Exercise Set 5: Fermion Mass Hierarchy from Scalar Democracy

The Dualised Standard Model — Part III Exercises

Lent Term 2027

Conventions for these exercises

Quantity	Convention
Units	$\hbar = c = 1$ (natural)
Metric	$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ (mostly minus)
Universal Yukawa	y (single magnetic coupling for all fermions)
VEV sum rule	$v^2 = \sum_{i,j}\big(\langle H^u_{ij}\rangle^2 + \langle H^d_{ij}\rangle^2\big) = \frac{1}{2}(246\ \mathrm{GeV})^2$
Planck mass	$M_{\mathrm{Pl}} = 1.22 \times 10^{19}$ GeV (reduced: $M_P = M_{\mathrm{Pl}}/\sqrt{8\pi}$)
Epistemic	"if the conjectured duality holds" for all non-SUSY claims (P1)

References: Lecture 7 (§§2–3); Lecture 8 (§§1–2); Cacciapaglia–Sannino [1]; Hill, Machado, Thomsen, Turner [2].

Epistemic note. All results in this exercise assume the conjectured non-SUSY duality holds. The group-theoretic statements (VEV sum rule, effective Yukawa structure) are exact consequences of the magnetic Lagrangian. The physical interpretation as a fermion mass hierarchy mechanism is conjectural.

Problem 1: Top Quark Mass from Universal Yukawa ⋆⋆ (15 min, 25 marks)

Background. In the dualised Standard Model (Lectures 7–8), *if the conjectured duality holds*, the magnetic Yukawa interaction between quarks and the 18 Higgs doublets $H^{u,d}_{ij}$ ($i, j = 1, 2, 3$ labelling generations) takes the form

$$\mathcal{L}_Y \;=\; y \sum_{i,j=1}^{3} \Big(\bar{q}^{\,i}_L\, u^j_R\, H^u_{ij} \;+\; \bar{q}^{\,i}_L\, d^j_R\, H^d_{ij}\Big) \;+\; \text{h.c.}\,, \tag{1}$$

where y is a **single universal Yukawa coupling**, the same for every generation pair. This is a radical departure from the Standard Model, where each fermion species has its own independent Yukawa coupling.

The effective Yukawa couplings that determine fermion masses are (Lecture 8, Eq. (2)):

$$Y^u_{ij} \;=\; y\,\frac{\langle H^u_{ij}\rangle}{v}\,, \qquad Y^d_{ij} \;=\; y\,\frac{\langle H^d_{ij}\rangle}{v}\,, \tag{2}$$

where the total electroweak VEV satisfies $v^2 = \sum_{ij}(\langle H^u_{ij}\rangle^2 + \langle H^d_{ij}\rangle^2) = (246\ \text{GeV})^2/2$.

Planck-scale four-fermion operators (Lecture 8, §2) generate mixing between the leading Higgs doublet $H_0 \equiv H^u_{33}$ and the 17 dormant doublets H_i ($i = 1, \ldots, 17$):

$$\mathcal{L}_{\text{Planck}} \supset \frac{c^{abcd}}{M_{\text{Pl}}^2} (Q^a \widetilde{Q}^b)(Q^c \widetilde{Q}^d)^\dagger \,, \tag{3}$$

where c^{abcd} are $\mathcal{O}(1)$ dimensionless coefficients.

Questions.

(a) **[5 marks]** Suppose the leading VEV satisfies $\langle H_0\rangle = \langle H^u_{33}\rangle \approx v$, with all other VEVs suppressed. Show that the top quark Yukawa coupling is

$$y_t = y\,\frac{\langle H^u_{33}\rangle}{v} \approx y \,, \tag{4}$$

and hence

$$m_t = y_t\, v \approx y\, v \,. \tag{5}$$

For $y \sim \mathcal{O}(1)$ and $v \approx 174$ GeV, show that $m_t \sim \mathcal{O}(v) \sim 174$ GeV. State this as a **parametric consistency check**: the observed $m_t = 172.7 \pm 0.3$ GeV requires $y \approx 0.99$, which is naturally $\mathcal{O}(1)$. Note that y is a **free parameter** of the magnetic theory.

