Meson Mass Sum Rules in the Dual 6×6 Flavour Matrix Framework

Dualised Standard Model Research Programme

March 2026

Abstract

The dualised Standard Model identifies the $N_f \times N_f$ meson flavour matrix $\mathcal{M}_{ij} = q_i \bar{q}_j$ with the elementary mes-Higgs field of the magnetic dual theory. We systematically derive the algebraic mass sum rules that follow from this 6×6 matrix structure, combined with chiral perturbation theory in the light sector, heavy quark symmetry in the heavy-light sector, and the scalar democracy VEV hierarchy. We identify six classes of sum rules: (I) the Gell-Mann–Oakes–Renner relation and its kaon extension; (II) the Gell-Mann–Okubo mass formula with η–η' mixing; (III) heavy quark symmetry flavour splittings; (IV) the B_c mass sum rule $m_{B_c} \approx (m_{J/\psi} + m_\Upsilon)/2$; (V) scalar democracy geometric hierarchy relations; and (VI) heavy-light analogues of the Gell-Mann–Okubo formula. All sum rules are tested against PDG 2024 data. We find that sum rules I–IV and VI are consequences of standard QCD symmetries (chiral symmetry, heavy quark symmetry, and the additive quark model) and are well satisfied; the dual meson language makes the flavour structure manifest but does not yield numerically new predictions beyond ChPT and HQET. Sum rule V—the scalar democracy geometric hierarchy $m_s/m_t \approx (m_b/m_t)^2$—is the only genuinely framework-specific test and holds at the $\sim 8\%$ level.

Contents

1 Introduction

In the dualised Standard Model (DSM) [1,2], the QCD sector is described by an SU(N) gauge theory that admits a non-supersymmetric Seiberg-like dual [3]. The magnetic theory contains an elementary gauge-singlet field—the *mes-Higgs*—which is an $N_f \times N_f$ matrix transforming as $(N_f, \bar{N}_f)$ under the chiral flavour symmetry $\mathrm{SU}(N_f)_L \times \mathrm{SU}(N_f)_R$. For $N_f = 6$ quark flavours (u, d, s, c, b, t), this is a 6×6 matrix.

The scalar democracy framework [4] posits a single universal Yukawa coupling y and a hierarchical pattern of vacuum expectation values (VEVs) for the 18 Higgs doublets encoded in the mes-Higgs matrix. The fermion mass hierarchy $m_t \gg m_b \gg m_c \gg m_s \gg m_d \gg m_u$ arises entirely from the VEV hierarchy, not from a hierarchy in the Yukawa couplings.

The meson flavour matrix $\mathcal{M}_{ij} = q_i \bar{q}_j$ is the same object whether viewed from the electric theory (where it is a composite operator) or the magnetic theory (where, *if the duality holds*, it is an elementary field). The question we address in this paper is: *what algebraic sum rules among meson masses follow from the matrix structure of $\mathcal{M}_{ij}$ and the scalar democracy VEV hierarchy?*

The answer involves three regimes, each governed by a different effective theory:

- **Light sector** (u, d, s): governed by chiral perturbation theory [5, 6], which gives the GMOR and Gell-Mann–Okubo relations;
- **Heavy-light sector** (c or b with u, d, s): governed by heavy quark effective theory [11], which gives flavour-independent mass splittings;
- **Heavy-heavy sector** ($c\bar{c}$, $b\bar{b}$, $c\bar{b}$): governed by non-relativistic QCD, where the additive quark model provides the leading-order mass formula.

The top quark decays before hadronising ($\Gamma_t \gg \Lambda_{\mathrm{QCD}}$), so the 6th row and column of the meson matrix are formal entries only.

We present each sum rule, derive it from the appropriate effective theory, identify which constraints are standard QCD results and which encode additional structure from the dual meson matrix, and test all predictions against PDG 2024 data [17].

2 The meson mass matrix

2.1 Definition and structure

The 6×6 meson flavour matrix has entries

$$\mathcal{M}_{ij} \sim q_i \bar{q}_j \,, \qquad i, j \in \{u, d, s, c, b, t\} \,, \tag{1}$$

where q_i denotes a quark of flavour i and $\bar{q}_j$ an antiquark of flavour j. Each entry corresponds to a meson (or formal meson for the top quark) with well-defined quantum numbers determined by the quark content.

In the pseudoscalar sector, the physical 5×5 meson mass matrix (excluding the top row and column) is:

$q_i \backslash \bar{q}_j$	$\bar{u}$	$\bar{d}$	$\bar{s}$	$\bar{c}$	$\bar{b}$
u	π^0	π^+	K^+	$\bar{D}^0$	B^+
d	π^-	π^0	K^0	D^-	B^0
s	K^-	$\bar{K}^0$	η_s	D_s^-	B_s^0
c	D^0	D^+	D_s^+	η_c	B_c^+
b	B^-	$\bar{B}^0$	$\bar{B}_s^0$	B_c^-	η_b

(2)

The diagonal entries involve flavour mixing (π^0–η–η' in the light sector, and the quarkonium states η_c, η_b in the heavy sector).

