = 0$). An FI D-term with parameter ξ is present.

Questions.

(a) Write down the D-term scalar potential:

$$V_D = \frac{1}{2}\big(g\,|\phi|^2 + \xi\big)^2.$$

Find the vacuum configuration (the field value $\langle|\phi|^2\rangle$ that minimises V_D) for:

(i) $\xi > 0$;

(ii) $\xi < 0$.

In each case, determine whether SUSY is preserved ($V_D = 0$) or broken ($V_D > 0$), and whether the U(1) gauge symmetry is broken.

(b) **Dimensional analysis.** Verify:

(i) $[\xi] = [\text{mass}]^2$.

(ii) For the SUSY-preserving case ($\xi < 0$), the VEV $\langle|\phi|^2\rangle = |\xi|/g$ has the correct dimensions.

(c) For the SUSY-preserving case ($\xi < 0$), compute the gauge boson mass m_A arising from the Higgs mechanism. Show that

$$m_A^2 = 2g^2\,\langle|\phi|^2\rangle = 2g\,|\xi|\,,$$

and verify $[m_A] = [\text{mass}]$.

(d) Now add a second chiral superfield $\widetilde{\Phi}$ of charge $q = -1$, still with $W = 0$. The D-term potential becomes:

$$V_D = \frac{1}{2}\big(g(|\phi|^2 - |\tilde{\phi}|^2) + \xi\big)^2.$$

Show that for *any* sign of ξ, the D-flatness condition $g(|\phi|^2 - |\tilde{\phi}|^2) + \xi = 0$ can be satisfied. Give explicit VEV configurations for $\xi > 0$ and $\xi < 0$.

(e) **Connection to the $\mathcal{N} = 2$ Coulomb branch.** In the $\mathcal{N} = 2$ context, the $\mathrm{U}(1)^{N_c-1}$ factors on the Coulomb branch each carry effective FI parameters. When the APS deformation $W_{\text{def}} = \mu\,\mathrm{Tr}\Phi^2$ lifts the Coulomb branch and drives the theory to singular points, the effective FI parameters force the massless monopoles to condense (acquire VEVs), analogous to the $\xi < 0$ case of part (a)(ii). This monopole condensation confines the original electric charges by the dual Meissner effect. Summarise in one or two sentences why this provides a mechanism for confinement.

Hints:

- For (a): the D-flatness condition is $D = g|\phi|^2 + \xi = 0$. When $\xi > 0$, the required $|\phi|^2$ would be negative, which is impossible.
- For (c): the gauge boson mass arises from the kinetic term $|D_\mu\phi|^2 = |(\partial_\mu - igA_\mu)\phi|^2$ evaluated at the VEV $\langle\phi\rangle = v$.
- For (d): with two fields of opposite charge, you have two VEVs to choose. The D-flatness condition is one equation in two unknowns.

Solution to Problem 2. **(a) Vacuum analysis.**

The D-term scalar potential is:

$$V_D = \frac{1}{2}\big(g\,|\phi|^2 + \xi\big)^2.$$

SUSY requires $V_D = 0$, *i.e.* $D = g\,|\phi|^2 + \xi = 0$, which gives $|\phi|^2 = -\xi/g$.

(i) $\xi > 0$: The equation $|\phi|^2 = -\xi/g < 0$ has no solution (since $|\phi|^2 \geq 0$). The minimum of V_D is at $\phi = 0$:

$$V_D^{\min} = \frac{\xi^2}{2} > 0\,.$$

$$\boxed{\xi > 0: \quad \text{SUSY broken } (V_D > 0), \quad \text{U(1) unbroken } (\langle\phi\rangle = 0).}$$

(ii) $\xi < 0$: Now $|\phi|^2 = -\xi/g = |\xi|/g > 0$ has a valid solution:

$$\langle|\phi|^2\rangle = \frac{|\xi|}{g}\,, \qquad V_D = 0\,.$$

$$\boxed{\xi < 0: \quad \text{SUSY preserved } (V_D = 0), \quad \text{U(1) broken } (\langle\phi\rangle \neq 0).}$$

(b) Dimensional analysis.