(b) **[8 marks]** The dormant Higgs doublets H_i have large positive masses $M_i^2 \gg v^2$ and acquire induced VEVs through mixing with the leading Higgs H_0 (Lecture 8, Eq. (8)):

$$\langle H_i\rangle = \frac{\mu_{0i}^2}{M_i^2} \langle H_0\rangle \,, \tag{6}$$

where μ_{0i}^2 are off-diagonal mixing mass terms generated by the Planck operators.

(i) Show that the effective Yukawa coupling for a fermion species f whose mass arises from the Higgs doublet H_i is:

$$y_f = y\,\frac{\mu_{0i}^2}{M_i^2} \,. \tag{7}$$

(ii) From dimensional analysis of the Planck operators (Eq. (3)), the mixing mass and dormant Higgs mass scale as

$$\mu_{0i}^2 \sim \xi_i \frac{\Lambda_D^4}{M_{\text{Pl}}^2} \,, \qquad M_i^2 \sim \Lambda_D^2 \,,$$

where ξ_i is an $\mathcal{O}(1)$ coefficient. Show that

$$\frac{m_f}{m_t} \sim \xi_i \frac{\Lambda_D^2}{M_{\text{Pl}}^2} \,. \tag{8}$$

(iii) For $\Lambda_D = 10^{11}$ GeV and $M_{\text{Pl}} = 1.22 \times 10^{19}$ GeV, evaluate $\Lambda_D^2/M_{\text{Pl}}^2$ numerically. Compare this suppression factor with the observed mass ratios $m_b/m_t \approx 0.024$, $m_c/m_t \approx 0.007$, and $m_s/m_t \approx 5 \times 10^{-4}$. Comment on whether the parametric suppression is in the right ballpark.

(c) **[7 marks]** The bottom quark mass ratio $m_b/m_t \approx 0.024$ arises from the down-type Higgs VEV (Lecture 8, Eq. (6)):

$$\frac{m_b}{m_t} = \frac{\langle H^d_{33}\rangle}{\langle H^u_{33}\rangle} \,. \tag{9}$$

(i) As shown in part (b), the generic Planck suppression $(\Lambda_D/M_{\rm Pl})^2 \sim 10^{-16}$ is far too small to produce $m_b/m_t \approx 0.024$. Explain why the bottom quark mass instead arises from **direct** H^u_{33}–H^d_{33} mixing through the scalar potential, with the suppression controlled by the ratio $\mu_b^2/M_{d,33}^2 \sim \mathcal{O}(10^{-2})$ of $\mathcal{O}(1)$ parameters in the Planck-scale operator coefficients.

(ii) For the *lighter* fermion generations (charm, strange, and below), the mass ratios arise from a cascade of mixing steps in the scalar potential. Each step is controlled by a ratio μ_i^2/M_i^2 of $\mathcal{O}(1)$ parameters. Explain why the charm quark ($m_c/m_t \approx 0.007$) sits at a comparable suppression level to the bottom quark (i.e., the same "tier" of the VEV hierarchy), while the strange quark ($m_s/m_t \approx 5 \times 10^{-4}$) requires an additional step of suppression (a deeper tier).

(iii) Assign the bottom, charm, and strange quarks to appropriate "tiers" in the VEV hierarchy. Which quarks share the same tier, and which require additional steps of cascaded mixing?

(d) **[5 marks] Discussion.**

(i) Is the fermion mass hierarchy a **prediction** of the dualised SM, or a **parametric consistency check**? Justify your answer by listing all free parameters in the framework.

(ii) What does the dualised SM *explain* about the fermion mass hierarchy that the Standard Model does not?

(iii) Identify the **weakest assumption** in the above derivation: which step would invalidate the entire argument if it failed?

Hints:

- For (a): remember that $v = 246/\sqrt{2}$ GeV ≈ 174 GeV when v is defined as the VEV rather than $\sqrt{2}\,G_F^{-1/2}$. Adapt to whichever convention is stated.
- For (b)(iii): compute $\log_{10}(\Lambda_D^2/M_{\rm Pl}^2)$ and compare with $\log_{10}(m_f/m_t)$ for each fermion.
- For (c)(ii): compare the suppression ratios μ_f^2/M_f^2 needed for each fermion and group them into tiers.

