Lecture 1: Anomaly Matching and the Conformal Window

The Dualised Standard Model — Part III Lecture Notes

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Contents

1 Introduction and overview

These lecture notes develop the idea that the Standard Model of particle physics might be understood as a *magnetic dual* theory—the low-energy, weakly-coupled description of a more fundamental, strongly-coupled *electric* theory at high energies. The programme, due primarily to Sannino and collaborators [1], builds on Seiberg's electric–magnetic duality for supersymmetric gauge theories [2] and extends it, with appropriate caveats, to non-supersymmetric gauge theories.

Prerequisites. We assume the Michaelmas QFT course: gauge theory (classical and quantised), anomalies (ABJ anomaly, triangle diagrams), and the representation theory of SU(N_c) at the level of Peskin and Schroeder [3]. *No knowledge of supersymmetry is required* for this lecture; SUSY will be introduced from scratch in Lecture 2.

Course goal. This lecture establishes the tools (anomaly matching, conformal window) that will be used in later lectures to analyse the dualised SM spectrum.

Outline of Lecture 1. We begin with the mathematical tools that underpin all subsequent lectures: 't Hooft anomaly matching and the conformal window of gauge theories. Specifically:

- §2: conventions and notation.
- §3: review of gauge anomaly cancellation (recap from Michaelmas).
- §4: 't Hooft anomaly matching for global symmetries.
- §5: worked example—SU(3) QCD with $N_f = 2$.
- §6: general anomaly matching for SU(N_c) with N_f flavours.
- §7: the beta function and the conformal window.
- §8: summary and preview of Lecture 2.

Remark 1.1 (Chan–Tsou disambiguation). The term "Dualised Standard Model" has been used in a *different* sense by Chan and Tsou [12, 13]. Their programme applies a loop-space *Hodge duality* within the SM gauge group $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ itself, identifying dual colour as a generation symmetry. This is conceptually and technically distinct from the Seiberg-type *electric–magnetic duality between different theories* pursued by Sannino and collaborators. Throughout these notes, "dualised Standard Model" always refers to the Cacciapaglia–Sannino gauge-fermion duality programme [1], not the Chan–Tsou construction.[1]

2 Conventions and notation

All conventions follow Seiberg [2] unless otherwise noted.

[1]For the Chan–Tsou programme, see [12] and the lecture notes [13]. Their construction has no direct connection to Seiberg duality or the anomaly matching machinery developed in these lectures.

Conventions for these lecture notes

Quantity	Convention	Comment
Units	$\hbar = c = 1$ (natural)	
Metric	$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$	mostly minus
Dynkin index	$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\,\delta^{ab}$	fundamental of $\mathrm{SU}(N_c)$
Covariant derivative	$D_\mu = \partial_\mu - i g_s A_\mu^a T^a$	
Anomaly coefficient	$A(\Box) = 1$	cubic; per fundamental Weyl fermion
Beta function	$b_0 = \dfrac{11N_c - 2N_f}{3}$	N_f Dirac flavours
N_f counting	N_f = number of Dirac pairs	each pair = 2 Weyl fermions
Coupling	$\alpha_s = g_s^2/(4\pi)$	
R-charge (SUSY)	$R[Q] = (N_f - N_c)/N_f$	non-anomalous R-symmetry

Weyl vs. Dirac counting

Throughout these notes, N_f counts the number of **Dirac flavour pairs**. Each Dirac fermion consists of two Weyl fermions (one left-handed, one right-handed). In the SUSY literature, one often encounters N_f chiral superfield *pairs* $(Q_i, \widetilde{Q}_i)$, $i = 1, \ldots, N_f$—each such pair corresponds to one Dirac flavour and contains *two* Weyl fermions. Some references use $T(\Box) = 1$ per *Dirac* fermion; we use $T(\Box) = \frac{1}{2}$ per *Weyl* fermion (equivalently, per fundamental of $\mathrm{SU}(N_c)$). The two conventions differ by a factor of 2 that propagates to all anomaly coefficients and the conformal window boundaries.

Group theory data. We collect the Dynkin indices and quadratic Casimirs for common representations of $\mathrm{SU}(N_c)$:

Representation R	Dimension $d(R)$	Dynkin index $T(R)$	Quadratic Casimir $C_2(R)$
Fundamental N_c	N_c	$\frac{1}{2}$	$\frac{N_c^2-1}{2N_c}$
Antifundamental $\overline{N_c}$	N_c	$\frac{1}{2}$	$\frac{N_c^2-1}{2N_c}$
Adjoint	$N_c^2 - 1$	N_c	N_c
Antisymmetric $\wedge^2$	$\frac{N_c(N_c-1)}{2}$	$\frac{N_c-2}{2}$	$\frac{(N_c-1)(N_c+2)}{2N_c}$
Symmetric S_2	$\frac{N_c(N_c+1)}{2}$	$\frac{N_c+2}{2}$	$\frac{(N_c-1)(N_c+4)}{2N_c}$

Check: For SU(3), the fundamental has $C_2(\Box) = (9 - 1)/6 = 4/3$. This is the standard QCD result.

