Lecture 9: The Landscape of SUSY Dualities

The Dualised Standard Model — Part III Lecture Notes

Lent Term 2027

Contents

Conventions for Lecture 9

All conventions from Lectures 1–8 remain in force. This lecture introduces orthogonal and symplectic gauge groups, which require additional index conventions.

Quantity	Convention	Comment
Units	$\hbar = c = 1$ (natural)	
Metric	$\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$	mostly minus
Dynkin index (SU)	$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\,\delta^{ab}$	fundamental of $\mathrm{SU}(N_c)$
Cubic anomaly	$A(\Box) = 1,\ A(\overline{\Box}) = -1$	per fundamental Weyl
N_f counting (SU)	Dirac flavour pairs	each $= (Q_i, \widetilde{Q}_i)$
New in this lecture:		
$\mathrm{SO}(N_c)$ convention	N_c-dimensional vector representation	
$T(\text{vector } \mathrm{SO}(N_c))$	1	**not** $\frac{1}{2}$; see [5]
$\mathrm{Sp}(N_c)$ convention	**rank** N_c; fund. dim. $2N_c$	$\mathrm{Sp}(1) = \mathrm{SU}(2)$
$T(\text{fund } \mathrm{Sp}(N_c))$	$\frac{1}{2}$	same as SU
$T(\wedge^2\ \mathrm{Sp}(N_c))$	$N_c - 1$	antisymmetric of $\mathrm{Sp}(N_c)$
Dual Coxeter number	$h_{\mathrm{SU}(N)} = N,\ \ h_{\mathrm{SO}(N)} = N-2,\ \ h_{\mathrm{Sp}(N)} = N+1$	

The $\mathrm{Sp}(N)$ naming convention

The symplectic group is named inconsistently across the literature. We follow **Intriligator and Seiberg** [4,5]: $\mathrm{Sp}(N_c)$ denotes the group of **rank** N_c whose fundamental representation has dimension $2N_c$. Thus $\mathrm{Sp}(1) = \mathrm{SU}(2)$ and $\mathrm{Sp}(2)$ has a 4-dimensional fundamental. Some mathematics references write $\mathrm{Sp}(2N_c)$ for the same group (labelling by the dimension of the fundamental rather than the rank). **Verify:** $\mathrm{Sp}(1)$ has $\dim(\text{fund}) = 2$, matching $\mathrm{SU}(2)$. ✓

1 Recap and Motivation

In Lecture 3, we established Seiberg duality [1] for $\mathcal{N} = 1$ $\mathrm{SU}(N_c)$ SQCD with N_f fundamental flavours: the IR physics of the electric theory is equivalently described by a *magnetic* $\mathrm{SU}(N_f - N_c)$ gauge theory with N_f dual quarks and an elementary $N_f \times N_f$ meson matrix M. We verified this statement through six independent 't Hooft anomaly coefficients, mass deformation consistency, and moduli space matching. The conformal window $\frac{3}{2}N_c < N_f < 3N_c$ was identified as the regime where both descriptions flow to the same IR fixed point.

A natural question arises: *is Seiberg duality a peculiarity of* $\mathrm{SU}(N_c)$ *with fundamental matter, or does it extend to other gauge groups and representations?* The answer, developed through an intensive programme in the mid-1990s, is that electric–magnetic duality is a *pervasive structural feature* of strongly coupled $\mathcal{N} = 1$ gauge theories. Dualities have been established for:

- $\mathrm{SU}(N_c)$ with fundamental *and adjoint* matter (Kutasov–Schwimmer [2]);
- orthogonal gauge groups $\mathrm{SO}(N_c)$ (Intriligator–Seiberg [4]);
- symplectic gauge groups $\mathrm{Sp}(N_c)$ (Intriligator–Seiberg [4]);

- product gauge groups and quiver gauge theories;
- theories exhibiting *s-confinement* without a dual gauge group (Csáki–Schmaltz–Skiba [7]);
- cascading gauge theories in which repeated Seiberg dualities drive a renormalisation group flow (Klebanov–Strassler [8]).

This lecture surveys the established landscape of $\mathcal{N}=1$ SUSY dualities, organised by gauge group (following the pedagogical approach of Peskin [10] and Intriligator–Seiberg [5]). The unifying thread is **'t Hooft anomaly matching**: in every case, the anomaly coefficients of the global symmetry group match exactly between the electric and magnetic descriptions. We verify one anomaly coefficient explicitly for each new gauge group, confirming that the general framework extends consistently beyond $\mathrm{SU}(N_c)$.

Connection to the dualised SM

The dualised Standard Model of Lectures 7–8 employs Seiberg duality for an $\mathrm{SU}(N)$ electric theory with fundamental and adjoint matter (the Pati–Salam embedding). The Kutasov–Schwimmer duality of §2 is therefore the most directly relevant generalisation for the dualised SM programme. The SO and Sp dualities demonstrate the *breadth* of the duality phenomenon.

2 Kutasov–Schwimmer Duality: $\mathrm{SU}(N_c)$ with Adjoint Matter

2.1 Electric theory

Consider $\mathcal{N}=1$ $\mathrm{SU}(N_c)$ with the following matter content:

- N_f chiral superfields Q^i in the fundamental ($\square$) and N_f chiral superfields $\widetilde{Q}_i$ in the anti-fundamental ($\overline{\square}$), where $i=1,\ldots,N_f$.
- One chiral superfield Φ in the adjoint representation (adj).
- A tree-level superpotential:
$$W_{\mathrm{el}} = \frac{g}{k+1}\,\mathrm{Tr}\left(\Phi^{k+1}\right)\,, \tag{1}$$
where $k \geq 1$ is a positive integer and g is a coupling constant.

The superpotential (1) is a relevant deformation that lifts the flat directions associated with the adjoint field. The integer k parametrises a *family* of theories, each with a different conformal window and a different dual description.

