Lecture 3: Seiberg Duality for $\mathcal{N} = 1$ SQCD

The Dualised Standard Model — Part III Lecture Notes

Lent Term 2027

Contents

1 $\mathcal{N} = 1$ SQCD Recap and Phase Structure **2**

1.1 Phase structure of $\mathrm{SU}(N_c)$ SQCD . . . 2

2 The Seiberg Duality Statement **3**

2.1 Scale matching relation . . . 4

3 Quantum Numbers of the Magnetic Theory **4**

3.1 Electric theory field content . . . 4

3.2 Magnetic theory field content . . . 5

3.3 Derivation of the magnetic baryon charge . . . 5

3.4 R-charge consistency of the magnetic superpotential . . . 5

4 't Hooft Anomaly Matching Verification **5**

4.1 Coefficient 1: $\mathrm{SU}(N_f)_L^3$ . . . 6

4.2 Coefficient 2: $\mathrm{SU}(N_f)_R^3$ . . . 6

4.3 Coefficient 3: $\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_B$ . . . 6

4.4 Coefficient 4: $\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_R$ . . . 7

4.5 Coefficient 5: $\mathrm{U}(1)_B^2\,\mathrm{U}(1)_R$ . . . 7

4.6 Coefficient 6: $\mathrm{U}(1)_R$ (gravitational) and $\mathrm{U}(1)_R^3$ . . . 8

5 Mass Deformation Consistency **9**

5.1 Electric side: $N_f \to N_f - 1$ . . . 10

5.2 Magnetic side: $\mathrm{SU}(N_f - N_c) \to \mathrm{SU}(N_f - N_c - 1)$ . . . 10

5.3 Scale matching under mass deformation . . . 10

6 Moduli Space Matching and Further Evidence **11**

6.1 Classical moduli space . . . 11

6.2 Conformal window from duality . . . 11

6.3 Summary: evidence for Seiberg duality . . . 12

7 Summary and Preview of Lecture 4 **12**

Conventions for Lecture 3

All conventions from Lectures 1 and 2 remain in force. Additional conventions specific to Seiberg duality (following Seiberg [1] and Intriligator–Seiberg [2]):

Quantity	Convention	Comment
Units	$\hbar = c = 1$ (natural)	
Metric	$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$	mostly minus
Dynkin index	$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\,\delta^{ab}$	fundamental of $\mathrm{SU}(N_c)$
Cubic anomaly	$A(\square) = 1,\ A(\overline{\square}) = -1$	normalised per fundamental Weyl
R-charge (superfield)	$R[Q] = (N_f - N_c)/N_f$	anomaly-free $\mathrm{U}(1)_R$
R-charge (fermion)	$R_{\text{ferm}} = R_{\text{sf}} - 1$	**chiral multiplet fermions only**
Gaugino R	$R_\lambda = +1$	directly; not shifted
Baryon charge	$B[Q] = +1,\ \ B[\widetilde{Q}] = -1$	electric theory
N_f counting	N_f = Dirac pairs	each pair $= (Q_i, \widetilde{Q}_i)$

1 $\mathcal{N} = 1$ SQCD Recap and Phase Structure

We recall the setup from Lecture 2. The *electric theory* is $\mathcal{N} = 1$ supersymmetric QCD:

- **Gauge group:** $\mathrm{SU}(N_c)$.
- **Matter:** N_f chiral superfields Q^i in the fundamental ($\square$) and N_f chiral superfields $\widetilde{Q}_i$ in the antifundamental ($\overline{\square}$), where $i = 1, \ldots, N_f$.
- **Superpotential:** $W_{\text{el}} = 0$ (massless quarks).
- **Global symmetry:** $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$.

The anomaly-free R-charge, derived in Lecture 2 from the condition $\mathrm{SU}(N_c)^2\,\mathrm{U}(1)_R$ anomaly $= 0$, is:

$$R[Q] = R[\widetilde{Q}] = \frac{N_f - N_c}{N_f}\,. \tag{1}$$

1.1 Phase structure of $\mathrm{SU}(N_c)$ SQCD

The dynamics depends dramatically on the ratio N_f/N_c. We summarise the complete picture:

1. **$N_f = 0$:** Pure $\mathrm{SU}(N_c)$ super-Yang–Mills. Gaugino condensation produces N_c isolated SUSY vacua.
2. **$1 \leq N_f < N_c$:** The Affleck–Dine–Seiberg (ADS) superpotential is generated non-perturbatively:

$$W_{\text{ADS}} = (N_c - N_f)\left(\frac{\Lambda^{3N_c - N_f}}{\det M}\right)^{1/(N_c - N_f)}, \tag{2}$$

   where ${M^i}_j = Q^i \widetilde{Q}_j$ is the meson matrix and Λ is the dynamical scale. This generates a runaway potential with no stable vacuum at finite field values.

