Lecture 11: The Koide Relation and the Dualised Standard Model

The Dualised Standard Model — Part III Lecture Notes

Lent Term 2027

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1 Introduction and motivation

In Lectures 7–8, we constructed the dualised Standard Model: a magnetic dual of SU(3) QCD with $N_f = 6$ flavours, where 18 Higgs doublets emerge from the mes-Higgs field Φ_H and the fermion mass hierarchy arises from hierarchical VEVs among these doublets. The Yukawa coupling y is *universal* (a prediction of the duality), and the effective Yukawa matrix is

$$y_{ij}^{\mathrm{eff}} = y\,\frac{\langle H_{ij}\rangle}{v}\,, \tag{1}$$

where H_{ij} are the Higgs doublet components and v is the electroweak VEV.

In this lecture we ask: *does the dualised SM predict or constrain the fermion mass eigenvalues?* We establish one solid result and explore two conjectures:

- **(R)** The Koide relation $Q = 3/2$ is algebraically equivalent to a **vacuum alignment condition** on the Higgs doublet VEV ratios—a reformulation that introduces no new parameters and follows from the universal Yukawa structure of Lecture 8.
- **(C1)** The *numerical value* $Q = 3/2$ coincides with the SU(5) Casimir ratio $C_2(\mathbf{10})/C_2(\mathbf{5}) = 2(N-2)/(N-1)$, which uniquely selects $N = 5$. Whether the Higgs potential enforces this value is *conjectural.*

The Koide angle $\delta = 2/9$ remains an empirically determined parameter; its origin is an open problem (see §4).

2 The Koide relation as a VEV condition

This section contains the lecture's main result. It requires only the universal Yukawa coupling of Lecture 8 and standard algebra.

2.1 Setup: from VEVs to masses

From Lecture 8, the effective Yukawa coupling for the k-th generation in a given charge sector is

$$y_k^{\mathrm{eff}} = y\,\frac{\langle H_k\rangle}{v} = y\,\frac{\mu_{0k}^2}{M_k^2}\,, \tag{2}$$

where μ_{0k}^2 is the mixing mass between the SM Higgs H_0 and the k-th dormant doublet, and M_k is the diagonal mass of the k-th doublet. The fermion mass is then

$$m_k = y_k^{\mathrm{eff}}\,\frac{v}{\sqrt{2}} = y\,\frac{v}{\sqrt{2}}\,\frac{\mu_{0k}^2}{M_k^2}\,. \tag{3}$$

Define the *mixing ratio*:

$$x_k \equiv \frac{\mu_{0k}}{M_k}\,. \tag{4}$$

Then $m_k = m_0\,x_k^2$ where $m_0 = y\,v/\sqrt{2}$ is the universal scale, and

$$\sqrt{m_k} = \sqrt{m_0}\,x_k\,. \tag{5}$$

2.2 The Koide condition on mixing ratios

The Koide parameter for three fermion masses m_1, m_2, m_3 is

$$Q = \frac{(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2}{m_1 + m_2 + m_3}\,. \tag{6}$$

Substituting (5):

$$\boxed{Q = \frac{(x_1 + x_2 + x_3)^2}{x_1^2 + x_2^2 + x_3^2}\,.} \tag{7}$$

The universal scale m_0 cancels. The Koide condition $Q = 3/2$ is a **pure constraint on the mixing ratios** $x_k = \mu_{0k}/M_k$. This is the central observation of this lecture: in any model with a universal Yukawa coupling and multiple Higgs doublets, the Koide relation reduces to a condition on the *Higgs sector* alone.

Important

The Koide condition is independent of the universal Yukawa coupling y and the Higgs VEV v. It constrains only the *ratios* μ_{0k}/M_k of the Higgs sector parameters. This reformulation is exact. What follows in Sections 3–4 is conjectural.

2.3 Decomposition into common and traceless parts

Write $x_k = z_0 + z_k$ with

$$z_0 = \frac{x_1 + x_2 + x_3}{3}\,, \qquad z_1 + z_2 + z_3 = 0\,. \tag{8}$$

Then:

$$Q = \frac{9\,z_0^2}{3\,z_0^2 + \sum_k z_k^2}\,. \tag{9}$$

The condition $Q = 3/2$ requires

$$z_0^2 = \frac{1}{3}\sum_{k=1}^{3} z_k^2\,. \tag{10}$$

This is the **trace condition**: the squared common piece equals one-third of the sum of squared traceless pieces. It is identical to Koide's original 1983 constraint $\mathrm{Tr}[U^2] = \mathrm{Tr}[V^2]$ [4].

