The Koide ratio as a Casimir invariant: a Diophantine selection of three families

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Abstract

The Koide ratio $K = \sum m_i / (\sum \sqrt{m_i})^2$ equals 2/3 for charged leptons to 0.001%. We show that this value is the mean-square weight $\langle J_3^2 \rangle$ of the principal $\mathfrak{sl}(2)$ subalgebra of $\mathfrak{sl}(N)$ evaluated in the fundamental representation, and that the equality $K_N = \langle J_3^2 \rangle$ holds if and only if $N = 3$. The condition reduces to the Diophantine equation $N(N^2-1) = 24$, whose unique positive integer solution selects three families. The operator connecting the two natural mass branches of an SU(3) confining theory—the direct ($\mathrm{adj}(M) \propto m$) and the inverse ($M \propto 1/m$)—is identified as the spin-2 component of $\mathrm{End}(V)$ under the same principal embedding, linking the mass structure to Kostant's exponents of $\mathfrak{sl}(3)$.

Keywords: Koide formula, family replication, principal $\mathfrak{sl}(2)$, Casimir invariant

1. Introduction

The charged-lepton mass ratio

$$K \;\equiv\; \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2} \;=\; 0.666661 \quad (1)$$

was noted by Koide [?] to lie within 0.001% of the value 2/3. The precision is striking: among all triplets drawn from a random mass spectrum, the probability of this level of agreement is $p < 10^{-5}$ [?]. Numerous models have been proposed to explain the relation [? ? ?], but none has derived the value 2/3 from first principles without introducing it as an input.

In this Letter we observe that 2/3 is a representation-theoretic invariant: it is the *mean-square weight* of the fundamental representation of SU(N) under its principal $\mathfrak{sl}(2)$ subalgebra [?], and that the equality with the Koide ratio holds *if and only if* $N = 3$. This provides a Diophantine selection of three families that is independent of any dynamical model.

We also show that the operator connecting the direct and inverse mass branches of an SU(3) confining theory [? ?] is the *spin-2* component of End(V) under the principal embedding, establishing a connection between the Koide mass structure and Kostant's exponents of $\mathfrak{sl}(3)$.

2. The Koide ratio for U(N)

Let $\hat{z}_i = z_0 + z_i$ ($i = 1, \ldots, N$) be the signed charges of a U(N) family symmetry, with z_0 the U(1) generator and z_i the SU(N) Cartan components. Canonical normalisation gives:

$$\sum_{i=1}^{N} z_i = 0\,, \qquad \sum_{i=1}^{N} z_i^2 = \frac{1}{2}\,, \qquad z_0 = \frac{1}{\sqrt{2N}}\,. \quad (2)$$

Theorem 1 (Generalised Koide ratio)**.** *For the monomial mass law $m_i = \mu\, \hat{z}_i^2$ and the ratio*

$$K_N \;\equiv\; \frac{\sum_{i=1}^{N} \hat{z}_i^2}{\left(\sum_{i=1}^{N} \hat{z}_i\right)^2}\,, \quad (3)$$

one has $K_N = 2/N$, independently of the Cartan direction.

Proof. Using (??): $\sum \hat{z}_i^2 = N z_0^2 + \sum z_i^2 = \frac{1}{2} + \frac{1}{2} = 1$ and $\sum \hat{z}_i = N z_0 = \sqrt{N/2}$, so $K_N = 1/(N/2) = 2/N$. □

For $N = 3$: $K_3 = 2/3$, recovering the Koide value as an algebraic identity of U(3) normalisation. This is the $n = 2$ uniqueness theorem [? ?]: among monomial mass functions $m \propto \hat{z}^n$, only $n = 2$ gives a direction-independent ratio.

3. The principal $\mathfrak{sl}(2)$ and its Casimir

Every complex simple Lie algebra $\mathfrak{g}$ contains a distinguished three-dimensional simple subalgebra, the *principal* $\mathfrak{sl}(2)$, defined (up to conjugacy) as the unique $\mathfrak{sl}(2)$ for which the fundamental representation of $\mathfrak{g}$ is irreducible [**?**].

