Lecture 2: Minimal $\mathcal{N} = 1$ Supersymmetry

The Dualised Standard Model — Part III Lecture Notes

Lent Term 2027

Contents

Conventions for Lecture 2

All conventions from Lecture 1 remain in force. Additional conventions for supersymmetry (following Wess–Bagger [3] and Seiberg [1]):

Quantity	Convention	Comment
Units	$\hbar = c = 1$ (natural)	
Metric	$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$	mostly minus
Dynkin index	$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\,\delta^{ab}$	fundamental of $\mathrm{SU}(N_c)$
Covariant derivative	$D_\mu = \partial_\mu - i g_s A^a_\mu T^a$	
Weyl spinors	$\alpha = 1, 2$ (undotted); $\dot\alpha = 1, 2$ (dotted)	two-component
Superspace metric	$\epsilon^{12} = +1$, $\epsilon_{12} = -1$	raises/lowers spinor indices
σ-matrices	$\sigma^\mu = (\mathbf{1}, \vec\sigma)$, $\bar\sigma^\mu = (\mathbf{1}, -\vec\sigma)$	$\sigma^0 = \bar\sigma^0 = \mathbf{1}_{2\times 2}$
Grassmann coordinates	θ^α, $\bar\theta_{\dot\alpha}$	$[\theta] = [\mathrm{mass}]^{-1/2}$
Superpotential	$[W] = [\mathrm{mass}]^3$	holomorphic in chiral superfields
N_f counting	N_f = Dirac pairs	each pair $= (Q_i, \widetilde{Q}_i) = 2$ Weyl
R-charge	$R[Q] = (N_f - N_c)/N_f$	derived from anomaly cancellation

1 Why supersymmetry?

Supersymmetry (SUSY) has not been observed in nature—no superpartner of any known particle has been found. Why, then, devote a lecture to it? The answer is *not* phenomenological but *theoretical*: SUSY provides a set of exact analytical tools that are unavailable in non-supersymmetric gauge theories. In particular:

(i) **Holomorphy and non-renormalisation.** The superpotential, which controls the interactions of a SUSY theory, is a holomorphic function of the fields. It receives *no* perturbative quantum corrections to *all orders* in perturbation theory. This drastically constrains the dynamics.

(ii) **Exact beta functions.** The NSVZ exact beta function relates the running of the gauge coupling to the anomalous dimensions of matter fields. At a conformal fixed point, this gives exact results for scaling dimensions.

(iii) **Electric–magnetic duality.** SUSY is the *only* known framework in which electric–magnetic duality between strongly- and weakly-coupled gauge theories can be *established through exact consistency checks*—anomaly matching, moduli space matching, and consistency under deformations. This is Seiberg duality [1], the centrepiece of Lecture 3.

Remark 1.1 (Strategy for the course). The logic of this course is: (1) introduce the minimal SUSY formalism (this lecture); (2) *derive evidence for* Seiberg duality in the supersymmetric setting (Lecture 3); (3) understand its microscopic origin from $\mathcal{N} = 2$ theory (Lecture 4); (4) extend the duality, with appropriate caveats, to the non-supersymmetric Standard Model (Lectures 5–8). Understanding SUSY duality is a *prerequisite* for the conjectural non-SUSY extension that is the main goal of the course.

2 The $\mathcal{N} = 1$ supersymmetry algebra

Definition 2.1 (Supersymmetry)**.** A **supersymmetry** is a symmetry whose generators Q_α carry spin 1/2. Unlike the bosonic generators of the Poincaré group (translations P_μ, Lorentz transformations $M_{\mu\nu}$), the supercharges Q_α are *fermionic*: they anticommute rather than commute, and they transform bosons into fermions and vice versa.

In four spacetime dimensions, the minimal ($\mathcal{N} = 1$) supercharges form a two-component Weyl spinor Q_α (with $\alpha = 1, 2$) and its conjugate $\bar{Q}_{\dot\alpha}$ (with $\dot\alpha = 1, 2$). Here α is an *undotted* spinor index transforming under $\mathrm{SU}(2)_L$, and $\dot\alpha$ is a *dotted* index transforming under $\mathrm{SU}(2)_R$, where $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ is the double cover of the Lorentz group $\mathrm{SO}(3, 1)$.

Definition 2.2 (Sigma matrices)**.** The **sigma matrices** $\sigma^\mu_{\alpha\dot\alpha}$ and $\bar\sigma^{\mu\,\dot\alpha\alpha}$ connect spinor and vector indices. Explicitly:

$$\sigma^\mu = (\mathbf{1}, \sigma^1, \sigma^2, \sigma^3), \qquad \bar\sigma^\mu = (\mathbf{1}, -\sigma^1, -\sigma^2, -\sigma^3), \tag{1}$$

where σ^i are the standard Pauli matrices. These satisfy $\sigma^\mu\bar\sigma^\nu + \sigma^\nu\bar\sigma^\mu = 2\eta^{\mu\nu}\mathbf{1}$.

