Exercise Set 8: Physical Mesons as Dual Mesons

The Dualised Standard Model — Part III Exercises

Lent Term 2027

Conventions for these exercises

Quantity	**Convention**
Units	$\hbar = c = 1$ (natural)
Metric	$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ (mostly minus)
Flavour ordering	$(u, d, s, c, b, t) = (1, 2, 3, 4, 5, 6)$ (PDG)
Meson naming	PDG 2024 [?]
Pion decay constant	$f_\pi = 92.1$ MeV (physical convention)
Epistemic	"if the conjectured duality holds" for all non-SUSY claims (P1)

References: Lecture 10 (all sections); PDG 2024 [?]; Cacciapaglia–Sannino [?].

Total marks: 65. **Suggested time: 90 minutes.**

Epistemic note. The quantum number assignments and the GMOR relation in these exercises are standard QCD results, independent of any duality. The *interpretation* of the meson matrix as an elementary field of the magnetic theory is conjectural and is marked explicitly where it arises.

Problem 1: Meson Quantum Numbers from the Flavour Matrix

⋆⋆ (30 min, 25 marks)

Background. The meson matrix $\mathcal{M}_{ij}$ $(i, j = 1, \ldots, 6)$ has entries $\mathcal{M}_{ij} \sim q_i\,\bar{q}_j$, where q_i is a quark of flavour i and $\bar{q}_j$ an antiquark of flavour j. Each entry corresponds to a meson whose electric charge is (Lecture 10, Eq. (10)):

$$Q_{\mathrm{em}}(\mathcal{M}_{ij}) = Q_{\mathrm{em}}(q_i) - Q_{\mathrm{em}}(q_j)\,. \tag{1}$$

The quark quantum numbers are:

$$\begin{array}{c|cccccc} & u & d & s & c & b & t \\ \hline Q_{\mathrm{em}} & +\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} & +\frac{2}{3} & -\frac{1}{3} & +\frac{2}{3} \\ I_3 & +\frac{1}{2} & -\frac{1}{2} & 0 & 0 & 0 & 0 \\ S & 0 & 0 & -1 & 0 & 0 & 0 \\ C & 0 & 0 & 0 & +1 & 0 & 0 \\ B_q & 0 & 0 & 0 & 0 & -1 & 0 \end{array} \tag{2}$$

For a meson with $B = 0$ (baryon number), the generalised Gell-Mann–Nishijima formula gives (Lecture 10, Eq. (9)):

$$Q_{\text{em}} = I_3 + \tfrac{1}{2}(S + C + B_q)\,. \tag{3}$$

Questions.

(a) **[5 marks]** Write out the 3×3 meson matrix for the light quarks (u, d, s). Label each entry with its quark content $q_i\,\bar{q}_j$. Identify the off-diagonal entries with physical pseudoscalar mesons ($\pi^{\pm}$, $K^{\pm}$, K^0, $\bar{K}^0$).

(b) **[10 marks]** For each of the six off-diagonal mesons in the 3×3 block:

  (i) Determine the quark content.

  (ii) Assign the quantum numbers I, I_3, and S of the meson (using $I_3^{\text{meson}} = I_3^{q_i} - I_3^{q_j}$ and $S^{\text{meson}} = S^{q_i} - S^{q_j}$).

  (iii) Verify the electric charge using the Gell-Mann–Nishijima formula (**??**).

  (iv) Also verify the charge directly from $Q = Q(q_i) - Q(q_j)$.

  Present your results in a table.

(c) **[5 marks]** The three diagonal entries $\mathcal{M}_{11} = u\bar{u}$, $\mathcal{M}_{22} = d\bar{d}$, $\mathcal{M}_{33} = s\bar{s}$ are flavour-neutral. Explain why the physical states are not simply $u\bar{u}$, $d\bar{d}$, $s\bar{s}$ but rather π^0, η, and η'. What symmetry principle determines the mixing? Briefly explain the role of the $\mathrm{U}(1)_A$ anomaly in making η' heavy.

(d) **[5 marks]** Extend the meson matrix to $N_f = 4$ by including the charm quark. Identify the new entries $\mathcal{M}_{14}$, $\mathcal{M}_{24}$, $\mathcal{M}_{34}$, $\mathcal{M}_{41}$, $\mathcal{M}_{42}$, $\mathcal{M}_{43}$ with physical mesons (the D meson family). State their quark content, electric charge, and PDG mass (to the nearest MeV).

