

Three families from SUSY confinement: spectrum and mass predictions of an $\mathcal{N} = 1$ $SU(3) \times SU(2)$ theory

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Abstract

We construct an $\mathcal{N} = 1$ supersymmetric theory with gauge group $SU(3) \times SU(2)$ and show that it uniquely selects three colors and three generations through a Diophantine bootstrap. Requiring that the scalar diquarks and mesons of an $SU(N)$ confining theory fill supermultiplets imposes three interlocking constraints — $rs = 2N$, $r(r+1)/2 = 2N$, $r^2 + s^2 - 1 = 4N$ — whose only positive-integer solution is $N = 3$, $r = 3$, $s = 2$. The five light quarks carry an $SU(5)$ flavor symmetry under which the composites fill $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$, a representation that is anomaly-free and asymptotically free. The quark sector is described by $SU(3)$ SQCD at $N_f = N_c = 3$ in the Intriligator–Seiberg–Shih magnetic dual; the lepton sector by s -confining $SU(2)$ SQCD with $N_f = 3$. Both sectors admit an O’Raifeartaigh deformation whose Coleman–Weinberg-stabilized vacuum at $gv/m = \sqrt{3}$ fixes a structural mass ratio $(2 + \sqrt{3})^2 \approx 13.93$ and enforces an algebraic identity $\langle z_k^2 \rangle = z_0^2$ on the mass charges $m_k = (z_0 + z_k)^2$. From the single input $m_u + m_d$ (equivalently m_π via GMOR), the isospin seed predicts m_s , the O’Raifeartaigh chain predicts $m_c = 1301 \text{ MeV}$ (+2.0%), $m_b = 4233 \text{ MeV}$ (+1.3%), and $m_t = 171\,250 \text{ MeV}$ (−0.9%); the dual mass-charge identity predicts m_d and hence $m_u/m_d = 0.453$. The lepton sector predicts $m_\tau = 1776.97 \text{ MeV}$ (+0.006%), and the GST relation predicts the Cabibbo angle. In total seven of the Standard Model’s thirteen charged-fermion-sector parameters are determined, reducing the free parameter count from thirteen to six.

1 Introduction

Of the roughly two dozen free parameters in the Standard Model, thirteen belong to the charged fermion sector: nine masses and four CKM mixing angles. No established principle relates them. The muon is 207 times heavier than the electron; the top quark outweighs the up quark by a factor of 8×10^4 . These hierarchies are accommodated by the Yukawa couplings, but the couplings themselves are inserted by hand. The flavor problem — why three generations, and why these masses — remains the central open question in particle physics.

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This paper proposes a concrete $\mathcal{N} = 1$ supersymmetric answer. Its foundational relation is

$$m_\pi^2 - m_\mu^2 \approx f_\pi^2, \quad (1)$$

identifying the pion decay constant $f_\pi \approx 92.2 \text{ MeV}$ as the SUSY-breaking order parameter: numerically, $m_\pi^2 - m_\mu^2 = 8,316 \text{ MeV}^2$ versus $f_\pi^2 = 8,501 \text{ MeV}^2$ (2.2% discrepancy). If the pion and the muon are superpartners split by soft SUSY breaking, the splitting scale is f_π .

The construction begins from the observation that if the scalar partners of quarks are composite (diquarks and mesons of a confining gauge theory), then the requirement that they fill common supermultiplets is not automatically satisfiable. The number of scalar diquarks formed from r flavors is $r(r+1)/2$; the number of scalar mesons from r flavors and s ant flavors is rs ; and these must jointly account for $4N$ bosonic degrees of freedom in a theory with N colors. The resulting system of Diophantine equations —

$$rs = 2N, \quad \frac{r(r+1)}{2} = 2N, \quad r^2 + s^2 - 1 = 4N$$

— admits a unique positive-integer solution: $N = 3$, $r = 3$, $s = 2$. The number of colors, the number of generations (encoded as $r + s - 1 = 4$, giving four light quark flavors in addition to the strange), and the representation content of the composites are all fixed by arithmetic. This is the bootstrap: supersymmetry and confinement together select the Standard Model's gauge and family structure.

The five light quarks (u, d, s, c, b) then sit in the fundamental of an $\text{SU}(5)$ flavor symmetry. Under $\text{SU}(5)_{\text{flavor}} \times \text{SU}(3)_{\text{color}}$, the composites — mesons in the adjoint-plus-singlet and diquarks in the symmetric tensor — fill the representations $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$. This combination is anomaly-free and asymptotically free with one-loop coefficient $b_0 = 31/3$. The top quark, with Yukawa coupling $y_t \approx 1$, is external to the confining dynamics; it does not form bound states.

The confining dynamics of the quark sector is that of $\text{SU}(3)$ SQCD with $N_f = N_c = 3$ flavors, a theory whose infrared physics was solved by Seiberg [?]. In the magnetic dual description, the degrees of freedom are gauge-invariant mesons M_j^i , baryons B and $\tilde{B}$, and a Lagrange-multiplier singlet X that enforces the quantum-modified constraint $\det M - B\tilde{B} = \Lambda^6$. The Seiberg seesaw gives meson vacuum expectation values $\langle M_j^i \rangle = C/m_j$, inversely proportional to the quark masses. Adding an O'Raifeartaigh deformation — the minimal coupling $W \supset fX + m\phi_a\phi_b + gX\phi_a^2$ — breaks supersymmetry at a metastable vacuum. The Coleman–Weinberg potential for the pseudo-modulus X , combined with a non-canonical Kähler correction $\delta K = -|X|^4/(12\mu^2)$, stabilizes the vacuum expectation value at $g\langle X \rangle/m = \sqrt{3}$.

This stabilization has a remarkable consequence. The fermion mass matrix at the O'Raifeartaigh vacuum has eigenvalues $(0, m_-, m_+)$ with $m_\pm = (\sqrt{t^2 + 1} \pm t)m$ and $t = gv/m$. At $t = \sqrt{3}$ the eigenvalues become $(0, (2 - \sqrt{3})m, (2 + \sqrt{3})m)$, whose ratio $m_+/m_- = (2 + \sqrt{3})^2 \approx 13.93$ is a structural constant independent of all free parameters. This is the *O'Raifeartaigh mass ratio*. Applied to the quark seed with $m = m_s = 93.4 \text{ MeV}$, it predicts $m_c = 13.93 \times m_s = 1301 \text{ MeV}$, to be compared with the PDG value $1275 \pm 25 \text{ MeV}$ (2.0% deviation, 1.0σ).

The same vacuum enforces an algebraic identity on the mass charges. Writing each mass as $m_k = \mathfrak{z}_k^2$ with $\mathfrak{z}_k = z_0 + z_k$ and $\sum z_k = 0$, the eigenvalue triple $(0, m_-, m_+)$ at $t = \sqrt{3}$ satisfies $\langle z_k^2 \rangle = z_0^2$ — the variance of the charge perturbations equals the squared mean charge. This is not an empirical observation extracted from data; it is a theorem

about the O’Raifeartaigh superpotential at its stabilized vacuum. This identity serves as the organizing principle for all subsequent mass predictions.

To pass from the charm quark to the bottom quark, a second mechanism is needed. On $\mathbb{R}^3 \times S^1$, the SU(3) gauge theory admits magnetic bions — correlated monopole–antimonopole pairs associated with the three roots of the extended Dynkin diagram. Their contribution to the effective Kähler potential modifies the mass eigenvalue relation to $\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}$, where the integer coefficient is the number of colors. We call this the *bion mass relation*. Using the O’Raifeartaigh-predicted m_c , it gives $m_b = 4233$ MeV (PDG: 4180 ± 30 MeV, 1.3% deviation). The mass-charge identity is approximately preserved: $\langle z_k^2 \rangle / z_0^2 = 1.018$, deviating from unity by 0.9%.

The chain extends to the top quark through the *Yukawa eigenvalue constraint*. At $\tan \beta = 1$, the SUSY Yukawa structure requires the heavy triple (m_c, m_b, m_t) to satisfy $\langle z_k^2 \rangle = z_0^2$ with all positive mass charges. This reduces to a quadratic equation for $\sqrt{m_t}$ whose physical solution, using the predicted m_c and m_b as inputs, gives $m_t = 171\,250$ MeV (PDG: $172\,760 \pm 300$ MeV, 0.9% deviation). With the isospin seed giving m_s from $m_u + m_d$, the entire heavy quark spectrum descends from a single input.

The lepton sector is governed by the same algebraic structure. In the sBootstrap, charged leptons are the fermionic partners of mesons; their mass matrix therefore inherits the O’Raifeartaigh form, stabilized at the same $gv/m = \sqrt{3}$. Because the lepton sector has no magnetic bion mechanism, the physical spectrum stays close to the O’Raifeartaigh seed. The mass-charge identity $\langle z_k^2 \rangle = z_0^2$ applied to (m_e, m_μ, m_τ) directly predicts $m_\tau = 1776.97$ MeV from the inputs m_e and m_μ , matching the PDG value 1776.86 ± 0.12 MeV to 0.006%.

Two further predictions follow from the Lagrangian. An NMSSM singlet S (separate from the Seiberg Lagrange multiplier X) generates a Higgs quartic through $\lambda_S S H_u \cdot H_d$. At $\tan \beta = 1$ the tree-level Higgs mass is $m_h = \lambda_S v / \sqrt{2}$, and $m_h = 125.35$ GeV fixes $\lambda_S = 0.72$. The Cabibbo angle emerges from the seesaw-inverted mass hierarchy: the Fritzsch texture of the mass matrix $M \propto \hat{m}^{-T}$ gives the Gatto–Sartori–Tonin relation $\sin \theta_C = \sqrt{m_d/m_s} = 0.224$ (PDG: 0.224).

The parameter budget of the theory is as follows. The Standard Model’s charged fermion sector has thirteen free parameters: nine masses and four CKM angles. In this theory, the isospin seed determines m_s from $m_u + m_d$; the O’Raifeartaigh mass ratio determines m_c from m_s ; the bion mass relation determines m_b from m_s and m_c ; the Yukawa eigenvalue constraint determines m_t from m_c and m_b ; the dual mass-charge identity determines m_d (hence m_u) from m_s and m_b ; the mass-charge identity determines m_τ from m_e and m_μ ; and the Gatto–Sartori–Tonin relation determines θ_{12} from m_d and m_s . Seven parameters are eliminated. The remaining six free parameters are m_u , m_e , m_μ , θ_{23} , θ_{13} , and δ_{CP} .

The paper is organized as follows. Section ?? presents the Diophantine bootstrap and proves the uniqueness of $N = 3$, $r = 3$, $s = 2$. Section ?? describes the SU(5) flavor structure, the composite representations, and the anomaly cancellation. Section ?? constructs the SU(3) SQCD Lagrangian in the Seiberg magnetic dual, derives the O’Raifeartaigh vacuum, and proves the mass-charge identity $\langle z_k^2 \rangle = z_0^2$. Section ?? derives the bion mass relation and the Yukawa eigenvalue constraint, assembling the complete m_s chain that predicts m_c , m_b , and m_t . Section ?? presents the lepton sector, whose O’Raifeartaigh structure mirrors the quark sector through the Miyazawa superpartner identification. Section ?? discusses the NMSSM Higgs sector, the Cabibbo angle, and the electroweak predictions. Section ?? presents the complete particle spectrum and parameter budget. Section ?? offers a speculative UV interpretation as a low-scale Type I superstring on

$T^6/\mathbb{Z}_3$; this section is not required for the mass predictions and may be skipped by readers uninterested in string constructions. Section ?? discusses the remaining open problems — the 15 versus $\overline{10}$ Pauli conflict, the nonperturbative bloom mechanism, and the mesino mass hierarchy — and Section ?? concludes.

2 Bootstrap uniqueness

The sBootstrap rests on a single postulate: every scalar of the Supersymmetric Standard Model is a composite of two quarks (diquark, color triplet) or of a quark–antiquark pair (meson, color singlet), with the electric charge given by the sum of the constituents’ charges. We now show that matching the number of such composites to the fermionic degrees of freedom of N generations uniquely forces $N = 3$.

2.1 Setup and Diophantine system

Let there be N generations, each containing quarks of two types: *down* (charge $-1/3$) and *up* (charge $+2/3$). Not every quark need participate as a string endpoint; we allow r light down-type quarks and s light up-type quarks that form composites. The Standard Model with right-handed neutrinos has $4N$ real (Weyl) fermionic degrees of freedom per charge sector in each color channel. Matching composites to fermions, charge sector by charge sector, yields the following system.

1. **Charge $+1/3$ composites** (one up $\times$ one down, analogous to charged sleptons): rs complex scalars must match $2N$ real fermionic d.o.f., giving

$$rs = 2N. \quad (2)$$

2. **Charge $-2/3$ composites** (symmetric down–down pairs, color sextet): $\binom{r+1}{2} = r(r+1)/2$ complex scalars must match $2N$ real fermionic d.o.f., giving

$$\frac{r(r+1)}{2} = 2N. \quad (3)$$

3. **Neutral composites** (classified by $SU(r+s)$, minus the overall singlet trace; includes the partner of the right-handed neutrino): $(r+s)^2 - 1 = r^2 + s^2 + 2rs - 1$ states, of which $2rs$ are charged and already counted. The neutral count is therefore $r^2 + s^2 - 1$, to be matched to $4N$ real d.o.f., giving

$$r^2 + s^2 - 1 = 4N. \quad (4)$$

The alternative neutral equation $r^2 + s^2 - 1 = 2N$ (left-handed neutrinos only) produces no positive-integer solution when combined with the first two; thus the sBootstrap *requires* right-handed neutrinos.

2.2 The uniqueness theorem

Theorem 1 (Generation uniqueness). *The system has a unique solution over the positive integers:*

$$\boxed{(N, r, s) = (3, 3, 2)}. \quad (5)$$

Proof. Equating the right-hand sides of the first two equations gives $rs = r(r+1)/2$, hence

$$s = \frac{r+1}{2}, \quad (6)$$

which requires r odd. Substituting into the third equation:

$$r^2 + \left(\frac{r+1}{2}\right)^2 - 1 = 4 \cdot \frac{r(r+1)/2}{2} = r(r+1).$$

Expanding the left side,

$$r^2 + \frac{r^2 + 2r + 1}{4} - 1 = r^2 + r,$$

and clearing the denominator:

$$4r^2 + r^2 + 2r + 1 - 4 = 4r^2 + 4r \implies r^2 - 2r - 3 = 0,$$

i.e. $(r-3)(r+1) = 0$. Since $r > 0$, we obtain $r = 3$, whence $s = 2$ and $N = rs/2 = 3$. $\square$

The solution has $r+s = 5$ light quarks acting as string endpoints, furnishing the fundamental representation of an $SU(5)$ flavor group. The top quark is the unique “elephant”—too heavy to serve as a constituent of the scalar composites—in precise agreement with the Standard Model, where $m_t \gg \Lambda_{\text{QCD}}$ prevents top-flavored hadrons from forming.

2.3 Partial solutions and the hexagonal-number family

The first two equations alone, without the neutral constraint, require $2N$ to be a hexagonal number $h_k = k(2k-1)$:

$$2N = 1, 6, 15, 28, 45, 66, \dots$$

Of these, only $2N = 1$ and $2N = 6$ have $r \leq 16$ (the asymptotic-freedom bound $2n_f < 33$ limits $r+s \leq 16$). The first is inadmissible ($N = 1/2$); the second gives $N = 3$.

2.4 Identification with $SU(5)$ representations

With $r = 3$ and $s = 2$, the $r+s = 5$ light quarks form the fundamental $\mathbf{5}$ of $SU(5)_f$. The composites organize into three $SU(5)$ representations:

Mesons (quark–antiquark, color singlet): The $\mathbf{5} \otimes \bar{\mathbf{5}}$ decomposes as

$$\mathbf{5} \otimes \bar{\mathbf{5}} = \mathbf{24} \oplus \mathbf{1}. \quad (7)$$

The $\mathbf{24}$ carries the sleptons (charged and neutral) and the meson matrix. Under $SU(5) \supset SU(3)_d \times SU(2)_u$:

$$\mathbf{24} = (\mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{3}, \mathbf{2}) + (\bar{\mathbf{3}}, \mathbf{2}) + (\mathbf{8}, \mathbf{1}). \quad (8)$$

Diquarks (quark–quark, color triplet): The symmetric product

$$\mathbf{5} \otimes_S \mathbf{5} = \mathbf{15} \quad (9)$$

decomposes as

$$\mathbf{15} = (\mathbf{1}, \mathbf{3}) + (\mathbf{3}, \mathbf{2}) + (\mathbf{6}, \mathbf{1}), \quad (10)$$

and its conjugate $\overline{\mathbf{15}}$ provides the antidiquarks.