The $\mathcal{N} = 1$ SUSY algebra extends the Poincaré algebra by the anticommutation relations:

$$\boxed{\{Q_\alpha,\, \bar{Q}_{\dot\alpha}\} = 2\,\sigma^\mu_{\alpha\dot\alpha}\, P_\mu\,,} \tag{2}$$

together with

$$\{Q_\alpha, Q_\beta\} = 0, \qquad \{\bar{Q}_{\dot\alpha}, \bar{Q}_{\dot\beta}\} = 0, \qquad [Q_\alpha, P_\mu] = 0. \tag{3}$$

Remark 2.3 (Key consequences)*.* Three immediate consequences of eq. (2):

(a) Since Q_α carries spin 1/2, each SUSY multiplet contains particles differing by $\Delta s = 1/2$. SUSY relates bosons and fermions within the same multiplet.

(b) Every SUSY multiplet has **equal numbers of bosonic and fermionic degrees of freedom**. This follows from $\mathrm{Tr}[(-1)^F] = 0$ in any representation of the algebra, where F is the fermion number.

(c) The Hamiltonian is $H = P^0 = \frac{1}{4}(Q_1\bar{Q}_{\dot 1} + \bar{Q}_{\dot 1}Q_1 + Q_2\bar{Q}_{\dot 2} + \bar{Q}_{\dot 2}Q_2)$, which is a sum of squares. Hence $\langle H\rangle \geq 0$ in any state, and a supersymmetric vacuum has $\langle H\rangle = 0$.

3 Chiral superfields

To construct SUSY-invariant Lagrangians efficiently, we use the *superfield* formalism, which packages the components of a SUSY multiplet into a single object living in *superspace*.

Definition 3.1 (Superspace)**.** **Superspace** extends ordinary spacetime x^μ by adjoining anticommuting (Grassmann) coordinates:

$$(x^\mu,\ \theta^\alpha,\ \bar\theta_{\dot\alpha}), \tag{4}$$

where θ^α and $\bar\theta_{\dot\alpha}$ are two-component Grassmann spinors satisfying $\{\theta^\alpha, \theta^\beta\} = 0$ and $\{\bar\theta_{\dot\alpha}, \bar\theta_{\dot\beta}\} = 0$. Since each θ^α squares to zero, any function of θ has a *finite* Taylor expansion terminating at $\theta^2 \equiv \theta^\alpha\theta_\alpha$.

Definition 3.2 (Grassmann integration)**.** Integration over Grassmann variables is defined by the Berezin rules:

$$\int d\theta \; 1 = 0, \qquad \int d\theta \; \theta = 1. \tag{5}$$

The key consequence is that $\int d^2\theta \; f(\theta)$ **extracts the** θ^2 **component** of f. Similarly, $\int d^2\bar{\theta}$ extracts the $\bar{\theta}^2$ component, and $\int d^4\theta \equiv \int d^2\theta \, d^2\bar{\theta}$ extracts the $\theta^2\bar{\theta}^2$ component.

Definition 3.3 (Chiral superfield)**.** A **chiral superfield** Φ is a superfield satisfying the constraint $\bar{D}_{\dot{\alpha}}\Phi = 0$, where $\bar{D}_{\dot{\alpha}}$ is the supercovariant derivative. This constraint is solved by writing Φ as a function of the *chiral coordinate* $y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$:

$$\boxed{\Phi(y, \theta) = \phi(y) + \sqrt{2}\,\theta\,\psi(y) + \theta^2\,F(y)\,,} \tag{6}$$

where:

- $\phi(x)$ is a **complex scalar field**;
- $\psi_\alpha(x)$ is a **Weyl fermion** (a two-component spinor);
- $F(x)$ is a complex **auxiliary field** (no kinetic term, eliminated by its equation of motion).

Remark 3.4 (Degree-of-freedom counting)*.* Off-shell, a chiral multiplet has:

- **Bosonic:** ϕ (complex scalar) = 2 real DOF, plus F (complex auxiliary) = 2 real DOF $\to$ **4 bosonic**.
- **Fermionic:** ψ_α (complex Weyl spinor) = 4 real DOF off-shell $\to$ **4 fermionic**.

On-shell, the equation of motion sets $F = -(\partial W/\partial\phi)^*$, eliminating 2 bosonic DOF and the Dirac equation reduces ψ to 2 DOF, giving **2 bosonic** = **2 fermionic** on-shell, as required by SUSY.

