Three generations from SU(3) confinement and U(3) family symmetry

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Abstract

We study the flavour sector of the Standard Model using two ingredients: the moduli space of $\mathcal{N} = 1$ SU(3) SQCD with three flavours, and a U(3) family symmetry with canonical normalization. The Seiberg confined description gives a meson moduli space with a $\mathbb{Z}_2$ involution $M \to \Lambda^4 M^{-1}$ that exchanges the $\mathbf{15}$ and $\overline{\mathbf{15}}$ of $\mathrm{SU}(5)_{\text{flavour}}$. The U(3) structure determines the mass deformations through a charge ansatz $m_i = k(z_0 + z_i)^2$, with $z_0 = 1/\sqrt{6}$ fixed by group theory. The self-dual charge spectrum has algebraic discriminant 21, which equals the dynamical discriminant of the family Higgs potential only for $N = 3$ generations. Extension to $N_f = 4$ true s-confinement recovers the Pati–Salam decomposition from the baryon sector, with the top quark as the decoupling fourth flavour. The effective Lagrangian involves a single meson field: both electroweak Higgs doublets emerge from M and its adjugate $\mathrm{adj}(M)$, which is holomorphic in M alone, constraining the ratio v_u/v_d at the superpotential level (subject to Kähler corrections). Electroweak symmetry breaking necessarily breaks SUSY: the F-term equations are overconstrained, and the SUSY-breaking auxiliary fields *are* the lepton masses, $|F_{2j}| = m_j$. The family confinement scale $\Lambda = f_\pi\sqrt{3}/2$ relates the electroweak VEV to the lepton mass hierarchy and the Kähler normalization of the meson condensate. The framework has four continuous inputs—a scale k, an angle α (both fixed by m_e and m_μ), m_d, and m_s—together with a set of structural assumptions (Table **??**). From these, it is consistent with thirteen observables including the predicted m_τ (0.001% in K), four quark masses (1–4%, with m_b sensitive to the m_d input), the Cabibbo angle (0.01%), $|V_{cb}| = 1/(5R)$ (1%), $|V_{ub}| = 2\sqrt{2}/(7R^3)$ (0.2%), and $\delta_{CP} = \arctan\sqrt{R}$ (0.2σ), where $R = (5 + \sqrt{21})/2$. The inverse Koide ratio for down quarks, $K^{-1}(d, s, b)$, agrees to 0.25% at its conventional evaluation scale ($m_{d,s}$ at 2 GeV, m_b at m_b) and improves to 0.13% at M_Z.

1 Introduction

The Standard Model contains 13 free parameters in the flavour sector—six quark masses, three charged-lepton masses, three CKM angles, and a CP phase—with no known relations among them.

In 1983, Koide [**?**] constructed a preon model aiming to relate the Cabibbo angle to the mass ratios between generations. The model produced, as a byproduct, a mass matrix

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ansatz $m_i \propto (a + b_i)^2$ where a is a universal constant and the b_i are signed quantities summing to zero, with the constraint that the average of the b_i^2 equals a^2. This predicted the tau mass as $m_\tau = 1776.97\,\text{MeV}$—confirmed years later by experiment. The original preon model was subsequently ruled out by the absence of compositeness signals at high-energy colliders, but the mass matrix relation survived as an empirical fact in search of a theoretical home. The formula is a statement about the mass matrix, not about square roots: the b_i can be positive or negative, and it is only their *squares* that must be positive (as befits physical masses).

Seiberg's exact results for $\mathcal{N} = 1$ SQCD [?, ?] provide a controlled setting in which the infrared dynamics of a strongly coupled gauge theory is described by gauge-invariant composites (mesons and baryons) subject to an exact quantum constraint. For SU(3) with $N_f = 3$ flavours, this constraint defines a moduli space with a non-trivial involution. In this work we show that, when the mass deformations of this moduli space are parameterized by the charges of a U(3) family symmetry—where $a = z_0$ is the U(1) normalization and the $b_i = z_i$ are the SU(3) Cartan charges—the Koide relation becomes an algebraic identity:

- The charged-lepton Koide ratio $K \equiv \sum m_i / (\sum \sqrt{m_i})^2 = 2/3$ to 0.001% [?], evaluated at pole masses (here all $z_0 + z_i > 0$, so $\sqrt{m_i} = \sqrt{k}\,(z_0 + z_i)$ with the positive root).
- The down-quark masses satisfy $K(1/m_d, 1/m_s, 1/m_b) = 2/3$ to 0.25% [?], using $\overline{\text{MS}}$ masses at their conventional scales ($m_{d,s}$ at $2\,\text{GeV}$, m_b at m_b).
- The Cabibbo angle obeys $|V_{us}| \approx \sqrt{m_d/m_s}$ [?, ?, ?].

The paper is organized as follows and includes pedagogical explanations of the SUSY techniques for readers more familiar with the Standard Model than with superfield methods. Section **??** establishes the Seiberg confined description: the meson moduli space, its involution, and the off-diagonal meson mass matrix, with introductory material on holomorphy, anomaly matching, and the classification of SQCD theories. Section **??** introduces the U(3) family charges that parameterize the mass deformations. Section **??** shows how the involution exchanges the charged-lepton and down-quark charge assignments, identified with the $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$ exchange of $\text{SU}(5)_{\text{flavour}}$, and explains the adjugate mechanism. Section **??** proves a uniqueness theorem for $N = 3$ generations via discriminant matching. Section **??** extends to $N_f = 4$ true s-confinement and the Pati–Salam structure. Section **??** collects the numerical results. Section **??** discusses open problems and structural implications.

2 SU(3) SQCD: Confinement, Symmetry Breaking, and the Moduli Space

2.1 Why SUSY exact results are trustworthy

Before presenting the technical setup, we briefly explain why supersymmetric gauge theories allow exact non-perturbative statements—a feature absent in ordinary QCD.

In a supersymmetric theory, the superpotential W is a holomorphic function of the chiral superfields and the coupling constants. Holomorphy—the requirement that W depends only on the fields Φ and not on their complex conjugates $\bar{\Phi}$—is extremely constraining.

(For readers unfamiliar with the term: a holomorphic function is a "complex-analytic" function, one that can be expanded in a Taylor series in its complex variables. A function like $|\Phi|^2 = \Phi\bar{\Phi}$ is *not* holomorphic because it depends on $\bar{\Phi}$; a function like $\Phi^3 + m\Phi$ *is* holomorphic.) This constraint forbids most quantum corrections. Seiberg's key insight [**?**] was that holomorphy, combined with symmetries and known limiting behaviour (weak coupling, large masses), often determines W uniquely. The result is an exact effective superpotential valid at *all* coupling strengths, including the strongly-coupled regime where perturbation theory fails.

The second pillar is 't Hooft anomaly matching. In quantum field theory, an *anomaly* arises when a symmetry of the classical theory is violated by quantum effects. The most familiar example is the chiral anomaly in QED, which is responsible for the decay $\pi^0 \to \gamma\gamma$. 't Hooft showed that anomaly coefficients—numbers computed from one-loop triangle diagrams of global symmetry currents—are *exact*: they receive no higher-order corrections and are independent of the energy scale. This means that any proposed low-energy description of a theory must reproduce the same anomaly coefficients as the UV quarks and gluons. This is a powerful consistency check: most candidate effective theories fail it. For SU(N_c) with N_f flavours, these conditions—together with holomorphy—uniquely fix the effective superpotential in terms of composite fields (mesons and baryons), up to an overall normalization set by the dynamical scale Λ.

The physical picture is analogous to QCD: the strong $\mathrm{SU}(3)_F$ interaction confines the "preons" (Q, $\tilde{Q}$) into colour-singlet bound states. In QCD, the quark bilinear $\bar{q}_i q_j$ forms a 3×3 matrix of bound states—the scalar mesons (σ, π, K, η, a_0, ...). In our SUSY theory, the analogous bilinear $Q^i\tilde{Q}_j$ forms a 3×3 matrix of meson *superfields* $M^i{}_j$, each containing a complex scalar and a Weyl fermion (the "mesino"). Similarly, the colour-antisymmetric combination of three quarks (the baryon) has a SUSY counterpart $B = \epsilon_{abc}Q^{a1}Q^{b2}Q^{c3}$.

The crucial difference from QCD is that supersymmetry and holomorphy allow *exact* control of the resulting low-energy theory. In QCD, the chiral Lagrangian for pions is only an approximation—corrections are incalculable at strong coupling. In SUSY QCD, the superpotential of the confined theory is exact to all orders. This is what makes the present framework non-trivially predictive: the relations between meson VEVs and mass deformations (eq. **??** below) are not approximations but exact consequences of holomorphy and symmetry.

