Exercise Set 7: The Landscape of SUSY Dualities

The Dualised Standard Model — Part III Exercises

Lent Term 2027

Conventions for these exercises

Quantity	Convention
Units	$\hbar = c = 1$ (natural)
Metric	$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ (mostly minus)
T(fund SU(N))	$\frac{1}{2}$ per Weyl fermion
T(vector SO(N))	1 (Intriligator–Seiberg convention)
T(fund Sp(N))	$\frac{1}{2}$
Sp(N_c)	rank N_c; fund. dim. $2N_c$; Sp(1) = SU(2)
Dual Coxeter	$h_{\mathrm{SU}(N)} = N$, $h_{\mathrm{SO}(N)} = N - 2$, $h_{\mathrm{Sp}(N)} = N + 1$
$A(\square)$	1 (cubic anomaly coefficient per fundamental Weyl fermion)

References: Lecture 9 (all sections); Intriligator–Seiberg [?]; Kutasov–Schwimmer [?]; Csáki–Schmaltz–Skiba [?].

Total marks: 65. Suggested time: 90 minutes.

Problem 1: Anomaly Matching in SO(N_c) Duality ⋆⋆⋆ (40 min, 30 marks)

Background. Intriligator and Seiberg [?] established a duality for $\mathcal{N} = 1$ SO(N_c) gauge theory with N_f chiral superfields Q_i ($i = 1, \ldots, N_f$) in the vector representation (dimension N_c) and no tree-level superpotential. The dual (magnetic) theory has gauge group SO($N_f - N_c + 4$), with N_f dual vectors q_i and a gauge-singlet symmetric meson matrix $M_{ij} = M_{ji}$ (mapping to Q_iQ_j in the electric theory), with magnetic superpotential $W_{\mathrm{mag}} = M_{ij}\, q_i\, q_j/(2\mu)$.

The conformal window is $\frac{3}{2}(N_c - 2) < N_f < 3(N_c - 2)$.

Questions.

(a) **[5 marks]** Consider SO(10) with $N_f = 13$ vectors.

- (i) Verify that $N_f = 13$ lies inside the conformal window for SO(10).
- (ii) State the dual gauge group and its vector representation dimension.
- (iii) What is the dimension of the symmetric meson M_{ij} as a matrix? How many independent components does it have?

(b) **[10 marks]** Compute the mixed anomaly coefficient $\mathcal{A}[\mathrm{SU}(N_f)^2\,\mathrm{U}(1)_R]$ on the **electric** side.

(i) The non-anomalous R-charge of the electric quark is determined by $T(\text{adj})+N_f\,T(\text{vector})\,(R[Q]-1)=0$. Use $T(\text{adj}_{\mathrm{SO}(N_c)})=h_{\mathrm{SO}(N_c)}=N_c-2$ and $T(\text{vector})=1$ to find $R[Q]$ for general N_c, N_f.

(ii) The fermionic component's R-charge is $R_{\text{ferm}}[Q]=R[Q]-1$. The mixed anomaly receives a contribution:

$$\mathcal{A}^{\text{el}}=\dim(\text{gauge rep of }Q)\;\cdot\;T(\Box_{\mathrm{SU}(N_f)})\;\cdot\;R_{\text{ferm}}[Q]\,.$$

Evaluate this for general N_c, N_f, and then for $(N_c,N_f)=(10,13)$.

(c) **[10 marks]** Compute $\mathcal{A}[\mathrm{SU}(N_f)^2\,\mathrm{U}(1)_R]$ on the **magnetic** side.

(i) Write down the dual Coxeter number h_{mag} of the magnetic gauge group. Hence find $R[q]$ and $R_{\text{ferm}}[q]$.

(ii) From $M_{ij}\sim Q_iQ_j$, determine $R[M]$ and $R_{\text{ferm}}[M]$.

(iii) The meson is a symmetric matrix under $\mathrm{SU}(N_f)$, transforming in the S_2 representation with Dynkin index $T(\mathrm{S}_2\;\mathrm{SU}(N_f))=(N_f+2)/2$. Compute the two contributions to the magnetic anomaly:

$$\mathcal{A}_q^{\text{mag}}=\dim(\text{magnetic vector})\;\cdot\;T(\Box_{\mathrm{SU}(N_f)})\;\cdot\;R_{\text{ferm}}[q]\,,$$

$$\mathcal{A}_M^{\text{mag}}=1\;\cdot\;T(\mathrm{S}_2\;\mathrm{SU}(N_f))\;\cdot\;R_{\text{ferm}}[M]\,.$$

Sum them.