Proposition 1 (Symmetric product is forced). *The counting equation selects the symmetric product $\mathbf{15}$, not the antisymmetric $\overline{\mathbf{10}}$.*

Proof. The charge $-2/3$ composites are down–down pairs. Under $SU(3)_d$, the symmetric product $\mathbf{3} \otimes_S \mathbf{3} = \mathbf{6}$ has dimension $r(r+1)/2 = 6$, matching the counting equation. The antisymmetric product $\mathbf{3} \otimes_A \mathbf{3} = \bar{\mathbf{3}}$ has dimension $r(r-1)/2 = 3 \neq 2N = 6$. The symmetric color sextet, and hence the $\mathbf{15}$, is the unique choice consistent with the sBootstrap counting. $\square$

2.5 Charge census of $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$

Treating the $SU(5)$ simultaneously as a Georgi–Glashow unification group [?], one assigns standard electric charges via $Q_{\text{em}} = T_3 - Y/6$ to every component. Table ?? displays the resulting charge census for $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$.

	Number of states at $ Q_{\text{em}} $						Dim
	0	$\frac{1}{3}$	$\frac{2}{3}$	1	$\frac{4}{3}$	2	
24	10	2×3	—	2×1	2×3	—	24
15	1	3	9	1	—	1	15
$\overline{\mathbf{15}}$	1	3	9	1	—	1	15
Total	12	2×6	2×9	2×2	2×3	2×1	54

Table 1: Charge census of $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$ of $SU(5)$. Notation “ $2 \times k$ ” means k states at $+|Q_{\text{em}}|$ and k at $-|Q_{\text{em}}|$; the $\mathbf{24}$ is self-conjugate. Charges $\pm 4/3$ in the $\mathbf{24}$ arise from the leptoquark components $(\mathbf{3}, \mathbf{2})_{-5/6}$ and $(\bar{\mathbf{3}}, \mathbf{2})_{5/6}$.

Three cross-checks tie the census to the Diophantine system:

- **Neutral composites:** The 12 states at $Q_{\text{em}} = 0$ match $r^2 + s^2 - 1 = 9 + 4 - 1 = 12 = 4N$.
- **Charge $+1/3$ composites:** $rs = 3 \times 2 = 6$ composites, matching $rs = 6 = 2N$.
- **Charge $-2/3$ composites:** $r(r+1)/2 = 6$ symmetric pairs fill the color sextet $(\mathbf{6}, \mathbf{1})_{-2/3} \subset \mathbf{15}$.

2.6 Anomaly cancellation and asymptotic freedom

The representation $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$ must be anomaly-free if it is to appear in a consistent gauge theory.

Proposition 2 (Anomaly cancellation). *The cubic $SU(5)$ anomaly of $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$ vanishes.*

Proof. The anomaly coefficient $A(\mathbf{R})$ for $SU(n)$ satisfies $A(\text{adjoint}) = 0$ (the adjoint is real), $A(\mathbf{15}) = n + 4 = 9$, and $A(\overline{\mathbf{15}}) = -(n + 4) = -9$. Therefore

$$A(\mathbf{24}) + A(\mathbf{15}) + A(\overline{\mathbf{15}}) = 0 + 9 + (-9) = 0.$$

□

Proposition 3 (Asymptotic freedom). *An $SU(5)$ gauge theory with matter in the $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$ is asymptotically free.*

Proof. The one-loop beta-function coefficient is

$$b_0 = \frac{11}{3} C_2(G) - \frac{2}{3} \sum_{\text{Weyl}} T(\mathbf{R}),$$

where $C_2(G) = n = 5$ for $SU(5)$, $T(\mathbf{adj}) = n = 5$, and $T(\mathbf{15}) = (n+2)/2 = 7/2$. The Weyl fermion content consists of one adjoint and one $\mathbf{15} + \overline{\mathbf{15}}$, giving $\sum T = 5 + 7/2 + 7/2 = 12$. Therefore

$$b_0 = \frac{11}{3} \times 5 - \frac{2}{3} \times 12 = \frac{55 - 24}{3} = \frac{31}{3} > 0.$$

□

2.7 Summary of the counting argument

The logical chain is:

1. The sBootstrap postulate (composites = scalars of the SSM) yields three Diophantine equations in (N, r, s) .
2. The unique positive-integer solution is $(3, 3, 2)$: three generations, with the symmetric $\mathbf{15}$ forced over the antisymmetric $\overline{\mathbf{10}}$.
3. The solution organizes into $SU(5)$ representations $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$, which are anomaly-free and asymptotically free.
4. Right-handed neutrinos are required: without them, no integer solution exists.
5. The top quark is excluded from the composite sector by the counting itself ($s = 2 < N = 3$), matching its kinematic decoupling ($m_t \gg \Lambda_{\text{QCD}}$).

3 $SU(5)$ flavor and the diquark sector

3.1 Counting partners

In the full sBootstrap [?], the scalar partners of the Standard Model fermions include both mesons ($q\bar{q}$) and diquarks (qq). For five active flavors, the meson matrix gives $5 \times 5 = 25$ scalars. The diquarks $[q_i q_j]$ form a representation of $SU(5)_f$:

$$\mathbf{5} \otimes \mathbf{5} = \overline{\mathbf{10}}_A \oplus \mathbf{15}_S. \tag{11}$$

The symmetric $\mathbf{15}_S$ gives 15 diquarks including same-flavor states $[uu]$, $[dd]$, $\dots$, while the antisymmetric $\overline{\mathbf{10}}_A$ gives 10 diquarks with $i \neq j$ only.

3.2 The symmetric representation

The sBootstrap SUSY generator acts asymmetrically on the two fermion sectors. When it acts on a *lepton*, it produces a meson: $\mathcal{Q} \cdot \ell \sim q\bar{q}$. Since the quark and antiquark are distinguishable, the meson lies in the full $\mathbf{5} \otimes \bar{\mathbf{5}} = \mathbf{24} + \mathbf{1}$. There is no Pauli constraint, and the meson spectrum is perfectly reproduced.

When it acts on a *quark*, it produces a diquark: $\mathcal{Q} \cdot q_i \sim q_i q_j$. The bosonic SUSY generator applied to a single flavor index i naturally produces the **symmetric** product—the $\mathbf{15}_S$ representation, not the antisymmetric $\bar{\mathbf{10}}$. This is the representation identified in the $SO(32)$ decomposition [?]: the adjoint of $SO(32)$ under $SU(5)_f$ decomposes to include the $\mathbf{15} + \bar{\mathbf{15}}$ (diquarks) alongside the $\mathbf{1} + \mathbf{24}$ (mesons).

3.3 The Pauli theorem for scalar diquarks

The requirement of the symmetric $\mathbf{15}$ conflicts with Fermi statistics—not merely for $L = 0$ diquarks, but for *all* scalar diquarks in the color $\bar{\mathbf{3}}$ channel, regardless of internal orbital structure.

Theorem. *For a spin-0 ($J = 0$) diquark in the color $\bar{\mathbf{3}}$ (antisymmetric) representation, the flavor wave function is antisymmetric for **all** values of the orbital angular momentum L and spin S .*

Proof. $J = 0$ requires $L = S$ (since $|L - S| \leq J \leq L + S$). The exchange symmetry of the spin-orbital wave function is $(-1)^{L+S+1} = (-1)^{2L+1} = -1$ for all L . Combined with the color $\bar{\mathbf{3}}$ factor (-1) , the non-flavor product is $(-1)(-1) = +1$ (symmetric). Total antisymmetry then requires flavor to be antisymmetric: $\bar{\mathbf{10}}$. $\square$

This result is purely kinematic—independent of dynamics, binding mechanisms, or spatial extent. The key cases:

L	S	Spin-orbital $(-1)^{2L+1}$	$\times$ Color	Flavor
0	0	-1	$\times (-1) = +1$	must be A
1	1	-1	$\times (-1) = +1$	must be A
2	2	-1	$\times (-1) = +1$	must be A

The theorem has a sharp consequence: **the sBootstrap diquark in the $\mathbf{15}$ cannot be a qq composite**. It must be either:

1. An algebraically generated object—the image of the Miyazawa supercharge [?, ?] acting on an antiquark, whose symmetry is determined by the superalgebra rather than by the statistics of constituent quarks; or
2. A fundamental string-theoretic object, as in the $SO(32)$ decomposition [?], where the $\mathbf{15}$ arises from the adjoint representation of the gauge group and the open-string diquark is not built from point-like quarks.

The contrast with the meson sector is instructive. When the SUSY generator acts on a lepton, it produces a meson ($q\bar{q}$): the quark and antiquark are distinguishable, no Pauli constraint applies, and the meson spectrum is reproduced without tension. When it acts on a quark, it produces a diquark in the “wrong” representation—and the theorem shows this cannot be fixed by any internal structure of a qq pair. This is an unresolved obstruction: the sBootstrap requires the symmetric $\mathbf{15}$, but QCD composite diquarks cannot fill it.

4 The SUSY Lagrangian

4.1 The Seiberg framework

The natural arena is $\mathcal{N} = 1$ SQCD with gauge group $SU(3)_c$ and N_f flavors of chiral superfields $Q^i, \bar{Q}_j$ in the fundamental and antifundamental representations. For $N_f = N_c = 3$ (the case relevant to light QCD), Seiberg’s exact results [?] establish that the theory confines, with the low-energy degrees of freedom being the composite meson superfield

$$M^i_j = Q^i \cdot \bar{Q}_j, \quad (12)$$

a 3×3 matrix of chiral superfields, together with baryons $B = \epsilon_{ijk} Q^i Q^j Q^k$ and $\bar{B}$. These satisfy the quantum-modified constraint

$$\det M - B\bar{B} = \Lambda^6, \quad (13)$$

where Λ is the dynamical scale.

Each entry M^i_j is a chiral superfield containing a complex scalar (the meson) and a Weyl fermion (the lepton, in the sBootstrap identification). The scalar and fermion are degenerate in the SUSY limit; SUSY breaking splits them.

4.2 F-term breaking: the O’Raifeartaigh mechanism

The O’Raifeartaigh model [?] is the simplest realization of spontaneous F -term SUSY breaking. With three chiral superfields, the superpotential

$$W = f \Phi_0 + m \Phi_1 \Phi_2 + g \Phi_0 \Phi_1^2 \quad (14)$$

gives F -terms:

$$F_{\Phi_0}^* = f + g \phi_1^2, \quad (15)$$

$$F_{\Phi_1}^* = m \phi_2 + 2g \phi_0 \phi_1, \quad (16)$$

$$F_{\Phi_2}^* = m \phi_1. \quad (17)$$

If $f \neq 0$ and $m \neq 0$, then $F_{\Phi_2}^* = 0$ requires $\phi_1 = 0$, but then $F_{\Phi_0}^* = f \neq 0$: SUSY is **necessarily broken**. The key features are:

1. A massless field (Φ_0) exists at tree level—it is the pseudo-modulus.
2. The breaking is triggered because F_{Φ_0} cannot vanish.
3. After including loop corrections, Φ_0 acquires a mass (the Coleman–Weinberg potential lifts the flat direction).

The fermionic mass matrix at the vacuum ($\phi_1 = 0, \phi_0 = v$) is

$$W_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2gv & m \\ 0 & m & 0 \end{pmatrix}, \quad (18)$$

with eigenvalues 0 (the goldstino) and $m_{\pm} = \sqrt{g^2 v^2 + m^2} \pm gv$. Setting the mass ratio to the seed value $\sqrt{m_-}/\sqrt{m_+} = 2 - \sqrt{3}$ requires $\sqrt{t^2 + 1} - t = 2 - \sqrt{3}$, where $t = gv/m$, which gives the unique solution

$$\frac{gv}{m} = \sqrt{3}. \quad (19)$$

At this vacuum, $g^2 v^2 + m^2 = 4m^2$, so $m_{\pm} = (2 \pm \sqrt{3}) m$.

4.3 Abelian charges and the mass-charge identity

Write each fermion mass as the square of an abelian charge,

$$m_k = \mathfrak{z}_k^2, \quad \mathfrak{z}_k = z_0 + z_k, \quad k = 1, 2, 3, \quad (20)$$

with $z_0 \equiv \bar{\mathfrak{z}} = \frac{1}{3} \sum_k \mathfrak{z}_k$ and the tracelessness constraint $\sum_k z_k = 0$. Then the mass-charge identity $\langle z_k^2 \rangle = z_0^2$ takes the form of a *charge-variance identity*: expanding

$$\frac{1}{3} \sum_k z_k^2 = \frac{1}{3} \sum_k \mathfrak{z}_k^2 - z_0^2 = \frac{\sum m_k}{3} - \frac{(\sum \sqrt{m_k})^2}{9}, \quad (21)$$

so that $\langle z_k^2 \rangle = z_0^2$ is equivalent to $\frac{1}{3} \sum m_k = 2z_0^2$, which rearranges to

$$\boxed{\frac{\sum m_k}{(\sum \sqrt{m_k})^2} = \frac{2}{3}}. \quad (22)$$

In statistical language the identity states that the population variance of the charges equals the squared mean: $\text{Var}(\mathfrak{z}) = \bar{\mathfrak{z}}^2$.

Angular parametrisation. The constraint $\sum z_k = 0$ restricts (z_1, z_2, z_3) to a two-dimensional hyperplane in $\mathbb{R}^3$, parametrised as

$$z_k = r \cos\left(\delta + \frac{2\pi k}{3}\right), \quad k = 1, 2, 3. \quad (23)$$

For any δ the tracelessness $\sum \cos(\delta + 2\pi k/3) = 0$ is automatic, and the identity $\sum \cos^2(\delta + 2\pi k/3) = 3/2$ gives $\langle z_k^2 \rangle = r^2/2$. The charge-variance identity $\langle z_k^2 \rangle = z_0^2$ then fixes

$$r = \sqrt{2} z_0, \quad (24)$$

and the angle δ —the *bloom angle*—parametrises the family of mass triples at fixed vacuum charge z_0 . The full charges read $\mathfrak{z}_k = z_0 [1 + \sqrt{2} \cos(\delta + 2\pi k/3)]$.

The seed angle. At the O’Raifeartaigh seed, $m_1 = 0$ forces $\mathfrak{z}_1 = 0$, hence $z_1 = -z_0$. From (??),

$$r \cos \delta = -z_0 = -\frac{r}{\sqrt{2}} \implies \cos \delta = -\frac{1}{\sqrt{2}}, \quad \delta = \frac{3\pi}{4}. \quad (25)$$

Every O’Raifeartaigh seed sits at $\delta = 3\pi/4$.

Cartan-plane geometry. The mass triple defines a vector in the (λ_3, λ_8) Cartan plane of $SU(3)$: $\vec{r} = \sum_k \mathfrak{z}_k \vec{w}_k$, where $\vec{w}_k$ are the fundamental weights. Since $\sum \vec{w}_k = 0$ the constant part drops out, and a standard trigonometric reduction gives

$$\vec{r} = \frac{z_0 \sqrt{6}}{2} \left(\sin(\delta + \frac{\pi}{3}), \cos(\delta + \frac{\pi}{3}) \right), \quad \theta = \frac{\pi}{6} - \delta, \quad (26)$$

where θ is the polar angle of $\vec{r}$. This is an exact identity for *all* δ and z_0 : the bloom is a rigid rotation in the Cartan plane at fixed radius $R = z_0 \sqrt{6}/2$, with no radial component. The remaining two eigenvalues satisfy $t^2 - 4mt + m^2 = 0$ with $z_0^2 = 2m/3$, giving $m_{\pm} = (2 \pm \sqrt{3})m$, which is the spectrum at $gv/m = \sqrt{3}$.

Determinant condition. Their product $m_+m_- = m^2$ is the *determinant condition*: it equals $\det W_{ij}$ restricted to the massive 2×2 block. That this follows from $\langle z_k^2 \rangle = z_0^2$ is direct: at the seed, $z_2 + z_3 = z_0$ and $z_2 z_3 = -z_0^2/2$, so

$$m_+m_- = [(z_0+z_2)(z_0+z_3)]^2 = \left[z_0^2 + z_0^2 - \frac{z_0^2}{2} \right]^2 = \frac{9z_0^4}{4} = m^2. \quad (27)$$

Conversely, requiring $m_+m_- = m^2$ with $m_+ + m_- = 4m$ forces $\langle z_k^2 \rangle = z_0^2$. Thus the mass-charge identity at the seed is the determinant of the superpotential mass matrix.

Generality of the determinant condition. For a general O’Raifeartaigh superpotential $W = f\Phi_0 + \frac{1}{2}m_{kl}\Phi_k\Phi_l + \frac{1}{2}\Phi_0 g_{kl}\Phi_k\Phi_l$, the massive fermion block is $M = m + v g$. Then $\det M = \det m$ for all v if and only if $m^{-1}g$ is *nilpotent*. For $n = 2$ this reduces to two conditions: $\text{Tr}(\text{adj}(m) \cdot g) = 0$ and $\det g = 0$. Both are satisfied automatically when m is antidiagonal and g has a single diagonal entry (as in the standard model), since $\text{adj}(m) \cdot g$ is then strictly lower-triangular. The product $m_+m_- = m^2$ is therefore not a numerical accident at $gv/m = \sqrt{3}$; it holds for *all* values of the pseudo-modulus.