3. $N_f = N_c$**:** The classical moduli space is *quantum-modified*. The classical constraint $\det M - B\widetilde{B} = 0$ is replaced by the exact quantum relation:

$$\det M - B\widetilde{B} = \Lambda^{2N_c}\,, \tag{3}$$

where $B = \epsilon^{i_1\cdots i_{N_c}} Q_{i_1}\cdots Q_{i_{N_c}}$ and $\widetilde{B} = \epsilon_{i_1\cdots i_{N_c}} \widetilde{Q}^{i_1}\cdots \widetilde{Q}^{i_{N_c}}$ are the baryonic operators. The origin $M = B = \widetilde{B} = 0$ is *removed* from the quantum moduli space.

4. $N_f = N_c + 1$**:** The theory *s-confines*: it confines without chiral symmetry breaking. The low-energy degrees of freedom are mesons M, baryons B_i, $\widetilde{B}^i$, with the exact confining superpotential:

$$W = \frac{1}{\Lambda^{2N_c-1}}\left(B_i\, M^i{}_j\, \widetilde{B}^j - \det M\right). \tag{4}$$

5. $N_c + 2 \le N_f \le \frac{3}{2}N_c$**:** The *free magnetic phase*. The electric theory confines, and the low-energy physics is described by a *weakly-coupled magnetic dual* — an IR-free $\mathrm{SU}(N_f - N_c)$ gauge theory. This is the regime where Seiberg duality is most powerful.

6. $\frac{3}{2}N_c < N_f < 3N_c$**:** The *conformal window* (from Lecture 1, §6). Both the electric and magnetic theories flow to the same interacting superconformal fixed point in the IR.

7. $N_f \ge 3N_c$**:** The electric theory loses asymptotic freedom ($b_0^{\mathrm{el}} = 3N_c - N_f \le 0$) and is IR-free.

Remark 1.1 (Self-duality at the conformal window)*.* Within the conformal window $\frac{3}{2}N_c < N_f < 3N_c$, the electric and magnetic descriptions are *equally valid*: neither is weakly coupled in the UV, and both flow to the same fixed point. The duality exchanges strong and weak coupling while preserving all IR physics.

2 The Seiberg Duality Statement

Seiberg's electric–magnetic duality [1] asserts that the IR physics of the electric $\mathrm{SU}(N_c)$ SQCD with N_f flavours (in the range $N_c + 1 < N_f < 3N_c$) is equivalently described by a **magnetic theory**:

Seiberg Duality for $\mathcal{N} = 1$ SQCD

Electric theory: $\mathrm{SU}(N_c)$ with $N_f \times (Q, \widetilde{Q})$, $\quad W_{\mathrm{el}} = 0$.
Magnetic theory:

- **Gauge group:** $\mathrm{SU}(N_f - N_c)$.
- **Magnetic quarks:** N_f chiral superfields q_i in the fundamental ($\square$) and N_f chiral superfields $\widetilde{q}^{\,i}$ in the antifundamental ($\overline{\square}$) of $\mathrm{SU}(N_f - N_c)$.
- **Gauge-singlet mesons:** $M^i{}_j$, an $N_f \times N_f$ matrix of chiral superfields, singlets under the magnetic gauge group.
- **Magnetic superpotential:**

$$W_{\mathrm{mag}} = \frac{1}{\mu}\, M^i{}_j\, q_i\, \widetilde{q}^{\,j}\,, \tag{5}$$

where μ is a dimensionful scale ($[\mu] = [\mathrm{mass}]$).

The meson M is elementary in the magnetic theory

Although the meson field $M^i{}_j$ in the magnetic theory maps to the *composite* operator $Q^i\widetilde{Q}_j$ in the electric theory, it is an *elementary* gauge-singlet field in the magnetic description. Its inclusion is essential: without M, the magnetic theory would not reproduce the correct anomaly matching, and the moduli spaces would not agree.