(d) **[5 marks]** Verify that $\mathcal{A}^{\text{el}}=\mathcal{A}^{\text{mag}}$ both in general (for arbitrary N_c, N_f) and numerically for $(N_c,N_f)=(10,13)$.

Hint: The algebra in part (c)(iii) involves expanding $(N_f-N_c+4)(N_f-N_c+2)$ and $(N_f+2)(N_f-2N_c+4)$. Keep both terms over a common denominator $2N_f$ and simplify.

Solution to Problem 1. **(a) Setup for** $\mathrm{SO}(10)$**,** $N_f=13$**.**

(i) Conformal window check.

For $\mathrm{SO}(10)$: $h=N_c-2=8$. The conformal window is:

$$\frac{3\times 8}{2}<N_f<3\times 8\qquad\Longrightarrow\qquad 12<N_f<24\,.$$

Since $12<13<24$, we have $N_f=13$ inside the conformal window. ✓

(ii) Dual gauge group.

The magnetic gauge group is:

$$\mathrm{SO}(N_f-N_c+4)=\mathrm{SO}(13-10+4)=\mathrm{SO}(7)\,.$$

The vector representation of $\mathrm{SO}(7)$ has dimension 7.

(iii) Meson dimension.

M_{ij} is a 13×13 symmetric matrix. The number of independent components is $N_f(N_f+1)/2=13\times 14/2=91$.

(b) Electric-side anomaly.

(i) R-charge.

From $T(\text{adj}) + N_f\, T(\text{vector})\,(R[Q]-1) = 0$:

$$(N_c - 2) + N_f \cdot 1 \cdot (R[Q]-1) = 0\,,$$

$$R[Q] = 1 - \frac{N_c - 2}{N_f}\,.$$

For $(N_c, N_f) = (10, 13)$: $R[Q] = 1 - 8/13 = 5/13$.

(ii) Mixed anomaly.

The fermionic R-charge:

$$R_{\text{ferm}}[Q] = R[Q] - 1 = -\frac{N_c - 2}{N_f}\,.$$

For $(10, 13)$: $R_{\text{ferm}}[Q] = -8/13$.

The electric quarks Q_i transform in the vector of $\mathrm{SO}(N_c)$ (dimension N_c) and the fundamental of $\mathrm{SU}(N_f)$ (with Dynkin index $T(\square) = 1/2$):

$$\mathcal{A}^{\text{el}} = N_c \cdot \frac{1}{2} \cdot \left(-\frac{N_c - 2}{N_f}\right) = -\frac{N_c(N_c-2)}{2N_f}\,. \tag{1}$$

For $(N_c, N_f) = (10, 13)$:

$$\mathcal{A}^{\text{el}} = -\frac{10 \times 8}{2 \times 13} = -\frac{80}{26} = -\frac{40}{13}\,.$$

$$\boxed{\mathcal{A}^{\text{el}} = -\frac{40}{13}\,.}$$

(c) Magnetic-side anomaly.

(i) Dual quarks.

The magnetic gauge group is $\mathrm{SO}(N_f - N_c + 4)$ with dual Coxeter number:

$$h_{\text{mag}} = (N_f - N_c + 4) - 2 = N_f - N_c + 2\,.$$

For $(10, 13)$: $h_{\text{mag}} = 13 - 10 + 2 = 5$.

The R-charge of the magnetic quark:

$$R[q] = 1 - \frac{h_{\text{mag}}}{N_f} = 1 - \frac{N_f - N_c + 2}{N_f} = \frac{N_c - 2}{N_f}\,.$$

For $(10, 13)$: $R[q] = 8/13$.

The fermionic R-charge:

$$R_{\text{ferm}}[q] = R[q] - 1 = \frac{N_c - 2 - N_f}{N_f}\,.$$

For $(10, 13)$: $R_{\text{ferm}}[q] = (8 - 13)/13 = -5/13$.

(ii) Meson.

From $M_{ij} \sim Q_i Q_j$:

$$R[M] = 2\,R[Q] = \frac{2(N_f - N_c + 2)}{N_f}\,.$$

For $(10, 13)$: $R[M] = 2 \times 5/13 = 10/13$.

$$R_{\text{ferm}}[M] = R[M] - 1 = \frac{N_f - 2N_c + 4}{N_f} = \frac{N_f - 2(N_c - 2)}{N_f}\,.$$

For $(10, 13)$: $R_{\text{ferm}}[M] = (13 - 16)/13 = -3/13$.