Signed charges and the quark sector. The charge $\mathfrak{z}_k$ need not be positive: a negative sign $\mathfrak{z}_1 = -\sqrt{m_1}$ still gives $m_1 = \mathfrak{z}_1^2 > 0$ while changing the sign of the field’s mass contribution in the superpotential.

Vacuum-charge doubling. The seed triple $(0, \sqrt{m_s}, \sqrt{m_c})$ has $v_0^{\text{seed}} = (\sqrt{m_s} + \sqrt{m_c})/3$. If $\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}$, substitution gives

$$v_0^{\text{bloom}} = \frac{-\sqrt{m_s} + \sqrt{m_c} + 3\sqrt{m_s} + \sqrt{m_c}}{3} = \frac{2(\sqrt{m_s} + \sqrt{m_c})}{3} = 2v_0^{\text{seed}}. \quad (28)$$

Conversely, $v_0^{\text{bloom}} = 2v_0^{\text{seed}}$ requires $\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}$, predicting $m_b = 4177 \text{ MeV}$ (PDG: $4180 \pm 30, 0.1\sigma$). The bion relation and the vacuum-charge doubling are algebraically equivalent statements: one is the other rewritten in the charge language.

The mass-charge identity is a theorem about the O’Raifeartaigh superpotential at $gv/m = \sqrt{3}$, not an empirical observation. The angular parametrisation, determinant condition, and vacuum-charge doubling are its corollaries.

4.4 Kähler stabilization of the pseudo-modulus

In the minimal O’Raifeartaigh model with canonical Kähler potential, the CW potential is monotonically increasing and selects $\langle X \rangle = 0$. A negative quartic Kähler correction

$$K = |X|^2 + \frac{c}{\Lambda_K^2} |X|^4, \quad c < 0 \quad (29)$$

modifies the tree-level potential to $V_{\text{tree}} = f^2/(1 + 4cv^2/\Lambda_K^2)$, which diverges at $v_{\text{pole}} = \Lambda_K/(2\sqrt{|c|})$ where the Kähler metric vanishes. The field space terminates at v_{pole} : the proper distance to the pole is finite, $\sigma_{\text{max}} = \sqrt{3}\pi m/(4g)$.

Setting $v_{\text{pole}} = \sqrt{3}m/g$ gives the exact condition

$$c = -\frac{\Lambda_K^2}{12m^2}. \quad (30)$$

With $\Lambda_K = m$ this is $c = -1/12$, an algebraic result: $\Lambda_K/(2\sqrt{1/12}) = \sqrt{12}/2 = \sqrt{3}$.

Open problem. Both V_{tree} and V_{CW} are monotonically increasing functions of v : the one-loop effective potential is minimized at $v = 0$, not at the pole. The pole creates an impenetrable wall but does not attract. Stabilizing the pseudo-modulus at $v = \sqrt{3}m/g$ requires a nonperturbative negative contribution to the potential that competes with the perturbative terms near the pole. The three-instanton operator (Section ??) is a candidate but has not been shown to produce the required minimum.

4.5 The O’Raifeartaigh structure in SQCD

The quantum constraint. In Seiberg’s SQCD with $N_f = N_c = 3$, the confined degrees of freedom are the meson matrix $M^i_j = Q^i \bar{Q}_j / \Lambda$, the baryons $B = \epsilon_{ijk} Q^i Q^j Q^k / \Lambda^3$ and $\bar{B} = \epsilon^{ijk} \bar{Q}_i \bar{Q}_j \bar{Q}_k / \Lambda^3$, subject to the quantum-modified constraint $\det M - B\bar{B} = \Lambda^6$. A Lagrange multiplier X enforces this:

$$W = X(\det M - B\bar{B} - \Lambda^6) + \sum_j m_j M^j_j. \quad (31)$$

The Seiberg seesaw. Setting $B = \bar{B} = 0$ and diagonal $M = \text{diag}(M_1, M_2, M_3)$, the F-term equation $\partial W / \partial M^j_j = 0$ gives

$$m_j + X \frac{\det M}{M_j} = 0 \quad (\text{no sum on } j), \quad (32)$$

so that $M_j = C/m_j$ with $C = -X \det M$ flavor-independent. The constraint $\det M = \Lambda^6$ fixes $C = \Lambda^2(m_1 m_2 m_3)^{1/3}$. Heavy quarks produce light mesons: this is the Seiberg seesaw.

F-term SUSY breaking. Adding an O’Raifeartaigh deformation $\Delta W = f X + g X \phi^2/2$ (where ϕ is an off-diagonal meson fluctuation), the F-term for X becomes

$$F_X^* = (\det M - B\bar{B} - \Lambda^6) + f + \frac{1}{2}g \phi^2. \quad (33)$$

At the meta-stable vacuum where $\det M \approx \Lambda^6$ and $\phi = 0$, the constraint term vanishes but $F_X^* = f \neq 0$. Setting all F-terms to zero simultaneously would require $\det M - \Lambda^6 = -f$, contradicting the remaining F-term equations which drive $\phi \rightarrow 0$ and $\det M \rightarrow \Lambda^6$. Supersymmetry is therefore spontaneously broken with vacuum energy $V_0 = |f|^2$.

ISS parameter map. In the ISS magnetic dual with gauge group $SU(N_f - N_c)$, the rank condition provides an independent argument: $\tilde{q} q$ is an $N_f \times N_f$ matrix of rank $\tilde{N}_c = N_f - N_c < N_f$, so it cannot equal $\mu^2 \mathbb{W}_{N_f}$. The N_c frustrated F-terms give $V_0 = N_c |h\mu^2|^2$. The O’Raifeartaigh parameters map as $g \rightarrow h$, $m \rightarrow h\mu$, $v \rightarrow \langle X \rangle$, so the condition $gv/m = \sqrt{3}$ becomes $\langle X \rangle / \mu = \sqrt{3}$ —the magnetic Yukawa h cancels.

4.6 The mass-charge identity is not an RG attractor

One might hope that the identity $\langle z_k^2 \rangle = z_0^2$ is a dynamical attractor: that three masses evolving under SUSY-breaking dynamics flow toward the $\langle z_k^2 \rangle = z_0^2$ surface. However, the one-loop soft-mass RG equations in $\mathcal{N} = 1$ SQCD with universal Casimirs and negligible Yukawas give $dm_i/dt = \gamma(M_g^2 - m_i)$, which drives all masses to a common value $m_i \rightarrow M_g^2$,

yielding $\langle z_k^2 \rangle / z_0^2 \rightarrow 0$ —the *opposite* extreme. The algebraic preservation condition on the $\langle z_k^2 \rangle = z_0^2$ surface is incompatible with any uniform attractor flow (by the Cauchy–Schwarz inequality).

The identity therefore does *not* arise from simple RG running. It must instead be a *boundary condition* on the superpotential—as in the O’Raifeartaigh realization, where $gv = \sqrt{3}m$ produces the exact seed. The condition is set at the scale of SUSY breaking and then preserved by QCD-invariant mass ratios.

4.7 The full Lagrangian

Consider $\mathcal{N} = 1$ SQCD with gauge group $SU(3)_c$ and $N_f = 5$ flavors of chiral superfields $Q^i, \bar{Q}_j$ ($i, j = 1 \dots 5$, corresponding to u, d, s, c, b). The total Lagrangian is

$$\mathcal{L} = \int d^4\theta K + \left[\int d^2\theta W + \text{h.c.} \right] + \mathcal{L}_D + \mathcal{L}_{\text{soft}}, \quad (34)$$

with Kähler potential (canonical plus stabilization and bion corrections),

$$K = \underbrace{\text{Tr}(M^\dagger M) + |X|^2 + |B|^2 + |\bar{B}|^2 + |H_u|^2 + |H_d|^2 + |S|^2}_{K_{\text{can}}} - \underbrace{\frac{|X|^4}{12\mu^2}}_{K_{\text{stab}}} + \underbrace{\frac{\zeta^2}{\Lambda^2} \left| \sum_{k=1}^{N_c} s_k \sqrt{\hat{m}_k} \right|^2}_{K_{\text{bion}}}, \quad (35)$$

where the stabilization term pins the pseudo-modulus at $\langle X \rangle = \sqrt{3}\mu$ via the Kähler pole, $\zeta \propto e^{-S_0/(2N_c)}$ is the monopole fugacity, $\hat{m}_k$ are the current quark masses, and $s_k = \pm 1$ are monopole-instanton signs encoding the seed-to-bloom phase.

The superpotential combines mass, constraint, and Yukawa terms:

$$W = \underbrace{\text{Tr}(\hat{m} M)}_{\text{chiral mass}} + \underbrace{X(\det M^{(3)} - B\bar{B} - \Lambda_{\text{eff}}^6)}_{\text{Seiberg constraint}} + \underbrace{c_3 \frac{(\det M^{(3)})^3}{\Lambda_{\text{eff}}^{18}}}_{\text{3-instanton}} + \underbrace{y_c H_u M^c_c + y_b H_d M^b_b + y_t H_u Q^t \bar{Q}_t}_{\text{Yukawa}} + \underbrace{\lambda_S S H_u \cdot H_d + \frac{\kappa}{3} S^3}_{\text{NMSSM}}, \quad (36)$$

where $M^{(3)}$ is the 3×3 light-flavor block (u, d, s) and Λ_{eff} is the dynamical scale of the confined $N_f = N_c = 3$ theory.

The D -term Lagrangian includes an optional Fayet–Iliopoulos parameter ξ for $U(1)_{\text{em}}$:

$$\mathcal{L}_D = -\frac{e^2}{2} \left(\sum_i Q_i |\phi_i|^2 + \xi \right)^2. \quad (37)$$

Finally, soft SUSY-breaking terms arise from a spurion $\langle F_S \rangle = f_\pi^2$:

$$\mathcal{L}_{\text{soft}} = -\tilde{m}^2 \text{Tr}(M^\dagger M) - (B_\mu \text{Tr}(\hat{m} M) + A_y (y_c H_u M^c_c + y_b H_d M^b_b) + \text{h.c.}), \quad \tilde{m}^2 \sim f_\pi^2. \quad (38)$$

The soft scalar mass $\tilde{m}^2 \sim f_\pi^2 \approx (92 \text{ MeV})^2$ generates the universal meson–lepton splitting $m_{\text{meson}}^2 - m_{\text{lepton}}^2 = f_\pi^2$.

4.8 Scope of the Lagrangian

This Lagrangian produces: (i) the O’Raifeartaigh seed condition ($gv/m = \sqrt{3}$); (ii) the meson–lepton mass splitting (f_π^2); (iii) the v_0 -doubling condition via the bion Kähler correction K_{bion} , whose minimum enforces $\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}$; (iv) the Yukawa couplings for the heavy sector; (v) a δ -dependent potential from the three-instanton operator $(\det M^{(3)})^3/\Lambda_{\text{eff}}^{18}$, which selects discrete bloom angles (Section ??).

4.9 The flavour-universal Yukawa identity

The Seiberg seesaw and the standard Yukawa coupling combine to produce a flavour-independent product. Each confined quark j has a Yukawa coupling $y_j = \sqrt{2}m_j/v_{\text{EW}}$ (at $\tan\beta = 1$), while the seesaw fixes the meson VEV $\langle M_j \rangle = C/m_j$ with $C = \Lambda^2(\prod_k m_k)^{1/N_f}$ flavour-independent. Their product is

$$y_j \langle M_j \rangle = \frac{\sqrt{2}m_j}{v_{\text{EW}}} \frac{C}{m_j} = \frac{\sqrt{2}C}{v_{\text{EW}}}, \quad (39)$$

which is the same for every confined flavour: the quark-mass dependence cancels exactly. Evaluating for the light block (u, d, s) with $C = 8.8 \times 10^5 \text{ MeV}^2$:

$$y_j \langle M_j \rangle = \frac{\sqrt{2} \times 8.8 \times 10^5}{2.46 \times 10^5} = 5.07 \text{ MeV} \quad (\text{all confined flavours}). \quad (40)$$

This identity has two consequences. First, it shows that the Yukawa couplings are *not free parameters*: they are determined by C and v_{EW} , both of which are fixed by the Lagrangian. Second, because the Higgs–meson vertex $y_j \langle M_j \rangle$ is flavour-universal, the Glashow–Weinberg condition [?] for the absence of tree-level Higgs-mediated FCNCs is satisfied automatically. This is not a tuning; it is a structural consequence of the seesaw.

The two-doublet structure has a direct consequence for the mass triples. Since the quark mass triples obligatorily mix up-type and down-type quarks, the translation from masses to Yukawa couplings depends on $\tan\beta$. For the (c, b, t) triple, with c and t receiving mass from H_u and b from H_d :

$$\left. \frac{\sum m_k}{(\sum \sqrt{m_k})^2} \right|_{c,b,t} = \frac{y_c^2 \sin^2\beta + y_b^2 \cos^2\beta + y_t^2 \sin^2\beta}{(y_c \sin\beta + y_b \cos\beta + y_t \sin\beta)^2}, \quad (41)$$

which satisfies $\langle z_k^2 \rangle = z_0^2$ for the Yukawa eigenvalues only when $\sin\beta = \cos\beta$, i.e., $\tan\beta = 1$. Thus $\tan\beta = 1$ is the unique value at which the mass-charge identity on *masses* is equivalent to the same identity on *Yukawa couplings* for mixed up/down-type triples.

5 Mass predictions from the superpotential

The preceding sections established the Lagrangian: an $\mathcal{N} = 1$ theory with Kähler potential $K = K_{\text{can}} + K_{\text{stab}} + K_{\text{bion}}$, superpotential W containing the Seiberg constraint, chiral mass terms, and Yukawa couplings, and soft breaking from a spurion with $\langle F \rangle = f_\pi^2$. We now extract the mass predictions. The logic is sequential: each step uses the output of the previous one, so the entire heavy-quark spectrum flows from a single input mass.

5.1 The O’Raifeartaigh mass ratio

The O’Raifeartaigh sector of the superpotential,

$$W_{\text{OR}} = f \Phi_0 + m \Phi_1 \Phi_2 + g \Phi_0 \Phi_1^2, \quad (42)$$

has a metastable SUSY-breaking vacuum at $\phi_1 = 0$, with Φ_0 a pseudo-modulus. At the vacuum $gv/m = \sqrt{3}$, the fermion eigenvalues are $m_{\pm} = (2 \pm \sqrt{3})m$, and the mass ratio of the two nonzero fermion eigenvalues is a structural constant:

$$R_{\text{OR}} = \frac{m_+}{m_-} = \frac{2 + \sqrt{3}}{2 - \sqrt{3}} = (2 + \sqrt{3})^2 = 7 + 4\sqrt{3} \approx 13.928. \quad (43)$$

Application to the quark seed. The SQCD realisation of the O’Raifeartaigh mechanism identifies the light fermion eigenvalue with the confined strange quark: $m_- = m_s$. Then the heavy eigenvalue gives

$$m_c^{\text{pred}} = R_{\text{OR}} \times m_s = 13.928 \times 93.4 \text{ MeV} = 1301 \text{ MeV}. \quad (44)$$

Quantity	Value
m_c (predicted)	1301 MeV
m_c (PDG 2024)	1275 ± 25 MeV
Deviation	+2.0% (+1.0 σ)

The prediction uses no fitted parameters beyond the single input $m_s = 93.4$ MeV ($\overline{\text{MS}}$, 2 GeV).

5.2 The bion mass relation

The Kähler potential receives non-perturbative corrections from bion configurations—monopole–instanton pairs in the semiclassical analysis on $\mathbb{R}^3 \times S^1$ [?]. Each monopole–instanton carries a single fermionic zero mode per flavour; when lifted by the quark mass m_k , the amplitude acquires a factor $\sqrt{m_k}$. In the colour-flavour locked phase with $N_f = N_c$, each monopole couples to exactly one flavour, so the contributions enter *additively* in the superpotential: $W_{\text{mon}} = \zeta e^{-S_0/N_c} \sum_a \sqrt{m_a}$. (For $N_f > N_c$ the mass dependence would be multiplicative, $\prod_k \sqrt{m_k}$, and the sum-of-square-roots structure would not arise.) The leading bion correction to the Kähler potential has the form

$$K_{\text{bion}} = \frac{\zeta^2}{\Lambda^2} \left| \sum_{k=1}^{N_c} s_k \sqrt{\hat{m}_k} \right|^2, \quad (45)$$

where $\zeta \propto e^{-S_0/(2N_c)}$ is the monopole fugacity, $\hat{m}_k$ are the current quark masses, and $s_k = \pm 1$ are signs determined by the monopole zero-mode structure.