2.1 Scale matching relation

The dynamical scales of the electric and magnetic theories are related by:

$$\Lambda_{\text{el}}^{3N_c-N_f}\,\Lambda_{\text{mag}}^{3(N_f-N_c)-N_f} = (-1)^{N_f-N_c}\,\mu^{N_f}\,. \tag{6}$$

Dimensional consistency check. The exponent on the left-hand side is:

$$(3N_c - N_f) + (2N_f - 3N_c) = N_f\,,$$

matching the right-hand side. Since $[\Lambda] = [\text{mass}]$ and $[\mu] = [\text{mass}]$, both sides have mass dimension N_f. ✓

Remark 2.1 (Magnetic beta function)*.* The one-loop beta function coefficient for $\mathrm{SU}(N_f - N_c)$ with N_f flavours is $b_0^{\text{mag}} = 3(N_f - N_c) - N_f = 2N_f - 3N_c$. In the free magnetic phase ($N_c + 2 \le N_f \le \frac{3}{2}N_c$), we have $b_0^{\text{mag}} \le 0$, confirming that the magnetic theory is IR-free. At $N_f = \frac{3}{2}N_c$, we have $b_0^{\text{mag}} = 0$ — the boundary of the conformal window from the magnetic side.

3 Quantum Numbers of the Magnetic Theory

To verify the duality, we need the complete quantum numbers of all fields under the global symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$. We list both the *superfield* R-charge and the *fermion* R-charge, since anomaly matching uses the latter.

3.1 Electric theory field content

Field	$\mathrm{SU}(N_c)$	$\mathrm{SU}(N_f)_L$	$\mathrm{SU}(N_f)_R$	$\mathrm{U}(1)_B$	R_{sf}	R_{ferm}
Q	$\square$	$\square$	$\mathbf{1}$	$+1$	$\frac{N_f - N_c}{N_f}$	$-\frac{N_c}{N_f}$
$\widetilde{Q}$	$\overline{\square}$	$\mathbf{1}$	$\overline{\square}$	-1	$\frac{N_f - N_c}{N_f}$	$-\frac{N_c}{N_f}$
λ	adj	$\mathbf{1}$	$\mathbf{1}$	0	$+1$	$+1$

(7)

The fermion R-charge is obtained from the superfield R-charge by the shift $R_{\text{ferm}} = R_{\text{sf}} - 1$ for chiral multiplet components (Q, $\widetilde{Q}$). The gaugino λ is a fermion in the *vector* multiplet; its R-charge is $+1$ directly and does *not* receive the -1 shift.

3.2 Magnetic theory field content

$$
\begin{array}{l|c|c|c|c|c|c}
\textbf{Field} & \mathrm{SU}(N_f-N_c) & \mathrm{SU}(N_f)_L & \mathrm{SU}(N_f)_R & \mathrm{U}(1)_B & R_{\rm sf} & R_{\rm ferm} \\
\hline
q & \square & \overline{\square} & \mathbf{1} & +\dfrac{N_c}{N_f-N_c} & \dfrac{N_c}{N_f} & \dfrac{N_c-N_f}{N_f} \\
\widetilde{q} & \overline{\square} & \mathbf{1} & \square & -\dfrac{N_c}{N_f-N_c} & \dfrac{N_c}{N_f} & \dfrac{N_c-N_f}{N_f} \\
M & \mathbf{1} & \square & \overline{\square} & 0 & \dfrac{2(N_f-N_c)}{N_f} & \dfrac{N_f-2N_c}{N_f} \\
\lambda_{\rm mag} & \text{adj} & \mathbf{1} & \mathbf{1} & 0 & +1 & +1
\end{array}
\tag{8}
$$

Fermion R-charges in anomaly matching

All anomaly triangle computations below use the **fermion** R-charges $R_{\rm ferm}$ from the rightmost column. Using the superfield R-charge $R_{\rm sf}$ without the -1 shift is the single most common error in anomaly matching and gives *wrong* results.

3.3 Derivation of the magnetic baryon charge

The baryon charge of the magnetic quarks is *not* $+1$; it is determined by the baryon operator map. The electric baryon is:

$$B = \epsilon^{i_1\cdots i_{N_c}}\, Q_{i_1}\cdots Q_{i_{N_c}}\,, \qquad B[B] = N_c\,.$$

The magnetic baryon is:

$$b = \epsilon^{i_1\cdots i_{N_f-N_c}}\, q_{i_1}\cdots q_{i_{N_f-N_c}}\,, \qquad B[b] = (N_f-N_c)\,B[q]\,.$$

The duality identifies $b \leftrightarrow B$, so we require $B[b] = B[B]$:

$$(N_f-N_c)\,B[q] = N_c \qquad \Longrightarrow \qquad \boxed{B[q] = \frac{N_c}{N_f-N_c}}\,. \tag{9}$$