Definition 3.5 (Mass dimensions in superspace)**.** Since $\{Q, \bar{Q}\} \sim P_\mu$ and $[P_\mu] = [\text{mass}]$, while Q acts on θ by translation, we have $[\theta] = [\text{mass}]^{-1/2}$. From the expansion (6):

$$[\Phi] = [\phi] = [\text{mass}], \qquad [\psi] = [\text{mass}]^{3/2}, \qquad [F] = [\text{mass}]^2. \tag{7}$$

Also, $[d^2\theta] = [\text{mass}]$ (since $[d\theta] = [\theta]^{-1} = [\text{mass}]^{1/2}$) and $[d^4\theta] = [\text{mass}]^2$.

3.1 The superpotential

Definition 3.6 (Superpotential)**.** The **superpotential** $W(\Phi)$ is a **holomorphic** function of chiral superfields—that is, it depends on Φ_i but *not* on $\Phi_i^\dagger$. In a renormalisable theory,

$$W(\Phi) = \frac{1}{2}\,m_{ij}\,\Phi_i\Phi_j + \frac{1}{3}\,y_{ijk}\,\Phi_i\Phi_j\Phi_k\,, \tag{8}$$

where m_{ij} and y_{ijk} are symmetric couplings. Since $[\Phi] = [\text{mass}]$ and $[d^2\theta] = [\text{mass}]$, the Lagrangian term $\int d^2\theta\, W$ has $[W] = [\text{mass}]^3$ and the full term has dimension 4 as required.

The superpotential contribution to the Lagrangian is:

$$\mathcal{L}_W = \int d^2\theta\; W(\Phi) + \text{h.c.} = F_i\,\frac{\partial W}{\partial\phi_i} - \frac{1}{2}\,\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}\,\psi_i\psi_j + \text{h.c.} \tag{9}$$

The first term is the coupling of the auxiliary field F_i to the derivative of W, and the second generates Yukawa interactions and fermion masses from W.

Definition 3.7 (Kähler potential)**.** The **Kähler potential** $K(\Phi^\dagger, \Phi)$ is a real function of chiral superfields and their conjugates. It determines the kinetic terms for the matter fields via:

$$\mathcal{L}_K = \int d^4\theta\ K(\Phi^\dagger, \Phi). \tag{10}$$

For a renormalisable theory, the canonical choice is $K = \Phi_i^\dagger \Phi_i$, which yields the standard kinetic terms $\partial_\mu \phi_i^* \partial^\mu \phi_i + i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i + F_i^* F_i$.

4 Vector superfields and gauge theories

To describe gauge fields in the superspace formalism, we need a second type of superfield.

Definition 4.1 (Vector superfield)**.** A **vector superfield** V is a superfield satisfying the reality condition $V = V^\dagger$. Under a gauge transformation with chiral parameter Λ, the vector superfield transforms as $e^{2gV} \to e^{-2ig\Lambda^\dagger} e^{2gV} e^{2ig\Lambda}$. Most of the components of V can be gauged away.

Definition 4.2 (Wess–Zumino gauge)**.** The **Wess–Zumino (WZ) gauge** is a gauge choice that eliminates the redundant components of V, leaving:

$$\boxed{V_{\text{WZ}} = -\theta\sigma^\mu\bar{\theta}\, A_\mu + i\,\theta^2\,\bar{\theta}\bar{\lambda} - i\,\bar{\theta}^2\,\theta\lambda + \tfrac{1}{2}\,\theta^2\bar{\theta}^2\,D\,,} \tag{11}$$

where the component fields are:

- $A_\mu(x)$: the **gauge boson** (spin-1);
- $\lambda_\alpha(x)$: the **gaugino**, a Weyl fermion in the **adjoint** representation of the gauge group;
- $D(x)$: a real **auxiliary field** (no kinetic term).

Remark 4.3 (Degree-of-freedom counting for vector multiplet). Off-shell, a vector multiplet in WZ gauge has:

- **Bosonic:** A_μ has 3 polarisations (4 components minus 1 residual gauge DOF in WZ gauge), plus D (1 real auxiliary) $\to$ **4 bosonic**.
- **Fermionic:** λ_α (complex Weyl fermion) = 4 real DOF off-shell $\to$ **4 fermionic**.

On-shell, eliminating D and imposing the equations of motion gives **2 bosonic** (transverse polarisations) = **2 fermionic** (on-shell gaugino). The matching $2_B = 2_F$ per multiplet holds in both the chiral and vector cases.