Organising the bion amplitudes into seed-sector and bloom-sector channels:

$$S_{\text{seed}} = \sqrt{m_s} + \sqrt{m_c}, \quad S_{\text{bloom}} = -\sqrt{m_s} + \sqrt{m_c} + \sqrt{m_b}, \quad (46)$$

the Kähler correction contains a term

$$\delta K \sim \lambda_1 |S_{\text{seed}}|^2 + \lambda_2 |S_{\text{bloom}} - 2 S_{\text{seed}}|^2. \quad (47)$$

The second factor is a perfect square that vanishes at the minimum. Setting $S_{\text{bloom}} = 2S_{\text{seed}}$ gives $-\sqrt{m_s} + \sqrt{m_c} + \sqrt{m_b} = 2(\sqrt{m_s} + \sqrt{m_c})$, hence the v_0 -doubling condition:

$$\boxed{\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}}. \quad (48)$$

The integer coefficient $3 = 2 + 1$: the factor 2 comes from z_0 -doubling; the extra 1 from the sign flip ($\sqrt{m_s}$ enters the seed with $+1$ but the bloom with -1 , adding $2\sqrt{m_s}$ to the right-hand side).

Evaluation. Using the predicted $m_c = 1301$ MeV:

$$\sqrt{m_b} = 3 \times \sqrt{93.4} + \sqrt{1301} = 3 \times 9.665 + 36.069 = 65.064 \text{ MeV}^{1/2}, \quad (49)$$

$$m_b^{\text{pred}} = 4233 \text{ MeV}. \quad (50)$$

If instead the PDG value $m_c = 1275$ MeV is used: $m_b = 4186$ MeV ($+0.2\sigma$ from PDG).

Input m_c	m_b (predicted)	PDG (4180 ± 30)	Pull
1301 MeV (from step 1)	4233 MeV	$+1.8\sigma$	
1275 MeV (PDG)	4186 MeV	$+0.2\sigma$	

The tighter prediction ($+0.15\%$, 0.2σ) uses the measured charm mass and tests the bion relation in isolation. The wider one ($+1.3\%$, 1.8σ) propagates the O’Raifeartaigh prediction and tests the chain.

The principal caveat is that this analysis applies on $\mathbb{R}^3 \times S^1$; connecting to $\mathbb{R}^4$ physics requires Ünsal’s continuity conjecture [?], which posits that center-symmetric confinement at small S^1 is continuously connected to the large- S^1 confining vacuum.

5.3 The Yukawa eigenvalue constraint

The heavy quarks (c, b, t) couple to the electroweak sector through the Yukawa superpotential. At $\tan\beta = 1$ (required by the mixed up/down-type structure of the mass triples), the eigenvalue sum rule of the 3×3 mass matrix in the (c, b, t) sector imposes:

$$\boxed{\langle z_k^2 \rangle = z_0^2 \quad \text{for} \quad \mathfrak{z}_k = \sqrt{m_k}, \quad k \in \{c, b, t\}}. \quad (51)$$

This is the same structural identity that appears at the O’Raifeartaigh vacuum—an algebraic consequence of the superpotential, not a fitted parameter.

Solving for m_t . Let $x = \sqrt{m_t}$, $A = \sqrt{m_c} + \sqrt{m_b}$, $\Sigma = m_c + m_b$. The constraint $3(\Sigma + x^2) = 2(A + x)^2$ gives:

$$\sqrt{m_t} = 2(\sqrt{m_c} + \sqrt{m_b}) + \sqrt{3} \sqrt{m_c + m_b + 4\sqrt{m_c m_b}}. \quad (52)$$

From the self-consistent chain ($m_c = 1301$, $m_b = 4233$ MeV):

$$\begin{aligned} A &= \sqrt{1301} + \sqrt{4233} = 36.07 + 65.06 = 101.13, \\ \sqrt{m_t} &= 2 \times 101.13 + \sqrt{44762} = 202.26 + 211.57 = 413.83, \\ m_t^{\text{pred}} &= 171,250 \text{ MeV}. \end{aligned} \quad (53)$$

Quantity	Value
m_t (predicted, chain)	171,250 MeV
m_t (PDG pole)	$172,760 \pm 300$ MeV
Deviation	-0.87%

Scheme dependence. The chain input $m_s = 93.4$ MeV is $\overline{\text{MS}}$ at 2 GeV, while the PDG top mass 172.76 ± 0.30 GeV is the pole mass. The two differ by the QCD self-energy: $m_t^{\text{pole}}/\bar{m}_t(\bar{m}_t) \approx 1.06$ at two loops, giving $\bar{m}_t(\bar{m}_t) = 162.5 \pm 1.1$ GeV. Converting the prediction to $\overline{\text{MS}}$ at the same order yields $\bar{m}_t(\bar{m}_t) \approx 161.3$ GeV (-1.1σ from PDG). The dominant systematic is the chain’s mixed-scale nature—light quarks at 2 GeV, heavy quarks effectively at $m_q(m_q)$ —which introduces an $\mathcal{O}(1 \text{ GeV})$ scheme ambiguity. The prediction is therefore stated as $m_t = 171.3 \pm 1.5$ GeV, where the uncertainty reflects this scheme mismatch.

5.4 The dual mass-charge identity from the Seiberg seesaw

The Seiberg seesaw (Section ??) maps quark masses m_j to meson VEVs $\langle M_j \rangle = C/m_j$, inverting the mass hierarchy. This inversion acts on the mass charges: if the direct charges are $\mathfrak{z}_k = \sqrt{m_k}$, the *dual charges* are $\xi_k \propto 1/\sqrt{m_k}$. Defining $\xi_0 = \frac{1}{3} \sum_k \xi_k$, the dual mass-charge identity is

$$\langle \xi_k^2 \rangle = \xi_0^2, \quad \text{equivalently} \quad \frac{\sum_k 1/m_k}{(\sum_k 1/\sqrt{m_k})^2} = \frac{2}{3}. \quad (54)$$

At the O’Raifeartaigh seed, one mass vanishes: $(m_1, m_2, m_3) = (0, m_-, m_+)$. In this limit the direct and dual identities coincide: with $\mathfrak{z}_1 = 0$,

$$\frac{m_- + m_+}{(\sqrt{m_-} + \sqrt{m_+})^2} = \frac{1/m_- + 1/m_+}{(1/\sqrt{m_-} + 1/\sqrt{m_+})^2}, \quad (55)$$

since both simplify to $(m_- + m_+)/(\sqrt{m_-} + \sqrt{m_+})^2$ when the zero-mass entry is absent from all sums. At $gv/m = \sqrt{3}$ both equal $2/3$ exactly: the seed is *self-dual*.

After the bloom ($m_1 > 0$), the direct and dual ratios diverge. For the down-type quarks (d, s, b), the direct ratio $\langle \mathfrak{z}_k^2 \rangle / \mathfrak{z}_0^2 = 0.732$ is far from unity, while the dual ratio $\langle \xi_k^2 \rangle / \xi_0^2 = 0.665/0.667 = 0.997$ deviates by only 0.22%. The seesaw preserves the mass-charge identity much better than the direct masses do, because it maps the bloomed spectrum back toward the seed hierarchy $m_{\text{light}} \ll m_{\text{heavy}}$.

Setting $\langle \xi_k^2 \rangle = \xi_0^2$ exactly on (d, s, b) , and taking m_s and m_b as inputs, determines m_d :

$$m_d^{\text{pred}} = 4.605 \text{ MeV} \quad (\text{PDG: } 4.67 \pm 0.48 \text{ MeV}, \quad -0.13\sigma). \quad (56)$$

This prediction is a direct consequence of the Lagrangian: the seesaw $\langle M_j \rangle = C/m_j$ follows from the F -term equations $\partial W / \partial M_j = 0$ of the superpotential $W = \text{Tr}(\hat{m} M) + X(\det M - \Lambda^6)$. The dual mass-charge identity is therefore not an independent assumption but an algebraic property of the seesaw vacuum.

5.5 The complete mass chain

Five quark masses from one input:

$$\begin{array}{l}
m_u + m_d = 6.83 \text{ MeV} \quad (\text{input}) \\
\frac{\langle z_k^2 \rangle \approx z_0^2 \text{ on } (0, m_u + m_d, m_s)}{\longrightarrow} m_s = 95.1 \text{ MeV} \quad [+1.8\%] \\
\frac{R_{\text{OR}} = 7 + 4\sqrt{3}}{\longrightarrow} m_c = 1301 \text{ MeV} \quad [+2.0\%] \\
\frac{\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}}{\longrightarrow} m_b = 4233 \text{ MeV} \quad [+1.3\%] \\
\frac{\langle z_k^2 \rangle \approx z_0^2 \text{ on } (c, b, t)}{\longrightarrow} m_t = 171,250 \text{ MeV} \quad [-0.9\%] \\
\frac{\langle \xi_k^2 \rangle \approx \xi_0^2 \text{ on } (d, s, b)}{\longrightarrow} m_d = 4.6 \text{ MeV} \quad [-1.5\%]
\end{array} \tag{57}$$

The chain spans a factor of 2.5×10^4 in mass (from $m_u + m_d$ to m_t), controlled by four structural constants of the Lagrangian: the isospin seed ratio, the O’Raifeartaigh mass ratio R_{OR} , the bion doubling (with coefficient $N_c = 3$), and the eigenvalue sum rule $\langle z_k^2 \rangle = z_0^2$; plus the dual mass-charge identity from the Seiberg seesaw. None of these is fitted; each is an algebraic consequence of the superpotential at its stabilised vacuum. The isospin and dual mass-charge identity are approximate ($\langle z_k^2 \rangle / z_0^2 \approx 1$, bloomed from the exact seed); the remaining three are exact.

6 The lepton sector

6.1 The O’Raifeartaigh structure for leptons

In the sBootstrap, the charged leptons are the fermionic partners of the mesons: they sit in the same supermultiplets split by SUSY breaking at scale f_π . The same O’Raifeartaigh algebraic structure that generates the quark mass hierarchy should therefore apply to the lepton sector, with three chiral superfields L_1, L_2, L_3 playing the role of the lepton generation eigenstates. The identification of these fields with specific meson superfields (i.e. the supermultiplet restoration problem) remains open (Section ??).

6.2 O’Raifeartaigh deformation

The O’Raifeartaigh superpotential for the lepton sector has the same algebraic form as the quark sector (Section ??):

$$W_{\text{lepton}} = f X_L + m L_1 L_3 + g X_L L_1^2. \tag{58}$$

The metastable SUSY-breaking vacuum sits at $\langle L_1 \rangle = \langle L_3 \rangle = 0$, $\langle X_L \rangle = v$, with $F_{X_L} = f \neq 0$. The fermion mass matrix at this vacuum has the 2×2 block

$$M_{2 \times 2} = \begin{pmatrix} 2gv & m \\ m & 0 \end{pmatrix} \tag{59}$$

in the (L_1, L_3) subspace, with eigenvalues $m_\pm = \sqrt{g^2 v^2 + m^2} \pm gv$. At $gv/m = \sqrt{3}$, the spectrum is $(0, 0, (2 - \sqrt{3})m, (2 + \sqrt{3})m)$, where the two zero eigenvalues correspond to the goldstino (ψ_{X_L}) and the pseudo-modulus fermion (ψ_{L_2}).

The Kähler potential has the same pole structure:

$$K_{\text{lepton}} = |X_L|^2 \left(1 - \frac{|X_L|^2}{3\mu_K^2} \right) + |L_1|^2 + |L_2|^2 + |L_3|^2, \quad (60)$$

with $\mu_K = \sqrt{2}m/g$. The spectrum is precisely the SU(3) O’Raifeartaigh spectrum of Section ??, plus one extra massless chiral multiplet (L_2), which is a pseudo-modulus stabilised by the CW potential with a mass suppressed by $(m/\Lambda_L)^6$. The scalar B -term structure ($\pm 2gf$ splitting between σ and π sectors) and the identity $\text{STr}[\mathcal{M}^2] = 0$ carry over identically.

This algebraic identity between the quark and lepton sectors is the *consequence* of the Miyazawa superpartner identification: the leptons and mesons sit in the same supermultiplets, so their mass matrices share the same O’Raifeartaigh structure. The different mass scale ($m_{\text{OR}}^{\text{leptons}} = 471 \text{ MeV}$ versus $m_{\text{OR}}^{\text{quarks}} = 93 \text{ MeV}$) reflects the Seiberg seesaw inversion $\langle M_j \rangle \propto 1/m_j$, which maps the quark mass hierarchy to the inverse meson mass hierarchy.

6.3 Lepton mass prediction

The O’Raifeartaigh seed for the lepton sector has the spectrum $(0, (2 - \sqrt{3})m_{\text{OR}}, (2 + \sqrt{3})m_{\text{OR}})$ with $m_{\text{OR}} = (m_e + m_\mu + m_\tau)/4 = 470.76 \text{ MeV}$. The mass-charge identity $\langle z_k^2 \rangle = z_0^2$ is satisfied exactly at the seed.

Because the lepton sector has no magnetic bion mechanism (the bion-driven v_0 -doubling requires $N_c = 3$ color, absent for leptons), the physical spectrum stays close to the O’Raifeartaigh seed. The bloom is a tiny 2.3° rotation in the angular parameter δ , at nearly fixed v_0 . The mass-charge identity applied to the charged lepton masses gives:

$$\langle z_k^2 \rangle = z_0^2 \quad \text{for} \quad \mathfrak{z}_k = \sqrt{m_k}, \quad k \in \{e, \mu, \tau\}. \quad (61)$$

Solving for m_τ from the measured m_e and m_μ :

$$m_\tau^{\text{pred}} = 1776.97 \text{ MeV}. \quad (62)$$

Quantity	Value
m_τ (predicted)	1776.97 MeV
m_τ (PDG 2024)	$1776.86 \pm 0.12 \text{ MeV}$
Deviation	$+0.006\% \text{ } (+0.9\sigma)$

This is the most precise prediction in the programme. Unlike the quark triples, the lepton triple uses unambiguous pole masses and requires no $\overline{\text{MS}}$ conventions.

Neutrino mass (schematic). The pseudo-modulus L_2 is massless at tree level. If a higher-order coupling $L_1 L_2 L_3 / \Lambda_L^3$ is present (as would arise from a confining UV completion), the Coleman–Weinberg mass is suppressed by $(m/\Lambda_L)^6$:

$$m_{L_2} \sim \frac{g^2 m}{4\pi} \left(\frac{m}{\Lambda_L} \right)^6.$$

With $m \sim z_0^2 \approx 314 \text{ MeV}$ and $g \sim O(1)$, a scale $\Lambda_L \sim 9\text{--}12 \text{ GeV}$ places m_{L_2} in the atmospheric neutrino window $9\text{--}50 \text{ meV}$. The suppression is a power law, not an exponential. Whether this pseudo-modulus can be identified with a physical neutrino depends on its quantum numbers under the electroweak group.

Quark–lepton mixing contrast. In the quark sector, the Seiberg seesaw $\langle M_j \rangle = C/m_j$ creates a hierarchical texture (the Fritzsch matrix) that produces small CKM angles via $\sin \theta_C = \sqrt{m_d/m_s}$. In the lepton sector, if the cubic coupling $L_1 L_2 L_3$ is present, it treats all three fields democratically (invariant under S_3), providing a natural source of large PMNS mixing angles:

Sector	Texture	Mixing
Quarks	Seiberg seesaw (hierarchical)	small CKM
Leptons	$L_1 L_2 L_3$ coupling (democratic)	large PMNS

6.4 Parameter counting for the lepton sector

Two parameters determine three lepton masses:

Parameter	What it determines	Count
$m_{\text{OR}} = 470.76 \text{ MeV}$	Overall scale: $M_0 = 4 m_{\text{OR}}$	1
$gv/m = \sqrt{3}$	Energy-balance condition (constraint)	0
$\delta_{\text{bloom}} = -2.27^\circ$	Electron mass generation	1

Equivalently: specifying any two of (m_e, m_μ, m_τ) determines the third, via $\langle z_k^2 \rangle = z_0^2$.

7 Electroweak predictions

7.1 The Higgs mass at $\tan \beta = 1$

At $\tan \beta = 1$, the D -term quartic $(g^2 + g'^2)(|H_u|^2 - |H_d|^2)^2/8$ vanishes on the D -flat direction $H_u = H_d$. The tree-level lightest CP-even Higgs mass is $m_h = m_Z |\cos 2\beta| = 0$ —a known MSSM result.

In the NMSSM, a singlet coupling $\lambda_S S H_u \cdot H_d$ generates an extra quartic $\lambda_S^2 |H_u \cdot H_d|^2$; at $\sin 2\beta = 1$, $m_h^2 = \lambda_S^2 v^2/2$, and $\lambda_S = 0.72$ gives $m_h = 125 \text{ GeV}$ at tree level.

The Seiberg Lagrange multiplier X cannot serve as the NMSSM singlet: $[X] = \text{mass}^{-3}$, so the coupling $\lambda_0 X H_u \cdot H_d$ requires $[\lambda_0] = \text{mass}^4$. Dimensional analysis gives $\lambda_{\text{NMSSM}} = \hat{\lambda}$ with $\hat{\lambda} \approx 0.51$, but positive meson VEVs require $\hat{\lambda} < 2\Lambda^2/v^2 \approx 3 \times 10^{-6}$: a five-order-of-magnitude conflict. Therefore S must be a *separate* elementary singlet.

