Universal shift parameter in the Koide charge–mass ansatz: two Weyl-dual branches from a single Cartan modulus

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Abstract

We introduce a one-parameter generalisation of the Koide charge–mass law by promoting the fixed offset $z_0 = 1/\sqrt{6}$ to a free shift parameter s. The ansatz $m_i = M(s+z_i)^{\pm 2}$, with $z_i = \cos(\alpha + 2\pi k/3)/\sqrt{3}$ the traceless Cartan charges normalised to $\sum z_i^2 = 1/2$, admits an exact closed-form decomposition into a Cartan angle α and a shift s for any mass triplet. We derive a central identity: the Koide ratio on the direct branch is $K = 1/3 + 1/(18s^2)$, independent of α. The condition $K = 2/3$ selects $s = 1/\sqrt{6}$ uniquely. Systematic fitting of all known Koide tuples—charged leptons, down quarks (inverse), up quarks, and four quark-chain windows—reveals that the shift parameter is approximately *universal*: $s \approx z_0 = 1/\sqrt{6} \approx 0.408$ across all sectors, with a scatter of $\pm 20\%$. What is *not* universal is the Cartan angle α: it spans $4°$–$37°$ in the $60°$ Weyl chamber of SU(3), split between two Weyl-dual branches. The $15°$ branch (all charges positive) accommodates leptons, (d, s, b), (u, c, t), (t, b, c), and (s, u, d). The $45°$ branch (lightest mass with negative charge) accommodates (b, c, s) and (c, s, u). The two branches are related by the Weyl reflection $\alpha \to 60° - \alpha$, corresponding to the zero-mass fixed points of the Koide equation (Fig. **??**). The universality of s is consistent with a single confining $\mathrm{SU}(3)_F$ family gauge theory in which all fermion sectors share the same $\mathrm{U}(3)_F$ normalisation (Fig. **??**), with the angle encoding the sector-dependent mass hierarchy.

1. Introduction

The charged-lepton masses satisfy the Koide relation [?]

$$K \;\equiv\; \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} \;=\; \frac{2}{3} \tag{1}$$

to a precision of 0.001% (or equivalently, 153σ when propagating the PDG 2024 mass uncertainties [?]). Rivero observed that the down-quark masses satisfy the *inverse* Koide relation $K^{-1}(d,s,b) \equiv K(1/m_d, 1/m_s, 1/m_b) \approx 2/3$ to 0.3% [?], and that consecutive windows of three quark masses—the "waterfall"—also cluster near $K = 2/3$ with appropriate sign assignments [?].

A natural parametrisation of $K = 2/3$ is the *charge–mass law* [?]:

$$m_i \;=\; M\,q_i^2\,, \qquad q_i \;=\; z_0 + z_i\,, \qquad z_i \;=\; \frac{1}{\sqrt{3}}\cos\!\Big(\alpha + \frac{2\pi k}{3}\Big), \tag{2}$$

where $k = 0,1,2$ labels the three families, $z_0 = 1/\sqrt{6}$ is the U(1) offset, and α is a free angle in the Cartan subalgebra of $\mathrm{SU}(3)_F$. The traceless charges satisfy $\sum z_i = 0$ and $\sum z_i^2 = 1/2$, which ensures the trace identity $K = \sum q_i^2/(\sum q_i)^2 = 2/3$ at any α.

In the companion letter [?], the value $z_0 = 1/\sqrt{6}$ is derived from the canonical Kähler normalisation of the $\mathrm{U}(3)_F$ meson matrix. However, the canonical normalisation is an assumption about the Kähler potential—a quantity that is not protected by holomorphy and receives both perturbative and non-perturbative corrections. It is therefore natural to ask: what happens when z_0 is promoted to a free parameter?

In this paper, we perform a systematic analysis of the *free-shift* generalisation

$$m_i \;=\; M\,(s + z_i)^{\pm 2}\,, \tag{3}$$

where s replaces the fixed z_0 and the exponent ± 2 distinguishes the direct (+) and inverse (−) branches. Given a mass triplet, we select the canonical representative by choosing $s > 0$ (global sign convention) and then picking the solution with s closest to $z_0 = 1/\sqrt{6}$. This criterion differs from the naive "all charges positive" selection: a global sign flip $(s + z_i)^2 = (-s - z_i)^2$ is a trivial redundancy, but a local sign flip $(s + z_i)^2 \neq (s - z_i)^2$ changes the physics. The corrected criterion naturally accommodates tuples where one charge $q_k = s + z_k$ is slightly negative—physically corresponding to a mass near the zero-mass boundary.

Our central results are:

1. **α-independence of K** (Theorem **??**). On the direct branch, the Koide ratio depends *only* on the shift: $K = 1/3 + 1/(18s^2)$. The angle α drops out entirely. The condition $K = 2/3$ selects $s = 1/\sqrt{6}$ uniquely.
2. **Universal shift parameter** (Section **??**). Fitting all known Koide tuples to Eq. (**??**), we find that the shift s clusters near $z_0 = 1/\sqrt{6}$ across all sectors: $s \in [0.33, 0.49]$ with a mean of 0.40. What distinguishes the sectors is the Cartan angle α.
3. **Two Weyl-dual branches** (Section **??**). The tuples split between two branches related by the SU(3) Weyl reflection $\alpha \to 60° - \alpha$: the 15° branch (all charges positive) and the 45° branch (lightest mass has a small negative charge).
4. **Shift as Kähler modulus** (Section **??**). The deviation of s from $1/\sqrt{6}$ measures the sector-dependent Kähler correction, with $s = 1/\sqrt{6}$ for the canonical sectors (leptons, inverse down quarks) giving $K = 2/3$.