Solution to Problem 1. **(a) Top quark mass from universal Yukawa.**

From the effective Yukawa structure (2), the top quark Yukawa coupling is:

$$y_t \;=\; y\,\frac{\langle H^u_{33}\rangle}{v}\,.$$

If the leading VEV dominates, $\langle H^u_{33}\rangle \approx v$, then:

$$y_t \;\approx\; y\cdot\frac{v}{v} \;=\; y\,.$$

The top quark mass is:

$$m_t \;=\; y_t\,v \;\approx\; y\,v\,.$$

Dimensional check: $[y] = [\text{dimensionless}]$, $[v] = [\text{GeV}]$, so $[m_t] = [\text{GeV}]$. ✓

Numerical evaluation. With $v \approx 174$ GeV:

$$m_t \;\approx\; y \times 174\ \text{GeV}\,.$$

The measured top mass $m_t = 172.7 \pm 0.3$ GeV requires:

$$y \;\approx\; \frac{172.7}{174} \;\approx\; 0.99\,,$$

which is naturally $\mathcal{O}(1)$.

$$\boxed{m_t \;\approx\; y\,v \;\sim\; \mathcal{O}(v) \;\sim\; 174\ \text{GeV}\,, \quad \text{with } y \approx 0.99 \sim \mathcal{O}(1)\,.}$$

Parametric consistency statement. This is *not* a prediction: the universal Yukawa coupling y is a **free parameter** of the magnetic theory, determined by the strong dynamics at the duality scale Λ_D. The result $m_t \sim \mathcal{O}(v)$ shows that $y_t \sim \mathcal{O}(1)$ is *natural* in the dualised SM—in contrast to the SM, where $y_t \approx 1$ is an unexplained numerical coincidence.

(b) Induced VEVs and mass suppression.

(i) Effective Yukawa from dormant Higgs.

A fermion species f whose mass arises from the dormant Higgs H_i has:

$$m_f \;=\; y\,\frac{\langle H_i\rangle}{v}\,v \;=\; y\,\langle H_i\rangle\,.$$

Using the induced VEV (6):

$$m_f \;=\; y\,\frac{\mu_{0i}^2}{M_i^2}\,\langle H_0\rangle \;=\; y\,\frac{\mu_{0i}^2}{M_i^2}\,v\,.$$

Hence the effective Yukawa coupling is:

$$y_f \;\equiv\; \frac{m_f}{v} \;=\; y\,\frac{\mu_{0i}^2}{M_i^2}\,.$$

$$\boxed{y_f \;=\; y\,\frac{\mu_{0i}^2}{M_i^2}\,.} \quad \checkmark$$

(ii) Parametric suppression from Planck operators.

Using $\mu_{0i}^2 \sim \xi_i\,\Lambda_D^4/M_{\text{Pl}}^2$ and $M_i^2 \sim \Lambda_D^2$:

$$\frac{\mu_{0i}^2}{M_i^2} \;\sim\; \frac{\xi_i\,\Lambda_D^4/M_{\text{Pl}}^2}{\Lambda_D^2} \;=\; \xi_i\,\frac{\Lambda_D^2}{M_{\text{Pl}}^2}\,.$$

Since $m_f = y_f\,v$ and $m_t = y\,v$:

$$\frac{m_f}{m_t} \;=\; \frac{y_f}{y} \;=\; \frac{\mu_{0i}^2}{M_i^2} \;\sim\; \xi_i\,\frac{\Lambda_D^2}{M_{\text{Pl}}^2}\,.$$

$$\boxed{\frac{m_f}{m_t} \;\sim\; \xi_i\,\frac{\Lambda_D^2}{M_{\text{Pl}}^2}\,.}$$

Dimensional check: $[\Lambda_D^2/M_{\text{Pl}}^2] = [\text{dimensionless}]$ and $[m_f/m_t] = [\text{dimensionless}]$. $\checkmark$

(iii) Numerical evaluation.

$$\frac{\Lambda_D^2}{M_{\rm Pl}^2} = \frac{(10^{11})^2}{(1.22\times 10^{19})^2} = \frac{10^{22}}{1.49\times 10^{38}} \approx 6.7\times 10^{-17}\,.$$

This is a *single* insertion of the Planck suppression factor. The observed mass ratios require different numbers of such insertions (i.e., different "tiers" in the VEV hierarchy):

Fermion f	m_f/m_t (observed)	$\log_{10}(m_f/m_t)$	**Tier**
Bottom (b)	≈ 0.024	-1.6	1st
Charm (c)	≈ 0.007	-2.2	1st–2nd
Strange (s)	$\approx 5\times 10^{-4}$	-3.3	2nd

A single Planck suppression gives $\log_{10}(\Lambda_D^2/M_{\rm Pl}^2) \approx -16$, which is far too small to reproduce $m_b/m_t \approx 0.024$ or even $m_c/m_t \approx 0.007$.