Quantity	Value
m_h (tree-level, $\lambda_S = 0.72$)	125.35 GeV
m_h (PDG)	$125.25 \pm 0.17 \text{ GeV}$
Pull	0.6σ

7.2 The Cabibbo angle

The empirical relation $\sin \theta_C \approx \sqrt{m_d/m_s}$, noticed by Gatto, Sartori and Tonin [?] shortly after the Cabibbo hypothesis, was one of the original motivations for quark compositeness—the idea that if the Cabibbo angle is a ratio of constituent masses, the quarks themselves may have substructure. In the sBootstrap the substructure is not preons but the SQCD

seesaw: the Fritzsch texture emerges from the inverse mass hierarchy $\langle M_j \rangle \propto 1/m_j$, and the Cabibbo angle becomes a derived quantity of the Lagrangian.

The UV quark mass matrix has a nearest-neighbour (Fritzsch) texture

$$M = \begin{pmatrix} 0 & A & 0 \\ A^* & 0 & B \\ 0 & B^* & C \end{pmatrix}, \quad (63)$$

a structural property of the seesaw vacuum $M \propto \hat{m}^{-T}$ that inverts the mass hierarchy while preserving eigenvectors. In the hierarchy $|A| \ll |B| \ll C$, the eigenvalues are $m_1 \approx |A|^2 C / |B|^2$, $m_2 \approx |B|^2 / C$, $m_3 \approx C$, and the 1–2 mixing angle satisfies the Gatto–Sartori–Tonin relation [?]:

$$\sin \theta_C = \sqrt{\frac{m_d}{m_s}} = \sqrt{\frac{4.67}{93.4}} = 0.2236. \quad (64)$$

Up-type correction. With separate Fritzsch textures for up-type and down-type quarks, the CKM element receives a phase-dependent correction:

$$|V_{us}| = \left| \sqrt{\frac{m_d}{m_s}} - e^{i\phi} \sqrt{\frac{m_u}{m_c}} \right|. \quad (65)$$

The second term $\sqrt{m_u/m_c} = 0.041$ is an 18% correction. With $\phi = \pi/2$ the classic Fritzsch prediction gives $|V_{us}| = 0.2274$ (3.8σ high); the PDG value 0.2243 ± 0.0008 is reproduced at $\phi \approx 86^\circ$.

Method	$\sin \theta_C$	θ_C
$\sqrt{m_d/m_s}$ (GST)	0.2236	12.92°
$ V_{us} $ (PDG 2024)	0.2243 ± 0.0008	12.96°
Deviation	-0.3%	(-0.9σ)

Charge-language interpretation. In the angular parametrisation (??) with $k = 0, 1, 2$ labelling (d, s, b) , the GST ratio becomes

$$\sin \theta_C = \frac{|1 + \sqrt{2} \cos \delta|}{|1 + \sqrt{2} \cos(\delta + 2\pi/3)|}. \quad (66)$$

At the seed $\delta = 3\pi/4$, the numerator vanishes ($1 + \sqrt{2} \cos(3\pi/4) = 0$), forcing $m_d = 0$ and $\theta_C = 0$. A bloom $\delta = 3\pi/4 + \varepsilon$ opens both a nonzero m_d and a nonzero Cabibbo angle; they are the *same degree of freedom*. The mass-charge identity $\langle z_k^2 \rangle = z_0^2$ is an identity $\sum \cos^2(\delta + 2\pi k/3) = 3/2$ for all δ and imposes no constraint on ε . The Cabibbo angle is thus controlled entirely by δ : it measures how far the down-type charge configuration has bloomed from the massless seed. The scale parameter z_0 remains separate.

This is not an independent prediction of the superpotential but a structural property of the seesaw-inverted mass hierarchy. The free parameters m_u and m_d sit outside the mass chain; the GST relation connects them to the CKM matrix. The ISS model with $N_f = 4$ confines four flavors (u, d, s, c) . The CKM matrix is determined by off-diagonal entries in the UV mass matrix $\hat{\mu}$ *within* each electroweak sector: cross-block entries connecting up-type to down-type give zero CKM mixing (the Schur complement of the seesaw-inverted

matrix is diagonal). With only four flavors, the maximum CKM structure is 2×2 (the Cabibbo angle). The 2–3 mixing V_{cb} requires the top and bottom quarks, but $N_f = 5$ lies outside the ISS conformal window $N_c < N_f < 3N_c/2 = 4.5$. The six-texture-zero Fritzsch ansatz, applied formally to the 3×3 case, gives $|V_{cb}| = |\sqrt{m_s/m_b} - e^{i\psi}\sqrt{m_c/m_t}| \geq 0.064$, overshooting the PDG value 0.0422 by more than 50%. This structural failure (known since the 1990s) signals that the 2–3 sector requires either a modified texture, RG running, or a separate mechanism for the top–bottom coupling. The $\mathbb{Z}_3$ orbifold of Section ?? provides a candidate: Yukawa couplings between different twisted sectors are suppressed by $\epsilon_1 = e^{-2\pi\sqrt{3}/12} \approx 0.40$ per T^2 factor. The product over the three tori in T^6 gives $\epsilon_1^3 = e^{-\pi\sqrt{3}/2} \approx 0.066$, which is in the right range for $|V_{cb}|$ but overshoots by 56%. The precise value depends on the worldsheet instanton calculation and the fixed-point assignment of quarks — details that await a concrete Type I model.

7.3 The Casimir eigenvalue equation

Consider the eigenvalue equation

$$x^2 + C_2 x - C_2 = 0, \quad C_2 = s(s+1), \quad (67)$$

where C_2 is the quadratic Casimir of $SU(2)$ at spin s . Three structural constraints pin the polynomial: (i) no spin-independent mass term, (ii) a massless eigenstate at $s = 0$, (iii) coefficient antisymmetry. The solutions are

$$x_{\pm}(s) = \frac{-C_2 \pm \sqrt{C_2^2 + 4C_2}}{2}. \quad (68)$$

Defining the ratio $R \equiv 1 - x_+(1/2)/x_+(1)$ and evaluating at $s = 1/2$ ($C_2 = 3/4$) and $s = 1$ ($C_2 = 2$):

$$R = \frac{(\sqrt{19} - 3)(\sqrt{19} - \sqrt{3})}{16} \approx 0.2231013\dots \quad (69)$$

This should be compared with the on-shell electroweak mixing parameter $\sin^2\theta_W^{\text{os}} = 0.22320 \pm 0.00026$ (PDG 2024).

The W -mass prediction from $M_W = M_Z\sqrt{1-R}$:

$$M_W^{\text{pred}} = 91.1876 \times \sqrt{1 - 0.22310} = 80.374 \pm 0.002 \text{ GeV}, \quad (70)$$

which lies 0.39σ from the PDG average $M_W = 80.369 \pm 0.013 \text{ GeV}$ and strongly disfavors the anomalous CDF-II measurement (6.2σ tension).

The dynamical origin of this equation and the choice of renormalization scheme (on-shell vs. $\overline{\text{MS}}$) remain open problems.

8 Complete particle spectrum

8.1 O’Raifeartaigh model spectrum

At general $t = gv/m$, the scalar potential $V = \sum |F_k|^2$ depends on a second dimensionless parameter $y = gf/m^2$ (the SUSY-breaking strength, with $y < 1/2$ for stability). The

6×6 real scalar mass matrix in the basis $(\sigma_0, \sigma_1, \sigma_2, \pi_0, \pi_1, \pi_2)$ is block-diagonal, $\mathcal{M}_S^2 = \text{diag}(\mathcal{M}_\sigma^2, \mathcal{M}_\pi^2)$, with

$$\frac{\mathcal{M}_\sigma^2}{m^2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4t^2+2y+1 & 2t \\ 0 & 2t & 1 \end{pmatrix}, \quad \frac{\mathcal{M}_\pi^2}{m^2} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 4t^2-2y+1 & 2t \\ 0 & 2t & 1 \end{pmatrix}. \quad (71)$$

The B -term $B_{11} = 2gf$ is the sole nonvanishing element of $B_{ij} = \sum_k F_k W_{kij}$, contributing $+2y$ to σ and $-2y$ to π . The four nonzero eigenvalues are

$$\frac{m_{\sigma,\pm}^2}{m^2} = 2t^2 + y + 1 \pm \sqrt{(2t^2 + y)^2 + 4t^2}, \quad \frac{m_{\pi,\pm}^2}{m^2} = 2t^2 - y + 1 \pm \sqrt{(2t^2 - y)^2 + 4t^2}. \quad (72)$$

At $t = \sqrt{3}$, each fermion mass is bracketed by its σ (heavier) and π (lighter) scalar partners. In the SUSY limit $y \rightarrow 0$, the σ - π splittings vanish and each fermion mass-squared $m_\pm^2 = 7 \pm 4\sqrt{3}$ is doubly degenerate.

The supertrace vanishes identically: $\text{STr}[\mathcal{M}^2] = m^2(8t^2+4) - 2m^2(4t^2+2) = 0$ for all t and y , because the B -term contributions $\pm 2y$ cancel between sectors.

8.2 Scalar meson masses

At the Seiberg vacuum with soft breaking $\tilde{m}^2 \sim f_\pi^2$:

- 12 off-diagonal real scalar modes: all degenerate at $m = \sqrt{2} f_\pi \approx 130$ MeV.
- 6 diagonal real scalar modes: all at $m = f_\pi = 92$ MeV.

All scalar meson masses are positive. The Yukawa-induced F -terms are nonzero at the seesaw vacuum but do not couple to the off-diagonal meson directions: the off-diagonal cofactors vanish at a diagonal matrix. A fundamental block-diagonal obstruction prevents cross-flavour meson mixing.

8.3 Mesino fermion masses

The off-diagonal mesinos form three Dirac pairs with masses set by the inverted seesaw hierarchy $|X_0|\langle M_k \rangle \propto 1/m_k$:

Pair	Spectator	Dirac mass
$\psi(M_d^u, M_u^d)$	s	1.1×10^{-5} MeV
$\psi(M_s^u, M_u^s)$	d	2.3×10^{-4} MeV
$\psi(M_s^d, M_d^s)$	u	4.9×10^{-4} MeV

The mass hierarchy is inverted relative to quark masses: off-diagonal mesinos involving the lightest quarks are heaviest. The ISS Coleman–Weinberg potential from these loops transmits the quark mass ratios through the confined spectrum.

8.4 Baryons

With $m_B = \Lambda = 300$ MeV in the superpotential $W_B = m_B B\tilde{B}$: scalar baryons at 300 MeV and a Dirac baryonino pair at 300 MeV. SUSY is approximately preserved in the baryon sector.

8.5 Heavy quark sector

Quark	Q_{em}	Mass	Origin
charm	+2/3	1275 MeV (1301 predicted)	O’Raifeartaigh seed
bottom	-1/3	4180 MeV (4233 predicted)	Bion mass relation
top	+2/3	172,760 MeV	Direct Yukawa y_t

8.6 Higgs sector

The lightest CP-even Higgs at $m_h = \lambda_S v / \sqrt{2} = 125.35 \text{ GeV}$ (0.6σ from PDG). Heavy NMSSM states (H , A , $H^\pm$, singlino) depend on soft parameters not specified by the confined sector.

8.7 Supertrace

$\text{STr}[M^2] = 2 \text{Tr}(M_{\text{soft}}^2) = 18 f_\pi^2 = 152,352 \text{ MeV}^2$: the W_{IJ}^2 piece cancels between scalar and fermion sectors, and the holomorphic third-derivative piece cancels between real and imaginary scalar components, leaving only the nine soft meson masses.

8.8 Spectrum flow

The spectrum evolves through six stages as successive Lagrangian terms are switched on. Table ?? tracks the scalar and fermion content at each stage, all evaluated at $t = \sqrt{3}$. Mass-squareds are in units of m^2 (the chiral mass parameter).

Stage	Fermions	Scalars	STr
0. Full SUSY ($y = 0$)	$(0, m_-, m_+)$	$\sigma = \pi = (0, m_-^2, m_+^2)$; exact pairs	0
1. F -breaking ($f \neq 0$)	unchanged	B -term splits: $\pi_k^2 < m_k^2 < \sigma_k^2$; splittings $\propto y$	0
2. CW stabilization	unchanged	$\sigma_0 \rightarrow m_{\text{CW}} \approx 13 \text{ MeV}$; π_0 remains light	0
3. Bion K	v_0 -doubling: $(m_s, m_c) \rightarrow (m_s, m_c, m_b)$	scalars track fermion masses	0
4. 3-instanton	δ selected; bloom angle fixed	angular potential adds δ -dependent shift	0
5. Soft ($\tilde{m}^2 = f_\pi^2$)	unchanged	all scalars $\rightarrow m^2 + f_\pi^2$; mesons at $\sqrt{2} f_\pi$	$18 f_\pi^2$

Table 2: Spectrum flow from SUSY to the physical vacuum. $m_\pm = (2 \pm \sqrt{3})m$, so $m_-^2 = (7-4\sqrt{3})m^2 \approx 0.072m^2$, $m_+^2 = (7+4\sqrt{3})m^2 \approx 13.93m^2$. The supertrace vanishes at every stage except the last, where only the nine soft meson masses contribute.

At Stage 0, the three chiral multiplets form exact $\mathcal{N} = 1$ supermultiplets: each Weyl fermion of mass m_k pairs with a complex scalar (one σ and one π component) of mass-squared m_k^2 . At Stage 1, the B -term $B_{11} = 2gf$ splits each pair into a heavier σ and a lighter π partner while preserving the supertrace. The ordering $\pi_k^2 < m_k^2 < \sigma_k^2$ holds at all $y < 1/2$ and $t = \sqrt{3}$, with splittings proportional to y . The CW potential (Stage 2) lifts

only the goldstino multiplet’s flat directions; Stages 3–4 constrain the mass ratios (bion: radial; instanton: angular) without altering the multiplet structure. Only at Stage 5 does the supertrace become nonzero: $\text{STr}[M^2] = 18 f_\pi^2$, because the soft scalar masses $\tilde{m}^2 = f_\pi^2$ break the boson–fermion mass balance.

8.9 Supermultiplet identification

The sBootstrap identification maps the fermion component of each meson superfield M_j^i to a lepton, and the scalar component to the physical meson. In the SUSY limit ($y \rightarrow 0$, $\tilde{m}^2 \rightarrow 0$), each lepton is exactly degenerate with its meson partner:

Multiplet	Fermion (lepton)	Scalar (meson)	SUSY-limit mass
Φ_0 (goldstino)	e (0.511 MeV)	π (135 MeV)	$\rightarrow 0$ (seed)
Φ_1 (light)	μ (105.7 MeV)	K (494 MeV)	$m_- = (2 - \sqrt{3}) m$
Φ_2 (heavy)	τ (1,777 MeV)	D_s (1,968 MeV)	$m_+ = (2 + \sqrt{3}) m$

In the physical spectrum, the soft mass $\tilde{m}^2 = f_\pi^2$ dominates the lightest multiplet—lifting π far above e ($m_\pi/m_e \approx 264$)—is comparable to the intermediate one ($m_K/m_\mu \approx 4.7$), and negligible for the heaviest ($m_{D_s}/m_\tau \approx 1.11$). This pattern—large fractional splitting for the lightest generation, approximate degeneracy for the heaviest—is characteristic of a universal soft mass $\tilde{m}^2 = f_\pi^2$ applied to a hierarchical spectrum.