The **cubic anomaly coefficient** is defined by

$$A(R)\, d^{abc} = \mathrm{Tr}_R\big[T^a\{T^b, T^c\}\big], \tag{1}$$

where d^{abc} is the totally symmetric d-symbol of $\mathrm{SU}(N_c)$, normalised so that $A(\Box) = 1$.

Representation	$A(R)$
Fundamental N_c	$+1$
Antifundamental $\overline{N_c}$	-1
Adjoint	$2N_c$
Antisymmetric $\wedge^2$	$N_c - 4$
Symmetric S_2	$N_c + 4$

Note that $A(\overline{R}) = -A(R)$, and that $A(\text{adj}) = 0$ for SU(2) (since SU(2) has no cubic anomaly: all representations are (pseudo-)real).

3 Review: gauge anomaly cancellation

We begin with a brief review of material from the Michaelmas QFT course. The purpose is to set up the contrast with 't Hooft anomaly matching in §4.

3.1 The ABJ anomaly

Consider a triangle Feynman diagram with three external gauge bosons coupled to a chiral fermion in representation R of the gauge group. The one-loop amplitude contains an *anomaly*: the classical symmetry $\partial_\mu j_5^\mu = 0$ is violated at the quantum level,

$$\partial_\mu j_5^\mu = \frac{1}{16\pi^2} F^a_{\mu\nu} \widetilde{F}^{a\,\mu\nu}, \tag{2}$$

where $\widetilde{F}^{a\,\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F^a_{\rho\sigma}$. This is the Adler–Bell–Jackiw (ABJ) anomaly [6, 7].

3.2 Gauge anomaly cancellation: the consistency requirement

For a gauge theory to be *quantum mechanically consistent*, all gauge anomalies must vanish. This means that for every triple of gauge generators (T^a, T^b, T^c), the anomaly coefficient

$$\mathcal{A}^{abc}_{\text{gauge}} = \sum_{\text{Weyl fermions } i} \mathrm{Tr}\big[T^a_{R_i}\{T^b_{R_i}, T^c_{R_i}\}\big] \tag{3}$$

must *vanish*: $\mathcal{A}^{abc}_{\text{gauge}} = 0$. The sum runs over all left-handed Weyl fermions in the theory (with right-handed Weyl fermions contributing with a minus sign, since $\overline{R}$ has $A(\overline{R}) = -A(R)$).

3.3 Example: the Standard Model

In the Standard Model, gauge anomaly cancellation is a non-trivial constraint on the fermion content. For a single generation, the fermion representations under $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ are:

Field	$\mathrm{SU}(3)_c$	$\mathrm{SU}(2)_L$	Y	Chirality
$Q_L = (u_L, d_L)$	**3**	**2**	$+1/6$	L
u_R	**3**	**1**	$+2/3$	R
d_R	**3**	**1**	$-1/3$	R
$L_L = (\nu_L, e_L)$	**1**	**2**	$-1/2$	L
e_R	**1**	**1**	-1	R

We state without rederivation that all gauge anomalies cancel. For instance:

- $\mathrm{SU}(3)_c^2\,\mathrm{U}(1)_Y$: contributions from Q_L, u_R, d_R cancel: $2 \times \frac{1}{2} \times (+\frac{1}{6}) - 1 \times \frac{1}{2} \times (+\frac{2}{3}) - 1 \times \frac{1}{2} \times (-\frac{1}{3}) = \frac{1}{2}(+\frac{1}{3} - \frac{2}{3} + \frac{1}{3}) = 0$.
- $\mathrm{U}(1)_Y^3$: sum of Y^3 over all left-handed Weyl fermions (converting right-handed to left-handed with a sign flip in Y) vanishes.

Key point: Gauge anomaly cancellation is a *consistency requirement*. If it fails, the theory is inconsistent at the quantum level. We now contrast this with a very different use of anomalies.

4 'Hooft anomaly matching

4.1 Gauge anomaly cancellation vs. 't Hooft anomaly matching

The following distinction is **critical** and is a source of widespread confusion (see *e.g.* Peskin [4], Sec. 3).

Gauge anomaly cancellation vs. 't Hooft anomaly matching

	Gauge anomaly cancellation	**'t Hooft anomaly matching**
Symmetry type	*Gauged* symmetry	*Global* symmetry
Condition	Anomaly must **vanish** (= 0)	Anomaly must **match** (UV = IR)
Physical meaning	Quantum consistency of the gauge theory	Constraint on the IR spectrum
What if violated?	Theory is **inconsistent**	Proposed IR spectrum is **wrong**
Role of RG flow	Holds at every scale	Relates two different scales (UV and IR)
Computational object	$\sum_i \mathrm{Tr}[T^a_{R_i}\{T^b_{R_i}, T^c_{R_i}\}] = 0$	$\sum_{\mathrm{UV}} \mathrm{Tr}[T^a\{T^b, T^c\}] = \sum_{\mathrm{IR}} \mathrm{Tr}[T^a\{T^b, T^c\}]$

The triangle diagrams and the group-theoretic computation are *identical*—both involve $\mathrm{Tr}[T^a\{T^b, T^c\}]$. But the *generators* are different (gauged vs. global), and the *interpretation* is different (vanishing vs. matching). Never confuse the two.