2.2 The dual (magnetic) theory

Kutasov and Schwimmer [2], with further developments by Kutasov, Schwimmer, and Seiberg [3], showed that the IR physics of the electric theory is equivalently described by the following **magnetic theory**:

Kutasov–Schwimmer Duality

Electric: $\mathrm{SU}(N_c)$ with $N_f \times (Q,\widetilde{Q})$ + adjoint Φ, $\quad W_{\mathrm{el}} = \dfrac{g}{k+1}\,\mathrm{Tr}(\Phi^{k+1})$.

Magnetic:

- **Gauge group:** $\mathrm{SU}(kN_f - N_c)$.
- **Dual quarks:** N_f chiral superfields q_i ($\Box$) and N_f chiral superfields $\widetilde{q}^{\,i}$ ($\overline{\Box}$) of $\mathrm{SU}(kN_f - N_c)$.
- **Dual adjoint:** one chiral superfield ϕ in the adjoint of $\mathrm{SU}(kN_f - N_c)$.
- **Mesons:** k gauge-singlet chiral superfields $M_j{}^i{}_{\tilde{\imath}}$ ($j = 0, 1, \ldots, k-1$), defined by the map from the electric theory:

$$M_j \;\sim\; Q\,\Phi^j\,\widetilde{Q}\,, \qquad j = 0, 1, \ldots, k-1\,. \tag{2}$$

  Each M_j is an $N_f \times N_f$ matrix transforming as $(\Box, \overline{\Box})$ under $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$.
- **Magnetic superpotential:**

$$W_{\mathrm{mag}} = \frac{g'}{k+1}\,\mathrm{Tr}(\phi^{k+1}) + \sum_{j=0}^{k-1} \frac{1}{\mu_j}\,\mathrm{Tr}\Big(M_j\, q\, \phi^{k-1-j}\, \widetilde{q}\Big)\,. \tag{3}$$

The key features distinguishing Kutasov–Schwimmer (KS) duality from basic Seiberg duality are:

1. The dual gauge group is $\mathrm{SU}(kN_f - N_c)$, with the integer k appearing as a *multiplier* of N_f.
2. The dual theory retains an adjoint chiral superfield ϕ with the same polynomial superpotential.
3. There are k distinct meson species $M_0, M_1, \ldots, M_{k-1}$, arising from the electric operators $Q\Phi^j\widetilde{Q}$ for $j = 0, \ldots, k-1$. The operator $Q\Phi^k\widetilde{Q}$ and higher powers are related to lower ones via the equation of motion $\partial W_{\mathrm{el}}/\partial\Phi = 0$, which gives the chiral ring relation $\Phi^k \sim 0$ (up to lower powers).

2.3 Conformal window

Both the electric and magnetic theories must be asymptotically free for the duality to produce a non-trivial interacting fixed point. The one-loop beta function coefficient for $\mathrm{SU}(N_c)$ with N_f fundamentals and one adjoint is:

$$b_0^{\mathrm{el}} = 3N_c - N_f - N_c = 2N_c - N_f\,, \tag{4}$$

using $T(\mathrm{adj}) = N_c$ and $T(\Box) = \frac{1}{2}$. (The factor of N_c from the adjoint replaces N_c in the usual $3N_c - N_f$ by $3N_c - N_c = 2N_c$, because the adjoint contributes as if it were N_c additional flavours.)

However, the superpotential $W = \mathrm{Tr}(\Phi^{k+1})$ constrains the anomalous dimension of Φ at the fixed point to $\gamma_\Phi^* = -2/(k+1)$ (so that W has R-charge 2). Incorporating this constraint, the effective conformal window becomes [2, 3]:

$$\boxed{\frac{3N_c}{k+1} < N_f < \frac{3N_c}{k}}\,. \tag{5}$$

The upper bound $N_f < 3N_c/k$ ensures $b_0^{\mathrm{el}} > 0$ after accounting for the adjoint's anomalous dimension at the fixed point; the lower bound ensures the same for the magnetic theory.

2.4 The $k = 1$ limit: recovering Seiberg duality

The crucial consistency check is that the $k = 1$ case reduces to the Seiberg duality of Lecture 3.

KS at $k = 1$: Recovering Seiberg Duality

Step 1: Electric theory at $k = 1$. The superpotential becomes

$$W_{\rm el} = \frac{g}{2}\,\mathrm{Tr}(\Phi^2)\,.$$

This is simply a **mass term** for the adjoint field Φ. In the IR (below the mass scale $m_\Phi = g$), we integrate out Φ, leaving pure $\mathrm{SU}(N_c)$ SQCD with N_f fundamental flavours and $W = 0$ — exactly the electric theory of Lecture 3.

Step 2: Magnetic gauge group at $k = 1$.

$$\mathrm{SU}(kN_f - N_c)\big|_{k=1} = \mathrm{SU}(N_f - N_c)\,.$$

This is precisely the Seiberg dual gauge group from Lecture 3. ✓

Step 3: Meson content at $k = 1$. With $k = 1$, there is only one meson:

$$M_0 = Q\,\Phi^0\,\widetilde{Q} = Q\,\widetilde{Q}\,,$$

which is the standard Seiberg meson $M^i{}_j = Q^i\widetilde{Q}_j$. ✓

Step 4: Magnetic superpotential at $k = 1$. The dual adjoint ϕ also acquires a mass term $W \supset \frac{g'}{2}\mathrm{Tr}(\phi^2)$ and can be integrated out. The meson coupling $\mathrm{Tr}(M_0\,q\,\phi^0\,\widetilde{q}) = \mathrm{Tr}(M_0\,q\,\widetilde{q})$ reduces to the standard Seiberg magnetic superpotential:

$$W_{\rm mag} \xrightarrow{\text{integrate out } \phi} \frac{1}{\mu}\,M\,q\,\widetilde{q}\,.$$

This matches $W_{\rm mag}$ from Lecture 3, Eq. (3.5). ✓

Step 5: Conformal window at $k = 1$.

$$\frac{3N_c}{2} < N_f < 3N_c\,,$$

which is the standard Seiberg conformal window. ✓

The KS duality thus provides a one-parameter family of dualities (parametrised by k) that interpolates between Seiberg duality ($k = 1$) and theories with increasingly rich meson spectra ($k \geq 2$). For the dualised Standard Model, the $k = 1$ case is most directly relevant, but the general framework shows that the duality structure is robust under the addition of adjoint matter.