8.10 Parameter budget

Relation	Predicts	From
Isospin seed: $m_s \approx m_s$ from m_u+m_d $(2+\sqrt{3})^2(m_u+m_d)$		$\langle z_k^2 \rangle \approx z_0^2$ on $(0, m_u+m_d, m_s)$; bloomed
O’Raifeartaigh: $m_c/m_s = m_c$ from m_s $(2+\sqrt{3})^2$		W : $gv/m = \sqrt{3}$
Bion: $\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}$	m_b from m_s, m_c	K_{bion} : minimum
Yukawa: $\langle z_k^2 \rangle = z_0^2$ on (c, b, t)	m_t from m_c, m_b	W_{Yukawa} at $\tan \beta = 1$
Dual: $\langle \xi_k^2 \rangle = \xi_0^2$ on (d, s, b)	m_d from m_s, m_b	Seiberg seesaw
Lepton: $\langle z_k^2 \rangle = z_0^2$ on (e, μ, τ)	m_τ from m_e, m_μ	W_{lepton} : O’Raifeartaigh
GST: $\sin \theta_C = \sqrt{m_d/m_s}$	θ_{12} from m_d, m_s	Seesaw structure
Remaining free:	$m_u, m_e, m_\mu, \theta_{23}, \theta_{13}, \delta_{\text{CP}}$ (6 parameters)	

Renormalisation scale. The quark masses appearing in the mass-charge triples are $\overline{\text{MS}}$ masses at the PDG conventional scales: $m_{u,d,s}$ at $\mu = 2 \text{ GeV}$, $m_c(m_c)$, $m_b(m_b)$, $m_t(m_t)$. Running all six quarks to a common scale $\mu = 2 \text{ GeV}$ degrades the ratios sharply: $\langle z_k^2 \rangle/z_0^2$ for $(-s, c, b)$ worsens from 1.2% to 3.2%, and for (c, b, t) from 0.4% to 6.2%. In fact, the mass-charge ratio is *exactly* invariant under pure QCD running. Under one-loop QCD, all quark masses run with a universal anomalous dimension: $m_i(\mu) = m_i(\mu_0) [\alpha_s(\mu)/\alpha_s(\mu_0)]^{\gamma_m}$. The common factor cancels in the ratio:

$$\frac{\sum c m_k^{(0)}}{(\sum \sqrt{c m_k^{(0)}})^2} = \frac{c \sum m_k^{(0)}}{c (\sum \sqrt{m_k^{(0)}})^2} = \frac{\sum m_k^{(0)}}{(\sum \sqrt{m_k^{(0)}})^2}, \quad (73)$$

where $c = [\alpha_s(\mu)/\alpha_s(\mu_0)]^{2\gamma_m}$. The mass-charge ratio depends only on mass *ratios*, and QCD preserves all mass ratios. Electroweak corrections break this universality (up-type and down-type quarks have different anomalous dimensions above M_W), but in the confined sector where the identity is set, only QCD is active. The isospin seed is a special case: $m_1 = 0$, $m_2 = m_u + m_d$, $m_3 = m_s$, immune to running because all entries share the same anomalous dimension and the ratio $m_s/(m_u + m_d)$ is an RG invariant. The Gell-Mann–Oakes–Renner relation [?] $m_\pi^2 f_\pi^2 = (m_u + m_d) \langle \bar{q}q \rangle$ makes this ratio a statement about physical hadron observables (m_π , f_π , m_K), explaining why the isospin seed is scheme-independent. None of the quark triples sits exactly at $\langle z_k^2 \rangle = z_0^2$: the bloom shifts each triple away from the seed, by 0.27% (isospin), 0.4% (c, b, t), and 1.2% ($-s, c, b$). The lattice ratio $m_s/m_{ud} = 27.23 \pm 0.10$ (FLAG 2021) resolves the isospin bloom at 6.3σ from the exact seed value $2(2+\sqrt{3})^2 = 27.86$. For the cross-threshold triples ($-s, c, b$) and (c, b, t), the preferred convention—each quark at its own mass—is natural in the ISS framework, where each meson M_j has dynamics at scale $\mu \sim m_j$.

9 Summary of predictions

Quantity	Predicted	PDG 2024	Source
m_s	95.1 MeV	93.4 ± 8.6 (0.2σ)	Isospin seed
m_c	1,301 MeV	$1,275 \pm 25$ (1.0σ)	O’Raifeartaigh: $gv/m = \sqrt{3}$
m_b	4,233 MeV	$4,180 \pm 30$ (1.8σ)	Bion Kähler
m_t	171.3 ± 1.5 GeV	172.76 ± 0.30	Yukawa eigenvalue constraint
m_d	4.6 MeV	4.67 ± 0.48 (0.1σ)	Dual mass-charge identity
m_u/m_d	0.453	0.46 ± 0.05 (0.2σ)	Self-consistency
m_τ	1776.97 MeV	1776.86 ± 0.12 (0.9σ)	Lepton mass-charge identity
$\sin^2 \theta_W$	0.22310	0.22320 ± 0.00026 (0.4σ)	Casimir equation
V_{us}	0.2236	0.2243 ± 0.0008 (0.9σ)	GST relation
m_h	125.35 GeV	125.25 ± 0.17 (0.6σ)	NMSSM: $\lambda_S v/\sqrt{2}$
$\tan \beta$	1	— (not directly measured)	Mixed-triple structure
$\text{STr}[M^2]$	$18 f_\pi^2$	— (not directly measured)	Soft mass only

Table 3: Predictions from the sBootstrap Lagrangian. All quark predictions follow from a single input $m_u + m_d = 6.83$ MeV (equivalently m_π via GMOR); the isospin seed gives m_s , the O’Raifeartaigh chain gives m_c , the bion gives m_b , and the dual mass-charge identity gives m_d (hence m_u). Cross-threshold masses are $\overline{\text{MS}}$ at $\mu = m_q$. The lepton prediction uses m_e and m_μ as inputs.

10 The Lagrangian from the viewpoint of hadronic superstrings

The composites of the sBootstrap — mesons in the **24** and diquarks in the $\mathbf{15} \oplus \overline{\mathbf{15}}$ — fill exactly the representations that appear when the adjoint of $SO(32)$ is decomposed under $SU(5) \times SU(3)$. The group $SO(32)$ is the gauge group of Type I open superstring theory, and this coincidence invites a literal interpretation: the sBootstrap Lagrangian may be the low-energy effective theory of a superstring at the hadronic scale. Such a

string would not live at the Planck scale (10^{19} GeV) but at the QCD scale (~ 1 GeV) — returning string theory to the hadronic setting in which it was originally conceived [?, ?].

This section develops that interpretation in detail.

10.1 Open strings and Chan-Paton labels

A relativistic string is a one-dimensional extended object whose quantised vibrations produce particles. An *open* string has two free endpoints; among its massless vibrational modes are a spin-1 particle (a gauge boson) and, in the supersymmetric case, a spin-1/2 partner. The mass gap to the first excited string mode is set by the string tension $T = 1/(2\pi\alpha')$; the *string scale* is $M_s = 1/\sqrt{\alpha'}$.

Each endpoint of an open string carries a discrete label $i \in \{1, \dots, N\}$, called a *Chan-Paton label* [?]. A string state $|i, j\rangle$ (one label per endpoint) transforms under $U(N)$: the N^2 massless gauge bosons fill the adjoint of $U(N)$.

In *Type I* string theory, the string is unoriented: the orientation-reversal projection Ω identifies $|i, j\rangle$ with $|j, i\rangle$. This halves the gauge degrees of freedom and projects $U(N)$ down to $O(N)$ or $Sp(N)$, depending on the sign of the orientifold projection. Consistency of the ten-dimensional quantum theory — specifically, cancellation of gauge and gravitational anomalies via the Green–Schwarz mechanism [?] — requires exactly

$$N = 32, \quad G = SO(32). \quad (74)$$

The adjoint representation is the antisymmetric product of two fundamentals:

$$\mathbf{496} = \wedge^2(\mathbf{32}), \quad (75)$$

containing $32 \times 31/2 = 496$ gauge bosons.

10.2 The $SO(32)$ adjoint contains the sBootstrap

The fundamental $\mathbf{32}$ of $SO(32)$ is a real representation; it decomposes under the maximal subgroup $SU(16) \times U(1)_A$ as

$$\mathbf{32} = \mathbf{16}_{+1} \oplus \overline{\mathbf{16}}_{-1}. \quad (76)$$

The 16-dimensional complex representation of $SU(16)$ further decomposes under $SU(5) \times SU(3) \times U(1)_B$ as

$$\mathbf{16} = (\mathbf{5}, \mathbf{3})_{+1} \oplus (\mathbf{1}, \mathbf{1})_{-15}. \quad (77)$$

The 15 entries $(\mathbf{5}, \mathbf{3})$ are precisely the five quark flavors (u, d, s, c, b) in three colors (r, g, b). The remaining singlet $(\mathbf{1}, \mathbf{1})$ is the top quark, which does not participate in the composite sector. The identity

$$5 \times 3 + 1 = 16 \quad (78)$$

is the Chan-Paton dimension of a single D-brane stack.

Adjoint decomposition. The adjoint $\mathbf{496} = \wedge^2(\mathbf{32})$ decomposes under $SU(16) \times U(1)_A$ as

$$\mathbf{496} = (\mathbf{255} \oplus \mathbf{1})_0 \oplus \mathbf{120}_{+2} \oplus \overline{\mathbf{120}}_{-2}. \quad (79)$$

The further decomposition under $SU(5) \times SU(3)$ is given in Table ??.

Sector	$SU(5) \times SU(3)$	dim	q_A	q_B	Origin
255 ⊕ 1	(24, 8)	192	0	0	adj($SU(15)$)
	(24, 1)	24	0	0	mesons
	(1, 8)	8	0	0	adj($SU(3)$)
	(5, 3) + ($\bar{5}$, $\bar{3}$)	30	0	±16	cross-terms
	$2 \times (1, 1)$	2	0	0	$U(1)_{A,B}$
120	(15, $\bar{3}$)	45	+2	+2	diquarks
	($\bar{10}$, 6)	60	+2	+2	$\wedge^2(\mathbf{5}) \otimes S^2(\mathbf{3})$
	(5, 3)	15	+2	−14	cross-term
$\bar{120}$	($\bar{15}$, 3)	45	−2	−2	diquarks
	(10, $\bar{6}$)	60	−2	−2	conjugate
	($\bar{5}$, $\bar{3}$)	15	−2	+14	conjugate
Total		496			

Table 4: Decomposition of the $SO(32)$ adjoint under $SU(5)_{\text{flavour}} \times SU(3)_{\text{colour}}$. The boldface entries are the sBootstrap composites: mesons $(\mathbf{24}, \mathbf{1})$ and diquarks $(\mathbf{15}, \bar{\mathbf{3}}) \oplus (\bar{\mathbf{15}}, \mathbf{3})$, accounting for $24 + 45 + 45 = 114$ of the 496 states.

Why the symmetric 15 appears. The key algebraic content of $\wedge^2(\mathbf{16})$ comes from $\wedge^2[(\mathbf{5}, \mathbf{3})]$ — the antisymmetric product of two quarks, each carrying flavour and colour indices. The antisymmetric product of a tensor-product space obeys the identity

$$\wedge^2(V \otimes W) = [S^2(V) \otimes \wedge^2(W)] \oplus [\wedge^2(V) \otimes S^2(W)]. \quad (80)$$

Overall antisymmetry of the composite index (ia) — one flavour index i and one colour index a per quark — forces a correlation: symmetric in one factor implies antisymmetric in the other. Setting $V = \mathbf{5}$ (flavour), $W = \mathbf{3}$ (colour):

$$\wedge^2[(\mathbf{5}, \mathbf{3})] = \underbrace{(\mathbf{15}, \bar{\mathbf{3}})}_{\text{flavour-symmetric, colour-antisymmetric}} \oplus \underbrace{(\bar{\mathbf{10}}, \mathbf{6})}_{\text{flavour-antisymmetric, colour-symmetric}}. \quad (81)$$

The symmetric flavour $\mathbf{15} = S^2(\mathbf{5})$ is forced into antisymmetric colour $\bar{\mathbf{3}} = \wedge^2(\mathbf{3})$; the pairing is not a modeling choice but a consequence of the tensor-product structure of open-string states.

Why not $E_8 \times E_8$. The heterotic string in ten dimensions admits two possible gauge groups: $SO(32)$ or $E_8 \times E_8$. The second is excluded by the sBootstrap. Under the chain $E_8 \supset SU(5) \times SU(5)$, every E_8 representation decomposes into exterior powers of $\mathbf{5}$ ($\mathbf{5}$, $\bar{\mathbf{10}} = \wedge^2 \mathbf{5}$, $\mathbf{24}$, ...). The symmetric product $S^2(\mathbf{5}) = \mathbf{15}$ never appears. This is not a matter of choosing a different embedding; it is an algebraic obstruction rooted in the structure of E_8 . In the $SO(16) \times SO(16)$ subgroup of $SO(32)$, the obstruction is equally sharp: no single $\mathbf{16}$ of $SO(16)$ can hold the 15-dimensional block $(\mathbf{5}, \mathbf{3})$, so $SU(5)$ and $SU(3)$ embed as separate direct summands whose $\wedge^2$ produces only $\bar{\mathbf{10}}$, never $\mathbf{15}$. The sBootstrap thus selects $SO(32)$ over $E_8 \times E_8$ as the unique string-theoretic gauge group compatible with its composite spectrum.

10.3 Compactification: the $\mathbb{Z}_3$ orbifold and three generations

Type I superstring theory is consistent in ten spacetime dimensions. To make contact with four-dimensional physics, the six extra dimensions must be compact. The simplest compact space is a six-torus $T^6 = (T^2)^3$, where each factor T^2 is a two-dimensional torus defined by a lattice $\Lambda \subset \mathbb{C}$.

A torus alone preserves too much supersymmetry ($\mathcal{N} = 4$ in 4d) and produces no chiral fermions. Both problems are solved by an *orbifold*: the quotient of T^6 by a discrete symmetry group G that acts simultaneously on the compact coordinates and on the Chan-Paton labels. The surviving states are G -invariant.

For the sBootstrap, the natural choice is $G = \mathbb{Z}_3$, acting on the three complex extra coordinates as

$$\theta : (z_1, z_2, z_3) \longrightarrow (\omega z_1, \omega z_2, \omega z_3), \quad \omega = e^{2\pi i/3}. \quad (82)$$

This is the unique $\mathbb{Z}_3$ action that preserves exactly $\mathcal{N} = 1$ SUSY in four dimensions (it embeds into $SU(3)$ holonomy, satisfying the Calabi–Yau condition).

Chan-Paton embedding and tadpole cancellation. The $\mathbb{Z}_3$ must also act on the 16 D9-brane Chan-Paton labels (before orientifold images) via a matrix γ_θ in the fundamental. Two consistency conditions must be satisfied:

1. The orientifold Ω exchanges branes with eigenvalue ω and those with $\omega^2 = \bar{\omega}$, so their multiplicities must be equal: $n_1 = n_2$.
2. The *twisted tadpole cancellation* condition requires $\text{Tr}(\gamma_\theta) = 1$ (in the absence of D5-branes).

With n_0 branes at eigenvalue 1, and $n_1 = n_2$ branes at eigenvalues ω, ω^2 :

$$\text{Tr}(\gamma_\theta) = n_0 + n_1\omega + n_2\omega^2 = n_0 + n_1(\omega + \omega^2) = n_0 - n_1, \quad (83)$$

using $\omega + \omega^2 = -1$. The conditions $n_0 + 2n_1 = 16$ and $n_0 - n_1 = 1$ give the unique solution

$$n_0 = 6, \quad n_1 = n_2 = 5. \quad (84)$$

The Chan-Paton matrix is therefore

$$\gamma_\theta = \text{diag}(\underbrace{\omega, \dots, \omega}_5, \underbrace{\omega^2, \dots, \omega^2}_5, \underbrace{1, \dots, 1}_6). \quad (85)$$

The five ω -branes and five ω^2 -branes are exchanged by Ω , forming an $SU(5)$ gauge factor. The six ω^0 -branes must be partitioned into Ω -exchanged pairs; the partition $[3, 3]$ gives

$$G = SU(5)_{\text{flavour}} \times SU(3)_{\text{colour}} \times U(1)^2. \quad (86)$$

The partition $[3, 3]$ is the *unique* choice yielding $SU(3)$ as a direct gauge factor: $[6]$ gives $SO(6)$, $[4, 2]$ gives $SU(4) \times SU(2)$, $[2, 2, 2]$ gives $SU(2)^3$.

Three chiral generations. The chiral spectrum from the D9-brane sector in the $\mathbb{Z}_3$ orbifold is

$$3 \times [(\mathbf{10}, \mathbf{1}) + (\bar{\mathbf{5}}, \mathbf{3}) + (\mathbf{5}, \bar{\mathbf{3}})] , \quad (87)$$

where the multiplicity 3 is the number of $\mathbb{Z}_3$ twisted sectors, and the $\mathbf{10}$ is the antisymmetric two-index representation of $SU(5)_f$. This antisymmetric projection ($\varepsilon = +1$) is forced by the odd dimension of the charge-0 blocks ($n = 3$), which cannot support an Sp-type projection (Sp requires even dimension). Whether the symmetric $\mathbf{15}$ can be obtained through a modified orientifold remains an open question (Section ??).

10.4 The string scale and extra dimensions

In string theory, gauge interactions (open strings) are confined to the D-branes, while gravity (closed strings) propagates in all ten dimensions. The effective four-dimensional Planck mass M_{Pl} depends on the total volume V_6 of the compact space.

Derivation. In ten dimensions, Newton’s constant is $G_{10} \sim g_s^2/M_s^8$. In four dimensions after compactification on a volume V_6 , Gauss’s law gives $G_4 = G_{10}/V_6$, hence

$$M_{\text{Pl}}^2 = \frac{1}{G_4} \sim \frac{M_s^8 V_6}{g_s^2} . \quad (88)$$

If n of the six dimensions have a common radius R (the “large” dimensions) and the remaining $6 - n$ have the string radius $R_s \sim 1/M_s$ (the “small” dimensions):

$$V_6 \sim R^n \times M_s^{-(6-n)} , \quad \text{so} \quad M_{\text{Pl}}^2 \sim \frac{M_s^{n+2} R^n}{g_s^2} . \quad (89)$$

Solving for R :

$$R \sim \left(\frac{g_s^2 M_{\text{Pl}}^2}{M_s^{n+2}} \right)^{1/n} . \quad (90)$$

A hadronic string scale. In the sBootstrap, the dynamical scale is $\Lambda_{\text{QCD}} \sim 300$ MeV. Setting the string scale at $M_s \sim 1$ GeV places the string at the *hadronic* scale — historically, the setting in which string theory was invented [?, ?] as a model for the Regge trajectories of hadrons.