4.2 Statement of the result

Theorem 4.1 ('t Hooft anomaly matching [5]). *Let G_{gauge} be the gauge group of an asymptotically free gauge theory, and let G_{global} be its exact global symmetry group. For every triple of generators (T^a, T^b, T^c) of G_{global} (and/or gravity), the anomaly coefficient*

$$\mathcal{A}^{abc} = \sum_{\text{massless fermions}} \mathrm{Tr}\big[T^a\{T^b, T^c\}\big] \tag{4}$$

*takes the **same value** when computed using the* UV *degrees of freedom (quarks, gluons, ...) and when computed using the* IR *degrees of freedom (hadrons, Goldstone bosons, ...):*

$$\boxed{\mathcal{A}^{abc}_{\mathrm{UV}} = \mathcal{A}^{abc}_{\mathrm{IR}}.} \tag{5}$$

This is an *exact* result: by the Adler–Bardeen theorem [8], the anomaly coefficient (1) receives contributions only at one loop and is not renormalised by higher-order corrections. Therefore the anomaly computed at any scale is the same, and in particular the UV and IR values must agree.

4.3 The spectator argument

We now present 't Hooft's elegant proof via spectator fields [4, 5]. The argument proceeds in five steps:

Step 1: Promote the global symmetry to a gauge symmetry. Take the global symmetry G_{global} and weakly gauge it: introduce gauge bosons B^a_μ for G_{global} with a very small coupling $g_{\text{w}} \to 0$. (This is a thought experiment; g_{w} is an auxiliary parameter that will be sent to zero at the end.)

Step 2: Add spectator fermions to cancel the new gauge anomaly. The original fermions will generically produce a non-vanishing G^3_{global} gauge anomaly (since we have now gauged what was previously a global symmetry). Add *spectator fermions*—free fermions that are *singlets* under the original gauge group G_{gauge} but carry charges under the newly gauged G_{global}—chosen so that the total gauge anomaly of the combined system vanishes. This is always possible by appropriate choice of spectator representations.

Step 3: The combined theory is consistent at all scales. Since the total G^3_{global} gauge anomaly now vanishes, the combined theory (original fermions + spectators, with both G_{gauge} and G_{global} gauged) is a consistent quantum theory at all energy scales.

Step 4: Spectators are free and unchanged by RG flow. The spectator fermions are singlets under G_{gauge} and interact only via the G_{global} gauge bosons at coupling $g_{\text{w}} \to 0$. Therefore they are free fields: they do not participate in the strong dynamics of G_{gauge}, and their anomaly contribution is the *same* in the UV and in the IR.

Step 5: Anomaly matching follows. The total anomaly is zero at all scales. The spectator contribution is the same at all scales. Therefore the anomaly from the *original* fermions must also be the same at all scales:

$$\underbrace{\mathcal{A}^{\mathrm{UV}}_{\text{original}}}_{\text{quarks}} + \underbrace{\mathcal{A}_{\text{spectators}}}_{\text{unchanged}} = 0 = \underbrace{\mathcal{A}^{\mathrm{IR}}_{\text{original}}}_{\text{hadrons}} + \underbrace{\mathcal{A}_{\text{spectators}}}_{\text{unchanged}},$$

hence $\mathcal{A}^{\mathrm{UV}}_{\text{original}} = \mathcal{A}^{\mathrm{IR}}_{\text{original}}$. □

4.4 Important caveats

(a) **Anomaly matching is necessary but not sufficient.** Matching the anomaly coefficients constrains the IR spectrum but does *not* uniquely determine it. There may be multiple anomaly-consistent proposals for the IR content; additional physical input (dynamics, lattice evidence, SUSY, ...) is needed to select among them.

(b) **Only exact global symmetries participate.** Anomaly matching applies to global symmetries that are *exact*—not explicitly or spontaneously broken. Spontaneously broken symmetries are still exact symmetries of the Lagrangian and *do* participate in anomaly matching (the Goldstone bosons are massless and must be counted in the IR spectrum). However, *anomalous* symmetries—those broken by quantum effects, such as $\mathrm{U}(1)_A$ in QCD—must *not* be included.

(c) **The result is exact.** By the Adler–Bardeen theorem [8], the anomaly is one-loop exact. There are no higher-order corrections, no non-perturbative corrections, and no scheme dependence. This is what makes anomaly matching such a powerful tool in strongly coupled gauge theories, where perturbation theory fails.

5 Warm-up: SU(3) QCD with $N_f = 2$

We now illustrate anomaly matching with a concrete and familiar example: QCD with two light flavours.

