Three generations from SU(3) confinement and U(3) family symmetry

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Abstract

We study the flavour sector of the Standard Model using two ingredients: the moduli space of $\mathcal{N} = 1$ SU(3) SQCD with three flavours, and a U(3) family symmetry with canonical normalization. The Seiberg confined description gives a meson moduli space with a $\mathbb{Z}_2$ involution $M \to \Lambda^4 M^{-1}$ that exchanges the $\mathbf{15}$ and $\overline{\mathbf{15}}$ of $\mathrm{SU(5)}_{\text{flavour}}$. The U(3) structure determines the mass deformations through a charge ansatz $m_i = k(z_0 + z_i)^2$, with $z_0 = 1/\sqrt{6}$ fixed by group theory. The self-dual charge spectrum has algebraic discriminant 21, which equals the dynamical discriminant of the family Higgs potential only for $N = 3$ generations. Extension to $N_f = 4$ true s-confinement recovers the Pati–Salam decomposition from the baryon sector, with the top quark as the decoupling fourth flavour. The effective Lagrangian involves a single meson field: both electroweak Higgs doublets emerge from M and its adjugate $\mathrm{adj}(M)$, which is holomorphic in M alone, constraining the ratio v_u/v_d at the superpotential level (subject to Kähler corrections). Electroweak symmetry breaking necessarily breaks SUSY: the F-term equations are overconstrained, and the SUSY-breaking auxiliary fields *are* the lepton masses, $|F_{2j}| = m_j$. The family confinement scale $\Lambda = f_\pi\sqrt{3}/2$ relates the electroweak VEV to the lepton mass hierarchy and the Kähler normalization of the meson condensate. From four continuous input parameters—a scale k, an angle α, m_d, and m_s—together with a set of structural assumptions (Table **??**), the framework is consistent with thirteen observables including three charged-lepton masses (0.001% in K), four quark masses (1–4%, with m_b sensitive to the m_d input), the Cabibbo angle (0.01%), $|V_{cb}| = 1/(5R)$ (1%), $|V_{ub}| = 2\sqrt{2}/(7R^3)$ (0.2%), and $\delta_{CP} = \arctan\sqrt{R}$ (0.2σ), where $R = (5 + \sqrt{21})/2$.

1 Introduction

The Standard Model contains 13 free parameters in the flavour sector—six quark masses, three charged-lepton masses, three CKM angles, and a CP phase—with no known relations among them.

In 1983, Koide [**?**] constructed a preon model aiming to relate the Cabibbo angle to the mass ratios between generations. The model produced, as a byproduct, a mass matrix ansatz $m_i \propto (a + b_i)^2$ where a is a universal constant and the b_i are signed quantities summing to zero, with the constraint that the average of the b_i^2 equals a^2. This predicted the tau mass as $m_\tau = 1776.97\,\mathrm{MeV}$—confirmed years later by experiment. The original

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preon model was subsequently ruled out by the absence of compositeness signals at high-energy colliders, but the mass matrix relation survived as an empirical fact in search of a theoretical home. The formula is a statement about the mass matrix, not about square roots: the b_i can be positive or negative, and it is only their *squares* that must be positive (as befits physical masses).

Seiberg's exact results for $\mathcal{N} = 1$ SQCD [?, ?] provide a controlled setting in which the infrared dynamics of a strongly coupled gauge theory is described by gauge-invariant composites (mesons and baryons) subject to an exact quantum constraint. For SU(3) with $N_f = 3$ flavours, this constraint defines a moduli space with a non-trivial involution. In this work we show that, when the mass deformations of this moduli space are parameterized by the charges of a U(3) family symmetry—where $a = z_0$ is the U(1) normalization and the $b_i = z_i$ are the SU(3) Cartan charges—the Koide relation becomes an algebraic identity:

- The charged-lepton Koide ratio $K \equiv \sum m_i/(\sum \sqrt{m_i})^2 = 2/3$ to 0.001% [?], evaluated at pole masses (here all $z_0 + z_i > 0$, so $\sqrt{m_i} = \sqrt{k}\,(z_0+z_i)$ with the positive root).
- The down-quark masses satisfy $K(1/m_d, 1/m_s, 1/m_b) = 2/3$ to 0.2% [?], using $\overline{\mathrm{MS}}$ masses at their conventional scales ($m_{d,s}$ at 2 GeV, m_b at m_b).
- The Cabibbo angle obeys $|V_{us}| \approx \sqrt{m_d/m_s}$ [?, ?, ?].

The paper is organized as follows and includes pedagogical explanations of the SUSY techniques for readers more familiar with the Standard Model than with superfield methods. Section **??** establishes the Seiberg confined description: the meson moduli space, its involution, and the off-diagonal meson mass matrix, with introductory material on holomorphy, anomaly matching, and the classification of SQCD theories. Section **??** introduces the U(3) family charges that parameterize the mass deformations. Section **??** shows how the involution exchanges the charged-lepton and down-quark charge assignments, identified with the $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$ exchange of $\mathrm{SU}(5)_{\text{flavour}}$, and explains the adjugate mechanism. Section **??** proves a uniqueness theorem for $N = 3$ generations via discriminant matching. Section **??** extends to $N_f = 4$ true s-confinement and the Pati–Salam structure. Section **??** collects the numerical results. Section **??** discusses open problems, including a self-contained primer on F-terms, D-terms, and SUSY breaking.

2 SU(3) SQCD: Confinement, Symmetry Breaking, and the Moduli Space

2.1 Why SUSY exact results are trustworthy

Before presenting the technical setup, we briefly explain why supersymmetric gauge theories allow exact non-perturbative statements—a feature absent in ordinary QCD.

In a supersymmetric theory, the superpotential W is a holomorphic function of the chiral superfields and the coupling constants. Holomorphy—the requirement that W depends only on the fields Φ and not on their complex conjugates $\bar{\Phi}$—is extremely constraining. (For readers unfamiliar with the term: a holomorphic function is a "complex-analytic" function, one that can be expanded in a Taylor series in its complex variables. A function like $|\Phi|^2 = \Phi\bar{\Phi}$ is *not* holomorphic because it depends on $\bar{\Phi}$; a function like $\Phi^3 + m\Phi$

is holomorphic.) This constraint forbids most quantum corrections. Seiberg's key insight [**?**] was that holomorphy, combined with symmetries and known limiting behaviour (weak coupling, large masses), often determines W uniquely. The result is an exact effective superpotential valid at *all* coupling strengths, including the strongly-coupled regime where perturbation theory fails.

The second pillar is 't Hooft anomaly matching. In quantum field theory, an *anomaly* arises when a symmetry of the classical theory is violated by quantum effects. The most familiar example is the chiral anomaly in QED, which is responsible for the decay $\pi^0 \to \gamma\gamma$. 't Hooft showed that anomaly coefficients—numbers computed from one-loop triangle diagrams of global symmetry currents—are *exact*: they receive no higher-order corrections and are independent of the energy scale. This means that any proposed low-energy description of a theory must reproduce the same anomaly coefficients as the UV quarks and gluons. This is a powerful consistency check: most candidate effective theories fail it. For $\mathrm{SU}(N_c)$ with N_f flavours, these conditions—together with holomorphy—uniquely fix the effective superpotential in terms of composite fields (mesons and baryons), up to an overall normalization set by the dynamical scale Λ.

The physical picture is analogous to QCD: the strong $\mathrm{SU}(3)_F$ interaction confines the "preons" (Q, $\tilde{Q}$) into colour-singlet bound states. In QCD, the quark bilinear $\bar{q}_i q_j$ forms a 3×3 matrix of bound states—the scalar mesons (σ, π, K, η, a_0, ...). In our SUSY theory, the analogous bilinear $Q^i\tilde{Q}_j$ forms a 3×3 matrix of meson *superfields* $M^i{}_j$, each containing a complex scalar and a Weyl fermion (the "mesino"). Similarly, the colour-antisymmetric combination of three quarks (the baryon) has a SUSY counterpart $B = \epsilon_{abc}Q^{a1}Q^{b2}Q^{c3}$.

The crucial difference from QCD is that supersymmetry and holomorphy allow *exact* control of the resulting low-energy theory. In QCD, the chiral Lagrangian for pions is only an approximation—corrections are incalculable at strong coupling. In SUSY QCD, the superpotential of the confined theory is exact to all orders. This is what makes the present framework non-trivially predictive: the relations between meson VEVs and mass deformations (eq. **??** below) are not approximations but exact consequences of holomorphy and symmetry.

Why $N_f = N_c$ is special: the quantum-modified moduli space. Seiberg's analysis [**?**, **?**] classifies SQCD theories by the ratio N_f/N_c:

- $N_f > 3N_c$: the theory is not asymptotically free.
- $\frac{3}{2}N_c < N_f < 3N_c$: the theory flows to an interacting conformal fixed point ("conformal window").
- $N_c + 1 \leq N_f \leq \frac{3}{2}N_c$: the theory confines and can be described by dual variables (Seiberg duality).
- $N_f = N_c + 1$: *s-confinement.* The confined theory has a smooth moduli space with a SUSY-preserving origin. The effective superpotential is a polynomial in the composites.
- $N_f = N_c$ (our case): the confined theory has a *quantum-modified* moduli space. The classical moduli space (where $\det M = 0$ is allowed) is deformed by instantons—non-perturbative tunnelling events between topologically inequivalent gauge-field configurations—to $\det M - B\tilde{B} = \Lambda^{2N_c}$, removing the origin. This is a genuinely quantum-mechanical effect with no classical analogue.

The $N_f = N_c$ case is the borderline between the s-confining window and the ADS (Affleck–Dine–Seiberg) regime [?], where a dynamical superpotential $W \propto 1/\det M$ is generated and no stable vacuum exists. At $N_f = N_c$, the ADS superpotential is replaced by a constraint—a qualitative change that occurs precisely at the threshold where the baryon becomes an independent degree of freedom (for $N_f < N_c$, baryons cannot be formed).

The reader unfamiliar with SUSY gauge theories may consult the reviews [?, ?] for pedagogical treatments; we now summarise the essential results.

2.2 The UV theory and its symmetries

Consider $\mathcal{N} = 1$ SU(3) SQCD with $N_f = 3$ flavours of chiral superfields Q^i, $\tilde{Q}_j$ $(i, j = 1, 2, 3)$ [?, ?]. (A *chiral superfield* is the basic building block of $\mathcal{N} = 1$ SUSY: it packages a complex scalar ϕ, a Weyl fermion ψ, and an auxiliary field F into a single object. The scalar and fermion have the same gauge quantum numbers—they are "superpartners." The field Q^i is the SUSY analogue of a quark: it transforms as a fundamental $\mathbf{3}$ of the gauge group SU(3), and carries a flavour index $i = 1, 2, 3$. The conjugate field $\tilde{Q}_j$ transforms as $\bar{\mathbf{3}}$.)

The classical global symmetry is

$$G_{\rm cl} = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A \times \mathrm{U}(1)_R \,. \tag{1}$$

The axial $\mathrm{U}(1)_A$ is anomalous: although it is a symmetry of the classical Lagrangian, quantum effects (instantons) break it. This is exactly analogous to the axial anomaly in QCD, which is responsible for the large mass of the η' meson.[1] The surviving non-anomalous symmetry includes the R-symmetry—a continuous symmetry unique to SUSY theories in which different components of a superfield carry different charges. (The R-charge is the SUSY analogue of chirality: the scalar component of a superfield carries R-charge r, while the fermion component carries R-charge $r-1$. The superpotential must have total R-charge 2, which severely constrains its form.) With the standard assignment $R_Q = R_{\tilde{Q}} = 0$ and $R_\Lambda = 2/3$, the R-symmetry constrains the effective superpotential.

2.3 Confined spectrum

The low-energy degrees of freedom are gauge-invariant composites:

- 9 mesons: $M^i{}_j = Q^i \tilde{Q}_j \quad (\mathbf{3}_L \otimes \bar{\mathbf{3}}_R)$,
- a baryon: $B = \epsilon_{abc}\, \epsilon_{ijk}\, Q^{ai} Q^{bj} Q^{ck} \quad (\mathbf{1}_L,\, B = +1)$,
- an antibaryon: $\tilde{B} = \epsilon^{abc}\, \epsilon^{ijk}\, \tilde{Q}_{ai} \tilde{Q}_{bj} \tilde{Q}_{ck} \quad (\mathbf{1}_R,\, B = -1)$.

Each meson superfield $M^i{}_j$ contains a complex scalar and a Weyl fermion (the "mesino") sharing the same quantum numbers. The 9 mesino fermions transform as $(\mathbf{3}, \bar{\mathbf{3}})$ under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$: 3 diagonal (neutral under $\mathrm{SU}(3)_{\rm diag}$) and 6 off-diagonal (charged). This counting—9 Weyl fermions—matches the minimal SM lepton sector ($L_i + e^c_i$, no ν_R) [?], a point we return to in the discussion.

These composites satisfy the exact quantum constraint

$$\det M - B\tilde{B} = \Lambda^6 \,, \tag{2}$$

[1] In QCD, the axial $\mathrm{U}(1)_A$ symmetry $q \to e^{i\gamma_5\theta} q$ is broken by the triangle anomaly to $\mathbb{Z}_{2N_f}$. The same mechanism operates here.

where Λ is the dynamical scale. Equation (**??**) is enforced by a Lagrange multiplier superfield X through the effective superpotential

$$W_{\text{dyn}} = X\big(\det M - B\tilde{B} - \Lambda^6\big). \tag{3}$$

This is the quantum-modified moduli space of the $N_f = N_c$ theory [**?**]. (The $N_f = N_c$ theory is *not* s-confining in the strict sense of [**?**]: the origin is removed and the moduli space is modified, whereas true s-confinement at $N_f = N_c + 1$ has a smooth origin. See Section **??**.)

Physical meaning of the quantum constraint. The constraint $\det M = \Lambda^6$ on the mesonic branch says that the product of the three diagonal meson VEVs is *fixed* by the strong dynamics:

$$\langle M_1\rangle \langle M_2\rangle \langle M_3\rangle = \Lambda^6 \,. \tag{4}$$

This is the SUSY analogue of the QCD chiral condensate $\langle \bar{q}q\rangle \sim \Lambda^3_{\text{QCD}}$, but elevated from an approximate order parameter to an exact constraint. While in QCD the value of the chiral condensate is only known numerically (from lattice calculations or sum rules), the SUSY constraint is exact.

The constraint has a dramatic physical consequence: if one meson VEV is made small (by giving the corresponding preon a large mass), the others must grow to compensate. This is the origin of the seesaw-like relation $\langle M_i\rangle \propto 1/m_i$ in eq. (**??**)—the same structure that produces the Koide mass hierarchy from a hierarchy of input charges. Removing the origin ($\det M = 0$ is forbidden) means that the confined theory has no symmetric vacuum: at least one meson must condense, and chiral symmetry is always broken.

2.4 Branches and symmetry breaking

What is a moduli space? In ordinary quantum field theory, the vacuum is typically unique (up to symmetry transformations). In supersymmetric theories, there are often *families* of degenerate vacua parameterised by the expectation values of scalar fields. This family of vacua is the *moduli space*—a manifold whose coordinates are the vacuum expectation values (VEVs) of the light scalars, and whose geometry encodes the low-energy physics. Moving along the moduli space costs no energy (it is "flat" in the potential), but the spectrum of fluctuations—the masses and couplings—changes from point to point. In the $N_f = N_c$ theory, the moduli space is the constraint surface $\det M - B\tilde{B} = \Lambda^6$: a 5-complex-dimensional manifold (9 meson entries plus B and $\tilde{B}$, minus the constraint, minus 1 from the Lagrange multiplier X).

The moduli space has two branches:

- **Mesonic branch** ($B = \tilde{B} = 0$): $\det M = \Lambda^6$, so at least one meson VEV is nonzero. The diagonal vacuum $\langle M\rangle = \text{diag}(\Lambda^2\omega_1, \Lambda^2\omega_2, \Lambda^2\omega_3)$ with $\omega_1\omega_2\omega_3 = 1$ breaks $\text{SU}(3)_L \times \text{SU}(3)_R \to \text{SU}(3)_V$ (or further, depending on the ω_i).
- **Baryonic branch** ($\det M > \Lambda^6$): $B\tilde{B} \neq 0$, breaking $\text{U}(1)_B$. We do not use this branch, but it exists as an integral part of the moduli space.

2.5 F-term equations and the mass-deformed vacuum

We now add explicit masses for the preons. In a SUSY theory, mass terms live in the superpotential: $W_{\text{tree}} = m_i M^i{}_i$ (with $m_i > 0$). Physically, giving a preon a mass m_i

means giving both the scalar and the fermion inside the superfield Q^i the same mass (SUSY enforces this). The mass deformation explicitly breaks the flavour symmetry: different generations get different masses, just as in the Standard Model.

The total superpotential is

$$W = X\big(\det M - B\tilde{B} - \Lambda^6\big) + m_i M^i{}_i\,. \tag{5}$$

To find the vacuum, we set all F-terms to zero (which, as explained in the SUSY-breaking primer in §??, minimises the scalar potential). The F-term equations are:

$$F_X = \det M - B\tilde{B} - \Lambda^6 = 0\,, \tag{6}$$

$$F_{M^i{}_j} = X\,(\operatorname{cof} M)^j{}_i + m_i\,\delta^j_i = 0\,, \qquad (\operatorname{cof} M = \text{cofactor matrix} \equiv \operatorname{adj}(M)^T) \tag{7}$$

$$F_B = -X\,\tilde{B} = 0\,, \tag{8}$$

$$F_{\tilde{B}} = -X\,B = 0\,. \tag{9}$$

From (??)–(??): either $X = 0$ (which contradicts (??) since $m_i \neq 0$) or $B = \tilde{B} = 0$. Taking the latter, we are on the mesonic branch with $\det M = \Lambda^6$.

The off-diagonal components of (??) force $\langle M^i{}_j\rangle = 0$ for $i \neq j$ (since the cofactor matrix of a diagonal M is diagonal). The diagonal components give

$$\langle M_i\rangle = \frac{\Lambda^2 P^{1/3}}{m_i}\,, \qquad P = m_1 m_2 m_3\,. \tag{10}$$

This is the central equation of the framework. It says: *heavy preons produce light mesons, and vice versa.* The meson VEV is *inversely proportional* to the preon mass, like a seesaw. This inverse relation is not assumed but derived from the F-term equations and the quantum constraint. It is exact in the SUSY limit and holds non-perturbatively.

Symmetry-breaking pattern. The mass deformation breaks $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ explicitly. In the vacuum (??):

- If $m_1 = m_2 = m_3$: $\mathrm{SU}(3)_V$ is preserved.
- If all m_i are distinct: the global symmetry is broken to $\mathrm{U}(1)^2$.
- $\mathrm{U}(1)_B$ is preserved ($B = \tilde{B} = 0$).
- $\mathrm{U}(1)_R$ is broken by $\langle M_i\rangle \neq 0$.

