Lecture 4: $\mathcal{N} = 2$ SUSY and the Origin of Magnetic Quarks

The Dualised Standard Model — Part III Lecture Notes

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Conventions for Lecture 4

All conventions from Lectures 1–3 remain in force. Additional conventions for $\mathcal{N}=2$ theories (following Seiberg–Witten [1] and Argyres–Plesser–Seiberg [3]):

Quantity	**Convention**	**Comment**
Units	$\hbar = c = 1$ (natural)	
Metric	$\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$	mostly minus
Dynkin index	$\mathrm{Tr}(T^aT^b) = \frac{1}{2}\,\delta^{ab}$	$T(\square) = \frac{1}{2}$
N_f counting ($\mathcal{N}=2$)	N_f hypermultiplets	each $= (Q_i, \widetilde{Q}_i)$
Adjoint	Φ_{adj} from $\mathcal{N}=2$ vector	NOT counted in N_f
FI parameter	$\xi,\ \ [\xi] = [\mathrm{mass}]^2$	real parameter
Deformation	$W_{\mathrm{def}} = \mu\,\mathrm{Tr}\Phi^2,\ \ [\mu] = [\mathrm{mass}]$	adjoint mass
Coulomb branch	parametrised by $u_k = \langle \mathrm{Tr}\Phi^k \rangle$	$k = 2, \ldots, N_c$

$\mathcal{N}=1$ / $\mathcal{N}=2$ counting

The decomposition of $\mathcal{N}=2$ multiplets into $\mathcal{N}=1$ components is the single most common source of counting errors when relating $\mathcal{N}=2$ and $\mathcal{N}=1$ results. See the table in §1 and use it every time a formula crosses from one language to the other.

1 Extended Supersymmetry: $\mathcal{N}=2$ Multiplets

In Lectures 2–3 we worked with $\mathcal{N}=1$ supersymmetry: a single supercharge Q_α (plus its conjugate $\bar{Q}_{\dot\alpha}$). $\mathcal{N}=2$ supersymmetry has *two* supercharges Q^I_α, $I = 1, 2$, and its multiplets are correspondingly larger. We shall not develop the $\mathcal{N}=2$ superfield formalism; instead, we decompose every $\mathcal{N}=2$ multiplet into $\mathcal{N}=1$ language, which is sufficient for our purposes.

1.1 The $\mathcal{N}=2$ vector multiplet

The $\mathcal{N}=2$ vector multiplet contains, in $\mathcal{N}=1$ language:

- An $\mathcal{N}=1$ **vector multiplet** $V = (A_\mu, \lambda)$, containing the gauge boson and one Weyl gaugino. Both transform in the **adjoint** of the gauge group.
- An $\mathcal{N}=1$ **chiral multiplet** $\Phi = (\phi, \psi, F)$, also in the **adjoint** representation. The scalar component ϕ is a complex scalar field valued in the Lie algebra.

The two gauginos λ and ψ are related by the second supersymmetry. The key point is that the adjoint chiral Φ is *part of the gauge sector*, not a matter field.

1.2 The $\mathcal{N}=2$ hypermultiplet

The $\mathcal{N}=2$ hypermultiplet contains, in $\mathcal{N}=1$ language:

- Two $\mathcal{N}=1$ **chiral multiplets**: Q_i in the fundamental ($\square$) and $\widetilde{Q}_i$ in the antifundamental ($\overline{\square}$) of the gauge group, for $i = 1, \ldots, N_f$.

Each hypermultiplet contributes one unit to N_f.

1.3 Decomposition table

$\mathcal{N}=2 \to \mathcal{N}=1$ Decomposition

$\mathcal{N}=2$ **multiplet**	$\mathcal{N}=1$ **content**	**Representation**	N_f **counting**
Vector multiplet	$V+\Phi_{\text{adj}}$	both in adj	Φ_{adj} **not** counted in N_f
Hypermultiplet	$Q_i+\widetilde{Q}_i$	$\square+\overline{\square}$	each hyper $=1$ unit of N_f

1.4 $\mathcal{N}=2$ SQCD

$\mathcal{N}=2$ SQCD with gauge group $\mathrm{SU}(N_c)$ and N_f hypermultiplets is specified by:

- **Gauge sector:** one $\mathcal{N}=2$ vector multiplet ($=V+\Phi_{\text{adj}}$ in $\mathcal{N}=1$ language).
- **Matter:** N_f hypermultiplets ($=N_f$ pairs $(Q_i,\widetilde{Q}_i)$ in $\mathcal{N}=1$ language).
- **Superpotential:** The $\mathcal{N}=2$ algebra requires

$$W_{\mathcal{N}=2}=\sqrt{2}\,Q_i\,\Phi\,\widetilde{Q}^i\,, \tag{1}$$

where Φ is the adjoint chiral from the vector multiplet. This Yukawa coupling is *not* a choice—it is dictated by the extended supersymmetry.