3.4 R-charge consistency of the magnetic superpotential

The magnetic superpotential (5) must have R-charge 2 (since $\int d^2\theta\, W$ requires $R[W]=2$). Using the *superfield* R-charges from the table:

$$R[W_{\rm mag}] = R[M] + R[q] + R[\widetilde{q}] = \frac{2(N_f-N_c)}{N_f} + \frac{N_c}{N_f} + \frac{N_c}{N_f} = \frac{2N_f-2N_c+2N_c}{N_f} = 2\,. \quad \checkmark \tag{10}$$

The scale μ carries $R[\mu]=0$ (it is a dimensionful parameter, not a dynamical field).

4 't Hooft Anomaly Matching Verification

The strongest evidence for Seiberg duality is the exact matching of all *independent* 't Hooft anomaly coefficients between the electric and magnetic theories. Recall from Lecture 1 that 't Hooft anomaly matching is an *exact* constraint: the anomaly coefficients of global symmetries, computed from the UV degrees of freedom, must equal those computed from the IR degrees of freedom.

The global symmetry group is $G = \mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$, giving the following independent anomaly coefficients. We use:

- $T(R)$: Dynkin index. $T(\square) = T(\overline{\square}) = \frac{1}{2}$.
- $A(R)$: cubic anomaly coefficient. $A(\square) = 1$, $A(\overline{\square}) = -1$.
- **Fermion** R-charges throughout (not superfield R-charges).

4.1 Coefficient 1: $\mathrm{SU}(N_f)_L^3$

This is the cubic anomaly for the $\mathrm{SU}(N_f)_L$ factor. Only fermions transforming under $\mathrm{SU}(N_f)_L$ contribute.

Electric side. Only Q transforms under $\mathrm{SU}(N_f)_L$ (as $\square$), with N_c gauge-colour copies:

$$\mathcal{A}_1^{\mathrm{el}} = N_c \cdot A(\square) = N_c \cdot 1 = N_c\,.$$

Magnetic side. Two fields transform under $\mathrm{SU}(N_f)_L$:

- q transforms as $\overline{\square}$ under $\mathrm{SU}(N_f)_L$, with $(N_f - N_c)$ gauge-colour copies: contributes $(N_f - N_c) \cdot A(\overline{\square}) = (N_f - N_c)(-1) = -(N_f - N_c)$.
- M transforms as $\square$ under $\mathrm{SU}(N_f)_L$ (with N_f copies from the $\mathrm{SU}(N_f)_R$ antifundamental index): contributes $N_f \cdot A(\square) = N_f \cdot 1 = N_f$.

Total:

$$\boxed{\mathcal{A}_1^{\mathrm{el}} = N_c = -(N_f - N_c) + N_f = \mathcal{A}_1^{\mathrm{mag}}}\,. \quad \checkmark \tag{11}$$

4.2 Coefficient 2: $\mathrm{SU}(N_f)_R^3$

By the left–right symmetry of the theory ($Q \leftrightarrow \widetilde{Q}$, $q \leftrightarrow \widetilde{q}$, $M^i{}_j \leftrightarrow (M^j{}_i)^*$), this is identical to Coefficient 1:

$$\boxed{\mathcal{A}_2^{\mathrm{el}} = N_c = \mathcal{A}_2^{\mathrm{mag}}}\,. \quad \checkmark \tag{12}$$

4.3 Coefficient 3: $\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_B$

The mixed anomaly is $\sum_f T(R_L^{(f)}) \cdot B_f$, summed over all fermions f charged under $\mathrm{SU}(N_f)_L$.

Electric side. Q: N_c colours, $\square$ under $\mathrm{SU}(N_f)_L$, $B = +1$:

$$\mathcal{A}_3^{\mathrm{el}} = N_c \cdot T(\square) \cdot (+1) = N_c \cdot \tfrac{1}{2} \cdot 1 = \frac{N_c}{2}\,.$$

Magnetic side.

- q: $(N_f - N_c)$ colours, $\overline{\square}$ under $\mathrm{SU}(N_f)_L$, $B = N_c/(N_f - N_c)$:

$$(N_f - N_c) \cdot T(\overline{\square}) \cdot \frac{N_c}{N_f - N_c} = (N_f - N_c) \cdot \tfrac{1}{2} \cdot \frac{N_c}{N_f - N_c} = \frac{N_c}{2}\,.$$

- M: $B[M] = 0$, so no contribution.

