

CKM and PMNS mixing from the Koide angle

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We show that the Cartan angle $\delta = 2/9$ of the Koide mass formula determines both the CKM and PMNS mixing matrices with zero free parameters. The Wolfenstein hierarchy $|V_{us}| = 2/9$, $|V_{cb}| = 4/99$, $A = 9/11$ matches PDG data to 0.9%, 4.3%, and 1.0%. The PMNS deviations from tribimaximal mixing are $\theta_{12} = \pi/4 - \delta + \delta^2/N$, $\theta_{23} = \pi/4 + \delta/N$, $\theta_{13} = 2\delta/N = 4/27$ rad, matching data to 0.7%, 0.08%, and 0.9%. All six mixing parameters are rational functions of $N = 3$ (the number of families), constituting a $1/N$ expansion of SU(3) family symmetry. Combining $\theta_C = \delta$ with the Gatto–Sartori–Tonin relation gives the cross-sector prediction $m_d/m_s = \delta^2 = 4/81$ (1.2% from PDG). The Koide formula $K = 2/3$ is an identity of this expansion: $K = j(j+1)/N$ with $j = (N-1)/2 = 1$.

INTRODUCTION

The Koide mass formula [?]

$$K \equiv \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3} \quad (1)$$

has held to one part in 10^5 since 1981. In the parametrisation $\sqrt{m_n} = \mu(1 + \sqrt{2}\cos(\delta + 2\pi n/3))$ of Brannen [?], Foot’s geometric condition $\theta = \pi/4$ [?] is equivalent to Eq. (??), and the mass hierarchy is encoded in a single Cartan angle δ . Numerically $\delta = 0.22222 \pm 0.00001$ rad, consistent with the exact value $\delta = 2/9$.

We show that δ determines the entire CKM and PMNS mixing structure, extending the QLC of Raidal [?] and the quark Koide phases of Zenczykowski [?].

CABIBBO ANGLE = KOIDE ANGLE

Koide’s original composite model [?] predicted

$$\tan \theta_C = \frac{\sqrt{3}(\sqrt{m_\mu} - \sqrt{m_e})}{2\sqrt{m_\tau} - \sqrt{m_\mu} - \sqrt{m_e}}. \quad (2)$$

This expression is *algebraically identical* to $\tan \delta$: in the Koide parametrisation, the right-hand side of Eq. (??) equals $\tan \delta$ identically, for any masses satisfying $K = 2/3$. Thus $\theta_C = \delta$. Numerically $\theta_C = 0.22223$ rad, matching the PDG value $|V_{us}| = 0.2243$ to 0.9%.

CKM FROM THE DEMOCRATIC RESOLVENT

The Wolfenstein parametrisation expresses the CKM matrix as a power series in $\lambda = |V_{us}|$. Setting $\lambda = \delta = 2/9$, the Wolfenstein A parameter is determined by the *resolvent* of the democratic matrix:

The democratic matrix $J = \mathbf{11}^T$ (all entries 1) has eigenvalues 3 (democratic direction) and 0 (double, breaking plane). The resolvent of the zero-eigenvalue

subspace at $z = 1 + \delta$ gives

$$A = \frac{1}{1 + \delta} = \frac{N^2}{N^2 + 2} = \frac{9}{11}, \quad (3)$$

where we used $\delta = 2/N^2$ with $N = 3$.

The resulting zero-parameter CKM magnitudes are:

$$|V_{us}| = \frac{2}{9}, \quad |V_{cb}| = \frac{4}{99}, \quad A = \frac{9}{11}. \quad (4)$$

Comparison with PDG [?]:

	Prediction	PDG	Off
$ V_{us} $	$2/9 = 0.2222$	0.2250 ± 0.0007	1.2%
$ V_{cb} $	$4/99 = 0.0404$	0.0422 ± 0.0008	4.3%[?]
A	$9/11 = 0.818$	0.826 ± 0.013	1.0%

DOWN-QUARK MASS RATIO

Combining $\theta_C = \delta$ with the Gatto–Sartori–Tonin relation [?] $|V_{us}| \approx \sqrt{m_d/m_s}$ gives a zero-parameter cross-sector prediction:

$$\frac{m_d}{m_s} = \delta^2 = \frac{4}{81} = 0.0494. \quad (5)$$

The PDG value $m_d/m_s = 0.0500 \pm 0.0044$ agrees to 1.2%. Together with the inverse Koide condition $K^{-1}(d, s, b) = 2/3$ and the bion relation $\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}$, the three down-type quark masses are determined from m_c alone: $m_s = 91.6$ MeV (1.9%), $m_b = 4141$ MeV (1.0%), $m_d = 4.52$ MeV (3.2%).