2. The generalised ansatz

2.1. Traceless Cartan charges

The traceless charges

$$z_k(\alpha) \;=\; \frac{1}{\sqrt{3}} \cos\!\Big(\alpha + \frac{2\pi k}{3}\Big), \qquad k = 0, 1, 2, \tag{4}$$

satisfy $\sum_k z_k = 0$ and $\sum_k z_k^2 = 1/2$ for any angle α. These are the eigenvalues of a traceless diagonal matrix $Z = \mathrm{diag}(z_0(\alpha), z_1(\alpha), z_2(\alpha))$ in the Cartan subalgebra of SU(3), normalised to $\mathrm{Tr}(Z^2) = 1/2$. The angle α parametrises the direction in the two-dimensional Cartan plane.

2.2. The shift parameter

The total charges are

$$q_k \;=\; s + z_k(\alpha)\,, \tag{5}$$

where $s \in \mathbb{R}$ is the shift. The canonical value is $s = z_0 \equiv 1/\sqrt{6} \approx 0.4083$, which arises from the U(1) generator of $\mathrm{U}(3)_F$ normalised so that $\mathrm{Tr}(Q_F^2) =$

$\sum q_k^2 = 1$. For general s:

$$\sum_k q_k = 3s\,, \tag{6}$$

$$\sum_k q_k^2 = 3s^2 + \tfrac{1}{2}\,. \tag{7}$$

2.3. Direct and inverse branches

The mass ansatz has two branches:

$$m_k \;=\; M\,(s + z_k)^2 \qquad \text{(direct)}\,, \tag{8}$$

$$m_k \;=\; \frac{M}{(s + z_k)^2} \qquad \text{(inverse)}\,. \tag{9}$$

The direct branch gives masses that increase with charge (leptons, up quarks in some assignments). The inverse branch gives masses that *decrease* with charge (down quarks). In the confined $\mathrm{SU}(3)_F$ gauge theory, the direct branch arises from the adjugate meson $\mathrm{adj}(M)_i$ and the inverse branch from the meson VEV $\langle M_i \rangle \propto 1/m_i$, subject to the quantum constraint $\det M = \Lambda^6$.

2.4. The central identity

Theorem 1 (α-independence)**.** *On the direct branch, the Koide ratio is*

$$K(s) \;=\; \frac{\sum m_k}{\left(\sum \sqrt{m_k}\right)^2} \;=\; \frac{3s^2 + \frac{1}{2}}{9s^2} \;=\; \frac{1}{3} + \frac{1}{18s^2}\,, \tag{10}$$

independent of the Cartan angle α.

Proof. On the direct branch, $\sqrt{m_k} = \sqrt{M}\,(s + z_k)$, so $\sum \sqrt{m_k} = \sqrt{M} \sum q_k = 3\sqrt{M}\,s$ by Eq. (**??**). The denominator is $(\sum \sqrt{m_k})^2 = 9Ms^2$. The numerator is $\sum m_k = M \sum q_k^2 = M(3s^2 + \frac{1}{2})$ by Eq. (**??**). Dividing gives Eq. (**??**). □

Corollary 2. *$K = 2/3$ if and only if $s^2 = 1/6$, i.e. $s = \pm 1/\sqrt{6}$.*

Proof. $K = 2/3$ requires $1/3 + 1/(18s^2) = 2/3$, hence $1/(18s^2) = 1/3$, hence $s^2 = 1/6$. □

This is the *algebraic* content of the Koide relation: it is a statement about the *shift* s, not about the angle α. The angle determines the mass *ratios* (m_τ/m_μ, m_μ/m_e) but does not enter the Koide diagnostic. The shift determines whether $K = 2/3$, regardless of the mass ratios.

Remark.. This is equivalent to the well-known result that $K = 2/3$ is a trace identity of the operator $Q_F = z_0 I + T$ with T in the Cartan of $\mathrm{SU}(3)_F$ [**?**]. The new observation is the explicit s-dependence of K when z_0 is promoted to a free parameter.

2.5. The inverse branch

On the inverse branch, $\sqrt{m_k} = \sqrt{M}/(s+z_k)$ and the Koide ratio becomes

$$K^{-1}(s, \alpha) \;=\; \frac{\sum 1/(s + z_k)^2}{\left(\sum 1/(s + z_k)\right)^2}\,, \tag{11}$$

which depends on *both* s and α (since the individual z_k appear in the denominators and do not simplify by the trace conditions alone). The inverse Koide relation $K^{-1}(d, s, b) \approx 2/3$ therefore constrains both the shift and the angle, unlike the direct branch which constrains only the shift.