Total:

$$\boxed{\mathcal{A}_3^{\mathrm{el}} = \frac{N_c}{2} = \mathcal{A}_3^{\mathrm{mag}}}\,. \quad \checkmark \tag{13}$$

4.4 Coefficient 4: $\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_R$

The mixed anomaly is $\sum_f T(R_L^{(f)}) \cdot R_{\mathrm{ferm},f}$.

Electric side. Q: N_c colours, $\square$ under $\mathrm{SU}(N_f)_L$, $R_{\mathrm{ferm}} = -N_c/N_f$:

$$\mathcal{A}_4^{\mathrm{el}} = N_c \cdot \tfrac{1}{2} \cdot \left(-\frac{N_c}{N_f}\right) = -\frac{N_c^2}{2N_f}\,.$$

Magnetic side.

- q: $(N_f - N_c)$ colours, $\overline{\square}$ under $\mathrm{SU}(N_f)_L$, $R_{\mathrm{ferm}} = (N_c - N_f)/N_f$:

$$(N_f - N_c) \cdot \tfrac{1}{2} \cdot \frac{N_c - N_f}{N_f} = -\frac{(N_f - N_c)^2}{2N_f}\,.$$

- M: N_f copies of $\square$ under $\mathrm{SU}(N_f)_L$, $R_{\mathrm{ferm}} = (N_f - 2N_c)/N_f$:

$$N_f \cdot \tfrac{1}{2} \cdot \frac{N_f - 2N_c}{N_f} = \frac{N_f - 2N_c}{2}\,.$$

Total magnetic:

$$\begin{aligned}
\mathcal{A}_4^{\mathrm{mag}} &= -\frac{(N_f - N_c)^2}{2N_f} + \frac{N_f - 2N_c}{2} \\
&= \frac{-(N_f - N_c)^2 + N_f(N_f - 2N_c)}{2N_f} \\
&= \frac{-N_f^2 + 2N_cN_f - N_c^2 + N_f^2 - 2N_cN_f}{2N_f} = -\frac{N_c^2}{2N_f}\,.
\end{aligned}$$

$$\boxed{\mathcal{A}_4^{\mathrm{el}} = -\frac{N_c^2}{2N_f} = \mathcal{A}_4^{\mathrm{mag}}}\,. \quad \checkmark \tag{14}$$

4.5 Coefficient 5: $\mathrm{U}(1)_B^2\,\mathrm{U}(1)_R$

The anomaly is $\sum_f n_f \cdot B_f^2 \cdot R_{\mathrm{ferm},f}$, where n_f is the total multiplicity (gauge $\times$ flavour) of fermion species f.

Electric side.

- Q: N_cN_f fermions, $B = +1$, $R_{\mathrm{ferm}} = -N_c/N_f$: $N_cN_f \cdot 1 \cdot (-N_c/N_f) = -N_c^2$.
- $\widetilde{Q}$: N_cN_f fermions, $B = -1$, $R_{\mathrm{ferm}} = -N_c/N_f$: $N_cN_f \cdot 1 \cdot (-N_c/N_f) = -N_c^2$.
- Gaugino: $B = 0$, no contribution.

Total:

$$\mathcal{A}_5^{\mathrm{el}} = -2N_c^2\,.$$

Magnetic side.

- q: $(N_f - N_c)N_f$ fermions, $B = N_c/(N_f - N_c)$, $R_{\mathrm{ferm}} = (N_c - N_f)/N_f$:

$$(N_f - N_c)N_f \cdot \frac{N_c^2}{(N_f - N_c)^2} \cdot \frac{N_c - N_f}{N_f} = \frac{N_c^2(N_c - N_f)}{(N_f - N_c)} = -N_c^2\,.$$

- $\widetilde{q}$: same as q (with $B = -N_c/(N_f - N_c)$, but B^2 is the same): $= -N_c^2$.
- M: $B[M] = 0$, no contribution.
- Gaugino: $B = 0$, no contribution.

Total:

$$\boxed{\mathcal{A}_5^{\text{el}} = -2N_c^2 = \mathcal{A}_5^{\text{mag}}}\,. \quad \checkmark \tag{15}$$

4.6 Coefficient 6: $\mathrm{U}(1)_R$ (gravitational) and $\mathrm{U}(1)_R^3$

The gravitational anomaly $\mathrm{Tr}[R_{\text{ferm}}]$ counts the total weighted R-charge of all fermions in the theory.

$\mathrm{U}(1)_R$ **gravitational anomaly:** $\mathrm{Tr}[R]$

Electric side.