3 $\mathrm{SO}(N_c)$ Duality

3.1 Electric theory

The electric theory is $\mathcal{N} = 1$ $\mathrm{SO}(N_c)$ gauge theory with:

- N_f chiral superfields Q_i ($i = 1, \ldots, N_f$) in the **vector** representation (dimension N_c) of $\mathrm{SO}(N_c)$.

- No tree-level superpotential: $W_{\rm el} = 0$.

Since the vector representation of $\mathrm{SO}(N_c)$ is *real* ($\square \cong \overline{\square}$), there is no distinction between Q and $\widetilde{Q}$. The global symmetry is therefore

$$G_{\rm global} = \mathrm{SU}(N_f) \times \mathrm{U}(1)_R\,,$$

with a single $\mathrm{SU}(N_f)$ factor rather than $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$ (which appears for $\mathrm{SU}(N_c)$ with complex fundamental representations).

3.2 The dual (magnetic) theory

Intriligator and Seiberg [4] established the following duality:

Intriligator–Seiberg Duality for $\mathrm{SO}(N_c)$

Electric: $\mathrm{SO}(N_c)$ with N_f vectors Q_i, $\quad W_{\rm el} = 0$.

Magnetic:

- **Gauge group:**
$$\boxed{\mathrm{SO}(N_f - N_c + 4)}\,. \tag{6}$$
- **Dual quarks:** N_f chiral superfields q_i in the vector (dimension $N_f - N_c + 4$) of the magnetic gauge group.
- **Meson:** a gauge-singlet chiral superfield $M_{ij} = M_{ji}$ (an $N_f \times N_f$ *symmetric* matrix), mapping to the electric composite $Q_i Q_j$.
- **Magnetic superpotential:**
$$W_{\rm mag} = \frac{1}{2\mu}\, M_{ij}\, q_i\, q_j\,. \tag{7}$$

Note the crucial **shift by** $+4$ in the magnetic gauge group rank: $\mathrm{SO}(N_f - N_c + 4)$, compared to $\mathrm{SU}(N_f - N_c)$ for SU duality. This is not a convention; it is a physical consequence of the different group-theoretic structure of $\mathrm{SO}(N_c)$.

The meson M_{ij} is symmetric because the electric quarks Q_i are in a *real* representation. The product $Q_i Q_j$ is symmetric under $i \leftrightarrow j$ (there is no antisymmetric invariant for the SO vector).

3.3 Conformal window

The one-loop beta function coefficient for $\mathrm{SO}(N_c)$ with N_f vectors is:

$$b_0^{\rm el} = 3\,T(\mathrm{adj}) - N_f\,T(\mathrm{vector}) = 3(N_c - 2) - N_f\,, \tag{8}$$

using the dual Coxeter number $h_{\mathrm{SO}(N_c)} = N_c - 2$ (which equals $T(\mathrm{adj})$ for $\mathrm{SO}(N_c)$) and $T(\mathrm{vector}) = 1$.

Asymptotic freedom requires $N_f < 3(N_c - 2)$. The conformal window is:

$$\boxed{\frac{3(N_c - 2)}{2} < N_f < 3(N_c - 2)}\,. \tag{9}$$

Test: $\mathrm{SO}(10)$: conformal window $12 < N_f < 24$. $\checkmark$

3.4 Anomaly matching: $\mathrm{SU}(N_f)^2\,\mathrm{U}(1)_R$

We verify one anomaly coefficient explicitly. The non-anomalous R-charge is determined by the condition $T(\text{adj}) + N_f\,T(\text{vector})\,(R[Q]-1) = 0$:

$$R[Q] = 1 - \frac{N_c-2}{N_f}\,, \qquad R_{\text{ferm}}[Q] = -\frac{N_c-2}{N_f}\,. \tag{10}$$

The mixed anomaly coefficient $\mathcal{A} = \sum_f \dim(\text{gauge rep}_f)\,T(\mathrm{SU}(N_f)\ \text{rep}_f)\,R_{\text{ferm},f}$ is computed as follows.

Electric side. Only Q_i contributes (the gaugino is a singlet under $\mathrm{SU}(N_f)$):

$$\mathcal{A}^{\text{el}} = N_c \cdot \frac{1}{2} \cdot \left(-\frac{N_c-2}{N_f}\right) = -\frac{N_c(N_c-2)}{2N_f}\,. \tag{11}$$

Here N_c is the dimension of the vector representation of $\mathrm{SO}(N_c)$, and $T(\square_{\mathrm{SU}(N_f)}) = \frac{1}{2}$.

Magnetic side. The dual Coxeter number is $h_{\text{mag}} = (N_f - N_c + 4) - 2 = N_f - N_c + 2$, giving:

$$R[q] = 1 - \frac{N_f - N_c + 2}{N_f} = \frac{N_c-2}{N_f}\,, \qquad R_{\text{ferm}}[q] = \frac{N_c - 2 - N_f}{N_f}\,. \tag{12}$$

The meson R-charge (from $M \sim Q^2$):

$$R[M] = 2\,R[Q] = \frac{2(N_f - N_c + 2)}{N_f}\,, \qquad R_{\text{ferm}}[M] = \frac{N_f - 2N_c + 4}{N_f}\,. \tag{13}$$

The meson M_{ij} is a symmetric matrix, transforming in the S_2 representation of $\mathrm{SU}(N_f)$ with Dynkin index $T(\mathrm{S}_2\ \mathrm{SU}(N_f)) = (N_f+2)/2$.