Solution to Problem 1. **(a) The 3×3 light-quark meson matrix.**

$$\mathcal{M}_{ij} = \begin{pmatrix} u\bar{u} & u\bar{d} & u\bar{s} \\ d\bar{u} & d\bar{d} & d\bar{s} \\ s\bar{u} & s\bar{d} & s\bar{s} \end{pmatrix} = \begin{pmatrix} \pi^0/\eta/\eta' & \pi^+ & K^+ \\ \pi^- & \pi^0/\eta/\eta' & K^0 \\ K^- & \bar{K}^0 & \eta_s \end{pmatrix}$$

The diagonal entries involve flavour mixing (discussed in part (c)). The off-diagonal entries each correspond to a single physical pseudoscalar meson.

(b) Quantum numbers of off-diagonal mesons.

For each meson $\mathcal{M}_{ij} = q_i\,\bar{q}_j$, the quantum numbers are obtained from:

$$\begin{aligned} I_3^{\text{meson}} &= I_3(q_i) - I_3(q_j)\,, \\ S^{\text{meson}} &= S(q_i) - S(q_j) = S(q_i) + S(\bar{q}_j)\,. \end{aligned}$$

Meson	**Entry**	**Content**	I	I_3	S	Q (GMN)	Q (direct)
π^+	$\mathcal{M}_{12}$	$u\bar{d}$	1	$+1$	0	$+1+0=+1$	$\frac{2}{3}+\frac{1}{3}=+1$
π^-	$\mathcal{M}_{21}$	$d\bar{u}$	1	-1	0	$-1+0=-1$	$-\frac{1}{3}-\frac{2}{3}=-1$
K^+	$\mathcal{M}_{13}$	$u\bar{s}$	$\frac{1}{2}$	$+\frac{1}{2}$	$+1$	$+\frac{1}{2}+\frac{1}{2}=+1$	$\frac{2}{3}+\frac{1}{3}=+1$
K^0	$\mathcal{M}_{23}$	$d\bar{s}$	$\frac{1}{2}$	$-\frac{1}{2}$	$+1$	$-\frac{1}{2}+\frac{1}{2}=0$	$-\frac{1}{3}+\frac{1}{3}=0$
K^-	$\mathcal{M}_{31}$	$s\bar{u}$	$\frac{1}{2}$	$-\frac{1}{2}$	-1	$-\frac{1}{2}-\frac{1}{2}=-1$	$-\frac{1}{3}-\frac{2}{3}=-1$
$\bar{K}^0$	$\mathcal{M}_{32}$	$s\bar{d}$	$\frac{1}{2}$	$+\frac{1}{2}$	-1	$+\frac{1}{2}-\frac{1}{2}=0$	$-\frac{1}{3}+\frac{1}{3}=0$

Verification:

- For kaons: $S=+1$ for $\bar{s}$ (antiquark carries opposite strangeness to quark; s has $S=-1$, so $\bar{s}$ has $S=+1$). The Gell-Mann–Nishijima formula $Q=I_3+S/2$ correctly gives $Q(K^+)=1/2+1/2=1$. ✓
- All six charges agree between the two methods. ✓

(c) Diagonal mixing and the η' problem.

The diagonal entries $u\bar{u}$, $d\bar{d}$, $s\bar{s}$ all have $Q=0$, $I_3=0$, $S=0$. Therefore they can mix. The physical states are determined by the **flavour** SU(3) **representation decomposition**:

$$\mathbf{3}\otimes\bar{\mathbf{3}}=\mathbf{8}\oplus\mathbf{1}\,.$$

- The **octet** (**8**) contains π^0 and η_8:

$$\pi^0=\frac{u\bar{u}-d\bar{d}}{\sqrt{2}}\,,\qquad \eta_8=\frac{u\bar{u}+d\bar{d}-2s\bar{s}}{\sqrt{6}}\,.$$

- The **singlet** (**1**) is:

$$\eta_1=\frac{u\bar{u}+d\bar{d}+s\bar{s}}{\sqrt{3}}\,.$$

If SU(3) flavour symmetry were exact, η_8 and η_1 would be mass eigenstates. In reality, SU(3) is broken by $m_s\gg m_u,m_d$, so η_8 and η_1 mix to produce the physical η (548 MeV) and η' (958 MeV).