- Q: $N_c N_f$ fermions with $R = -N_c/N_f$: $\to -N_c^2$.
- $\widetilde{Q}$: $N_c N_f$ fermions with $R = -N_c/N_f$: $\to -N_c^2$.
- Gaugino: $(N_c^2 - 1)$ fermions with $R = +1$: $\to N_c^2 - 1$.

Total:

$$\mathcal{A}_{6\text{a}}^{\text{el}} = -2N_c^2 + N_c^2 - 1 = -N_c^2 - 1\,.$$

Magnetic side.

- q: $(N_f - N_c)N_f$ fermions with $R = (N_c - N_f)/N_f$: $\to -(N_f - N_c)^2$.
- $\widetilde{q}$: same: $\to -(N_f - N_c)^2$.
- M: N_f^2 fermions with $R = (N_f - 2N_c)/N_f$: $\to N_f(N_f - 2N_c)$.
- Gaugino: $((N_f - N_c)^2 - 1)$ fermions with $R = +1$: $\to (N_f - N_c)^2 - 1$.

Total:

$$\begin{aligned}
\mathcal{A}_{6\text{a}}^{\text{mag}} &= -2(N_f - N_c)^2 + N_f^2 - 2N_cN_f + (N_f - N_c)^2 - 1 \\
&= -(N_f - N_c)^2 + N_f^2 - 2N_cN_f - 1 \\
&= -(N_f^2 - 2N_cN_f + N_c^2) + N_f^2 - 2N_cN_f - 1 \\
&= -N_c^2 - 1\,.
\end{aligned}$$

$$\boxed{\mathcal{A}_{6\text{a}}^{\text{el}} = -N_c^2 - 1 = \mathcal{A}_{6\text{a}}^{\text{mag}}}\,. \quad \checkmark \tag{16}$$

$\mathrm{U}(1)_R^3$ **anomaly:** $\mathrm{Tr}[R^3]$

Electric side.

$$\mathcal{A}_{6\mathrm{b}}^{\mathrm{el}} = 2 \cdot N_c N_f \cdot \left(-\frac{N_c}{N_f}\right)^3 + (N_c^2 - 1) \cdot 1^3$$
$$= -\frac{2N_c^4}{N_f^2} + N_c^2 - 1\,.$$

Magnetic side.

$$\mathcal{A}_{6\mathrm{b}}^{\mathrm{mag}} = 2 \cdot (N_f - N_c)N_f \cdot \left(\frac{N_c - N_f}{N_f}\right)^3 + N_f^2 \cdot \left(\frac{N_f - 2N_c}{N_f}\right)^3 + \left((N_f - N_c)^2 - 1\right) \cdot 1$$
$$= -\frac{2(N_f - N_c)^4}{N_f^2} + \frac{(N_f - 2N_c)^3}{N_f} + (N_f - N_c)^2 - 1\,.$$

The equality $\mathcal{A}_{6\mathrm{b}}^{\mathrm{el}} = \mathcal{A}_{6\mathrm{b}}^{\mathrm{mag}}$ can be verified by expanding and simplifying; this is a polynomial identity in N_c and N_f. We verify it computationally in the companion script `scripts/verify_anomaly_matching.py`:

$$\boxed{\mathcal{A}_{6\mathrm{b}}^{\mathrm{el}} = -\frac{2N_c^4}{N_f^2} + N_c^2 - 1 = \mathcal{A}_{6\mathrm{b}}^{\mathrm{mag}}}\,. \quad \checkmark \tag{17}$$

Anomaly Matching Summary

All six independent 't Hooft anomaly coefficients match exactly between the electric $\mathrm{SU}(N_c)$ theory and the magnetic $\mathrm{SU}(N_f - N_c)$ theory, as polynomial identities in N_c and N_f:

Coefficient	Electric	Magnetic
$\mathrm{SU}(N_f)_L^3$	N_c	N_c
$\mathrm{SU}(N_f)_R^3$	N_c	N_c
$\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_B$	$N_c/2$	$N_c/2$
$\mathrm{SU}(N_f)_L^2\,\mathrm{U}(1)_R$	$-N_c^2/(2N_f)$	$-N_c^2/(2N_f)$
$\mathrm{U}(1)_B^2\,\mathrm{U}(1)_R$	$-2N_c^2$	$-2N_c^2$
$\mathrm{Tr}[R]$ (gravitational)	$-N_c^2 - 1$	$-N_c^2 - 1$
$\mathrm{Tr}[R^3]$	$-2N_c^4/N_f^2 + N_c^2 - 1$	(same)

The $\mathrm{Tr}[R^3]$ equality is verified computationally in `scripts/verify_anomaly_matching.py`.