2.2 Mass generation mechanisms by sector

The pseudoscalar meson masses arise from different mechanisms depending on the quark masses involved:

(i) **Light–light** (u, d, s): pseudo-Goldstone bosons of chiral symmetry breaking. Masses given by leading-order ChPT:
$$m^2_{q_i\bar{q}_j} = B_0\,(m_i + m_j)\,, \tag{3}$$
where $B_0 = -\langle\bar{q}\,q\rangle/f_\pi^2$ and $\langle\bar{q}\,q\rangle$ is the chiral condensate.

(ii) **Heavy–light** ($Q = c, b$ with $q = u, d, s$): mass dominated by the heavy quark. In HQET:
$$m_{Qq} = m_Q + \bar{\Lambda}_q + \frac{\lambda_1 + d_Q\,\lambda_2}{2m_Q} + \mathcal{O}\left(\frac{1}{m_Q^2}\right)\,, \tag{4}$$
where $\bar{\Lambda}_q \sim 0.4$–0.5 GeV is the light-quark binding energy and $\lambda_{1,2}$ are HQET matrix elements.

(iii) **Heavy–heavy** ($c\bar{c}$, $b\bar{b}$, $c\bar{b}$): non-relativistic bound states. To leading order in the additive quark model:
$$m_{Q_1\bar{Q}_2} \approx m_{Q_1} + m_{Q_2} + \Delta(m_{Q_1}, m_{Q_2})\,, \tag{5}$$
where Δ is the binding energy.

2.3 The scalar democracy connection

In the dualised SM, the quark masses arise from the universal Yukawa coupling y and the VEV hierarchy [1]:
$$m_i = y\,\frac{\langle H_{ii}\rangle}{v}\,v = y\,\langle H_{ii}\rangle\,, \qquad \frac{m_i}{m_j} = \frac{\langle H_{ii}\rangle}{\langle H_{jj}\rangle}\,. \tag{6}$$
The VEV hierarchy is organised into tiers [1]: tier 0 (m_t), tier 1 (m_b, m_c), tier 2 (m_s), with each tier suppressed by a factor $\epsilon \equiv \mu^2/M^2 \ll 1$ relative to the preceding one, where $\mu^2 \sim \Lambda_D^4/M_{\rm Pl}^2$. This geometric structure implies:
$$\frac{m_s}{m_t} \sim \epsilon^2\,, \qquad \frac{m_b}{m_t} \sim \epsilon\,, \qquad \frac{m_c}{m_t} \sim \epsilon\,, \tag{7}$$
which we test in Section 7.

3 Sum Rule I: GMOR and kaon mass relation

3.1 The GMOR relation

The Gell-Mann–Oakes–Renner (GMOR) relation [7] is the foundational sum rule of chiral symmetry breaking. From the leading-order ChPT mass formula (3):
$$\boxed{m_\pi^2\,f_\pi^2 = -(m_u + m_d)\,\langle\bar{q}\,q\rangle\,,} \tag{8}$$
with the identification $B_0 = -\langle\bar{q}\,q\rangle/f_\pi^2$.

Numerical check. Using the PDG 2024 inputs $f_\pi = 92.1$ MeV, $m_u + m_d = 6.83$ MeV ($\overline{\mathrm{MS}}$, $\mu = 2$ GeV), and $\langle \bar{q}\, q\rangle \approx -(270\ \mathrm{MeV})^3$:

$$m_\pi^2 = \frac{(m_u + m_d)\, |\langle \bar{q}\, q\rangle|}{f_\pi^2} = \frac{6.83 \times 10^{-3} \times (0.270)^3}{(0.0921)^2}\ \mathrm{GeV}^2 \approx 0.0159\ \mathrm{GeV}^2\,, \tag{9}$$

giving $m_\pi \approx 126$ MeV, compared to the experimental value $m_{\pi^\pm} = 139.6$ MeV. The $\sim 10\%$ discrepancy is within the $\sim 15\%$ uncertainty of the condensate value [18].

Status. This is a well-established QCD sum rule. The meson matrix structure adds nothing new: GMOR follows from the $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ chiral symmetry of QCD alone.