Why $N_f = N_c$ is special: the quantum-modified moduli space. Seiberg's analysis [**?**, **?**] classifies SQCD theories by the ratio N_f/N_c:

- $N_f > 3N_c$: the theory is not asymptotically free.
- $\frac{3}{2}N_c < N_f < 3N_c$: the theory flows to an interacting conformal fixed point ("conformal window").
- $N_c + 1 \leq N_f \leq \frac{3}{2}N_c$: the theory confines and can be described by dual variables (Seiberg duality).
- $N_f = N_c + 1$: *s-confinement.* The confined theory has a smooth moduli space with a SUSY-preserving origin. The effective superpotential is a polynomial in the composites.
- $N_f = N_c$ (our case): the confined theory has a *quantum-modified* moduli space. The classical moduli space (where $\det M = 0$ is allowed) is deformed by instantons—non-perturbative tunnelling events between topologically inequivalent gauge-field

configurations—to $\det M - B\tilde{B} = \Lambda^{2N_c}$, removing the origin. This is a genuinely quantum-mechanical effect with no classical analogue.

The $N_f = N_c$ case is the borderline between the s-confining window and the ADS (Affleck–Dine–Seiberg) regime [**?**], where a dynamical superpotential $W \propto 1/\det M$ is generated and no stable vacuum exists. At $N_f = N_c$, the ADS superpotential is replaced by a constraint—a qualitative change that occurs precisely at the threshold where the baryon becomes an independent degree of freedom (for $N_f < N_c$, baryons cannot be formed).

The reader unfamiliar with SUSY gauge theories may consult the reviews [**?**, **?**] for pedagogical treatments; we now summarise the essential results.

2.2 The UV theory and its symmetries

Consider $\mathcal{N} = 1$ SU(3) SQCD with $N_f = 3$ flavours of chiral superfields Q^i, $\tilde{Q}_j$ $(i, j = 1, 2, 3)$ [**?**, **?**]. (A *chiral superfield* is the basic building block of $\mathcal{N} = 1$ SUSY: it packages a complex scalar ϕ, a Weyl fermion ψ, and an auxiliary field F into a single object. The scalar and fermion have the same gauge quantum numbers—they are "superpartners." The field Q^i is the SUSY analogue of a quark: it transforms as a fundamental **3** of the gauge group SU(3), and carries a flavour index $i = 1, 2, 3$. The conjugate field $\tilde{Q}_j$ transforms as $\bar{\mathbf{3}}$.)

The classical global symmetry is

$$G_{\rm cl} = \mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_A \times \mathrm{U}(1)_R\,. \tag{1}$$

The axial $\mathrm{U}(1)_A$ is anomalous: although it is a symmetry of the classical Lagrangian, quantum effects (instantons) break it. This is exactly analogous to the axial anomaly in QCD, which is responsible for the large mass of the η' meson.[1] The surviving non-anomalous symmetry includes the R-symmetry—a continuous symmetry unique to SUSY theories in which different components of a superfield carry different charges. (The R-charge is the SUSY analogue of chirality: the scalar component of a superfield carries R-charge r, while the fermion component carries R-charge $r-1$. The superpotential must have total R-charge 2, which severely constrains its form.) With the standard assignment $R_Q = R_{\tilde{Q}} = 0$ and $R_\Lambda = 2/3$, the R-symmetry constrains the effective superpotential.

2.3 Confined spectrum

The low-energy degrees of freedom are gauge-invariant composites:

- 9 mesons: $M^i{}_j = Q^i \tilde{Q}_j \quad (\mathbf{3}_L \otimes \bar{\mathbf{3}}_R)$,
- a baryon: $B = \epsilon_{abc}\,\epsilon_{ijk}\,Q^{ai}Q^{bj}Q^{ck} \quad (\mathbf{1}_L,\, B = +1)$,
- an antibaryon: $\tilde{B} = \epsilon^{abc}\,\epsilon^{ijk}\,\tilde{Q}_{ai}\tilde{Q}_{bj}\tilde{Q}_{ck} \quad (\mathbf{1}_R,\, B = -1)$.

Each meson superfield $M^i{}_j$ contains a complex scalar and a Weyl fermion (the "mesino") sharing the same quantum numbers. The 9 mesino fermions transform as $(\mathbf{3}, \bar{\mathbf{3}})$ under $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$: 3 diagonal (neutral under $\mathrm{SU}(3)_{\rm diag}$) and 6 off-diagonal (charged). This counting—9 Weyl fermions—matches the minimal SM lepton sector ($L_i + e^c_i$, no ν_R) [**?**], a point we return to in the discussion.

[1] In QCD, the axial $\mathrm{U}(1)_A$ symmetry $q \to e^{i\gamma_5\theta}q$ is broken by the triangle anomaly to $\mathbb{Z}_{2N_f}$. The same mechanism operates here.

These composites satisfy the exact quantum constraint

$$\det M - B\tilde{B} = \Lambda^6\,, \tag{2}$$

where Λ is the dynamical scale. Equation (**??**) is enforced by a Lagrange multiplier superfield X through the effective superpotential

$$W_{\rm dyn} = X\big(\det M - B\tilde{B} - \Lambda^6\big). \tag{3}$$

This is the quantum-modified moduli space of the $N_f = N_c$ theory [?]. (The $N_f = N_c$ theory is *not* s-confining in the strict sense of [?]: the origin is removed and the moduli space is modified, whereas true s-confinement at $N_f = N_c + 1$ has a smooth origin. See Section **??**.)

Physical meaning of the quantum constraint. The constraint $\det M = \Lambda^6$ on the mesonic branch says that the product of the three diagonal meson VEVs is *fixed* by the strong dynamics:

$$\langle M_1\rangle\,\langle M_2\rangle\,\langle M_3\rangle = \Lambda^6\,. \tag{4}$$

This is the SUSY analogue of the QCD chiral condensate $\langle\bar{q}q\rangle \sim \Lambda^3_{\rm QCD}$, but elevated from an approximate order parameter to an exact constraint. While in QCD the value of the chiral condensate is only known numerically (from lattice calculations or sum rules), the SUSY constraint is exact.

The constraint has a dramatic physical consequence: if one meson VEV is made small (by giving the corresponding preon a large mass), the others must grow to compensate. This is the origin of the seesaw-like relation $\langle M_i\rangle \propto 1/m_i$ in eq. (**??**)—the same structure that produces the Koide mass hierarchy from a hierarchy of input charges. Removing the origin ($\det M = 0$ is forbidden) means that the confined theory has no symmetric vacuum: at least one meson must condense, and chiral symmetry is always broken.

2.4 Branches and symmetry breaking

Moduli space. In SUSY theories, families of degenerate vacua parameterised by scalar VEVs form a *moduli space.* For $N_f = N_c$, it is the constraint surface $\det M - B\tilde{B} = \Lambda^6$: a 5-complex-dimensional manifold.

The moduli space has two branches:

- **Mesonic branch** ($B = \tilde{B} = 0$): $\det M = \Lambda^6$, so at least one meson VEV is nonzero. The diagonal vacuum $\langle M\rangle = \mathrm{diag}(\Lambda^2\omega_1,\, \Lambda^2\omega_2,\, \Lambda^2\omega_3)$ with $\omega_1\omega_2\omega_3 = 1$ breaks $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R \to \mathrm{SU}(3)_V$ (or further, depending on the ω_i).

- **Baryonic branch** ($\det M > \Lambda^6$): $B\tilde{B} \neq 0$, breaking $\mathrm{U}(1)_B$. We do not use this branch, but it exists as an integral part of the moduli space.

2.5 F-term equations and the mass-deformed vacuum

We now add explicit masses for the preons. In a SUSY theory, mass terms live in the superpotential: $W_{\rm tree} = m_i {M^i}_i$ (with $m_i > 0$). Physically, giving a preon a mass m_i means giving both the scalar and the fermion inside the superfield Q^i the same mass (SUSY enforces this). The mass deformation explicitly breaks the flavour symmetry: different generations get different masses, just as in the Standard Model.

The total superpotential is

$$W = X\big(\det M - B\tilde{B} - \Lambda^6\big) + m_i M^i{}_i \,. \tag{5}$$

To find the vacuum, we set all F-terms to zero (which minimises the scalar potential; see Appendix **??**). The F-term equations are:

$$F_X = \det M - B\tilde{B} - \Lambda^6 = 0\,, \tag{6}$$

$$F_{M^i{}_j} = X\,(\mathrm{cof}\,M)^j{}_i + m_i\,\delta_i^j = 0\,, \qquad (\mathrm{cof}\,M = \text{cofactor matrix} \equiv \mathrm{adj}(M)^T) \tag{7}$$

$$F_B = -X\,\tilde{B} = 0\,, \tag{8}$$

$$F_{\tilde{B}} = -X\,B = 0\,. \tag{9}$$

From (**??**)–(**??**): either $X = 0$ (which contradicts (**??**) since $m_i \neq 0$) or $B = \tilde{B} = 0$. Taking the latter, we are on the mesonic branch with $\det M = \Lambda^6$.