5.1 UV spectrum

The UV theory is SU(3) gauge theory with $N_f = 2$ massless Dirac quarks—the u and d quarks. In terms of Weyl fermions:

- $q^{\alpha}_{L,i}$: left-handed quarks, $\alpha = 1, 2, 3$ (colour), $i = 1, 2$ (flavour). Representation: $(\mathbf{3}, \mathbf{2}, 1)$ under $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.
- $q^{\alpha}_{R,i}$: right-handed quarks (equivalently, left-handed antiquarks $\bar{q}_{L,i\,\alpha}$). Representation: $(\bar{\mathbf{3}}, 1, \mathbf{2})$ under $\mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$.

The *exact global symmetry* (for massless quarks) is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \times \mathrm{U}(1)_B$, where $\mathrm{U}(1)_B$ is baryon number with $B[q] = 1/3$.

$\mathrm{U}(1)_A$ is anomalous

The axial symmetry $\mathrm{U}(1)_A$ is explicitly broken by the ABJ anomaly (2). It is *not* an exact global symmetry and must **not** be included in the anomaly matching conditions.

5.2 UV anomaly coefficients

Vanishing of $\mathrm{SU}(2)^3_L$ and $\mathrm{SU}(2)^3_R$. Since SU(2) has no cubic anomaly—$d^{abc} = 0$ because all SU(2) representations are (pseudo-)real (cf. the conventions in §2)—the purely cubic anomaly coefficients vanish identically:

$$\mathcal{A}_{\mathrm{UV}}\left[\mathrm{SU}(2)^3_L\right] = \mathcal{A}_{\mathrm{UV}}\left[\mathrm{SU}(2)^3_R\right] = 0. \tag{6}$$

The non-trivial matching condition comes instead from the *mixed* anomaly $\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B$. The only fermions charged under $\mathrm{SU}(2)_L$ are the left-handed quarks $q^\alpha_{L,i}$, which form $N_c = 3$ copies (one per colour) of the fundamental of $\mathrm{SU}(2)_L$, each carrying baryon number $B = 1/3$:

$$\mathcal{A}_{\mathrm{UV}}\big[\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B\big] = N_c \times T(\Box_{\mathrm{SU}(2)}) \times B[q] = 3 \times \tfrac{1}{2} \times \tfrac{1}{3} = \frac{1}{2}. \tag{7}$$

By an identical argument (with $B[\bar{q}] = -1/3$), $\mathcal{A}_{\mathrm{UV}}[\mathrm{SU}(2)_R^2\,\mathrm{U}(1)_B] = -1/2$.

5.3 IR spectrum: confinement and chiral symmetry breaking

At low energies, $\mathrm{SU}(3)_c$ confines and chiral symmetry breaks spontaneously:

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \xrightarrow{\langle \bar{q}q\rangle \neq 0} \mathrm{SU}(2)_V \quad \text{(isospin)}.$$

The low-energy degrees of freedom are:

- **3 Goldstone bosons** (pions): π^+, π^-, π^0, in the adjoint (= triplet) of $\mathrm{SU}(2)_V$. These are $(\mathbf{3},\mathbf{3})_0$ under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ in a non-linear sigma model representation. Goldstone bosons are *scalars* and do not contribute to the anomaly coefficient $\mathcal{A}^{abc}$ (which involves only fermions in perturbation theory).
- **Baryons** (proton and neutron): p and n, forming an isospin doublet. As composite fermions made of three quarks, the baryons are in $(\mathbf{2},\mathbf{1})$ under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ after chiral symmetry breaking—more precisely, in $(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})$ as Dirac fermions. The left-handed component transforms as $\mathbf{2}$ under $\mathrm{SU}(2)_L$; the right-handed component is a singlet under $\mathrm{SU}(2)_L$.

5.4 IR anomaly matching

Since $\mathrm{SU}(2)_L^3$ and $\mathrm{SU}(2)_R^3$ vanish identically ($d^{abc} = 0$), the non-trivial check is the mixed anomaly $\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B$.

Naive fermion counting. The only IR fermions charged under $\mathrm{SU}(2)_L$ are the left-handed baryons (the nucleon doublet) in the $\mathbf{2}$ of $\mathrm{SU}(2)_L$, with baryon number $B = 1$:

$$\mathcal{A}_{\mathrm{IR}}^{(\text{fermions})}\big[\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B\big] = 1 \times T(\Box_{\mathrm{SU}(2)}) \times B[N] = 1 \times \tfrac{1}{2} \times 1 = \frac{1}{2}. \tag{8}$$

This already matches $\mathcal{A}_{\mathrm{UV}} = 1/2$ (**??**)—the mixed anomaly is saturated by the nucleon doublet alone.

Role of the WZW term. Although the mixed anomaly happens to match from fermions alone, the Goldstone bosons still play a crucial role: the *Wess–Zumino–Witten (WZW) term* in the chiral Lagrangian [10,11] encodes the anomaly of the underlying microscopic (quark) theory in the effective low-energy (pion) theory. Its coefficient is *not* a free parameter—it is fixed by anomaly matching to equal $N_c = 3$. The WZW term ensures that processes such as $\pi^0 \to \gamma\gamma$ have the correct rate; it provides an independent consistency check on the chiral Lagrangian description of QCD.

$$\mathcal{A}_{\mathrm{IR}}\big[\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B\big] = \frac{1}{2} = \mathcal{A}_{\mathrm{UV}}\big[\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B\big]. \tag{9}$$

Remark 5.1. This example illustrates two important points. First, for SU(2) flavour symmetries the purely cubic anomaly vanishes identically ($d^{abc} = 0$), so the non-trivial constraints come from mixed anomalies such as $\mathrm{SU}(2)_L^2\,\mathrm{U}(1)_B$. Second, in theories with spontaneous chiral symmetry breaking, the Goldstone bosons carry anomaly information through the WZW term. The WZW coefficient is fixed by anomaly matching and equals N_c—a powerful consistency check on the low-energy effective theory.