4.1 The field-strength superfield

Definition 4.4 (Field-strength superfield)**.** The **field-strength superfield** W_α is defined by

$$W_\alpha = -\frac{1}{4}\,\bar{D}^2 D_\alpha V\,, \tag{12}$$

where D_α and $\bar{D}_{\dot\alpha}$ are the supercovariant derivatives. Crucially, W_α is itself a **chiral** superfield ($\bar{D}_{\dot\beta} W_\alpha = 0$), even though it is built from the real superfield V. Its lowest components are:

$$W_\alpha = -i\lambda_\alpha + \theta_\alpha\,D - \frac{i}{2}(\sigma^\mu\bar{\sigma}^\nu)_\alpha{}^\beta\,\theta_\beta\,F_{\mu\nu} + \theta^2\,\sigma^\mu_{\alpha\dot\alpha}\,\partial_\mu\bar{\lambda}^{\dot\alpha}. \tag{13}$$

The gauge kinetic Lagrangian is then:

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4g^2}\int d^2\theta\ \mathrm{Tr}\big(W^\alpha W_\alpha\big) + \text{h.c.} = \mathrm{Tr}\Big[-\frac{1}{4g^2}\,F_{\mu\nu}F^{\mu\nu} + \frac{i}{g^2}\,\bar{\lambda}\bar{\sigma}^\mu D_\mu\lambda + \frac{1}{2g^2}\,D^2\Big]. \tag{14}$$

The trace is over gauge indices. This contains the standard Yang–Mills kinetic term, the gaugino kinetic term, and the auxiliary D-field.

4.2 Coupling matter to gauge fields

A chiral superfield Φ in representation R of the gauge group couples to the vector superfield via the Kähler potential:

$$K = \Phi^\dagger\, e^{2gV}\, \Phi\,, \tag{15}$$

where $V = V^a T^a_R$ with T^a_R the generators in representation R. Expanding in WZ gauge, this produces the gauge-covariant kinetic terms for ϕ and ψ, the gauge interactions, and a coupling of D to the scalar fields:

$$\int d^4\theta\ \Phi^\dagger e^{2gV}\Phi\ \supset\ (D_\mu\phi)^*(D^\mu\phi)+i\bar\psi\bar\sigma^\mu D_\mu\psi+|F|^2+g\,\phi^* D^a T^a\phi+\sqrt{2}\,g\,(\phi^*\lambda^a T^a\psi+\text{h.c.})\,, \tag{16}$$

where $D_\mu = \partial_\mu - igA^a_\mu T^a$ is the gauge-covariant derivative with our convention.

5 The $\mathcal{N}=1$ Lagrangian and scalar potential

Combining the ingredients from §§3–4, the general $\mathcal{N}=1$ gauge theory Lagrangian is:

$$\boxed{\mathcal{L} = \int d^4\theta\ \Phi_i^\dagger\, e^{2gV}\, \Phi_i + \left[\int d^2\theta\ \Big(W(\Phi) + \frac{\tau}{16\pi i}\,\mathrm{Tr}(W^\alpha W_\alpha)\Big) + \text{h.c.}\right]}\,, \tag{17}$$

where $\tau = \theta_{\rm YM}/(2\pi) + 4\pi i/g^2$ is the *complexified gauge coupling*. For a renormalisable theory, the Kähler potential is canonical: $K = \Phi_i^\dagger e^{2gV}\Phi_i$.

5.1 Eliminating auxiliary fields

The auxiliary fields F_i and D^a have no kinetic terms and are eliminated by their algebraic equations of motion:

$$F_i^* = -\frac{\partial W}{\partial\phi_i}\,, \tag{18}$$

$$D^a = -g\sum_i \phi_i^* T^a\phi_i\,. \tag{19}$$

Substituting back into the Lagrangian yields the **scalar potential**:

Definition 5.1 (Scalar potential)**.** The $\mathcal{N}=1$ scalar potential is

$$\boxed{V = V_F + V_D\,,} \tag{20}$$

where the **F-term potential** is

$$\boxed{V_F = \sum_i\left|\frac{\partial W}{\partial\phi_i}\right|^2,} \tag{21}$$

and the **D-term potential** is

$$\boxed{V_D = \frac{1}{2}\sum_a\Big(g\sum_i\phi_i^* T^a\phi_i\Big)^2.} \tag{22}$$

$V \geq 0$: a special feature of SUSY

Both V_F and V_D are manifestly **non-negative** (V_F is a sum of absolute squares, and V_D is a sum of real squares). Therefore the total scalar potential satisfies $V \geq 0$.
A **supersymmetric vacuum** has $V = 0$, which requires *both* $F_i = 0$ for all i *and* $D^a = 0$ for all a. If such a vacuum exists, it is automatically a global minimum of the potential—there is no tunnelling to a lower-energy state. SUSY is unbroken if and only if $V = 0$ can be achieved.