The off-diagonal components of (**??**) force $\langle M^i{}_j\rangle = 0$ for $i \neq j$ (since the cofactor matrix of a diagonal M is diagonal). The diagonal components give

$$\langle M_i\rangle = \frac{\Lambda^2 P^{1/3}}{m_i}\,, \qquad P = m_1 m_2 m_3\,. \tag{10}$$

This is the central equation of the framework. It says: *heavy preons produce light mesons, and vice versa.* The meson VEV is *inversely proportional* to the preon mass, like a seesaw. This inverse relation is not assumed but derived from the F-term equations and the quantum constraint. It is exact in the SUSY limit and holds non-perturbatively.

Symmetry-breaking pattern. The mass deformation breaks $\mathrm{SU}(3)_L \times \mathrm{SU}(3)_R$ explicitly. In the vacuum (**??**):

- If $m_1 = m_2 = m_3$: $\mathrm{SU}(3)_V$ is preserved.
- If all m_i are distinct: the global symmetry is broken to $\mathrm{U}(1)^2$.
- $\mathrm{U}(1)_B$ is preserved ($B = \tilde{B} = 0$).
- $\mathrm{U}(1)_R$ is broken by $\langle M_i\rangle \neq 0$.

This is a rich scaffold for flavour symmetry breaking, driven entirely by the mass parameters m_i—which in Section **??** we fix using $\mathrm{U}(3)_F$ charges.

2.6 The meson mass matrix

The off-diagonal meson masses are

$$m_{\mathrm{phys}}(M^i{}_j) = \frac{m_i m_j}{P^{1/3}}\,, \tag{11}$$

which are Λ-independent: they depend only on ratios of the input masses. The diagonal meson mass matrix, obtained from second derivatives of W on the constraint surface, is

$$W_S = P^{1/3}\big(\mathrm{diag}(v_i^2) - v \otimes v^T\big)\,, \qquad v_i = \frac{m_i}{P^{1/3}}\,. \tag{12}$$

This matrix has the properties:

1. $\mathrm{Tr}(W_S) = 0$: the eigenvalues sum to zero.

2. $(W_S)_{kk} = 0$ for all k: the diagonal entries vanish (a consequence of $\partial^2 \det M/\partial M_k^2 = 0$, i.e., the multilinearity of the determinant).

3. Rank 2 (one zero eigenvalue from the constraint).

Property (2) is the key structural result: the Seiberg dynamics is *transparent* to the mass parameters. If the input masses m_i satisfy a sum rule, the physical spectrum preserves it at leading order in the off-diagonal perturbation.

Eigenvalue structure. The eigenvalues of W_S at the physical lepton mass scale sum to zero: $\lambda_1+\lambda_2+\lambda_3 = 0$. This immediately rules out identifying the Seiberg eigenvalues directly with lepton masses (which are all positive). Numerically, $\lambda \approx \{+3229, -3229, -0.4\}$ MeV for physical lepton mass inputs. The Seiberg dynamics preserves the charge structure but does not generate the mass spectrum—the mechanism lives in the family sector.

Summary of the confined framework. To summarise the physical picture before proceeding to the charge ansatz: three "preons" (Q_i and $\tilde{Q}_j$) bind under a family $\mathrm{SU}(3)_F$ gauge force to form 9 meson superfields $M^i{}_j$ and a baryon/antibaryon pair. The mesons contain the Higgs doublets and the lepton fields. The vacuum is determined by an exact quantum constraint (the product of meson VEVs is fixed) plus mass deformations m_i that break the flavour symmetry. Heavy preons produce light mesons: $\langle M_i\rangle \propto 1/m_i$. The off-diagonal meson mass matrix is "transparent"—it does not spoil the charge structure. The remaining question is: *what determines the mass deformations m_i?*

3 Family Charges from U(3)

The mass parameters m_i in the tree-level superpotential $W_{\mathrm{tree}} = m_i M^i{}_i$ are free inputs. We now specify them using a $\mathrm{U}(3)_F$ family symmetry.

Let the three generations transform as the fundamental **3** of $\mathrm{U}(3)_F = \mathrm{SU}(3)_F \times \mathrm{U}(1)$, with all generators in canonical normalization $\mathrm{Tr}(T_A T_B) = \frac{1}{2}\delta_{AB}$.

The SU(3) Cartan charges of the fundamental components satisfy

$$\sum_{i=1}^{3} z_i = 0\,, \qquad \sum_{i=1}^{3} z_i^2 = \frac{1}{2}\,, \tag{13}$$

and the U(1) charge is fixed:

$$z_0 = \frac{1}{\sqrt{6}}\,. \tag{14}$$

Physical interpretation of m_i. The parameter m_i is the tree-level superpotential mass deformation for the preon bilinear $Q^i\tilde{Q}_i$: $W_{\mathrm{tree}} = \sum_i m_i\, Q^i\tilde{Q}_i$. After confinement, m_i appears simultaneously as: (i) the mass deformation that lifts the moduli space; (ii) the effective Yukawa coupling that determines fermion hierarchies (Section **??**); and (iii) the determinant factor in the meson VEV via $\langle M_i\rangle = \Lambda^2 P^{1/3}/m_i$. As a superpotential coupling, m_i has mass dimension one and is protected by holomorphy against perturbative corrections in the $\mathcal{N} = 1$ theory.

Charge ansatz. We *postulate* a quadratic relation between the mass deformation and the family quantum numbers:

$$m_i = k\,(z_0 + z_i)^2\,. \tag{15}$$

This is the norm-squared on the U(3) charge lattice—the Killing-form inner product restricted to the Cartan subalgebra. The quadratic form is distinguished on three grounds: (i) it is the unique W-invariant bilinear on the weight lattice (the Weyl group permutes the z_i, leaving $\sum(z_0+z_i)^2$ invariant); (ii) in heterotic string compactifications on a Narain lattice, the BPS mass-squared is $M^2 = |p_L|^2/\alpha'$ [?], where p_L is the left-moving momentum on the charge lattice; a Wilson line provides the universal shift z_0, the z_i are gauge Cartan charges; (iii) in the $\mathcal{N}=2$ attractor mechanism, the BPS mass at the attractor point is $M^2 = |Z|^2$ with Z linear in charges [?]; on the U(3) lattice this gives (**??**). Note that the standard BPS formula $M^2 \propto Q^2$ yields a mass-*squared* quadratic in charges, whereas the ansatz (**??**) has a mass (dimension 1) quadratic in charges. The meson ${M^i}_j = Q^i\tilde{Q}_j$ resolves this: as a quark–antiquark bilinear it has mass dimension 2, so its VEV $\langle M_i\rangle$ plays the role of M^2, not M. Physically, the meson is a flavour "capacitor"—a bound pair of charges Q^i and $\tilde{Q}_i$ separated at the confinement scale Λ_F. The quadratic form has a gauge-theoretic origin: by Gauss's law the chromoelectric field of a static preon with family charge q_i is $|\mathbf{E}_i| \propto q_i$; since the gauge Lagrangian is $\mathcal{L} \supset -(1/4g^2)F_{\mu\nu}^2$, the field energy is $U_i = (2g^2)^{-1}\int d^3x\,\mathbf{E}_i^2 \propto q_i^2\,\Lambda_F/g_F^2$, i.e. quadratic in the charge, just as for a classical capacitor $E = Q^2/(2C)$. The same result follows from the $\mathcal{N}=1$ effective Lagrangian for the residual $\mathrm{U}(1)_F^2$ after confinement. On the mesonic vacuum with $B = \tilde{B} = 0$, the static energy functional $\mathcal{E} = (2g^2)^{-1}(\mathbf{E}^2{+}\mathbf{B}^2) + \frac{g^2}{2}J_F^2 + \cdots$ contains the D-eliminated family current $J_F = \sum_i q_i\,|\phi_i|^2 + \xi$ (where ϕ_i are the scalar meson fields). For composite states whose spatial profile is generation-independent (the confinement scale Λ_F sets the tube width for all i) but whose field strength scales as $\mathbf{E}_i = q_i\,\mathbf{E}_0$, the integrated energy is $\Delta M_i = q_i^2\,\mu$ with $\mu = \int d^3x\big[(2g^2)^{-1}\mathbf{E}_0^2 + (g^2/2)\,J_0^2\big]$. The overall scale $\mu = k$ is non-perturbative and depends on the Kähler metric; the q_i^2 pattern is fixed by gauge theory. In the M-theory lift (Appendix **??**), this bound pair maps to an M2-brane whose BPS mass is $T_2 \times$ Area; the quadratic form would follow if both worldvolume directions scale linearly with $(z_0{+}z_i)$—a geometric restatement of the same field-energy picture.