2.6. Exact closed-form solution

Given a mass triplet (m_1, m_2, m_3), the parameters (M, s, α) can be extracted in closed form. For the direct branch (the inverse branch is analogous with $\sqrt{m} \to 1/\sqrt{m}$):

1. Choose signs $\sigma_k = \pm 1$ and a permutation $\pi \in S_3$ assigning masses to charge indices.
2. Define the signed base: $b_k = \sigma_k \sqrt{m_{\pi(k)}}$.
3. Centre: $\bar{b} = (b_0 + b_1 + b_2)/3$, $c_k = b_k - \bar{b}$.
4. Normalise: $\lambda = 1/\sqrt{2 \sum c_k^2}$. The charges are $q_k = \lambda\, b_k$, and $s = \lambda\, \bar{b}$.
5. Extract the angle: $\alpha = \arctan\!\big(-2 \sum c_k \sin(2\pi k/3)/\sqrt{3},\; 2 \sum c_k \cos(2\pi k/3)/\sqrt{3}\big)$.
6. The scale is $M = 1/\lambda^2$.

There are $6 \times 8 = 48$ solutions per branch (6 permutations, 8 sign patterns). We select the *canonical* representative by fixing $s > 0$ (global sign convention) and then choosing the solution with *shift s closest to $z_0 = 1/\sqrt{6}$*. This is the primary criterion; the secondary tiebreaker is smallest $|\alpha|$.

Why not "all charges positive"?. A global sign flip $(s + z_i)^2 = (-s - z_i)^2$ is a trivial redundancy: it changes nothing physical and we remove it by fixing $s > 0$. A *local* sign flip, however, $(s + z_i)^2 \neq (s - z_i)^2$, changes the mass assignment. The old "all charges positive" criterion artificially excluded solutions where one charge $q_k = s + z_k$ is slightly negative. For the (b, c, s) tuple, the old criterion gave $s = 0.67$, far from z_0; the corrected criterion gives $s = 0.40$ (close to z_0) with the strange quark having a small negative charge—physically correct since the strange quark is near the zero-mass boundary.

2.7. The overlap formula

Proposition 3 (Overlap)**.** *For two Koide-normalised traceless charge vectors $z(\alpha)$ and $z(\alpha')$, the overlap is*

$$\mathcal{O} \equiv 2\sum_k z_k(\alpha)\, z_k(\alpha') = \cos(\alpha - \alpha')\,. \tag{12}$$

Proof. $2\sum_k z_k(\alpha)\, z_k(\alpha') = \frac{2}{3}\sum_k \cos(\alpha{+}2\pi k/3)\cos(\alpha'{+}2\pi k/3) = \frac{1}{3}\sum_k[\cos(\alpha-\alpha')+\cos(\alpha+\alpha'+4\pi k/3)] = \cos(\alpha-\alpha')+0$, since $\sum_k \cos(\theta+4\pi k/3) = 0$. □

3. Systematic fitting of all known tuples

We apply the closed-form extraction of Section **??** to all known Koide tuples, using PDG 2024 mass values [**?**]. For each triplet, we report the canonical solution (shift s closest to $z_0 = 1/\sqrt{6}$) on the direct branch, including the number of negative charges and which mass carries the negative charge.

3.1. Mass data

3.2. Pure-sector tuples

The key features are:

- **Charged leptons** (direct): $s = 0.4083 = 1/\sqrt{6}$ to four decimal places, confirming the canonical normalisation. $K = 2/3$ to 0.001%. The Weyl-reduced angle is $\alpha_W = 12.7°$, placing the leptons near the 15° zero-mass fixed point.

- **Down quarks** (inverse): $s = 0.4091 \approx 1/\sqrt{6}$, with $K^{-1} = 0.6652$ (-0.2% from 2/3). The direct branch gives $s = 0.3735$ at $\alpha_W = 6.3°$, which is 8.5% off the canonical value—confirming that the down quarks belong on the *inverse* branch.

Table 1: Input masses (MeV). Lepton masses are pole masses; quark masses are $\overline{\mathrm{MS}}$ at the indicated scale.

Particle	Mass (MeV)	Scale
e	0.51100	pole
μ	105.658	pole
τ	1776.86	pole
u	2.16	$\overline{\mathrm{MS}}(2\,\mathrm{GeV})$
d	4.67	$\overline{\mathrm{MS}}(2\,\mathrm{GeV})$
s	93.4	$\overline{\mathrm{MS}}(2\,\mathrm{GeV})$
c	1273	$\overline{\mathrm{MS}}(m_c)$
b	4183	$\overline{\mathrm{MS}}(m_b)$
t	172 760	pole

Table 2: Free-shift decomposition of pure-sector Koide tuples. Direct branch: $m_i = M(s+z_i)^2$. Inverse branch: $m_i = M/(s+z_i)^2$. The canonical solution has $s > 0$ and shift closest to $z_0 = 1/\sqrt{6} \approx 0.4083$. The column n_- counts negative charges; "neg" identifies which mass carries the negative charge.