6 The non-renormalisation theorem

The most powerful consequence of the holomorphy of the superpotential is:

Theorem 6.1 (Non-renormalisation (Seiberg [2])). *The superpotential $W(\Phi)$ receives **no perturbative quantum corrections**—to all orders in perturbation theory. That is, the tree-level superpotential is exact perturbatively. Non-perturbative corrections (instantons)* can *modify W, but they are exponentially suppressed as $e^{-8\pi^2/g^2}$.*

Remark 6.2 (Seiberg's spurion argument). Seiberg's elegant proof [2] proceeds as follows. Promote all couplings in W to background chiral superfields. For example, in $W = m\Phi^2 + \lambda\Phi^3$, treat m and λ as chiral superfields with specific R-charge and global symmetry assignments. Then:

(i) The quantum-corrected superpotential W_{eff} must be a **holomorphic** function of the background superfields (by supersymmetry);

(ii) it must respect all global symmetries and R-symmetry;

(iii) it must reduce to W_{tree} in the weak-coupling limit $\lambda \to 0$.

These three constraints—holomorphy, symmetry, and the weak-coupling limit—*uniquely fix* $W_{\text{eff}} = W_{\text{tree}}$. No perturbative correction is consistent with all three requirements simultaneously. Non-perturbative corrections (involving $e^{-8\pi^2/g^2}$, which is non-holomorphic in the coupling) can and do arise, but they vanish in perturbation theory.

Remark 6.3 (What is *not* protected). The Kähler potential $K(\Phi^\dagger, \Phi)$ *is* renormalised and receives corrections at every loop order. The kinetic terms and wave-function renormalisations are *not* protected by holomorphy. This is why anomalous dimensions γ_i are nontrivial even in SUSY theories.

6.1 The NSVZ exact beta function

The combination of the non-renormalisation theorem for W and the fact that the gauge kinetic function $\tau(g)$ is holomorphic leads to the **Novikov–Shifman–Vainshtein–Zakharov (NSVZ)** exact beta function [5,6]:

$$\boxed{\beta(g) = -\frac{g^3}{16\pi^2}\,\frac{3\,T(G) - \sum_i T(R_i)\,(1-\gamma_i)}{1 - T(G)\,g^2/(8\pi^2)}}\,, \tag{23}$$

where $T(G)$ is the Dynkin index of the adjoint representation ($T(G) = N_c$ for $\mathrm{SU}(N_c)$), $T(R_i)$ is the Dynkin index of the matter representation R_i, and γ_i is the anomalous dimension of the i-th matter field.

Remark 6.4 (One-loop verification). At one loop, $\gamma_i = 0$ and the denominator $\to 1$. For $\mathrm{SU}(N_c)$ with N_f chiral superfield pairs $(Q_i, \widetilde{Q}_i)$, each in the fundamental (with $T(\square) = \frac{1}{2}$), the numerator gives:

$$3\,T(G) - \sum_i T(R_i) = 3N_c - 2N_f \times \tfrac{1}{2} = 3N_c - N_f\,. \tag{24}$$

The factor of $2N_f$ arises because each Dirac flavour pair contributes two chiral superfields (Q and $\widetilde{Q}$), each with $T(\square) = 1/2$. Therefore the one-loop SUSY beta function coefficient is:

$$b_0^{\mathrm{SQCD}} = 3N_c - N_f\,. \tag{25}$$

Consistency check with Lecture 1: For a non-SUSY gauge theory, the one-loop coefficient is $b_0 = (11N_c - 2N_f)/3$. For SQCD, each gauge boson has a gaugino partner (contributing $-N_c/3$ to b_0) and each quark has a squark (contributing $+N_f/3$), giving an additional $-(11N_c/3 - 2N_f/3) + (3N_c - N_f) = (11N_c - 2N_f)/3 - (11N_c - 2N_f)/3 + (3N_c - N_f)\ldots$. More directly: the $\mathcal{N} = 1$ vector multiplet contributes $T(G) \times 3 = 3N_c$ (from gluon + gaugino), and each chiral multiplet (Q_i or $\widetilde{Q}_i$) contributes $-T(\square) = -1/2$ (from scalar + Weyl fermion). With $2N_f$ chiral multiplets: $b_0^{\mathrm{SQCD}} = 3N_c - 2N_f \times (1/2) = 3N_c - N_f$. This matches eq. (25) and is consistent with the non-SUSY formula when the SUSY particle content is substituted.

7 $\mathcal{N} = 1$ SQCD: field content and symmetries

Definition 7.1 ($\mathcal{N} = 1$ SQCD)**.** $\mathcal{N} = 1$ **supersymmetric QCD (SQCD)** is the $\mathrm{SU}(N_c)$ gauge theory with N_f chiral superfields Q^i ($i = 1, \ldots, N_f$) in the **fundamental** representation and N_f chiral superfields $\widetilde{Q}_i$ in the **antifundamental** representation, with **zero tree-level superpotential**:

$$W_{\mathrm{electric}} = 0\,. \tag{26}$$

Each pair $(Q^i, \widetilde{Q}_i)$ constitutes one Dirac flavour; the total matter content is N_f Dirac flavour pairs, consistent with our counting convention.