What is excluded—and why the square is unique. A linear ansatz $m_i = k'(z_0{+}z_i)$ is excluded by positivity (some $m_i < 0$ for $|z_i| > z_0$). A quartic ansatz $m_i = k''(z_0 + z_i)^4$ does not produce $K = 2/3$. More generally, consider the monomial family $m_i = c\,(z_0{+}z_i)^n$. The signed Koide ratio

$$K_n \;=\; \frac{\sum_i (z_0{+}z_i)^n}{\big(\sum_i (z_0{+}z_i)^{n/2}\big)^2} \tag{16}$$

is *direction-independent* in the Cartan subalgebra—i.e. independent of the sector angle α—if and only if $n = 2$. For $n = 2$, the numerator is $S_2 = 3z_0^2 + \sum z_i^2 = 1$ and the denominator is $S_1^2 = (3z_0)^2 = 3/2$, both independent of α; hence $K_2 = 2/3$ identically. For $n \geq 3$, S_n depends on the cubic and higher Casimirs $\sum z_i^p$ $(p \geq 3)$, which vary with direction. The cubic Casimir $d_{abc}\,T^aT^bT^c \propto \mathbf{1}$ for the fundamental, so it shifts z_0 but not the generation-dependent z_i: the leading correction that could spoil $K = 2/3$ is the direction-dependent power sum $S_3 \supset \cos(3\alpha)/(4\sqrt{3})$, suppressed by an additional insertion of the family-breaking background field. Mixed invariants of the form $m_i m_j$ for $i \neq j$ are off-diagonal mass terms forbidden by the diagonal $\mathrm{U}(1)^3$ unbroken in the Cartan subalgebra.

The uniqueness of $n = 2$ has a physical interpretation: the family charge q_i is an *amplitude*; the mass is its *norm-square*. Every second-order mechanism that produces mass from a linear precursor—seesaw mixing $\mu_i \propto q_i$ giving $m_i \propto \mu_i^2/M$, bilinear composites $M^i{}_i = Q^i\tilde{Q}_i$ giving a VEV $\propto q_i^2$, capacitor energy $E = Q^2/(2C)$, or M2-brane area $\propto$ (distance)^2—yields $n = 2$ and therefore direction-independent $K = 2/3$.

Radiative stability. Since m_i is a superpotential coupling, it receives no perturbative corrections from gauge loops (the non-renormalization theorem of [?]). The leading violation arises from $\mathrm{U}(3)_F$-breaking soft terms at the scale m_{soft}; writing $\delta m_i \sim m_{\mathrm{soft}}\, c_i$, the correction to K is $\delta K \sim (m_{\mathrm{soft}}/k)\,(\delta c/c) \sim 10^{-2}$ for $m_{\mathrm{soft}} \sim 1\,\mathrm{TeV}$, $k \sim 80\,\mathrm{GeV}$. Thus the $K = 2/3$ relation is radiatively stable to per-cent accuracy whenever the family symmetry breaking scale exceeds the soft mass scale. We emphasise that the ansatz (**??**) is *uniquely selected*: it is the only monomial mass formula for which $K = 2/3$ is a group-theoretic identity rather than a direction-dependent numerical coincidence. The field-energy argument above provides a gauge-theoretic derivation of the pattern (q_i^2); the overall scale $k = \mu$ remains a non-perturbative quantity tied to the Kähler metric (Section **??**).

It follows from (**??**)–(**??**) that the Koide ratio

$$K \;\equiv\; \frac{\sum m_i}{\left(\sum \sqrt{m_i}\right)^2} = \frac{\sum (z_0{+}z_i)^2}{\left(\sum |z_0{+}z_i|\right)^2} = \frac{2}{3} \tag{17}$$

identically for *any* direction in the Cartan subalgebra. Note that $\sqrt{m_i}$ in the standard formula means $|z_0 + z_i|$—the absolute value of a signed charge. When all charges are positive ($z_0 + z_i > 0$ for each i), the absolute value is redundant. For sectors where one charge crosses zero (as in the (b, c, s) bridge tuple of Section **??**), the sign matters: the Koide ratio is computed with signed charges, $K = \sum(z_0{+}z_i)^2/(\sum(z_0{+}z_i))^2$, not with positive square roots.

The proof uses only $\sum z_i = 0$ (tracelessness), $\sum z_i^2 = 1/2$ (Casimir), and $z_0 = 1/\sqrt{6}$ (canonical U(1) normalization). In $\mathrm{SU}(3) \times \mathrm{U}(1)$, the value of z_0 is free; in U(3), it is fixed by the requirement $\mathrm{Tr}(T_0^2) = 1/2$. String constructions generate $\mathrm{U}(N)$, not $\mathrm{SU}(N)$, from N D-branes, with the U(1) normalization protected by holomorphy [?].

Angle parameterization. The Cartan charges are

$$z_i = \frac{1}{\sqrt{3}}\cos\!\left(\alpha + \tfrac{2\pi(i-1)}{3}\right), \qquad i = 1, 2, 3\,. \tag{18}$$

Each mass sector is parameterized by one angle α and one scale k: two parameters for three masses.

Transparency. Combined with property (2) of W_S (eq. **??**), the ratio $K = 2/3$ is preserved to first order through the Seiberg dynamics. To see this, write the full meson mass matrix as $\mathcal{M} = \mathrm{diag}(m_i) + W_S$, where $m_i = k(z_0{+}z_i)^2$ are the UV masses and $W_S = P^{1/3}(\mathrm{diag}(v_i^2) - v \otimes v^T)$ is the Seiberg perturbation. Since $(W_S)_{kk} = 0$ (diagonal entries vanish by det multilinearity), the diagonal of $\mathcal{M}$ is m_k itself. By standard first-order perturbation theory, the eigenvalues are $\lambda_k = m_k + O(|W_S^{\mathrm{off}}|^2/\Delta m)$, and $K(\lambda_k) = K(m_k){+}O(v^4/m^4) = 2/3{+}O(v^4/m^4)$. Numerically, $|v_{\max}^2/m_{\min}| \sim 10^{-3}$, so the correction is negligible. The confining dynamics neither generates nor destroys the charge structure; it is *transparent* to the Koide relation.[2]

4 The Involution and the $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$ Exchange

The constraint hypersurface $\det M = \Lambda^6$ has a remarkable symmetry: it is invariant under the exchange

$$M_i \;\to\; \frac{\Lambda^4}{M_i}\,, \qquad \text{mapping} \qquad m_i \;\to\; \tilde{m}_i = \frac{P^{2/3}}{m_i}\,. \tag{19}$$

This is a $\mathbb{Z}_2$ *involution*—a map that, applied twice, returns to the starting point: ($M_i \to \Lambda^4/M_i \to M_i$). Physically, it exchanges large and small meson VEVs while preserving the product constraint. Each point on the moduli space has a "mirror" point related to it by this involution; the self-dual locus ($M_i = \Lambda^4/M_i$, i.e., $M_i = \Lambda^2$) is the fixed-point set.

Since K is scale-invariant, $K(\tilde{m}) = K(1/m)$: the involution exchanges the charge assignment of the standard sector (charged leptons, $K(m) = 2/3$) with that of the inverse sector (down quarks, $K(1/m) = 2/3$). This is the key to the lepton–quark connection: the *same* UV charge structure produces both Koide relations, related by the moduli-space involution.

4.1 The adjugate mechanism

The dynamical origin of this exchange is the *adjugate* of the meson matrix—a construction that may be unfamiliar to readers outside linear algebra, but plays a central role in the framework.

For any $n \times n$ matrix A, the adjugate $\mathrm{adj}(A)$ is the transpose of the cofactor matrix. For a 3×3 diagonal matrix $M = \mathrm{diag}(M_1, M_2, M_3)$, it takes the simple form

$$\mathrm{adj}(M)_i = \prod_{j \neq i} M_j\,, \qquad M \cdot \mathrm{adj}(M) = \det(M)\,\mathbf{1}\,. \tag{20}$$

Explicitly: $\mathrm{adj}(M)_1 = M_2M_3$, $\mathrm{adj}(M)_2 = M_1M_3$, $\mathrm{adj}(M)_3 = M_1M_2$. The key point is that $\mathrm{adj}(M)$ is a *polynomial* (quadratic) in the meson entries—it involves only multiplication, no division. This makes it holomorphic: it is a legitimate superpotential coupling. Yet from the identity $M \cdot \mathrm{adj}(M) = \det(M)\,\mathbf{1}$, we see that on the constraint surface $\det M = \Lambda^6$, the adjugate *behaves* like the matrix inverse: $\mathrm{adj}(M) = \Lambda^6\, M^{-1}$.

This is the mathematical magic of the framework: the adjugate provides a holomorphic realisation of the matrix inverse—an operation that is *not* holomorphic in general ($1/M$ involves M in the denominator). The quantum constraint "trades" the non-holomorphic inverse for a holomorphic polynomial.