This is a rich scaffold for flavour symmetry breaking, driven entirely by the mass parameters m_i—which in Section ?? we fix using $\mathrm{U}(3)_F$ charges.

2.6 The meson mass matrix

The off-diagonal meson masses are

$$m_{\text{phys}}(M^i{}_j) = \frac{m_i m_j}{P^{1/3}}\,, \tag{11}$$

which are Λ-independent: they depend only on ratios of the input masses. The diagonal meson mass matrix, obtained from second derivatives of W on the constraint surface, is

$$W_S = P^{1/3}\big(\operatorname{diag}(v_i^2) - v \otimes v^T\big), \qquad v_i = \frac{m_i}{P^{1/3}}\,. \tag{12}$$

This matrix has the properties:

1. $\mathrm{Tr}(W_S) = 0$: the eigenvalues sum to zero.
2. $(W_S)_{kk} = 0$ for all k: the diagonal entries vanish (a consequence of $\partial^2 \det M/\partial M_k^2 = 0$, i.e., the multilinearity of the determinant).
3. Rank 2 (one zero eigenvalue from the constraint).

Property (2) is the key structural result: the Seiberg dynamics is *transparent* to the mass parameters. If the input masses m_i satisfy a sum rule, the physical spectrum preserves it at leading order in the off-diagonal perturbation.

Eigenvalue structure. The eigenvalues of W_S sum to zero: $\lambda_1 + \lambda_2 + \lambda_3 = 0$. This immediately rules out identifying the Seiberg eigenvalues directly with lepton masses (which are all positive). Numerically, $\lambda \approx \{+3229, -3229, -0.4\}$ MeV for physical lepton mass inputs. The Seiberg dynamics preserves the charge structure but does not generate the mass spectrum—the mechanism lives in the family sector.

Summary of the confined framework. To summarise the physical picture before proceeding to the charge ansatz: three "preons" (Q_i and $\tilde{Q}_j$) bind under a family $\mathrm{SU}(3)_F$ gauge force to form 9 meson superfields $M^i{}_j$ and a baryon/antibaryon pair. The mesons contain the Higgs doublets and the lepton fields. The vacuum is determined by an exact quantum constraint (the product of meson VEVs is fixed) plus mass deformations m_i that break the flavour symmetry. Heavy preons produce light mesons: $\langle M_i \rangle \propto 1/m_i$. The off-diagonal meson mass matrix is "transparent"—it does not spoil the charge structure. The remaining question is: *what determines the mass deformations m_i?*

3 Family Charges from U(3)

The mass parameters m_i in the tree-level superpotential $W_{\text{tree}} = m_i M^i{}_i$ are free inputs. We now specify them using a $\mathrm{U}(3)_F$ family symmetry.

Let the three generations transform as the fundamental **3** of $\mathrm{U}(3)_F = \mathrm{SU}(3)_F \times \mathrm{U}(1)$, with all generators in canonical normalization $\mathrm{Tr}(T_A T_B) = \frac{1}{2}\delta_{AB}$.

The SU(3) Cartan charges of the fundamental components satisfy

$$\sum_{i=1}^{3} z_i = 0\,, \qquad \sum_{i=1}^{3} z_i^2 = \frac{1}{2}\,, \tag{13}$$

and the U(1) charge is fixed:

$$z_0 = \frac{1}{\sqrt{6}}\,. \tag{14}$$

Physical interpretation of m_i. The parameter m_i is the tree-level superpotential mass deformation for the preon bilinear $Q^i\tilde{Q}_i$: $W_{\text{tree}} = \sum_i m_i\, Q^i\tilde{Q}_i$. After confinement, m_i appears simultaneously as: (i) the mass deformation that lifts the moduli space; (ii) the effective Yukawa coupling that determines fermion hierarchies (Section **??**); and (iii) the determinant factor in the meson VEV via $\langle M_i \rangle = \Lambda^2 P^{1/3}/m_i$. As a superpotential coupling, m_i has mass dimension one and is protected by holomorphy against perturbative corrections in the $\mathcal{N} = 1$ theory.

Charge ansatz. We *postulate* a quadratic relation between the mass deformation and the family quantum numbers:

$$m_i = k\,(z_0 + z_i)^2\,. \tag{15}$$

This is the norm-squared on the U(3) charge lattice—the Killing-form inner product restricted to the Cartan subalgebra. The quadratic form is distinguished on three grounds: (i) it is the unique W-invariant bilinear on the weight lattice (the Weyl group permutes the z_i, leaving $\sum(z_0+z_i)^2$ invariant); (ii) in heterotic string compactifications on a Narain lattice, the BPS mass-squared is $M^2 = |p_L|^2/\alpha'$ [?], where p_L is the left-moving momentum on the charge lattice; a Wilson line provides the universal shift z_0, the z_i are gauge Cartan charges; (iii) in the $\mathcal{N} = 2$ attractor mechanism, the BPS mass at the attractor point is $M^2 = |Z|^2$ with Z linear in charges [?]; on the U(3) lattice this gives (??). Note that the standard BPS formula $M^2 \propto Q^2$ yields a mass-*squared* quadratic in charges, whereas the ansatz (??) has a mass (dimension 1) quadratic in charges. The meson $M^i{}_j = Q^i\tilde{Q}_j$ resolves this: as a quark–antiquark bilinear it has mass dimension 2, so its VEV $\langle M_i\rangle$ plays the role of M^2, not M. Physically, the meson is a flavour "capacitor"—a bound pair of charges Q^i and $\tilde{Q}_i$ separated at the confinement scale Λ_F. The energy stored in such a pair is quadratic in the charge, $E \propto (z_0+z_i)^2$, just as the energy of a classical capacitor is $Q^2/(2C)$. In the M-theory lift (Appendix ??), this bound pair maps to an M2-brane whose BPS mass is $T_2 \times$ Area; the quadratic form would follow if both worldvolume directions scale linearly with (z_0+z_i)—a plausible but unproven geometric restatement of the same capacitor picture.

What is excluded. A linear ansatz $m_i = k'(z_0 + z_i)$ is excluded by positivity (some $m_i < 0$ for $|z_i| > z_0$). A quartic ansatz $m_i = k''(z_0 + z_i)^4$ does not produce $K = 2/3$. Higher-order U(3)-invariant deformations—for instance $\delta m_i \propto z_i^3$ from cubic Casimirs of the flavour group—vanish identically for SU(3) (the symmetric d-tensor contracts to zero in the fundamental). Mixed invariants of the form $m_i m_j$ for $i \neq j$ are off-diagonal mass terms forbidden by the diagonal $\mathrm{U}(1)^3$ unbroken in the Cartan subalgebra.

Radiative stability. Since m_i is a superpotential coupling, it receives no perturbative corrections from gauge loops (the non-renormalization theorem of [?]). The leading violation arises from $\mathrm{U}(3)_F$-breaking soft terms at the scale m_{soft}; writing $\delta m_i \sim m_{\text{soft}}\,c_i$, the correction to K is $\delta K \sim (m_{\text{soft}}/k)\,(\delta c/c) \sim 10^{-2}$ for $m_{\text{soft}} \sim 1\,\text{TeV}$, $k \sim 80\,\text{GeV}$. Thus the $K = 2/3$ relation is radiatively stable to per-cent accuracy whenever the family symmetry breaking scale exceeds the soft mass scale. We emphasise that the ansatz (??) is a *structural postulate*—it is the central assumption of this framework, not a derived result.

It follows from (??)–(??) that the Koide ratio

$$K \;\equiv\; \frac{\sum m_i}{\left(\sum\sqrt{m_i}\right)^2} = \frac{\sum(z_0+z_i)^2}{\left(\sum|z_0+z_i|\right)^2} = \frac{2}{3} \tag{16}$$

identically for *any* direction in the Cartan subalgebra. Note that $\sqrt{m_i}$ in the standard formula means $|z_0 + z_i|$—the absolute value of a signed charge. When all charges are positive ($z_0 + z_i > 0$ for each i), the absolute value is redundant. For sectors where one charge crosses zero (as in the (b, c, s) bridge tuple of Section ??), the sign matters: the Koide ratio is computed with signed charges, $K = \sum(z_0+z_i)^2/(\sum(z_0+z_i))^2$, not with positive square roots.

The proof uses only $\sum z_i = 0$ (tracelessness), $\sum z_i^2 = 1/2$ (Casimir), and $z_0 = 1/\sqrt{6}$ (canonical U(1) normalization). In SU(3) × U(1), the value of z_0 is free; in U(3), it is fixed by the requirement $\mathrm{Tr}(T_0^2) = 1/2$. String constructions generate U(N), not SU(N), from N D-branes, with the U(1) normalization protected by holomorphy [**?**].

Angle parameterization. The Cartan charges are

$$z_i = \frac{1}{\sqrt{3}} \cos\!\big(\alpha + \tfrac{2\pi(i-1)}{3}\big), \qquad i = 1, 2, 3\,. \tag{17}$$

Each mass sector is parameterized by one angle α and one scale k: two parameters for three masses.

Transparency. Combined with property (2) of W_S (eq. **??**), the ratio $K = 2/3$ is preserved to first order through the Seiberg dynamics. To see this, write the full meson mass matrix as $\mathcal{M} = \mathrm{diag}(m_i) + W_S$, where $m_i = k(z_0{+}z_i)^2$ are the UV masses and $W_S = P^{1/3}(\mathrm{diag}(v_i^2) - v \otimes v^T)$ is the Seiberg perturbation. Since $(W_S)_{kk} = 0$ (diagonal entries vanish by det multilinearity), the diagonal of $\mathcal{M}$ is m_k itself. By standard first-order perturbation theory, the eigenvalues are $\lambda_k = m_k + O(|W_S^{\mathrm{off}}|^2/\Delta m)$, and $K(\lambda_k) = K(m_k) + O(v^4/m^4) = 2/3 + O(v^4/m^4)$. Numerically, $|v_{\max}^2/m_{\min}| \sim 10^{-3}$, so the correction is negligible. The confining dynamics neither generates nor destroys the charge structure; it is *transparent* to the Koide relation.[2]

4 The Involution and the $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$ Exchange

The constraint hypersurface $\det M = \Lambda^6$ has a remarkable symmetry: it is invariant under the exchange

$$M_i \;\to\; \frac{\Lambda^4}{M_i}\,, \qquad \text{mapping} \qquad m_i \;\to\; \tilde{m}_i = \frac{P^{2/3}}{m_i}\,. \tag{18}$$

This is a $\mathbb{Z}_2$ *involution*—a map that, applied twice, returns to the starting point: ($M_i \to \Lambda^4/M_i \to M_i$). Physically, it exchanges large and small meson VEVs while preserving the product constraint. Each point on the moduli space has a "mirror" point related to it by this involution; the self-dual locus ($M_i = \Lambda^4/M_i$, i.e., $M_i = \Lambda^2$) is the fixed-point set.

Since K is scale-invariant, $K(\tilde{m}) = K(1/m)$: the involution exchanges the charge assignment of the standard sector (charged leptons, $K(m) = 2/3$) with that of the inverse sector (down quarks, $K(1/m) = 2/3$). This is the key to the lepton–quark connection: the *same* UV charge structure produces both Koide relations, related by the moduli-space involution.

4.1 The adjugate mechanism

The dynamical origin of this exchange is the *adjugate* of the meson matrix—a construction that may be unfamiliar to readers outside linear algebra, but plays a central role in the framework.

[2] This argument assumes the off-diagonal entries of W_S are perturbative compared to the mass splittings. For SUSY-broken meson masses the relevant splittings are the lepton masses, and $W_S^{ij} = -v_i v_j$ with $v_i = m_i/P^{1/3}$; the largest entry is $|W_S^{23}| \approx 3.2\,\mathrm{GeV}$ while $m_\tau - m_\mu \approx 1.7\,\mathrm{GeV}$, so the perturbative expansion is only marginally valid for the (μ, τ) sector. A non-perturbative argument using the exact eigenvalue equation $\det(\mathcal{M} - \lambda) = 0$ confirms the result [**?**].

For any $n \times n$ matrix A, the adjugate $\mathrm{adj}(A)$ is the transpose of the cofactor matrix. For a 3×3 diagonal matrix $M = \mathrm{diag}(M_1, M_2, M_3)$, it takes the simple form

$$\mathrm{adj}(M)_i = \prod_{j \neq i} M_j\,, \qquad M \cdot \mathrm{adj}(M) = \det(M)\,\mathbf{1}\,. \tag{19}$$

Explicitly: $\mathrm{adj}(M)_1 = M_2 M_3$, $\mathrm{adj}(M)_2 = M_1 M_3$, $\mathrm{adj}(M)_3 = M_1 M_2$. The key point is that $\mathrm{adj}(M)$ is a *polynomial* (quadratic) in the meson entries—it involves only multiplication, no division. This makes it holomorphic: it is a legitimate superpotential coupling. Yet from the identity $M \cdot \mathrm{adj}(M) = \det(M)\,\mathbf{1}$, we see that on the constraint surface $\det M = \Lambda^6$, the adjugate *behaves* like the matrix inverse: $\mathrm{adj}(M) = \Lambda^6\, M^{-1}$.

This is the mathematical magic of the framework: the adjugate provides a holomorphic realisation of the matrix inverse—an operation that is *not* holomorphic in general ($1/M$ involves M in the denominator). The quantum constraint "trades" the non-holomorphic inverse for a holomorphic polynomial.

The F-term equations of the s-confined superpotential $W = (\det M - B\tilde{B})/\Lambda^3 + m_i M^i{}_i$ yield, for diagonal M:

$$\frac{\partial W}{\partial M^i{}_i} = \frac{\mathrm{adj}(M)_i}{\Lambda^3} + m_i = 0 \qquad \Longrightarrow \qquad \mathrm{adj}(M)_i = -\Lambda^3\, m_i\,. \tag{20}$$

The B, $\tilde{B}$ F-terms force $B = \tilde{B} = 0$ in the massive vacuum. Thus *the adjugate of the meson matrix is proportional to the UV mass matrix*, while $M_i \propto 1/m_i$. This is exact in the supersymmetric limit.

Standard Model fermion masses arise from effective Yukawa couplings to the confined composites. If the left chiral symmetry is partially gauged as $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \subset \mathrm{SU}(3)_L$, with the branching rule $\mathbf{3}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}$, then the meson doublet $M^{1,2}{}_i$ has the quantum numbers of the down-type Higgs: $(1,2)_{-1/2}$. The adjugate $\mathrm{adj}(M)$ transforms as $(\bar{\mathbf{3}}_L, \mathbf{3}_R)$; since $\bar{\mathbf{3}}_L \to \mathbf{2}_{+1/2} \oplus \mathbf{1}_{-1}$, its doublet component has $(1,2)_{+1/2}$—the up-type Higgs. Both electroweak Higgs doublets thus emerge from a single meson matrix M, with $\mathrm{adj}(M)$ a holomorphic polynomial in M (not an independent chiral superfield). At the superpotential level, this constrains $\tan\beta$: $\langle \mathrm{adj}(M)_i \rangle / \langle M_i \rangle = \Lambda^6 / \langle M_i \rangle^2$ is fixed by the quantum constraint.

In the Standard Model, fermion masses arise from Yukawa couplings $y\,\bar{\psi}\,H\,\psi$ between fermions and the Higgs. In the present framework, the Higgs is the meson doublet $M^{1,2}$, and there are two natural holomorphic Yukawa operators at the confinement scale—one using M directly, the other using its adjugate:

$$W_{\mathrm{down}} = y_d\, q_L\, d^c\, \frac{M^{1,2}}{\Lambda} + y_\ell\, L\, e^c\, M^{1,2}\,, \qquad [H_d\text{-type},\ Y{=}-\tfrac{1}{2}]\,, \tag{21}$$

$$W_{\mathrm{up}} = y_u\, q_L\, u^c\, \frac{\mathrm{adj}(M)^{1,2}}{\Lambda^3}\,, \qquad [H_u\text{-type},\ Y{=}+\tfrac{1}{2}]\,. \tag{22}$$

The W_{down} operator gives masses $\propto \langle M_i \rangle \propto 1/m_i$ ("standard" masses, large for light preons); the W_{up} operator gives masses $\propto \langle \mathrm{adj}(M)_i \rangle \propto m_i$ ("inverse" masses, large for heavy preons). This is why the lepton and down-quark sectors obey *different* Koide relations: they couple to different operators of the same meson field. All operators conserve hypercharge. The mass parameter m_i in $W_{\mathrm{tree}} = m_i M^i{}_i$ serves three simultaneous roles: (i) the UV mass deformation of $\mathrm{SU}(3)_F$ SQCD, (ii) the effective Yukawa coupling between preons Q_i and $\tilde{Q}_i$, and (iii) the determinant of the meson VEV via $\langle M_i \rangle = \Lambda^2 P^{1/3}/m_i$. There are no independent Yukawa parameters: the hierarchy of SM masses *is* the hierarchy of family charges z_i.

Species coupling through M acquire physical masses $m_{\text{phys}} \propto M_i \propto 1/m_i$; species coupling through $\text{adj}(M)$ acquire $m_{\text{phys}} \propto \text{adj}(M)_i \propto m_i$.

Charged leptons, being the mesino components of $M^{1,2}{}_i$ itself, have $m_\ell \propto 1/m$ and therefore $K(m_\ell) = K(1/m) = 2/3$ (standard Koide). Down quarks, as elementary fields, couple to M through partial compositeness [?] with generation-dependent mixing $\lambda_i \propto m_i^{\gamma_*}$. If $\gamma_* \geq 1$, the net mass scales as $m_d \propto m_i^{2\gamma_*-1}$, and the physical Koide parameter depends on the exponent. In particular, $\gamma_* = 1$ gives $m_d \propto m_i$, producing inverse Koide:

$$K(m_\ell) = K(1/m_d) = K(1/m) = \frac{2}{3}\,, \tag{23}$$

unifying the two empirical relations as a single UV condition.

After soft SUSY breaking, the meson VEVs shift from the F-flat locus. The zero-diagonal property $(W_S)_{kk} = 0$ of the off-diagonal meson mass matrix (Section **??**) ensures that the Koide condition is preserved to first order in m_{soft}/Λ: the meson sector is *transparent* to the family-sector structure.

4.2 SU(5) assignment and arithmetic

In the SU(5) flavour decomposition of [?], the meson $M^{ij} \in \text{Sym}^2(5) = \mathbf{15}$ and the adjugate $\text{adj}(M)_{ij} \in \text{Sym}^2(\bar{5}) = \overline{\mathbf{15}}$. The involution (**??**) maps $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$, with the adjoint $\mathbf{24}$ fixed. Standard and inverse charge assignments are related by charge conjugation on the scalar composites. This generalises the Georgi–Jarlskog [?] mass relations between quarks and leptons from a GUT boundary condition to a moduli-space involution.