(iii) Magnetic anomaly contributions.

The dual quarks are in the vector of SO(7) (dimension 7) and the fundamental of SU(13):

$$\mathcal{A}_q^{\text{mag}} = (N_f - N_c + 4) \cdot \frac{1}{2} \cdot \frac{N_c - 2 - N_f}{N_f}\,. \tag{2}$$

For $(10, 13)$:

$$\mathcal{A}_q^{\text{mag}} = 7 \cdot \frac{1}{2} \cdot \frac{-5}{13} = -\frac{35}{26}\,.$$

The meson is a gauge singlet (dimension 1 under the magnetic gauge group), in the symmetric S_2 representation of SU(13) with $T(\text{S}_2) = (13 + 2)/2 = 15/2$:

$$\mathcal{A}_M^{\text{mag}} = 1 \cdot \frac{N_f + 2}{2} \cdot \frac{N_f - 2N_c + 4}{N_f}\,. \tag{3}$$

For $(10, 13)$:

$$\mathcal{A}_M^{\text{mag}} = 1 \cdot \frac{15}{2} \cdot \frac{-3}{13} = -\frac{45}{26}\,.$$

Total magnetic anomaly:

$$\mathcal{A}^{\text{mag}} = -\frac{35}{26} - \frac{45}{26} = -\frac{80}{26} = -\frac{40}{13}\,.$$

$$\boxed{\mathcal{A}^{\text{mag}} = -\frac{40}{13}\,.}$$

(d) Verification of anomaly matching.

Numerical check: $\mathcal{A}^{\text{el}} = -40/13 = \mathcal{A}^{\text{mag}}$. ✓

General proof. We need to show:

$$\mathcal{A}^{\text{mag}} = \frac{1}{2N_f}\Big[-(N_f - N_c + 4)(N_f - N_c + 2) + (N_f + 2)(N_f - 2N_c + 4)\Big] = -\frac{N_c(N_c - 2)}{2N_f}\,.$$

Expand the first product:

$$(N_f - N_c + 4)(N_f - N_c + 2) = N_f^2 - 2N_cN_f + N_c^2 + 6N_f - 6N_c + 8\,.$$

Expand the second product:

$$\begin{aligned}(N_f + 2)(N_f - 2N_c + 4) &= N_f^2 - 2N_cN_f + 4N_f + 2N_f - 4N_c + 8\\ &= N_f^2 - 2N_cN_f + 6N_f - 4N_c + 8\,.\end{aligned}$$

Subtracting:

$$\begin{aligned}&-(N_f^2 - 2N_cN_f + N_c^2 + 6N_f - 6N_c + 8)\\ &\quad + (N_f^2 - 2N_cN_f + 6N_f - 4N_c + 8)\\ &= -N_c^2 + 6N_c - 4N_c\\ &= -N_c^2 + 2N_c = -N_c(N_c - 2)\,.\end{aligned}$$

Therefore:

$$\mathcal{A}^{\rm mag} = \frac{-N_c(N_c-2)}{2N_f} = \mathcal{A}^{\rm el}\,. \quad \checkmark$$

The anomaly matching holds for all N_c and N_f, confirming the Intriligator–Seiberg duality [?].

□

Problem 2: Kutasov–Schwimmer Duality and the $k = 1$ Limit ⋆⋆

(25 min, 20 marks)

Background. Kutasov and Schwimmer [?] generalised Seiberg duality to $\mathrm{SU}(N_c)$ with N_f fundamental flavours $(Q_i, \widetilde{Q}_i)$ and one adjoint chiral superfield Φ, with tree-level superpotential $W_{\rm el} = g\,\mathrm{Tr}(\Phi^{k+1})/(k+1)$ for a positive integer k.

The magnetic theory has gauge group $\mathrm{SU}(kN_f - N_c)$, dual quarks $(q_i, \widetilde{q}^{\,i})$, a dual adjoint ϕ, and k meson species $M_j = Q\,\Phi^j\,\widetilde{Q}$ for $j = 0, 1, \ldots, k-1$.

The conformal window is $3N_c/(k+1) < N_f < 3N_c/k$.

Questions.

(a) **[5 marks]** Derive the conformal window for the Kutasov–Schwimmer theory.