For $\mathfrak{g} = \mathfrak{sl}(N)$, the fundamental $\mathbf{N}$ is the spin-j representation of the principal $\mathfrak{sl}(2)$ with

$$j = \frac{N-1}{2}\,, \qquad \dim = 2j+1 = N\,. \tag{4}$$

The weight operator J_3 has eigenvalues $m = j, j-1, \ldots, -j$. The quadratic Casimir is $C_2 = j(j+1)$, and the mean-square weight is

$$\langle J_3^2\rangle \equiv \frac{\mathrm{Tr}(J_3^2)}{N} = \frac{1}{N}\sum_{m=-j}^{j} m^2 = \frac{j(j+1)(2j+1)}{3N}\,. \tag{5}$$

Using $j = (N-1)/2$ and $2j+1 = N$:

$$\langle J_3^2\rangle = \frac{(N-1)(N+1)}{12} = \frac{N^2-1}{12}\,. \tag{6}$$

For $N = 3$: $\langle J_3^2\rangle = 8/12 = 2/3$.

4. The Diophantine condition

Setting $K_N = \langle J_3^2\rangle$:

$$\frac{2}{N} = \frac{N^2-1}{12} \iff N(N^2-1) = 24\,. \tag{7}$$

Factoring:

$$N(N-1)(N+1) = 24 = 2\times 3\times 4\,. \tag{8}$$

The left-hand side is the product of three consecutive integers centred at N. The only positive integer solution is $\boxed{N=3}$ (since $2\times3\times4 = 24$ and $3\times4\times5 = 60 > 24$).

Equivalently, the factored form $(N-3)(N^2+3N+8) = 0$ has discriminant $\Delta = 9-32 < 0$ for the quadratic factor, so there are no other real roots.

Physical interpretation.. At $N = 3$ and *only* $N = 3$, the representation dimension $2j+1 = N$ coincides with the algebra dimension $\dim \mathfrak{sl}(2) = 3$. The Casimir per state (C_2/N) then equals the Casimir per generator ($C_2/3$), giving

$$\frac{C_2}{N} = \frac{C_2}{3} = \langle J_3^2\rangle = K = \frac{2}{3}\,. \tag{9}$$

This is the representation-theoretic content of the Koide value: $K = 2/3$ is the statement that the fundamental representation of SU(3) *is* the adjoint of its principal $\mathfrak{sl}(2)$.

5. The spin-2 operator and the mass branches

In SU(3) SQCD with $N_f = N_c = 3$, the quantum-modified constraint $\det M = \Lambda^6$ [**?**] produces two holomorphic mass branches from one meson matrix [**?**]:

$$\langle M_i\rangle \propto \frac{1}{m_i}\,, \qquad \mathrm{adj}(M)_i \propto m_i\,. \tag{10}$$

Define the mixing operator R by

$$R_{ij} = \langle M_k\rangle \quad (k \neq i,j;\ i\neq j)\,, \qquad R_{ii} = 0\,. \tag{11}$$

This operator satisfies the exact identity

$$\boxed{R\cdot\mathbf{M} = 2\,\mathrm{adj}(\mathbf{M})\,,} \tag{12}$$

where $\mathbf{M} = (M_1, M_2, M_3)^T$ is the meson VEV vector. On the constraint surface, the Cayley–Hamilton identity gives:

$$R^3 = \mathrm{Tr}(D^2)\,R + 2\Lambda^6\,I\,, \tag{13}$$

where $D = \mathrm{diag}(M_1, M_2, M_3)$. After conjugation by D:

$$D\,R\,D = \Lambda^6\,(J-I)\,, \tag{14}$$

where J is the 3×3 all-ones matrix. The right-hand side is the *democratic matrix*, independent of the mass hierarchy: the individual masses are entirely absorbed by the D conjugation.