The involution has an arithmetic interpretation: $R = (5 + \sqrt{21})/2$ is the fundamental unit of the ring of integers $\mathbb{Z}[(1 + \sqrt{21})/2]$ of the real quadratic field $\mathbb{Q}(\sqrt{21})$, with $\text{Norm}(R) = R \cdot R' = 1$. The map $R \to 1/R = R' = (5 - \sqrt{21})/2$ is the non-trivial element of $\text{Gal}(\mathbb{Q}(\sqrt{21})/\mathbb{Q})$: the moduli-space involution *is* the Galois conjugation $\sqrt{21} \to -\sqrt{21}$.

5 Self-Duality, the Discriminant 21, and Uniqueness of $N = 3$

This section addresses the question: *why three generations?* The Standard Model works for any number of generations (anomaly cancellation requires each generation to be complete, but does not fix their number). We show that the charge structure of the preceding sections uniquely selects $N = 3$.

5.1 The self-dual charge spectrum

A natural question to ask about the involution of Section **??** is: what charge spectrum is *self-dual*—invariant under $m_i \to P^{2/3}/m_i$? On the self-dual locus, the charges form a geometric progression $Z_1 : Z_2 : Z_3 = 1 : R : R^2$—the same structure that arises from periodic solutions of Schwinger–Dyson gap equations, where Blumhofer and Hutter [?] found geometric mass ratios $\sim e^\pi$ between generations without free parameters. Imposing both $K(m) = 2/3$ and $K(1/m) = 2/3$ is algebraically equivalent to requiring

$$3(1 + R^2 + R^4) = 2(1 + R + R^2)^2\,, \tag{24}$$

which, using $1{+}R{+}R^2 = 6R$ if $R^2 = 5R{-}1$, reduces to $72R^2 = 72R^2$ identically. The self-dual Koide identity (**??**) is therefore equivalent to the minimal polynomial

$$t^2 - 5t + 1 = 0\,, \qquad R = \frac{5+\sqrt{21}}{2}\,, \qquad \mathrm{disc} = 21\,. \tag{25}$$

The self-dual angle follows from the geometric-progression condition. Writing $Z_i = (1 + \sqrt{2}\,c_i)/\sqrt{6}$ with $c_i = \cos(\alpha + 2\pi(i{-}1)/3)$, the condition $Z_2^2 = Z_1 Z_3$ expands to

$$3\sqrt{2}\,c_2 + 2c_2^2 - 2\,c_1 c_3 = 0\,. \tag{26}$$

The combination $2c_2^2 - 2c_1c_3$ is evaluated by product-to-sum identities (using $c_1c_3 = [-\frac{1}{2} + \cos(2\alpha{+}\frac{2\pi}{3})]/2$ and $c_2^2 = [1 + \cos(2\alpha{+}\frac{2\pi}{3})]/2$): it equals $\frac{3}{2}$ independently of α. Hence (**??**) reduces to $c_2 = -1/(2\sqrt{2})$. Since $c_2 = \cos(\alpha - 2\pi/3) = \sin(\alpha - \pi/6)$, we obtain $\sin(\alpha{-}\pi/6) = -1/(2\sqrt{2})$, whence $\cos(\alpha{-}\pi/6) = \sqrt{7/8}$ and $\tan(\pi/6{-}\alpha) = 1/\sqrt{7}$. Therefore

$$\alpha_{\rm sd} = \frac{\pi}{6} - \arctan\frac{1}{\sqrt{7}}\,, \qquad 7 = \frac{D}{N} = \frac{21}{3}\,. \tag{27}$$

The structural form $\alpha_{\rm sd} = \pi/(2N) - \arctan\sqrt{N/D}$ links the self-dual angle to the same trio $(\pi, N, D{=}21)$ that controls the uniqueness theorem and the CP phase.

5.2 General-N uniqueness

The construction of the preceding sections generalises to N generations with a $\mathrm{U}(N)_F$ family symmetry. Two numbers arise naturally:

- The **algebraic discriminant** $D_{\rm alg}$: the discriminant of the self-dual quadratic $t^2 - (N{+}2)t + 1 = 0$, which is $D_{\rm alg} = (N{+}2)^2 - 4$. This characterises the number field $\mathbb{Q}(\sqrt{D_{\rm alg}})$ in which the self-dual charge ratios live.
- The **dynamical discriminant** $D_{\rm dyn}$: from the family Higgs potential at the enhanced symmetry point $\lambda_0 = \lambda_1 = \lambda_{\rm eff}$, the coupling ratio $\mu_0^2/\mu_a^2 = (N{+}2)/(2N{+}1)$ defines $D_{\rm dyn} = N(2N{+}1)$. This counts the degrees of freedom in the family sector.

For the charge structure and the dynamics to be compatible, these two numbers must coincide: $D_{\rm alg} = D_{\rm dyn}$. This condition reduces to $N^2 - 3N = 0$:

$$\boxed{N = 0 \ \text{ or } \ N = 3\,.} \tag{28}$$

A complementary constraint comes from the sector angle. The requirement $\alpha < \alpha_{\rm crit} = \pi/(4N)$ (necessary for positive fermion masses) restricts the allowed range; for the fitted lepton angle $\alpha_{\rm lep} = 0.22222(1)$ rad, only $N \geq 3$ admits a viable Koide spectrum—a criterion independent of discriminant matching. A third criterion: $N^2 + 4 = N^2 + N + 1$ only at $N = 3$. The left-hand side appears in the self-dual discriminant ($D_{\rm alg} = N(N{+}4)$); the right-hand side is the dimension of the projective plane $\mathbb{P}^2(\mathbb{F}_N)$. The coincidence is equivalent to $D_{\rm alg} = D_{\rm dyn}$. A fourth, M-theoretic criterion (Appendix **??**): $\binom{3N}{N} = (N{+}1) \cdot N(2N{+}1)$ holds uniquely at $N = 3$, giving $84 = 4 \times 21$. The factors are the number of C_3 polarisations in 11d supergravity ($\binom{9}{3} = 84$), the $N_f = N_c{+}1 = 4$ Pati–Salam flavours, and $\dim \mathrm{Sp}(6) = 21$—the charge-lattice discriminant.

N	$D_{\rm alg}$	$D_{\rm dyn}$	dim adj SO($2N$+1)	Match?
1	5	3	3	No
2	12	10	10	No
3	**21**	**21**	**21**	**Yes**
4	32	36	36	No
5	45	55	55	No

Table 1: Discriminant matching uniquely selects $N = 3$, where $D = 21 = \dim \mathrm{adj}\, \mathrm{SO}(7) = \dim \mathrm{Sp}(6)$. The Lie algebras B_N and C_N have the same dimension $N(2N+1)$; the symplectic form $\dim \mathrm{Sp}(2N)$ counts degrees of freedom in the electric–magnetic charge lattice of N families.

6 $N_f = 4$, Pati–Salam Structure, and the Hybrid Scenario

As noted in Section **??**, the $N_f = N_c$ theory has a quantum-modified moduli space—the origin is removed, and the constraint $\det M = \Lambda^6$ is enforced by a Lagrange multiplier. *True* s-confinement [?, ?]—where the confined theory has a smooth moduli space with a SUSY-preserving origin, and the effective superpotential is a polynomial in the composites without any Lagrange multiplier—requires $N_f = N_c + 1$. For SU(3), this gives $N_f = 4$. The s-confining superpotential is uniquely determined by holomorphy, symmetries, and the requirement that it reproduces the correct anomaly matching coefficients (all 11/11 independent 't Hooft conditions).

Global symmetry and gauged subgroup. The UV theory has global symmetry $\mathrm{SU}(4)_L \times \mathrm{SU}(4)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$. The confined spectrum consists of:

- 16 mesons $M^i{}_j = Q^i \tilde{Q}_j$, $i, j = 1, \ldots, 4$, transforming as $(\mathbf{4}, \bar{\mathbf{4}})$;
- 4 baryons $B_i = \epsilon^{abc} Q^j_a Q^k_b Q^l_c \epsilon_{ijkl}$, transforming as $(\bar{\mathbf{4}}, \mathbf{1})$;
- 4 antibaryons $\tilde{B}^j = \epsilon_{abc} \tilde{Q}^a_k \tilde{Q}^b_l \tilde{Q}^c_m \epsilon^{jklm}$, transforming as $(\mathbf{1}, \mathbf{4})$.

The confined superpotential is $W = (M^i{}_j B_i \tilde{B}^j - \det M)/\Lambda^5$, with all 't Hooft anomalies matching between UV and IR (11/11 independent conditions, vs. 7/11 for $N_f = 3$).

The Pati–Salam model [?] is a partial unification in which quarks and leptons are placed in the same multiplet: the gauge group SU(4) treats the three colours and the lepton number as four "colours," so a quark triplet plus a lepton singlet form a $\mathbf{4}$. In the present context, the $\mathrm{SU}(4)_R$ flavour symmetry of the $N_f = 4$ theory plays the role of the Pati–Salam colour group. Gauging $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{B-L} \subset \mathrm{SU}(4)_R$ gives:

$$\bar{\mathbf{4}} \;\to\; \bar{\mathbf{3}}_{-1/3} \,+\, \mathbf{1}_{+1} \;: \quad 3 \text{ quarks} + 1 \text{ lepton}\,. \tag{29}$$

The baryon B_i thus decomposes into three colour-triplet states (identified with right-handed quarks) and one singlet (a right-handed neutrino). The remaining SM gauge factors $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ are embedded in $\mathrm{SU}(4)_L$ via $\mathbf{4}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1} \oplus \mathbf{1}_0$, as in Section **??**.

The four baryoninos ψ_{B_i} are the candidate right-handed neutrinos. Under $\mathrm{SU}(4) \to \mathrm{SU}(3) \times \mathrm{U}(1)$, the $\bar{\mathbf{4}}$ decomposes into three $\nu_{R,i}$ plus one sterile singlet. On the mesonic branch ($\langle B \rangle = \langle \tilde{B} \rangle = 0$), the baryonino mass matrix is $\mathcal{M}_R^{ij} = \partial^2 W / \partial B_i \partial \tilde{B}^j = \tilde{M}^i{}_j / \Lambda^5$, where $\tilde{M}$ is the cofactor matrix of M.[3] For diagonal $\langle M \rangle$, the F-term equations $\prod_{k\neq i} v_k = m_i\,\Lambda^5$ imply

$$\mathcal{M}_R^{ii} = m_i\,, \tag{30}$$

so the baryonino masses equal the mass-deformation parameters. In particular, $M_R(\nu_e) = m_e$ and $M_R(\nu_4) = m_t$. Meanwhile, no tree-level Dirac coupling between mesinos and baryoninos exists on this branch ($\partial^2 W/\partial M\,\partial B \propto \langle \tilde{B} \rangle = 0$), so neutrino masses arise only from radiative corrections, schematically $m_\nu \sim m_e^2/(16\pi^2\,\Lambda) \approx 0.02$ eV—the correct order of magnitude. This provides a structural explanation for the smallness of neutrino masses in the sBootstrap [?] (the "SUSY-bootstrap" framework where hadronic SUSY relates mesons to fermions), without invoking a separate seesaw scale.

A *hybrid scenario* reconciles $N_f = 4$ with the $N = 3$ uniqueness of Section **??**: at high energy, SU(3) SQCD with 4 flavours exhibits Pati–Salam structure. When the heaviest flavour decouples ($m_4 = m_t \gg \Lambda$), holomorphic scale matching gives $\Lambda_3^6 = \Lambda_4^5 \cdot m_t$, and the low-energy theory is effective $N_f = 3$ where the U(3) charge structure applies. The top quark plays the role of the spectator in the turtle/elephant partition of [?]: the one quark that does not participate in the charge binding. With $\Lambda_3 = f_\pi\sqrt{3}/2 \approx 80\,\mathrm{MeV}$ (Section **??**) and $m_t = 173\,\mathrm{GeV}$, scale matching gives $\Lambda_4 \approx 17\,\mathrm{MeV}$. This scale is well below $\Lambda_{\mathrm{QCD}} \approx 220\,\mathrm{MeV}$ and might seem dangerously low. However, the family $\mathrm{SU}(3)_F$ is a *separate* gauge group from QCD: its strong coupling scale governs the composite Higgs/lepton masses, not hadronic binding. The physical role of Λ_4 is to set the normalisation of the $N_f = 4$ confined superpotential relative to the mass deformations. Since $\Lambda_4 \ll m_t$, the $N_f = 4$ theory is deeply in its confined phase when the fourth flavour is integrated out, and the transition to the $N_f = 3$ effective theory is smooth.

PMNS mixing: an open problem. A significant phenomenological limitation of the mesonic-branch construction is the neutrino mixing pattern. On this branch, $\mathcal{M}_R$ is diagonal (**??**) and the Dirac coupling vanishes at tree level, so the PMNS matrix is the identity to all orders in perturbation theory within the superpotential. This is in clear conflict with the observed large lepton mixing angles ($\theta_{12} \approx 34°$, $\theta_{23} \approx 49°$, $\theta_{13} \approx 8.5°$ [?]). Potential sources of large PMNS mixing include: (i) Kähler-potential rotations, which are unconstrained in the strongly-coupled regime and can be $O(1)$; (ii) SUSY-breaking contributions from the baryonic branch, where $\langle B \rangle \neq 0$ generates off-diagonal Dirac couplings; (iii) threshold corrections at the $N_f = 4 \to N_f = 3$ matching scale. We do not claim to explain PMNS mixing in this paper and leave its computation as the most pressing open problem of the framework.

7 The Effective Lagrangian

Below the family confinement scale Λ, the sBootstrap effective theory is specified by a single chiral superfield $M^\alpha{}_i$ (the meson matrix), plus the elementary SM fermions. The UV theory has gauge group $\mathrm{SU}(3)_F \times \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, with preons $Q_a^\alpha \in (\mathbf{3}_F,\, \mathbf{3}_L)$ and $\tilde{Q}_{a,i} \in (\bar{\mathbf{3}}_F,\, \mathbf{1}_{\mathrm{SM}})$. After $\mathrm{SU}(3)_F$ confinement:

[3] The R-charge assignment $R(B) = R(\tilde{B}) = 3/4$, $R(M) = 1/2$ selects $B\,M\,\tilde{B}$ as the marginal coupling. The cofactor form $\tilde{M}/\Lambda^5$ arises after canonically normalising the composite baryonic operators via their Kähler metric, $K_B \sim |B|^2/\Lambda^4$.

Superpotential.

$$W = \frac{\det M - B\tilde{B}}{\Lambda^3} + m_i\, M^i{}_i + \frac{y_\ell}{\Lambda}\, L_i\, e^c_j\, M^{1,2}{}_k\, \epsilon^{ijk} + \frac{y_u}{\Lambda^3}\, q_L\, u^c\, \mathrm{adj}(M)^{1,2} + \frac{y_d}{\Lambda}\, q_L\, d^c\, M^{1,2}\,, \quad (31)$$

where $M^{1,2}$ denotes the doublet ($Y{=}{-}1/2$) components and $\mathrm{adj}(M)^{1,2}$ the doublet ($Y{=}{+}1/2$) components of the adjugate. The lepton Yukawa is automatic: L_i is the mesino $\psi_{M^{1,2}{}_i}$ and e^c_j is $\psi_{M^3{}_j}$, so the y_ℓ term is contained in $\det M/\Lambda^3$ as a meson self-coupling. (In the confined Hessian, the doublet–singlet block is antisymmetric in generation indices; the symmetric SUSY-breaking contributions that complete the physical mass matrix are discussed below.) Quark Yukawas arise from partial compositeness mixing.

Field content.

Field	$\mathrm{SU}(3)_c$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$	Origin
$L_i = \psi_{M^{1,2}{}_i}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	mesino (composite)
$e^c_i = \psi_{M^3{}_i}$	$\mathbf{1}$	$\mathbf{1}$	$+1$	mesino (composite)
$\nu_R = \psi_B$	$\mathbf{1}$	$\mathbf{1}$	0	baryonino (composite)
$H_d = M^{1,2}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	meson scalar
$H_u = \mathrm{adj}(M)^{1,2}$	$\mathbf{1}$	$\mathbf{2}$	$+1/2$	adjugate (function of M)
q_L, u^c, d^c	$\mathbf{3}$, $\bar{\mathbf{3}}$, $\bar{\mathbf{3}}$	—	—	elementary

Key structural features.

1. *One chiral field*: $\mathrm{adj}(M)$ is a quadratic polynomial in M; the superpotential (**??**) is holomorphic in M alone. No independent H_u superfield is needed.

2. *No free Yukawas*: The mass parameter m_i simultaneously sets the UV mass deformation, the effective Yukawa coupling, and the meson VEV. The Yukawa hierarchy *is* the family charge hierarchy.

3. *Determined* $\tan\beta$: At the superpotential level, the VEV ratio $\langle \mathrm{adj}(M)\rangle/\langle M\rangle = \Lambda^6/\langle M\rangle^2$ is fixed by the quantum constraint. However, $\mathrm{adj}(M)$ is a composite operator, quadratic in M; its physical VEV depends on the Kähler metric $K(M, M^\dagger)$, which is not protected by holomorphy. Whether v_u/v_d remains fully determined at the quantum level requires computing the Kähler potential on the constraint surface—an open problem (cf. Table **??**).

4. *Fermi scale*: The meson condensate $\langle M_\tau\rangle = \Lambda^2 m_\tau/P_\ell^{1/3}$ relates v_{EW} to Λ through the Kähler normalization Z (see §**??**). Matching $\Lambda = f_\pi\sqrt{3}/2 \approx 80\,\mathrm{MeV}$.

8 Numerical Results

8.1 Charged leptons

With $K = 2/3$ exact, the two lighter masses m_e and m_μ fix $k_{\mathrm{lep}} z_0^2 = 313.8\,\mathrm{MeV}$ and $\alpha_{\mathrm{lep}} = 0.22222(1)$ rad; the third mass m_τ is then a *prediction*. The charge ansatz (**??**)–(**??**) gives:

	Predicted	PDG (MeV)	Deviation
m_e	0.511	0.511	input
m_μ	105.66	105.66	input
m_τ	1776.97	1776.86 ± 0.12	$+0.91\sigma$

Table 2: Charged-lepton masses from the U(3) charge ansatz. The two inputs (m_e, m_μ) fix k and α; the predicted m_τ deviates by 0.91σ from PDG.

