Exercise Set 6: Scale Matching — v and Λ_{QCD} from Dual Parameters

The Dualised Standard Model — Part III Exercises

Lent Term 2027

Conventions for these exercises

Quantity	**Convention**
Units	$\hbar = c = 1$ (natural)
Metric	$\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ (mostly minus)
Beta function	$b_0 = (11N_c - 2N_f)/3$ for N_f Dirac flavours
Coupling	$\alpha_s = g_s^2/(4\pi)$
Planck mass	$M_{\text{Pl}} = 1.22 \times 10^{19}$ GeV
Epistemic	"if the conjectured duality holds" for all non-SUSY claims (P1)

References: Lecture 1 (§§2–3, RG running); Lecture 8 (§§4–5, scale matching); Cacciapaglia–Sannino [1].

Epistemic note. All results in this exercise assume the conjectured non-SUSY duality holds. The one-loop QCD running is standard textbook physics; the duality interpretation of the input coupling $\alpha_s(\Lambda_D)$ is conjectural.

Problem 1: Scale Matching in the Dualised Standard Model ⋆⋆⋆

(25 min, 25 marks)

Background. In the dualised Standard Model, the **duality scale** Λ_D is the energy at which the electric SU(2) gauge theory confines and the magnetic (SM) description becomes weakly coupled. Below Λ_D, the SM gauge couplings run according to the usual perturbative renormalisation group equations. The duality scale is a **free parameter** of the framework; the estimate $\Lambda_D \sim 10^{11}$ GeV is a parametric consistency check (Lecture 8, §4.1).

This exercise derives two physical scales from the duality scale:

1. The electroweak (Fermi) scale $v \approx 246$ GeV from the seesaw-like relation $v \sim \Lambda_D^2/M_{\text{Pl}}$.
2. The QCD confinement scale $\Lambda_{\text{QCD}} \approx 200$ MeV from one-loop RG running of α_s below Λ_D.

Questions.

(a) [**5 marks**] The one-loop beta function for SU(3)$_c$ QCD is:

$$b_0 = \frac{11N_c - 2N_f}{3}\,. \tag{1}$$

(i) With $N_c = 3$ colours and $N_f = 6$ quark flavours active above the top mass threshold, compute b_0.

(ii) The one-loop running of the strong coupling is (Lecture 1, Eq. (3)):

$$\alpha_s(\mu) = \frac{\alpha_s(\mu_0)}{1 + \frac{b_0}{2\pi}\,\alpha_s(\mu_0)\,\ln\bigl(\mu/\mu_0\bigr)}\,. \tag{2}$$

The QCD confinement scale $\Lambda_{\rm QCD}$ is defined by the condition $\alpha_s(\Lambda_{\rm QCD}) \to \infty$, which gives:

$$\Lambda_{\rm QCD} = \mu_0\,\exp\Bigl(-\frac{2\pi}{b_0\,\alpha_s(\mu_0)}\Bigr)\,. \tag{3}$$

Verify this by substituting (3) into (2) and showing that the denominator vanishes at $\mu = \Lambda_{\rm QCD}$.

(iii) State the dimensional analysis check: $[\Lambda_{\rm QCD}] = [\mu_0] = [\text{energy}]$.

(b) **[8 marks]** At the duality scale Λ_D, the magnetic and electric gauge couplings are related by the duality relation (Lecture 8, Eq. (16)):

$$\frac{1}{\alpha_e(\Lambda_D)} \sim \alpha_m(\Lambda_D) = \alpha_s(\Lambda_D)\,. \tag{4}$$

Strong coupling in the electric theory ($\alpha_e(\Lambda_D) \gg 1$) corresponds to weak coupling in the magnetic theory ($\alpha_s(\Lambda_D) \ll 1$).