2.4 What the potential must do

The effective potential for the dormant Higgs doublets (Lecture 8) is

$$V_{\text{eff}} = \sum_k M_k^2\,|H_k|^2 + \sum_k \mu_{0k}^2\,(H_0^\dagger H_k + \text{h.c.}) + \text{quartic terms}\,. \tag{11}$$

Minimising with respect to $\langle H_k \rangle$ gives $x_k = \mu_{0k}/M_k$ as before. The Koide condition (10) is then a **constraint on the potential minimum**: the quartic terms must be structured so that the minimum satisfies $z_0^2 = \frac{1}{3}\sum z_k^2$.

We do *not* derive that the potential has this property. The question of whether the emergent quasi-SUSY of the magnetic theory constrains the quartic couplings sufficiently is the central open problem of this programme.

3 Conjecture C1: $Q = 3/2$ and the SU(5) Casimir ratio

3.1 The numerical coincidence

For $\mathrm{SU}(N)$, the quadratic Casimir operators of the fundamental representation ($\square$) and the antisymmetric two-tensor ($\wedge^2\square$) satisfy:

$$\frac{C_2(\wedge^2\square)}{C_2(\square)} = \frac{2(N-2)}{N-1}\,. \tag{12}$$

This follows from $C_2(\wedge^p N) = p(N-p)(N+1)/(2N)$. Setting the ratio to 3/2:

$$\frac{2(N-2)}{N-1} = \frac{3}{2} \qquad \Longrightarrow \qquad N = 5\,. \tag{13}$$

This is the unique positive integer solution.

N	$C_2(\wedge^2 N)/C_2(N)$	
3	1	(degenerate)
4	4/3	
5	**3/2**	$= Q_{\mathrm{Koide}}$
6	8/5	
∞	2	

3.2 Why $N = 5$?

In the dualised SM of Lectures 7–8, the flavour group is SU(6) (from $N_f = 6$). The value $N = 5$ does not correspond to SU(6) but to $\mathrm{SU}(5) \subset \mathrm{SU}(6)$. The five lightest quarks (u, c, d, s, b) form the fundamental of an SU(5) flavour subgroup, with the top quark as the sixth flavour playing a special role.

This SU(5) flavour structure is precisely the *sBootstrap* framework [2,3], where the top is called the "elephant" and the five light quarks are "turtles." The decomposition

$$\mathrm{SU}(6)_{\mathrm{flavour}} \supset \mathrm{SU}(5)_{\mathrm{turtle}} \times \mathrm{U}(1)_{\mathrm{top}} \tag{14}$$

is natural from the mass hierarchy alone ($m_t \gg m_b$), but the duality does not *derive* it: nothing in the CS framework selects SU(5) over SU(6) as the relevant Casimir group.

Important

The coincidence $C_2(\mathbf{10})/C_2(\mathbf{5}) = 3/2 = Q_{\mathrm{Koide}}$ is a *mathematical fact* about SU(5). Whether the Higgs potential minimum is determined by this Casimir ratio is a *conjecture without proof*. No derivation exists connecting the one-loop potential of the 18-doublet sector to the eigenvalue condition (10).

3.3 Dynkin index integers

If one takes the SU(5) Casimir connection seriously, the Dynkin index ratios provide further numerical coincidences:

$$\frac{T(\mathbf{15})}{T(\mathbf{5})} = N + 2 = 7\,, \tag{15}$$

$$\frac{T(\mathbf{24})}{T(\mathbf{5})} = 2N = 10\,, \tag{16}$$

$$\frac{T(\mathbf{10})}{T(\mathbf{5})} = N - 2 = 3 = N_c\,. \tag{17}$$

The integers 7 and 10 appear in the empirical "Z-ladder" mass relations, and $N - 2 = 3$ is the number of QCD colours. These coincidences are suggestive but do not constitute a derivation.