Observable	Method	Predicted	PDG	Dev.
$m_u + m_d$	zero-charge ($\alpha{=}\pi/4$)	6.7 MeV	6.8 MeV	1.4%
m_c	charge chain	1301 MeV	1270 MeV	2.4%
m_b	inverse sector	4567 MeV	4180 MeV	9.3%
m_t	standard chain	180 GeV	172.8 GeV	3.9%
$\lvert V_{us}\rvert$	$\sqrt{m_d/m_s}$	0.2236	0.2243 ± 0.0008	0.9σ
$\lvert V_{cb}\rvert$	$1/(5R)$	0.0417	0.0422 ± 0.0008	0.6σ
$\lvert V_{ub}\rvert$	$2\sqrt{2}/(7R^3)$	0.00367	0.00368 ± 0.00011	0.1σ
δ_{CP}	$\arctan\sqrt{R}$	$65.4°$	$65.7 \pm 3.4°$	0.1σ
J	from above	$3.04{\times}10^{-5}$	$3.08 \pm 0.15{\times}10^{-5}$	0.3σ
$\sin 2\beta$	from above	0.723	0.699 ± 0.017	1.4σ

Table 3: Quark sector and CKM elements. Inputs: $m_d = 4.67\,\text{MeV}$, $m_s = 93.4\,\text{MeV}$ ($\overline{\text{MS}}$ at 2 GeV). Here $R = (5 + \sqrt{21})/2$. The CKM expressions involving R are *empirical pattern identifications*; derivations from the operator basis remain open (see text). The m_b and m_t entries have elasticities $\varepsilon \approx 6$ and 4 to m_d; at $m_d = 4.60\,\text{MeV}$ they shift to 4180 and 170 GeV respectively.

8.2 Quark sector and CKM elements

Chain structure and SUSY breaking. The quark predictions in Table ?? arise from a chain of overlapping Koide triples [?] that splits into two regimes separated by the SUSY-breaking scale:
Fermi side (above the breaking scale):

$$(t,\, b,\, c): \quad K(m_t,\, m_b,\, m_c) \approx \tfrac{2}{3} \qquad (\text{standard Koide, } \overline{\text{MS}} \text{ masses})\,. \tag{32}$$

Chiral side (below the breaking scale):

$$(c,\, s,\, 0): \quad \frac{m_c}{m_s} = (2{+}\sqrt{3})^2\,; \qquad (s,\, 0,\, m_u{+}m_d): \quad \frac{m_s}{m_u{+}m_d} = (2{+}\sqrt{3})^2\,. \tag{33}$$

The chiral-side links use zero-charge triples ($z_2 = 0$, i.e., $\alpha = \pi/4$), where one Koide charge vanishes and $K = 2/3$ reduces to a mass ratio $(2{+}\sqrt{3})^2 = 13.93$. Crucially, the lightest quark entry uses the *isospin reparametrization* $u \to 0$, $d \to m_u + m_d$: only the total light-quark mass enters, because below Λ_{QCD} the individual m_u and m_d are unobservable—the Gell-Mann–Oakes–Renner relation $m_\pi^2 = B_0(m_u{+}m_d)$ determines only the sum. This reparametrization is natural in the sBootstrap interpretation, where Koide is a scalar (meson) mass formula: $(c, s, 0)$ connects to the zero-charge meson Koide $K(0,\, m_\pi,\, m_D) = 2/3$ (with $\sqrt{m_D/m_\pi} \approx 2{+}\sqrt{3}$ to 2%). The two sides are connected by the tuple (b, c, s) with a *signed* square root: $K(\sqrt{m_b} + \sqrt{m_c} - \sqrt{m_s}) = 0.675$, deviating from 2/3 by 1.2%.

The negative sign on $\sqrt{m_s}$ corresponds to $z_0 + z_s < 0$ in the charge ansatz—the sector angle has crossed the critical value where the lightest charge flips sign. The complete chain is therefore

$$(t, b, c) \;-\; (b, c, s)^{\pm} \;-\; (c, s, 0) \;-\; (s, 0, m_u{+}m_d)\,, \tag{34}$$

with (b, c, s) as the bridge link. The signed tuple hits $K = 2/3$ exactly at $m_s \approx 84\,\text{MeV}$—between the PDG value $m_s(2\,\text{GeV}) = 93.4\,\text{MeV}$ and the higher-scale estimate $m_s(m_b) \approx 81\,\text{MeV}$—suggesting that its natural evaluation scale is $\mu \sim 3$–$5\,\text{GeV}$. At 1.2% the bridge tuple is not compelling on its own, but its structural role as the link between Fermi and chiral regimes is significant.

The Fermi/chiral split coincides with the transition from fermionic to scalar mass formulae: above the breaking scale the SUSY mass relation $m_{\text{fermion}} \approx m_{\text{scalar}}$ makes the chain applicable to quark masses directly; below it, the chain applies to meson masses, with the isospin reparametrization reflecting the fact that chiral symmetry breaking groups u and d into a single Goldstone sector.

IR/UV scale preference. The two Yukawa operators of Section **??**—M (meson VEV, masses $\propto 1/m_{\text{UV}}$) and $\text{adj}(M)$ (adjugate, masses $\propto m_{\text{UV}}$)—make a structural prediction about scale dependence: the W_1 sector (species coupling through M) should be evaluated at *infrared* scales (pole masses or low $\overline{\text{MS}}$ scales), while the W_2 sector (species coupling through $\text{adj}(M)$) should be evaluated at *ultraviolet* scales (high $\overline{\text{MS}}$). Using the AHS running masses [**?**]:

Tuple	Best $K - 2/3$	At M_Z	Preferred scale
$K(e, \mu, \tau)$	0.001% (pole)	0.18%	IR
$K^{-1}(d, s, b)$	0.3% (m_i)	0.11%	UV
$K(t, b, c)$	0.4% (m_i)	8%	IR
$K(c, s, 0)$	0.3% (m_i)	2.6%	IR

Every chain tuple except the inverse down quarks worsens at M_Z, confirming they belong to the W_1 sector ($M \propto 1/m$). The inverse Koide *improves* at M_Z and passes through $2/3$ near $1\,\text{TeV}$ (§**??**), consistent with $\text{adj}(M) \propto m$ evaluated at UV scales. This IR/UV dichotomy is a structural consequence of the two operators, not a tuning of evaluation scales.

The empirical relation $|V_{cb}| = 1/(R^2+1) = (5-\sqrt{21})/10$ is algebraically natural given $R^2 + 1 = 5R$ (a consequence of $R^2 - 5R + 1 = 0$), but a derivation from the operator basis of Section **??** has not been established. Combined with $|V_{us}| = \sqrt{m_d/m_s}$, this gives the Wolfenstein A parameter: $A = |V_{cb}|/|V_{us}|^2 = m_s/(5R\,m_d) = 0.835$, compared to the PDG value $A = 0.836 \pm 0.015$ (0.1%).

The CP phase $\delta_{CP} = \arctan\sqrt{R}$ is equivalent to $\cos^2 2\delta = 3/7$, or $\sin 2\delta = 2/\sqrt{7}$, since $(R{+}1)^2 = 7R$. This is not an independent prediction but a consequence of the $N = 3$ uniqueness: since $(R{-}1)^2 = NR$ and $(R{+}1)^2 = (N{+}4)R$ (from the self-dual quadratic), one has $\cos^2 2\delta = N/(N{+}4)$. This equals $N/(2N{+}1)$ only when $N{+}4 = 2N{+}1$, i.e., $N = 3$—the same condition as $D_{\text{alg}} = D_{\text{dyn}}$. The Jarlskog invariant has the closed form $J \approx 2\sqrt{2}/(5 \times 7^{5/4})\,\sqrt{m_d(m_s{-}m_d)}/m_s \times R^{-15/4} = 3.04 \times 10^{-5}$ (PDG: 3.08×10^{-5}). Figure **??** shows the predicted unitarity triangle, with the apex at $(\bar{\rho}, \bar{\eta}) = (0.164, 0.358)$ and the three angles $\alpha = 92.0°$, $\beta = 22.6°$, $\gamma = 65.4°$, all within 1.5σ of PDG constraints.

figures/ut_triangle.pdf

Figure 1: The CKM unitarity triangle from the framework's four inputs. The shaded bands show PDG constraints on $\sin 2\beta$ and γ. The predicted apex (black dot) lies at the intersection of the R_b and R_t circles.

8.3 Sensitivity and parameter counting

Of the thirteen observables, six are *parameter-free* predictions—independent of the quark mass inputs m_d and m_s: the three charged-lepton masses (determined by k and α), and $|V_{cb}|$, $|V_{ub}|$, δ_{CP} (determined by R). All six agree with PDG central values to better than 1.2%, with $\chi^2/n = 3.5/6 = 0.6$ against experimental uncertainties.

The remaining seven observables (m_b, m_c, m_t, $m_u{+}m_d$, $|V_{us}|$, J, plus the $K^{-1}(d,s,b) = 2/3$ consistency check) depend on the two quark mass inputs. The elasticities $\varepsilon = (\partial \log \text{obs})/(\partial \log \text{input})$ reveal a striking sensitivity structure:

$$\varepsilon(m_b, m_d) = 6.3\,, \qquad \varepsilon(m_t, m_d) \approx 4\,. \tag{35}$$

The inverse Koide formula amplifies the m_d uncertainty by a factor of 6 in m_b and 4 in m_t (Fig. **??**); given the PDG range $m_d = 4.7 \pm 0.4\,\text{MeV}$, the predicted m_b spans $\sim$3000–7000 MeV. Setting $m_d = 4.60\,\text{MeV}$ (within 0.24σ of PDG central) simultaneously gives $m_b = 4180\,\text{MeV}$ and $m_t = 170\,\text{GeV}$ (-1.7%); the quark predictions are consistent with experiment at every m_d in the PDG range.

In contrast, $m_c = m_s\,(2{+}\sqrt{3})^2$ has unit elasticity to m_s and is the most predictive quark relation. At PDG central $m_s = 93.4\,\text{MeV}$, the chain gives $m_c = 1301\,\text{MeV}$ (1.5σ above PDG); at the Rivero value $m_s = 95\,\text{MeV}$, it gives $m_c = 1323\,\text{MeV}$ (2.7σ). This is the largest tension in the framework, and it sharpens considerably once scale dependence is examined carefully.

figures/sensitivity_plot.pdf

Figure 2: Inverse Koide predictions for m_b (top) and the chain prediction for m_t (bottom) as functions of the input m_d. The red bands show PDG central values $\pm 1\sigma$; the grey band is the PDG range for m_d. The red dot marks $m_d = 4.60\,\text{MeV}$, where m_b and m_t simultaneously match PDG.

Holomorphic masses versus physical masses. The $\overline{\text{MS}}$ anomalous dimension γ_m is flavour-independent at all known loop orders (verified through five loops [**?**]), so the ratio $m_c(\mu)/m_s(\mu)$ *does not run* in QCD. The most precise lattice determination gives $m_c/m_s = 11.783 \pm 0.025$ [**?**], which is 18% below $(2+\sqrt{3})^2 = 13.93$. The frequently quoted "agreement" at the level of 13.6 arises only from comparing m_s at $2\,\text{GeV}$ with m_c at m_c—an inconsistent mixed-scale comparison that accidentally compensates for the discrepancy.

This tension has a natural interpretation in the present framework. The mass deformations m_i entering the superpotential $W_{\text{tree}} = \text{Tr}(m\,M)$ are *holomorphic parameters*: they are protected by the non-renormalisation theorem and do not run at any scale [**?**, **?**]. Physical (pole or $\overline{\text{MS}}$) masses, by contrast, are related to the holomorphic ones through the Kähler metric:

$$m_i^{\text{phys}} = \frac{m_i^{\text{hol}}}{\sqrt{Z_i}}\,, \tag{36}$$

where Z_i is the wavefunction normalisation of the i-th composite. In a strongly coupled s-confining theory, the Z_i are generation-dependent—different mass deformations break $\text{SU}(3)_F$ differently—and non-perturbatively unknown.

If the chain relation $(2+\sqrt{3})^2$ is a statement about holomorphic mass ratios, then the 18% discrepancy with the physical m_c/m_s requires $Z_c/Z_s \approx 1.37$: a 37% difference in wavefunction normalisation between charm and strange composites. For strongly coupled scalars this is not unreasonable, and it connects the chain tension directly to the central open problem of computing the Kähler metric (Section **??**).

By contrast, the *lepton* Koide relation compares pole masses of a single generation-universal composite type (mesinos), where gauge corrections to Z_i are generation-blind at leading order and the holomorphic prediction transfers to physical masses with only

$\mathcal{O}(\alpha/4\pi)$ corrections—consistent with the observed 0.04% accuracy.

(The 2024 PDG uncertainties on m_b and m_s are roughly 10 times smaller than the 2022 values, making the inverse Koide deviation of 0.3% a 2σ tension at the conventional scales. This tension is relieved at M_Z, where the deviation falls to 0.11%.) Figure **??** summarises the pulls of all CKM predictions and the lepton deviations.

figures/pull_plot.pdf

Figure 3: Left: pulls (prediction $-$ PDG central)/σ_{exp} for the seven CKM observables ($\chi^2/7 = 0.46$). Green bars are parameter-free predictions (from R only); orange bars depend on the quark mass inputs m_d, m_s. Right: the predicted m_τ ($+0.9\sigma$), with m_e and m_μ used as inputs to fix k and α. Total $\chi^2/8 = 0.50$.

8.4 Empirical relations involving f_π

The following numerical relations connecting the pion decay constant $f_\pi = 92.07\,\text{MeV}$ to the charge framework are *empirical observations* without derivations from first principles. They are included because their precision (especially eq. **??**) suggests an underlying structure, but they should not be read as predictions of the framework:

$$m_\pi^2 - m_\mu^2 \approx f_\pi^2 \qquad (1.9\%)\,, \tag{37}$$

$$\frac{kz_0^2}{f_\pi} = 3 + \frac{1}{\sqrt{6}} \qquad (0.013\%)\,, \tag{38}$$

$$\arctan\left(\frac{f_\pi}{m_\mu}\right) = \frac{R\pi}{21} \qquad (0.0012\%)\,. \tag{39}$$

Relation (??) is consistent with a soft SUSY splitting between scalar and fermionic components of a chiral multiplet [?, ?], with breaking scale f_π. Relation (??) is the most precise numerical result found in the framework (12 ppm), but no derivation from first principles exists; the deepest reformulation reduces it to an identity over the cyclotomic field $\mathbb{Q}(\zeta_{21})$.

If $kz_0^2/f_\pi = 3 + 1/\sqrt{6}$ is exact and the family confinement scale is $\Lambda = f_\pi\sqrt{3}/2$ (Section ??), then the meson condensate becomes a fourth f_π relation:

$$\langle M_\tau\rangle = \frac{3}{4}\, f_\pi^2\, \frac{m_\tau}{(m_e m_\mu m_\tau)^{1/3}}\,. \tag{40}$$

Numerically, $\langle M_\tau\rangle \approx 2.47 \times 10^5\,\mathrm{MeV}^2$. The physical electroweak VEV $v_{\mathrm{EW}} = Z^{1/2}\langle M_\tau\rangle$ depends on the Kähler normalization Z (with $[Z] = \mathrm{MeV}^{-2}$ for a dimension-2 condensate); see eq. (??). Numerically, $Z = \Lambda^2/P_\ell^{2/3} \approx 3.03$, suggesting the natural scaling $Z = N_c$ of the Kähler metric in the large-N_c limit. If $Z = 3$ exactly, the confinement scale and the pion decay constant are both determined by the lepton mass product:

$$\Lambda = \sqrt{N_c}\,(m_e m_\mu m_\tau)^{1/3}, \qquad f_\pi = 2\,(m_e m_\mu m_\tau)^{1/3}\,. \tag{41}$$

The latter holds to 0.56%; the former reproduces $\Lambda = f_\pi\sqrt{3}/2$ to the same accuracy.

Remark. The product $P_\ell = m_e m_\mu m_\tau$ has a Seiberg-dual interpretation. From the Yukawa coupling $W \supset y_1\, M\,\bar{\ell}\,\ell/\Lambda$ and the meson VEV $\langle M_i\rangle = \Lambda^2 P^{1/3}/m_i$, the physical lepton mass is $m_{\ell,i} = y_1 \Lambda P^{1/3}/m_i$. Taking the product over three generations: $\prod m_{\ell,i} = y_1^3\Lambda^3$, so $(m_e m_\mu m_\tau)^{1/3} = y_1\Lambda$. This is a direct consequence of $\det M = \Lambda^6$ (the quantum constraint) and requires no charge ansatz or angle. Eq. (??) then reads $f_\pi = 2\,y_1\,\Lambda$; combined with $\Lambda = f_\pi\sqrt{3}/2$, it fixes the Yukawa coupling: $y_1 = 1/\sqrt{3} = 1/\sqrt{N_c}$. The relation remains empirical—the identification of the lepton tuple is not derived—but its structural origin in the quantum constraint is worth noting.

The lepton mass hierarchy $m_\tau/P_\ell^{1/3} \approx 39$ is a geometric consequence of the family charge angle α_{lep}.

8.5 Lepton–quark angle shift

The lepton sector angle $\alpha_{\mathrm{lep}} = 0.22222(1)$ rad $= 12.73°$ and the inverse down-quark sector angle $\alpha_{\mathrm{inv}} = 10.59°$ (from $K^{-1}(d,s,b) = 2/3$ at the physical masses) differ by $\Delta\alpha = \alpha_{\mathrm{lep}} - \alpha_{\mathrm{inv}} = 2.14°$. Empirically,

$$\tan(\Delta\alpha) \;=\; \frac{\pi}{84} \;=\; \frac{\pi}{4\,D} \tag{42}$$

to 1.6%, where $D = 21$ is the discriminant. If exact, this formula determines the quark mass ratio $m_d/m_s = 0.0492$ (PDG: 0.050 ± 0.003), reducing the parameter count from four to three: only (k, α, m_s) are needed. No derivation of (??) is known.

8.6 Charge-plane geometry

The Koide parametrisation $\sqrt{m_i} = \sqrt{k}\,(z_0 + z_i)$ with $\sum z_i = 0$ places the charges z_i in a two-dimensional plane, whose orientation is specified by the sector angle α. Two Koide triples with angles α_1, α_2 have charge vectors with inner product $\cos(\alpha_1 - \alpha_2)$; they are orthogonal when $|\alpha_1 - \alpha_2| = \pi/2$.

The charge-plane angles (mod 120°) of the SM Koide triples cluster into three groups:

Triple	α mod 120°	Cluster
$K(e, \mu, \tau)$	107.3°	
$K^{-1}(d, s, b)$	109.3°	lepton/inv-down
$K(0, \mu, \tau)$	106.5°	
$K_{\pm}(c, s, b)$	82.7°	charm–strange–bottom
$K(0, c, b)$	86.5°	
$K(0, s, c)$	15.2°	chiral / inv-chain
$K_{\pm}^{-1}(s, c, t)$	16.2°	

Two observations:

1. $K(0, s, c) \approx K^{-1}(s, c, t)$ to $\sim 1°$: the chiral tuple $(0, m_s, m_c)$ has nearly the same charge-plane orientation as the inverse chain $(1/m_s, 1/m_c, 1/m_t)$. The hierarchy $1/m_t \ll 1/m_c \ll 1/m_s$ makes the top entry effectively zero, mirroring the $(0, s, c)$ parametrisation. Similarly, $K(0, \mu, \tau) \approx K(e, \mu, \tau)$ to $\sim 1°$: the electron is so light that it acts as approximate zero in angle space.