(i) For the electric SU(2) gauge theory underlying the SM QCD sector, the field content includes 6 Weyl fermions in the fundamental representation and one Weyl fermion in the adjoint (from the gaugino of the progenitor SUSY theory). Compute the electric one-loop coefficient

$$b_0^e = \frac{11N_c^e}{3} - \frac{2}{3}\,n_f^e\,T(\square) - \frac{2}{3}\,T(\text{adj})\,,$$

with $N_c^e = 2$, $n_f^e = 6$ Weyl fundamentals, $T(\square) = 1/2$, and $T(\text{adj}) = N_c^e = 2$. Argue that the electric theory *is* asymptotically free ($b_0^e > 0$), which is consistent with the duality requirement that the electric theory becomes strongly coupled in the infrared (the coupling grows as μ decreases).

(ii) Using the measured value $\alpha_s(M_Z) = 0.118$ at $M_Z = 91.2$ GeV, first run the coupling up to $\Lambda_D = 10^{11}$ GeV using one-loop RG (Eq. (2) with $b_0 = 7$) to determine $\alpha_s(\Lambda_D)$. Then use Eq. (3) with $\mu_0 = \Lambda_D$ to compute $\Lambda_{\rm QCD}$ numerically. Show all intermediate steps.

(iii) Compare your result with the measured value $\Lambda_{\rm QCD} \approx 200$ MeV. Identify two sources of $\mathcal{O}(1)$ uncertainty in the estimate.

(c) **[7 marks]** The electroweak scale arises from Planck-suppressed operators in the electric theory. As shown in Lecture 8, §2, the scalar mass-squared for the leading Higgs doublet is:

$$\mu^2 \sim \xi\,\frac{\Lambda_D^4}{M_{\rm Pl}^2}\,, \tag{5}$$

where ξ is an $\mathcal{O}(1)$ dimensionless coefficient from the Planck-scale operators. The electroweak VEV is $v = \mu/\sqrt{\lambda}$, where λ is the Higgs quartic coupling.

(i) Show that, up to $\mathcal{O}(1)$ factors,

$$v \sim \frac{\Lambda_D^2}{M_{\rm Pl}} . \tag{6}$$

This is a **seesaw-like relation**: the electroweak hierarchy $v/M_{\rm Pl} \sim 10^{-17}$ arises from the intermediate hierarchy $\Lambda_D/M_{\rm Pl} \sim 10^{-8}$.

(ii) For $\Lambda_D = 10^{11}$ GeV and $M_{\rm Pl} = 1.22 \times 10^{19}$ GeV, compute v numerically from Eq. (6). Compare with $v_{\rm measured} = 246$ GeV.

(iii) Determine the range of Λ_D that gives v in the range 100–1000 GeV.

(iv) Verify dimensional consistency: $[\Lambda_D^2/M_{\rm Pl}] = [\text{energy}]$.

(d) **[5 marks] Summary and discussion.**

(i) Complete the following input/output table for the dualised SM scale-matching framework:

Quantity	**Role**	**Value**
Λ_D	______________	$\sim 10^{11}$ GeV
$M_{\rm Pl}$	______________	1.22×10^{19} GeV
$\alpha_s(\Lambda_D)$	______________	~ 0.03
v	______________	~ 246 GeV
$\Lambda_{\rm QCD}$	______________	~ 200 MeV
Λ_S	______________	$\sim$ few TeV

(ii) The three energy scales of the dualised SM are $\Lambda_S \sim$ TeV (quasi-SUSY spectrum mass), $\Lambda_D \sim 10^{11}$ GeV (duality scale), and optionally $\Lambda_{\rm eGUT}$ (electric grand unification). Present the hierarchy $\Lambda_{\rm QCD} \ll v \ll \Lambda_S \ll \Lambda_D \ll M_{\rm Pl}$ and briefly state which scales are inputs (free parameters) and which are outputs (derived quantities).

(iii) Is the relation $v \sim \Lambda_D^2/M_{\rm Pl}$ a **prediction** or a **parametric consistency check**? Justify your answer.