With $M_s = 1$ GeV and $g_s = 1$, equation (??) gives the extra-dimension radius as a function of n . Using $M_{\text{Pl}} = 1.22 \times 10^{19}$ GeV:

Large dims n	R	Status
2	~ 400 m	excluded
3	~ 0.2 mm	excluded (short-range gravity tests)
4	~ 1 μm	marginally safe
5	~ 12 nm	safe
6	~ 0.6 nm	safe

Current torsion-balance experiments [?] constrain deviations from Newton’s law at distances down to ~ 40 μm . The scenario with $n \geq 4$ large extra dimensions is experimentally viable. The estimates above assume $g_s \sim 1$; a smaller coupling reduces R .

Equivalently, three of the six compact dimensions could be at the string scale and three large, giving $R \sim 0.2$ mm — already excluded. With four or more large dimensions the radii drop to the micrometre scale or below, safely out of reach.

This is a low-scale string theory in the sense of Arkani-Hamed–Dimopoulos–Dvali, but applied at $M_s \sim 1$ GeV rather than $M_s \sim \text{TeV}$. The gauge hierarchy is not solved — it is *inverted*: the puzzle is why gravity is so weak compared to the hadronic scale.

10.5 The $SU(3)$ root lattice and $gv/m = \sqrt{3}$

The $\mathbb{Z}_3$ orbifold requires each T^2 to have a lattice compatible with the three-fold rotational symmetry. The unique such lattice (up to scaling) is the $SU(3)$ root lattice, generated by the simple roots

$$e_1 = 1, \quad e_2 = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}, \quad (91)$$

in the complex plane. The modular parameter of the resulting torus is $\tau = e^{i\pi/3}$, and the area of the fundamental domain is

$$A = \text{Im}(\tau) = \frac{\sqrt{3}}{2} \quad (\text{in units of } \alpha'). \quad (92)$$

The $\mathbb{Z}_3$ orbifold has three fixed points on each T^2 , located at

$$z = 0, \quad \frac{1}{3}(1 + \tau), \quad \frac{1}{3}(2 + 2\tau). \quad (93)$$

These form an equilateral triangle whose area is $\sqrt{3}/12$ (one-sixth of the fundamental domain).

The number $\sqrt{3}$ appears here in a purely geometric role: it is the imaginary part of the modular parameter of the unique $\mathbb{Z}_3$ -compatible torus. In the sBootstrap Lagrangian, the pseudo-modulus is stabilised at $\langle X \rangle / \mu = \sqrt{3}$ (equivalently $gv/m = \sqrt{3}$). Two suggestive connections deserve mention:

- The Kähler modulus T of each T^2 satisfies $\text{Im}(T) = \sqrt{3}/2$, and $gv/m = \sqrt{3} = 2\text{Im}(T)$.
- The proper distance from $\langle X \rangle = 0$ to the Kähler pole at $\sqrt{3}\mu$ is $\sigma_{\text{max}} = \sqrt{3}\pi m/(4g)$ (Section ??), a finite extent that could correspond to the size of the fundamental domain in the compact space.

Whether these coincidences reflect a physical mechanism — the orbifold geometry fixing the pseudo-modulus VEV — is an open problem. A concrete realisation would require solving the moduli stabilisation problem on the $T^6/\mathbb{Z}_3$ orbifold with the sBootstrap D-brane configuration.

10.6 Fractional instantons and the bloom potential

At each $\mathbb{Z}_3$ fixed point of T^2 , the gauge bundle can carry a *fractional instanton*: an instanton configuration whose topological charge is $1/3$ of the bulk instanton charge. The instanton action is correspondingly reduced:

$$S_{\text{frac}} = \frac{S_0}{3}, \quad (94)$$

where $S_0 = 8\pi^2/g_{\text{YM}}^2$ is the full instanton action. The fractional instanton amplitude is $\sim e^{-S_0/3}$, parametrically larger than the full instanton amplitude $\sim e^{-S_0}$.

The three fixed points on a single T^2 each contribute a fractional instanton. In the effective superpotential, the $\mathbb{Z}_3$ phases of the twisted sectors produce contributions proportional to $e^{2\pi i k/3}$ ($k = 0, 1, 2$). The product over three eigenvalues (one per colour, in the colour-flavour locked phase) yields the determinant $\det M$, and the exact identity

$$\prod_{k=1}^3 [1 + \sqrt{2} \cos(\delta + 2\pi k/3)] = -\frac{1}{2} + \frac{\cos 3\delta}{\sqrt{2}} \quad (95)$$

(proved in Section ??) reflects the $\mathbb{Z}_3$ averaging over fixed points.

The three-instanton operator in the sBootstrap Lagrangian,

$$W_3 = c_3 \frac{(\det M^{(3)})^3}{\Lambda^{18}}, \quad (96)$$

has a natural interpretation as the product over the three copies of T^2 in $T^6 = (T^2)^3$:

Orbifold structure	Lagrangian term
3 eigenvalues per T^2	$\rightarrow \det M$ (product of 3)
3 copies of T^2	$\rightarrow (\det M)^3$ (cube)
$3 \times 3 = 9$ fractional instantons	$\rightarrow \mathbb{Z}_9 \subset \mathbb{Z}_{18}$ symmetry
$\cos(3\delta)$ per T^2	$\rightarrow \cos(9\delta)$ subdominant harmonic

The dominant $\cos(3\delta)$ harmonic is the $\mathbb{Z}_3$ harmonic of a single torus; the $\cos(9\delta)$ subdominant (34% of the $\cos(3\delta)$ amplitude) is the $\mathbb{Z}_9$ harmonic arising from the product over all three tori. The effective bloom potential inherits its harmonic structure from the orbifold geometry.

10.7 Self-referential structure

The sBootstrap is self-referential in a precise sense. The Chan-Paton labels of the Type I string are the quarks themselves:

$$\underbrace{u, d, s, c, b}_{5 \text{ flavours}} \times \underbrace{r, g, b}_{3 \text{ colours}} + \underbrace{t}_{\text{singlet}} = 16 \text{ labels}. \quad (97)$$

In Type I, each D9-brane and its orientifold image form a pair: 16 branes + 16 images = 32 Chan-Paton indices, giving $SO(32)$. The adjoint of this $SO(32)$ decomposes to contain $(\mathbf{24}, \mathbf{1}) \oplus (\mathbf{15}, \mathbf{\bar{3}}) \oplus (\mathbf{\bar{15}}, \mathbf{3})$ — exactly the mesons and diquarks that those same quarks form as composites.

The logical circle is:

1. The quarks define the Chan-Paton labels ($5 \times 3 + 1 = 16$).
2. The labels define the gauge group ($SO(32)$).
3. The adjoint of $SO(32)$ contains the scalar composites of the quarks.
4. Those composites fill the supermultiplets of the $\mathcal{N} = 1$ Lagrangian.

Whether this circularity is a deep consistency condition — a “bootstrap” in the original sense of the word — or a coincidence of small numbers is the central interpretive question raised by the $SO(32)$ embedding.

11 Open problems

11.1 The diquark sector

The Pauli theorem (Section ??) proves that no $J = 0$, color- $\bar{3}$ diquark can have symmetric flavor. The sBootstrap’s **15** diquark must therefore be non-composite: either an algebraic image of the Miyazawa supercharge or a fundamental $SO(32)$ string state (Section ??). However, the mixed anticommutator $\{Q_\alpha, \mathcal{Q}_{[ij],\beta}\} = \epsilon_{\alpha\beta} \mathcal{S}_{[ij]}$ produces a diquark generator in $\overline{\mathbf{10}}$ (antisymmetric), not the symmetric **15**; thus the Miyazawa route alone does not resolve the conflict. The exotic $+4/3$ -charged diquarks of the **15** would be a testable consequence.

Orientifold routes to the symmetric 15. The $\mathbb{Z}_3$ orbifold of Section ?? produces the antisymmetric **10** because the $SU(5)$ Chan-Paton block has odd dimension ($n_1 = 5$), which cannot support an Sp-type ($\varepsilon = -1$) projection. Three natural modifications fail to circumvent this:

1. *D5-branes.* Adding D5-branes at fixed points shifts the twisted tadpole. The only even- n_1 solution is $n_1 = 4$ (twisted tadpole = 4), giving $SU(4)$ rather than $SU(5)$.
2. *Discrete torsion.* Flipping the twisted-sector sign reverses the tadpole condition to $n_0 - n_1 = -1$, giving $n_1 = 17/3$ — not an integer.
3. *Changing the gauge group.* The constraint $n_0 + 2n_1 = 16$, $n_0 - n_1 = 1$ has the unique solution $n_1 = 5$; no other integer satisfies both.

Two viable routes remain, both involving deformations away from the strict orbifold point. First, in *magnetized brane models* (intersecting branes on T^6), the chiral spectrum arises from intersection numbers $I_{aa'}$ of a brane stack a with its orientifold image a' . The symmetric **15** appears when a wraps cycles intersecting O^- planes, and three generations require $I_{aa'} = 3$. Second, *local $O5^-$ planes* at the $\mathbb{Z}_3$ fixed points can flip the projection from antisymmetric to symmetric at the locus where the $SU(5)$ matter lives, even though the global gauge group remains $SO(32)$. Both mechanisms select a specific region of the Type I moduli space; the sBootstrap’s requirement of the symmetric **15** is thus a *moduli constraint* on the string vacuum.

11.2 The bloom mechanism

The mass-charge identity constrains but does not uniquely fix the angular parameter δ that controls how the seed develops into a full triple. The O’Raifeartaigh vacuum fixes the seed ratio $gv/m = \sqrt{3}$ but leaves δ undetermined.

The lowest-dimension R -breaking deformation that affects the spectrum is the bilinear $\Delta W = \epsilon \Phi_0 \Phi_2$, which forces $\langle \Phi_1 \rangle \neq 0$ and mixes the goldstino with the massive sector. (Cubic deformations in Φ_1, Φ_2 alone vanish at the vacuum and are inert.) To first order in $\varepsilon = \epsilon/m$, all eigenvalue shifts vanish, but the bloom angle shifts at $O(\varepsilon)$ because $\delta\delta \sim \sqrt{m_0} \sim \varepsilon$:

$$\delta\delta = 3^{5/4} |\varepsilon| \approx 3.95 |\varepsilon|.$$

The direction is *toward* the physical quark spectrum ($\delta > 3\pi/4$). The determinant condition $m_+ m_- = m^2$ is broken at $O(\varepsilon^2)$ with fractional deviation $49\varepsilon^2$. The seed is therefore

an unstable fixed point under any R -breaking, and the instability points in the right direction. The bloom magnitude, however, requires a nonperturbative mechanism.

A natural candidate for a dynamical origin is the instanton structure of SQCD: for $N_f = N_c = 3$, instantons break $U(1)_{R+2A} \rightarrow \mathbb{Z}_{18}$, which contains $\mathbb{Z}_9$ as a subgroup. The three-instanton operator $W_3 = c_3(\det M)^3/\Lambda^{18}$ generates a δ -dependent potential. To see the harmonic content, note the exact identity

$$\prod_{k=1}^3 [1 + \sqrt{2} \cos(\delta + 2\pi k/3)] = -\frac{1}{2} + \frac{\cos 3\delta}{\sqrt{2}}, \quad (98)$$

so $\det M = z_0^6 f(\delta)^2$ with $f = -1/2 + \cos(3\delta)/\sqrt{2}$. The potential $|W_3|^2 \propto f(\delta)^{12}$ contains all harmonics $\cos(3n\delta)$ for $n = 0, \dots, 6$; the $\cos(9\delta)$ term (the unique $\mathbb{Z}_9$ harmonic) carries only 34% of the amplitude of the dominant $\cos(3\delta)$ mode. The effective potential therefore has six minima per 2π cycle, not nine.

The bion Kähler potential $\delta K \sim |v_0 - 2v_0^{\text{seed}}|^2$ constrains the overall scale z_0 (radial direction in the Cartan plane), while the three-instanton F -term potential constrains the angle δ . For the physical $(-s, c, b)$ triple, the bloom angle $\delta_{\text{phys}} = 159^\circ$ lies 24° past the seed and 21° before the nearest instanton well at 180° , at a *maximum* of the combined effective potential $V_{\text{eff}} = -|f|^{2p} + r(S-3)^2$ for all powers p and positive r . The physical point is therefore a *saddle*: stable in the radial direction (bion minimum), unstable in the angular direction (instanton-driven). This is the same metastability structure as the ISS vacuum itself—the angular instability is slow and the configuration is long-lived. The bloom angle remains an open problem; it may be fixed by the overdetermination of the mass chain rather than by a simple potential minimum.

Dual self-consistency. The Seiberg seesaw maps direct charges $\mathfrak{z}_k = \sqrt{m_k}$ to dual charges $\xi_k = 1/\sqrt{m_k}$. In terms of the elementary symmetric polynomials e_i of the $\mathfrak{z}_k$, the dual variance ratio is $\langle \xi_k^2 \rangle / \xi_0^2 = 1 - 2e_1 e_3 / e_2^2$. In the angular parametrisation, requiring $\langle z_k^2 \rangle = z_0^2$ and $\langle \xi_k^2 \rangle = \xi_0^2$ simultaneously reduces to

$$\cos 3\delta = \frac{5\sqrt{2}}{8}, \quad (99)$$

an isolated angle distinct from the seed ($\delta = 3\pi/4$). At the seed itself the identity $\langle z_k^2 \rangle = z_0^2 \Leftrightarrow \langle \xi_k^2 \rangle = \xi_0^2$ is automatic (one mass vanishes). For the physical down-type quarks (d, s, b) , $\langle \xi_k^2 \rangle / \xi_0^2 = 0.997$ (0.22% from unity) while $\langle z_k^2 \rangle / z_0^2 = 1.10$ —far from unity. The seesaw thus creates an independent near-identity condition on the dual charges, constraining m_d from (m_s, m_b) .

11.3 Supermultiplet restoration

The central open problem of the sBootstrap is to identify the exact supermultiplet pairings: which meson is degenerate with which lepton in the SUSY limit ($f \rightarrow 0$, soft terms off)?

A complete Lagrangian for the sBootstrap would have the property that each SUSY-breaking parameter can be dialled continuously to zero, and at each stage the spectrum reorganises into progressively more degenerate fermion–scalar pairs. At $f = 0$, $\tilde{m}^2 = 0$, and bloom parameter $\varepsilon = 0$, the spectrum collapses to three exact $\mathcal{N} = 1$ chiral

multiplets at masses $(0, m_-, m_+)$. This “restoration” would be the definitive signature of the Miyazawa supersymmetry.

In the O’Raifeartaigh superpotential at $f = 0$, the fermion and scalar mass matrices become degenerate: each fermion eigenvalue m_k has a scalar partner with the same mass. Turning f back on splits each pair by the B -term $\pm 2gf$, and the founding relation $m_\pi^2 - m_\mu^2 \approx f_\pi^2$ is precisely this SUSY-breaking mass splitting for one such pair.

Within the $N_f = 4$ ISS sector, the O’Raifeartaigh eigenvalue structure groups quarks as (u, d) at eigenvalue 0 and (c, s) at eigenvalues (m_+, m_-) — consistent with the standard generations and with $|V_{ud}| \approx |V_{cs}| \approx 1$. The Cabibbo angle is the rotation between the eigenvalue basis and the mass eigenbasis within this 2×2 block. The bottom quark enters through the bion mechanism, with $\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}$ mixing the m_- and m_+ eigenvalues. The V_{cb} element of the CKM matrix measures this mixing: it arises because b is a coherent superposition of ISS eigenstates, not a pure eigenvalue. The cross-generation pairings (t, s) , (u, b) , (c, d) mentioned in the sBootstrap literature refer to the $SU(5)$ representation content, not the O’Raifeartaigh eigenvalue grouping.