(i) The D-term potential has $[V_D] = [\text{mass}]^4$. Since $V_D = (1/2)(g|\phi|^2+\xi)^2$ and $[g] = [\text{dimensionless}]$ in 4D, $[|\phi|^2] = [\text{mass}]^2$, we need:

$$[(g|\phi|^2 + \xi)^2] = [\text{mass}]^4 \qquad \Longrightarrow \qquad [g|\phi|^2 + \xi] = [\text{mass}]^2\,.$$

Since $[g|\phi|^2] = [\text{mass}]^2$, consistency requires:

$$\boxed{[\xi] = [\text{mass}]^2\,.} \quad \checkmark$$

(ii) The VEV $\langle|\phi|^2\rangle = |\xi|/g$ has dimensions $[\text{mass}]^2/[\text{dimensionless}] = [\text{mass}]^2$. Since $[|\phi|^2] = [\text{mass}]^2$, this is consistent. $\checkmark$

(c) Gauge boson mass.

The kinetic term for ϕ is:

$$|D_\mu\phi|^2 = |(\partial_\mu - igA_\mu)\phi|^2\,.$$

Expanding around the VEV $\phi = v + \delta\phi$ with $v^2 = \langle|\phi|^2\rangle = |\xi|/g$, the quadratic terms in A_μ give:

$$g^2v^2A_\mu A^\mu\,.$$

The standard identification from $\mathcal{L} \supset \frac{1}{2}m_A^2\, A_\mu A^\mu$ gives:

$$m_A^2 = 2g^2v^2 = 2g^2\,\frac{|\xi|}{g} = 2g\,|\xi|\,.$$

Dimensional check: $[g]$ = [dimensionless], $[|\xi|]$ = $[\text{mass}]^2$, so $[m_A^2]$ = $[\text{mass}]^2$. Therefore $[m_A]$ = [mass].

$$\boxed{m_A^2 = 2g\,|\xi|, \qquad [m_A] = [\text{mass}].} \quad \checkmark$$

Remark 0.2 (Factor of 2). The factor of 2 in $m_A^2 = 2g^2v^2$ arises because ϕ is a complex scalar: both the real and imaginary parts contribute equally to the gauge boson mass via the Higgs mechanism. (For a real scalar of charge q, the mass would be $m_A^2 = g^2q^2v^2$.)

(d) Two fields of opposite charge.

With ϕ (charge $+1$) and $\tilde{\phi}$ (charge -1), the D-flatness condition is:

$$g\big(|\phi|^2 - |\tilde{\phi}|^2\big) + \xi = 0 \qquad \Longrightarrow \qquad |\phi|^2 - |\tilde{\phi}|^2 = -\frac{\xi}{g}\,.$$

This is one equation in two unknowns, so it always has solutions:

$\xi > 0$*:* Set $\phi = 0$ and $|\tilde{\phi}|^2 = \xi/g > 0$:

$$0 - \xi/g = -\xi/g\,. \quad \checkmark$$

$\xi < 0$*:* Set $\tilde{\phi} = 0$ and $|\phi|^2 = |\xi|/g > 0$:

$$|\xi|/g - 0 = |\xi|/g = -\xi/g\,. \quad \checkmark$$

$$\boxed{\text{For either sign of } \xi,\ D\text{-flatness is achieved. SUSY is preserved and U(1) is broken.}}$$

The crucial point is that having fields of *both* charges allows D-flatness to be satisfied regardless of the sign of ξ. This is the generic situation in the $\mathcal{N} = 2$ context, where both monopoles and dyons (carrying opposite effective charges) are present at the singular points.

(e) Connection to confinement.

On the $\mathcal{N} = 2$ Coulomb branch, the effective U(1) theories at the singular points have massless monopoles that are charged under the residual U(1) gauge group. When the APS deformation drives the theory to these points and the effective FI parameter is nonzero, the monopoles are forced to condense (acquire VEVs to satisfy D-flatness), just as $\langle\phi\rangle \neq 0$ in part (a)(ii).

Monopole condensation produces confinement by the dual Meissner effect: a magnetic condensate confines electric charges into flux tubes, just as an electric superconductor (Cooper pair condensate) expels magnetic flux lines. This provides a concrete, semiclassically controlled mechanism for the confinement of the original quarks.