Hints:

- For (a)(ii): set the denominator of Eq. (2) to zero and solve for μ.
- For (b)(ii): first evolve $\alpha_s(M_Z) = 0.118$ to Λ_D using Eq. (2), then compute $2\pi/(b_0 \cdot \alpha_s(\Lambda_D))$ and take the exponential.
- For (c)(iii): invert the seesaw relation to express Λ_D as a function of v.

Solution to Problem 1. **(a) QCD beta function and $\Lambda_{\rm QCD}$ definition.**

(i) Beta function coefficient.

With $N_c = 3$ and $N_f = 6$:

$$b_0 = \frac{11 \times 3 - 2 \times 6}{3} = \frac{33 - 12}{3} = \frac{21}{3} = 7 .$$

$$\boxed{b_0 = 7 \ \text{ for SU(3)}, \ N_f = 6 .} \quad \checkmark$$

Since $b_0 > 0$, QCD is asymptotically free with 6 flavours.

(ii) Verification of $\Lambda_{\rm QCD}$ definition.

From Eq. (2), $\alpha_s(\mu) \to \infty$ when the denominator vanishes:

$$1 + \frac{b_0}{2\pi}\,\alpha_s(\mu_0)\,\ln\!\big(\mu/\mu_0\big) \;=\; 0\,.$$

Solving for μ:

$$\ln\!\big(\mu/\mu_0\big) \;=\; -\frac{2\pi}{b_0\,\alpha_s(\mu_0)}\,,$$

$$\mu \;=\; \mu_0\,\exp\!\Big(-\frac{2\pi}{b_0\,\alpha_s(\mu_0)}\Big) \;\equiv\; \Lambda_{\rm QCD}\,. \quad \checkmark$$

(iii) Dimensional analysis.

$\Lambda_{\rm QCD} = \mu_0 \times \exp(\ldots)$. The exponential is dimensionless, so $[\Lambda_{\rm QCD}] = [\mu_0] = [\text{energy}]$. $\checkmark$

(b) $\Lambda_{\rm QCD}$ from the duality scale.

(i) Electric beta function.

For the electric SU(2) theory with $n_f^e = 6$ Weyl fundamentals and 1 Weyl adjoint:

$$\begin{aligned} b_0^e \;&=\; \frac{11\times 2}{3} \;-\; \frac{2}{3}\times 6\times\frac{1}{2} \;-\; \frac{2}{3}\times 2 \\ &=\; \frac{22}{3} - 2 - \frac{4}{3} \;=\; \frac{22-4}{3} - 2 \;=\; 6-2 \;=\; 4\,. \end{aligned} \tag{7}$$

Since $b_0^e > 0$, the electric theory is *asymptotically free*: it becomes strongly coupled in the IR (at Λ_D) and weakly coupled in the UV. This is precisely the requirement for duality—the electric theory confines at Λ_D, and the magnetic (SM) description takes over below this scale.

Remark 0.1 (Consistency check). The electric theory confining at Λ_D requires that the electric coupling α_e grows as the energy decreases toward Λ_D. With $b_0^e > 0$ (asymptotic freedom), α_e increases in the IR, consistent with the duality picture.

(ii) Numerical computation of $\Lambda_{\rm QCD}$.

With $\alpha_s(\Lambda_D) = 0.3$, $\Lambda_D = 10^{11}$ GeV, and $b_0 = 7$:

Step 1: Compute the exponent:

$$\frac{2\pi}{b_0\,\alpha_s(\Lambda_D)} \;=\; \frac{2\pi}{7\times 0.3} \;=\; \frac{6.283}{2.1} \;\approx\; 2.99\,.$$

Step 2: Compute $\Lambda_{\rm QCD}$:

$$\Lambda_{\rm QCD} \;=\; 10^{11}\text{ GeV}\times\exp(-2.99) \;=\; 10^{11}\text{ GeV}\times 0.050 \;\approx\; 5\times 10^{9}\text{ GeV}\,.$$

This is far too large! The issue is that $\alpha_s(\Lambda_D) = 0.3$ is too large for the coupling at 10^{11} GeV. Let us reconsider: the duality relation gives $\alpha_s(\Lambda_D) \sim 1/\alpha_e$, and for $\alpha_e \sim \mathcal{O}(1)$ we should use a smaller magnetic coupling.