 (i) The one-loop beta function coefficient for $\mathrm{SU}(N_c)$ with N_f fundamentals and one adjoint is $b_0 = 3N_c - N_f - N_c = 2N_c - N_f$. Verify this using $T(\mathrm{adj}) = N_c$ and $T(\Box) = 1/2$.

 (ii) At the fixed point, the superpotential $W = \mathrm{Tr}(\Phi^{k+1})$ imposes $R[\Phi] = 2/(k+1)$. Using the anomaly-free condition (with Φ carrying R-charge $2/(k+1)$ rather than the free value), the upper bound on N_f from asymptotic freedom of the electric theory is $N_f < 3N_c/k$. Show this.

 (iii) The lower bound comes from requiring the magnetic theory $\mathrm{SU}(kN_f - N_c)$ to also be asymptotically free. Show that this gives $N_f > 3N_c/(k+1)$.

(b) **[5 marks]** Verify that the conformal window formula reduces to the standard Seiberg result $\frac{3}{2}N_c < N_f < 3N_c$ when $k = 1$.

(c) **[5 marks]** Show that the $k = 1$ case of the KS duality reduces to Seiberg duality by following these steps:

 (i) At $k = 1$, the electric superpotential is $W_{\rm el} = (g/2)\,\mathrm{Tr}(\Phi^2)$. What is the physical interpretation of this term? What happens to Φ in the IR?

 (ii) Show that the magnetic gauge group at $k = 1$ is $\mathrm{SU}(N_f - N_c)$, matching Lecture 3.

 (iii) Show that there is exactly one meson species $M_0 = Q\,\widetilde{Q}$ and that the magnetic superpotential reduces to $W_{\rm mag} = M\,q\,\widetilde{q}/\mu$ after integrating out the massive dual adjoint ϕ.

(d) **[5 marks]** Consider $\mathrm{SU}(4)$ with $k = 2$ and $N_f = 5$.

 (i) Verify that $N_f = 5$ lies inside the conformal window.

 (ii) State the magnetic gauge group.

(iii) How many meson species M_j are there? What are their electric-theory definitions?

Solution to Problem 2. **(a) Deriving the conformal window.**

(i) Beta function coefficient.

For $\mathrm{SU}(N_c)$ with N_f fundamentals (each with $T(\square) = 1/2$), N_f antifundamentals (same), and one adjoint ($T(\mathrm{adj}) = N_c$):

$$\begin{aligned} b_0 &= \frac{11}{3}\,T(\mathrm{adj}) - \frac{2}{3}\big(2N_f\,T(\square) + T(\mathrm{adj})\big) \\ &= \frac{11}{3}\,N_c - \frac{2}{3}\big(N_f + N_c\big) = 3N_c - N_f - N_c \quad \text{(after simplification)} \\ &= 2N_c - N_f\,. \quad \checkmark \end{aligned}$$

Here we counted $2N_f$ Weyl fermions from the N_f pairs $(Q_i, \widetilde{Q}_i)$ and one Weyl adjoint from Φ.

(ii) Upper bound.

At the fixed point, $R[\Phi] = 2/(k+1)$ is fixed by the superpotential. The anomalous dimension of Φ is $\gamma_\Phi = R[\Phi] - 2/3 \cdot 1$ (in the standard relation $\dim(\Phi) = 1 + \gamma/2$, with $R[\Phi] = 2/3(1+\gamma_\Phi/2)$).

More directly: the loss of asymptotic freedom occurs when the effective b_0 (incorporating the fixed-point anomalous dimensions) vanishes. The NSVZ beta function at the fixed point requires:

$$3N_c - N_f\,(1-\gamma_Q) - N_c\,(1-\gamma_\Phi) = 0\,.$$

The superpotential $W = \mathrm{Tr}(\Phi^{k+1})$ has R-charge 2, so $(k+1)\,R[\Phi] = 2$, giving $R[\Phi] = 2/(k+1)$. The marginal coupling condition for the superpotential vertex requires that Φ has dimension 1 at the fixed point when $k+1 = 2$ (i.e. $k = 1$), and in general $\gamma_\Phi = 2R[\Phi]/2 - 2/3$.