The U(1)$_A$ **anomaly** plays a critical role: the axial U(1)$_A$ symmetry is broken by instantons, giving the η_1 an anomalous mass contribution. Without this anomaly, the η' would be a ninth pseudo-Goldstone boson with mass comparable to the pion. The resolution of this "U(1) problem" was provided by 't Hooft [**?**], Witten [**?**], and Veneziano [**?**]: the anomaly generates a mass

$$m_{\eta'}^2\sim\frac{2N_f}{f_\pi^2}\,\chi_{\text{top}}\,,$$

where χ_{top} is the topological susceptibility of pure Yang–Mills theory. This pushes $m_{\eta'}$ to ~ 958 MeV, much heavier than the octet pseudoscalars.

(d) Extension to $N_f=4$: the D meson family.

The new entries involving charm quarks and light antiquarks:

Meson	Entry	Quark content	Q	Mass (MeV)
D^0	$\mathcal{M}_{41}$	$c\bar{u}$	$\frac{2}{3} - \frac{2}{3} = 0$	1864.8
D^+	$\mathcal{M}_{42}$	$c\bar{d}$	$\frac{2}{3} + \frac{1}{3} = +1$	1869.7
D_s^+	$\mathcal{M}_{43}$	$c\bar{s}$	$\frac{2}{3} + \frac{1}{3} = +1$	1968.3
$\bar{D}^0$	$\mathcal{M}_{14}$	$u\bar{c}$	$\frac{2}{3} - \frac{2}{3} = 0$	1864.8
D^-	$\mathcal{M}_{24}$	$d\bar{c}$	$-\frac{1}{3} - \frac{2}{3} = -1$	1869.7
D_s^-	$\mathcal{M}_{34}$	$s\bar{c}$	$-\frac{1}{3} - \frac{2}{3} = -1$	1968.3

Quantum number checks:

- $D^0 = c\bar{u}$: charm $C = +1$, no strangeness. $Q = I_3 + (C+S)/2 = 0 + 1/2 = 1/2$? No—$D^0$ has $I_3 = -1/2$ (it is an isodoublet partner of D^+).

  Actually, the charm meson isospin assignments depend on convention. For $c\bar{u}$: $Q = Q(c) - Q(u) = 2/3 - 2/3 = 0$. ✓
- $D_s^+ = c\bar{s}$: $C = +1$, $S = +1$ ($\bar{s}$ contributes $+1$). $Q = 2/3 + 1/3 = +1$. ✓

Note that $\mathcal{M}_{ij}$ and $\mathcal{M}_{ji}$ are charge conjugates: $D^0 = c\bar{u}$ and $\bar{D}^0 = u\bar{c}$.

□

Problem 2: The Scalar–Pseudoscalar Distinction ⋆⋆⋆ (30 min, 20 marks)

Background. In the SUSY dual theory, the meson M_{ij} is a chiral superfield whose lowest component is a complex scalar ($J^P = 0^+$). Physical pions, however, are pseudoscalars ($J^P = 0^-$). The resolution lies in the pattern of chiral symmetry breaking.

The lowest component ϕ_{ij} of the meson superfield can be decomposed around the chiral condensate as (Lecture 10, Eq. (15)):

$$\phi_{ij} = \frac{1}{\sqrt{2}}\big(v_{ij} + \sigma_{ij} + i\,\pi_{ij}\big)\,, \tag{4}$$

where v_{ij} is the vacuum expectation value, σ_{ij} is the scalar (amplitude) fluctuation ($J^P = 0^+$), and π_{ij} is the pseudoscalar (phase) fluctuation ($J^P = 0^-$).

Questions.

(a) **[5 marks]** Identify the physical particles in each sector for $N_f = 3$:

  (i) The scalar fluctuation σ_{12} corresponds to which physical meson? What is its mass?

  (ii) The pseudoscalar fluctuation π_{12} corresponds to which physical meson? What is its mass?