1.5 The $\mathcal{N}=2$ beta function

The one-loop beta function coefficient for $\mathcal{N}=2$ $\mathrm{SU}(N_c)$ with N_f hypermultiplets is:

$$\boxed{b_0^{\mathcal{N}=2}=2N_c-N_f\,.} \tag{2}$$

Verification via $\mathcal{N}=1$ decomposition. In $\mathcal{N}=1$ language, the matter content of $\mathcal{N}=2$ SQCD consists of:

- N_f chiral pairs $(Q_i,\widetilde{Q}_i)$ in the fundamental — contributing N_f to the $\mathcal{N}=1$ beta function.
- One chiral multiplet Φ in the adjoint — contributing N_c (since $T(\text{adj})=N_c$).

The $\mathcal{N}=1$ beta function gives:

$$b_0^{\mathcal{N}=1}=3N_c-N_f-N_c=2N_c-N_f\,. \tag{3}$$

This matches (2) exactly, as it must. ✓

Asymptotic freedom requires $b_0^{\mathcal{N}=2}>0$, *i.e.*:

$$N_f<2N_c \qquad (\mathcal{N}=2 \text{ asymptotic freedom}). \tag{4}$$

Remark 1.1 ($\mathcal{N}=2$ conformal window). For $N_f=2N_c$, the $\mathcal{N}=2$ theory is exactly conformal at the origin of moduli space. For $N_f<2N_c$ it is asymptotically free. There is no analogue of the Seiberg conformal window in the $\mathcal{N}=2$ setting; the extended supersymmetry is too constraining.

2 The Coulomb Branch

2.1 Definition

The adjoint scalar ϕ in the $\mathcal{N} = 2$ vector multiplet can acquire a vacuum expectation value (VEV). Since ϕ transforms in the adjoint, its VEV can be diagonalised by a gauge transformation:

$$\langle\phi\rangle = \mathrm{diag}(a_1, a_2, \ldots, a_{N_c}), \qquad \sum_{i=1}^{N_c} a_i = 0, \tag{5}$$

where the tracelessness condition follows from $\mathrm{SU}(N_c)$.

At a *generic* point where all a_i are distinct, the gauge group is broken to its maximal abelian subgroup:

$$\mathrm{SU}(N_c) \longrightarrow \mathrm{U}(1)^{N_c-1}. \tag{6}$$

The low-energy theory is a collection of $(N_c - 1)$ abelian $\mathcal{N} = 2$ gauge multiplets — a free $\mathrm{U}(1)^{N_c-1}$ gauge theory. This is an abelian **Coulomb phase**, hence the name.

2.2 Gauge-invariant coordinates

The moduli space of vacua is parametrised by gauge-invariant quantities:

$$u_k = \langle \mathrm{Tr}\phi^k \rangle, \qquad k = 2, 3, \ldots, N_c. \tag{7}$$

For SU(2), there is a single complex parameter:

$$u = \langle \mathrm{Tr}\phi^2 \rangle = a_1^2 + a_2^2 = 2a^2, \tag{8}$$

where $a_1 = -a_2 = a$ (tracelessness). The Coulomb branch is a complex plane parametrised by u.

2.3 Classical singularities

Classically, the Coulomb branch is smooth except at the **origin** $u = 0$ (equivalently $a = 0$), where the gauge symmetry is enhanced back to SU(2) and the W-bosons become massless.

Quantum mechanically, the story changes dramatically.

3 The Seiberg–Witten Solution: Executive Summary

Scope of this section

We **state** the key results of the Seiberg–Witten solution without derivation. The full derivation — involving an auxiliary elliptic curve, its periods, monodromies, and the instanton expansion — would require several additional lectures. We refer to [1,2] for the original papers and to Alvarez-Gaumé and Hassan [5] for a pedagogical treatment.