3.2 Kaon mass extension

The same leading-order ChPT formula gives the kaon mass:

$$m_{K^+}^2 = B_0\,(m_u + m_s)\,, \qquad m_{K^0}^2 = B_0\,(m_d + m_s)\,. \tag{10}$$

Combined with the pion mass formula, we obtain the **Weinberg mass ratio** [10]:

$$\frac{m_{K^+}^2}{m_\pi^2} = \frac{m_u + m_s}{m_u + m_d}\,. \tag{11}$$

Numerically: LHS $= (493.7)^2/(139.6)^2 = 12.51$; RHS $= (2.16+93.4)/(2.16+4.67) = 13.99$. The $\sim 12\%$ discrepancy is accounted for by next-to-leading-order (NLO) ChPT corrections, which are enhanced by $\ln(m_s/m_u)$.

Status. Standard ChPT. No new content from the meson matrix.

4 Sum Rule II: Gell-Mann–Okubo and η–η' mixing

4.1 The Gell-Mann–Okubo formula

For the pseudoscalar meson octet, the Gell-Mann–Okubo (GMO) mass formula [8, 9] follows from first-order SU(3) flavour symmetry breaking:

$$\boxed{4\, m_K^2 = 3\, m_{\eta_8}^2 + m_\pi^2\,,} \tag{12}$$

where m_{η_8} is the mass of the octet η state (before mixing with the singlet).

Numerical check. Using PDG values and replacing m_{η_8} by the physical m_η:

$$4 \times (493.7)^2 = 975{,}160\ \mathrm{MeV}^2\,, \qquad 3 \times (547.8)^2 + (139.6)^2 = 919{,}740\ \mathrm{MeV}^2\,. \tag{13}$$

The 5.7% discrepancy is due to η–η' mixing: the physical η is not a pure octet state.

4.2 The η–η' mass matrix

The η and η' are mixtures of the octet state η_8 and the singlet state η_0. The mass matrix in the (η_8, η_0) basis is:

$$\mathcal{M}^2_{\eta\eta'} = \begin{pmatrix} m^2_{88} & m^2_{80} \\ m^2_{80} & m^2_{00} \end{pmatrix} , \tag{14}$$

where $m^2_{88} = (4m^2_K - m^2_\pi)/3$ from the GMO relation and $m^2_{00} = (2m^2_K + m^2_\pi)/3 + m^2_0$ includes the U(1)$_A$ anomaly contribution m^2_0. The trace gives:

$$m^2_\eta + m^2_{\eta'} = m^2_{88} + m^2_{00} , \tag{15}$$

from which we extract the anomaly mass:

$$m^2_0 = m^2_\eta + m^2_{\eta'} - \frac{4m^2_K - m^2_\pi}{3} - \frac{2m^2_K + m^2_\pi}{3} = m^2_\eta + m^2_{\eta'} - 2m^2_K . \tag{16}$$

Numerically: $m^2_0 = (547.8)^2 + (957.8)^2 - 2 \times (493.7)^2 = 730{,}000$ MeV^2, giving $m_0 \approx 854$ MeV. This is consistent with the Witten–Veneziano formula [15, 16]:

$$m^2_0 = \frac{2N_f\, \chi_{\text{top}}}{f^2_\pi} , \tag{17}$$

where $\chi_{\text{top}} \approx (180\ \text{MeV})^4$ is the topological susceptibility from lattice QCD [18]. This gives $m_0 \approx 862$ MeV, in excellent agreement.

Status. Standard QCD. The GMO formula and its η–η' extension are well-known consequences of SU(3) flavour symmetry. The meson matrix makes the 2×2 mixing structure in the diagonal sector explicit, but adds no new sum rule.

5 Sum Rule III: Heavy quark symmetry flavour splittings

5.1 The HQS mass formula

In the heavy-light sector, heavy quark effective theory [11] gives the meson mass as an expansion in $1/m_Q$:

$$m(Q\bar{q}) = m_Q + \bar{\Lambda}_q + \frac{\lambda_1}{2m_Q} + \frac{d_Q\, \lambda_2}{2m_Q} + \mathcal{O}\left(\frac{1}{m^2_Q}\right) , \tag{18}$$

where $\bar{\Lambda}_q$ is the binding energy that depends on the light quark flavour but **not** on the heavy quark flavour (to leading order in $1/m_Q$), and $d_Q = 3$ for pseudoscalars.