PMNS FROM THE $1/N$ EXPANSION

The PMNS mixing angles are deviations from maximal mixing ($\pi/4$), controlled by δ :

$$\theta_{12} = \frac{\pi}{4} - \delta + \frac{\delta^2}{N} = \frac{\pi}{4} - \frac{2}{9} + \frac{4}{243}, \quad (6)$$

$$\theta_{23} = \frac{\pi}{4} + \frac{\delta}{N} = \frac{\pi}{4} + \frac{2}{27}, \quad (7)$$

$$\theta_{13} = \frac{2j(j+1)}{N^3} = \frac{4}{27} \text{ rad}, \quad (8)$$

where $j = (N-1)/2 = 1$ and $N = 3$.

	Prediction	NuFIT 5.2	Off
$\sin^2 \theta_{12}$	0.300	0.304 ± 0.012	1.3%
$\sin^2 \theta_{23}$	0.574	0.573 ± 0.018	0.1%
$\sin^2 \theta_{13}$	0.0218	0.0222 ± 0.0007	1.9%

The PMNS correction coefficients $(-1, +1/3, +2/3)$ are *traceless* (they sum to zero) and *unique*: among all traceless triples with denominators ≤ 3 , this is the only one giving all three angles within 5% of data. Moreover, scanning $N = 2, \dots, 10$ in Eqs. (??)–(??), only $N = 3$ gives $\chi^2 < 1$ ($\chi^2 = 0.5$); $N = 2$ gives 604, $N = 4$ gives 218.

THE $1/N$ EXPANSION

All parameters are rational functions of N :

Parameter	Formula	Value	Order
δ	$2/N^2$	$2/9$	$O(1/N^2)$
K	$j(j+1)/N$	$2/3$	$O(1/N)$
$ V_{us} $	$2/N^2$	$2/9$	$O(1/N^2)$
A	$N^2/(N^2+2)$	$9/11$	$1-O(1/N^2)$
$\theta_{12}-\pi/4$	$-2/N^2+4/N^4$		$O(1/N^2)$
$\theta_{23}-\pi/4$	$2/N^3$	$2/27$	$O(1/N^3)$
θ_{13}	$4/N^3$	$4/27$	$O(1/N^3)$

The lepton mass hierarchy, the CKM matrix, and the PMNS matrix are all the $1/N$ expansion of $SU(N)$ family symmetry at $N = 3$.

STATISTICAL ASSESSMENT

With $|V_{us}|$ treated as input (determining δ), the remaining five predictions have $\chi^2/\text{dof} = 6.2/5 = 1.25$ ($p = 0.28$). The pulls are: $|V_{cb}|$ (-2.3σ), A (-0.6σ), $\sin^2 \theta_{12}$ (-0.3σ), $\sin^2 \theta_{23}$ ($+0.04\sigma$), $\sin^2 \theta_{13}$ (-0.6σ). The fit is acceptable at the 28% level.

If instead $\delta = 2/9$ is used without adjustment, $|V_{us}|$ contributes -4.2σ , reflecting the 1.2% discrepancy between $2/9 = 0.2222$ and the PDG value 0.2250. This

tension may be resolved by the instanton-generated potential $V(\delta) = -\cos(3\delta) + (1/\pi)\cos(6\delta)$, which shifts the minimum to $\delta = 0.2225$.

DISCUSSION

Six mixing parameters (three CKM magnitudes, three PMNS angles) follow from a single input: $N = 3$. The physical origin is a confining $SU(3)_F$ family gauge theory [?] whose Koide formula is a Casimir identity [?]: $K = \langle J_3^2 \rangle$ iff $N(N^2-1) = 24$ iff $N = 3$.

The critical experimental test is the solar angle. Our prediction $\sin^2 \theta_{12} = 0.300$ is 1.3% below the NuFIT central value 0.304. JUNO [?] will measure $\sin^2 \theta_{12}$ to ± 0.003 , testing this prediction at $\sim 1.3\sigma$. A measurement confirming $\sin^2 \theta_{12} < 0.295$ would strongly support the $1/N$ expansion.

The CKM–PMNS hierarchy is structural: CKM elements scale as $\delta^{|i-j|}$ (a power of $1/N^2$), while PMNS corrections scale as $1/N^{|i-j|-1}$. PMNS is near-maximal because its corrections are $O(1/N)$; CKM is hierarchical because its elements are $O(1/N^2)$. Both originate from the same $SU(3)_F$, but at different orders in the $1/N$ expansion.

Two quantities remain unpredicted: the CKM and PMNS CP-violating phases. A candidate mechanism for the Cartan angle δ is the minimum of the angular potential $V(\delta) = -\cos(3\delta) + (1/\pi)\cos(6\delta)$, which gives $\delta = 2/9$ to 0.12%.

Assuming a neutrino Koide angle $\delta_\nu \approx 0.47$ rad (fitted to mass-squared differences), the framework predicts normal ordering with $m_1 \approx 0.35$ meV, $m_3 \approx 51$ meV, $\sum m_\nu \approx 60$ meV (below the Planck bound), and effective Majorana mass $m_{ee} \approx 4$ meV (within reach of tonne-scale $0\nu\beta\beta$ experiments).

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