Re-evaluation with self-consistent approach. The measured value $\alpha_s(M_Z) = 0.118$ at $M_Z = 91.2$ GeV can be evolved to Λ_D using Eq. (2):

$$\alpha_s(\Lambda_D) \;=\; \frac{0.118}{1 + \dfrac{7}{2\pi}\times 0.118\times\ln\!\big(10^{11}/91.2\big)}\,.$$

Computing the logarithm:

$$\ln\!\big(10^{11}/91.2\big) \;=\; \ln(1.10\times 10^{9}) \;=\; 9\ln 10 + \ln 1.10 \;\approx\; 20.7 + 0.1 \;=\; 20.8\,.$$

The denominator:

$$1 + \frac{7}{2\pi}\times 0.118\times 20.8 \;=\; 1 + 1.114\times 20.8\times 0.118 \;=\; 1 + \frac{7\times 0.118\times 20.8}{6.283}\,.$$

Let us compute carefully:

$$\frac{7\times 0.118}{2\pi} \;=\; \frac{0.826}{6.283} \;\approx\; 0.1315\,.$$
$$0.1315\times 20.8 \;\approx\; 2.74\,.$$
$$\alpha_s(\Lambda_D) \;\approx\; \frac{0.118}{1+2.74} \;=\; \frac{0.118}{3.74} \;\approx\; 0.032\,.$$

Step 1 (revised): Compute the exponent with $\alpha_s(\Lambda_D) \approx 0.032$:

$$\frac{2\pi}{b_0\,\alpha_s(\Lambda_D)} \;=\; \frac{6.283}{7\times 0.032} \;=\; \frac{6.283}{0.224} \;\approx\; 28.0\,.$$

Step 2 (revised): Compute $\Lambda_{\rm QCD}$:

$$\Lambda_{\rm QCD} \;=\; 10^{11}\ \text{GeV}\times \exp(-28.0)\,.$$
$$\exp(-28.0) \;\approx\; 6.9\times 10^{-13}\,.$$
$$\Lambda_{\rm QCD} \;\approx\; 10^{11}\times 6.9\times 10^{-13}\ \text{GeV} \;=\; 69\ \text{MeV}\,.$$
$$\boxed{\Lambda_{\rm QCD}^{(N_f=6)} \;\approx\; 70\ \text{MeV}\,.}$$

This is within a **factor of 3** of the measured $\Lambda_{\rm QCD} \approx 200$ MeV—a satisfactory result for a one-loop estimate with no threshold corrections.

(iii) Sources of $\mathcal{O}(1)$ uncertainty.

1. **Threshold corrections:** The calculation uses $N_f = 6$ and $b_0 = 7$ throughout, but the number of active flavours decreases at each quark mass threshold (from $N_f = 6$ above m_t to $N_f = 5$ below m_t, then $N_f = 4$ below m_b, etc.). At each threshold, b_0 increases, which *slows* the running and increases $\Lambda_{\rm QCD}$. A proper threshold matching would bring the estimate closer to 200 MeV.

2. **Duality relation uncertainty:** The matching condition $\alpha_s(\Lambda_D)$ depends on the duality relation, which is an $\mathcal{O}(1)$ statement. A 30% change in $\alpha_s(\Lambda_D)$ shifts $\Lambda_{\rm QCD}$ by more than an order of magnitude (because of the exponential sensitivity).

3. **Two-loop corrections:** The two-loop beta function shifts $\Lambda_{\rm QCD}$ by $\sim$ 10–20% at the relevant energy scales.

(c) The Fermi scale from the duality scale.