5.2 Flavour-independent mass splittings

Proposition 1 (HQS flavour splitting sum rule)**.** *For heavy mesons containing the same heavy quark Q and different light quarks q_1, q_2:*

$$\boxed{m(Q\bar{q}_1) - m(Q\bar{q}_2) = \bar{\Lambda}_{q_1} - \bar{\Lambda}_{q_2} + \mathcal{O}\left(\frac{\Lambda^2_{\text{QCD}}}{m_Q}\right) ,} \tag{19}$$

*which is **independent of the heavy quark identity** to leading order.*

This gives the concrete prediction:

$$\Delta^{(c)}_s \equiv m_{D_s} - m_{D^0} = \Delta^{(b)}_s \equiv m_{B_s} - m_{B^0} + \mathcal{O}\left(\frac{\Lambda^2_{\text{QCD}}}{m_c} - \frac{\Lambda^2_{\text{QCD}}}{m_b}\right) . \tag{20}$$

Numerical check.

$$\Delta_s^{(c)} = m_{D_s} - m_{D^0} = 1968.3 - 1864.8 = 103.5 \text{ MeV}\,, \tag{21}$$
$$\Delta_s^{(b)} = m_{B_s} - m_{B^0} = 5366.9 - 5279.7 = 87.2 \text{ MeV}\,. \tag{22}$$

The difference $\Delta_s^{(c)} - \Delta_s^{(b)} = 16.3$ MeV reflects the $1/m_Q$ corrections.

Corollary 1 ($1/m_Q$ correction estimate)**.** *From $\Delta_s^{(c)} - \Delta_s^{(b)} = a\,(1/m_c - 1/m_b)$, we extract $a \approx 30$ MeV $\cdot$ GeV, which is $\mathcal{O}(\Lambda^2_{\text{QCD}})$ as expected ($\Lambda_{\text{QCD}} \sim 300$ MeV gives $\Lambda^2_{\text{QCD}} \approx 0.09$ GeV^2, and the discrepancy of one third is accounted for by the $\mathcal{O}(1)$ Wilson coefficient in the $1/m_Q$ expansion).*

5.3 Heavy-quark mass difference sum rule

A second HQS sum rule relates the same-light-quark mass differences across different heavy quarks:

$$m(B_q) - m(D_q) = m_b - m_c + \mathcal{O}\left(\frac{\Lambda^2_{\text{QCD}}}{m_c}\right)\,, \tag{23}$$

independently of the light quark q.

Numerical check.

$$\begin{aligned} m_{B^+} - m_{D^+} &= 5279.3 - 1869.6 &&= 3409.7 \text{ MeV}\,,\\ m_{B^0} - m_{D^0} &= 5279.7 - 1864.8 &&= 3414.9 \text{ MeV}\,,\\ m_{B_s} - m_{D_s} &= 5366.9 - 1968.3 &&= 3398.6 \text{ MeV}\,. \end{aligned} \tag{24}$$

These differ by at most 16 MeV, confirming that the difference is independent of the light quark to $< 0.5\%$. The common value ~ 3408 MeV should equal $m_b - m_c$; comparing with the pole mass difference $m_b^{\text{pole}} - m_c^{\text{pole}} = 4180 - 1270 = 2910$ MeV reveals a ~ 500 MeV offset, which is the $\mathcal{O}(\Lambda_{\text{QCD}})$ contribution from the binding energy difference $\bar{\Lambda} \sim 0.5$ GeV in Eq. (18). (The pole masses are scheme-dependent; in the kinetic scheme the agreement is better.)

Status. These are standard HQET results, well-known since the 1990s [11, 12]. The meson matrix structure makes the heavy quark flavour independence manifest (it corresponds to invariance under independent row and column permutations of the heavy-quark submatrix), but does not produce a new quantitative prediction.

6 Sum Rule IV: The B_c mass sum rule

6.1 Derivation from the additive quark model

For the heavy quarkonium and B_c system, the additive quark model gives:

$$m_{J/\psi} = 2\hat{m}_c + \Delta_{c\bar{c}}\,, \qquad m_\Upsilon = 2\hat{m}_b + \Delta_{b\bar{b}}\,, \qquad m_{B_c} = \hat{m}_c + \hat{m}_b + \Delta_{c\bar{b}}\,, \tag{25}$$

where $\hat{m}_Q$ denotes the constituent quark mass and Δ is the binding energy correction. If the binding energies are equal ($\Delta_{c\bar{c}} \approx \Delta_{b\bar{b}} \approx \Delta_{c\bar{b}}$), then:

$$\boxed{m_{B_c} = \frac{m_{J/\psi} + m_\Upsilon}{2}\,.} \tag{26}$$

This "equal binding" assumption is crude—the Coulomb-plus-linear potential gives binding energies that depend on the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$. Nevertheless, the logarithmic dependence of the Coulombic binding energy on the reduced mass, $\Delta \sim -\alpha_s^2 \mu/2$, is weak for the $c\bar{b}$ system where $\mu_{c\bar{b}} \approx m_c m_b/(m_c + m_b)$ lies between $\mu_{c\bar{c}} = m_c/2$ and $\mu_{b\bar{b}} = m_b/2$.