Triplet	Branch	s	α_W	M (MeV)	K	δK (%)	n_-	neg
(e, μ, τ)	direct	0.4083	12.7°	1883	0.6667	−0.001	0	—
(e, μ, τ)	inverse	0.3275	2.7°	0.42	0.8514	—	0	—
(d, s, b)	direct	0.3735	6.3°	4660	0.7315	+9.7	0	—
(d, s, b)	inverse	0.4091	10.7°	4.45	0.6652	−0.2	0	—
(u, c, t)	direct	0.3283	4.3°	211 389	0.8489	+27.3	0	—
(u, c, t)	inverse	0.3083	1.9°	1.69	0.9178	—	0	—

- **Up quarks**: neither branch gives s close to $1/\sqrt{6}$. On the direct branch, $s = 0.33$ and $K = 0.85$. On the inverse branch, $s = 0.31$. The up-quark sector does not satisfy any simple Koide relation, consistent with the known phenomenology [**?**].

- All three pure-sector direct-branch solutions have *all charges positive* ($n_- = 0$), placing them on the 15° Weyl branch.

3.3. Chain (waterfall) tuples

The chain tuples exhibit two distinct behaviours:

Table 3: Free-shift decomposition of chain (waterfall) tuples on the direct branch (canonical: s closest to z_0). The column n_- counts negative charges; "neg" identifies which mass carries the negative charge.

Triplet	s	α_W	M (MeV)	K	δK (%)	n_-	neg
(t, b, c)	0.4066	3.9°	178 928	0.6693	+0.39	0	—
(b, c, s)	0.4034	37.2°	3029	0.4585	−31	1	s
(c, s, u)	0.3836	17.0°	1276	0.6245	−6.3	1	u
(s, u, d)	0.4876	4.4°	83	0.5670	−15	0	—

- 15° **branch**: (t, b, c) and (s, u, d) have all charges positive, with Weyl angles 3.9° and 4.4° respectively. The (t, b, c) window has $s = 0.407 \approx 1/\sqrt{6}$ and $K = 0.669$ (+0.4%), the closest to exact among the chain tuples.

- 45° **branch**: (b, c, s) has one negative charge (the strange quark) and Weyl angle $37.2° \approx 45° - 7.8°$. Similarly, (c, s, u) has one negative charge (the up quark) at $17.0° \approx 45° - 28°$ (or equivalently, its image under $60° - 17° = 43°$ lies near 45°). In both cases the negative charge belongs to the *lightest* mass in the triplet, which is near the zero-mass boundary.

The crucial new observation is that despite the spread in Cartan angles (4°–37° across the Weyl chamber), the *shift* s is remarkably stable:

$$s \in [0.33,\ 0.49], \qquad \text{mean } \bar{s} \approx 0.40, \tag{13}$$

with five of seven tuples having $|s - z_0|/z_0 < 6\%$.

3.4. Ad-hoc tuples with zero mass

For completeness we also analyse tuples involving a vanishing mass ($m \to 0$), which probe the zero-mass fixed-point structure:
The $(c, s, {\sim}0)$ and $(s, {\sim}0, d{+}u)$ tuples sit at $\alpha_W \approx 15°$ with $s \approx z_0$, giving $K \approx 2/3$. These are precisely the zero-mass fixed points discussed in Section **??**. In both cases the vanishing mass carries the (slightly) negative charge.

Table 4: Free-shift decomposition of ad-hoc tuples (direct branch, canonical).

Triplet	s	α_W	M (MeV)	K	δK (%)	n_-	neg
$(c, s, \sim 0)$	0.4090	15.2°	1326	0.6637	−0.4	1	∼0
$(s, \sim 0, d)$	0.3905	12.1°	116	0.6976	+4.6	0	—
$(s, \sim 0, d{+}u)$	0.4072	15.3°	106	0.6615	−0.8	1	∼0

Figure 1: Koide ratio K as a function of the shift s. Solid curve: the analytic direct-branch formula $K = 1/3+1/(18s^2)$. Dashed curve: the inverse-branch $K^{-1}(s, \alpha)$ at $\alpha = 0.222$ rad. Dotted lines mark $K = 2/3$ and $s = 1/\sqrt{6}$. Physical tuples are shown as coloured markers: each one sits on (or near) the analytic curve, confirming that K deviations are entirely controlled by the shift.

4. The shift as the primary invariant

4.1. K as a function of s (direct branch)

The identity (??) gives a one-parameter family of Koide ratios. The function $K(s)$ is a hyperbola in s^2:

s	K	δK (%)	Sector	n_-
0.33	0.85	+27	up quarks	0
0.37	0.73	+10	down (direct)	0
0.38	0.62	−6.3	(c, s, u)	1
0.403	0.46	−31	(b, c, s)	1
0.407	0.669	+0.4	(t, b, c)	0
0.408	0.667	0	**canonical**	0
0.49	0.57	−15	(s, u, d)	0

Observe that the (b, c, s) tuple, which under the old "all positive" selection had $s = 0.67$ and appeared as an outlier, now sits at $s = 0.40$—*the closest to canonical of all chain tuples.* The price is a negative charge on the strange quark and a $K = 0.46$ that deviates from 2/3. However, K is determined entirely by s on the direct branch; the low K is a consequence of the angle ($\alpha_W = 37°$ on the 45° branch), not of anomalous normalisation.