The F-term equations of the s-confined superpotential $W = (\det M - B\tilde{B})/\Lambda^3 + m_i M^i{}_i$ yield, for diagonal M:

$$\frac{\partial W}{\partial M^i{}_i} = \frac{\mathrm{adj}(M)_i}{\Lambda^3} + m_i = 0 \qquad \Longrightarrow \qquad \mathrm{adj}(M)_i = -\Lambda^3\, m_i\,. \tag{21}$$

[2]This argument assumes the off-diagonal entries of W_S are perturbative compared to the mass splittings. For SUSY-broken meson masses the relevant splittings are the lepton masses, and $W_S^{ij} = -v_iv_j$ with $v_i = m_i/P^{1/3}$; the largest entry is $|W_S^{23}| \approx 3.2\,\mathrm{GeV}$ while $m_\tau - m_\mu \approx 1.7\,\mathrm{GeV}$, so the perturbative expansion is only marginally valid for the (μ, τ) sector. A non-perturbative argument using the exact eigenvalue equation $\det(\mathcal{M} - \lambda) = 0$ confirms the result [**?**].

The B, $\tilde{B}$ F-terms force $B = \tilde{B} = 0$ in the massive vacuum. Thus *the adjugate of the meson matrix is proportional to the UV mass matrix*, while $M_i \propto 1/m_i$. This is exact in the supersymmetric limit.

Standard Model fermion masses arise from effective Yukawa couplings to the confined composites. If the left chiral symmetry is partially gauged as $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \subset \mathrm{SU}(3)_L$, with the branching rule $\mathbf{3}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}$, then the meson doublet $M^{1,2}{}_i$ has the quantum numbers of the down-type Higgs: $(1, 2)_{-1/2}$. The adjugate $\mathrm{adj}(M)$ transforms as $(\bar{\mathbf{3}}_L, \mathbf{3}_R)$; since $\bar{\mathbf{3}}_L \to \mathbf{2}_{+1/2} \oplus \mathbf{1}_{-1}$, its doublet component has $(1, 2)_{+1/2}$—the up-type Higgs. Both electroweak Higgs doublets thus emerge from a single meson matrix M, with $\mathrm{adj}(M)$ a holomorphic polynomial in M (not an independent chiral superfield). At the superpotential level, this constrains $\tan\beta$: $\langle \mathrm{adj}(M)_i \rangle / \langle M_i \rangle = \Lambda^6 / \langle M_i \rangle^2$ is fixed by the quantum constraint.

In the Standard Model, fermion masses arise from Yukawa couplings $y\,\bar{\psi}\,H\,\psi$ between fermions and the Higgs. In the present framework, the Higgs is the meson doublet $M^{1,2}$, and there are two natural holomorphic Yukawa operators at the confinement scale—one using M directly, the other using its adjugate:

$$W_{\text{down}} = y_d\, q_L\, d^c\, \frac{M^{1,2}}{\Lambda} + y_\ell\, L\, e^c\, M^{1,2}\,, \qquad [H_d\text{-type},\ Y{=}-\tfrac{1}{2}]\,, \tag{22}$$

$$W_{\text{up}} = y_u\, q_L\, u^c\, \frac{\mathrm{adj}(M)^{1,2}}{\Lambda^3}\,, \qquad [H_u\text{-type},\ Y{=}+\tfrac{1}{2}]\,. \tag{23}$$

The W_{down} operator gives masses $\propto \langle M_i \rangle \propto 1/m_i$ ("standard" masses, large for light preons); the W_{up} operator gives masses $\propto \langle \mathrm{adj}(M)_i \rangle \propto m_i$ ("inverse" masses, large for heavy preons). This is why the lepton and down-quark sectors obey *different* Koide relations: they couple to different operators of the same meson field. All operators conserve hypercharge. The mass parameter m_i in $W_{\text{tree}} = m_i M^i{}_i$ serves three simultaneous roles: (i) the UV mass deformation of $\mathrm{SU}(3)_F$ SQCD, (ii) the effective Yukawa coupling between preons Q_i and $\tilde{Q}_i$, and (iii) the determinant of the meson VEV via $\langle M_i \rangle = \Lambda^2 P^{1/3} / m_i$. There are no independent Yukawa parameters: the hierarchy of SM masses *is* the hierarchy of family charges z_i.

Species coupling through M acquire physical masses $m_{\text{phys}} \propto M_i \propto 1/m_i$; species coupling through $\mathrm{adj}(M)$ acquire $m_{\text{phys}} \propto \mathrm{adj}(M)_i \propto m_i$.

Charged leptons, being the mesino components of $M^{1,2}{}_i$ itself, have $m_\ell \propto 1/m$ and therefore $K(m_\ell) = K(1/m) = 2/3$ (standard Koide). Down quarks, as elementary fields, couple to M through partial compositeness [?] with generation-dependent mixing $\lambda_i \propto m_i^{\gamma_*}$. If $\gamma_* \geq 1$, the net mass scales as $m_d \propto m_i^{2\gamma_* - 1}$, and the physical Koide parameter depends on the exponent. In particular, $\gamma_* = 1$ gives $m_d \propto m_i$, producing inverse Koide:

$$K(m_\ell) = K(1/m_d) = K(1/m) = \frac{2}{3}\,, \tag{24}$$

unifying the two empirical relations as a single UV condition.

After soft SUSY breaking, the meson VEVs shift from the F-flat locus. The zero-diagonal property $(W_S)_{kk} = 0$ of the off-diagonal meson mass matrix (Section **??**) ensures that the Koide condition is preserved to first order in m_{soft}/Λ: the meson sector is *transparent* to the family-sector structure.

4.2 SU(5) assignment and arithmetic

In the SU(5) flavour decomposition of [?], the meson $M^{ij} \in \mathrm{Sym}^2(5) = \mathbf{15}$ and the adjugate $\mathrm{adj}(M)_{ij} \in \mathrm{Sym}^2(\bar{5}) = \overline{\mathbf{15}}$. The involution (**??**) maps $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$, with the adjoint

24 fixed. Standard and inverse charge assignments are related by charge conjugation on the scalar composites. This generalises the Georgi–Jarlskog [?] mass relations between quarks and leptons from a GUT boundary condition to a moduli-space involution.

The involution has an arithmetic interpretation: $R = (5+\sqrt{21})/2$ is the fundamental unit of the ring of integers $\mathbb{Z}[(1+\sqrt{21})/2]$ of the real quadratic field $\mathbb{Q}(\sqrt{21})$, with $\mathrm{Norm}(R) = R \cdot R' = 1$. The map $R \to 1/R = R' = (5-\sqrt{21})/2$ is the non-trivial element of $\mathrm{Gal}(\mathbb{Q}(\sqrt{21})/\mathbb{Q})$: the moduli-space involution *is* the Galois conjugation $\sqrt{21} \to -\sqrt{21}$.

5 Self-Duality, the Discriminant 21, and Uniqueness of $N = 3$

This section addresses the question: *why three generations?* The Standard Model works for any number of generations (anomaly cancellation requires each generation to be complete, but does not fix their number). We show that the charge structure of the preceding sections uniquely selects $N = 3$.

5.1 The self-dual charge spectrum

A natural question to ask about the involution of Section **??** is: what charge spectrum is *self-dual*—invariant under $m_i \to P^{2/3}/m_i$? On the self-dual locus, the charges form a geometric progression $Z_1 : Z_2 : Z_3 = 1 : R : R^2$—the same structure that arises from periodic solutions of Schwinger–Dyson gap equations, where Blumhofer and Hutter [?] found geometric mass ratios $\sim e^{\pi}$ between generations without free parameters. Imposing both $K(m) = 2/3$ and $K(1/m) = 2/3$ is algebraically equivalent to requiring

$$3(1 + R^2 + R^4) = 2(1 + R + R^2)^2\,, \tag{25}$$

which, using $1+R+R^2 = 6R$ if $R^2 = 5R-1$, reduces to $72R^2 = 72R^2$ identically. The self-dual Koide identity (**??**) is therefore equivalent to the minimal polynomial

$$t^2 - 5t + 1 = 0\,, \qquad R = \frac{5+\sqrt{21}}{2}\,, \qquad \mathrm{disc} = 21\,. \tag{26}$$

The self-dual angle follows from the geometric-progression condition. Writing $Z_i = (1 + \sqrt{2}\,c_i)/\sqrt{6}$ with $c_i = \cos(\alpha + 2\pi(i-1)/3)$, the condition $Z_2^2 = Z_1 Z_3$ expands to

$$3\sqrt{2}\,c_2 + 2c_2^2 - 2\,c_1 c_3 = 0\,. \tag{27}$$

The combination $2c_2^2 - 2c_1c_3$ is evaluated by product-to-sum identities (using $c_1c_3 = [-\frac{1}{2} + \cos(2\alpha+\frac{2\pi}{3})]/2$ and $c_2^2 = [1 + \cos(2\alpha+\frac{2\pi}{3})]/2$): it equals $\frac{3}{2}$ independently of α. Hence (**??**) reduces to $c_2 = -1/(2\sqrt{2})$. Since $c_2 = \cos(\alpha - 2\pi/3) = \sin(\alpha - \pi/6)$, we obtain $\sin(\alpha-\pi/6) = -1/(2\sqrt{2})$, whence $\cos(\alpha-\pi/6) = \sqrt{7/8}$ and $\tan(\pi/6-\alpha) = 1/\sqrt{7}$. Therefore

$$\alpha_{\mathrm{sd}} = \frac{\pi}{6} - \arctan\frac{1}{\sqrt{7}}\,, \qquad 7 = \frac{D}{N} = \frac{21}{3}\,. \tag{28}$$

The structural form $\alpha_{\mathrm{sd}} = \pi/(2N) - \arctan\sqrt{N/D}$ links the self-dual angle to the same trio $(\pi, N, D{=}21)$ that controls the uniqueness theorem and the CP phase.