Numerical check.

$$\frac{m_{J/\psi} + m_\Upsilon}{2} = \frac{3096.9 + 9460.3}{2} = 6278.6 \text{ MeV}\,, \tag{27}$$

compared to $m_{B_c} = 6274.9$ MeV. The agreement is **remarkable**: the discrepancy is only **3.7 MeV**, or 0.06%.

6.2 Connection to the meson matrix

In the 6×6 meson matrix, the B_c occupies the off-diagonal entry $\mathcal{M}_{45}$ (charm row, bottom column). The quarkonium states $J/\psi = \mathcal{M}_{44}$ and $\Upsilon = \mathcal{M}_{55}$ are diagonal entries. The sum rule (26) relates the off-diagonal entry to the average of the adjacent diagonal entries.

This is precisely the structure of a matrix whose entries depend *additively* on the row and column indices. If $m_{\mathcal{M}_{ij}} = f(i) + g(j)$ for some functions f, g, then:

$$m_{\mathcal{M}_{ij}} + m_{\mathcal{M}_{kl}} = m_{\mathcal{M}_{il}} + m_{\mathcal{M}_{kj}}\,, \tag{28}$$

which is the "rectangular sum rule" for the meson mass matrix.

6.3 Nussinov–Wetzel inequality

The meson matrix structure also implies the Nussinov–Wetzel inequality [13]:

$$m_{B_c}^2 \leq \frac{m_{J/\psi}^2 + m_\Upsilon^2}{2}\,. \tag{29}$$

Numerically: RHS $= (9.591 + 89.50)/2 = 49.54$ GeV^2, $m_{B_c}^2 = 39.37$ GeV^2. The inequality is satisfied with a comfortable margin.

Status. The B_c sum rule (26) is known from the additive quark model and was used to predict m_{B_c} before its discovery [14]. The meson matrix gives it a *structural* interpretation (additivity of a matrix) but does not produce a sharper prediction. The Nussinov–Wetzel inequality follows from the Schwarz inequality applied to the quark propagator and pre-dates the duality framework.

7 Sum Rule V: Scalar democracy hierarchy predictions

This section contains the sum rules that are *specific to the scalar democracy framework* and not consequences of standard QCD symmetries alone.

7.1 Geometric VEV hierarchy

In the scalar democracy framework of the dualised SM [1,4], the Higgs VEVs are organised into tiers, with each successive tier suppressed by a factor $\epsilon = \mu^2/M^2$ relative to the preceding one

(Section 2). If both up-type and down-type quark masses follow the same geometric progression, we predict:

$$\boxed{\frac{m_s}{m_t} \approx \left(\frac{m_b}{m_t}\right)^2,} \tag{30}$$

i.e., $m_s/m_t \sim \epsilon^2$ if $m_b/m_t \sim \epsilon$.

Numerical check.

$$\frac{m_s}{m_t} = \frac{93.4 \times 10^{-3}}{172.69} = 5.41 \times 10^{-4}, \qquad \left(\frac{m_b}{m_t}\right)^2 = (0.02421)^2 = 5.86 \times 10^{-4}. \tag{31}$$

The ratio is $(m_s/m_t)/(m_b/m_t)^2 = 0.92$, confirming the geometric hierarchy at the 8% level.

7.2 Tier assignment consistency

Proposition 2 (Scalar democracy tier prediction)**.** *If the quark mass hierarchy is generated by a single expansion parameter* ϵ*, the tier structure predicts:*

$$\begin{array}{llll} \textit{Tier 0:} & m_t & \sim yv & \epsilon^0 \\ \textit{Tier 1:} & m_b,\, m_c & \sim \epsilon\, yv & \epsilon^1 \\ \textit{Tier 2:} & m_s & \sim \epsilon^2\, yv & \epsilon^2 \\ \textit{Tier 3:} & m_d,\, m_u & \sim \epsilon^3\, yv & \epsilon^3 \end{array} \tag{32}$$

Numerical check. We test this against PDG quark masses. Taking $\epsilon = m_b/m_t = 0.0242$ as the expansion parameter:

Quark	Mass (GeV)	m_q/m_t	Tier	ϵ^n	Ratio
t	172.69	1	0	1	1.00
b	4.18	0.0242	1	0.0242	1.00
c	1.27	0.00735	1	0.0242	0.30
s	0.0934	5.41×10^{-4}	2	5.86×10^{-4}	0.92
d	0.00467	2.70×10^{-5}	3	1.42×10^{-5}	1.91
u	0.00216	1.25×10^{-5}	3	1.42×10^{-5}	0.88

The "Ratio" column gives $m_q/(m_t\,\epsilon^n)$, which should be $\mathcal{O}(1)$ if the tier assignment is correct. The results are:

- **Tier 0** (t): exact by definition.
- **Tier 1** (b): exact by definition of ϵ.
- **Tier 1** (c): ratio $= 0.30$, indicating m_c is at the lower end of tier 1. The c–b mass ratio $m_c/m_b = 0.30$ is $\mathcal{O}(1)$, consistent.
- **Tier 2** (s): ratio $= 0.92$, **excellent** agreement.
- **Tier 3** (d): ratio $= 1.91$, consistent within $\mathcal{O}(1)$ uncertainties.
- **Tier 3** (u): ratio $= 0.88$, **good** agreement.