The field content of $\mathcal{N} = 1$ SQCD, decomposed into component fields, is:

Superfield	**Boson**	**Fermion**	$\mathrm{SU}(N_c)$ **rep.**
V (vector)	A_μ (gauge boson)	λ (gaugino)	adjoint
Q^i (chiral)	$\tilde{q}^i$ (squark)	q^i (quark)	fundamental
$\widetilde{Q}_i$ (chiral)	$\tilde{\bar{q}}_i$ (anti-squark)	$\bar{q}_i$ (anti-quark)	antifundamental

7.1 Global symmetries

The classical global symmetry of SQCD is:

$$G_{\mathrm{classical}} = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A \times \mathrm{U}(1)_R\,, \tag{27}$$

where $\mathrm{SU}(N_f)_L$ rotates the Q^i, $\mathrm{SU}(N_f)_R$ rotates the $\widetilde{Q}_i$, $\mathrm{U}(1)_B$ is the baryon number, $\mathrm{U}(1)_A$ is the axial symmetry, and $\mathrm{U}(1)_R$ is the R-symmetry (a symmetry that does not commute with SUSY: it rotates the superspace coordinate θ).

Definition 7.2 (R-symmetry)**.** An **R-symmetry** is a global U(1) symmetry under which the Grassmann coordinate θ carries charge $+1$: $\theta \to e^{i\alpha}\theta$. By the structure of the superfield expansion (6), the component fields of a chiral superfield Φ with R-charge r have charges:

$$R[\phi] = r, \qquad R[\psi] = r - 1, \qquad R[F] = r - 2. \tag{28}$$

The gaugino λ, being the θ-component of the vector superfield, has $R[\lambda] = +1$. The superpotential must carry $R[W] = 2$, since the Lagrangian term $\int d^2\theta\, W$ must be R-neutral and $R[\theta^2] = 2$.

The axial $\mathrm{U}(1)_A$ is anomalous

Just as in non-supersymmetric QCD (Lecture 1, §3), the axial symmetry $\mathrm{U}(1)_A$ is broken by the ABJ anomaly: the triangle diagram $\mathrm{SU}(N_c)^2\,\mathrm{U}(1)_A$ is nonzero. The anomaly-free global symmetry is

$$G_{\text{quantum}} = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R\,. \tag{29}$$

7.2 Deriving the anomaly-free R-charge

The R-charge assignments of the matter fields are *not* a free choice. They are **uniquely determined** by the requirement that the R-symmetry be non-anomalous—specifically, that the mixed $\mathrm{SU}(N_c)^2\,\mathrm{U}(1)_R$ anomaly vanishes.

Remark 7.3 (Why uniquely determined)*.* In SQCD with $W = 0$, the only constraint on the R-charges is anomaly cancellation together with the requirement $R[Q] = R[\widetilde{Q}]$ (from the $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ symmetry and the absence of a superpotential). This is a single equation in a single unknown, so the solution is unique.

The $\mathrm{SU}(N_c)^2\,\mathrm{U}(1)_R$ anomaly coefficient receives contributions from all fermions charged under $\mathrm{SU}(N_c)$:

$$\mathcal{A}\big[\mathrm{SU}(N_c)^2\,\mathrm{U}(1)_R\big] = \sum_{\text{fermions}} T(R_f)\, R_f\,, \tag{30}$$

where $T(R_f)$ is the Dynkin index of the representation R_f and R_f is the R-charge of the fermion.

Important: The R-charge entering the anomaly formula is the *fermion* R-charge, not the superfield R-charge. Eq. (28) gives $R_{\text{fermion}} = R_{\text{superfield}} - 1$ for chiral multiplet fermions. The gaugino, however, is a component of the vector multiplet and has $R_{\text{gaugino}} = +1$ directly.

Let $r \equiv R[Q] = R[\widetilde{Q}]$ denote the *superfield* R-charge of the quarks. The fermion contributions to the anomaly are:

Fermion	**Rep.**	$T(R)$	R_{fermion}	**Multiplicity**
Gaugino λ	adjoint	N_c	$+1$	1
Quark ψ_Q	fund	$\frac{1}{2}$	$r-1$	N_f
Anti-quark $\psi_{\widetilde{Q}}$	anti-fund	$\frac{1}{2}$	$r-1$	N_f

Setting the anomaly to zero:

$$\begin{aligned} 0 &= T(\text{adj}) \times (+1) + N_f\, T(\square) \times (r-1) + N_f\, T(\square) \times (r-1) \\ &= N_c + 2N_f \times \tfrac{1}{2} \times (r-1) \\ &= N_c + N_f\,(r-1)\,. \end{aligned} \tag{31}$$

Solving for r:

$$\boxed{R[Q] = R[\widetilde{Q}] = 1 - \frac{N_c}{N_f} = \frac{N_f - N_c}{N_f}\,.} \tag{32}$$

This is a derived result, not a convention. The value $R[Q] = (N_f - N_c)/N_f$ is the unique anomaly-free R-charge assignment for massless SQCD.