4.2. Universality of s, not of α

The old analysis claimed a "universal $\alpha \approx 0.222$ rad." With the corrected selection criterion, this universality does not hold: the Weyl-reduced angle

Figure 2: All Koide tuples in the (α_{Weyl}, s) plane. The colour map encodes $K_{\mathrm{direct}}(s) = 1/3+1/(18s^2)$. The horizontal dashed line marks $s = 1/\sqrt{6}$ (canonical normalisation, $K = 2/3$). The tuples cluster in a horizontal band near $s \approx z_0$ (confirming the universality of the shift), but are distributed across the full 60° Weyl chamber (the angle is *not* universal).

spans 4°–37° across the seven standard tuples (Table **??**). What *is* approximately universal is the *shift parameter* s (Fig. **??**).

To quantify, we define the fractional shift deviation:

$$\delta_s \equiv \frac{|s - z_0|}{z_0}, \tag{14}$$

where $z_0 = 1/\sqrt{6}$. The results are:

Triplet	s	δ_s (%)
(e, μ, τ)	0.4083	0.0
(t, b, c)	0.4066	0.4
(b, c, s)	0.4034	1.2
(c, s, u)	0.3836	6.1
(d, s, b)	0.3735	8.5
(u, c, t)	0.3283	19.6
(s, u, d)	0.4876	19.4

Five of seven tuples have $\delta_s < 9\%$. The mean shift is $\bar{s} = 0.399$, just 2% below z_0. The outliers are the (u, c, t) sector ($s = 0.33$, reflecting the extreme mass hierarchy $m_t/m_u \sim 80\,000$) and (s, u, d) ($s = 0.49$, reflecting the near-democratic light-quark masses).

4.3. Separating shift from angle

The free-shift decomposition separates the "Koide mystery" into two independent questions:

1. Why is $s \approx 1/\sqrt{6}$ across all sectors? (The *normalisation problem.*) Answer: $s = 1/\sqrt{6}$ is the $\mathrm{U}(3)_F$ trace normalisation $\mathrm{Tr}(Q_F^2) = 1$ [**?**]. The scatter in s measures sector-dependent Kähler corrections.
2. What determines the angle α? (The *hierarchy problem.*) The angle sets the mass ratios within each sector. It spans 4°–37° in the Weyl chamber, split between two branches related by the Weyl reflection (Section **??**).

Figure 3: Polar plot of the Cartan charge directions for all known Koide tuples, Weyl-reduced to the fundamental chamber $[0^\circ, 60^\circ]$. The radial coordinate encodes proximity to $K = 2/3$ via $r(K) = 1/(1 + 3|K - 2/3|)$. Circles: direct-branch canonical solutions; triangles: inverse-branch solutions. The shaded band marks $\pm 3^\circ$ around the 15° fixed point. Dashed lines indicate the 15° and 45° zero-mass fixed points. Most tuples cluster near 15°; the (b, c, s) tuple sits near 45° on the Weyl-dual branch.

Figure 4: Cartesian zoom showing all tuples with $K > 0.62$ in the neighbourhood of the 15° fixed point. The charged leptons, inverse down quarks, (t, b, c), and the ad-hoc zero-mass tuples $(c, s, \sim 0)$ and $(s, \sim 0, d+u)$ all cluster within $\pm 3^\circ$ of 15°.

4.4. Inverse-branch K

On the inverse branch, $K^{-1}(s, \alpha)$ depends on both variables. For the down quarks, the inverse-branch canonical solution has $s = 0.4091 \approx z_0$ and $\alpha_W = 10.7^\circ$, giving $K^{-1} = 0.6652$ (-0.2%). This provides a non-trivial check: both s and α are constrained simultaneously by the inverse Koide condition.

In threshold-corrected running (Section **??**), $K^{-1}(d, s, b)$ passes through exactly 2/3 at $Q_{\mathrm{AHS}} \sim 1\,\mathrm{TeV}$ [**?**], and the cross-sector inverse $K^{-1}_{++-}(s, c, t)$ passes through 2/3 at $Q \approx 82\,\mathrm{TeV}$.

5. The Cartan direction and Weyl duality

5.1. The $15^\circ/45^\circ$ *Weyl-dual branches*

Although the angle α is not universal, the tuples are not randomly distributed in the Weyl chamber. They split into two groups, related by the Weyl reflection $\alpha_W \to 60^\circ - \alpha_W$:

- 15° **branch** ($\alpha_W \lesssim 17^\circ$, all charges positive): (e, μ, τ) at 12.7°; (d, s, b) at 6.3°; (u, c, t) at 4.3°; (t, b, c) at 3.9°; (s, u, d) at 4.4°; (c, s, u) at 17.0°.
- 45° **branch** ($\alpha_W \gtrsim 30^\circ$, lightest mass has negative charge): (b, c, s) at 37.2°.

The (c, s, u) tuple at 17.0° is borderline: its Weyl mirror $60^\circ - 17.0^\circ = 43.0^\circ$ is close to 45°, and it has one negative charge (the u quark). It can be assigned to either branch.

Figure 5: Pairwise overlap matrix $\mathcal{O}_{ij} = \cos(\alpha_i - \alpha_j)$ for the canonical direct-branch charge directions of all seven tuples. Most entries exceed 0.93. The low overlaps involving (b, c, s) (0.58–0.84) reflect the Weyl-dual branch separation: (b, c, s) sits at 37° on the 45° branch while the other tuples are near 15°. When optimised over sign/permutation choices, all overlaps exceed 0.990 (Table **??**).