  (iii) Why is $m_\sigma \gg m_\pi$? Give a symmetry-based explanation.

(b) **[10 marks]** The pseudoscalar mesons are pseudo-Goldstone bosons of chiral symmetry breaking $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V$.

(i) How many Goldstone bosons are there for $N_f = 2$? Identify them with physical mesons.

(ii) How many Goldstone bosons for $N_f = 3$? Identify them.

(iii) The pions are massive ($m_\pi \neq 0$) despite being Goldstone bosons. Explain why, and state what would happen to m_π in the chiral limit $m_u, m_d \to 0$.

(iv) The Gell-Mann–Oakes–Renner (GMOR) relation [?] connects the pion mass to the quark masses and the chiral condensate:

$$m_\pi^2 f_\pi^2 = -(m_u + m_d) \langle \bar{q}\, q \rangle \,. \tag{5}$$

Using the PDG/FLAG inputs [?, ?]:

$$\begin{aligned} f_\pi &= 92.1 \text{ MeV}\,, \\ \langle \bar{q}\, q \rangle &\approx -(270 \text{ MeV})^3\,, \\ m_u + m_d &\approx 8.5 \text{ MeV} \quad (\overline{\text{MS}},\ \mu = 2 \text{ GeV})\,, \end{aligned}$$

compute m_π numerically and compare with the observed value $m_{\pi^\pm} = 139.6$ MeV.

(c) **[5 marks]** Explain why the statement "the pion IS the dual meson" is misleading without qualification. Your explanation should address:

(i) What is the J^P of the lowest component of the SUSY meson superfield?

(ii) What is the J^P of the pion?

(iii) How does the pion arise from the decomposition (**??**)?

(iv) What is the **correct** qualified statement relating the dual meson to the pion?

(*All claims about the duality interpretation should carry the qualifier "if the conjectured duality holds."* The QCD statements about chiral symmetry breaking are standard.)

Solution to Problem 2. **(a) Physical identification of scalar and pseudoscalar modes.**

(i) Scalar mode σ_{12}.

The entry $(1, 2)$ corresponds to $u\bar{d}$. The scalar ($J^P = 0^+$) fluctuation σ_{12} is the $a_0(980)^+$ meson, with mass ~ 980 MeV.

(ii) Pseudoscalar mode π_{12}.

The pseudoscalar ($J^P = 0^-$) fluctuation π_{12} is the π^+ pion, with mass 139.6 MeV.

(iii) Why $m_\sigma \gg m_\pi$.

The pion is a **pseudo-Goldstone boson** of spontaneous chiral symmetry breaking $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \to \mathrm{SU}(N_f)_V$. Goldstone's theorem guarantees that it would be exactly massless if the symmetry were exact; its mass arises only from the small explicit breaking by quark masses ($m_u, m_d \sim 5 \text{ MeV} \ll \Lambda_\chi$).

The scalar meson a_0, by contrast, is **not** protected by any Goldstone mechanism. It is a massive excitation of the condensate whose mass is set by the strong interaction scale $\sim \Lambda_\chi \sim 1$ GeV. Hence $m_{a_0} \sim 1 \text{ GeV} \gg m_\pi \sim 140$ MeV.

(b) Goldstone boson counting and the GMOR relation.

(i) $N_f = 2$.

$SU(2)_L \times SU(2)_R \to SU(2)_V$. The number of broken generators is $N_f^2 - 1 = 3$. The three Goldstone bosons are π^+, π^-, π^0.

(ii) $N_f = 3$.

$SU(3)_L \times SU(3)_R \to SU(3)_V$. The number of broken generators is $N_f^2 - 1 = 8$. The eight Goldstone bosons are π^+, π^-, π^0, K^+, K^-, K^0, $\bar{K}^0$, η. (The η' is not a Goldstone boson of $SU(3) \times SU(3)$ breaking; it is the $U(1)_A$ partner made heavy by instantons.)

(iii) Why pions are massive.

The pions are **pseudo-Goldstone bosons**: the chiral symmetry $SU(N_f)_L \times SU(N_f)_R$ is not exact but is *explicitly* broken by the quark mass terms $m_q\,\bar{q}\,q$ in the QCD Lagrangian. This explicit breaking gives the pions a mass proportional to the quark masses.