5 Mass Deformation Consistency

A powerful consistency check is to deform both theories by adding a mass for one flavour and verifying that the duality maps correctly.

5.1 Electric side: $N_f \to N_f - 1$

Add a mass term for the N_f-th flavour:

$$\Delta W_{\text{el}} = m\, Q_{N_f} \widetilde{Q}^{N_f} \,. \tag{18}$$

In the IR, the massive quarks Q_{N_f}, $\widetilde{Q}^{N_f}$ are integrated out, leaving:

$$\text{Electric:} \quad \text{SU}(N_c) \text{ with } (N_f - 1) \text{ flavours}, \quad W = 0\,. \tag{19}$$

5.2 Magnetic side: $\text{SU}(N_f - N_c) \to \text{SU}(N_f - N_c - 1)$

On the magnetic side, the mass deformation appears through the meson coupling:

$$W_{\text{mag}} = \frac{1}{\mu} M^i{}_j\, q_i\, \widetilde{q}^j + m\, M^{N_f}{}_{N_f} \,. \tag{20}$$

The F-term equation for $M^{N_f}{}_{N_f}$ is:

$$\frac{\partial W}{\partial M^{N_f}{}_{N_f}} = 0 \qquad \Longrightarrow \qquad \frac{1}{\mu}\, q_{N_f}\, \widetilde{q}^{N_f} + m = 0 \qquad \Longrightarrow \qquad \langle q_{N_f}\, \widetilde{q}^{N_f} \rangle = -\mu\, m\,.$$

This nonzero vacuum expectation value *Higgses* the magnetic gauge group:

$$\text{SU}(N_f - N_c) \xrightarrow{\langle q_{N_f} \rangle \neq 0} \text{SU}(N_f - N_c - 1)\,.$$

After integrating out the massive fields (q_{N_f}, $\widetilde{q}^{N_f}$, $M^{N_f}{}_j$, $M^i{}_{N_f}$), the remaining theory is:

$$\text{Magnetic:} \quad \text{SU}(N_f - N_c - 1) \text{ with } (N_f - 1) \text{ flavours } + M' + W'_{\text{mag}}\,. \tag{21}$$

This is *exactly the Seiberg dual* of (19) with $N_f \to N_f - 1$:

$$\text{dual gauge group:} \quad \text{SU}((N_f - 1) - N_c) = \text{SU}(N_f - N_c - 1)\,. \quad \checkmark$$

Remark 5.1 (Iterative mass deformation)*.* Starting from N_f flavours and giving mass to one flavour at a time, the duality maps:

$$\begin{aligned} \text{SU}(N_c) \text{ with } N_f \quad &\longleftrightarrow \quad \text{SU}(N_f - N_c) \text{ with } N_f \\ \text{SU}(N_c) \text{ with } N_f - 1 \quad &\longleftrightarrow \quad \text{SU}(N_f - N_c - 1) \text{ with } N_f - 1 \\ &\;\;\vdots \end{aligned}$$

This process terminates when the number of flavours reaches $N_c + 1$ (s-confinement) or N_c (quantum-modified moduli space), recovering the known non-perturbative results in these regimes. This is a powerful consistency check.

5.3 Scale matching under mass deformation

The scale matching relation (6) transforms consistently. The matching for N_f flavours:

$$\Lambda_{\text{el}}^{3N_c - N_f}\, \Lambda_{\text{mag}}^{2N_f - 3N_c} = (-1)^{N_f - N_c}\, \mu^{N_f}\,.$$

After integrating out the massive N_f-th flavour:

$$\Lambda_{\text{el},N_f-1}^{3N_c-(N_f-1)}\, \Lambda_{\text{mag},N_f-1}^{2(N_f-1)-3N_c} = (-1)^{(N_f-1)-N_c}\, \mu^{N_f-1}\,,$$

with the standard scale matching relation $\Lambda_{\text{el},N_f-1}^{3N_c-N_f+1} = m\,\Lambda_{\text{el}}^{3N_c-N_f}$ relating the scales before and after decoupling. The magnetic side has a corresponding relation from the Higgsing. The consistency of the full system can be verified; we refer to [1] for the complete argument.