4 The Koide angle δ: an open problem

The Koide condition (10) constrains the sum $\sum z_k^2$ relative to z_0^2, but does not fix the *individual* z_k values. The mass spectrum is parametrised by a single angle δ in the Z_3 representation:

$$x_k = z_0\left(1 + \sqrt{2}\,\cos\left(\frac{2\pi k}{3} + \delta\right)\right), \qquad k = 1, 2, 3\,. \tag{18}$$

The value $\delta = 2/9$ rad reproduces the charged lepton masses to 10^{-5} precision [5].

Neither the dualised SM nor the sBootstrap currently derives this value. Within the multi-Higgs framework, δ is determined by the relative magnitudes of the mixing masses μ_{0k} and the diagonal masses M_k. These originate from Planck-scale operators in the electric theory (Lecture 8), and their structure is not sufficiently constrained by the duality to fix δ.

Deriving δ from a dynamical principle—whether from modular flavour symmetries, conformal fixed-point dynamics, or some other mechanism—remains the central open problem of the Koide programme.

5 What $Q = 3/2$ determines and what it leaves free

The Koide condition $Q = 3/2$ is one constraint on three eigenvalues, leaving a two-parameter family: the overall scale m_0 and the angle δ. The scale m_0 sets the heaviest mass; δ sets the ratios.

With the empirical $\delta = 2/9$, the Koide formula predicts

$$\frac{m_e}{m_\mu} = 0.004836\,, \qquad \text{(measured: 0.004836)} \tag{19}$$

$$\frac{m_\mu}{m_\tau} = 0.05946\,, \qquad \text{(measured: 0.05946)} \tag{20}$$

Both agree to better than 0.002%. But this precision is the content of the original Koide formula [4]; it is not a prediction of the dualised SM, which so far provides $Q = 3/2$ as a VEV condition but does not derive δ.

6 The chain angle and colour

The Koide chain extends the lepton relation to quarks: $(0, s, c) \to (-s, c, b) \to (c, b, t)$, with each link satisfying $Q \approx 3/2$ and the chain angle $\delta_{scb} \approx 3\,\delta_\ell$.

One interpretation of the factor 3: the quark sector carries $SU(3)_c$ colour, which multiplies the vacuum alignment angle:

$$\delta_{\text{chain}} = N_c\,\delta_\ell = 3 \times \frac{2}{9} = \frac{2}{3}\ \text{rad} = 38.2^\circ\,. \tag{21}$$

The measured value $\delta_{scb} \approx 37.2^\circ$ agrees to 2.7%. No calculation within the dualised SM or the sBootstrap currently produces the factor N_c from dynamics. Whether the 2.7% agreement is evidence for the colour interpretation or a numerical coincidence cannot be decided without a derivation of the vacuum alignment in the coloured sector.

7 The inverse Koide from the seesaw

7.1 Q is preserved under inversion

The Seiberg seesaw (Lecture 3) gives the meson VEV $\langle M_{kk}\rangle = \Lambda^2/m_k$. Since Q is homogeneous of degree zero:

$$Q\left(\frac{\Lambda^2}{m_1}, \frac{\Lambda^2}{m_2}, \frac{\Lambda^2}{m_3}\right) = Q\left(\frac{1}{m_1}, \frac{1}{m_2}, \frac{1}{m_3}\right). \tag{22}$$

This is a theorem, not a conjecture.

7.2 Empirical status

With PDG 2024 masses [6]:

$$Q(m_e, m_\mu, m_\tau) = 1.50000\,, \qquad Q(1/m_d, 1/m_s, 1/m_b) = 1.505\,. \tag{23}$$

The 0.3% agreement is consistent with the seesaw relating the two Q values, but the *individual mass ratios* do not match through a simple seesaw $m_\ell = c/m_q$ (see §7.3).

7.3 The hierarchy problem of the seesaw

The seesaw $m_\ell \propto 1/m_q$ gives an *inverted* hierarchy: the lightest quark maps to the heaviest lepton. With the standard generation pairing ($d \leftrightarrow e,\ s \leftrightarrow \mu,\ b \leftrightarrow \tau$): $m_e/m_\mu = m_s/m_d = 19.9$, while the actual ratio is 0.0048—a factor of 4000 discrepancy.

With the reversed pairing ($d \leftrightarrow \tau,\ s \leftrightarrow \mu,\ b \leftrightarrow e$): m_τ is recovered by construction, m_μ is 15% off, and m_e is a factor 4 wrong.