In the **chiral limit** $m_u, m_d \to 0$, the explicit breaking vanishes and $m_\pi \to 0$. The pions become true Goldstone bosons.

(iv) GMOR numerical computation.

$$\begin{aligned} m_\pi^2 &= \frac{(m_u + m_d)\,|\langle \bar{q}\,q \rangle|}{f_\pi^2} \\ &= \frac{8.5 \text{ MeV} \times (270 \text{ MeV})^3}{(92.1 \text{ MeV})^2}\,. \end{aligned}$$

Step 1: $(270)^3 = 270 \times 270 \times 270 = 19{,}683{,}000 \text{ MeV}^3$.

Step 2: Numerator: $8.5 \times 19{,}683{,}000 = 167{,}306{,}000 \text{ MeV}^4 \approx 1.673 \times 10^8 \text{ MeV}^4$.

Step 3: Denominator: $(92.1)^2 = 8482 \text{ MeV}^2$.

Step 4:

$$m_\pi^2 = \frac{1.673 \times 10^8}{8482} \text{ MeV}^2 \approx 19{,}720 \text{ MeV}^2\,.$$

Step 5:

$$m_\pi = \sqrt{19{,}720} \text{ MeV} \approx 140 \text{ MeV}\,.$$

$$\boxed{m_\pi \approx 140 \text{ MeV}\,.}$$

This matches the measured value $m_{\pi^\pm} = 139.6$ MeV to better than 1%, well within the $\sim 10\%$ uncertainties on the input parameters $m_u + m_d$ and $\langle \bar{q}\,q \rangle$. ✓

(c) Why "the pion IS the dual meson" is misleading.

(i) The lowest component ϕ_{ij} of the SUSY chiral superfield M_{ij} has $J^P = 0^+$ (scalar).

(ii) The pion has $J^P = 0^-$ (pseudoscalar).

(iii) The pion arises as the **imaginary part** (phase fluctuation) of the decomposition $\phi_{ij} = (v_{ij} + \sigma_{ij} + i\,\pi_{ij})/\sqrt{2}$ around the chiral condensate. It is not the scalar field ϕ_{ij} itself, but a fluctuation of ϕ_{ij} around its VEV.

(iv) The **correct qualified statement** is:

> *If the conjectured non-SUSY duality holds, the pseudoscalar meson π^+ corresponds to the phase fluctuation of the chiral condensate $\langle \phi_{12} \rangle$ of the dual meson field. The scalar lowest component ϕ_{12} of the chiral superfield M_{12} maps instead to the scalar meson $a_0(980)^+$. The SUSY meson superfield contains both sectors; the physical distinction arises from chiral symmetry breaking.*

The key point is that a single complex scalar field ϕ_{ij} yields **two** physical sectors after symmetry breaking: the scalars σ and the pseudoscalars π.

□

Problem 3: Mass Hierarchy from the VEV Structure ⋆⋆ (30 min, 20 marks)

Background. In the dualised Standard Model (Lectures 7–8), *if the conjectured duality holds*, the effective Yukawa couplings are (Lecture 7, Eq. (14)):

$$Y_{ij} = y\,\frac{\langle H_{ij}\rangle}{v}\,, \tag{6}$$

where y is a single universal coupling and $\langle H_{ij}\rangle$ is the VEV of the (i,j)-th Higgs doublet. The fermion masses are therefore:

$$m^q_{ij} = y\,\langle H^q_{ij}\rangle\,, \qquad q \in \{u,d\}\,. \tag{7}$$

The physical quark masses (diagonal in the mass basis) arise from diagonalising m_{ij}, with the CKM matrix encoding the mismatch between the up-type and down-type diagonalisations.

The VEV hierarchy suggested by Cacciapaglia and Sannino [**?**] is:

$$\langle H_{33}\rangle \gg \langle H_{22}\rangle \gg \langle H_{11}\rangle\,, \tag{8}$$

arising from higher-dimensional (Planck-suppressed) operators in the electric theory with $\mathcal{O}(1)$ dimensionless coefficients.

Questions.

(a) **[5 marks]** If $\langle H^u_{33}\rangle$ is the dominant VEV (comparable to the electroweak scale v), and the universal Yukawa coupling y is $\mathcal{O}(1)$:

  (i) Show that the top quark mass is $m_t = y\,\langle H^u_{33}\rangle \sim v \sim 174$ GeV.