3.1 The exact prepotential

Seiberg and Witten [1] showed that the *exact* low-energy effective action of $\mathcal{N} = 2$ SU(2) pure gauge theory is determined by a holomorphic function $\mathcal{F}(a)$, the **prepotential**, which encodes all couplings and interactions of the U(1) theory on the Coulomb branch. The key constraint is that $\mathcal{F}$ must be consistent with the global structure of the moduli space, including its singularities and monodromies.

The solution is encoded in an auxiliary **elliptic curve** (the Seiberg–Witten curve), whose periods determine a and $a_D = \partial\mathcal{F}/\partial a$ as functions of u. We state the result without deriving it.

3.2 Singular points of the Coulomb branch

The central result of [1] is:

Seiberg–Witten Solution for SU(2)

The quantum Coulomb branch of $\mathcal{N} = 2$ SU(2) pure gauge theory has **two** singular points:

1. At $u = +\Lambda^2_{\mathcal{N}=2}$: a magnetic **monopole** (with charges $(n_e, n_m) = (0, 1)$) becomes massless.
2. At $u = -\Lambda^2_{\mathcal{N}=2}$: a **dyon** (with charges $(n_e, n_m) = (1, 1)$) becomes massless.

Here $\Lambda_{\mathcal{N}=2}$ is the dynamical scale of the $\mathcal{N} = 2$ theory. The classical singularity at $u = 0$ (massless W-bosons) is **split** into two singularities by quantum effects. At each singular point, the low-energy theory is a U(1) gauge theory coupled to the massless monopole or dyon.

3.3 Physical picture

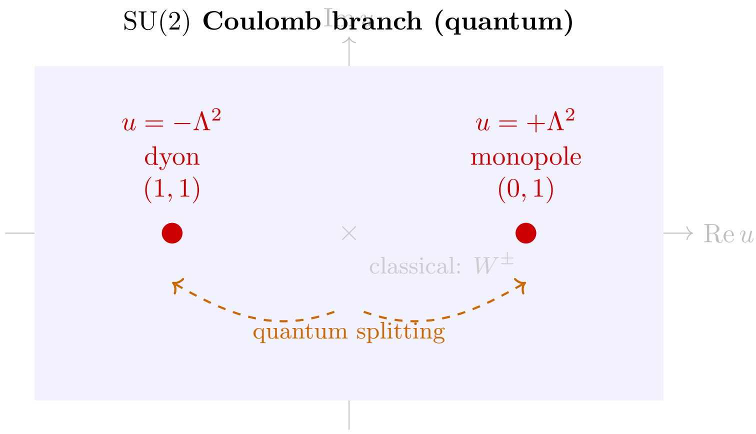

Figure 1: The Coulomb branch of $\mathcal{N} = 2$ SU(2) pure gauge theory. Classically, there is a single singularity at $u = 0$ where W-bosons become massless. Quantum mechanically, this splits into two singularities at $u = \pm\Lambda^2$, where a monopole and a dyon (respectively) become massless.

The physical picture is remarkable: the theory dynamically generates new light states — monopoles and dyons — that are *not* present in the perturbative spectrum. These are solitonic states whose mass is determined by the BPS formula:

$$M = \sqrt{2}\,|n_e\, a + n_m\, a_D|\,, \tag{9}$$

where (n_e, n_m) are the electric and magnetic charges, and a, a_D are the special coordinates on the Coulomb branch. At the singular points, $a_D \to 0$ (monopole point) or $a - a_D \to 0$ (dyon point), making the corresponding state massless.

3.4 $\mathrm{SU}(N_c)$ with N_f hypermultiplets

For $\mathrm{SU}(N_c)$ with N_f hypermultiplets (Seiberg–Witten II [2]), the Coulomb branch has a richer structure:

- The moduli space is $(N_c - 1)$-dimensional, parametrised by $u_2, \ldots, u_{N_c}$.
- There are multiple singular loci where different magnetic and dyonic states become massless.
- The monopoles carry **flavour quantum numbers** because the BPS mass formula (9) involves the quark masses (which enter through the hypermultiplet VEVs), and the monopole wavefunctions "feel" the flavour structure of the quarks via their fermionic zero modes.

The last point is crucial: the monopoles are not flavour-singlets. They transform non-trivially under the global flavour symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$.