7.3 Cross-sector mass ratio predictions

The tier structure also predicts relations between up-type and down-type quark masses within the same generation:

$$\frac{m_s}{m_b} \sim \epsilon \sim \frac{m_b}{m_t}\,, \tag{33}$$

i.e., the second-generation down-type to third-generation down-type mass ratio should equal the expansion parameter ϵ.

Numerical check.

$$\frac{m_s}{m_b} = 0.0223\,, \qquad \frac{m_b}{m_t} = 0.0242\,. \tag{34}$$

The ratio $(m_s/m_b)/(m_b/m_t) = 0.92$, consistent at the 8% level.

Status. These are **new predictions of the scalar democracy framework**. They are not consequences of QCD symmetries (chiral or heavy quark) and cannot be derived from ChPT or HQET alone. The geometric hierarchy $m_q \sim \epsilon^n$ with a single parameter $\epsilon \approx 0.024$ provides a non-trivial $\mathcal{O}(10\%)$ description of the quark mass hierarchy spanning five orders of magnitude. However, the $\mathcal{O}(1)$ coefficients at each tier are free parameters (the Planck-scale operator coefficients c^{abcd}), so the framework is explanatory rather than predictive at the level of precise mass values.

8 Sum Rule VI: Heavy-light meson mass relations from the matrix

8.1 Column sum rule

The meson matrix structure, combined with the HQET mass formula (18), implies a "column sum rule" for heavy-light mesons. Consider the D mesons (c row) and B mesons (b row) with the same set of light antiquarks. From HQET:

$$m_{D_s} - m_{D^0} = \bar{\Lambda}_s - \bar{\Lambda}_u + \mathcal{O}(1/m_c)\,, \tag{35}$$

$$m_{B_s} - m_{B^0} = \bar{\Lambda}_s - \bar{\Lambda}_u + \mathcal{O}(1/m_b)\,. \tag{36}$$

Subtracting:

$$\boxed{(m_{D_s} - m_{D^0}) - (m_{B_s} - m_{B^0}) = \mathcal{O}\left(\frac{\Lambda^2_{\text{QCD}}}{m_c} - \frac{\Lambda^2_{\text{QCD}}}{m_b}\right)\,.} \tag{37}$$

This is the statement that the *light-quark mass splitting is universal across the c and b rows of the meson matrix*, up to $1/m_Q$ corrections.

Numerical check. $(m_{D_s} - m_{D^0}) - (m_{B_s} - m_{B^0}) = 103.5 - 87.2 = 16.3$ MeV, which is $\mathcal{O}(\Lambda^2_{\text{QCD}}/m_c) \sim (300\text{ MeV})^2/1.27\text{ GeV} \sim 70$ MeV. The observed value is smaller than this naive estimate, consistent with a suppressed Wilson coefficient.

8.2 Rectangular identity and its violations

For a perfectly additive mass matrix, Eq. (28) gives the rectangular identity:

$$m(q_i\bar{q}_j) + m(q_k\bar{q}_l) = m(q_i\bar{q}_l) + m(q_k\bar{q}_j)\,. \tag{38}$$

This is exact for the heavy-heavy sector (where masses are approximately additive in constituent quark masses) but breaks down when one or more quarks are light, because the binding dynamics depends non-additively on the quark masses.

Heavy-sector test. Taking $(i, j, k, l) = (c, c, b, b)$:

$$m_{J/\psi} + m_\Upsilon = 3096.9 + 9460.3 = 12{,}557.2 \text{ MeV}\,, \tag{39}$$

$$2\, m_{B_c} = 2 \times 6274.9 = 12{,}549.8 \text{ MeV}\,. \tag{40}$$

Discrepancy: 7.4 MeV, or 0.06%. The rectangular identity holds at remarkable precision.

Mixed-sector test. Taking $(i, j, k, l) = (u, s, c, b)$ so that the four mesons are $K^+ = u\bar{s}$, $B_c = c\bar{b}$, $B^+ = u\bar{b}$, $D_s^+ = c\bar{s}$:

$$m_K + m_{B_c} = 493.7 + 6274.9 = 6768.6 \text{ MeV}\,, \tag{41}$$

$$m_B + m_{D_s} = 5279.3 + 1968.3 = 7247.6 \text{ MeV}\,. \tag{42}$$

Discrepancy: 479 MeV, or 7%. The failure of the rectangular identity in the mixed sector reflects the breakdown of mass additivity for light quarks, where the binding energy is *not* a small correction to the quark mass.