2. **Clustering**: the three groups—lepton/inverse-down (107–109°), signed charm–strange–bottom (83–87°), and chiral/inverse-chain (15–16°)—are separated by $\sim 25°$, $\sim 65°$, and $\sim 30°$ around the 120° circle. The grouping reflects overlapping masses between adjacent chain links (Section **??**).

In the preon language $\sqrt{m_i} = \sqrt{k}\,(z_0 + z_i)$, a vanishing mass $m_i = 0$ means $z_i = -z_0$: one preon exactly cancels the other. The equivalence $K(0, s, c) \approx K^{-1}(s, c, t)$ then says that the top quark is so heavy that $1/m_t \approx 0$, i.e. the inverse-top entry has $z \approx -z_0$—the same preon cancellation as the literal zero. Similarly $K(0, \mu, \tau) \approx K(e, \mu, \tau)$ because the electron is already so light that $z_e \approx -z_0$. Different sectors thus correspond to different orientations of the family charges z_i within a common rank-2 lattice.

8.7 Scale dependence

Using the running Yukawa couplings of Antusch, Hinze, and Saad [**?**] (SM, $\overline{\mathrm{MS}}$ scheme, 2024 PDG inputs), the Koide ratio for the charged leptons is $K_{\mathrm{lep}} = 0.6678$ at all scales from M_Z to 10^7 GeV (0.17% from 2/3). The larger deviation compared to the pole-mass value (0.001%) reflects QED running corrections $\delta m \propto \alpha\, \log(\mu^2/m_{\mathrm{pole}}^2)$, which affect each lepton differently. The approximate scale invariance of K under RG flow supports Sumino's observation [**?**] that gauge interactions approximately preserve the Koide relation.

For the inverse Koide ratio of the down quarks, the running reveals an interesting structure: $K(1/m_d,\, 1/m_s,\, 1/m_b)$ passes through 2/3 at $\mu \approx 1$ TeV, with the central value $K - 2/3 = +2 \times 10^{-6}$ at 10^3 GeV. The deviation grows to +0.1% at M_Z and −0.2% at 10^7 GeV. Although the current experimental uncertainties on m_d and m_s imply a $\sim 0.4\%$

band on K, the trend is robust: the inverse Koide for down quarks is best satisfied near the TeV scale, potentially connected to the adjoint mass μ of the $\mathcal{N} = 2$ parent theory (§**??**). Figure **??** shows both Koide ratios as a function of scale, using the AHS running Yukawa couplings in the SM $\overline{\text{MS}}$ scheme.

`figures/running_koide.pdf`

Figure 4: Koide ratios vs. renormalization scale in the SM. Blue: $K(e, \mu, \tau)$; red: $K^{-1}(d, s, b)$; dashed: $K = 2/3$. The shaded band shows the uncertainty from m_d. Data from Antusch, Hinze, and Saad [**?**].

9 Discussion

What is derived. The charge ratio $K = 2/3$ follows from U(3) normalization. The standard–inverse duality follows from the F-term structure of the s-confined theory: the adjugate $\mathrm{adj}(M) \propto m$ and the meson $M \propto 1/m$ are the two natural Yukawa operators at the confinement scale, leading to standard and inverse Koide for the respective SM species. The duality is identified with the $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$ exchange of SU(5) and the Galois conjugation of $\mathbb{Q}(\sqrt{21})$. Three generations is uniquely selected by discriminant matching. These are exact results contingent on the charge ansatz (**??**) and the family symmetry structure.

The $\mathcal{N} = 2$ origin and the Seiberg–Witten framework. The $N_f = 3$ $\mathcal{N} = 1$ theory used in this paper is not an arbitrary starting point: it descends naturally from a *more* constrained parent—$\mathcal{N} = 2$ supersymmetric QCD. The extra supersymmetry ($\mathcal{N} = 2$ vs.

$\mathcal{N} = 1$) doubles the supercharges and provides even more powerful non-renormalization theorems.

An $\mathcal{N} = 2$ gauge theory contains, in addition to the gauge multiplet (gluon + gluino), a chiral "adjoint" multiplet Φ in the adjoint representation of the gauge group. This field parameterises the *Coulomb branch*—a moduli space where the gauge group is broken to its Cartan subalgebra by $\langle\Phi\rangle$. The celebrated Seiberg–Witten solution [**?**, **?**] determined the exact low-energy effective action on this Coulomb branch, including the Kähler potential and the gauge couplings, by identifying the solution with the periods of an auxiliary algebraic curve.

The key physical idea is as follows. In an $\mathcal{N} = 2$ theory, the low-energy gauge coupling $\tau(\langle\Phi\rangle)$ varies over the Coulomb branch. Seiberg and Witten showed that τ is not an arbitrary function: it is the *period ratio* of a one-parameter family of elliptic curves (tori), one for each value of $\langle\Phi\rangle$. This is powerful because the mathematics of elliptic curves is highly constrained—there are only finitely many singular points (where the torus degenerates), and these correspond to physical particles (monopoles and dyons) becoming massless.

At these singular points, the light particles are BPS states—particles whose mass equals their central charge, $M = |Z|$. BPS states are the "atoms" of the $\mathcal{N} = 2$ spectrum: their masses are exactly determined by the charges they carry. The Seiberg–Witten curve encodes all of this information in a single algebraic equation, and the exact prepotential—the generating function for all couplings—is computed from its periods. This was the first complete, non-perturbative solution of an interacting four-dimensional quantum field theory, and it established that strongly-coupled physics in four dimensions can be exactly solvable.

The $\mathcal{N} = 2$ theory is physically distinct from the $\mathcal{N} = 1$ theory we use: it has the extra adjoint field Φ, a Coulomb branch, and no confinement in the usual sense. To arrive at $\mathcal{N} = 1$ SQCD, one gives Φ a mass:

$$W_{\mathcal{N}=2\to 1} = \sqrt{2}\,\tilde{Q}\Phi Q + \mu\,\mathrm{Tr}(\Phi^2)\,. \tag{43}$$

When the adjoint mass μ is much larger than the dynamical scale Λ, Φ is too heavy to be produced at low energies and can be "integrated out"—solved for in terms of the remaining fields and substituted back into the action. (This is the field-theory analogue of eliminating a heavy particle from a circuit: its effects are encoded in effective interactions between the remaining light fields.) Solving the equation of motion gives $\Phi = -Q\tilde{Q}/(\sqrt{2}\,\mu)$, and substituting back produces an effective quartic meson interaction:

$$W_{\mathrm{eff}} \supset -\frac{1}{2\mu}\mathrm{Tr}(M^2)\,, \qquad M = Q\tilde{Q}\,. \tag{44}$$

This $\mathrm{Tr}(M^2)$ term is the *fingerprint of the* $\mathcal{N} = 2$ *parent*: it lifts some of the flat directions of the $\mathcal{N} = 1$ moduli space.

In the s-confined description, the full superpotential becomes

$$W = \frac{\det M - B\tilde{B}}{\Lambda^3} + m_i M^i{}_i - \frac{1}{2\mu}\mathrm{Tr}(M^2)\,, \tag{45}$$

The limit $\mu \to \infty$ recovers the pure $\mathcal{N} = 1$ theory of Section **??**: the $\mathrm{Tr}(M^2)$ term vanishes and the moduli space is the full quantum-modified constraint surface. At finite μ, the $\mathrm{Tr}(M^2)$ term modifies the vacuum and the Koide relation receives corrections.

The modified F-term equation is

$$\frac{\mathrm{adj}(M)_i}{\Lambda^3} + m_i - \frac{M_i}{\mu} = 0\,. \tag{46}$$

This reduces to (??) when $\mu \to \infty$. The adjoint mass μ has a clear physical meaning: it controls how "close" the theory is to the $\mathcal{N} = 2$ parent. When μ is small, the theory remembers the $\mathcal{N} = 2$ structure and the $\mathrm{Tr}(M^2)$ term significantly distorts the Koide vacuum. When μ is large, the adjoint is heavy, decouples, and the pure $\mathcal{N} = 1$ dynamics—with exact Koide—is recovered.

At finite μ, the Koide vacuum ($M_i \propto 1/m_i$) exists only above a critical adjoint mass

$$\mu_{\mathrm{crit}} = \frac{4\,\Lambda^2\,P_\ell^{1/3}}{m_e^2} \approx 4.5\,\mathrm{TeV}\,. \tag{47}$$

Below this scale, the $\mathrm{Tr}(M^2)$ correction overwhelms the quantum constraint for the electron: the electron meson VEV—which must be the *largest* (since m_e is the smallest mass)—can no longer satisfy both the product constraint $M_1M_2M_3 = \Lambda^6$ and the F-term equation (??). The true phase boundary, where the Koide branch ceases to exist as a real solution, lies at $\mu_{\mathrm{bif}} = \mu_{\mathrm{crit}}/2^{1/3} \approx 3.5\,\mathrm{TeV}$: at this point the electron's quadratic discriminant vanishes, and the product constraint forces $\mu_{\mathrm{bif}}^3 = \mu_{\mathrm{crit}}^3/2$. Above μ_{bif}, the Koide ratio converges as $K - 2/3 \approx A/\mu$ where $A/\mu_{\mathrm{crit}} = (z_0^2 - z_e^2)/6$ is a pure family-charge quantity ($A \approx 9.8\,\mathrm{GeV}$); the observed accuracy $K = 2/3$ to 0.001% requires $\mu \gtrsim 37\,\mathrm{TeV}$ ($\mu/\mu_{\mathrm{crit}} \approx 8$). This constrains the $\mathcal{N} = 2$ SUSY-breaking scale in the family sector to be at least $\sim$40 TeV. Independently, the inverse down-quark Koide ratio $K^{-1}(d, s, b)$ crosses 2/3 at $\mu \approx$ 1–1.6 TeV under standard $\overline{\mathrm{MS}}$ running [?], the same order of magnitude as μ_{crit}.

The Fermi scale and Kähler normalization. In supersymmetric theories, the kinetic energy of a chiral superfield Φ is not simply $\partial_\mu\Phi^\dagger\partial^\mu\Phi$ as in a canonical scalar field. Instead, it is determined by the *Kähler potential* $K(\Phi, \Phi^\dagger)$, a real function of the fields: the kinetic term is $K_{i\bar{\jmath}}\,\partial_\mu\phi^i\partial^\mu\bar{\phi}^{\bar{\jmath}}$ where $K_{i\bar{\jmath}} = \partial^2K/\partial\phi^i\partial\bar{\phi}^{\bar{\jmath}}$ is the Kähler *metric*. Unlike the superpotential, the Kähler potential is *not* protected by holomorphy: it receives corrections at all loop orders and from strong-coupling effects. In the strongly-coupled confined phase of SQCD, the Kähler metric of the composite mesons is incalculable by current methods—this is the main source of uncontrolled corrections in the framework.

If the lepton mass is $m_\ell \propto 1/m_{\mathrm{UV}}$ (from the adjugate mechanism), then $\langle M_i\rangle = \Lambda^2 m_{\ell,i}/P_\ell^{1/3}$, where $P_\ell = (m_e m_\mu m_\tau)^{1/3}$. The meson condensate $\langle M_i\rangle$ has mass dimension 2 (since $M^i{}_j = Q^i\tilde{Q}_j$). The canonical Higgs field is $h_i = Z^{1/2}\,M_i$ where $Z = K_{M\bar{M}}$ is the Kähler wave-function renormalization. Thus

$$v_{\mathrm{EW}}^2 = Z\,\langle M_\tau\rangle^2 = Z\,\frac{\Lambda^4\,m_\tau^2}{P_\ell^{2/3}}\,. \tag{48}$$

The three doublet VEVs are hierarchical: $v_i/v_\tau = m_{\ell,i}/m_\tau$, so $v_\tau/\sqrt{\sum v_i^2} = 0.998$—the electroweak VEV is dominated by the heaviest lepton. Matching $v_{\mathrm{EW}} = 246.2\,\mathrm{GeV}$ determines Z:

$$Z^{1/2} = \frac{v_{\mathrm{EW}}\,P_\ell^{1/3}}{\Lambda^2\,m_\tau}\,. \tag{49}$$

If $\Lambda = f_\pi\sqrt{3}/2 \approx 79.7\,\text{MeV}$ (the family confinement scale set by chiral symmetry breaking, with the $\sqrt{3}/2$ characteristic of SU(3)), the canonical Kähler normalization for a dimension-2 condensate, $Z = 1/\Lambda^2$, gives

$$v_{\text{EW}} = \frac{\Lambda\, m_\tau}{P_\ell^{1/3}} = 3.09\,\text{GeV}\,, \tag{50}$$

which is two orders of magnitude below the observed value. The discrepancy requires a large anomalous Kähler correction $Z \gg 1/\Lambda^2$, characteristic of the strongly-coupled confined phase. By naïve dimensional analysis (NDA), the Kähler metric for a composite operator of dimension d scales as $Z \sim (4\pi)^{2(d-1)}/\Lambda^{2(d-1)}$ [?]; for $d = 2$ this gives $Z \sim (4\pi)^2/\Lambda^2$, an enhancement factor of ~ 160 over canonical. An independent estimate comes from QCD, where the pion decay constant $f_\pi \approx 93\,\text{MeV}$ governs the Kähler metric of the chiral condensate: $Z_{\text{QCD}} \sim (4\pi f_\pi)^2/\Lambda^4_{\text{QCD}} \sim 30/\Lambda^2_{\text{QCD}}$. Both estimates are parametrically consistent with the required $Z/Z_{\text{canon}} \approx 6{,}300$, but off by an $O(1)$ coefficient that cannot be fixed without a non-perturbative computation—either from the Seiberg–Witten prepotential on the Coulomb branch (which is known for SU(3) with 3 flavours [?]) or from lattice SQCD. Computing the Kähler potential on the quantum-modified moduli space remains the most important quantitative open problem in the framework.

Electroweak embedding and the CST mechanism. The $\text{SU}(3)_L$ chiral symmetry of the UV theory admits the subgroup $\text{SU}(2)_L \times \text{U}(1)_Y$ with $\mathbf{3}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}$—the branching rule of the 3-3-1 models [?, ?, ?]. A natural question is: where does $\text{SU}(2)_L$ come from? Csaki, Shirman, and Terning [?] provided a striking answer: $\text{SU}(2)_L$ can emerge as the *magnetic dual gauge group* of a confining SUSY theory.

The idea of Seiberg duality is that a strongly-coupled "electric" gauge theory with group $\text{SU}(N_e)$ and N_f flavours has an equivalent weakly-coupled description—the "magnetic dual"—with gauge group $\text{SU}(N_f - N_e)$ and the *same* N_f flavours. The two descriptions are not approximations of each other; they are exact reformulations that become useful in different regimes. (The analogy to electric–magnetic duality in electrodynamics is the origin of the names.) When $N_f - N_e = 2$, the magnetic gauge group is SU(2)—and CST identified this SU(2) with the Standard Model $\text{SU}(2)_L$.

For the $N_f = 4$ Pati–Salam extension (Section **??**), the electric theory is $\text{SU}(2)_e$ with 4 flavours—the self-dual point $N_f = 2N_c$, inside the conformal window. More generally, Komargodski [?] showed that the magnetic gauge group is the hidden local symmetry of the flavour group—the same structure that produces ρ mesons in QCD. In the present context, $\text{SU}(2)_L$ is the hidden local symmetry of $\text{SU}(3)_L$, arising automatically from confinement without being imposed by hand.

The full scale hierarchy is

$$\mu_{\mathcal{N}=2} \gtrsim 37\,\text{TeV} > \Lambda_{\text{CST}} > v_{\text{EW}} = 246\,\text{GeV} > \Lambda \approx 80\,\text{MeV}\,, \tag{51}$$

where Λ_{CST} is the CST duality scale (see Appendix **??** for the full construction). The upper bound on $\mu_{\mathcal{N}=2}$ comes from the Koide accuracy requirement (eq. **??**). In the Hanany–Witten brane construction [?], Seiberg duality is realised as NS5-brane exchange; the scale hierarchy (**??**) corresponds to the D4-brane separations along the interval.

With $\text{SU}(2)_L$ weakly gauged, the 9 mesino Weyl fermions decompose into 3 lepton doublets L_i and 3 charged singlets e^c_i: the complete SM lepton sector. The baryon B becomes a sterile singlet $(1, 1)_0$ (candidate right-handed neutrino).

Partial compositeness and quark masses. Quarks are elementary fields, external to the confining sector. How do they acquire mass? The mechanism is *partial compositeness* [?]: each elementary quark mixes linearly with a composite operator of the same quantum numbers: $\mathcal{L} \supset \lambda_L \, \bar{q}_L \, \mathcal{O}_R + \lambda_R \, \bar{q}_R \, \mathcal{O}_L + \text{h.c.}$ At low energies, the composite operators $\mathcal{O}$ are replaced by their VEVs, and the physical quark mass becomes $m_q \propto \lambda_L \, \lambda_R \, \langle M \rangle / \Lambda$. The crucial feature is that the mixing coefficients λ_i are generation-dependent: they are determined by the anomalous dimensions of the composite operators at the confining fixed point [?, ?]. Different generations mix with different strengths, producing the observed mass hierarchies without fine-tuning the Yukawa couplings.

Composite interpretation and the single-field structure. In the framework of [?], the objects obeying the charge ansatz are scalar composites (sleptons, squarks) built from pairs of SM fermions. Supersymmetry relates scalar and fermion masses within each chiral multiplet, with the splitting set by f_π (??). This is the hadronic supersymmetry of Miyazawa [?]: the SUSY operator maps elementary fermions (leptons) to composite scalars (mesons), not to elementary sfermions as in the MSSM.

Crucially, the effective Lagrangian involves a single meson field M. The "up-type Higgs" $\text{adj}(M)$ is a quadratic polynomial in M, not an independent chiral superfield, so the superpotential $W = y_d \, q_L \, d^c \, M/\Lambda + y_u \, q_L \, u^c \, \text{adj}(M)/\Lambda^3 + m_i M^i{}_i + (\det M - B\tilde{B})/\Lambda^3$ is holomorphic in M alone. The VEV ratio $\langle \text{adj}(M) \rangle / \langle M \rangle = \Lambda^6 / \langle M \rangle^2$ is dynamically determined at the superpotential level, constraining $\tan\beta$. The MSSM μ-problem is absent because there is no separate H_u chiral superfield. However, the physical ratio v_u/v_d receives corrections from the Kähler potential, which can in principle reintroduce an independent parameter. Whether $\tan\beta$ remains fully determined awaits a computation of the Kähler metric.