  (ii) Explain why this is a parametric consistency check (not a prediction): what are the free parameters?

(b) **[10 marks]** The observed quark masses at $\mu = 2$ GeV in the $\overline{\text{MS}}$ scheme are approximately [**?**]:

$$\begin{aligned} m_u &\approx 2.2\text{ MeV}\,, & m_c &\approx 1270\text{ MeV}\,, & m_t &\approx 173{,}000\text{ MeV}\,, \\ m_d &\approx 4.7\text{ MeV}\,, & m_s &\approx 95\text{ MeV}\,, & m_b &\approx 4180\text{ MeV}\,. \end{aligned} \tag{9}$$

(Note: m_t is the pole mass; the others are running masses.)

  (i) Compute the quark mass ratios m_t/m_c, m_c/m_u, m_b/m_s, m_s/m_d from the values above.

  (ii) If the VEV hierarchy is geometric: $\langle H_{33}\rangle/\langle H_{22}\rangle \sim 1/\epsilon$, $\langle H_{22}\rangle/\langle H_{11}\rangle \sim 1/\epsilon$, determine the value of ϵ needed to reproduce each of the ratios in (i) (treating up-type and down-type sectors independently).

(iii) Can a **single** value of ϵ simultaneously reproduce all four ratios? What does this tell you about the $\mathcal{O}(1)$ coefficients?

(c) **[5 marks]** In the light-quark sector, the pseudoscalar meson masses ($m_\pi \approx 140$ MeV, $m_K \approx 494$ MeV, $m_\eta \approx 548$ MeV) are determined by QCD dynamics (quark masses + chiral symmetry breaking), not directly by the Higgs VEVs.

(i) Explain why the dualised SM framework is **consistent with**, but does not **improve upon**, standard QCD predictions for light meson masses.

(ii) What **does** the dualised SM add to the picture beyond standard QCD? List at least two structural insights (assuming the conjectured duality holds).

Solution to Problem 3. **(a) Top quark mass.**

(i) With the dominant VEV $\langle H^u_{33}\rangle \sim v$ (the electroweak scale) and $y \sim \mathcal{O}(1)$:

$$m_t = y\,\langle H^u_{33}\rangle \sim y \times v \sim 1 \times 174 \text{ GeV} = 174 \text{ GeV}\,.$$

(Here $v = (\sqrt{2}\,G_F)^{-1/2} \approx 246$ GeV and $m_t = y_t\,v/\sqrt{2} \approx y_t \times 174$ GeV. Since $y_t = y\,\langle H^u_{33}\rangle/v$ and $\langle H^u_{33}\rangle \sim v$, we get $y_t \sim y \sim \mathcal{O}(1)$.)

This matches the observed $m_t \approx 173$ GeV. ✓

(ii) This is a **parametric consistency check**, not a prediction, because:

- The coupling y is a free parameter (it is $\mathcal{O}(1)$ by assumption, not predicted to be a specific number).
- The VEV $\langle H^u_{33}\rangle$ is set by the minimisation of the scalar potential, which depends on unknown $\mathcal{O}(1)$ coefficients from Planck-suppressed operators.
- The statement "$m_t \sim v$" is equivalent to "$y_t \sim \mathcal{O}(1)$", which is an input assumption, not an output.

The non-trivial content is that the framework *naturally accommodates* $m_t \sim v$ with $\mathcal{O}(1)$ couplings, without requiring fine-tuning.

(b) Quark mass ratios and the geometric hierarchy.

(i) Mass ratios.

Ratio	**Value**
m_t/m_c	$173{,}000/1270 \approx 136$
m_c/m_u	$1270/2.2 \approx 577$
m_b/m_s	$4180/95 \approx 44$
m_s/m_d	$95/4.7 \approx 20$

(ii) Geometric hierarchy parameter.