8.3 The meson mass matrix eigenvalue sum rule

The squared masses of the 5×5 physical meson matrix (excluding the top row/column) can be organised into a symmetric matrix $\mathbf{M}^2$ with entries $(\mathbf{M}^2)_{ij} = m^2(q_i\bar{q}_j)$. This matrix has the structure:

$$\mathbf{M}^2 \approx \begin{pmatrix} m_\pi^2 & m_\pi^2 & m_K^2 & m_D^2 & m_B^2 \\ m_\pi^2 & m_\pi^2 & m_K^2 & m_D^2 & m_B^2 \\ m_K^2 & m_K^2 & m_\eta^2 & m_{D_s}^2 & m_{B_s}^2 \\ m_D^2 & m_D^2 & m_{D_s}^2 & m_{J/\psi}^2 & m_{B_c}^2 \\ m_B^2 & m_B^2 & m_{B_s}^2 & m_{B_c}^2 & m_\Upsilon^2 \end{pmatrix}, \tag{43}$$

where we have suppressed the isospin splitting $m_\pi^2 \approx m_K^2$ in the upper-left block for clarity. The trace of this matrix is:

$$\mathrm{Tr}(\mathbf{M}^2) = m_\pi^2 + m_\pi^2 + m_\eta^2 + m_{J/\psi}^2 + m_\Upsilon^2 \approx 99.4 \text{ GeV}^2\,, \tag{44}$$

which is overwhelmingly dominated by $m_\Upsilon^2 = 89.5 \text{ GeV}^2$.

The **strong hierarchy** in the eigenvalues of this matrix—the largest eigenvalue ($\sim m_\Upsilon^2$) exceeds the smallest ($\sim m_\pi^2$) by a factor of ~ 4600—reflects the quark mass hierarchy and is consistent with the near-rank-1 structure expected from the scalar democracy mechanism, where the leading VEV dominates.

9 Numerical summary and comparison with PDG data

We collect all sum rules and their numerical tests in Table 1.

Table 1: Summary of meson mass sum rules tested against PDG 2024 data [17]. "Std. QCD" indicates the sum rule follows from standard QCD symmetries; "DSM" indicates it encodes additional structure from the dualised SM / scalar democracy framework.

#	Sum Rule	LHS	RHS	Accuracy	Origin
I a	$m_\pi^2 f_\pi^2 = -(m_u + m_d)\langle\bar{q}\,q\rangle$	(see text)		$\sim 10\%$	Std. QCD
I b	$m_K^2/m_\pi^2 = (m_u + m_s)/(m_u + m_d)$	12.5	14.0	12%	Std. QCD
II	$4m_K^2 = 3m_\eta^2 + m_\pi^2$	975	920	5.7%	Std. QCD
III a	$\Delta_s^{(c)} = \Delta_s^{(b)}$	103.5	87.2	16 MeV	Std. QCD
III b	$m_B - m_D$ indep. of q	3399–3415		$<$ 17 MeV	Std. QCD
IV	$m_{B_c} = (m_{J/\psi} + m_\Upsilon)/2$	6279	6275	**4 MeV**	Std. QCD
V a	$m_s/m_t \approx (m_b/m_t)^2$	5.41×10^{-4}	5.86×10^{-4}	8%	**DSM**
V b	$m_s/m_b \approx m_b/m_t$	0.0223	0.0242	8%	**DSM**
VI	$(m_{D_s} - m_{D^0}) - (m_{B_s} - m_{B^0})$	16.3 MeV		$\mathcal{O}(\Lambda_{\mathrm{QCD}}^2/m_c)$	Std. QCD

Assessment.

1. Sum rules **I–IV** and **VI** are consequences of standard QCD symmetries: chiral symmetry (I, II), heavy quark symmetry (III, VI), and the additive quark model (IV). The meson matrix of the dualised SM makes the flavour structure *manifest* (*e.g.*, the rectangular identity, the column sum rule) but does not produce numerically new predictions beyond what ChPT and HQET already provide.
2. Sum rule **V** (geometric VEV hierarchy) is **specific to the scalar democracy framework**. The relation $m_s/m_t \approx (m_b/m_t)^2$ is not a consequence of any QCD symmetry and represents a genuine consistency check of the dualised SM. It holds at the 8% level, which is remarkable for a framework with $\mathcal{O}(1)$ free parameters.
3. The most precise sum rule is **IV**: $m_{B_c} = (m_{J/\psi} + m_\Upsilon)/2$ to 4 MeV. This is a consequence of the approximate additivity of the heavy-heavy meson mass matrix and was used to predict m_{B_c} before its experimental measurement.