Radiative stability. The Koide ratio $K = 2/3$ is a tree-level relation at the compositeness scale Λ. Gauge-loop corrections from $\text{SU}(2)_L \times \text{U}(1)_Y$ are generation-independent (all leptons share the same quantum numbers) and therefore multiply all masses by a common wave-function factor: $\delta m_i / m_i = c \, \alpha/(4\pi)$, independent of i. Such a multiplicative rescaling preserves K exactly. QCD corrections to quark masses are likewise generation-independent. The leading generation-dependent corrections are two-loop electroweak effects $\sim \alpha^2 m_i^2 / M_W^2$, which shift K by $\lesssim 10^{-8}$—far below the observed 0.001% accuracy. Running from the pole masses to $\overline{\text{MS}}$ at M_Z increases δK to 0.17% [?], consistent with a tree-level relation receiving small QED logarithms $\propto \alpha \log(m_\tau / m_e)$.

The charge ansatz as a lattice norm. The mass parameters $m_i = k(z_0 + z_i)^2$ are norm-squared on the U(3) charge lattice. In the superpotential, m_i has mass dimension 1; if it encodes scalar mass-squared (as befits the composite scalars of [?]), then $\sqrt{m_i} \propto z_0 + z_i$ is the physical mass and the charge z_i is the constituent mass. The transition from multiplicative Seiberg masses ($m_i m_j / P^{1/3}$) to additive physical masses ($Z_i + Z_j$) is simply the square root.

The sector angle. The self-dual angle (??) is $\alpha_{\text{sd}} \approx 0.162$ rad. The physical lepton angle $\alpha_{\text{lep}} = 0.22222(1)$ rad, fitted from m_τ / m_μ, predicts m_τ to 0.9σ. Its dynamical origin—whether from RG flow away from the self-dual fixed point, from non-perturbative Kähler corrections, or from a UV completion—remains open.

The critical angle and the electron mass. At the critical angle $\alpha_{\mathrm{crit}} = \pi/12$, the lightest Koide mass vanishes: $z_0 + z_1 = 0$ (since $\cos(\pi/12 + 2\pi/3)/\sqrt{3} = -1/\sqrt{6} = -z_0$). Expanding around this zero,

$$z_0 + z_1 \;=\; \frac{\varepsilon}{\sqrt{6}} + \frac{\varepsilon^2}{2\sqrt{6}} + \mathcal{O}(\varepsilon^3)\,, \qquad \varepsilon \equiv \frac{\pi}{12} - \alpha_{\mathrm{lep}} \approx 0.040\,, \tag{52}$$

so the electron mass is controlled by the proximity of α to the critical angle:

$$m_e \;\approx\; k\,\frac{\varepsilon^2}{6}\,. \tag{53}$$

The first-order approximation reproduces m_e/k to 3.8%; including the quadratic term gives 0.05% accuracy. The lepton mass hierarchy $m_e/m_\tau \sim 3 \times 10^{-4}$ is not a fine-tuning problem: it is a geometric consequence of $\alpha_{\mathrm{lep}}/\alpha_{\mathrm{crit}} \approx 0.85$.

This admits a 't Hooft naturalness interpretation: at the boundary $\alpha = \alpha_{\mathrm{crit}}$, the first-generation mass vanishes exactly and the symmetry is enhanced—the off-diagonal mesons ${M^1}_j$ become massless (they are the Goldstone bosons of the enhanced chiral symmetry at $z_0 + z_1 = 0$). In the sBootstrap, the pion *is* the would-be Goldstone of this enhanced symmetry, given its small mass by the small departure $\varepsilon = \pi/12 - \alpha_{\mathrm{lep}} \approx 0.040$ from the boundary. The question "why is the electron light?" is thus replaced by "what sets α_{lep}?"—the proximity to the boundary is the explanation, not the mystery.

Primer: F-terms, D-terms, and SUSY breaking. This subsection reviews the mechanics of SUSY breaking for readers more familiar with the Standard Model than with superfield formalism. The key ideas are simple; the formalism is secondary.

In any $\mathcal{N} = 1$ supersymmetric theory, the energy of a field configuration is controlled by two types of auxiliary fields:

- **F-terms**: one for each chiral (matter) superfield ϕ_i, determined by the superpotential W:

$$F_i = -\frac{\partial W}{\partial \phi_i}\,. \tag{54}$$

  Each F_i is a complex scalar. The superpotential encodes all the Yukawa couplings and mass terms of the theory, so the F-terms know about the interactions.

- **D-terms**: one for each generator T^a of each gauge group, determined by the gauge charges of the matter fields:

$$D^a = -g\sum_i \phi_i^\dagger T^a \phi_i + \xi^a\,, \tag{55}$$

  where ξ^a is a Fayet–Iliopoulos parameter (nonzero only for abelian U(1) factors). Each D^a is a real scalar. The D-terms know about the gauge structure.

The scalar potential—the energy as a function of the scalar field values—is simply the sum of squares:

$$V = \sum_i |F_i|^2 + \frac{1}{2}\sum_a (D^a)^2\,. \tag{56}$$

This is non-negative by construction. Supersymmetry is *unbroken* if and only if $V = 0$ at the minimum, i.e., if all $F_i = 0$ and all $D^a = 0$ simultaneously. If no such point exists—if the F- and D-flatness conditions are incompatible—then SUSY is spontaneously broken, and $V > 0$ at the minimum.

The physical consequences of SUSY breaking are immediate:

1. **Mass splittings**: in an unbroken SUSY theory, every boson has a fermionic partner of equal mass. When SUSY breaks, these degeneracies split. The splitting is of order $\Delta m \sim |F|/M$ or $\Delta m \sim g|D|^{1/2}$, where M is the mediation scale and $|F|$, $|D|$ are the magnitudes of the non-vanishing auxiliary fields.

2. **The goldstino**: by the Goldstone theorem for fermionic symmetries, a massless spin-1/2 particle—the *goldstino*—must appear in the spectrum, the Nambu–Goldstone fermion of broken SUSY [**?**]. Its wave function is

$$\tilde{G} \,\propto\, \sum_i F_i\,\psi_i + \sum_a D^a\,\lambda^a\,, \tag{57}$$

where ψ_i are the matter fermions (partners of ϕ_i) and λ^a are the gauginos (partners of the gauge bosons). The goldstino "points in the direction of SUSY breaking." In supergravity, the goldstino is absorbed ("eaten") by the gravitino, which acquires a mass $m_{3/2} = |F|/(\sqrt{3}\,M_{\mathrm{Pl}})$.

3. **The supertrace**: the tree-level boson–fermion mass splitting satisfies $\mathrm{Tr}(m_B^2 - m_F^2) = 0$ within each gauge representation—the *supertrace rule*. This means SUSY breaking redistributes mass within multiplets but does not create mass out of nothing.

Volkov and Akulov [**?**] showed in 1973—before the development of superfield formalism—that the goldstino can be described by a nonlinear realisation of SUSY, analogous to the nonlinear sigma model for pions. In their construction, the goldstino is the *only* particle below the SUSY-breaking scale; all other superpartners are heavy. This is precisely the limit relevant to the present framework: below the family confinement scale Λ, the SUSY-breaking F-terms are the lepton masses (see below), and the goldstino is a linear combination of the mesino fermions.

F-type vs. D-type breaking in our framework. In the Seiberg confined theory, the scalar potential is computed from W_{eff} of eq. (**??**). Because W_{eff} depends on both the mesons M and the baryons B, $\tilde{B}$, each field contributes an F-term. On the mesonic branch ($B = \tilde{B} = 0$), the baryonic F-terms vanish automatically, and all SUSY breaking comes from the mesonic F-terms $F_M = \partial W/\partial M$—this is *$F$-type breaking*.

The D-terms for $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ play a different role: they enforce *vacuum alignment*. In a multi-Higgs model, the D-terms are $D^a = -g \sum_i v_i^\dagger\, T^a\, v_i$, where v_i are the VEVs of the three Higgs doublets. The D-flatness condition $D^a = 0$ requires all VEVs to point in the same $\mathrm{SU}(2)_L$ direction, preserving $\mathrm{U}(1)_{\mathrm{em}}$. The *magnitude* hierarchy $v_i \propto m_{\ell,i}$ is set by the F-terms (the adjugate mechanism), while the *direction* is aligned by the D-terms.

The ISS mechanism and metastable SUSY breaking. Intriligator, Seiberg, and Shih [**?**] made a remarkable discovery: SQCD with massive flavours in the regime $N_c < N_f < \frac{3}{2}N_c$ has a *metastable* SUSY-breaking vacuum. In the magnetic (dual) description, the F-term equations $F_\phi = -\mu\, q\tilde{q} = 0$ cannot be satisfied because the meson matrix has more rows than columns ($N_f > N_c$); the rank condition forces $F \neq 0$. SUSY is restored only by tunnelling to a distant vacuum, with a lifetime exponentially long compared to the age of the universe.

The ISS mechanism showed that SUSY breaking in gauge theories is generic, not fine-tuned: it requires only massive flavours and the right number of colours. In the present framework, an analogous overconstrained F-term system arises when EWSB is imposed: the 9 complex F-term equations for the 3×3 meson matrix cannot all vanish when the off-diagonal $\mathrm{SU}(2)_L$-breaking VEVs are turned on. SUSY breaks spontaneously, with $|F| \sim m_\tau$.

SUSY breaking from electroweak symmetry breaking. We now apply the F-term formalism of the primer above to the meson sector. The SUSY-preserving vacuum is diagonal: $\langle M^\alpha{}_i \rangle = v_\alpha \, \delta_{\alpha i}$, which does not break $\mathrm{SU}(2)_L$. This vacuum has $F_i = 0$ for all mesons, so $V = 0$ and SUSY is exact—but the electroweak symmetry is unbroken, contradicting experiment.

When $\mathrm{SU}(2)_L$ is broken by giving $\langle M^2{}_i \rangle = v_i$ for all three generations (the Higgs doublet components acquire VEVs), the system of F-term equations becomes overconstrained: 18 real equations for 12 real variables. There is no solution with all $F_i = 0$, so by the criterion of eq. (**??**), SUSY necessarily breaks.

Concretely, in the vacuum where the muon-generation F-term $F_{22} = m_2 - v_2/\mu$ vanishes (fixing the adjoint mass $\mu = \Lambda^2/P_\ell^{1/3} \approx 139\,\mathrm{MeV}$), the remaining F-terms are $F_{2j} = -v_j/\mu = -m_j$ for $j \neq 2$. The SUSY-breaking F-terms *are* the lepton masses:

$$|F_{21}| = m_e \,, \qquad |F_{22}| = 0 \,, \qquad |F_{23}| = m_\tau \,, \tag{58}$$

and the total SUSY-breaking scale is $|F| = \sqrt{\sum |F_{2j}|^2} \approx m_\tau \approx 1.8\,\mathrm{GeV}$.

This is a striking result: SUSY breaking and EWSB are structurally unified. The mass hierarchy $m_\tau/m_e \approx 3500$ is simultaneously the F-term hierarchy. The goldstino (eq. **??**) is a linear combination of the mesino fermions ψ_M, weighted by their F-terms: $\tilde{G} \propto m_e \, \psi_{M^2{}_1} + m_\tau \, \psi_{M^2{}_3}$; it is predominantly the τ-mesino, since $m_\tau/m_e \approx 3500$. The gravitino mass is $m_{3/2} = |F|/(\sqrt{3}\, M_{\mathrm{Pl}}) \approx 4 \times 10^{-16}\,\mathrm{GeV}$—negligibly small.

The three-doublet Higgs sector. The three Higgs doublets $H_i = M^{1,2}{}_i$ acquire hierarchical VEVs $v_i \propto m_{\ell,i}$ from the adjugate mechanism: v_τ dominates $(v_\tau/\sqrt{\sum v_i^2} = 0.998)$, so the SM Higgs is predominantly the τ-generation doublet, with $\sqrt{\sum v_i^2} = 247\,\mathrm{GeV}$. The SUSY-preserving mesino mass matrix (the Hessian $\partial^2 W/\partial M^2$) has an antisymmetric doublet–singlet block $\propto \epsilon_{ijk} \langle M^2{}_k \rangle$, which alone would give one massless lepton. The physical lepton spectrum requires SUSY-breaking contributions proportional to the mass deformation m_i; in the broken phase the lepton masses are set by $m_i = k(z_0 + z_i)^2$ with $K = 2/3$, while the antisymmetric Hessian provides a sub-leading mixing.

The forward path: we like our $K = 2/3$ derived, not assumed. Given the quadratic U(3) charge ansatz $m_i = k(z_0 + z_i)^2$ and the canonical U(1) normalization $z_0 = 1/\sqrt{2N}$, the identity $K = 2/3$ follows algebraically for *every* angle α (Section **??**). Throughout this paper, K therefore plays two logically distinct roles. *Role A*: $K(m_i) = 2/3$ is a *theorem* on the charge lattice; it is never used as input for any subsequent step. *Role B*: $K(1/m_i) = 2/3$ is a *constraint* that selects the self-dual angle—equivalently, the requirement that the involution maps the lattice to itself—and determines R. All thirteen predictions listed in Table **??** follow from the charge ansatz, the involution, and the discriminant condition; none of them require K as an assumption. Koide's 1983 observation [**?**] is the *last* thing derived by the theory, not the first thing assumed. This is the structural distinction between numerology and a dynamical framework.

What remains open.

1. The dynamical origin of α_{lep}. All SU(3)-invariant potentials for the traceless diagonal field depend on α only through $\cos 3\alpha$, whose extrema lie at $\alpha = 0$ and $\pi/3$. The one-loop Coleman–Weinberg effective potential introduces $\log M_i^2$ terms but remains $\mathbb{Z}_3$-invariant and monotonic in α at every renormalization scale μ_R. The quantum constraint $\det M = \Lambda^6$ is α-independent, so instanton effects cannot help. Moreover, the $N_f = 4 \to 3$ threshold correction is itself a sum $\sum f(m_a(\alpha))$—still $\mathbb{Z}_3$-invariant—because the N_f=4 light–heavy mode masses depend on products $m_b m_c = P^3/m_a \propto m_a$. Fixing α_{lep} therefore requires non-perturbative Kähler effects, cross-sector constraints, or input from a UV completion.

2. $|V_{ub}| = 2\sqrt{2}/((2N+1)R^3)$ gives 0.003674, matching the *exclusive* determination $|V_{ub}|_{\text{excl}} = 0.003682 \pm 0.000110$ (0.08σ) but deviating from the inclusive value $|V_{ub}|_{\text{incl}} = 0.00425 \pm 0.00010$ (5.8σ). This is a *prediction*, not a problem: the exclusive channel $B \to \pi \ell \nu$ is a pure adj(M)-to-adj(M) meson transition—precisely the operator that generates the CKM element—while the inclusive rate sums over all final states including baryonic channels. If the exclusive/inclusive tension is resolved in favour of the inclusive value, the framework is falsified. The CP phase $\delta = \arctan\sqrt{R}$ is algebraically equivalent to N=3 (Section **??**). Together these give the Jarlskog invariant $J = 3.04 \times 10^{-5}$ (PDG: 3.08×10^{-5}).

3. The third generation: the (η, τ) pair fails the SUSY-splitting test (**??**); an inverted hierarchy ($m_\tau > m_{\text{scalar}}$) places the scalar partner near $\eta(1760)$, with a breaking scale $\sim 244\,\text{MeV} \approx f_D$.

4. The partial compositeness exponent: $\gamma = 0$ for leptons (composite mesinos, standard Koide) and $\gamma = 1$ for down quarks (inverse Koide). The restriction $\gamma \in \{0, 1\}$ is not arbitrary: the superpotential is holomorphic in M and adj(M), so the most general Yukawa operator is $W \supset q\,\bar{q}\,M^a\,\text{adj}(M)^b/\Lambda^n$ with $a, b \in \mathbb{Z}_{\geq 0}$. On the constraint surface $\text{adj}(M) = \Lambda^6 M^{-1}$, the generation-dependent mass scaling is $m_q \propto m_{\text{UV}}^{a-b}$: the exponent $p = a - b$ is automatically an integer. EFT power counting selects the two leading operators: $(a, b) = (1, 0)$ (renormalisable, p=+1) and $(a, b) = (0, 1)$ (p=−1), related by the $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$ exchange under $\text{SU}(5)_{\text{flav}}$. The remaining question is narrower: what determines the *assignment* (why do down quarks couple to M rather than adj(M)?)—this is fixed by hypercharge ($Y(M^{1,2}) = -\frac{1}{2}$, $Y(\text{adj}(M)^{1,2}) = +\frac{1}{2}$).

5. PMNS mixing: the baryonino Majorana matrix $\mathcal{M}_R$ is diagonal on the mesonic branch (**??**), and the mesino–baryonino Dirac coupling vanishes at tree level. The resulting PMNS matrix is the identity—in conflict with the observed large mixing angles. The exact baryonic branch ($B \neq 0$) does not exist for nonzero mass deformations (the adjugate $\text{adj}(M)_{4\times4}$ has rank 1 when $\det M = 0$, incompatible with the F-term equations). Off-diagonal UV masses and one-loop meson corrections give mixing angles $\lesssim 0.1°$. The most promising path is non-diagonal Kähler corrections to the baryonic sector: the composite operator $B = \epsilon\,QQQ$ involves all three flavours symmetrically, so its Kähler metric need not inherit the meson hierarchy. A non-diagonal Z_B rotates the diagonal $\mathcal{M}_R$ into a near-democratic matrix whose eigenvectors can reproduce large PMNS angles. This naturally explains the CKM/PMNS split: mesons (hierarchical VEVs $\to$ small CKM) vs. baryoninos (flavour-symmetric Kähler $\to$ large PMNS).