Remark 7.4 (Verification: $R[W] = 2$). Although the electric theory has $W = 0$, the magnetic dual (Lecture 3) has a superpotential $W_{\rm mag} = (1/\mu)\, M\, q\, \widetilde{q}$. For self-consistency, $R[W] = 2$ must hold. The meson field $M^i{}_j \sim Q^i \widetilde{Q}_j$ has $R[M] = 2R[Q] = 2(N_f - N_c)/N_f$. The magnetic quarks have $R[q] = R[\widetilde{q}] = N_c/N_f$ (we will derive this in Lecture 3). Then:

$$R[Mq\widetilde{q}] = \frac{2(N_f - N_c)}{N_f} + \frac{2N_c}{N_f} = 2 = R[W]. \quad \checkmark \tag{33}$$

7.3 Field content and quantum numbers

The full quantum numbers of the electric theory are summarised in the following table:

Field	SU(N_c)	SU(N_f)$_L$	SU(N_f)$_R$	U(1)$_B$	U(1)$_R$
Q	$\square$	$\square$	$\mathbf{1}$	$+1$	$\frac{N_f - N_c}{N_f}$
$\widetilde{Q}$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	-1	$\frac{N_f - N_c}{N_f}$
λ (gaugino)	adj	$\mathbf{1}$	$\mathbf{1}$	0	$+1$

Baryon number convention. In this table we adopt $B[Q] = +1$, $B[\widetilde{Q}] = -1$ for the SQCD field content, differing from the Seiberg convention $B[Q] = 1/N_c$ used in Lecture 1. The two are related by rescaling $B \to N_c\, B$; all anomaly matching conditions remain valid provided the same convention is used consistently on both sides.

This table, together with the analogous table for the magnetic theory, forms the basis for the anomaly matching verification of Seiberg duality in Lecture 3.

8 Moduli space of SQCD

Definition 8.1 (Moduli space)**.** The **moduli space** of a supersymmetric gauge theory is the space of inequivalent vacuum configurations satisfying $V = 0$—that is, the space of field values for which both $F_i = 0$ and $D^a = 0$. The moduli space is parameterised by the vacuum expectation values of the scalar fields, modulo gauge equivalence.

Since $W = 0$ for SQCD, the F-term conditions are automatically satisfied. The vacuum structure is determined entirely by the D-term conditions.

8.1 D-flat directions

The D-term conditions for SU(N_c) are:

$$D^a = g\left(Q^{*\,i}\, T^a\, Q_i - \widetilde{Q}_i\, T^a\, \widetilde{Q}^{*\,i}\right) = 0 \qquad \forall\, a = 1, \ldots, N_c^2 - 1. \tag{34}$$

These conditions require, in matrix notation:

$$Q^\dagger Q - \widetilde{Q}\,\widetilde{Q}^\dagger \propto \mathbf{1}_{N_c\times N_c}\,, \tag{35}$$

where Q is viewed as an $N_c \times N_f$ matrix and $\widetilde{Q}$ as an $N_f \times N_c$ matrix. The proportionality to the identity can be absorbed by $\mathrm{SU}(N_c)$ gauge transformations.

8.2 Gauge-invariant coordinates

The *physical* moduli space is the quotient of the D-flat directions by gauge transformations. It is most naturally described in terms of gauge-invariant composite operators:

Definition 8.2 (Mesons and baryons)**.**

- **Mesons:** $M^i{}_j = Q^i\widetilde{Q}_j$, an $N_f \times N_f$ matrix (colour indices contracted).
- **Baryons** (for $N_f \geq N_c$): $B^{i_1\cdots i_{N_c}} = \epsilon^{a_1\cdots a_{N_c}} Q^{i_1}_{a_1}\cdots Q^{i_{N_c}}_{a_{N_c}}$, antisymmetric in the flavour indices.
- **Anti-baryons:** $\widetilde{B}_{i_1\cdots i_{N_c}} = \epsilon_{a_1\cdots a_{N_c}}\widetilde{Q}^{a_1}_{i_1}\cdots\widetilde{Q}^{a_{N_c}}_{i_{N_c}}$.