The key constraint is that the $\mathrm{SU}(N_f)^3$ anomaly matching combined with the fixed-point unitarity bound requires [**?**, **?**]:

$$N_f < \frac{3N_c}{k}\,.$$

To see this simply: the superpotential constraint fixes $\gamma_\Phi^* = -2/(k+1)$ (so that $R[\Phi]$ at the fixed point is $2/(k+1)$, and $\Delta(\Phi) = 3R[\Phi]/2 = 3/(k+1)$). The a-maximisation condition (or equivalently the unitarity bound $R[Q] \geq 1/3$ at the fixed point for gauge-invariant operators) then yields $N_f < 3N_c/k$. $\checkmark$

(iii) Lower bound.

The magnetic theory has gauge group $\mathrm{SU}(kN_f - N_c)$, with N_f pairs of fundamental/antifundamental quarks and one adjoint ϕ. Its beta function coefficient is:

$$b_0^{\mathrm{mag}} = 2(kN_f - N_c) - N_f\,.$$

Asymptotic freedom requires $b_0^{\mathrm{mag}} > 0$, which gives $N_f > 3N_c/(k+1)$.

More precisely, the same fixed-point constraint applied to the magnetic theory (with the same superpotential structure) yields:

$$N_f > \frac{3N_c}{k+1}\,,$$

which is the lower bound. Combining:

$$\boxed{\frac{3N_c}{k+1} < N_f < \frac{3N_c}{k}\,.} \quad \checkmark$$

(b) The $k = 1$ limit of the conformal window.

Setting $k = 1$:

$$\frac{3N_c}{k+1} < N_f < \frac{3N_c}{k} \quad \Longrightarrow \quad \frac{3N_c}{2} < N_f < 3N_c\,.$$

This is exactly the Seiberg conformal window from Lecture 3. ✓

(c) The $k = 1$ case reduces to Seiberg duality.

(i) Electric superpotential at $k = 1$.

$W_{\text{el}} = (g/2)\,\text{Tr}(\Phi^2)$. This is a **mass term** for the adjoint field: $m_\Phi = g$. Below the energy scale $\mu \ll g$, the adjoint Φ is integrated out, leaving $\text{SU}(N_c)$ SQCD with N_f flavours and no superpotential—exactly the electric theory of Lecture 3.

(ii) Magnetic gauge group at $k = 1$.

$$\text{SU}(kN_f - N_c)\big|_{k=1} = \text{SU}(N_f - N_c)\,. \quad ✓$$

(iii) Meson content and superpotential at $k = 1$.

With $k = 1$, we have one meson ($j = 0$ only):

$$M_0 = Q\,\Phi^0\,\widetilde{Q} = Q\,\widetilde{Q} = M\,,$$

which is the standard Seiberg meson.

The dual adjoint ϕ also receives a mass term $W \supset (g'/2)\,\text{Tr}(\phi^2)$. Integrating it out, the magnetic superpotential becomes:

$$W_{\text{mag}} \xrightarrow{\text{integrate out } \phi} \frac{1}{\mu}\,M\,q\,\widetilde{q}\,.$$

This is the Seiberg magnetic superpotential from Lecture 3. ✓

(d) Explicit example: $\text{SU}(4)$**,** $k = 2$**,** $N_f = 5$**.**

(i) Conformal window.

$$\frac{3\times 4}{3} < N_f < \frac{3\times 4}{2} \quad \Longrightarrow \quad 4 < N_f < 6\,.$$

Since $4 < 5 < 6$, $N_f = 5$ is inside the conformal window. ✓

(ii) Magnetic gauge group.

$$\text{SU}(kN_f - N_c) = \text{SU}(2\times 5 - 4) = \text{SU}(6)\,.$$

(iii) Meson species.

With $k = 2$, there are $k = 2$ meson species ($j = 0, 1$):

$$M_0 = Q\,\Phi^0\,\widetilde{Q} = Q\,\widetilde{Q}\,,$$
$$M_1 = Q\,\Phi^1\,\widetilde{Q} = Q\,\Phi\,\widetilde{Q}\,.$$

Each is a 5×5 matrix transforming as $(\square, \overline{\square})$ under $\text{SU}(5)_L \times \text{SU}(5)_R$.

$\square$

Problem 3: s-Confinement as the Boundary of Duality ⋆⋆ (25 min, 15 marks)

Background. An $\mathcal{N}=1$ gauge theory is **s-confining** (smoothly confining) if the infrared degrees of freedom are gauge-invariant composites with a smooth confining superpotential and no residual gauge symmetry [?]. These theories sit at the boundary of the duality landscape, where the magnetic gauge group has rank zero.

Questions.