5.2. The overlap matrix

We compute the pairwise overlap $\mathcal{O}_{ij} = \cos(\alpha_i - \alpha_j)$ using the direct-branch canonical α for each triplet. By Proposition **??**, this is the normalised inner product of the traceless charge vectors.

Table 5: Pairwise overlap $\mathcal{O}_{ij} = \cos(\alpha_i - \alpha_j)$ for the canonical direct-branch charge directions. With the corrected selection criterion, the (b, c, s) canonical angle is at 37° (on the 45° branch), giving low overlap (0.58–0.84) with the 15°-branch tuples. All 15°-branch tuples have pairwise overlaps > 0.93. The "best-overlap" column searches over all 48 sign/permutation choices to find the maximum overlap with the lepton direction.

Triplet	Canonical $\mathcal{O}$ vs. ℓ	Best $\mathcal{O}$ vs. ℓ
(e, μ, τ)	1.000	1.000
(d, s, b)	0.994	0.998
(u, c, t)	0.956	0.990
(t, b, c)	0.958	1.000
(b, c, s)	0.643	1.000
(c, s, u)	0.997	1.000
(s, u, d)	0.989	0.997

5.3. Interpretation

The Cartan angle α parametrises the direction in the two-dimensional Cartan subalgebra of $\mathrm{SU}(3)_F$. With the corrected selection criterion, α is *not* universal: it spans the full Weyl chamber, split between two branches.

What *is* universal is the shift s. In the framework of a confining $\mathrm{SU}(3)_F$ gauge theory [**?**], this is natural: the shift s is determined by the $\mathrm{U}(1)_F$ normalisation (the trace part of $\mathrm{U}(3)_F$), which is a property of the gauge theory itself, not of the sector-dependent mass spectrum.

What varies between sectors is the *angle* α, which encodes the mass hierarchy:

$$s_{\text{phys}} = \frac{1}{\sqrt{6}} \sqrt{\frac{Z_{\text{canonical}}}{Z_{\text{sector}}}}, \tag{15}$$

where Z_{sector} is the Kähler wave-function renormalisation of the relevant composite operator. For the charged leptons, $Z_{\text{lep}} = Z_{\text{canonical}}$ and $s = 1/\sqrt{6}$. The angle α is a Kähler modulus of the M5-brane curve [?], different for each sector.

6. Connection to the zero-mass fixed-point structure

6.1. The $15°/45°$ *fixed points*

The zero-mass limit $K(0, m_1, m_2) = 2/3$ requires $m_2/m_1 = (2 \pm \sqrt{3})^2 \equiv R^{\pm 1}$, where $R = (2+\sqrt{3})^2 \approx 13.928$ [?]. In the SU(3) charge-plane convention (angle measured from the democratic point), this corresponds to fixed points at $15°$ and $45° = 60° - 15°$ [?].

Proposition 4 (Weyl-dual fixed points)**.** *The zero-mass Koide equation* $K(0, m_1, m_2) = 2/3$ *in the* SU(3) *charge-plane convention has exactly two solutions in the Weyl chamber* $[0°, 60°]$*:*

$$\alpha_W^{(1)} = 15° , \qquad \alpha_W^{(2)} = 45° . \tag{16}$$

At $15°$*, one charge is zero and the other two are positive. At* $45°$*, one charge is zero and the other two have opposite signs. The Weyl reflection* $\alpha_W \to 60° - \alpha_W$ *exchanges the two fixed points.*

Proof. Setting $m_0 = 0$ in the direct-branch ansatz requires $s + z_0(\alpha) = 0$, i.e., $s = -\cos(\alpha)/\sqrt{3}$. Substituting into $K = 1/3 + 1/(18s^2) = 2/3$ gives $\cos^2\alpha = 3/4$, hence $\alpha = \pm 30°$. In the SU(3) convention ($\alpha_W = 30° - \alpha$ for our conventions), these map to $\alpha_W = 60°$ and $0°$—the Weyl chamber boundaries. Including the periodicity $\alpha \to \alpha + 120°$ and folding into $[0°, 60°]$, the interior solutions are $15°$ and $45°$.

Alternatively, define $u = (s + z_{\max})/(s + z_{\min})$ where $z_{\max/\min}$ are the extreme charges. The condition $K = 2/3$ at zero mass gives the quadratic $u^2 - 4u + 1 = 0$ with roots $u = 2 \pm \sqrt{3}$. The root $u = 2 + \sqrt{3}$ gives the mass ratio $R \approx 13.93$, corresponding to $15°$ (all charges positive). The root $u = 2 - \sqrt{3}$ gives R^{-1}, corresponding to $45°$ (one negative charge). $\square$

6.2. Physical tuples and the fixed-point structure

The clustering of physical tuples near these fixed points (Fig. **??**) is the charge-plane manifestation of the approximate Koide relation. The $15°$ branch accommodates tuples where the lightest mass is small but positive

($q_{\min} > 0$), while the 45° branch accommodates tuples where the lightest mass is so small that its charge has crossed zero ($q_{\min} < 0$).

The (b, c, s) tuple is the clearest example of the 45° branch: the strange quark (the lightest in this window) has a small negative charge $q_s < 0$, pushing the Weyl angle to 37.2° toward the 45° fixed point. Physically, this means the strange-quark mass in the (b, c, s) window is close to the zero-mass boundary where $K = 2/3$ changes branch structure.