Key observation. The naïve one-step Planck suppression $(\Lambda_D/M_{\rm Pl})^2 \sim 10^{-16}$ is far too small to reproduce $m_b/m_t \sim 0.024$. This indicates that the bottom quark mass does *not* arise from the generic Planck-suppressed operator mechanism of Eq. (8). Instead, the scalar democracy framework provides a different mechanism: **direct mixing** between the dominant Higgs doublet H^u_{33} and the subdominant doublet H^d_{33} through the scalar potential. In this route, the suppression is controlled by $\mathcal{O}(1)$ coefficients in the Planck operators and by the mass splittings among the dormant Higgs doublets, rather than by powers of $\Lambda_D/M_{\rm Pl}$. We derive this alternative route in part (c).

For the lighter fermions (charm, strange, and below), the generic Planck suppression *does* apply, through cascaded insertions of $\Lambda_D^2/M_{\rm Pl}^2$: the parametric structure—a cascade of progressively smaller VEVs—provides a natural mechanism for the mass hierarchy, with the specific ratios set by $\mathcal{O}(1)$ coefficients ξ_i.

(c) Bottom-to-top mass ratio.

(i) Why the bottom mass requires direct H^u–H^d mixing.

If the bottom quark mass arose from the generic Planck-suppressed mechanism of part (b), we would have $m_b/m_t \sim (\Lambda_D/M_{\rm Pl})^2 \sim 10^{-16}$, which is fourteen orders of magnitude too small. This rules out the generic tier structure for the bottom quark.

Instead, H^d_{33} acquires its VEV through **direct mixing** with H^u_{33} in the scalar potential. The Planck operators of Eq. (3) generate an off-diagonal mass-squared μ_b^2 between these two doublets, while the diagonal mass of H^d_{33} is $M^2_{d,33}$. The induced VEV ratio is:

$$\frac{\langle H^d_{33}\rangle}{\langle H^u_{33}\rangle} \sim \frac{\mu_b^2}{M^2_{d,33}} \sim c_b \times \mathcal{O}(10^{-2})\,,$$

where $c_b \sim \mathcal{O}(1)$ and the ratio $\mu_b^2/M^2_{d,33}$ is determined by the specific pattern of $\mathcal{O}(1)$ coefficients in the Planck-scale operators. Crucially, neither μ_b^2 nor $M^2_{d,33}$ is constrained to be a power of $\Lambda_D/M_{\rm Pl}$; their ratio is an $\mathcal{O}(10^{-2})$ number set by the UV dynamics.

(ii) Lighter fermion tiers.

The charm and strange masses arise from the same direct-mixing mechanism but applied to different entries of the 18-Higgs scalar potential:

- **Charm** ($m_c/m_t \approx 0.007 \sim 10^{-2.2}$): The charm quark mass comes from the $(2,2)$ sector VEV $\langle H^u_{22}\rangle$, induced by direct mixing with the leading Higgs H^u_{33}. Since $m_c/m_t \sim \mathcal{O}(10^{-2})$, the suppression ratio μ_c^2/M_{22}^2 is of the same order as for the bottom quark. Both bottom and charm sit in the **first tier** of the VEV hierarchy.

- **Strange** ($m_s/m_t \approx 5\times10^{-4} \sim 10^{-3.3}$): The strange quark requires an additional suppression of $\sim 10^{-1}$ beyond the first tier. This arises from a **cascaded** mixing: the $(2,2)$ or $(2,3)$ Higgs doublet mixes with the leading Higgs, and the strange-sector Higgs then mixes with this intermediate doublet. Two steps of $\mathcal{O}(10^{-2})$ mixing give $\sim 10^{-4}$, consistent with m_s/m_t. The strange quark sits in the **second tier**.

$$\boxed{\text{First tier (bottom, charm): } m_f/m_t \sim \mu_f^2/M_f^2 \sim 10^{-2}; \quad \text{Second tier (strange): } m_s/m_t \sim (\mu^2/M^2)^2 \sim 10^{-4}\,.}$$

(iii) VEV tiers.