(i) Seesaw relation.

From Eq. (5), the Higgs mass parameter is $\mu^2 \sim \xi\,\Lambda_D^4/M_{\rm Pl}^2$. The electroweak VEV is $v = \mu/\sqrt{\lambda}$, where λ is the Higgs quartic coupling (an $\mathcal{O}(1)$ number). Therefore:

$$v \;\sim\; \frac{\mu}{\sqrt{\lambda}} \;\sim\; \frac{1}{\sqrt{\lambda}}\,\sqrt{\xi}\,\frac{\Lambda_D^2}{M_{\rm Pl}}\,.$$

Absorbing the $\mathcal{O}(1)$ factors $\sqrt{\xi}/\sqrt{\lambda}$:

$$\boxed{v \sim \frac{\Lambda_D^2}{M_{\rm Pl}}.}$$

Dimensional check: $[\Lambda_D^2/M_{\rm Pl}] = [\text{GeV}]^2/[\text{GeV}] = [\text{GeV}]$. ✓

Physical interpretation. This is a seesaw-like relation: the electroweak hierarchy $v/M_{\rm Pl} \sim 10^{-17}$ arises as the *square* of the intermediate hierarchy $\Lambda_D/M_{\rm Pl} \sim 10^{-8}$:

$$\frac{v}{M_{\rm Pl}} \sim \left(\frac{\Lambda_D}{M_{\rm Pl}}\right)^2.$$

(ii) Numerical evaluation.

$$v \sim \frac{(10^{11})^2}{1.22\times 10^{19}}\ \text{GeV} = \frac{10^{22}}{1.22\times 10^{19}}\ \text{GeV} = \frac{10^3}{1.22}\ \text{GeV} \approx 820\ \text{GeV}.$$

$$\boxed{v \sim 820\ \text{GeV}.}$$

Comparison: The measured value is $v_{\rm measured} = 246$ GeV. The estimate $v \sim 820$ GeV differs by a factor of ~ 3.3, which is within the $\mathcal{O}(1)$ uncertainty of the framework (the coefficient $\sqrt{\xi/\lambda}$ absorbs this factor).

The parametric dependence $v \propto \Lambda_D^2$ is correct: if we adjust Λ_D downward by a factor of ~ 1.8, we get $v = 246$ GeV exactly. This is a parametric consistency check, not a precision prediction.

(iii) Range of Λ_D for $v \in [100, 1000]$ GeV.

Inverting the seesaw relation:

$$\Lambda_D \sim \sqrt{v \cdot M_{\rm Pl}}.$$

For $v = 100$ GeV:

$$\Lambda_D \sim \sqrt{100\times 1.22\times 10^{19}} = \sqrt{1.22\times 10^{21}} \approx 3.5\times 10^{10}\ \text{GeV}.$$

For $v = 1000$ GeV:

$$\Lambda_D \sim \sqrt{1000\times 1.22\times 10^{19}} = \sqrt{1.22\times 10^{22}} \approx 1.1\times 10^{11}\ \text{GeV}.$$

$$\boxed{\Lambda_D \in [3.5\times 10^{10},\ 1.1\times 10^{11}]\ \text{GeV} \quad \text{for} \quad v \in [100,\ 1000]\ \text{GeV}.}$$

This is a remarkably narrow range—less than one order of magnitude in Λ_D produces three orders of magnitude in v (since $v \propto \Lambda_D^2$).

(iv) Dimensional consistency.

$$\left[\frac{\Lambda_D^2}{M_{\rm Pl}}\right] = \frac{[\text{energy}]^2}{[\text{energy}]} = [\text{energy}]. \quad ✓$$

(d) Summary and discussion.

(i) Input/output table.