6. Vacuum alignment: the meson potential involves three Higgs doublets $H_i = M^{1,2}{}_i$. The D-term potential aligns their VEVs in a common $\mathrm{SU}(2)_L$ direction, preserving $\mathrm{U}(1)_{\mathrm{em}}$. The VEV hierarchy $v_i \propto m_{\ell,i}$ is set by the adjugate mechanism, giving $v_\tau/v_{\mathrm{total}} \approx 0.998$: the SM Higgs is predominantly the τ-generation doublet, with $\sqrt{v_e^2 + v_\mu^2 + v_\tau^2} = 247\,\mathrm{GeV}$. The $\mathrm{Tr}(M^2)/(2\mu)$ term from the $\mathcal{N} = 2$ flow shifts all doublet masses uniformly and does not split the spectrum. At tree level the F-term self-quartic $\partial^2 \mathrm{adj}(M)_i/\partial M_i^2$ vanishes, so $m_h \leq m_Z$ as in the MSSM; lifting m_h to 125 GeV requires composite-loop (Coleman–Weinberg) corrections or higher-order operators from the strong sector. As a parametric estimate we use the MSSM one-loop stop formula, which has the same structure as the composite-sector loop (the parametric dependence on M_S and X_t is fixed by quantum numbers and mass splittings, not by whether the states are elementary or composite):

$$\Delta m_h^2 = \frac{3m_t^4}{4\pi^2 v^2}\left[\log\frac{M_S^2}{m_t^2} + \frac{X_t^2}{M_S^2}\left(1 - \frac{X_t^2}{12M_S^2}\right)\right], \tag{59}$$

where M_S is the stop mass and X_t the stop mixing parameter. For small mixing ($X_t \approx 0$), $m_h = 125\,\mathrm{GeV}$ requires $M_S \approx 4.6\,\mathrm{TeV}$. This matches the $\mathcal{N}$=2 deformation scale $\mu_{\mathrm{crit}} = 4\Lambda^2 P_\ell^{1/3}/m_e^2 \approx 4.5\,\mathrm{TeV}$ to 3%—the same parameter that controls the accuracy of the Koide relation through the $\mathcal{N}$=2 $\to$ $\mathcal{N}$=1 flow. If the composite stop mass equals μ_{crit}, the Higgs mass is fully determined by the lepton spectrum and confinement scale, with no additional parameters. At maximal mixing ($X_t = \sqrt{6}\,M_S$), the required $M_S \approx 1\,\mathrm{TeV}$ coincides with the scale where $K^{-1}(d,s,b) = 2/3$ (cf. §??). In fact, the three TeV-range scales form a geometric progression: v_{EW}, $\sqrt{v_{\mathrm{EW}}\,\mu_{\mathrm{crit}}} \approx 1\,\mathrm{TeV}$, $\mu_{\mathrm{crit}} \approx 4.5\,\mathrm{TeV}$, with common ratio $r = 2P_\ell^{1/3}/(m_e\sqrt{m_\tau}) \approx 4.26$. Analytically, $\sqrt{v_{\mathrm{EW}}\,\mu_{\mathrm{crit}}} = 3f_\pi^2\sqrt{m_\tau}/(2m_e) \approx 1.05\,\mathrm{TeV}$—a parameter-free expression.

7. Flavour-changing neutral currents: three Higgs doublets H_i with generation-dependent VEVs generically mediate tree-level FCNC through the exchange of the two heavy neutral scalars [?]. In the present framework the VEV hierarchy $v_e : v_\mu : v_\tau \propto m_e : m_\mu : m_\tau$ provides a natural Cheng–Sher suppression [?]: off-diagonal Yukawa couplings scale as $\sqrt{m_i m_j}/v$, with the FCNC amplitude further suppressed by $1/M_{H^\pm}^2$. For the most stringent constraint (K^0–$\bar{K}^0$ mixing), the tree-level exchange amplitude scales as $\Delta m_K^{\mathrm{tree}} \propto f_K^2\, m_K\,(m_d m_s/v^2)/M_H^2$; requiring $\Delta m_K^{\mathrm{tree}} < \Delta m_K^{\mathrm{exp}}$ gives $M_H \gtrsim O(\text{few hundred GeV})$ with Cheng–Sher couplings. Including QCD corrections and B_K factors strengthens the bound to the TeV scale, consistent with μ_{crit}. A full computation of the scalar mass spectrum (which depends on the SUSY-breaking sector) and the resulting FCNC rates is needed to make this quantitative.

8. The composite/elementary asymmetry: leptons are composites (mesinos) while quarks are elementary. This identification is *forced by colour*: mesons of the colour-singlet family group $\mathrm{SU}(3)_F$ are themselves colour singlets, so their fermionic partners can be leptons but not quarks. It is not an arbitrary choice, but it is also not derived from first principles—the deeper question is why the family group commutes with colour. The Fradkin–Shenker complementarity of Appendix ?? renders the distinction frame-dependent: in the Higgs phase, "dressed quarks" are smoothly connected to mesinos, and the Koide relation (a ratio) is invariant under the change of frame. A colour–family unification ($\mathrm{SU}(3)_F \times \mathrm{SU}(3)_C \subset \mathrm{SU}(9)$, hinted at by the

$E_8 \to \mathrm{SU}(9)$ branching of Appendix ??) could promote quarks to composites as well, but this goes beyond the scope of the present framework.

9. Proton stability: SM baryon number $B_{\rm SM}$ and family baryon number B_F ($\subset \mathrm{U}(3)_F$) are independent symmetries. All family-sector composites (M, B, $\tilde{B}$) carry $B_{\rm SM} = 0$ because the preons Q, $\tilde{Q}$ are colour singlets. Every operator in the effective superpotential conserves $B_{\rm SM}$: the dangerous dimension-6 operator $QQQL/M^2$ cannot be generated because there is no $\mathrm{SU}(3)_F$-invariant coupling of three $B_{\rm SM} = 1/3$ quarks to one $B_{\rm SM} = 0$ composite lepton. The $N_f = 4$ Pati–Salam structure is safe because $\mathrm{SU}(4)_R$ is a *global* flavour symmetry (never gauged): there are no leptoquark gauge bosons. Proton stability is therefore an *exact structural prediction*—it does not require tuning a unification scale. If proton decay is observed, the framework is falsified.

Status of key claims. Table ?? classifies the main results by their logical status. “Proven” means derived algebraically from the stated assumptions. “Conditional” means derived contingent on a specific additional hypothesis. “Empirical” means a numerical relation that holds to stated accuracy but lacks a derivation. “Assumed” means a foundational input.

Claim	Status	Comment
$K = 2/3$ from U(3)	Proven	derived as theorem; never used as input
Seiberg transparency $(W_S)_{kk} = 0$	Proven	det multilinearity
Involution $M_i \leftrightarrow \Lambda^4/M_i$	Proven	moduli space
$\tan\beta$ elimination	Proven*	adj(M) holomorphic
$N = 3$ uniqueness	Conditional	equal quartic couplings
$\delta_{CP} = \arctan\sqrt{R}$	Conditional	equivalent to N=3
$\lvert V_{us}\rvert = \sqrt{m_d/m_s}$	Empirical	Gatto–Sartori–Tonin [?]
$\lvert V_{cb}\rvert = 1/(5R)$	Empirical	no derivation yet
Inverse Koide for down quarks	Empirical	assumes $\gamma_* = 1$
$K^{-1}(d, s, b) = 2/3$ at $\mu \approx 1$ TeV	Empirical	AHS running [?]
$\mathcal{M}_R^{ii} = m_i$ (baryonino)	Conditional‡	N_f=4 F-terms
$\Lambda = f_\pi\sqrt{3}/2$	Empirical	0.2%
$M_S(X_t{=}0) \approx \mu_{\rm crit}$	Empirical	3%, gives m_h=125 GeV
$m_e \approx k\varepsilon^2/6$, $\varepsilon{=}\pi/12{-}\alpha_{\rm lep}$	Exact	Taylor expansion of (??), 3.8% (0.05% at $O(\varepsilon^2)$)
$K(b, c, s)^{\pm} = 2/3$ (bridge tuple)	Tentative	1.2% at PDG, exact at $m_s \approx 84$ MeV
$\gamma \in \{0, 1\}$ (holomorphic quantisation)	Proven	W polynomial in M, adj(M)
Proton stability	Proven	$B_{\rm SM}$ exact; no dim-6 $QQQL$
Nelson–Barr strong CP	Conditional	involution = CP
$m_i = k(z_0{+}z_i)^2$	Assumed	charge ansatz

Table 4: Logical status of the main results. *At the superpotential level; the Kähler potential can in principle reintroduce an independent ratio v_u/v_d. ‡Requires canonical Kähler normalization of composite baryonic operators.

10 Conclusion

The flavour sector of the Standard Model can be organised by two structural ingredients: the moduli space of SU(3) SQCD, whose quantum constraint provides an involution connecting the charged-lepton and down-quark sectors; and the canonical normalization of a U(3) family symmetry, which fixes $z_0 = 1/\sqrt{6}$ and guarantees $K = 2/3$. The discriminant $D = 21$ appears simultaneously in the self-dual charge spectrum and the family Higgs potential, uniquely at $N = 3$ generations. Extension to $N_f = 4$ recovers the Pati–Salam decomposition from the baryon sector, with the top quark as the decoupling fourth flavour.

The effective Lagrangian involves a single meson field M: both electroweak Higgs doublets ($H_d = M^{1,2}$ and $H_u = \mathrm{adj}(M)^{1,2}$) are holomorphic in M alone, constraining $\tan\beta$ at the superpotential level (subject to Kähler corrections). The mass parameter m_i in the UV superpotential serves simultaneously as mass deformation, Yukawa coupling, and meson-VEV determinant—there are no independent Yukawa parameters. The family confinement scale $\Lambda \approx f_\pi\sqrt{3}/2$ relates the meson condensate to the electroweak VEV through the Kähler normalization (eq. **??**).

From four continuous parameters and the structural assumptions of Table **??**, the framework is consistent with thirteen observables. Six involve only R and are independent of the quark mass inputs: the three lepton masses (from k and α), $|V_{cb}|$, $|V_{ub}|$, and δ_{CP}—all within 1.2% of PDG values. The CKM expressions involving R are currently empirical identifications; derivations from the confined operator basis remain open. The remaining seven observables (m_b, m_c, m_t, $m_u{+}m_d$, $|V_{us}|$, J, $K^{-1}(d,s,b){=}2/3$) depend on two quark mass inputs. The overall CKM fit has $\chi^2/7 = 0.46$ (Fig. **??**), and the lepton Koide ratio agrees to 0.001%. Beyond the four continuous inputs, the framework relies on discrete structural choices: the charge ansatz $m_i = k(z_0 + z_i)^2$ (the foundational postulate), the partial compositeness exponents $\gamma_\ell = 0$ and $\gamma_d = 1$, and the Kähler normalization Z required to match v_{EW}. The paper's contribution is showing that these assumptions, combined with exact Seiberg dynamics, produce a large number of correlated observables from a small set of inputs. In particular, the Koide relation $K = 2/3$ is a *consequence* of U(3) normalisation, not an input: the derivation proceeds from the charge ansatz to the predictions without ever assuming it.

Three structural features merit emphasis. First, the $N_f = 4$ baryon sector provides three right-handed neutrinos via the Pati–Salam decomposition $\bar{4} \to \bar{3} \oplus 1$, and the tree-level decoupling $\partial^2 W/\partial M\,\partial B = 0$ naturally suppresses neutrino masses. The PMNS mixing pattern remains unexplained (Section **??**). Second, the EWSB vacuum necessarily breaks SUSY (the F-term system is overconstrained), with the SUSY-breaking F-terms equal to the lepton masses: $|F_{2j}| = m_j$. SUSY breaking and the lepton mass hierarchy are structurally unified. Combined with $\Lambda \approx f_\pi$, this means the SUSY breaking in the family sector occurs at the chiral scale—not at the TeV scale assumed in the MSSM. Third, the quark chain splits into a "Fermi side" (t, b, c)—connected through the signed bridge tuple (b, c, s)—to a "chiral side" $(c, s, 0)$, $(s, 0, m_u{+}m_d)$ where zero-charge triples connect to meson physics through the isospin reparametrization $u \to 0$, $d \to m_u{+}m_d$. This split reflects the transition from fermionic to scalar degrees of freedom across the SUSY-breaking scale, and is confirmed by the IR/UV scale preferences of the respective tuples (Section **??**).

Fourth, three TeV-range scales emerge from the framework with no TeV-scale input: the $\mathcal{N} = 2$ adjoint mass $\mu_{\mathrm{crit}} \approx 4.5\,\mathrm{TeV}$, the composite stop mass $M_S(X_t{=}0) \approx 4.6\,\mathrm{TeV}$ required for $m_h = 125\,\mathrm{GeV}$ (§**??**), and the inverse Koide crossing scale $\mu_K \approx 1\,\mathrm{TeV}$. These three scales are related by the geometric progression v_{EW}, $\sqrt{v\,\mu_{\mathrm{crit}}} \approx 1\,\mathrm{TeV}$, μ_{crit}, with common ratio determined entirely by the lepton mass product. If composite stops exist, this framework predicts them at 1–5 TeV.

The M-theory lift (Appendix ??) provides a geometric interpretation: mesons are M2-branes, baryons are M5-branes, the involution is electromagnetic duality of the C-field, and $K = 2/3$ is singled out by the membrane dimensionality ($p = 2$). The top quark is an asymptotic Kaluza–Klein monopole whose 12 degrees of freedom reside in the SU(9) adjoint **80** rather than the $\Lambda^3(\mathbf{9}) = \mathbf{84}$ that houses the family-structured fermions. Identifying the moduli-space involution with CP yields a Nelson–Barr solution to the strong CP problem, with $\delta_{CP} = \arctan\sqrt{R}$ arising from spontaneous CP violation.

The most pressing open problems are: (i) computing the Kähler metric on the quantum-modified moduli space (needed to fix $v_{\rm EW}$, $\tan\beta$, and the scalar spectrum); (ii) identifying a source of large PMNS mixing; (iii) deriving the CKM relations involving R from the confined operator basis.

Acknowledgments

A Alternative: $\mathrm{SU}(2)_L$ as a magnetic dual gauge group

The embedding $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \subset \mathrm{SU}(3)_L$ used in the main text (Section ??) is the minimal route to the Higgs identification $H_d = M^{1,2}$, $H_u = \mathrm{adj}(M)^{1,2}$. An alternative, due to Csaki, Shirman, and Terning [?], derives $\mathrm{SU}(2)_L$ dynamically as the magnetic dual gauge group of a confining theory at a scale $\Lambda_{\rm CST} > v_{\rm EW}$. We summarise the construction here; it is not used elsewhere in the paper.

The mechanism. Consider an electric theory with gauge group $\mathrm{SU}(N_e)$ and N_f flavours. Seiberg duality [?] gives a magnetic description with gauge group $\mathrm{SU}(N_f - N_e)$. Setting $N_f - N_e = 2$, the magnetic gauge group is SU(2), identified with $\mathrm{SU}(2)_L$. The magnetic description contains:

1. Magnetic quarks q_i, $\tilde{q}^j$ ($i = 1, \ldots, N_f$), transforming as doublets under $\mathrm{SU}(2)_{\rm mag}$. These are the preons that subsequently confine under $\mathrm{SU}(3)_F$.
2. The electric meson $\mathcal{M}^i{}_j = Q^{i,{\rm el}}\tilde{Q}^{\rm el}_j/\Lambda_{\rm CST}$, a gauge singlet in the magnetic description. A component of $\mathcal{M}$ serves as the Higgs.
3. The magnetic superpotential $W_{\rm mag} = q\,\mathcal{M}\,\tilde{q}/\mu$, which generates Yukawa couplings between the magnetic quarks and the Higgs.

Scale hierarchy. The construction requires three scales:

$$\Lambda_{\rm CST} \; > \; v_{\rm EW} \; > \; \Lambda_F \, . \tag{60}$$

At energies above $\Lambda_{\rm CST}$, the electric theory confines and is replaced by its magnetic dual. Between $v_{\rm EW}$ and $\Lambda_{\rm CST}$, the magnetic quarks are $\mathrm{SU}(2)_L$ doublets, and $\mathrm{SU}(3)_F$ sees $2N_f = 6$ effective flavours—inside the conformal window ($\frac{3}{2}N_c < 6 < 3N_c$), so $\mathrm{SU}(3)_F$ does not confine. Below $v_{\rm EW}$, the Higgs VEV gives the doublet partners masses $\sim y\,v_{\rm EW}$; they decouple, leaving $N_f = 3$ effective flavours and $\mathrm{SU}(3)_F$ confines at $\Lambda_F \approx f_\pi\sqrt{3}/2$.

Anomaly cancellation. The 't Hooft anomaly matching conditions for $\mathrm{SU}(2)_L$ (treated as a global symmetry of $\mathrm{SU}(3)_F$) are automatically satisfied between the electric quarks and the confined composites, as guaranteed by Seiberg duality. When $\mathrm{SU}(2)_L$ is promoted to a gauge symmetry, anomaly cancellation between quarks and leptons is inherited—the Standard Model quark–lepton anomaly cancellation follows from the duality rather than being assumed.

Connection to the Pati–Salam extension. For $N_f = 4$ (Section **??**), the electric theory is $\mathrm{SU}(N_e) = \mathrm{SU}(2)$ with 4 flavours. Since $N_f = 2N_c$, this is a self-dual point: the electric and magnetic theories have the same gauge group. The duality exchanges strong and weak coupling while preserving the gauge group, giving a non-trivial self-consistency check.

Comparison with the main text. The CST construction provides a dynamical origin for $\mathrm{SU}(2)_L$ and explains the Higgs as a composite of the electric theory. It costs at least two additional parameters (Λ_{CST} and the electric gauge coupling) that do not enter any prediction of the framework. The branching-rule embedding of the main text ($\mathbf{3}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}$) achieves the same Higgs identification with zero additional parameters. The two approaches are compatible: the branching rule describes the low-energy quantum numbers, while the CST mechanism (if realised) would explain why those quantum numbers exist. The choice between them does not affect any numerical result in the paper.

B Higgs–confinement complementarity

If the mesino (the fermionic component of the meson superfield M) *is* a Standard Model lepton, a conceptual question arises: the composites are built from preons Q, $\tilde{Q}$, but those preons carry the same quantum numbers as the composites themselves. How does one write a consistent action when the constituents and the bound states are the same particles? This appendix shows that the question is resolved by *Higgs–confinement complementarity* [**?**, **?**]: the two descriptions—elementary preons with a VEV (Higgs phase) and confined composites (confinement phase)—are smoothly connected, with no phase boundary separating them.

The Fradkin–Shenker theorem. For a gauge theory with matter in the fundamental representation, Fradkin and Shenker [**?**] proved that the Higgs and confinement regimes are analytically connected: no local order parameter distinguishes them, and physical observables vary smoothly between the two limits. Osterwalder and Seiler [**?**] rigorised this for lattice gauge theories. The theorem applies whenever the fundamental matter can screen all centre charges. For SU(3) with $N_f = 3$ fundamentals, the centre $\mathbb{Z}_3$ is fully screened, and the theory is s-confining—precisely the regime in which complementarity holds [**?**].

Higgs-phase spectrum. To see the matching explicitly, give the squarks a diagonal VEV, $\langle Q^i_a \rangle = v_i\, \delta^i_a$, $\langle \tilde{Q}^a_j \rangle = \tilde{v}_j\, \delta^a_j$. All 8 generators of SU(3) are broken; the 8 gauge bosons acquire masses $m_A^2 \propto g^2(v_i^2 + v_j^2)$ (for off-diagonal) or linear combinations of v_i^2 (for diagonal), and all 8 are non-zero for generic v_i.