5.2 General-N uniqueness

The construction of the preceding sections generalises to N generations with a $\mathrm{U}(N)_F$ family symmetry. Two numbers arise naturally:

- The **algebraic discriminant** D_{alg}: the discriminant of the self-dual quadratic $t^2 - (N+2)t + 1 = 0$, which is $D_{\mathrm{alg}} = (N+2)^2 - 4$. This characterises the number field $\mathbb{Q}(\sqrt{D_{\mathrm{alg}}})$ in which the self-dual charge ratios live.
- The **dynamical discriminant** D_{dyn}: from the family Higgs potential at the enhanced symmetry point $\lambda_0 = \lambda_1 = \lambda_{\mathrm{eff}}$, the coupling ratio $\mu_0^2/\mu_a^2 = (N+2)/(2N+1)$ defines $D_{\mathrm{dyn}} = N(2N+1)$. This counts the degrees of freedom in the family sector.

For the charge structure and the dynamics to be compatible, these two numbers must coincide: $D_{\mathrm{alg}} = D_{\mathrm{dyn}}$. This condition reduces to $N^2 - 3N = 0$ (since $(N+2)^2 - 4 = N^2 + 4N = N(2N+1)$ iff $N^2 = 3N$):

$$\boxed{N = 0 \ \text{ or } \ N = 3\,.} \tag{29}$$

N	D_{alg}	D_{dyn}	$\dim \mathrm{adj}\, \mathrm{SO}(2N+1)$	Match?
1	5	3	3	No
2	12	10	10	No
3	**21**	**21**	**21**	**Yes**
4	32	36	36	No
5	45	55	55	No

Table 1: Discriminant matching uniquely selects $N = 3$, where $D = 21 = \dim \mathrm{adj}\, \mathrm{SO}(7) = \dim \mathrm{Sp}(6)$. The Lie algebras B_N and C_N have the same dimension $N(2N+1)$; the symplectic form $\dim \mathrm{Sp}(2N)$ counts degrees of freedom in the electric–magnetic charge lattice of N families.

A complementary constraint comes from the sector angle. The requirement $\alpha < \alpha_{\mathrm{crit}} = \pi/(4N)$ (necessary for positive fermion masses) restricts the allowed range; for the fitted lepton angle $\alpha_{\mathrm{lep}} = 0.22222(1)$ rad, only $N \geq 3$ admits a viable Koide spectrum—a criterion independent of discriminant matching. A third criterion: $N^2 + 4 = N^2 + N + 1$ only at $N = 3$. The left-hand side appears in the self-dual discriminant ($D_{\mathrm{alg}} = N(N+4)$); the right-hand side is the dimension of the projective plane $\mathbb{P}^2(\mathbb{F}_N)$. The coincidence is equivalent to $D_{\mathrm{alg}} = D_{\mathrm{dyn}}$. A fourth, M-theoretic criterion (Appendix **??**): $\binom{3N}{N} = (N+1) \cdot N(2N+1)$ holds uniquely at $N = 3$, giving $84 = 4 \times 21$. The factors are the number of C_3 polarisations in 11d supergravity ($\binom{9}{3} = 84$), the $N_f = N_c{+}1 = 4$ Pati–Salam flavours, and $\dim \mathrm{Sp}(6) = 21$—the charge-lattice discriminant.

6 $N_f = 4$, Pati–Salam Structure, and the Hybrid Scenario

As noted in Section **??**, the $N_f = N_c$ theory has a quantum-modified moduli space—the origin is removed, and the constraint $\det M = \Lambda^6$ is enforced by a Lagrange multiplier. *True* s-confinement [**?**, **?**]—where the confined theory has a smooth moduli space with a

SUSY-preserving origin, and the effective superpotential is a polynomial in the composites without any Lagrange multiplier—requires $N_f = N_c + 1$. For SU(3), this gives $N_f = 4$. The s-confining superpotential is uniquely determined by holomorphy, symmetries, and the requirement that it reproduces the correct anomaly matching coefficients (all 11/11 independent 't Hooft conditions).

Global symmetry and gauged subgroup. The UV theory has global symmetry $\mathrm{SU}(4)_L \times \mathrm{SU}(4)_R \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R$. The confined spectrum consists of:

- 16 mesons $M^i{}_j = Q^i \tilde{Q}_j$, $i, j = 1, \ldots, 4$, transforming as $(\mathbf{4}, \bar{\mathbf{4}})$;
- 4 baryons $B_i = \epsilon^{abc} Q^j_a Q^k_b Q^l_c \epsilon_{ijkl}$, transforming as $(\bar{\mathbf{4}}, \mathbf{1})$;
- 4 antibaryons $\tilde{B}^j = \epsilon_{abc} \tilde{Q}^a_k \tilde{Q}^b_l \tilde{Q}^c_m \epsilon^{jklm}$, transforming as $(\mathbf{1}, \mathbf{4})$.

The confined superpotential is $W = (M^i{}_j B_i \tilde{B}^j - \det M)/\Lambda^5$, with all 't Hooft anomalies matching between UV and IR (11/11 independent conditions, vs. 7/11 for $N_f = 3$).

The Pati–Salam model [**?**] is a partial unification in which quarks and leptons are placed in the same multiplet: the gauge group SU(4) treats the three colours and the lepton number as four "colours," so a quark triplet plus a lepton singlet form a $\mathbf{4}$. In the present context, the $\mathrm{SU}(4)_R$ flavour symmetry of the $N_f = 4$ theory plays the role of the Pati–Salam colour group. Gauging $\mathrm{SU}(3)_c \times \mathrm{U}(1)_{B-L} \subset \mathrm{SU}(4)_R$ gives:

$$\bar{\mathbf{4}} \;\to\; \bar{\mathbf{3}}_{-1/3} \,+\, \mathbf{1}_{+1} \;: \quad 3 \text{ quarks} + 1 \text{ lepton}\,. \tag{30}$$

The baryon B_i thus decomposes into three colour-triplet states (identified with right-handed quarks) and one singlet (a right-handed neutrino). The remaining SM gauge factors $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ are embedded in $\mathrm{SU}(4)_L$ via $\mathbf{4}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1} \oplus \mathbf{1}_0$, as in Section **??**.

The four baryoninos ψ_{B_i} are the candidate right-handed neutrinos. Under $\mathrm{SU}(4) \to \mathrm{SU}(3) \times \mathrm{U}(1)$, the $\bar{\mathbf{4}}$ decomposes into three $\nu_{R,i}$ plus one sterile singlet. On the mesonic branch ($\langle B \rangle = \langle \tilde{B} \rangle = 0$), the baryonino mass matrix is $\mathcal{M}_R^{ij} = \partial^2 W/\partial B_i \partial \tilde{B}^j = \tilde{M}^i{}_j/\Lambda^5$, where $\tilde{M}$ is the cofactor matrix of M.[3] For diagonal $\langle M \rangle$, the F-term equations $\prod_{k \neq i} v_k = m_i \, \Lambda^5$ imply

$$\mathcal{M}_R^{ii} = m_i\,, \tag{31}$$

so the baryonino masses equal the mass-deformation parameters. In particular, $M_R(\nu_e) = m_e$ and $M_R(\nu_4) = m_t$. Meanwhile, no tree-level Dirac coupling between mesinos and baryoninos exists on this branch ($\partial^2 W/\partial M\, \partial B \propto \langle \tilde{B} \rangle = 0$), so neutrino masses arise only from radiative corrections, schematically $m_\nu \sim m_e^2/(16\pi^2\, \Lambda) \approx 0.02$ eV—the correct order of magnitude. This provides a structural explanation for the smallness of neutrino masses in the sBootstrap [**?**] (the "SUSY-bootstrap" framework where hadronic SUSY relates mesons to fermions), without invoking a separate seesaw scale.