R as a spin-2 operator.. Under the principal $\mathfrak{sl}(2)$ of Section **??**, the endomorphism algebra $\mathrm{End}(V)$ for $V =$ spin-1 decomposes as

$$\mathrm{End}(\text{spin-1}) = \text{spin-0}\oplus\text{spin-1}\oplus\text{spin-2}\,. \tag{15}$$

The three components correspond to:

- **Spin-0** (dim 1): $\propto I$ (trace part). Since $\mathrm{Tr}R = 0$, this is absent.
- **Spin-1** (dim 3): antisymmetric matrices. Since $R = R^T$, this is absent.
- **Spin-2** (dim 5): symmetric traceless matrices. **This is R.**

Therefore R is the *unique* spin-2 component of $\mathrm{End}(V)$ under the principal embedding.

In Kostant's theory [**?**], the adjoint representation of $\mathfrak{sl}(3)$ decomposes under the principal $\mathfrak{sl}(2)$ as $\mathbf{8} = \mathbf{3}\oplus\mathbf{5}$, with the spin content determined by the exponents $\{m_1, m_2\} = \{1, 2\}$ of $\mathfrak{sl}(3)$. The spin-1 piece ($\mathbf{3}$) *is* the principal $\mathfrak{sl}(2)$ itself; the spin-2 piece ($\mathbf{5}$) is the complementary subspace. The operator R lives in this exponent-2 representation: it is the "second exponent" of $\mathfrak{sl}(3)$.

6. Comparison with other $N = 3$ criteria

The Casimir–Koide criterion joins four previously known conditions that each uniquely select $N = 3$:

Criterion	Equation	Solution
Diophantine [?]	$rs = 2N$, $r(r+1)/2 = 2N$	$N = 3$
Discriminant	$(N+2)^2 - 4 = N(2N+1)$	$N = 3$
Binomial	$\binom{3N}{N} = (N+1) \cdot N(2N+1)$	$N = 3$
CP-phase	$N+4 = 2N+1$	$N = 3$
Casimir–Koide	$\mathbf{N(N^2-1) = 24}$	$\mathbf{N = 3}$

The first four arise from compositeness counting, algebraic geometry, 11d SUGRA, and the CKM phase, respectively. The Casimir–Koide criterion is the most algebraically natural: it identifies the Koide value 2/3 with a standard invariant of the principal $\mathfrak{sl}(2)$ embedding, and the $N = 3$ selection follows from the coincidence of the representation and algebra dimensions.

7. Discussion

The result presented here is purely algebraic: it connects the Koide ratio (an empirical mass relation) to the Casimir of the principal $\mathfrak{sl}(2)$ (a canonical object in Lie theory) through a Diophantine equation that selects $N = 3$. No dynamical model is assumed beyond the $U(N)$ charge structure and the quadratic mass law $m_i = \mu\, \hat{z}_i^2$.

The spin-2 identification of the branch-mixing operator R adds a structural layer: in the confining SU(3) theory, the operator that connects the direct (lepton-like) and inverse (quark-like) mass branches is not an arbitrary matrix but the unique spin-2 irreducible component of $\mathrm{End}(V)$ under Kostant's principal embedding. The democratic structure $DRD = \Lambda^6(J - I)$ shows that the R operator, when expressed in the canonical basis set by the meson VEVs, becomes mass-independent—the hierarchy is entirely encoded in the conjugation by D.

Two open questions deserve mention. First, the spin-2 nature of R is suggestive of a gravitational connection: in 11-dimensional supergravity, the metric g_{MN} is a spin-2 field, while the 3-form C_3 is spin-1. The composites of the SU(3) theory (mesons coupling to C_3, baryons coupling to C_6) are spin-1 objects in the M-theory lift [? ?]; the branch-mixing operator R is spin-2, matching the graviton sector. Whether this is a structural connection or a numerical coincidence remains to be determined.

Second, the cubic algebra (**??**) differs from the quadratic relation $J^2 = -1$ that characterises $SU(2)_R$ in $\mathcal{N} = 2$ supersymmetry. The emergent structure is therefore not standard $\mathcal{N} = 2$ but a cubic deformation controlled by the mass hierarchy ($\mathrm{Tr}D^2$) and the quantum constraint (Λ^6). Whether this cubic defines a consistent deformed superalgebra is an open problem.

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