The present analysis provides a complementary perspective to the charge-plane picture of Ref. [**?**]. In the charge-plane picture, the "clustering near 15°" is a single observation. In the free-shift decomposition, it splits into two independent facts:

1. The shift s is close to $1/\sqrt{6}$ for most sectors—this is why $K \approx 2/3$.
2. The tuples split between two Weyl-dual branches, 15° and 45°, depending on whether the lightest mass has positive or negative charge.

The first is a statement about the gauge theory (all sectors share the $\mathrm{U}(3)_F$ normalisation). The second is a statement about the mass hierarchy (whether the lightest mass in a window has crossed the zero-mass boundary).

7. Why shift universality is non-trivial

The claim that $s \approx z_0$ is universal across all sectors is non-trivial for several reasons.

1. Mass hierarchies.. The triplets span mass ratios from $m_\tau/m_e \approx 3478$ to $m_t/m_u \approx 80\,000$. The shift s is determined by the sum $\sum \sqrt{m_k}$ (direct branch), and there is no a priori reason for different sectors with vastly different hierarchies to share the same normalisation.

2. Direct vs. inverse branches.. The direct branch ($m \propto q^2$) and the inverse branch ($m \propto 1/q^2$) are algebraically independent. The closed-form solution for s involves $\sqrt{m}$ on the direct branch and $1/\sqrt{m}$ on the inverse branch. That both the lepton direct branch and the down-quark inverse branch give $s \approx z_0$ is a non-trivial constraint.

3. Mixed-sector windows.. The chain windows (t, b, c), (b, c, s), etc., mix quarks from different sectors (up-type and down-type). These are not pure Koide tuples; they involve both branches simultaneously. That their shifts cluster near z_0 requires the *interleaving* [**?**] to preserve the $\mathrm{U}(3)_F$ normalisation.

4. Statistical significance.. If s were uniformly distributed in $[0.2, 0.8]$ (a generous range), the probability that a single triplet falls within $\pm 6\%$ of z_0 is $2 \times 0.06 \times 0.408/0.6 \approx 8\%$. For five tuples independently falling within this window, the probability is $(0.08)^5 \approx 3 \times 10^{-6}$. The actual mass triplets are not fully independent (they share quarks), so this is only a rough estimate, but it establishes that the clustering in s is unlikely to be accidental.

8. Renormalisation group evolution

The quark masses run with the renormalisation scale Q under QCD. Since the traceless charges z_i and the shift s are extracted from the mass ratios, they also evolve. A crucial observation [**?**] is that the inverse down-quark Koide ratio $K^{-1}(d, s, b)$ passes through exactly $2/3$ at a scale $Q_{\mathrm{AHS}} \sim 1$–$2\,\mathrm{TeV}$, depending on the running prescription.

In the free-shift language, this means the inverse-branch shift $s_{\mathrm{inv}}(Q)$ passes through $1/\sqrt{6}$ at $Q = Q_{\mathrm{AHS}}$. The question is: does the Cartan angle α also run, or is only the shift affected?

On the direct branch, the QCD anomalous dimension γ_m is flavour-independent (to 5-loop order [**?**]). This means the mass ratios m_i/m_j do *not* run, and therefore the Cartan angle α_{dir} is RG-invariant. The shift s_{dir} is also RG-invariant (since K_{dir} is α-independent and the mass ratios are fixed, the only running is in the overall scale M, which does not affect s or α).

On the inverse branch, the situation is subtler. The inverse masses $1/m_i$ have different ratios from m_i, and K^{-1} depends on both s and α. However, since the anomalous dimension is flavour-independent, $m_i(Q)/m_j(Q) = m_i(\mu)/m_j(\mu)$ at all scales. This means that both α and s on the *inverse* branch are also RG-invariant: the $1/m_i$ ratios run the same way as the m_i ratios (same multiplicative factor cancels in all ratios).

Proposition 5. *Both the Cartan angle α and the shift s are RG-invariant at any loop order where the anomalous dimension γ_m is flavour-independent.*

The RG invariance of α and s strengthens the universality claim: it holds at all scales simultaneously, not just at a particular renormalisation point.

The AHS crossing.. The statement that $K^{-1}(d, s, b)$ "passes through 2/3 at Q_{AHS}" uses *threshold-corrected* running masses (freezing out heavy quarks below their mass shells). In the frozen-mass prescription, the effective $m_s(Q)/m_b(Q)$ ratio changes at the b-threshold, which does shift the effective s and α. This is

Figure 6: Polar plot of the Cartan charge directions at $Q = 100\,\mathrm{TeV}$ (5-loop SMDR running [?]), compared to PDG-scale ghost markers (open circles). The shifts and angles are RG-invariant within the $\overline{\mathrm{MS}}$ scheme: the pattern is stable from $Q = 2\,\mathrm{GeV}$ to $Q = 100\,\mathrm{TeV}$. The largest shift occurs for the (u, c, t) and (b, c, s) tuples, whose angles move by ~ 1–$5°$ due to threshold effects in the frozen-mass prescription.

a prescription-dependent effect, not a fundamental running. The invariance holds within a fixed mass scheme (e.g., $\overline{\mathrm{MS}}$ with all quarks active).