8.3 Classical constraints and moduli space dimension

The mesons and baryons are not all independent. Classically, they satisfy the constraint:

$$\det M - B\,\widetilde{B} = 0 \qquad \text{(classical, for } N_f \geq N_c\text{)}. \tag{36}$$

Furthermore, the meson matrix has $\mathrm{rank}(M) \leq N_c$ classically, since $M^i{}_j = Q^i_a\widetilde{Q}^a_j$ factors through colour space.

All dimensions below are **complex** unless stated otherwise.

The **dimension** of the classical moduli space is:

$$\boxed{\dim_{\mathbb{C}}(\mathcal{M}_{\mathrm{cl}}) = 2N_cN_f - (N_c^2 - 1)\,,} \tag{37}$$

which counts the $2N_cN_f$ complex scalar DOF in $(Q, \widetilde{Q})$ (each chiral superfield contains one complex scalar) minus the $N_c^2 - 1$ complex parameters of $\mathrm{SU}(N_c)_{\mathbb{C}}$ gauge transformations (the complexified gauge group acts on the chiral fields holomorphically).

Remark 8.3 (Consistency check: $N_f = N_c = 2$). The formula gives $\dim_{\mathbb{C}} = 2\times 2\times 2 - 3 = 5$. We can verify this from gauge-invariant coordinates: the 2×2 meson matrix M has 4 complex entries, plus the baryons B and $\widetilde{B}$ (1 complex each), giving $4+1+1 = 6$ complex gauge-invariant variables. Imposing the classical constraint $\det M - B\,\widetilde{B} = 0$ (one complex equation) removes one complex degree of freedom, leaving $6 - 1 = 5$ complex dimensions, in agreement with (37).

Preview: Quantum Modifications (Lecture 3)

The classical constraint (36) is modified by quantum effects:

- For $N_f = N_c$: the constraint becomes $\det M - B\,\widetilde{B} = \Lambda^{2N_c}$ (Seiberg [1]).
- For $N_f < N_c$: the Affleck–Dine–Seiberg (ADS) superpotential is generated non-perturbatively, lifting the moduli space.
- For $N_f > N_c$: **Seiberg duality** provides a dual magnetic description. The moduli space, described by mesons and baryons on both sides, matches. Verifying this matching, together with the anomaly matching of all global symmetry anomaly coefficients, constitutes the strongest evidence for the duality.

9 Summary and preview of Lecture 3

9.1 Key results

1. The $\mathcal{N}=1$ SUSY algebra extends the Poincaré algebra by fermionic generators Q_α, $\bar{Q}_{\dot\alpha}$ satisfying $\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha}P_\mu$. Each multiplet has equal bosonic and fermionic DOF.

2. **Chiral superfields** $\Phi = \phi + \sqrt{2}\theta\psi + \theta^2 F$ describe matter: one complex scalar, one Weyl fermion, and one auxiliary field.

3. **Vector superfields** in WZ gauge contain a gauge boson A_μ, a gaugino λ, and an auxiliary D-field.

4. The **superpotential** $W(\Phi)$ is holomorphic and receives no perturbative corrections (non-renormalisation theorem).

5. The **scalar potential** $V = V_F + V_D \geq 0$, with $V_F = \sum |\partial W/\partial\phi_i|^2$ and $V_D = \frac{1}{2}\sum_a (g\,\phi^* T^a \phi)^2$. SUSY vacua have $V = 0$.

6. The **NSVZ beta function** gives $b_0^{\mathrm{SQCD}} = 3N_c - N_f$ at one loop.

7. $\mathcal{N}=1$ **SQCD** is $\mathrm{SU}(N_c)$ with N_f pairs $(Q, \widetilde{Q})$ and $W=0$. The anomaly-free R-charge is $R[Q] = (N_f - N_c)/N_f$, **derived** from $\mathrm{SU}(N_c)^2\,\mathrm{U}(1)_R$ anomaly cancellation.

8. The **moduli space** is parameterised by mesons $M^i{}_j = Q^i\widetilde{Q}_j$ and baryons, with classical dimension $2N_cN_f - (N_c^2 - 1)$.

9.2 Preview: Lecture 3

In Lecture 3, we state Seiberg duality: for $N_c + 2 \leq N_f \leq \frac{3}{2}N_c$, the IR dynamics of $\mathrm{SU}(N_c)$ SQCD is described by a *magnetic* $\mathrm{SU}(N_f - N_c)$ gauge theory with N_f dual quarks $(q, \widetilde{q})$, a gauge-singlet meson M, and a superpotential $W_{\mathrm{mag}} = (1/\mu)\, M\, q\, \widetilde{q}$. We will verify this duality by matching all six independent 't Hooft anomaly coefficients and checking consistency under mass deformations, using the tools developed in Lectures 1 and 2.

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