4 The Fayet–Iliopoulos Mechanism

Before presenting the deformation argument, we need one more ingredient: the Fayet–Iliopoulos (FI) D-term [4].

4.1 Definition

For a U(1) gauge factor, there is a unique gauge-invariant and supersymmetric term that can be added to the Lagrangian:

$$\boxed{\mathcal{L}_{\mathrm{FI}} = \xi\, D\,,} \tag{10}$$

where D is the auxiliary field of the U(1) vector multiplet and ξ is a real parameter.

Dimensional analysis: $[D] = [\mathrm{mass}]^2$ (since D is an auxiliary field with no kinetic term, and $[\mathcal{L}] = [\mathrm{mass}]^4$), therefore:

$$[\xi] = [\mathrm{mass}]^2\,. \tag{11}$$

4.2 Effect on the D-term potential

In the presence of the FI term, the D-term scalar potential becomes:

$$V_D = \frac{1}{2}\left(g \sum_i q_i\,|\phi_i|^2 + \xi\right)^2, \tag{12}$$

where q_i are the U(1) charges of the scalar fields ϕ_i.

If $\xi \neq 0$, the minimum of V_D requires $g\sum_i q_i\,|\phi_i|^2 = -\xi$, which forces at least one scalar ϕ_i to acquire a non-zero VEV. This breaks the U(1) gauge symmetry.

4.3 Application to the Coulomb branch

On the Coulomb branch of $\mathcal{N} = 2$ theories, the unbroken gauge group is $\mathrm{U}(1)^{N_c-1}$. Each U(1) factor can, in principle, have an FI parameter. When the adjoint mass deformation (Section 5) lifts the Coulomb branch, the effective FI parameters at the surviving singular points can force the massless monopoles to *condense* (acquire VEVs). Monopole condensation is the dual mechanism to the Higgs effect: it produces **confinement** of the original electric charges, by the dual Meissner effect.

5 The APS Deformation: From $\mathcal{N} = 2$ to $\mathcal{N} = 1$

We now present the central argument of this lecture: the identification of the magnetic quarks of $\mathcal{N} = 1$ Seiberg duality with the monopoles and dyons of the $\mathcal{N} = 2$ Coulomb branch [3].

5.1 The deformation

Consider $\mathcal{N} = 2$ SU(N_c) SQCD with N_f hypermultiplets. Add a superpotential deformation:

$$W_{\text{def}} = \mu \, \text{Tr}\Phi^2 \,, \tag{13}$$

where Φ is the adjoint chiral from the $\mathcal{N} = 2$ vector multiplet and μ is a mass parameter with $[\mu] = [\text{mass}]$.

Dimensional check: $[W] = [\text{mass}]^3$, $[\Phi] = [\text{mass}]$, $[\text{Tr}\Phi^2] = [\text{mass}]^2$, so $[\mu \, \text{Tr}\Phi^2] = [\text{mass}]^3$. ✓

This deformation breaks $\mathcal{N} = 2 \to \mathcal{N} = 1$: the $\mathcal{N} = 2$ superpotential (1) plus the mass term (13) is not compatible with the second supercharge. The total $\mathcal{N} = 1$ superpotential is:

$$W = \sqrt{2}\, Q\Phi\widetilde{Q} + \mu \, \text{Tr}\Phi^2 \,. \tag{14}$$

5.2 The four-step argument

The logic of the APS argument [3] proceeds in four steps:

The APS Deformation Argument

Step 1. Start with $\mathcal{N} = 2$. Consider SU(N_c) $\mathcal{N} = 2$ SQCD with N_f hypermultiplets. The theory has a Coulomb branch with known singularities where monopoles and dyons become massless.

Step 2. Turn on the deformation. Add $W_{\text{def}} = \mu \, \text{Tr}\Phi^2$. For small μ, this is a soft perturbation of the $\mathcal{N} = 2$ theory. It generates a potential on the Coulomb branch:

$$V \supset |\mu|^2 \, |\text{Tr}\phi^2|^2 + \cdots$$

The Coulomb branch is **lifted**: the flat directions are no longer flat, and the theory is driven to specific minima.

Step 3. Identify the vacua. The vacua of the deformed theory are located at the **singular points** of the Coulomb branch — precisely the points where monopoles and dyons become massless. At these points, the monopoles condense (via the FI mechanism of §4), breaking the residual U(1) gauge symmetries and producing a confining vacuum.