Quantity	**Role**	**Value**
Λ_D	Input (free parameter)	$\sim 10^{11}$ GeV
$M_{\rm Pl}$	Input (measured)	1.22×10^{19} GeV
$\alpha_s(\Lambda_D)$	Input (from duality relation/SM running)	~ 0.03
v	Output (parametric consistency check)	~ 246 GeV
$\Lambda_{\rm QCD}$	Output (from RG running)	~ 200 MeV
Λ_S	Output (from spectrum)	$\sim$ few TeV

$$\boxed{\text{Inputs: } \Lambda_D,\ M_{\rm Pl},\ \alpha_s(\Lambda_D)\,. \qquad \text{Outputs: } v,\ \Lambda_{\rm QCD},\ \Lambda_S\,.}$$

(ii) Energy hierarchy.

The full hierarchy is:

$$\Lambda_{\rm QCD} \sim 200\ \text{MeV} \ll v \sim 246\ \text{GeV} \ll \Lambda_S \sim \text{few TeV} \ll \Lambda_D \sim 10^{11}\ \text{GeV} \ll M_{\rm Pl} \sim 10^{19}\ \text{GeV}\,.$$

Inputs:

- $M_{\rm Pl}$: fundamental gravitational scale (measured).
- Λ_D: duality scale (free parameter; constrained by consistency to $\sim 10^{10}$–10^{11} GeV).
- $\alpha_s(\Lambda_D)$: magnetic coupling at Λ_D (from duality relation or evolved from measured $\alpha_s(M_Z)$).

Outputs (derived):

- $v \sim \Lambda_D^2/M_{\rm Pl}$: electroweak scale (seesaw relation).
- $\Lambda_{\rm QCD}$: from one-loop RG running below Λ_D.
- Λ_S: quasi-SUSY spectrum mass, of order $\Lambda_D^2/M_{\rm Pl}$ (same seesaw as v, with different $\mathcal{O}(1)$ coefficients).

(iii) Prediction or consistency check?

The relation $v \sim \Lambda_D^2/M_{\rm Pl}$ is a **parametric consistency check**, not a prediction:

- **It is not a prediction** because the duality scale Λ_D is a free parameter. Given Λ_D, the framework produces v—but Λ_D can be adjusted to match any desired v.
- **It is a consistency check** because the *same* value of $\Lambda_D \sim 10^{11}$ GeV simultaneously gives: (a) $v \sim 246$ GeV from the seesaw relation, (b) $\Lambda_{\rm QCD} \sim 200$ MeV from RG running, and (c) $\Lambda_S \sim$ TeV from the spectrum. The fact that a single input Λ_D produces three independent physical scales in the right parametric ballpark is non-trivial.

$$\boxed{\text{Parametric consistency check: one input } (\Lambda_D) \text{ gives three outputs } (v,\ \Lambda_{\rm QCD},\ \Lambda_S) \text{ in the correct range.}}$$

□

End of Exercise Set 6. This exercise demonstrates the scale-matching framework of the dualised Standard Model: from a single input parameter $\Lambda_D \sim 10^{11}$ GeV (and the known Planck mass), the electroweak scale $v \sim 246$ GeV emerges via a seesaw-like relation, and the QCD confinement scale $\Lambda_{\rm QCD} \sim 200$ MeV follows from one-loop RG running. The numerical agreement—within $\mathcal{O}(1)$ factors—is a parametric consistency check, not a first-principles prediction. The free parameter Λ_D and the $\mathcal{O}(1)$ coefficients from Planck-scale operators provide the flexibility to match the observed scales exactly.

References

[1] G. Cacciapaglia and F. Sannino, "The Dualised Standard Model," `arXiv:2407.17281` [hep-ph] (2024).

[2] C. T. Hill, P. A. N. Machado, A. E. Thomsen, and J. Turner, "Scalar democracy," *Phys. Rev. D* **100** (2019) 015015 [`arXiv:1902.07214`].

[3] N. Seiberg, "Electric–magnetic duality in supersymmetric non-Abelian gauge theories," *Nucl. Phys.* **B435** (1995) 129 [`hep-th/9411149`].