The degree-of-freedom count proceeds as follows. In the UV, the 18 complex scalars in $Q \oplus \tilde{Q}$ and the 8 gauge bosons give 52 real bosonic degrees of freedom; the 18 Weyl fermions in $Q \oplus \tilde{Q}$ plus 8 gauginos give 26 Weyl fermionic degrees. After Higgsing, 8 massive vector multiplets each absorb one chiral multiplet, leaving $18 - 8 = 10$ light chiral multiplets. In the confined description, 9 mesons + 2 baryons − 1 quantum constraint = 10 chiral multiplets: an exact match.

Operator dictionary. The Higgs-phase and confinement-phase variables are related by the gauge-invariant map

$$M^i{}_j = Q^i_a \tilde{Q}^a_j \,, \qquad B = \epsilon_{abc} Q^{a\,1} Q^{b\,2} Q^{c\,3} \,, \qquad \tilde{B} = \epsilon^{abc} \tilde{Q}^1_a \tilde{Q}^2_b \tilde{Q}^3_c \,. \tag{61}$$

In the Higgs phase these are composite operators with expectation values $\langle M^i{}_j \rangle = v_i \tilde{v}_j$, $\langle B \rangle = v_1 v_2 v_3$, $\langle \tilde{B} \rangle = \tilde{v}_1 \tilde{v}_2 \tilde{v}_3$. The quantum constraint $\det M - B\tilde{B} = \Lambda^6$ is automatically satisfied—it is an operator identity deformed by one instanton. In the confinement phase, M, B, $\tilde{B}$ are the elementary fields, and the *same* constraint appears as the equation of motion from the s-confining superpotential (**??**). Complementarity states that these two descriptions are *equivalent parameterisations of the same physics*. The "mesino" ψ_M in the confined description is the *same fermion* as the "dressed quark" in the Higgs phase, $\psi^{i,j}_M \sim \tilde{v}_j \, \psi^i_Q + v_i \, \psi^j_{\tilde{Q}}$, viewed in different variables.

Anomaly matching. The 't Hooft anomaly coefficients must match between the UV (preon) and IR (composite) descriptions. Under the global symmetry $\mathrm{SU}(3)_F \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, the UV fields are $Q \in (\mathbf{3}_F,\, \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1})$ and $\tilde{Q} \in (\bar{\mathbf{3}}_F,\, \mathbf{1}_0)$. The IR composites are 9 mesinos (3 lepton doublets L_i and 3 charged singlets e^c_i) plus 2 baryoninos (sterile, $Y = 0$). All anomaly coefficients involving the electroweak quantum numbers—$\mathrm{SU}(3)^3_F$, $\mathrm{SU}(3)^2_F \times \mathrm{U}(1)_Y$, $\mathrm{SU}(2)^2_L \times \mathrm{U}(1)_Y$, $\mathrm{U}(1)^3_Y$, $\mathrm{U}(1)_Y \times \mathrm{grav}^2$, and the Witten $\mathrm{SU}(2)_L$ anomaly—match exactly (verified numerically in the ancillary script `anomaly_matching_uv_ir.py`). The mixed $\mathrm{SU}(N_f)^2 \times \mathrm{U}(1)_B$ anomalies require separate treatment because the baryons B, $\tilde{B}$ are $\mathrm{SU}(N_f)$ singlets when $N_f = N_c$ (the fully antisymmetric product $\epsilon_{i_1 \ldots i_N} Q^{i_1} \cdots Q^{i_N}$ is a singlet of $\mathrm{SU}(N)$); the matching is restored on the full quantum-modified moduli space [**?**].

Why the Koide structure survives. In the Higgs phase, the superpotential $W = m_i \, Q^i \tilde{Q}_i$ gives VEVs $v_i \propto 1/\sqrt{m_i}$ (from the F-term equations) and dressed-quark masses proportional to the VEVs. Since the Koide parameter K is a *ratio* (equation (**??**)), multiplicative factors from the Kähler metric cancel, and $K(m^{\text{dressed}}) = K(m^{\text{hol}}) = 2/3$. The same $K = 2/3$ holds in the confined description because the adjugate mechanism (Section **??**) preserves the charge structure. Complementarity guarantees the two computations agree: they are the same physical observable computed in two coordinate systems on the moduli space.

Dirac masses from mesons, Majorana masses from baryons. Complementarity also illuminates the two mass mechanisms operating in the framework. The meson bilinear $M^i{}_j = Q^i \tilde{Q}_j$ pairs *different* fields Q and $\tilde{Q}$—this is a **Dirac-type** structure, and produces the charged-lepton masses through the coupling $\mathrm{Tr}(m \cdot M)$. The baryon sector, by contrast, involves $B = \epsilon \, Q^3$ and $\tilde{B} = \epsilon \, \tilde{Q}^3$—trilinears of the *same* field type. The baryonino mass matrix $\partial^2 W/\partial B\,\partial\tilde{B}$ is a **Majorana-type** coupling, and in the $N_f = 4$ extension (Section **??**) produces the right-handed neutrino mass.

On the quantum-modified moduli space, $\det M = \Lambda^6 + B\tilde{B}$: the meson determinant and the baryon bilinear are conjugate variables. The involution $M_i \to \Lambda^4/M_i$ simultaneously exchanges the relative weight of $\det M$ and $B\tilde{B}$ in the constraint—mapping the Dirac face of the theory to its Majorana face. This duality between the two mass mechanisms is a geometric property of the moduli space, and in the M-theory lift (where the entire brane configuration becomes a single M5-brane wrapping a holomorphic curve Σ), both mechanisms are encoded as two regions of the same Σ, connected by the $\mathbb{Z}_2$ symmetry of the curve.

What about quarks? The self-referential structure applies only to the lepton sector. Quarks in this framework are elementary—they are the preons Q and $\tilde{Q}$ themselves, acquiring mass through partial compositeness (mixing with the composite operators M and $\mathrm{adj}(M)$, as described in Section **??**). The preon–composite loop closes only for leptons, and this asymmetry is the structural reason why leptons satisfy standard Koide ($K(m_\ell) = 2/3$, from $\mathrm{adj}(M) \propto m^{\mathrm{UV}}$) while quarks satisfy inverse Koide ($K(1/m_d) = 2/3$, from $M \propto 1/m^{\mathrm{UV}}$): the two sectors couple to different faces of the adjugate duality, determined by their partial-compositeness exponent γ_* (Section **??**).

C M-theory lift: mesons as M2-branes, baryons as M5-branes

The Hanany–Witten (HW) brane construction [?] realises SU(N_c) SQCD in Type IIA string theory using N_c D4-branes suspended between two NS5-branes, with N_f D6-branes providing flavours. When lifted to M-theory, the entire configuration—D4s, NS5s, and their intersections—merges into a *single* M5-brane wrapping a holomorphic curve Σ, and Seiberg duality becomes a change of variables on that curve [?].

What are branes? A p-brane is an extended object with p spatial dimensions (a point particle is a 0-brane, a string is a 1-brane). M-theory, the eleven-dimensional completion of string theory, contains two fundamental branes: the *M2-brane* (a membrane, coupling to the three-form potential C_3) and the *M5-brane* (coupling to the dual six-form $C_6 = {}^*C_3$). When compactified on a circle, M-theory reduces to Type IIA string theory, and M2-branes become fundamental strings while M5-branes become either D4-branes or NS5-branes, depending on orientation.

The curve for SU(3)**,** $N_f = 3$**.** In the $\mathcal{N} = 1$ limit ($\mu \to \infty$), the M5-brane curve for s-confining SU(3) with $N_f = 3$ splits into two genus-zero components [?]:

$$C_L: \quad \tilde{t} = (w - M_1)(w - M_2)(w - M_3), \quad v = 0\,, \tag{62}$$

$$C_R: \quad \tilde{t} = \Lambda^6, \quad w = 0\,, \tag{63}$$

where $v = x^4 + ix^5$ and $w = x^8 + ix^9$ are transverse coordinates, $\tilde{t} = e^{-(x^6 + ix^{10})/R_{11}}$ encodes the M-theory circle, $M_i = \Lambda^2 P^{1/3}/m_i$ are the meson eigenvalues (Section **??**), and Λ is the dynamical scale.

The s-confining nature of the theory is visible geometrically: all cycles have collapsed (genus zero), so there is no Coulomb branch—only the mesonic and baryonic moduli survive.

The quantum constraint from geometry. At $w = 0$, the left component gives $\tilde{t}\big|_{w=0} = (-M_1)(-M_2)(-M_3) = -\det M$, while the right component is $\tilde{t} = \Lambda^6$. The baryon condensate $B\tilde{B}$ measures the *separation* between the two components in the $\tilde{t}$-direction:

$$\det M - B\tilde{B} \;=\; \Lambda^6 \,. \tag{64}$$

The Seiberg quantum constraint (**??**) is not an input but a *consequence* of the geometric smoothness of the M5-brane configuration.

Mesons as M2-branes, baryons as M5-branes. In the HW construction, the composite operators have distinct brane realisations that persist into M-theory:

Field theory	Type IIA	M-theory	Mass type
Meson $M^i{}_j = Q^i\tilde{Q}_j$	open string (D6$_i$–D6$_j$)	**M2-brane**	Dirac
Baryon $B = \epsilon\, Q^3$	wrapped D4 (vertex)	**M5-brane**	Majorana

The meson, as a bilinear of *different* fields Q and $\tilde{Q}$, produces a **Dirac**-type mass. The baryon, as a trilinear of the *same* field Q, produces a **Majorana**-type mass for the baryonino. In M-theory, these two mass mechanisms are the two faces of *electromagnetic duality* of the C-field: the M2-brane couples to C_3, the M5-brane couples to $C_6 = {}^*C_3$ [**?**].

The involution as M2$\leftrightarrow$M5 duality. The $\mathbb{Z}_2$ involution $M_i \to \Lambda^4/M_i$ of Section **??** acts on the curve by exchanging the two components:

$$C_L \;\longleftrightarrow\; C_R \,.$$

This exchanges the meson moduli (positions M_i of M2-brane endpoints) with the baryon condensate (M5-brane wrapping mode). In eleven-dimensional language, it is the exchange of the C_3 and C_6 descriptions of the same C-field configuration—the *electromagnetic duality* of M-theory.

Why the mass is quadratic: the membrane has two dimensions. The quadratic mass form $m_i = k(z_0+z_i)^2$ of Section **??** is usually taken as a convenient parameterisation. In the brane picture, it is a *consequence* of the M2-brane's dimensionality.

With U(3) family symmetry, the three D6-brane positions are parameterised by the U(3) charges:

$$v_i \;=\; c\,(z_0 + z_i)\,, \tag{65}$$

where $\sum z_i = 0$ and $\langle z_i^2\rangle = z_0^2$ (canonical normalisation). In the $\mathcal{N} = 1$ rotated setup [**?**], each D6-brane acquires a displacement in *both* transverse directions v and w, with each displacement proportional to $(z_0 + z_i)$.

An M2-brane stretched from the colour D4-stack to the i-th D6-brane spans two spatial directions—one along v, one along w—so its worldvolume area, and hence its contribution to the four-dimensional mass, is

$$m_i \;\propto\; \mathrm{Area}_i \;=\; |\Delta v_i|\,|\Delta w_i| \;\propto\; (z_0 + z_i)^2\,. \tag{66}$$

This is the quadratic form. Because the square root $\sqrt{m_i} \propto |z_0 + z_i|$ is linear in the charges, the Koide parameter $K = \sum m_i/(\sum \sqrt{m_i})^2 = 2/3$ follows algebraically from $\langle z_i^2 \rangle = z_0^2$ (Section **??**).

The argument singles out the M2-brane. For a general p-brane whose p spatial dimensions are all proportional to $(z_0 + z_i)$, the mass scales as $(z_0 + z_i)^p$:

p	Object	$m_i \propto$	K
1	string	(z_0+z_i)	0.478
2	**M2-brane**	$(z_0+z_i)^2$	**2/3**
3	3-brane	$(z_0+z_i)^3$	0.805
5	M5-brane	$(z_0+z_i)^5$	0.945

Only $p = 2$ yields $K = 2/3$, because only the quadratic form makes $\sqrt{m}$ linear in the charges. The Koide formula is a *signature of the membrane*: a string ($p = 1$) or a five-brane ($p = 5$) in the same U(3) setup would not produce it.

The number 84 and mass protection. In eleven-dimensional supergravity, the massless little group is SO(9). The bosonic degrees of freedom decompose as $128 = 44 + 84$: the graviton (symmetric traceless 2-tensor, 44 components) plus the three-form C_3 (antisymmetric 3-tensor, $\binom{9}{3} = 84$ components). The dual six-form has the same count: $\binom{9}{6} = \binom{9}{3} = 84$. Under the maximal subgroup $\mathrm{SU}(9) \subset E_8$, the adjoint decomposes as $248 = 80 + 84 + \overline{84}$, with the two 84s corresponding to the C_3 and C_6 representations. Under the further branching $\mathrm{SU}(9) \to \mathrm{SU}(3)_F \times \mathrm{SU}(3)_L \times \mathrm{SU}(3)_C$ (trinification), the $84 = \Lambda^3(\mathbf{9})$ contains the component $(\mathbf{3}_F, \overline{\mathbf{3}}_L, \mathbf{1}_C)$—exactly the quantum numbers of the meson matrix $M^i{}_a$—as well as the singlets $\det_F$, $\det_L$, $\det_C$ that include the baryon B. The adjoint 80 contains $(\mathbf{1}, \overline{\mathbf{3}}_L, \mathbf{3}_C) \oplus (\mathbf{1}, \mathbf{3}_L, \overline{\mathbf{3}}_C)$: 12 family-singlet degrees of freedom with both electroweak and colour charges, matching the top quark.

In the Standard Model with right-handed neutrinos, the 96 fermionic degrees of freedom split as $84 + 12$ under two dual mass-protection symmetries [**?**]: one that leaves all states Dirac-massless except the top quark (12 degrees of freedom), and one that leaves all states Majorana-massless except the neutrinos (12 degrees of freedom). The E_8 structure provides a natural home for this split: the family-structured fermions reside in the antisymmetric $84 = \Lambda^3(\mathbf{9})$, the family-singlet top in the self-conjugate adjoint 80. That the protected count is $84 = \binom{9}{3}$ in both cases, matching the C_3/C_6 polarisation count, is consistent with the identification of the Dirac protection with the M2-brane sector and the Majorana protection with the M5-brane sector.

Relation to the angle shift. The appearance of $84 = |\mathrm{disc}(\mathbb{Q}(\sqrt{-21}))|$ in the angle-shift formula $\tan(\Delta\alpha) = \pi/84$ (Section **??**) now acquires a potential geometric interpretation: the discriminant of the companion imaginary quadratic field counts the same C_3 polarisations that govern the mass-protection split. Whether this is a coincidence or reflects a deeper connection between the Koide number field $\mathbb{Q}(\sqrt{21})$ and the M-theory C-field is a question for future work.

Top quark as Kaluza–Klein monopole. In the $N_f = 4$ hybrid scenario (Section **??**), the top quark is the 4th flavour that decouples at $m_4 = m_t$, reducing the UV theory to $N_f = 3$ where Koide applies. In the HW construction, the top corresponds to a D6-brane whose position in the v-plane is $v_4 = m_t$. When $m_t \to \infty$, the D6-brane recedes and the effective theory reduces to $N_f = 3$; the holomorphic scale matching

$$\Lambda_3^6 \;=\; \Lambda_4^5\, m_t \tag{67}$$

encodes the top's effect on the confined theory.

In the M-theory lift, D6-branes become *Kaluza–Klein monopoles*: centres of a multi-centre Taub-NUT geometry [?]. The receding $\mathrm{D6}_4$ moves its Taub-NUT centre to spatial infinity. Locally (at $r \ll m_t$) the geometry is a three-centre Taub-NUT—the $N_f = 3$ background—but the asymptotic NUT charge remains $N_f = 4$. The top mass enters the low-energy theory not as a local mode of the M5-brane curve but as a *boundary condition* on the global Taub-NUT topology:

$$\text{top quark} \;=\; \text{asymptotic KK monopole mode (12 d.o.f.)}\,.$$

The $12 = N_c \times 4$ real degrees of freedom (3 colours $\times$ Dirac spinor) live in the SU(9) adjoint **80**, not in the antisymmetric **84** (Section **??**, "The number 84"), consistent with the $96 = 84 + 12$ mass-protection split. This gives the split a brane-geometric origin: 84 fermions are local M2/M5 curve modes, while 12 are asymptotic KK monopole modes.

Kähler non-computability and the angle modulus. The Kähler potential $K(M, M^\dagger)$ of the confined meson fields determines the physical (canonical) mass matrix and hence the angle α. In the $\mathcal{N} = 2$ parent theory, the Kähler metric can be extracted from M5-brane periods via the de Boer–Hori–Ooguri–Oz (DHOO) formula [?], which reduces to the Seiberg–Witten special geometry. However, in the $\mathcal{N} = 1$ s-confined theory this formula fails for three independent reasons: (i) the genus-zero curve (**??**) has no holomorphic one-forms, so the period-integral method is inapplicable; (ii) the DBI probe computation suffers from string-scale contamination (a factor U^{-1} depending on R_{11} with no field-theory limit) [?]; (iii) the $\mathcal{N} = 1$ Kähler potential is not protected by non-renormalisation theorems and receives corrections at all orders. The non-computability of K is *consistent* with the observation that no scalar potential or instanton effect selects $\alpha_{\mathrm{lep}} = 2/9$ (§**??**, item 1): the angle is a Kähler modulus of the M5-brane embedding, invisible to the holomorphic sector.

Strong CP and the involution as CP. The moduli-space involution $M_i \to \Lambda^4/M_i$ (Section **??**) exchanges $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$ under $\mathrm{SU}(5)_{\mathrm{flav}}$ and acts as the Galois conjugation $\sqrt{21} \to -\sqrt{21}$. Identifying this involution with the discrete CP transformation yields a *Nelson–Barr* structure [?, ?]: the superpotential is manifestly real in the CP-symmetric basis (m_i real by the charge ansatz, Λ^6 real by convention), so $\arg\det m_q = 0$ at tree level. The CKM phase $\delta = \arctan\sqrt{R}$ arises from *spontaneous* CP violation through the relative orientation of the M and $\mathrm{adj}(M)$ sectors. Radiative corrections to $\bar{\theta}$ in Nelson–Barr models are generically two-loop suppressed [?]: $\delta\bar{\theta} \sim (\alpha/4\pi)^2 J \sim 10^{-11}$, where $J \approx 3 \times 10^{-5}$ is the Jarlskog invariant, at or below the experimental bound $|\bar{\theta}| < 10^{-10}$. This conditional solution depends on the UV Lagrangian respecting the CP = involution symmetry, and requires the partial compositeness mixings to be real—which they are if they descend solely from real superpotential couplings.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the author used Claude (Anthropic) in order to perform algebraic verification, numerical cross-checks, literature searches, and drafting assistance. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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