In the geometric hierarchy picture, the mass of the n-th generation scales as $m_n \propto \epsilon^{3-n}$ (with the 3rd generation having the largest mass). The ratio between consecutive generations is $m_{n+1}/m_n \sim 1/\epsilon$. Therefore:

Up-type sector:

$$\epsilon_u^{(t/c)} = \frac{m_c}{m_t} = \frac{1}{136} \approx 0.0074\,, \qquad \epsilon_u^{(c/u)} = \frac{m_u}{m_c} = \frac{1}{577} \approx 0.0017\,.$$

Down-type sector:

$$\epsilon_d^{(b/s)} = \frac{m_s}{m_b} = \frac{1}{44} \approx 0.023\,, \qquad \epsilon_d^{(s/d)} = \frac{m_d}{m_s} = \frac{1}{20} = 0.050\,.$$

(iii) Can a single ϵ work?

No. The four values of ϵ range from 0.0017 to 0.050—a factor of ~ 30. A **single** geometric ratio ϵ cannot simultaneously reproduce all four mass ratios.

This is **expected** in the Cacciapaglia–Sannino framework [**?**]: the VEV hierarchy is generated by higher-dimensional operators with $\mathcal{O}(1)$ dimensionless coefficients c^{abcd}. These coefficients are different for each entry of the Higgs matrix, so the effective ϵ varies between sectors:

$$m_i \sim y\, c_i\, \epsilon^{3-n_i}\, v\,,$$

where c_i are $\mathcal{O}(1)$ constants (not predicted by the framework).

The **parametric consistency check** is that $\epsilon \in [0.002, 0.05]$—spanning a factor of 25—can be absorbed into $\mathcal{O}(1)$ coefficients. Specifically, if $\epsilon \sim 0.01$ is a "typical" suppression per generation, then the actual ratios (0.002 to 0.05) differ from 0.01 by factors of 0.2 to 5, which are all $\mathcal{O}(1)$.

$$\boxed{\text{Parametric consistency: all ratios explained by a single } \epsilon \sim 0.01 \text{ with } \mathcal{O}(1) \text{ coefficients.}}$$

(c) Light meson masses and the dualised SM.

(i) The dualised SM framework is **consistent with** standard QCD for light meson masses because:

- The pseudoscalar meson masses (m_π, m_K, m_η) are determined by chiral symmetry breaking, which depends on the quark masses m_u, m_d, m_s and the chiral condensate $\langle \bar{q}\, q \rangle$.
- The quark masses enter through the VEV hierarchy (via $m_q = y\,\langle H \rangle$), but the meson masses depend on m_q *and* on the non-perturbative QCD dynamics that form the condensate.
- The dualised SM does not predict $\langle \bar{q}\, q \rangle$ or the individual quark masses; it only provides the hierarchical *structure* $m_t \gg m_c \gg m_u$ with $\mathcal{O}(1)$ free coefficients.

Therefore, the dualised SM reproduces the standard QCD predictions for light meson masses, but does not improve upon them: any set of quark masses can be accommodated by adjusting the $\mathcal{O}(1)$ coefficients.

(ii) What the dualised SM adds (assuming the conjectured duality holds):

1. **A UV origin for the meson matrix structure:** The 6×6 meson matrix arises as an elementary field in the magnetic theory, rather than being an ad hoc effective field. The $N_f \times N_f$ matrix structure is dictated by the gauge group of the electric theory.

2. **The universal Yukawa coupling:** All Yukawa couplings arise from a single coupling y in the magnetic superpotential, with the observed hierarchy coming from the VEV structure rather than from a hierarchy of couplings. This replaces 13 free Yukawa parameters (in the SM) with 1 coupling + Planck-scale operators (though the operators introduce their own free coefficients).

3. **Connection between the Higgs sector and strong dynamics:** The Higgs doublets are identified with entries of the meson matrix, providing a structural link between electroweak symmetry breaking and the strong interaction sector.

□

End of Exercise Set 8. These exercises illustrate three aspects of the physical meson identification: the quantum numbers of the flavour meson matrix are completely determined by quark content and the Gell-Mann–Nishijima formula (Problem 1); the distinction between scalar and pseudoscalar mesons arises from chiral symmetry breaking, with the GMOR relation confirming $m_\pi \approx 140$ MeV (Problem 2); and the fermion mass hierarchy can be parametrically accommodated by a VEV hierarchy with $\mathcal{O}(1)$ coefficients, consistent with but not predictive beyond standard QCD (Problem 3).

References

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