A *hybrid scenario* reconciles $N_f = 4$ with the $N = 3$ uniqueness of Section **??**: at high energy, SU(3) SQCD with 4 flavours exhibits Pati–Salam structure. When the heaviest flavour decouples ($m_4 = m_t \gg \Lambda$), holomorphic scale matching gives $\Lambda_3^6 = \Lambda_4^5 \cdot m_t$, and the low-energy theory is effective $N_f = 3$ where the U(3) charge structure applies. The top quark plays the role of the spectator in the turtle/elephant partition of [**?**]: the one

[3] The R-charge assignment $R(B) = R(\tilde{B}) = 3/4$, $R(M) = 1/2$ selects $B\, M\, \tilde{B}$ as the marginal coupling. The cofactor form $\tilde{M}/\Lambda^5$ arises after canonically normalising the composite baryonic operators via their Kähler metric, $K_B \sim |B|^2/\Lambda^4$.

quark that does not participate in the charge binding. With $\Lambda_3 = f_\pi\sqrt{3}/2 \approx 80\,\mathrm{MeV}$ (Section **??**) and $m_t = 173\,\mathrm{GeV}$, scale matching gives $\Lambda_4 \approx 17\,\mathrm{MeV}$. This scale is well below $\Lambda_{\mathrm{QCD}} \approx 220\,\mathrm{MeV}$ and might seem dangerously low. However, the family $\mathrm{SU}(3)_F$ is a *separate* gauge group from QCD: its strong coupling scale governs the composite Higgs/lepton masses, not hadronic binding. The physical role of Λ_4 is to set the normalisation of the $N_f = 4$ confined superpotential relative to the mass deformations. Since $\Lambda_4 \ll m_t$, the $N_f = 4$ theory is deeply in its confined phase when the fourth flavour is integrated out, and the transition to the $N_f = 3$ effective theory is smooth.

PMNS mixing: an open problem. A significant phenomenological limitation of the mesonic-branch construction is the neutrino mixing pattern. On this branch, $\mathcal{M}_R$ is diagonal (**??**) and the Dirac coupling vanishes at tree level, so the PMNS matrix is the identity to all orders in perturbation theory within the superpotential. This is in clear conflict with the observed large lepton mixing angles ($\theta_{12} \approx 34°$, $\theta_{23} \approx 49°$, $\theta_{13} \approx 8.5°$ [**?**]). Potential sources of large PMNS mixing include: (i) Kähler-potential rotations, which are unconstrained in the strongly-coupled regime and can be $O(1)$; (ii) SUSY-breaking contributions from the baryonic branch, where $\langle B \rangle \neq 0$ generates off-diagonal Dirac couplings; (iii) threshold corrections at the $N_f = 4 \to N_f = 3$ matching scale. We do not claim to explain PMNS mixing in this paper and leave its computation as the most pressing open problem of the framework.

7 The Effective Lagrangian

Below the family confinement scale Λ, the sBootstrap effective theory is specified by a single chiral superfield $M^\alpha{}_i$ (the meson matrix), plus the elementary SM fermions. The UV theory has gauge group $\mathrm{SU}(3)_F \times \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, with preons $Q^\alpha_a \in (\mathbf{3}_F, \mathbf{3}_L)$ and $\tilde{Q}_{a,i} \in (\bar{\mathbf{3}}_F, \mathbf{1}_{\mathrm{SM}})$. After $\mathrm{SU}(3)_F$ confinement:

Superpotential.

$$W = \frac{\det M - B\tilde{B}}{\Lambda^3} + m_i\, M^i{}_i + \frac{y_\ell}{\Lambda}\, L_i\, e^c_j\, M^{1,2}{}_k\, \epsilon^{ijk} + \frac{y_u}{\Lambda^3}\, q_L\, u^c\, \mathrm{adj}(M)^{1,2} + \frac{y_d}{\Lambda}\, q_L\, d^c\, M^{1,2}\,, \quad (32)$$

where $M^{1,2}$ denotes the doublet ($Y{=}-1/2$) components and $\mathrm{adj}(M)^{1,2}$ the doublet ($Y{=}+1/2$) components of the adjugate. The lepton Yukawa is automatic: L_i is the mesino $\psi_{M^{1,2}{}_i}$ and e^c_j is $\psi_{M^3{}_j}$, so the y_ℓ term is contained in $\det M/\Lambda^3$ as a meson self-coupling. (In the confined Hessian, the doublet–singlet block is antisymmetric in generation indices; the symmetric SUSY-breaking contributions that complete the physical mass matrix are discussed below.) Quark Yukawas arise from partial compositeness mixing.

Field content.

Field	$\mathrm{SU}(3)_c$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$	Origin
$L_i = \psi_{M^{1,2}{}_i}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	mesino (composite)
$e^c_i = \psi_{M^3{}_i}$	$\mathbf{1}$	$\mathbf{1}$	$+1$	mesino (composite)
$\nu_R = \psi_B$	$\mathbf{1}$	$\mathbf{1}$	0	baryonino (composite)
$H_d = M^{1,2}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$	meson scalar
$H_u = \mathrm{adj}(M)^{1,2}$	$\mathbf{1}$	$\mathbf{2}$	$+1/2$	adjugate (function of M)
q_L, u^c, d^c	$\mathbf{3}, \bar{\mathbf{3}}, \bar{\mathbf{3}}$	—	—	elementary

Key structural features.

1. *One chiral field*: adj(M) is a quadratic polynomial in M; the superpotential (**??**) is holomorphic in M alone. No independent H_u superfield is needed.

2. *No free Yukawas*: The mass parameter m_i simultaneously sets the UV mass deformation, the effective Yukawa coupling, and the meson VEV. The Yukawa hierarchy *is* the family charge hierarchy.

3. *Determined* $\tan\beta$: At the superpotential level, the VEV ratio $\langle \mathrm{adj}(M)\rangle/\langle M\rangle = \Lambda^6/\langle M\rangle^2$ is fixed by the quantum constraint. However, adj(M) is a composite operator, quadratic in M; its physical VEV depends on the Kähler metric $K(M, M^\dagger)$, which is not protected by holomorphy. Whether v_u/v_d remains fully determined at the quantum level requires computing the Kähler potential on the constraint surface—an open problem (cf. Table **??**).

4. *Fermi scale*: The meson condensate $\langle M_\tau\rangle = \Lambda^2 m_\tau/P_\ell^{1/3}$ relates v_{EW} to Λ through the Kähler normalization Z (see §**??**). Matching $\Lambda = f_\pi\sqrt{3}/2 \approx 80\,\mathrm{MeV}$.

8 Numerical Results

8.1 Charged leptons

With $K = 2/3$ exact, the two lighter masses m_e and m_μ fix $k_{\mathrm{lep}} z_0^2 = 313.8\,\mathrm{MeV}$ and $\alpha_{\mathrm{lep}} = 0.22222(1)$ rad; the third mass m_τ is then a *prediction*. The charge ansatz (**??**)–(**??**) gives:

	Predicted	PDG (MeV)	Deviation
m_e	0.511	0.511	input
m_μ	105.66	105.66	input
m_τ	1776.97	1776.86 ± 0.12	$+0.91\sigma$

Table 2: Charged-lepton masses from the U(3) charge ansatz. The two inputs (m_e, m_μ) fix k and α; the predicted m_τ deviates by 0.91σ from PDG.

8.2 Quark sector and CKM elements

Chain structure. The quark predictions in Table **??** arise from a chain of overlapping Koide triples [?] that splits into two regimes:
Fermi side (above the SUSY-breaking scale):

$$(t,\, b,\, c): \quad K(m_t,\, m_b,\, m_c) \approx \tfrac{2}{3} \qquad (\text{standard Koide, } \overline{\mathrm{MS}} \text{ masses})\,. \tag{33}$$

Chiral side (below the breaking scale):

$$(c,\, s,\, 0): \quad \frac{m_c}{m_s} = (2+\sqrt{3})^2\,; \qquad (s,\, 0,\, m_u+m_d): \quad \frac{m_s}{m_u+m_d} = (2+\sqrt{3})^2\,. \tag{34}$$

The chiral-side links use zero-charge triples ($z_2 = 0$, i.e., $\alpha = \pi/4$), where one Koide charge vanishes and $K = 2/3$ reduces to a mass ratio $(2+\sqrt{3})^2 = 13.93$. The lightest