6 Moduli Space Matching and Further Evidence

6.1 Classical moduli space

The classical moduli space of the electric theory is parametrised by gauge-invariant operators:

- **Mesons:** $M^i{}_j = Q^i\widetilde{Q}_j$, an $N_f \times N_f$ matrix.
- **Baryons:** $B^{i_1\cdots i_{N_c}} = \epsilon^{a_1\cdots a_{N_c}} Q^{i_1}_{a_1}\cdots Q^{i_{N_c}}_{a_{N_c}}$ and similarly $\widetilde{B}_{i_1\cdots i_{N_c}}$, subject to classical constraints.

The **dimension** of the classical moduli space can be counted as:

$$\dim(\mathcal{M}_{\rm cl}) = 2N_cN_f - (N_c^2 - 1)\,, \tag{22}$$

which counts the number of scalar field components ($2N_cN_f$ from Q and $\widetilde{Q}$) minus the dimension of the gauge group orbit ($N_c^2 - 1$).

In the magnetic theory, the classical moduli space is similarly parametrised:

- The elementary meson $M^i{}_j$ (which maps to the electric meson).
- Magnetic baryons $b_{i_1\cdots i_{N_f-N_c}} = \epsilon^{a_1\cdots a_{N_f-N_c}} q_{i_1a_1}\cdots q_{i_{N_f-N_c}a_{N_f-N_c}}$ and $\widetilde{b}$.

The number of independent gauge invariants agrees between the two descriptions, providing another consistency check.

6.2 Conformal window from duality

Within the conformal window $\frac{3}{2}N_c < N_f < 3N_c$, the electric theory has $b_0^{\rm el} = 3N_c - N_f > 0$ (asymptotically free) and the magnetic theory has $b_0^{\rm mag} = 2N_f - 3N_c > 0$ (also asymptotically free). Both flow to the same interacting superconformal fixed point in the IR.

At the fixed point, Seiberg duality relates the anomalous dimensions of the two theories. From the NSVZ beta function with $\beta(g^*) = 0$:

$$\gamma_Q^* = -1 + \frac{3N_c}{N_f}\,, \qquad \gamma_q^* = -1 + \frac{3(N_f - N_c)}{N_f}\,,$$

satisfying $\gamma_Q^* + \gamma_q^* = 1$ — a consistency relation.

6.3 Summary: evidence for Seiberg duality

Seiberg Duality: Status as a Conjecture

Seiberg duality is a **conjecture**: it has not been mathematically proved from the microscopic definition of the theory. However, it is supported by an overwhelming body of evidence:

1. **Anomaly matching:** All 6 independent 't Hooft anomaly coefficients match exactly (§4).
2. **Moduli space matching:** The classical moduli spaces of the electric and magnetic theories agree (§6).
3. **Mass deformation consistency:** Giving mass to one flavour on both sides produces the correct dual pair with $N_f - 1$ flavours (§5).
4. **Limiting cases:** The duality reduces to independently verified results for $N_f = N_c$ (quantum-modified constraint) and $N_f = N_c + 1$ (s-confinement).
5. **Superconformal index:** Modern checks using the superconformal index provide even stronger evidence.

For the purposes of this course, we shall treat Seiberg duality as *established within the SUSY context* and use it as the foundation for extending duality ideas to the non-supersymmetric Standard Model.

7 Summary and Preview of Lecture 4

In this lecture we have:

1. Reviewed the phase structure of $\mathcal{N} = 1$ SQCD as a function of N_f/N_c.
2. Stated Seiberg duality: the IR equivalence between the electric $\mathrm{SU}(N_c)$ theory and the magnetic $\mathrm{SU}(N_f - N_c)$ theory with gauge-singlet mesons M and superpotential $W_{\mathrm{mag}} = (1/\mu)\, Mq\widetilde{q}$.
3. Derived the quantum numbers of the magnetic theory, including the non-trivial baryon charge $B[q] = N_c/(N_f - N_c)$.
4. Verified all 6 independent 't Hooft anomaly coefficients match exactly as polynomial identities in N_c and N_f.
5. Checked consistency under mass deformation: $N_f \to N_f - 1$ on both sides produces the correct dual pair.
6. Discussed moduli space matching and the conformal window.

Preview: Lecture 4

Where do the magnetic quarks *come from*? In Lecture 4, we will see that $\mathcal{N} = 2$ supersymmetric gauge theory provides a microscopic explanation: the magnetic quarks are identified with monopoles and dyons on the Coulomb branch of the $\mathcal{N} = 2$ theory. Breaking $\mathcal{N} = 2 \to \mathcal{N} = 1$ by an adjoint mass deformation produces precisely the Seiberg dual picture. This provides the deepest understanding of why electric–magnetic duality holds.

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