Remark 5.2. An alternative perspective (following 't Hooft's original analysis [5]) considers the scenario where chiral symmetry is *not* broken. In that case, the IR spectrum must contain massless composite fermions whose anomaly coefficients match those of the quarks. This "confinement without chiral symmetry breaking" scenario requires very specific baryon representations and is believed not to occur for SU(3) with $N_f = 2$. The point is that anomaly matching *constrains* the possible IR dynamics: *either* chiral symmetry breaks (producing Goldstones with a WZW term) *or* massless composite fermions must appear. This is a non-trivial constraint that rules out certain confinement scenarios.

6 General anomaly matching for $\mathrm{SU}(N_c)$ with N_f flavours

We now set up the anomaly matching conditions for the general case that will be used throughout these lectures: $\mathrm{SU}(N_c)$ gauge theory with N_f Dirac flavours in the fundamental representation.

6.1 Global symmetry

The massless theory has the global symmetry

$$G_{\text{global}} = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A. \tag{10}$$

However, $\mathrm{U}(1)_A$ is anomalous (the ABJ anomaly) and is broken to a discrete subgroup $\mathbb{Z}_{2N_f}$. The $\mathrm{U}(1)_A$ anomaly is *not* matched—it is a genuine breaking of the symmetry, not a constraint.

Do not match the anomalous $\mathrm{U}(1)_A$

The axial $\mathrm{U}(1)_A$ symmetry is broken by the ABJ anomaly and must **not** be included in the 't Hooft anomaly matching conditions. Only the non-anomalous symmetries participate in matching.

The **exact** global symmetry group for anomaly matching purposes is

$$G_{\text{exact}} = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B. \tag{11}$$

The baryon number assignments follow the Seiberg convention [2]:

$$B[Q_i] = \frac{1}{N_c}, \qquad B[\widetilde{Q}^i] = -\frac{1}{N_c}, \tag{12}$$

where Q_i ($i = 1, \ldots, N_f$) are left-handed Weyl fermions in the fundamental of $\mathrm{SU}(N_c)$, and $\widetilde{Q}^i$ are left-handed Weyl fermions in the antifundamental. A baryon $B \sim Q^{N_c}$ then has $B[B] = 1$.

6.2 UV anomaly coefficients

The UV degrees of freedom are N_cN_f left-handed Weyl fermions Q_i^α in the $(\square, \square_L, \mathbf{1}_R)$ of $\mathrm{SU}(N_c) \times \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ and N_cN_f left-handed Weyl fermions $\widetilde{Q}^{i\,\alpha}$ in the $(\overline{\square}, \mathbf{1}_L, \square_R)$.

The independent anomaly coefficients to match are:

(1) $\mathrm{SU}(N_f)^3_L$**:** Only Q_i^α contributes. There are N_c copies (one per colour) of the fundamental of $\mathrm{SU}(N_f)_L$.

$$\mathcal{A}_{\mathrm{UV}}\big[\mathrm{SU}(N_f)^3_L\big] = N_c \times A(\square_{\mathrm{SU}(N_f)}) = N_c. \tag{13}$$

(2) $\mathrm{SU}(N_f)^3_R$**:** By $L \leftrightarrow R$ symmetry,

$$\mathcal{A}_{\mathrm{UV}}\big[\mathrm{SU}(N_f)^3_R\big] = N_c. \tag{14}$$

(3) $\mathrm{SU}(N_f)^2_L\,\mathrm{U}(1)_B$**:** This is a mixed anomaly. The Q_i^α contribute with $\mathrm{U}(1)_B$ charge $B = 1/N_c$, and there are N_c colours:

$$\mathcal{A}_{\mathrm{UV}}\big[\mathrm{SU}(N_f)^2_L\,\mathrm{U}(1)_B\big] = N_c \times T(\square_{\mathrm{SU}(N_f)}) \times \frac{1}{N_c} = \frac{1}{2}. \tag{15}$$

(Using $T(\square) = 1/2$.)