The mesino masses at the Seiberg vacuum ($\sim 10^{-3}$ – 10^{-2} MeV, with $\langle z_k^2 \rangle / z_0^2 = 0.44$) are too small and too degenerate to serve as lepton masses. This does not mean that a separate confining sector is needed; rather, it means that the lepton masses are the *tree-level* fermion eigenvalues of the O’Raifeartaigh mass matrix — the same eigenvalues $(0, m_-, m_+)$ that generate the quark spectrum — transmitted to the lepton sector through the Miyazawa superpartner identification. A preliminary calculation reveals that the B -term splitting is *non-universal*: it acts only on the Φ_1 direction of the O’Raifeartaigh superpotential. The mass eigenstates $\psi_\pm$ have different Φ_1 content: $|\langle \psi_- | \Phi_1 \rangle|^2 = 0.067$ (muon eigenstate) versus $|\langle \psi_+ | \Phi_1 \rangle|^2 = 0.933$ (tau eigenstate). The founding relation $m_\pi^2 - m_\mu^2 \approx f_\pi^2$ works precisely because the muon eigenstate has very little Φ_1 content, suppressing its splitting by a factor $\sim 1/14$ relative to the tau. Fixing the muon–pion splitting to f_π^2 predicts a tau meson partner 33 MeV above m_τ , i.e. at ~ 1810 MeV. The $D^\pm$ at 1870 MeV is 93 MeV above m_τ , a factor 2.8 larger than predicted; the m^2 splitting $m_D^2 - m_\tau^2 = 3.4 \times 10^5 \text{ MeV}^2$ exceeds the predicted $1.2 \times 10^5 \text{ MeV}^2$ by a factor ~ 3 . The apparent 3% mass-level agreement is an artefact of the dominance of m_τ^2 ; the splitting itself is only 3.7% of m_τ^2 , so any meson within ~ 200 MeV of the tau would “match” at the 10% level. Identifying the true tau partner requires either closing this factor-of-three gap with radiative corrections or finding a hitherto unobserved $J^P = 0^-$ state near 1810 MeV. The splitting ratio $\cos^2(\pi/12)/\sin^2(\pi/12) = (2 + \sqrt{3})^2 \approx 13.93$ is the *same* structural constant that controls the O’Raifeartaigh fermion mass ratio $m_+/m_- = (2 + \sqrt{3})^2$; both are consequences of the mass-charge condition $t = \sqrt{3}$.

In the mass-charge parametrisation, the seed spectrum $(0, m_-, m_+)$ sits at $\delta = 3\pi/4$ where the lightest eigenvalue vanishes. The bloom $\Delta W = \varepsilon \Phi_0 \Phi_2$ rotates δ by only 2.3° , creating the electron mass while shifting the muon eigenstate *down* by 16% and the tau eigenstate *up* by 1% relative to the seed values. At the seed, $m_+/m_- = (2 + \sqrt{3})^2 = 13.93$; at the physical point, $m_\tau/m_\mu = 16.82$. The entire difference is the bloom correction.

The supercharge and the two zeros. The supercharge Q that maps each lepton to its meson partner acts identically on quarks (mapping them to diquarks) and on leptons (mapping them to mesons). Its explicit action on the three O’Raifeartaigh multiplets is

$$Q|X\rangle = |e\rangle, \quad Q|\Phi_-\rangle = |\mu\rangle, \quad Q|\Phi_+\rangle = |\tau\rangle,$$

where $|X\rangle$ is the pseudo-modulus (a flavour singlet, not a standard meson), and $|\Phi_{\pm}\rangle$ are the scalar mass eigenstates identified with the pion and D meson respectively.

Two particles attempt to reach zero mass through independent protection mechanisms. The *electron*, sitting in the goldstino direction, has zero Φ_1 content and therefore zero B -term splitting; at the seed $\delta = 3\pi/4$ its mass vanishes exactly. The *pion*, a pseudo-Goldstone boson, would be massless in the chiral limit $m_u = m_d = 0$. The electron succeeds: $m_e = 0.511 \text{ MeV}$ is a residue of the 2.3° bloom. The pion fails: the GMOR relation $m_\pi^2 = (m_u + m_d) B_0$ lifts it to 140 MeV . The founding relation

$$m_\pi^2 - m_\mu^2 = f_\pi^2$$

links these two stories: the left-hand side is the B -term splitting inside the muon supermultiplet (SUSY breaking), while the right-hand side is the chiral symmetry breaking scale. The supercharge Q unifies them—the same operator that maps $\mu \leftrightarrow \pi$ connects the SUSY-breaking mass splitting to the chiral condensate.

The SUSY-breaking hierarchy among the three multiplets is inverted relative to mass: the tau- D pair is only 11% split in m^2 (nearest to degeneracy), the muon-pion pair is 74% split, and the electron- X pair is maximally broken (a 0.5 MeV fermion paired with a $\sim 13 \text{ MeV}$ Coleman-Weinberg scalar). Heavier multiplets are *more* supersymmetric. The massive spectrum—not the seed with its zero—is the structure that the meson sector displays: all four charged pseudoscalar mesons ($\pi^\pm$, $K^\pm$, $D^\pm$, $D_s^\pm$) are massive, with no near-massless member. The electron has “escaped” into near-masslessness because its direction in field space is the goldstino, orthogonal to the meson adjoint.

11.4 The baryon sector

The meson-baryon connection requires a Miyazawa supercharge $[?, ?]$ in the $\overline{\mathbf{10}}$ of $SU(5)_f$, with anticommutator decomposition $\overline{\mathbf{10}} \otimes \mathbf{10} = \mathbf{1} \oplus \mathbf{24} \oplus \mathbf{75}$. This evades the Haag-Łopuszański-Sohnius theorem because it is an approximate symmetry broken at scale Λ_{QCD} .

11.5 Vacuum stability and FCNC

The seesaw vacuum is cosmologically stable: Coleman-de Luccia bounce-action estimates give $S_4 \sim 2 \times 10^7$, far above the threshold $S_4 > 400$ for a lifetime exceeding the age of the universe. All off-diagonal meson masses are positive ($m^2 \approx 2f_\pi^2$), and a block-diagonal obstruction prevents cross-flavour condensation. The flavour-universal Yukawa identity $y_j \langle M_j \rangle = \sqrt{2} C / v_{\text{EW}}$ (derived in Section ??) automatically satisfies the Glashow-Weinberg condition for the absence of tree-level Higgs-mediated FCNCs.

11.6 Generation uniqueness

The Diophantine system uniquely selects $N = 3$ generations, but the third equation ($r^2 + s^2 - 1 = 4N$) has the weakest physical motivation. A dynamical derivation of the counting constraints would strengthen the argument considerably.

11.7 M-theory uplift and the top-neutrino duality

The Type I $SO(32)$ string theory of Section ?? admits a lift to M-theory. The route avoids the $E_8 \times E_8$ obstruction: Type I on a circle is T-dual to Type I' (Type IIA with

O8-planes and D8-branes), and Type IIA lifts to M-theory on S^1 . The path is

$$\text{Type I } (SO(32)) \xrightarrow{T\text{-dual}} \text{Type I}' \xrightarrow{g_s \rightarrow R_{11}} \text{M-theory}. \quad (100)$$

At no stage does $E_8 \times E_8$ appear.

The electric–magnetic dualities that permeate the sBootstrap — the Seiberg seesaw $\langle M_j \rangle = C/m_j$, the dual mass-charge identity $\xi_k \propto 1/\sqrt{m_k}$, the self-duality of the seed — are precisely the structures that become geometric in M-theory, where S-duality of Type IIB lifts to the $SL(2, \mathbb{Z})$ modular symmetry of the M-theory torus fiber. A theory built entirely on Seiberg duality is a theory whose natural home is eleven-dimensional.

A suggestive numerological structure supports this. In $D = 11$ supergravity, the on-shell bosonic degrees of freedom are

$$128 = \underbrace{44}_{\text{graviton } g_{MN}} + \underbrace{84}_{\text{3-form } C_{MNP}}, \quad (101)$$

where $44 = 9 \times 10/2 - 1$ (traceless symmetric tensor of $SO(9)$) and $84 = \binom{9}{3}$ (antisymmetric 3-index tensor). The graviton count decomposes as

$$44 = 32 + 12, \quad (102)$$

where 32 is the dimension of the $SO(32)$ fundamental — the Chan-Paton labels — and $12 = r^2 + s^2 - 1 = 4N$ is the neutral-composite count from the Diophantine bootstrap.

The “12” admits two dual interpretations corresponding to the two particles that sit outside the main composite spectrum:

Frame	The 12 is	Mass mechanism
Electric	top quark (singlet, $5 \times 3 + 1$)	Dirac ($y_t \sim 1$)
Magnetic	neutrino (pseudo-modulus)	Majorana ($\sim (m/\Lambda_L)^6$)

In the electric frame the top acquires its Dirac mass while the neutrino is massless; in the magnetic frame the neutrino acquires a Majorana mass while the top is massless. The two particles are at opposite extremes of the mass spectrum ($m_t/m_\nu \sim 10^{12}$), and their product $m_t \times m_\nu$ would be set by the duality scale — in M-theory, by the geometry of the compact 7-manifold.

Whether this duality can be made precise — and whether the G_2 -holonomy manifold required for $\mathcal{N} = 1$ in four dimensions from M-theory inherits the $\mathbb{Z}_3$ structure of the Type I orbifold — is an open problem that connects the sBootstrap to the deepest structures of string/M-theory.

12 Conclusions

We have constructed an explicit $\mathcal{N} = 1$ supersymmetric theory with gauge group $SU(3) \times SU(2)$ and demonstrated the following principal results.

1. The Diophantine counting uniquely selects $N = 3$. Three equations on the scalar content— $rs = 2N$, $r(r+1)/2 = 2N$, $r^2 + s^2 - 1 = 4N$ —have the unique positive-integer solution $(N, r, s) = (3, 3, 2)$, which maps to the $SU(5)$ representations $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$. This combination is anomaly-free and asymptotically free ($b_0 = 31/3$). Right-handed neutrinos are required; the top quark is automatically excluded from the composite sector.

2. The O’Raifeartaigh superpotential produces the mass-charge identity as a theorem. With three chiral superfields, the O’Raifeartaigh model at $gv/m = \sqrt{3}$ (stabilized by a Kähler pole at $c = -1/12$) produces the fermion spectrum $(0, (2-\sqrt{3})m, (2+\sqrt{3})m)$ satisfying $\langle z_k^2 \rangle = z_0^2$ exactly. This identity is a boundary condition on the superpotential, not an RG attractor (one-loop RG drives $\langle z_k^2 \rangle / z_0^2 \rightarrow 0$, the opposite extreme).

3. The quark mass chain from a single input. From the isospin combination $m_u + m_d = 6.83$ MeV (equivalently m_π via GMOR), the Lagrangian predicts $m_s = 95.1$ MeV (+1.8%), $m_c = 1301$ MeV (+2.0%), $m_b = 4233$ MeV (+1.3%), $m_t = 171.3 \pm 1.5$ GeV (−0.9%), and $m_d = 4.6$ MeV (−1.5%), via the isospin seed, the O’Raifeartaigh mass ratio $(2+\sqrt{3})^2$, the bion doubling with coefficient $N_c = 3$, the Yukawa eigenvalue constraint $\langle z_k^2 \rangle = z_0^2$, and the dual mass-charge identity from the Seiberg seesaw.

4. The lepton sector from O’Raifeartaigh. The same O’Raifeartaigh algebraic structure applies to the lepton sector — as expected from the Miyazawa superpartner identification of leptons with mesons — predicting $m_\tau = 1776.97$ MeV (+0.006%, 0.9σ) from m_e and m_μ .

5. Mixed mass triples require $\tan\beta = 1$. Since all quark mass triples satisfying $\langle z_k^2 \rangle \approx z_0^2$ obligatorily mix up-type and down-type quarks, the mass-charge identity on masses equals the identity on Yukawa couplings only at $\tan\beta = 1$. The NMSSM singlet coupling $\lambda_S = 0.72$ then gives $m_h = 125.35$ GeV at tree level (0.6σ from PDG).

6. Seven parameters eliminated. The isospin seed (m_s from $m_u + m_d$), the O’Raifeartaigh mass ratio (m_c from m_s), the bion mass relation (m_b from m_s, m_c), the Yukawa eigenvalue constraint (m_t from m_c, m_b), the dual mass-charge identity (m_d from m_s, m_b), the lepton mass-charge identity (m_τ from m_e, m_μ), and the GST relation (θ_{12} from m_d, m_s) together determine seven of the Standard Model’s thirteen charged-fermion parameters. The remaining six are $m_u, m_e, m_\mu, \theta_{23}, \theta_{13}$, and δ_{CP} . The entire quark mass spectrum (m_d through m_t) descends from the single isospin combination $m_u + m_d$, with the ratio $m_u/m_d = 0.453$ predicted by the self-consistency of the coupled system ($\chi^2/\text{dof} = 0.12$, $p = 0.97$ against PDG values).

7. The diquark obstruction and the hadronic superstring. The symmetric **15** required by the counting cannot be realized as a qq composite (by a kinematic theorem independent of dynamics). The $SO(32)$ adjoint of Type I string theory provides a natural UV completion containing both the **24** and **15**⊕**15**; the alternative $E_8 \times E_8$ is algebraically blocked. A $\mathbb{Z}_3$ orbifold on the $SU(3)$ root lattice gives three generations and connects the instanton structure to the bloom potential. The bloom mechanism and a Lagrangian derivation of the Diophantine counting remain the central open problems; the orientifold projection (which at the $\mathbb{Z}_3$ orbifold point produces the antisymmetric **10**) may be resolved by magnetized branes or local $O5^-$ planes that flip the projection to the symmetric **15**.

The empirical evidence for the mass-charge identity—the precision of the lepton prediction, the accuracy of the meson triple, and the statistical significance of the quark chains—will be presented in a companion paper [?].

References

- [1] A. Rivero, “An interpretation of scalars in $SO(32)$,” *Eur. Phys. J. C* **84**, 1058 (2024), arXiv:2407.05397 [hep-ph]; earlier versions: viXra:0503.001 (2005), viXra:0809.001 (2008).
- [2] A. Rivero, “The mass-charge identity $\langle z_k^2 \rangle = z_0^2$ in fermion mass triples: empirical evidence,” in preparation.
- [3] N. Seiberg, “Electric–magnetic duality in supersymmetric non-abelian gauge theories,” *Nucl. Phys. B* **435**, 129 (1995).
- [4] L. O’Raifeartaigh, “Spontaneous symmetry breaking for chiral scalar superfields,” *Nucl. Phys. B* **96**, 331 (1975).
- [5] K. Intriligator, N. Seiberg, D. Shih, “Dynamical SUSY breaking in meta-stable vacua,” *JHEP* **0604**, 021 (2006), arXiv:hep-ph/0602239.
- [6] M. Ünsal, “Abelian duality, confinement, and chiral symmetry breaking in $QCD(adj)$,” *Phys. Rev. Lett.* **100**, 032005 (2008), arXiv:0708.1772.
- [7] M. Ünsal, “Theta dependence, sign problems and topological interference,” *Phys. Rev. D* **86**, 105012 (2012), arXiv:1201.6426.
- [8] S. Catto and F. Gürsey, “Algebraic treatment of effective supersymmetry,” *Nuovo Cim. A* **86**, 201 (1985).
- [9] H. Miyazawa, “Baryon Number Changing Currents,” *Prog. Theor. Phys.* **36**, 1266 (1966).
- [10] R. Gatto, G. Sartori, and M. Tonin, “Weak self-masses, Cabibbo angle, and broken $SU(3)$,” *Phys. Lett. B* **28**, 128 (1968).
- [11] D. V. Volkov and V. P. Akulov, “Is the neutrino a Goldstone particle?,” *Phys. Lett. B* **46**, 109 (1973).
- [12] M. Gell-Mann, R. J. Oakes, B. Renner, “Behavior of current divergences under $SU(3) \times SU(3)$,” *Phys. Rev.* **175**, 2195 (1968).
- [13] P. Fayet and J. Iliopoulos, “Spontaneously broken supergauge symmetries and Goldstone spinors,” *Phys. Lett. B* **51**, 461 (1974).
- [14] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, J. E. Young, “Electromagnetic mass difference of pions,” *Phys. Rev. Lett.* **18**, 759 (1967).
- [15] A. Rivero, “The de Vries formula for $\sin^2 \theta_W$,” arXiv:hep-ph/0510068 (2005).
- [16] M. Bona et al. (UTfit Collaboration), “Model-independent constraints on $\Delta F = 2$ operators and the scale of new physics,” *JHEP* **03**, 049 (2008), arXiv:0707.0636.
- [17] G. Veneziano, “Construction of a crossing-symmetric, Regge-behaved amplitude for linearly rising trajectories,” *Nuovo Cim. A* **57**, 190 (1968).

- [18] Y. Nambu, “Quark model and the factorization of the Veneziano amplitude,” in *Symmetries and Quark Models*, ed. R. Chand (Gordon and Breach, 1970), p. 269.
- [19] J. E. Paton and H.-M. Chan, “Generalized Veneziano model with isospin,” Nucl. Phys. B **10**, 516 (1969).
- [20] M. B. Green and J. H. Schwarz, “Anomaly cancellations in supersymmetric $D = 10$ gauge theory and superstring theory,” Phys. Lett. B **149**, 117 (1984).
- [21] E. G. Adelberger, J. H. Gundlach, B. R. Heckel, S. Hoedl, S. Schlamminger, “Torsion balance experiments: a low-energy frontier of particle physics,” Prog. Part. Nucl. Phys. **62**, 102 (2009).