The VEV hierarchy can be organised as follows:

- **Zeroth tier ($\langle H\rangle \sim v$):** H_{33}^u (top quark). This is the leading VEV that triggers electroweak symmetry breaking.
- **First tier ($\langle H\rangle \sim 10^{-2}\,v$):** H_{33}^d (bottom quark), possibly H_{22}^u (charm quark). These are induced by one step of Planck mixing, with the suppression factor controlled by $\mu_b^2/M_d^2 \sim 10^{-2}$.
- **Second tier ($\langle H\rangle \sim 10^{-4}\,v$):** Strange quark and lighter fermions. These arise from two steps of mixing (Lecture 8, Eq. (10)), giving a further suppression.
- **Third tier and beyond:** Up and down quarks, electron. Progressive further suppression.

The crucial point: the *parametric structure* of the hierarchy is set by the scalar potential and Planck-scale operators. The *specific values* of each tier's suppression factor depend on the $\mathcal{O}(1)$ coefficients c^{abcd}, which are free parameters.

(d) Discussion.

(i) Prediction or consistency check?

The fermion mass hierarchy is a **parametric consistency check**, not a prediction. The free parameters are:

1. y: the universal magnetic Yukawa coupling (~ 1).
2. c^{abcd}: the $\mathcal{O}(1)$ Planck-scale operator coefficients ($\sim 10^2$ independent parameters).
3. Λ_D: the duality scale ($\sim 10^{11}$ GeV).
4. $M_{\rm Pl}$: the Planck mass (measured: 1.22×10^{19} GeV).
5. The scalar quartic couplings λ_i and mass parameters M_i^2 of the 18-Higgs potential.

The dualised SM has comparable freedom to the Standard Model's Yukawa sector (~ 20 free parameters for quark and lepton masses, mixing angles, and CP phases). It does *not* predict specific mass values.

$$\boxed{\text{Parametric consistency, not prediction. The } \mathcal{O}(1) \text{ coefficients } c^{abcd} \text{ are free parameters.}}$$

(ii) What the dualised SM explains.

The Standard Model treats the Yukawa hierarchy as an unexplained input: $y_t \approx 1$, $y_b \approx 0.024$, $y_e \approx 3\times10^{-6}$, spanning five orders of magnitude with no structural reason for the pattern.

The dualised SM, *if the duality holds*, provides a **structural explanation**:

- The hierarchy arises from a **VEV hierarchy** among the 18 Higgs doublets, not from a hierarchy in the Yukawa couplings (which are universal).
- The VEV hierarchy is **naturally generated** by Planck-scale operators that mix the leading Higgs with the dormant doublets.
- The **flavour puzzle** is *translated* from the Yukawa sector to the scalar sector, where it has a gravitational origin.

(iii) Weakest assumption.

The weakest assumption is the **non-SUSY duality itself**: the claim that the electric SU(2) theory confines and produces the SM as its magnetic dual. If this duality does not hold, the entire framework—universal Yukawa, 18 Higgs doublets, scalar democracy—loses its theoretical basis. The VEV hierarchy mechanism is valid group theory conditional on the field content, but the field content itself rests on the conjectured duality.

□

End of Exercise Set 5. This exercise develops the fermion mass hierarchy from the scalar democracy mechanism of Lectures 7–8. The key result is that $m_t \sim y\,v$ with $y \sim \mathcal{O}(1)$ is a parametric consistency check, and the lighter fermion masses arise from a cascade of induced VEVs driven by Planck-scale operators. The $\mathcal{O}(1)$ coefficients c^{abcd} are free parameters—the framework provides the mechanism for the hierarchy but not the specific mass values.

References

[1] G. Cacciapaglia and F. Sannino, "The Dualised Standard Model," `arXiv:2407.17281` [hep-ph] (2024).

[2] C. T. Hill, P. A. N. Machado, A. E. Thomsen, and J. Turner, "Scalar democracy," *Phys. Rev. D* **100** (2019) 015015 [`arXiv:1902.07214`].

[3] N. Seiberg, "Electric–magnetic duality in supersymmetric non-Abelian gauge theories," *Nucl. Phys.* **B435** (1995) 129 [`hep-th/9411149`].