(4) $\mathrm{SU}(N_f)^2_R\,\mathrm{U}(1)_B$**:** The $\widetilde{Q}^i$ have $B = -1/N_c$ and contribute similarly:

$$\mathcal{A}_{\mathrm{UV}}\big[\mathrm{SU}(N_f)^2_R\,\mathrm{U}(1)_B\big] = N_c \times T(\square_{\mathrm{SU}(N_f)}) \times \Big(-\frac{1}{N_c}\Big) = -\frac{1}{2}. \tag{16}$$

(5) $\mathrm{U}(1)^3_B$**:** The Q fields contribute N_cN_f Weyl fermions with $B = 1/N_c$; the $\widetilde{Q}$ fields contribute N_cN_f Weyl fermions with $B = -1/N_c$:

$$\mathcal{A}_{\mathrm{UV}}\big[\mathrm{U}(1)^3_B\big] = N_cN_f \times \Big(\frac{1}{N_c}\Big)^3 + N_cN_f \times \Big(-\frac{1}{N_c}\Big)^3 = 0. \tag{17}$$

(6) $\mathrm{U}(1)_B$ **(gravitational):** The gravitational anomaly is proportional to $\sum_i B_i$:

$$\mathcal{A}_{\mathrm{UV}}\big[\mathrm{U}(1)_B\text{-grav}\big] = N_cN_f \times \frac{1}{N_c} + N_cN_f \times \Big(-\frac{1}{N_c}\Big) = 0. \tag{18}$$

Collecting the results:

Anomaly triple	$\mathcal{A}_{\mathrm{UV}}$	**Comment**
$\mathrm{SU}(N_f)^3_L$	N_c	N_c colours in fundamental of $\mathrm{SU}(N_f)_L$
$\mathrm{SU}(N_f)^3_R$	N_c	by $L \leftrightarrow R$
$\mathrm{SU}(N_f)^2_L\,\mathrm{U}(1)_B$	$\frac{1}{2}$	uses $T(\square) = \frac{1}{2}$, $B = 1/N_c$
$\mathrm{SU}(N_f)^2_R\,\mathrm{U}(1)_B$	$-\frac{1}{2}$	uses $B[\widetilde{Q}] = -1/N_c$
$\mathrm{U}(1)^3_B$	0	Q and $\widetilde{Q}$ cancel
$\mathrm{U}(1)_B$-grav	0	Q and $\widetilde{Q}$ cancel

These six anomaly coefficients must be matched by *any* proposed IR spectrum. They will be our primary tool for verifying Seiberg duality in Lecture 3.

Exercise 6.1. Verify the entries in the table above by explicit computation. In particular, check that the $\mathrm{SU}(N_f)^2_L\,\mathrm{U}(1)_B$ anomaly gives 1/2 (not 1 or N_c) by carefully tracking the Dynkin index $T(\square) = 1/2$ and the baryon number normalisation $B = 1/N_c$. *(Exercise Set 1, Problem 1.)*

7 The beta function and the conformal window

We now turn to the second key ingredient for these lectures: the *conformal window*—the range of N_f for which an $\mathrm{SU}(N_c)$ gauge theory with fundamental matter flows to a non-trivial IR fixed point rather than confining.

7.1 One-loop beta function

The running of the gauge coupling is governed by the beta function. At one loop:

$$\mu \frac{d\alpha_s}{d\mu} = -\frac{b_0}{2\pi}\,\alpha_s^2 + \mathcal{O}(\alpha_s^3), \tag{19}$$

where the one-loop coefficient is

$$\boxed{b_0 = \frac{11N_c - 2N_f}{3}} \qquad (N_f \text{ Dirac flavours in the fundamental}). \tag{20}$$

Derivation sketch (not rederived here; see [3], Ch. 16). The coefficient b_0 receives two competing contributions:

- **Gluon self-coupling (antiscreening):** $+\frac{11}{3}\,T(\text{adj}) \times 2 = \frac{11}{3}\,N_c$. The non-Abelian gauge field self-interactions produce an *antiscreening* effect that makes the coupling decrease at short distances. This is the origin of asymptotic freedom.
- **Fermion loops (screening):** $-\frac{4}{3}\,T(\square) \times N_f = -\frac{4}{3} \times \frac{1}{2} \times N_f = -\frac{2N_f}{3}$. Each Dirac fermion contributes 2 Weyl fermions, each with $T(\square) = 1/2$, giving $T_{\text{Dirac}} = 1$. Fermion loops produce a conventional screening effect.

Combining: $b_0 = \frac{11N_c}{3} - \frac{2N_f}{3} = \frac{11N_c - 2N_f}{3}$.
Asymptotic freedom requires $b_0 > 0$, *i.e.*,

$$N_f < \frac{11N_c}{2}. \tag{21}$$

For SU(3): $N_f < 16.5$, so $N_f \leq 16$ preserves asymptotic freedom and $N_f \geq 17$ loses it.

Check: SU(3), $N_f = 6$ (physical QCD): $b_0 = (33 - 12)/3 = 7 > 0$. QCD is asymptotically free. ✓

7.2 Two-loop beta function and the Banks–Zaks fixed point

At two loops, the beta function acquires a second coefficient b_1:

$$\mu \frac{d\alpha_s}{d\mu} = -\frac{b_0}{2\pi}\,\alpha_s^2 - \frac{b_1}{4\pi^2}\,\alpha_s^3 + \mathcal{O}(\alpha_s^4), \tag{22}$$

where

$$b_1 = \frac{1}{3}\left[34N_c^2 - N_f\left(13N_c - \frac{3}{N_c}\right)\right]. \tag{23}$$

The beta function has a non-trivial zero (besides $\alpha_s = 0$) when $b_0 > 0$ and $b_1 < 0$. Setting $\beta(\alpha_s^*) = 0$ at two loops:

$$\alpha_s^* = -\frac{b_0}{b_1}\,\frac{2\pi}{1} = -\frac{2\pi\,b_0}{b_1}. \tag{24}$$

For this to be a physical (positive) fixed point, we need $b_1 < 0$ (since $b_0 > 0$).