Step 4. Take $\mu \to \infty$. As the adjoint mass μ becomes large, the adjoint chiral Φ decouples from the low-energy theory. Below the scale μ, the $\mathcal{N} = 1$ field content is:

- **Gauge group:** SU(N_c).
- **Matter:** N_f pairs $(Q_i, \widetilde{Q}_i)$ in the fundamental and antifundamental.
- **Superpotential:** $W = 0$ (the Yukawa coupling $Q\Phi\widetilde{Q}$ decouples with Φ).

This is precisely the **electric theory** of $\mathcal{N} = 1$ Seiberg duality from Lecture 3!

5.3 The punchline: monopoles become magnetic quarks

The conclusion of the four-step argument is:

Origin of Magnetic Quarks

The magnetic quarks q_i, $\widetilde{q}^j$ of the $\mathcal{N} = 1$ Seiberg dual are identified with the **monopoles and dyons** that become massless on the Coulomb branch of the parent $\mathcal{N} = 2$ theory. Seiberg duality for $\mathcal{N} = 1$ SQCD can be understood as a *consequence* of the exact $\mathcal{N} = 2$ results [3].

More specifically:

(a) The magnetic quarks are 't Hooft–Polyakov monopoles of the $\mathrm{SU}(N_c)$ theory, made light by the dynamics on the $\mathcal{N} = 2$ Coulomb branch.

(b) They carry flavour quantum numbers because the monopole wavefunctions have fermionic zero modes from the hypermultiplet quarks (see §3, final paragraph).

(c) The magnetic gauge group $\mathrm{SU}(N_f - N_c)$ arises from the pattern of monopole condensation at the singular loci.

(d) The meson field $M^i{}_j$ of the magnetic theory arises from the adjoint scalar Φ: as $\mu \to \infty$, the massive components of Φ decouple, but the composite operator $Q_i \, \Phi \, \widetilde{Q}^j$ survives as the elementary meson $M^i{}_j$ in the low-energy magnetic description.

Remark 5.1 (Continuity of the vacuum). The argument relies on the fact that the vacuum structure varies *continuously* as μ is increased from 0 (pure $\mathcal{N} = 2$) to ∞ (pure $\mathcal{N} = 1$). This is guaranteed by holomorphy of the superpotential: there are no phase transitions as a function of μ, so the identification of low-energy degrees of freedom at small μ (monopoles) persists to large μ (magnetic quarks).

6 Summary and Physical Picture

Lecture 4: Summary

1. $\mathcal{N} = 2$ SQCD has a **Coulomb branch** parametrised by $u_k = \langle \mathrm{Tr}\Phi^k \rangle$, where the gauge group $\mathrm{SU}(N_c)$ is broken to $\mathrm{U}(1)^{N_c-1}$.
2. The **Seiberg–Witten solution** (stated, not derived) shows that the quantum Coulomb branch has singular points where **monopoles** and **dyons** become massless.
3. The **Fayet–Iliopoulos mechanism** ($V_{\mathrm{FI}} = \xi D$, $[\xi] = [\mathrm{mass}]^2$) provides a mechanism for monopole condensation on the Coulomb branch.
4. The **APS deformation** $W = \mu \,\mathrm{Tr}\Phi^2$ breaks $\mathcal{N} = 2 \to \mathcal{N} = 1$ and drives the theory to the Coulomb branch singularities.
5. As $\mu \to \infty$, the adjoint Φ decouples, recovering $\mathcal{N} = 1$ SQCD. The massless **monopoles** and **dyons** of the $\mathcal{N} = 2$ theory become the **magnetic quarks** of $\mathcal{N} = 1$ Seiberg duality.
6. This explains *where* the magnetic degrees of freedom come from and *why* they carry flavour quantum numbers.

Preview: Lectures 5–6

In non-supersymmetric theories, we do not have the $\mathcal{N} = 2$ derivation. We cannot track the fate of monopoles as a function of a continuously tuneable parameter. Instead, we must rely on **anomaly matching** — verified in Lecture 3 — as the primary evidence for duality. This is why the SUSY case is foundational: it provides a microscopic *proof of concept* for electric–magnetic duality, which we then *conjecture* to extend beyond the SUSY setting. In Lectures 5–6, we will apply this duality framework to the Standard Model.

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