Two contributions to the magnetic anomaly:

$$\mathcal{A}_q^{\text{mag}} = (N_f - N_c + 4) \cdot \frac{1}{2} \cdot \frac{N_c - 2 - N_f}{N_f} = -\frac{(N_f - N_c + 4)(N_f - N_c + 2)}{2N_f}\,, \tag{14}$$

$$\mathcal{A}_M^{\text{mag}} = 1 \cdot \frac{N_f+2}{2} \cdot \frac{N_f - 2N_c + 4}{N_f} = \frac{(N_f+2)(N_f - 2N_c + 4)}{2N_f}\,. \tag{15}$$

The total magnetic anomaly is:

$$\begin{aligned}
\mathcal{A}^{\text{mag}} &= \frac{1}{2N_f}\Big[-(N_f - N_c + 4)(N_f - N_c + 2) + (N_f+2)(N_f - 2N_c + 4)\Big] \\
&= \frac{1}{2N_f}\Big[-(N_f^2 - 2N_cN_f + N_c^2 + 6N_f - 6N_c + 8) \\
&\qquad\qquad + (N_f^2 - 2N_cN_f + 4N_f + 2N_f - 4N_c + 8)\Big] \\
&= \frac{1}{2N_f}\big[-N_c^2 - 6N_f + 6N_c - 8 + 6N_f - 4N_c + 8\big] \\
&= \frac{1}{2N_f}\big[-N_c^2 + 2N_c\big] = -\frac{N_c(N_c-2)}{2N_f}\,.
\end{aligned} \tag{16}$$

$$\boxed{\mathcal{A}^{\text{el}} = -\frac{N_c(N_c-2)}{2N_f} = \mathcal{A}^{\text{mag}}}\,. \quad \checkmark \tag{17}$$

Numerical check: $\mathrm{SO}(10)$, $N_f = 14$: $\mathcal{A}^{\text{el}} = -10 \times 8/(2\times 14) = -20/7$. Magnetic $\mathrm{SO}(8)$: $\mathcal{A}_q^{\text{mag}} = -8 \times \frac{1}{2} \times (-6/14) = 24/14$; $\mathcal{A}_M^{\text{mag}} = 8 \times (-6/14) = -48/14$; total $= (24-48)/14 = -24/14 = -12/7$.

Wait — let me recompute: $\mathcal{A}_q^{\text{mag}} = 8 \cdot \frac{1}{2} \cdot \frac{8-14}{14} = 4 \cdot \frac{-6}{14} = -\frac{24}{14} = -\frac{12}{7}$, $\mathcal{A}_M^{\text{mag}} = \frac{16}{2} \cdot \frac{14-20+4}{14} = 8 \cdot \frac{-2}{14} = -\frac{16}{14} = -\frac{8}{7}$, total $= -\frac{12}{7} - \frac{8}{7} = -\frac{20}{7}$. ✓

3.5 Limiting cases

- $N_f = 0$ **(pure SYM):** The theory has $N_c - 2$ isolated SUSY vacua from gaugino condensation (using the dual Coxeter number $h = N_c - 2$), matching the Witten index computation.
- $N_f = N_c - 4$**:** The magnetic gauge group is SO(0), trivial. This is the boundary of s-confinement for $\mathrm{SO}(N_c)$ (see §5).

Remark 3.1 ($\mathrm{SO}(N_c)$ for even vs. odd N_c). The duality map holds for all N_c, but there are subtleties for even vs. odd N_c. When N_c is even, the centre of $\mathrm{SO}(N_c)$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ (for $N_c \equiv 0 \pmod 4$) or $\mathbb{Z}_4$ (for $N_c \equiv 2 \pmod 4$), and the theory admits spinor representations. When N_c is odd, the centre is $\mathbb{Z}_2$ and there is a single spinor. The duality map respects these discrete structures; see [5] for the complete treatment.

4 $\mathrm{Sp}(N_c)$ Duality

4.1 Electric theory

The electric theory is $\mathcal{N} = 1$ $\mathrm{Sp}(N_c)$ gauge theory with:

- $2N_f$ chiral superfields Q_i ($i = 1, \ldots, 2N_f$) in the **fundamental** representation of $\mathrm{Sp}(N_c)$ (dimension $2N_c$).
- No tree-level superpotential: $W_{\mathrm{el}} = 0$.

Since the fundamental of $\mathrm{Sp}(N_c)$ is *pseudo-real* (it is equivalent to its conjugate via the symplectic form J), the global symmetry is:

$$G_{\mathrm{global}} = \mathrm{SU}(2N_f) \times \mathrm{U}(1)_R \,.$$

The factor $\mathrm{SU}(2N_f)$ (rather than $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$) reflects the pseudo-reality of the fundamental representation.

4.2 The dual (magnetic) theory

Intriligator–Seiberg Duality for $\mathrm{Sp}(N_c)$

Electric: $\mathrm{Sp}(N_c)$ with $2N_f$ fundamentals Q_i, $\quad W_{\mathrm{el}} = 0$.

Magnetic:

- **Gauge group:**
$$\boxed{\mathrm{Sp}(N_f - N_c - 2)}\,. \tag{18}$$
- **Dual quarks:** $2N_f$ chiral superfields q_i in the fundamental (dimension $2(N_f - N_c - 2)$) of the magnetic gauge group.
- **Meson:** a gauge-singlet chiral superfield $M_{ij} = -M_{ji}$ (an $N_f \times N_f$ *antisymmetric* matrix under $\mathrm{SU}(2N_f)$), mapping to the electric composite $Q_i\,J\,Q_j$ where J is the symplectic form.
- **Magnetic superpotential:**
$$W_{\mathrm{mag}} = \frac{1}{\mu}\, M_{ij}\, q_i\, J\, q_j \,. \tag{19}$$

The shift is -2 (compare $+4$ for SO and $+0$ for SU). When $N_f = N_c + 2$, the magnetic gauge group is Sp(0) (trivial): the theory s-confines without a residual gauge symmetry.