This is the **Banks–Zaks fixed point** [9]: when N_f is just below $11N_c/2$, so that b_0 is small and positive while b_1 is negative, the coupling runs from $\alpha_s = 0$ in the UV to a small, perturbatively calculable value α_s^* in the IR. The theory flows to a *conformal field theory* (CFT) in the infrared.

Check: For SU(3), $N_f = 16$: $b_0 = (33 - 32)/3 = 1/3$, $b_1 = (34 \times 9 - 16 \times (39 - 1))/3 = (306 - 608)/3 = -302/3 < 0$. So $\alpha_s^* = -2\pi \times (1/3)/(-302/3) = 2\pi/302 \approx 0.021 \ll 1$. The fixed point is perturbatively reliable. ✓

7.3 The conformal window

The **conformal window** is the range of N_f for which the SU(N_c) gauge theory is asymptotically free but flows to a non-trivial IR fixed point (an interacting CFT) rather than confining.

- **Upper boundary:** $N_f = 11N_c/2$ (loss of asymptotic freedom). Above this value, $b_0 < 0$ and the theory is IR free (trivial).
- **Lower boundary (non-SUSY):** The lower boundary is where confinement and chiral symmetry breaking set in. This is *not* perturbatively calculable and is **debated**. Two-loop estimates give $N_f^* \sim 8$–12 for SU(3), but these are unreliable near the lower boundary where α_s^* is no longer small. Lattice simulations [17, 18] and functional RG methods give somewhat different values.

The non-SUSY lower boundary is model-dependent

The two-loop estimate of the conformal window lower boundary is only a rough guide. Different methods (ladder approximation, functional RG, lattice Monte Carlo) give different values for N_f^*. The *upper* boundary $N_f = 11N_c/2$ is exact (one-loop); the *lower* boundary is approximate and the subject of ongoing research.

7.4 The SUSY conformal window (preview)

One of the remarkable consequences of supersymmetry is that the conformal window boundaries can be determined *exactly*. This will be derived in Lectures 2–3 using the NSVZ beta function and Seiberg duality. For now, we state the result:

SUSY conformal window (exact, Seiberg 1994)

For $\mathcal{N} = 1$ SU(N_c) SQCD with N_f flavours (Dirac pairs of chiral superfields), the conformal window is

$$\boxed{\frac{3N_c}{2} < N_f < 3N_c.} \tag{25}$$

This result is *exact*: the upper bound $N_f = 3N_c$ follows from the loss of asymptotic freedom (the SUSY beta function has $b_0^{\text{SUSY}} = 3N_c - N_f$), and the lower bound $N_f = 3N_c/2$ follows from unitarity bounds on operator dimensions at the superconformal fixed point [2].

For SU(3): the SUSY conformal window is $4.5 < N_f < 9$, giving integer values $N_f = 5, 6, 7, 8$.

7.5 Phases of SU(N_c) gauge theory

The phase structure of SU(N_c) gauge theory with N_f fundamental flavours is summarised in Figure 1 and the following table:

Regime	N_f **range**	**IR behaviour**
Confinement	$N_f \lesssim N_f^*$	Chiral symmetry breaking, mass gap
Conformal window	$N_f^* < N_f < \frac{11N_c}{2}$	Interacting CFT (Banks–Zaks)
Free electric	$N_f = \frac{11N_c}{2}$	Marginal; AF boundary
Loss of AF	$N_f > \frac{11N_c}{2}$	IR free (trivial)

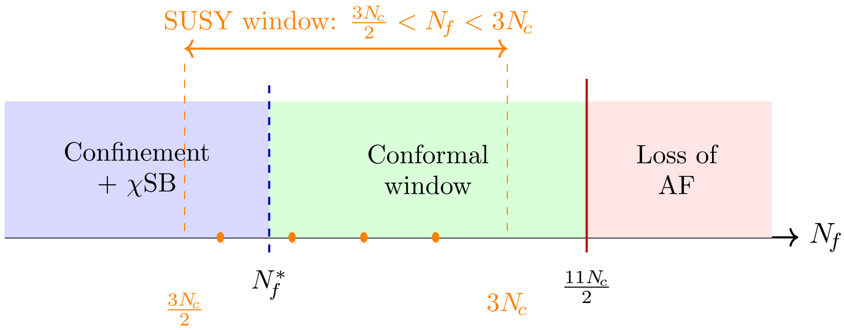

Figure 1: Phase structure of $SU(N_c)$ gauge theory with N_f fundamental Dirac flavours. The dashed blue line marks the lower boundary of the conformal window, N_f^*, which is model-dependent (for non-SUSY theories). The solid red line marks the loss of asymptotic freedom at $N_f = 11N_c/2$. The orange markers show the SUSY conformal window $3N_c/2 < N_f < 3N_c$, which is exact. For SU(3), the SUSY conformal window contains $N_f = 5, 6, 7, 8$ (orange dots).