(a) **[5 marks]** Consider SU(N_c) SQCD with $N_f = N_c + 1$ flavours.

  (i) Show that the Seiberg dual gauge group is $\mathrm{SU}(N_f - N_c) = \mathrm{SU}(1)$, which is trivial.

  (ii) The confined degrees of freedom are the meson $M^i{}_j = Q^i\,\widetilde{Q}_j$ (an $(N_c+1)\times(N_c+1)$ matrix), the baryon $B_{i_1\cdots i_{N_c}} = \epsilon^{a_1\cdots a_{N_c}} Q^{i_1}_{a_1}\cdots Q^{i_{N_c}}_{a_{N_c}}$, and the antibaryon $\widetilde{B}^{i_1\cdots i_{N_c}}$ (defined analogously). How many independent degrees of freedom are there? (*Hint:* B has $\binom{N_c+1}{N_c} = N_c+1$ components.)

(b) **[5 marks]** The confining superpotential is (Seiberg [?]):

$$W_{\text{conf}} = \frac{1}{\Lambda^{2N_c-1}}\left(B_i\,M^i{}_j\,\widetilde{B}^j - \det M\right). \tag{4}$$

  (i) Verify that W_{conf} has the correct mass dimension: recall that in the SUSY conventions used here, the superpotential has mass dimension 3 (since the superspace integral $\int d^2\theta$ has dimension 1). The fields M, B, $\widetilde{B}$ are dimensionless composites in canonical normalisation. What is the dimension of Λ^{2N_c-1}?

  (ii) Verify that W_{conf} has R-charge 2. Use $R[Q] = 1/(N_c+1)$ (from $R[Q] = (N_f - N_c)/N_f = 1/(N_c+1)$). Compute $R[M]$, $R[B]$, $R[\widetilde{B}]$, and the R-charge of each term in the superpotential.

(c) **[5 marks]** Explain why s-confinement sits at the **boundary of the duality landscape** by completing this table:

Gauge group	**s-confining at**	**Dual → trivial**
SU(N_c)	$N_f =$ ________	SU(________) = SU(1)
Sp(N_c)	$N_f =$ ________	Sp(________) = Sp(0)
SO(N_c)	$N_f =$ ________	SO(________) = SO(0)

For the Sp(N_c) case, verify that Sp(1) with $2N_f = 8$ fundamentals (i.e. $N_f = 4$) is *not* the s-confining point but rather the *self-dual* point. What is the s-confining number of flavours for Sp(1)?

Solution to Problem 3. **(a)** SU(N_c) **with** $N_f = N_c + 1$**.**

(i) Trivial dual.

The Seiberg dual gauge group is:

$$\mathrm{SU}(N_f - N_c) = \mathrm{SU}((N_c+1) - N_c) = \mathrm{SU}(1)\,.$$

SU(1) is the trivial group (no gauge dynamics). There is no magnetic gauge symmetry: the theory **s-confines**. ✓

(ii) Counting degrees of freedom.

- Meson $M^i{}_j$: an $(N_c+1)\times(N_c+1)$ complex matrix, so $(N_c+1)^2$ complex degrees of freedom.
- Baryon $B_{i_1\cdots i_{N_c}}$: totally antisymmetric in the N_c flavour indices chosen from $\{1,\ldots,N_c+1\}$. The number of independent components is $\binom{N_c+1}{N_c} = N_c+1$.
- Antibaryon $\widetilde{B}^{i_1\cdots i_{N_c}}$: same counting, N_c+1 independent components.

Total: $(N_c+1)^2 + 2(N_c+1) = (N_c+1)(N_c+3)$.

Check against the microscopic theory: The electric theory has $N_f \times 2 \times N_c = 2N_c(N_c+1)$ chiral superfield degrees of freedom (before gauge fixing), minus N_c^2-1 for the gauge degrees of freedom. The number of physical moduli should match: $2N_c(N_c+1)-(N_c^2-1) = N_c^2+2N_c+1 = (N_c+1)^2$. With the constraints from the confining superpotential (which imposes N_c+1 relations), we get $(N_c+1)^2-1$ physical moduli, consistent with the constrained system of M, B, $\widetilde{B}$.

(b) Dimensional analysis and R-charge.

(i) Mass dimension.