10 Discussion

10.1 New vs. known sum rules

Of the sum rules derived in this paper, only the **scalar democracy hierarchy predictions** (Sum Rule V) are genuinely new—they follow from the VEV tier structure of the dualised SM and have no counterpart in standard QCD. All other sum rules (I–IV, VI) are well-established consequences of chiral symmetry, heavy quark symmetry, and the additive quark model.

This is not a criticism of the dualised SM framework—rather, it reflects the framework's *consistency*: the meson mass relations that follow from the 6×6 flavour matrix structure are precisely those that are independently established by QCD effective theories. The dualised SM provides a *unified structural origin* for these relations (the matrix is elementary in the magnetic theory) but does not contradict or modify them.

10.2 The role of $\mathcal{O}(1)$ coefficients

The scalar democracy predictions are limited by the unknown $\mathcal{O}(1)$ coefficients c^{abcd} in the Planck-scale operators that generate the VEV hierarchy. With ~ 100 free parameters (the tensor c^{abcd}), the framework can accommodate any quark mass pattern. The non-trivial content of Sum Rule V is that a *single* expansion parameter $\epsilon \approx 0.024$ describes five orders of magnitude in quark masses, with $\mathcal{O}(1)$ ratios at each tier.

A sharper prediction would require a theory of the Planck-scale operator coefficients—*e.g.*, from a UV completion involving gravity or from a specific string compactification. This is beyond the scope of the dualised SM as currently formulated.

10.3 The top quark and the 6th row

The formal 6th row and column of the meson matrix (top-quark entries) do not correspond to physical mesons, since $\Gamma_t \gg \Lambda_{\rm QCD}$. However, they play a role in the algebraic structure: the VEV sum rule $v^2 = \sum_{ij} |\langle H_{ij}\rangle|^2 = (174 \text{ GeV})^2$ and the top quark mass $m_t = y\, v$ fix the overall scale of the meson matrix. The expansion parameter $\epsilon = m_b/m_t$ is defined relative to this scale.

10.4 Limitations and outlook

The main limitation of this analysis is that the meson mass sum rules derivable from the 6×6 matrix structure are *identical* to those from standard QCD effective theories (ChPT, HQET, additive quark model). The dualised SM provides a structural interpretation but not a calculational advantage.

A potentially more fruitful direction is the *scalar meson sector* ($J^P = 0^+$), where the dualised SM makes a concrete prediction: the scalar mesons (f_0, a_0, K_0^*, χ_{c0}, etc.) correspond to the *amplitude fluctuations* of the meson condensate, while the pseudoscalars correspond to the *phase fluctuations*. Mass relations between scalar and pseudoscalar partners (σ–π, κ–K, a_0–η) would provide stronger tests of the dual picture. However, the experimental status of light scalar mesons remains controversial.

11 Conclusions

We have derived and tested six classes of meson mass sum rules from the 6×6 flavour matrix of the dualised Standard Model:

1. The **GMOR relation** and **Weinberg mass ratio** (standard ChPT, verified at $\sim$ 10–12%).
2. The **Gell-Mann–Okubo formula** with η–η' mixing (standard SU(3) flavour, verified at 5.7%, with the anomaly mass $m_0 \approx 854$ MeV consistent with the Witten–Veneziano prediction).
3. **Heavy quark symmetry flavour splittings**: $m_{D_s} - m_{D^0}$ and $m_{B_s} - m_{B^0}$ agree to 16 MeV (standard HQET).
4. The B_c **mass sum rule** $m_{B_c} = (m_{J/\psi} + m_\Upsilon)/2$ to 4 MeV (additive quark model / rectangular identity of the meson matrix).
5. The **scalar democracy geometric hierarchy**: $m_s/m_t \approx (m_b/m_t)^2$ at 8% accuracy. This is the only sum rule *specific to the dualised SM* and not derivable from QCD symmetries alone.
6. **Heavy-light column sum rules** and rectangular identity (standard HQET, violated at 7% when mixing light and heavy sectors).

The honest conclusion is that the 6×6 meson matrix structure of the dualised SM reproduces the known QCD meson mass relations but adds only one genuinely new family of sum rules: the scalar democracy hierarchy predictions. These hold at the $\sim 10\%$ level, which is consistent with $\mathcal{O}(1)$ unknown coefficients in the Planck-scale operators. The framework is *consistent with* but does not *improve upon* standard QCD for meson masses.

Epistemic note. All statements about the dualised SM in this paper are conditional on the conjectured non-supersymmetric duality holding. The meson mass sum rules from standard QCD (I–IV, VI) are rigorous; the scalar democracy predictions (V) are framework-dependent.

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