The meson M_{ij} is **antisymmetric** because the symplectic form J provides an antisymmetric invariant tensor for contracting the gauge indices, so $Q_i J Q_j = -Q_j J Q_i$.

4.3 Conformal window

The one-loop beta function coefficient for $\mathrm{Sp}(N_c)$ with $2N_f$ fundamentals is:

$$b_0^{\mathrm{el}} = 3\,T(\mathrm{adj}) - 2N_f\,T(\mathrm{fund}) = 3(N_c+1) - 2N_f \cdot \frac{1}{2} = 3(N_c+1) - N_f\,, \tag{20}$$

using $h_{\mathrm{Sp}(N_c)} = N_c + 1$ and $T(\mathrm{fund}) = \frac{1}{2}$.
The conformal window is:

$$\boxed{\frac{3(N_c+1)}{2} < N_f < 3(N_c+1)}\,. \tag{21}$$

Test: Sp(2) (rank 2, fundamental dimension 4): conformal window $4.5 < N_f < 9$, i.e. $N_f \in \{5, 6, 7, 8\}$. ✓

4.4 Anomaly matching: $\mathrm{SU}(2N_f)^2\,\mathrm{U}(1)_R$

The non-anomalous R-charge is:

$$R[Q] = 1 - \frac{N_c+1}{N_f}\,, \qquad R_{\mathrm{ferm}}[Q] = -\frac{N_c+1}{N_f}\,. \tag{22}$$

We check the mixed anomaly $\mathcal{A} = \sum_f \dim(\text{gauge rep}_f)\,T(\mathrm{SU}(2N_f)\ \text{rep}_f)\,R_{\mathrm{ferm},f}$.
Electric side. Each Q_i lives in the fundamental of $\mathrm{Sp}(N_c)$ (dimension $2N_c$) and the fundamental of $\mathrm{SU}(2N_f)$:

$$\mathcal{A}^{\mathrm{el}} = 2N_c \cdot \frac{1}{2} \cdot \left(-\frac{N_c+1}{N_f}\right) = -\frac{N_c(N_c+1)}{N_f}\,. \tag{23}$$

Magnetic side. The dual Coxeter number is $h_{\mathrm{mag}} = (N_f - N_c - 2) + 1 = N_f - N_c - 1$:

$$R[q] = 1 - \frac{N_f - N_c - 1}{N_f} = \frac{N_c+1}{N_f}\,, \qquad R_{\mathrm{ferm}}[q] = \frac{N_c + 1 - N_f}{N_f}\,. \tag{24}$$

The meson:

$$R[M] = 2\,R[Q] = \frac{2(N_f - N_c - 1)}{N_f}\,, \qquad R_{\mathrm{ferm}}[M] = \frac{N_f - 2N_c - 2}{N_f}\,. \tag{25}$$

The meson M_{ij} is antisymmetric under $\mathrm{SU}(2N_f)$, with $T(\wedge^2\,\mathrm{SU}(2N_f)) = (2N_f - 2)/2 = N_f - 1$.
Two contributions:

$$\mathcal{A}_q^{\mathrm{mag}} = 2(N_f - N_c - 2) \cdot \frac{1}{2} \cdot \frac{N_c + 1 - N_f}{N_f} = -\frac{(N_f - N_c - 2)(N_f - N_c - 1)}{N_f}\,, \tag{26}$$

$$\mathcal{A}_M^{\mathrm{mag}} = 1 \cdot (N_f - 1) \cdot \frac{N_f - 2N_c - 2}{N_f} = \frac{(N_f - 1)(N_f - 2N_c - 2)}{N_f}\,. \tag{27}$$

Summing (with $a \equiv N_f - N_c$ for brevity):

$$\begin{aligned} N_f\,\mathcal{A}^{\text{mag}} &= -(a-2)(a-1) + (N_f-1)(N_f-2N_c-2) \\ &= -(a^2-3a+2) + (N_f^2 - 2N_cN_f - 2N_f - N_f + 2N_c + 2) \\ &= -(a^2-3a+2) + (N_f^2 - 2N_cN_f - 3N_f + 2N_c + 2)\,. \end{aligned} \tag{28}$$

Now $a^2 = N_f^2 - 2N_cN_f + N_c^2$ and $3a = 3N_f - 3N_c$, so:

$$\begin{aligned} N_f\,\mathcal{A}^{\text{mag}} &= -N_f^2 + 2N_cN_f - N_c^2 + 3N_f - 3N_c - 2 \\ &\quad + N_f^2 - 2N_cN_f - 3N_f + 2N_c + 2 \\ &= -N_c^2 - N_c = -N_c(N_c+1)\,. \end{aligned} \tag{29}$$

Therefore:

$$\boxed{\mathcal{A}^{\text{el}} = -\frac{N_c(N_c+1)}{N_f} = \mathcal{A}^{\text{mag}}}\,. \quad \checkmark \tag{30}$$

Numerical check: Sp(2) with $N_f = 5$ ($2N_f = 10$ fundamentals):
Electric: $\mathcal{A}^{\text{el}} = -2 \times 3/5 = -6/5$.
Magnetic Sp(1): $h_{\text{mag}} = 2$, $R_{\text{ferm}}[q] = (3-5)/5 = -2/5$, $R_{\text{ferm}}[M] = (5-6)/5 = -1/5$.
$\mathcal{A}_q^{\text{mag}} = 2 \cdot \frac{1}{2} \cdot (-2/5) = -2/5$; $\mathcal{A}_M^{\text{mag}} = 4 \cdot (-1/5) = -4/5$; total $= -6/5$. $\checkmark$

4.5 Limiting cases

- $N_f = 0$ **(pure SYM):** The theory has $N_c + 1$ isolated SUSY vacua from gaugino condensation (using $h = N_c + 1$).
- $N_f = N_c + 2$**:** The magnetic gauge group is Sp(0) (trivial). The theory s-confines: the IR degrees of freedom are the mesons M_{ij} and possibly additional composites, with no residual gauge symmetry.
- **Self-dual case:** Sp(1) with $N_f = 4$ ($2N_f = 8$ fundamentals): $\text{Sp}(N_f - N_c - 2) = \text{Sp}(4 - 1 - 2) = \text{Sp}(1)$. The theory is *self-dual*: the electric and magnetic descriptions have the same gauge group.