□

Problem 3: APS Deformation for SU(2) with $N_f = 4$ Hypers ⋆⋆⋆ (30 min)

Background. In Lecture 4, §5, we presented the Argyres–Plesser–Seiberg argument [2]: adding $W_{\text{def}} = \mu\,\text{Tr}\Phi^2$ to the $\mathcal{N} = 2$ superpotential breaks $\mathcal{N} = 2 \to \mathcal{N} = 1$. As $\mu \to \infty$, the adjoint Φ decouples and the theory reduces to $\mathcal{N} = 1$ SQCD with known Seiberg duals.

In this exercise you will work through the deformation explicitly for the *self-dual* case: SU(2) with $N_f = 4$ hypermultiplets. This case is special because $b_0^{\mathcal{N}=2} = 0$ (conformal), and the $\mathcal{N} = 1$ Seiberg dual is again SU(2) with $N_f = 4$—the theory is dual to *itself*.

Scope note. We accept the Seiberg–Witten solution [1] as a stated result (as in Lecture 4, §3). The exercises below test the deformation logic and the self-duality conclusion, not the derivation of the SW curve.

Questions.

(a) **Conformal check.** Verify that $b_0^{\mathcal{N}=2} = 0$ for SU(2) with $N_f = 4$.

(b) $\mathcal{N} = 1$ **decomposition.** Write down the complete $\mathcal{N} = 1$ field content of this theory. Verify the $\mathcal{N} = 1$ beta function:

$$b_0^{\mathcal{N}=1} = 3N_c - N_f - N_c$$

with $N_c = 2$ and $N_f = 4$, where the $-N_c$ contribution comes from the adjoint chiral Φ. Confirm $b_0 = 0$.

(c) **The deformation.** Add $W_{\text{def}} = \mu\,\text{Tr}\Phi^2$ to the $\mathcal{N} = 2$ superpotential.

  (i) Write the total $\mathcal{N} = 1$ superpotential W.

  (ii) Compute the F-term equation $\partial W/\partial\Phi = 0$ and solve for Φ.

  (iii) Show that as $\mu \to \infty$, $\Phi \to 0$ and decouples. What is the surviving $\mathcal{N} = 1$ theory?

(d) **Self-duality.** In the $\mu \to \infty$ limit, the theory is $\mathcal{N} = 1$ SU(2) with $N_f = 4$ flavours.

  (i) What is the Seiberg dual gauge group $\text{SU}(N_f - N_c)$?

  (ii) Describe the magnetic field content (magnetic quarks, mesons, magnetic superpotential).

  (iii) Explain why this is called *self-duality*: the dual theory has the same gauge group and matter content as the original.

(e) **Meson dimension and the conformal window.** At the IR fixed point, the conformal dimension of the meson $M = Q\widetilde{Q}$ is $\Delta[M] = \frac{3}{2}\,R[M]$, where $R[M] = 2(N_f - N_c)/N_f$ is the $\mathcal{N} = 1$ R-charge (using the non-anomalous R-symmetry after Φ decouples).

  (i) Compute $\Delta[M]$ for $N_c = 2$, $N_f = 4$.

  (ii) Verify that $N_f = 4$ lies in the $\mathcal{N} = 1$ conformal window $\frac{3}{2}N_c < N_f < 3N_c$ for $N_c = 2$.

  (iii) The meson is a "healthy" interacting operator if $\Delta[M] > 1$ (above the unitarity bound for a scalar). Check this.

(f) **Physical interpretation.** Explain briefly why the self-duality of SU(2) with $N_f = 4$ is a remnant of the $\text{SL}(2,\mathbb{Z})$ S-duality of the parent $\mathcal{N} = 2$ theory at the conformal point $b_0^{\mathcal{N}=2} = 0$.