In the frozen-mass prescription, 5-loop SMDR running [?] gives two crossing scales: $K^{-1}(d, s, b) = 2/3$ at $Q \approx 1\,\mathrm{TeV}$, and $K^{-1}_{++-}(s, c, t) = 2/3$ at $Q \approx 82\,\mathrm{TeV}$. Both inverse tuples converge toward exact $K = 2/3$ at high scale and then drift away, consistent with a *holomorphic* Koide condition that is masked by generation-dependent Kähler corrections except near the scale where those corrections happen to cancel.

9. Discussion

Summary.. The free-shift generalisation of the Koide ansatz decomposes the mass formula into two independent components:

1. The **shift** s, which determines the Koide ratio K (on the direct branch, via $K = 1/3 + 1/(18s^2)$) and is approximately universal across all sectors: $s \approx z_0 = 1/\sqrt{6}$.
2. The **Cartan angle** α, which determines mass ratios and varies between sectors, spanning $4°$–$37°$ in the $60°$ Weyl chamber.

The canonical value $s = 1/\sqrt{6}$ that gives $K = 2/3$ has a group-theoretic origin ($\mathrm{U}(3)_F$ trace normalisation). The tuples split between two Weyl-dual branches ($15°$ and $45°$) depending on whether the lightest mass has positive or negative charge.

What this achieves.. The universality of s reduces the "Koide mystery" from a set of apparently independent numerical coincidences (one per sector) to a *single* structural fact: all sectors share the same $\mathrm{U}(3)_F$ normalisation, with sector-dependent Kähler corrections at the $\sim 10\%$ level. The *angle* is not a mystery to be explained by a single parameter; it encodes the physical mass hierarchy specific to each sector.

The shift hierarchy.. The physical shifts form a hierarchy:

$$\begin{array}{lll} s \approx 1/\sqrt{6} & \Rightarrow & K = 2/3 \text{ (leptons, inv. down quarks, } (t,b,c)) \\ s \approx 0.38\text{–}0.40 & \Rightarrow & K \approx 0.46\text{–}0.66 \; ((b,c,s),\, (c,s,u)) \\ s \approx 0.37 & \Rightarrow & K \approx 0.73 \text{ (down quarks, direct)} \\ s \approx 0.33 & \Rightarrow & K \approx 0.85 \text{ (up quarks)} \\ s \approx 0.49 & \Rightarrow & K \approx 0.57 \; ((s,u,d)) \end{array}$$

The pattern $s_{\text{lep}} \approx s_{(t,b,c)} \approx s_{(b,c,s)} \approx z_0$ may reflect the Kähler metric hierarchy: composites involving only heavy quarks (t, b, c) have Kähler metrics closer to canonical, while composites involving light quarks (u, d, s) have larger corrections.

Predictive content.. The free-shift ansatz with universal s has **two** free parameters per sector: M (the overall scale) and α (the angle). These determine three masses, yielding **one prediction** per sector (the Koide ratio K is predicted by the universal s). If $s = z_0$ is exact, this predicts $K = 2/3$ for all sectors; deviations from $2/3$ measure the Kähler correction δ_s.

Relation to the quartic mixing.. In the waterfall analysis [**?**], the inter-branch coupling ε controls the mixing between the direct and inverse branches. The chain windows involve both branches, and their effective shifts and angles are determined by the quartic. The universality of s in the chain windows is therefore evidence that the quartic mixing *preserves* the $\mathrm{U}(3)_F$ normalisation while modifying the Cartan angle.

The Weyl duality.. The $15°/45°$ split has a natural interpretation. The $15°$ fixed point is the "standard" Koide configuration: one vanishing mass, the other two in the golden ratio $R = (2+\sqrt{3})^2$. The $45°$ fixed point is its Weyl mirror: one vanishing mass with a sign-flipped charge. Physical tuples near these fixed points inherit the branch structure. As a mass approaches zero from the $15°$ side, the charge $q = s + z$ passes through zero and the tuple transitions to the $45°$ branch. The (b,c,s) tuple, with its near-zero strange-quark charge, is precisely at this transition.

Open questions..

1. **Origin of $s = 1/\sqrt{6}$.** The group-theoretic derivation of $s = z_0$ from $\mathrm{U}(3)_F$ normalisation [**?**] explains the value but not why the physical Kähler potential is close to canonical. Can the $\sim 10\%$ sector-dependent corrections to s be computed from the Kähler metric of the confined $\mathrm{SU}(3)_F$ theory?

2. **Angle spectrum.** What determines the sector-dependent angles? The lepton angle ($12.7°$) is close to the $15°$ fixed point, while the (u, c, t) angle ($4.3°$) is far from it. In the M-theory lift, α is a Kähler modulus of the M5-brane curve. Can the quark angles be predicted?
3. **Neutrino sector.** The baryonino masses (Majorana neutrinos) should also decompose into (M, s, α). The universal s predicts that the neutrino shift is the same as the charged-lepton shift, but the angle may differ.
4. **Weyl branch selection rule.** Is there a dynamical rule that determines whether a given mass window sits on the $15°$ or $45°$ branch? The empirical pattern is that the lightest mass in the window determines the branch: if its charge is positive, $15°$; if negative (near the zero-mass boundary), $45°$.

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