5 s-Confining Theories

5.1 Definition and physical picture

An $\mathcal{N} = 1$ gauge theory is **s-confining** (smoothly confining) if:

1. All degrees of freedom at the *origin* of moduli space are gauge-invariant composites (mesons, baryons, or other invariants).
2. There is no dual gauge group: the confined description is a theory of gauge-singlet fields with a smooth, non-trivial confining superpotential W_{conf}.
3. The theory confines *without* chiral symmetry breaking.

The s-confining theories sit at the **boundary** of the duality landscape: they are the endpoint of Seiberg-type dualities where the magnetic gauge group has rank zero. For $\mathrm{SU}(N_c)$ SQCD, this occurs at $N_f = N_c + 1$, which we already encountered in Lecture 3, §1 (item 4): the confined degrees of freedom are the meson M, baryons B_i, $\widetilde{B}^i$, with the confining superpotential from Lecture 3, Eq. (3.4).

5.2 The canonical example: $\mathrm{SU}(N_c)$ with $N_f = N_c + 1$

We recall from Lecture 3 that for $N_f = N_c + 1$, the low-energy description is:

- Gauge-singlet fields: $M^i{}_j$ ($N_f \times N_f$ meson matrix), B_i, $\widetilde{B}^j$ (baryons).
- Confining superpotential:

$$W_{\mathrm{conf}} = \frac{1}{\Lambda^{2N_c-1}} \Big(B_i \, M^i{}_j \, \widetilde{B}^j - \det M \Big) \, .$$

- All 't Hooft anomalies match between the confined and microscopic descriptions.

There is no residual gauge group: the theory has *completely* confined.

5.3 Classification of s-confining theories

Csáki, Schmaltz, and Skiba [7] systematically classified all s-confining $\mathcal{N} = 1$ theories with simple gauge groups and no tree-level superpotential ($W_{\mathrm{tree}} = 0$). Their criteria are:

1. The one-loop beta function satisfies $b_0 > 0$ (asymptotic freedom).
2. The index condition $\sum_i \mu(R_i) = \mu(\mathrm{adj}){+}1$ is satisfied, where $\mu(R) = T(R){\cdot}\dim(G)/\dim(R)$ is the "index" of the representation. This ensures that the theory is at the boundary of the conformal window.
3. All flat directions are lifted by the confining superpotential.

The complete list for simple gauge groups is given in the following table, adapted from [7]:

Remark 5.1 (s-confinement as the boundary of duality). Each s-confining theory can be understood as the limiting case of a Seiberg-type duality where the magnetic gauge group has rank zero:

- $\mathrm{SU}(N_c)$ with $N_f = N_c + 1$: the dual is $\mathrm{SU}(N_f - N_c) = \mathrm{SU}(1)$ (trivial).
- $\mathrm{Sp}(N_c)$ with $N_f = N_c + 2$: the dual is $\mathrm{Sp}(N_f - N_c - 2) = \mathrm{Sp}(0)$ (trivial).
- $\mathrm{SO}(N_c)$ with $N_f = N_c - 4$: the dual is $\mathrm{SO}(N_f - N_c + 4) = \mathrm{SO}(0)$ (trivial).

The confining superpotential W_{conf} can be viewed as the "remnant" of the magnetic superpotential after the magnetic gauge group has been completely Higgsed/confined.

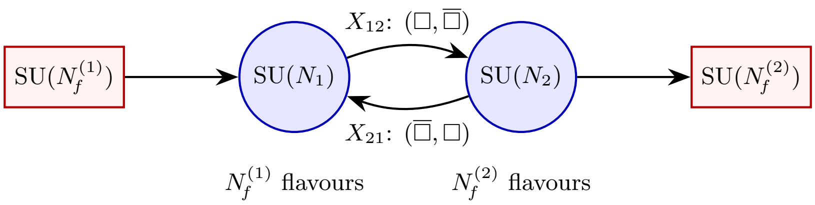
Gauge Group	Matter Content	Confined DOF
$\mathrm{SU}(N_c)$	$N_f = N_c + 1$ fundamentals	$M, B, \widetilde{B}$
$\mathrm{SU}(N_c)$	$1\ \wedge^2 + (N_c - 3)\ \square + 1\ \overline{\square}$	various composites
$\mathrm{SU}(N_c)$	$1\ \mathrm{S}_2 + 1\ \overline{\square}$ (for N_c odd)	composites from S_2
$\mathrm{SO}(N)$	$(N-2)$ vectors	M, Pfaffian (even N)
$\mathrm{SO}(7)$	5 spinors	composites from spinors
$\mathrm{Sp}(N_c)$	$2(N_c + 2)$ fundamentals	M_{ij} (antisymmetric)
$\mathrm{Sp}(N_c)$	$1\ \wedge^2 + 4\ \square$	composites from $\wedge^2$
G_2	4 fundamentals (**7**)	bilinear composites

Table 1: Selected s-confining theories with simple gauge groups and $W_{\text{tree}} = 0$, following Csáki–Schmaltz–Skiba [7]. The $\mathrm{SU}(N_c)$ case with $N_f = N_c + 1$ is the canonical example from Lecture 3. The exceptional groups F_4, E_6, E_7, E_8 also admit s-confining theories with specific matter representations.