7.6 R-charge at the conformal fixed point

In supersymmetric theories (to be developed in Lectures 2–3), the dimensions of chiral operators at a superconformal fixed point are determined by their R-charges. The non-anomalous R-charge of the quark superfield is

$$R[Q] = 1 - \frac{N_c}{N_f} = \frac{N_f - N_c}{N_f}. \tag{26}$$

The unitarity bound for a chiral operator $\mathcal{O}$ in a superconformal theory is $R[\mathcal{O}] \geq 2/3$ (with saturation corresponding to a free field). At the upper edge of the conformal window:

$$N_f = 3N_c: \quad R[Q] = \frac{3N_c - N_c}{3N_c} = \frac{2}{3}. \tag{27}$$

The unitarity bound is *saturated*: the quarks become free, and the theory flows to a trivial (free) fixed point. This is the upper boundary of the conformal window.

At the lower edge:

$$N_f = \frac{3N_c}{2}: \quad R[Q] = 1 - \frac{N_c}{3N_c/2} = 1 - \frac{2}{3} = \frac{1}{3}. \tag{28}$$

Below this value, the meson operator $M \sim Q\widetilde{Q}$ with $R[M] = 2R[Q] < 2/3$ falls below the unitarity bound, signalling that the conformal description breaks down and confinement sets in.

Check: For SU(3), $N_f = 9$: $R[Q] = (9-3)/9 = 2/3$. This is the upper edge; the SUSY conformal window is $4.5 < N_f < 9$. ✓

Exercise 7.1. Compute α_s^* at the Banks–Zaks fixed point (two-loop) for SU(3) with $N_f = 12, 14, 16$. For which values is the perturbative result $\alpha_s^* \ll 1$ reliable? Compare with the SUSY conformal window prediction for SU(3). *(Exercise Set 2, Problem 2.1.)*

Exercise 7.2. Show that the one-loop SUSY beta function coefficient is $b_0^{\text{SUSY}} = 3N_c - N_f$ for $SU(N_c)$ with N_f chiral pairs. Verify that asymptotic freedom requires $N_f < 3N_c$. *(Exercise Set 2, Problem 2.2. The derivation will use the NSVZ beta function introduced in Lecture 2.)*

8 Summary and preview of Lecture 2

8.1 What we have established

In this lecture, we have introduced the two foundational tools for the dualised Standard Model programme:

1. **'t Hooft anomaly matching** (§4–§6): For every exact global symmetry, the anomaly coefficient $\mathcal{A}^{abc} = \sum \mathrm{Tr}[T^a\{T^b, T^c\}]$ must be equal when computed using UV and IR degrees of freedom. This is exact (Adler–Bardeen) and provides powerful constraints on the low-energy spectrum of confining gauge theories.
   - This is distinct from gauge anomaly cancellation (which requires the anomaly to *vanish*, not merely match).
   - The warm-up (SU(3), $N_f = 2$) illustrated both the power and the subtlety of anomaly matching.
   - The six independent anomaly coefficients for SU(N_c) with N_f flavours are tabulated and ready for use.
2. **The conformal window** (§7): SU(N_c) with N_f fundamental flavours exhibits three phases: confinement ($N_f < N_f^*$), conformal window ($N_f^* < N_f < 11N_c/2$), and loss of asymptotic freedom ($N_f > 11N_c/2$). The SUSY conformal window is exact: $3N_c/2 < N_f < 3N_c$.

8.2 Preview: Lecture 2

In Lecture 2, we introduce minimal $\mathcal{N} = 1$ supersymmetry: superfields, the superpotential, F- and D-term conditions, and the holomorphic non-renormalisation theorem. The goal is to access the exact results (NSVZ beta function, Seiberg duality) that make the dualised Standard Model programme possible. No prior SUSY knowledge is assumed.

8.3 Exercises

The following exercises develop the material of this lecture:

- **Exercise Set 1** (Anomaly matching):
  - 1.1 Verify the six UV anomaly coefficients for SU(N_c) with N_f flavours (Table in §6).
  - 1.2 For SU(3) with $N_f = 3$, propose an IR baryon spectrum and check anomaly matching.
  - 1.3 Show that for $N_f = N_c$, there exists a unique baryon that can match all anomalies.
- **Exercise Set 2** (Conformal window):
  - 2.1 Compute the Banks–Zaks fixed-point coupling for SU(3) with $N_f = 12, 14, 16$.
  - 2.2 Derive $b_0^{\mathrm{SUSY}} = 3N_c - N_f$ from the matter content of $\mathcal{N} = 1$ SQCD (using the NSVZ beta function from Lecture 2).
  - 2.3 For SU(5), determine the SUSY conformal window and list the integer values of N_f.

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