The seesaw preserves Q (that is exact) but does *not* reproduce the individual lepton masses. The quark–lepton mass mapping is more complex than a simple inversion.

8 Connection to the sBootstrap

The sBootstrap framework [2,3] identifies sfermions with known mesons and diquarks via SU(5) flavour symmetry. The closure conditions [3]

$$2N = k_u\,k_d\,, \quad 2N = \frac{k_d(k_d+1)}{2}\,, \quad 4N = k_u^2 + k_d^2 - 1 \tag{24}$$

uniquely give $N = 3$ generations with $k_u = 2$, $k_d = 3$.

In the dualised SM, the ISS rank condition (Lecture 6) provides a *dynamical* derivation of the same result: $\mathrm{rank}(\tilde{\varphi}\varphi) = N_f - N_c = 2$ gives $k_u = 2$ massless directions and $k_d = N_c = 3$ massive directions.

The sBootstrap mass formula $m_k = (z_0 + z_k)^2 K_\Omega$ is algebraically equivalent to the Koide condition (10), as shown in [3]. The dualised SM provides the *physical context* for this formula: $z_0 + z_k$ is the mixing ratio x_k of the Higgs sector, and K_Ω is the universal scale m_0.

9 Summary and honest assessment

Result	Origin	Status
Koide = VEV alignment	Universal Yukawa + algebra	**Derived**
$k_u = 2$, $k_d = 3$	ISS rank condition	**Derived**
$Q(1/d, 1/s, 1/b) \approx 3/2$	Seesaw preserves Q	**Theorem**
$N_{\text{gen}} = 3$	Closure = duality constraint	**Two derivations**
$Q = C_2(\mathbf{10})/C_2(\mathbf{5})$	SU(5) Casimir	Observation
$\delta_{scb} = N_c\, \delta_\ell$	Colour multiplicity	2.7% match
$\delta = 2/9$	—	Open problem
VEV alignment mechanism	Quartic structure	*Not derived*
Quark–lepton mapping	Seesaw details	*Fails quantitatively*
Z-ladder formula	—	*Not derived*

What is established. The universal Yukawa coupling of the dualised SM translates the Koide relation into a VEV alignment condition on the multi-Higgs sector. The ISS rank condition derives the turtle/elephant decomposition. The seesaw preserves Q as a mathematical identity.

What is observed but not derived. The numerical coincidence $Q = C_2(\mathbf{10})/C_2(\mathbf{5}) = 3/2$ is an exact group-theoretic fact that matches the Koide value. It has no known dynamical origin within the dualised SM. The angle $\delta = 2/9$ has no known derivation at all.

What remains open. The quartic coupling structure of the 18-Higgs potential must be shown to have a minimum at the Koide point. The quark–lepton mass mapping through the seesaw preserves Q but not the individual mass ratios.

10 Exercises

Exercise 10.1. Verify that $C_2(\wedge^2\mathbf{N})/C_2(\mathbf{N}) = 2(N-2)/(N-1)$ using the general formula $C_2(\wedge^p N) = p(N-p)(N+1)/(2N)$. For which values of N is this ratio an integer?

Exercise 10.2. Compute Q for the three charged lepton pole masses ($m_e = 0.511$ MeV, $m_\mu = 105.658$ MeV, $m_\tau = 1776.93$ MeV). Compare with $Q(1/m_d,\ 1/m_s,\ 1/m_b)$ using the PDG values $m_d = 4.70$, $m_s = 93.5$, $m_b = 4183$ MeV.

Exercise 10.3. In the Z_3 parametrisation (18), show that $Q = 3/2$ is satisfied for *any* value of δ. Then verify that $\delta = 2/9$ reproduces the lepton mass ratios m_e/m_μ and m_μ/m_τ to better than 0.01%.

Exercise 10.4. The sBootstrap closure conditions give $k_u = 2$, $k_d = 3$ uniquely. The ISS rank condition gives $k_u = N_f - N_c$, $k_d = N_c$ for SQCD with N_f flavours and N_c colours. For which values of (N_c, N_f) do the two derivations agree?

Exercise 10.5 (Challenge)**.** Write the most general quartic potential for three complex scalar doublets H_1, H_2, H_3 with an S_3 permutation symmetry softly broken by the mixing masses μ_{0k}. Does the minimum satisfy (10) for any choice of quartic couplings? What additional conditions (if any) are needed?

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