6 Product Group Dualities and Quiver Mutation

6.1 Quiver gauge theories

Many UV completions of particle physics models involve *product* gauge groups $\prod_a G_a$ with matter fields transforming as *bifundamentals* — charged under two different gauge group factors. These are naturally represented by **quiver diagrams**: directed graphs where each node represents a gauge group factor and each arrow represents a bifundamental chiral superfield.



In this quiver, circular nodes represent gauge groups (dynamical) and square nodes represent global flavour symmetries (spectator).

6.2 Seiberg duality as quiver mutation

Seiberg duality can be applied to *one node at a time* in a quiver gauge theory [6]. The procedure is called **quiver mutation**:

1. Choose a node with gauge group $\mathrm{SU}(N_c)$.
2. Count the total number of *effective flavours* N_f^{eff} at that node: this includes both the external flavours and the bifundamentals from adjacent nodes.
3. Replace the node by $\mathrm{SU}(N_f^{\text{eff}} - N_c)$, add the appropriate meson fields (composites of the original bifundamentals passing through the dualised node), and modify the superpotential according to the standard Seiberg duality map.

The mutation rule preserves all 't Hooft anomaly coefficients and maps the moduli space correctly. Successive mutations at different nodes generate a *duality web* of physically equivalent descriptions of the same IR physics.

6.3 Product group dualities

Intriligator and Seiberg [6] extended the duality programme to product gauge groups, finding dualities and *trialities* (three equivalent descriptions) for specific product group theories. The key insight is that Seiberg duality is *local*: it acts on one gauge group factor at a time, treating the other factors as global symmetries from the perspective of the dualised node.

For the dualised Standard Model, the product group structure $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ (or its Pati–Salam extension $\mathrm{SU}(4) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$) is naturally embedded in a quiver. The duality acts on the colour factor $\mathrm{SU}(3)_C$ (or $\mathrm{SU}(4)$), treating the electroweak factors as spectator global symmetries.

7 The Duality Cascade

A remarkable application of Seiberg duality to product gauge groups is the **duality cascade** of Klebanov and Strassler [8].

7.1 Setup

Consider the gauge theory $\mathrm{SU}(N+M) \times \mathrm{SU}(N)$ with bifundamental chiral superfields A_1, A_2 in the $(\square, \overline{\square})$ and B_1, B_2 in the $(\overline{\square}, \square)$, together with a quartic superpotential:

$$W = h\, \epsilon^{ij} \epsilon^{kl}\, \mathrm{Tr}(A_i B_k A_j B_l)\,.$$

This theory arises from N regular and M fractional D3-branes placed at the tip of a conifold singularity in type IIB string theory, but the field theory can be analysed independently.

7.2 The cascade mechanism

When $M > 0$, the $\mathrm{SU}(N+M)$ factor has more colours than the $\mathrm{SU}(N)$ factor. As the theory flows to the IR:

1. The $\mathrm{SU}(N+M)$ factor becomes more strongly coupled. Applying Seiberg duality to the $\mathrm{SU}(N+M)$ node:

$$\mathrm{SU}(N+M) \times \mathrm{SU}(N) \xrightarrow{\text{dualise}} \mathrm{SU}(N-M) \times \mathrm{SU}(N)\,.$$

   (The effective number of flavours at the $\mathrm{SU}(N+M)$ node is $2N$ from the bifundamentals, giving a dual gauge group $\mathrm{SU}(2N-(N+M)) = \mathrm{SU}(N-M)$.)

2. After the duality, the $\mathrm{SU}(N)$ factor now has more colours. Dualising it next gives $\mathrm{SU}(N-M) \times \mathrm{SU}(N-2M)$.

3. The process repeats, *cascading*:

$$\mathrm{SU}(N+M) \times \mathrm{SU}(N) \to \mathrm{SU}(N-M) \times \mathrm{SU}(N) \to \mathrm{SU}(N-M) \times \mathrm{SU}(N-2M) \to \cdots$$

   Each step reduces the effective rank by M.

7.3 The endpoint

After approximately N/M steps, the cascade terminates when one gauge group factor reaches $\mathrm{SU}(M)$. The final theory is $\mathrm{SU}(2M) \times \mathrm{SU}(M)$, which confines. The IR physics exhibits **chiral symmetry breaking** and a mass gap, analogous to QCD confinement.

The cascade provides a concrete mechanism by which a theory that is UV-free (or conformal) at high energies flows through a sequence of Seiberg dualities to a confining theory at low energies. In the string theory realisation, the cascade corresponds to the radial evolution along a warped deformed conifold geometry, where the decreasing five-form flux corresponds to the decreasing rank of the gauge group.

Remark 7.1 (Beyond these lectures). The geometric interpretation of the duality cascade requires the framework of the AdS/CFT correspondence, which lies beyond the scope of these lectures. We refer to the original paper [8] and the review by Strassler [9] for the complete treatment.

8 Synthesis: The Duality Landscape and the Dualised SM

8.1 Master comparison table

We collect the dualities surveyed in this lecture and in Lecture 3 into a single comparison table.

Duality	**Electric**	**Magnetic**	**Mesons**	**Conf. Window**	**Key Feature**
Seiberg (Lec. 3)	$\mathrm{SU}(N_c)$, $N_f \times (\square, \overline{\square})$	$\mathrm{SU}(N_f - N_c)$	$M = Q\widetilde{Q}$ (sym. matrix)	$\frac{3N_c}{2} < N_f < 3N_c$	Prototype
KS (§2)	$\mathrm{SU}(N_c)$, N_f + adj, $W = \mathrm{Tr}\Phi^{k+1}$	$\mathrm{SU}(kN_f - N_c)$	$M_j = Q\Phi^j\widetilde{Q}$, $j = 0, \ldots, k-1$	$\frac{3N_c}{k+1} < N_f < \frac{3N_c}{k}$	$k = 1$ recovers Seiberg
SO (§3)	$\mathrm{SO}(N_c)$, N_f vectors	$\mathrm{SO}(N_f - N_c + 4)$	$M_{ij} = Q_i Q_j$ (symmetric)	$\frac{3(N_c-2)}{2} < N_f < 3(N_c - 2)$	Shift $+4$
Sp (§4)	$\mathrm{Sp}(N_c)$, $2N_f$ fund.	$\mathrm{Sp}(N_f - N_c - 2)$	$M_{ij} = Q_i J Q_j$ (antisym.)	$\frac{3(N_c+1)}{2} < N_f < 3(N_c + 1)$	Shift -2
s-confining (§5)	Various (Table 1)	*none* (trivial)	All composites	*boundary*	No dual gauge group
Cascade (§7)	$\mathrm{SU}(N+M) \times \mathrm{SU}(N)$, bifundamentals	$\mathrm{SU}(N-M) \times \mathrm{SU}(N)$ (iterates)	Generated at each step	*flows*	RG cascade