Hints:

- For (c)(ii): note that Φ is in the adjoint, so the F-equation involves a trace over colour indices. In the abelian limit on the Coulomb branch, Φ is diagonal.
- For (d): the Seiberg dual of SU(2) with N_f flavours has magnetic gauge group $\mathrm{SU}(N_f - 2)$.
- For (f): the $\mathcal{N} = 2$ theory at $b_0 = 0$ has an exact conformal symmetry enhanced by S-duality (exchange of electric and magnetic descriptions). The deformation $\mu\,\mathrm{Tr}\Phi^2$ preserves this exchange symmetry.

Solution to Problem 3. **(a) Conformal check.**

$$b_0^{\mathcal{N}=2} = 2N_c - N_f = 2(2) - 4 = 0\,.$$

$$\boxed{b_0^{\mathcal{N}=2} = 0: \quad \text{the theory is exactly conformal.}} \quad \checkmark$$

(b) $\mathcal{N} = 1$ decomposition and beta function.

The $\mathcal{N} = 1$ content of $\mathcal{N} = 2$ SU(2) with $N_f = 4$ hypermultiplets:

$\mathcal{N} = 1$ **multiplet**	SU(2) **rep.**	**Multiplicity**
Vector $V = (A_\mu, \lambda)$	adj	1
Chiral $\Phi = (\phi, \psi)$	adj	1
Chiral Q_i	$\square$	4
Chiral $\widetilde{Q}_i$	$\overline{\square}$	4

Beta function: Using $T(\mathrm{adj}) = N_c = 2$ for SU(2):

$$\begin{aligned} b_0 &= 3\,T(\mathrm{adj}) - \left[T(\mathrm{adj}) + 4 \times T(\square) + 4 \times T(\overline{\square})\right] \\ &= 3(2) - \left[2 + 4 \times \tfrac{1}{2} + 4 \times \tfrac{1}{2}\right] = 6 - (2 + 2 + 2) = 6 - 6 = 0\,. \end{aligned} \tag{2}$$

$$\boxed{b_0^{\mathcal{N}=1} = 0 = b_0^{\mathcal{N}=2}\,.} \quad \checkmark$$

(c) The deformation.

(i) The total $\mathcal{N} = 1$ superpotential is:

$$W = \sqrt{2}\, Q_i\, \Phi\, \widetilde{Q}^i + \mu\, \mathrm{Tr}\Phi^2\,. \tag{3}$$

(ii) The F-term equation for Φ:

$$\begin{aligned} \frac{\partial W}{\partial \Phi} &= 0\,, \\ \sqrt{2}\, Q_i\, \widetilde{Q}^i + 2\mu\, \Phi &= 0\,, \\ \Phi = -\frac{\sqrt{2}}{2\mu}\, Q_i\, \widetilde{Q}^i &= -\frac{1}{\sqrt{2}\,\mu}\, Q_i\, \widetilde{Q}^i\,. \end{aligned} \tag{4}$$

(iii) As $\mu \to \infty$:

$$\Phi \to -\frac{1}{\sqrt{2}\,\mu}\, Q_i\,\widetilde{Q}^i \;\longrightarrow\; 0\,.$$

The adjoint Φ decouples from the low-energy theory (it has acquired a mass of order μ from the deformation term). The surviving theory is:

$$\boxed{\mathcal{N}=1\;\; \mathrm{SU}(2) \text{ with } N_f = 4 \text{ flavours } (Q_i, \widetilde{Q}_i), \quad W = 0\,.}$$

Dimensional check of the deformation: $[\mu\,\mathrm{Tr}\Phi^2] = [\text{mass}] \times [\text{mass}]^2 = [\text{mass}]^3 = [W]$. ✓

(d) Self-duality.

(i) The Seiberg dual gauge group is:

$$\mathrm{SU}(N_f - N_c) = \mathrm{SU}(4-2) = \mathrm{SU}(2)\,.$$

(ii) The magnetic field content:

- **Magnetic gauge group:** $\mathrm{SU}(2)_{\text{mag}}$.
- **Magnetic quarks:** 4 pairs $(q_i, \widetilde{q}^i)$ in the $\square$ and $\overline{\square}$ of $\mathrm{SU}(2)_{\text{mag}}$.
- **Mesons:** $M^i{}_j = Q_i\,\widetilde{Q}_j$, a 4×4 matrix, gauge-singlet under $\mathrm{SU}(2)_{\text{mag}}$.
- **Magnetic superpotential:** $W_{\text{mag}} = \frac{1}{\Lambda}\, M^i{}_j\, q_i\,\widetilde{q}^j$.