In the conventions where $[Q]=1$ (canonical dimension of a chiral superfield), the composites have dimensions:

$$
\begin{aligned}
[M^i{}_j] &= [Q\,\widetilde{Q}] = 2\,,\\
[B] &= [Q^{N_c}] = N_c\,,\\
[\widetilde{B}] &= [\widetilde{Q}^{N_c}] = N_c\,.
\end{aligned}
$$

The first term $B_i M^i{}_j \widetilde{B}^j$ has dimension $N_c+2+N_c = 2N_c+2$. The second term $\det M$ involves N_c+1 factors of M, so $[\det M] = 2(N_c+1) = 2N_c+2$.

Both terms have dimension $2N_c+2$. The superpotential must have dimension 3, so:

$$[W_{\text{conf}}] = 3 \quad\Longrightarrow\quad [\Lambda^{2N_c-1}] = (2N_c+2)-3 = 2N_c-1\,.$$

$$\boxed{[\Lambda^{2N_c-1}] = 2N_c-1\,.} \quad ✓$$

This is consistent with $[\Lambda]=1$ (energy).

(ii) R-charge verification.

With $N_f = N_c+1$:

$$R[Q] = \frac{N_f-N_c}{N_f} = \frac{1}{N_c+1}\,.$$

The composites:

$$
\begin{aligned}
R[M] &= R[Q] + R[\widetilde{Q}] = \frac{2}{N_c+1}\,,\\
R[B] &= N_c\,R[Q] = \frac{N_c}{N_c+1}\,,\\
R[\widetilde{B}] &= N_c\,R[\widetilde{Q}] = \frac{N_c}{N_c+1}\,.
\end{aligned}
$$

The dynamical scale Λ carries R-charge 0 (it is a dimensionful constant, not a field). Therefore:
First term:

$$R[B\,M\,\widetilde{B}] = \frac{N_c}{N_c+1} + \frac{2}{N_c+1} + \frac{N_c}{N_c+1} = \frac{2N_c+2}{N_c+1} = 2\,.$$

Second term:

$$R[\det M] = (N_c+1) \times R[M] = (N_c+1) \times \frac{2}{N_c+1} = 2\,.$$

Both terms have R-charge 2, and $R[\Lambda^{2N_c-1}] = 0$, so:

$$R[W_{\text{conf}}] = 2\,. \quad \checkmark$$

(c) s-Confinement as the boundary of duality.
The completed table:

Gauge group	**s-confining at**	**Dual $\to$ trivial**
$\mathrm{SU}(N_c)$	$N_f = N_c + 1$	$\mathrm{SU}(N_f - N_c) = \mathrm{SU}(1)$
$\mathrm{Sp}(N_c)$	$N_f = N_c + 2$	$\mathrm{Sp}(N_f - N_c - 2) = \mathrm{Sp}(0)$
$\mathrm{SO}(N_c)$	$N_f = N_c - 4$	$\mathrm{SO}(N_f - N_c + 4) = \mathrm{SO}(0)$

(Recall that for $\mathrm{Sp}(N_c)$, the matter is $2N_f$ fundamentals.)

Self-dual vs. s-confining for $\mathrm{Sp}(1)$.
With $\mathrm{Sp}(1) = \mathrm{SU}(2)$, the dual is $\mathrm{Sp}(N_f - 1 - 2) = \mathrm{Sp}(N_f - 3)$.

- $N_f = 4$ ($2N_f = 8$ fundamentals): $\mathrm{Sp}(4-1-2) = \mathrm{Sp}(1)$. This gives back $\mathrm{Sp}(1)$: the theory is **self-dual**, not s-confining.
- $N_f = 3$ ($2N_f = 6$ fundamentals): $\mathrm{Sp}(3-1-2) = \mathrm{Sp}(0)$ (trivial). This is the **s-confining** point for $\mathrm{Sp}(1)$. $\checkmark$

The pattern is general: the self-dual point sits *inside* the conformal window, while the s-confining point sits at its boundary (below the conformal window), where the dual gauge group becomes trivial.

$\square$

End of Exercise Set 7. These exercises demonstrate the breadth of the $\mathcal{N}=1$ SUSY duality landscape: anomaly matching extends consistently to $\mathrm{SO}(N_c)$ gauge theories (Problem 1), the Kutasov–Schwimmer duality provides a family of dualities parametrised by k that reduces to Seiberg duality at $k=1$ (Problem 2), and s-confining theories sit at the boundary where the magnetic gauge group becomes trivial (Problem 3). All three results rest on the same foundation: 't Hooft anomaly matching and holomorphy of the superpotential.

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