Table 2: Master comparison table of $\mathcal{N} = 1$ SUSY dualities. The conformal window column gives the range of N_f for which both the electric and magnetic theories are asymptotically free and flow to the same interacting fixed point. The "shift" in the SO and Sp rows refers to the difference in the dual gauge group rank relative to the SU case: $\mathrm{SU}(N_f - N_c + 0)$ vs. $\mathrm{SO}(N_f - N_c + 4)$ vs. $\mathrm{Sp}(N_f - N_c - 2)$.

8.2 Unifying thread: anomaly matching

Across the entire landscape, the unifying consistency check is **'t Hooft anomaly matching**. In every case — SU, SO, Sp, product groups, KS — the anomaly coefficients of the global symmetry group match exactly between the electric and magnetic descriptions, as polynomial identities in N_c and N_f.

We have verified this explicitly:

- $\mathrm{SU}(N_c)$ (Seiberg): six independent anomaly coefficients matched in Lecture 3.
- $\mathrm{SO}(N_c)$: $\mathrm{SU}(N_f)^2\,\mathrm{U}(1)_R$ anomaly matched in §3.4.
- $\mathrm{Sp}(N_c)$: $\mathrm{SU}(2N_f)^2\,\mathrm{U}(1)_R$ anomaly matched in §4.4.

Each of these is an *exact* statement, valid for all values of N_c and N_f within the duality range. This universality of anomaly matching provides strong evidence that electric–magnetic duality is a deep structural feature of strongly coupled gauge theories, not a coincidence of specific group-theoretic numerology.

8.3 Relevance to the dualised SM

Which of these dualities is most relevant to the dualised Standard Model programme of Lectures 7–8 [11]?

1. **Kutasov–Schwimmer duality with adjoint matter** is the most directly relevant. The dualised SM embeds the SM gauge group in a Pati–Salam-type structure $\mathrm{SU}(4) \times \mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, where the SU(4) factor carries both fundamental quarks and potentially adjoint matter from the breaking of a larger gauge group. The KS duality with $k = 1$ (recovering standard Seiberg duality after integrating out the adjoint) provides the natural UV framework.
2. **Product group dualities** are relevant because the SM has a product gauge group structure. The quiver mutation technique shows that duality can be applied node-by-node: one can dualise the colour $\mathrm{SU}(3)_C$ factor while leaving the electroweak factors as spectators.
3. **The SO and Sp dualities** demonstrate the robustness of the duality phenomenon. While the SM gauge group involves SU-type factors (not SO or Sp), the existence of dualities for all classical Lie groups suggests that the mechanism is deeply structural.
4. **s-Confinement** is relevant to the QCD-like sector. If the electric theory has a number of flavours near the s-confining boundary, the confined degrees of freedom (composites) naturally map to the physical hadrons.

The central lesson of this lecture is that **duality is not a special trick for one particular gauge group**: it is a pervasive structural feature of strongly coupled $\mathcal{N} = 1$ gauge theories. The dualised Standard Model sits within this landscape, drawing on the specific features of $\mathrm{SU}(N)$ duality with fundamental (and potentially adjoint) matter.

Preview: Lecture 10

In Lecture 10, we will connect the abstract meson matrix $M^i{}_j$ of Seiberg duality to the physical meson spectrum of QCD. The question: if the SM quarks are "dual quarks" in a magnetic description, what are the corresponding mesons $M = Q\widetilde{Q}$, and how do they relate to the observed π, K, D, B mesons? This requires confronting the transition from the SUSY duality landscape to the non-supersymmetric real world — the central challenge of the dualised SM programme.

9 Summary

In this lecture we have:

1. Presented the **Kutasov–Schwimmer duality** for $\mathrm{SU}(N_c)$ with N_f flavours and an adjoint Φ with $W = \mathrm{Tr}(\Phi^{k+1})/(k+1)$: the dual is $\mathrm{SU}(kN_f - N_c)$ with k meson species $M_j = Q\Phi^j\widetilde{Q}$.
2. Verified the crucial $k = 1$ **limit**: integrating out the massive adjoint recovers standard Seiberg duality for $\mathrm{SU}(N_f - N_c)$.
3. Established the $\mathrm{SO}(N_c)$ **duality** with dual gauge group $\mathrm{SO}(N_f - N_c + 4)$ and verified the $\mathrm{SU}(N_f)^2\,\mathrm{U}(1)_R$ anomaly matching.
4. Established the $\mathrm{Sp}(N_c)$ **duality** with dual gauge group $\mathrm{Sp}(N_f - N_c - 2)$ and verified the $\mathrm{SU}(2N_f)^2\,\mathrm{U}(1)_R$ anomaly matching.
5. Classified **s-confining theories** following Csáki–Schmaltz–Skiba, identifying them as the boundary cases where the magnetic gauge group rank vanishes.
6. Described **product group dualities** (quiver mutation) and the **Klebanov–Strassler duality cascade**.
7. Synthesised the landscape into a **master comparison table** (Table 2) and identified the Kutasov–Schwimmer duality with adjoint matter as most relevant to the dualised SM programme.

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