(iii) **Self-duality.** The magnetic theory is SU(2) with 4 flavour pairs plus gauge-singlet mesons—the same gauge group and matter content as the electric theory (up to the mesons and superpotential coupling). The duality maps:

$$\mathrm{SU}(2),\; N_f = 4 \quad\longleftrightarrow\quad \mathrm{SU}(2)_{\text{mag}},\; N_f = 4\,.$$

$$\boxed{\text{The theory is } \textbf{self-dual}\text{: it is mapped to itself under Seiberg duality.}}$$

This is a non-trivial statement: the theory at the IR fixed point has an exact symmetry that exchanges the "electric" and "magnetic" descriptions. The coupling at the fixed point is mapped to itself under $g \to 1/g$ (schematically).

(e) Meson dimension and the conformal window.

(i) The $\mathcal{N}=1$ R-charge of the meson $M = Q\widetilde{Q}$ (after Φ decouples, using the non-anomalous R-symmetry) is:

$$R[M] = 2\,\frac{N_f - N_c}{N_f} = 2\,\frac{4-2}{4} = 1\,.$$

The conformal dimension is:

$$\Delta[M] = \frac{3}{2}\,R[M] = \frac{3}{2} \times 1 = \frac{3}{2}\,.$$

$$\boxed{\Delta[M] = \frac{3}{2}\,.}$$

(ii) The conformal window for $N_c = 2$ is:

$$\frac{3}{2}N_c < N_f < 3N_c \qquad\Longrightarrow\qquad 3 < N_f < 6\,.$$

Since $N_f = 4$ and $3 < 4 < 6$:

$$\boxed{N_f = 4 \text{ is inside the conformal window.}} \quad \checkmark$$

(iii) The meson dimension satisfies:

$$\Delta[M] = \frac{3}{2} > 1\,.$$

The meson is above the unitarity bound for scalar operators and is therefore a healthy interacting operator at the IR fixed point, not a free field. ✓

(f) Physical interpretation.

The parent $\mathcal{N} = 2$ SU(2) theory with $N_f = 4$ hypermultiplets has $b_0^{\mathcal{N}=2} = 0$ and is exactly conformal. At the conformal point, the $\mathcal{N} = 2$ theory possesses an exact $\mathrm{SL}(2, \mathbb{Z})$ S-duality symmetry—a generalisation of electromagnetic duality that exchanges strongly and weakly coupled descriptions. The S-duality transformation $\tau \to -1/\tau$ (where $\tau = \theta/(2\pi) + 4\pi i/g^2$ is the complexified coupling) exchanges the electric and magnetic degrees of freedom.

The APS deformation $W_{\text{def}} = \mu\,\mathrm{Tr}\Phi^2$ breaks $\mathcal{N} = 2 \to \mathcal{N} = 1$, but it does not break the self-duality: the deformation is invariant under the exchange of electric and magnetic descriptions (since Φ is the same field in both). As $\mu \to \infty$, the $\mathcal{N} = 2$ S-duality descends to the $\mathcal{N} = 1$ Seiberg self-duality of SU(2) with $N_f = 4$.

$$\boxed{\mathcal{N} = 1 \text{ self-duality of SU(2), } N_f = 4 \xleftarrow{\mu \to \infty} \mathcal{N} = 2 \text{ } S\text{-duality at } b_0 = 0\,.}$$

□

End of Exercise Set 4. These exercises provide hands-on practice with the $\mathcal{N} = 2$ to $\mathcal{N} = 1$ deformation framework from Lecture 4: multiplet decomposition, Fayet–Iliopoulos D-terms, and the APS argument identifying monopoles with magnetic quarks. The self-duality of SU(2) *with $N_f = 4$ illustrates how $\mathcal{N} = 2$ S-duality descends to $\mathcal{N} = 1$ Seiberg duality. In Lectures 5–6, we will apply this duality framework to the Standard Model.*

References

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