Observable	Method	Predicted	PDG	Dev.
$m_u + m_d$	zero-charge (α=$\pi/4$)	6.7 MeV	6.8 MeV	1.4%
m_c	charge chain	1301 MeV	1270 MeV	2.4%
m_b	inverse sector	4567 MeV	4180 MeV	9.3%
m_t	standard chain	180 GeV	172.8 GeV	3.9%
$\|V_{us}\|$	$\sqrt{m_d/m_s}$	0.2236	0.2243 ± 0.0008	0.9σ
$\|V_{cb}\|$	$1/(5R)$	0.0417	0.0422 ± 0.0008	0.6σ
$\|V_{ub}\|$	$2\sqrt{2}/(7R^3)$	0.00367	0.00368 ± 0.00011	0.1σ
δ_{CP}	$\arctan\sqrt{R}$	65.4°	$65.7 \pm 3.4°$	0.1σ
J	from above	$3.04{\times}10^{-5}$	$3.08 \pm 0.15{\times}10^{-5}$	0.3σ
$\sin 2\beta$	from above	0.723	0.699 ± 0.017	1.4σ

Table 3: Quark sector and CKM elements. Inputs: $m_d = 4.67\,\text{MeV}$, $m_s = 93.4\,\text{MeV}$ ($\overline{\text{MS}}$ at 2 GeV). Here $R = (5 + \sqrt{21})/2$. The CKM expressions involving R are *empirical pattern identifications*; derivations from the operator basis remain open (see text). The m_b and m_t entries have elasticities $\varepsilon \approx 6$ and 4 to m_d; at $m_d = 4.60\,\text{MeV}$ they shift to 4180 and 170 GeV respectively.

quark entry uses the *isospin reparametrization* $u \to 0$, $d \to m_u + m_d$: only the total light-quark mass enters, because below Λ_{QCD} the individual m_u and m_d are unobservable—the Gell-Mann–Oakes–Renner relation $m_\pi^2 = B_0(m_u{+}m_d)$ [?] determines only the sum. This reparametrization is natural in the sBootstrap interpretation, where Koide is a scalar (meson) mass formula: $(c, s, 0)$ connects to the zero-charge meson Koide $K(0,\, m_\pi,\, m_D) = 2/3$ (with $\sqrt{m_D/m_\pi} \approx 2{+}\sqrt{3}$ to 2%). The two sides are connected by the tuple (b, c, s) with a *signed* square root: $K(\sqrt{m_b} + \sqrt{m_c} - \sqrt{m_s}) = 0.675$, deviating from 2/3 by 1.2%. The negative sign on $\sqrt{m_s}$ corresponds to $z_0 + z_s < 0$ in the charge ansatz—the sector angle has crossed the critical value where the lightest charge flips sign. The complete chain is therefore

$$(t, b, c) \;-\; (b, c, s)^{\pm} \;-\; (c, s, 0) \;-\; (s, 0, m_u{+}m_d)\,, \tag{35}$$

with (b, c, s) as the bridge link. The signed tuple hits $K = 2/3$ exactly at $m_s \approx 84\,\text{MeV}$—between the PDG value $m_s(2\,\text{GeV}) = 93.4\,\text{MeV}$ and the higher-scale estimate $m_s(m_b) \approx 81\,\text{MeV}$—suggesting that its natural evaluation scale is $\mu \sim$ 3–5 GeV. At 1.2% the bridge tuple is not compelling on its own, but its structural role as the link between Fermi and chiral regimes is significant.

The Fermi/chiral split coincides with the transition from fermionic to scalar mass formulae: above the breaking scale the SUSY mass relation $m_{\text{fermion}} \approx m_{\text{scalar}}$ makes the chain applicable to quark masses directly; below it, the chain applies to meson masses, with the isospin reparametrization reflecting the fact that chiral symmetry breaking groups u and d into a single Goldstone sector.

IR/UV scale preference. The two Yukawa operators of Section **??**—M (meson VEV, masses $\propto 1/m_{\text{UV}}$) and $\text{adj}(M)$ (adjugate, masses $\propto m_{\text{UV}}$)—make a structural prediction about scale dependence: the W_1 sector (species coupling through M) should be evaluated at *infrared* scales (pole masses or low $\overline{\text{MS}}$ scales), while the W_2 sector (species coupling through $\text{adj}(M)$) should be evaluated at *ultraviolet* scales (high $\overline{\text{MS}}$). Using the AHS running masses [?]:

Tuple	Best $K-2/3$	At M_Z	Preferred scale
$K(e,\mu,\tau)$	0.001% (pole)	0.18%	IR
$K^{-1}(d,s,b)$	0.25% (m_i)	0.13%	UV
$K(t,b,c)$	0.4% (m_i)	8%	IR
$K(c,s,0)$	0.3% (m_i)	2.6%	IR

Every chain tuple except the inverse down quarks worsens at M_Z, confirming they belong to the W_1 sector ($M \propto 1/m$). The inverse Koide *improves* at M_Z and passes through 2/3 near 1 TeV (§**??**), consistent with $\mathrm{adj}(M) \propto m$ evaluated at UV scales. This IR/UV dichotomy is a structural consequence of the two operators, not a tuning of evaluation scales.

The empirical relation $|V_{cb}| = 1/(R^2+1) = (5-\sqrt{21})/10$ is algebraically natural given $R^2 + 1 = 5R$ (a consequence of $R^2 - 5R + 1 = 0$), but a derivation from the operator basis of Section **??** has not been established. Combined with $|V_{us}| = \sqrt{m_d/m_s}$, this gives the Wolfenstein A parameter: $A = |V_{cb}|/|V_{us}|^2 = m_s/(5R\, m_d) = 0.835$, compared to the PDG value $A = 0.836 \pm 0.015$ (0.1%).

The CP phase $\delta_{CP} = \arctan\sqrt{R}$ is equivalent to $\cos^2 2\delta = 3/7$, or $\sin 2\delta = 2/\sqrt{7}$, since $(R{+}1)^2 = 7R$. This is not an independent prediction but a consequence of the $N = 3$ uniqueness: since $(R{-}1)^2 = NR$ and $(R{+}1)^2 = (N{+}4)R$ (from the self-dual quadratic), one has $\cos^2 2\delta = N/(N{+}4)$. This equals $N/(2N{+}1)$ only when $N{+}4 = 2N{+}1$, i.e., $N = 3$—the same condition as $D_{\mathrm{alg}} = D_{\mathrm{dyn}}$. The Jarlskog invariant has the closed form $J \approx 2\sqrt{2}/(5 \times 7^{5/4})\, \sqrt{m_d(m_s{-}m_d)}/m_s \times R^{-15/4} = 3.04 \times 10^{-5}$ (PDG: 3.08×10^{-5}). Figure **??** shows the predicted unitarity triangle, with the apex at $(\bar\rho, \bar\eta) = (0.164, 0.358)$ and the three angles $\alpha = 92.0°$, $\beta = 22.6°$, $\gamma = 65.4°$, all within 1.5σ of PDG constraints.

8.3 Sensitivity and parameter counting

Input–output accounting. The framework has four continuous input parameters:

1. k (overall mass scale) — fixed by m_e;
2. α (sector angle) — fixed by m_μ;
3. m_d ($\overline{\mathrm{MS}}$ at 2 GeV) — taken from PDG;
4. m_s ($\overline{\mathrm{MS}}$ at 2 GeV) — taken from PDG.

In addition, the framework relies on discrete structural choices: the charge ansatz $m_i = k(z_0{+}z_i)^2$, the partial compositeness exponents $\gamma_\ell = 0$ and $\gamma_d = 1$, and the Kähler normalization Z required to match v_{EW}. These are not continuous parameters but foundational assumptions (Table **??**).

From these four inputs, the framework produces the following genuine predictions (outputs that are not used to fix any input):

- m_τ (from k, α, and $K = 2/3$): $+0.91\sigma$.
- $m_c = m_s(2{+}\sqrt{3})^2$: 2.4% above PDG.
- m_b from $K^{-1}(d,s,b) = 2/3$: sensitive to m_d ($\varepsilon = 6.3$).
- m_t from the chain: sensitive to m_d ($\varepsilon \approx 4$).

figures/ut_triangle.pdf

Figure 1: The CKM unitarity triangle from the framework's four inputs. The shaded bands show PDG constraints on $\sin 2\beta$ and γ. The predicted apex (black dot) lies at the intersection of the R_b and R_t circles.

- $m_u + m_d$ from $K(0, s, m_u{+}m_d) = 2/3$: 1.4%.
- $|V_{us}| = \sqrt{m_d/m_s}$: 0.9σ.

The CKM elements $|V_{cb}|$, $|V_{ub}|$, and δ_{CP} depend only on the self-dual ratio $R = (5{+}\sqrt{21})/2$ and are *independent of all four continuous inputs*. However, these are currently *empirical pattern identifications*, not derivations from the operator basis. The Jarlskog invariant J and $\sin 2\beta$ are derived quantities (not independent of $|V_{cb}|$, $|V_{ub}|$, δ_{CP}).

Figure **??** shows the sensitivity of m_b and m_t to the input m_d.

The inverse Koide formula amplifies the m_d uncertainty by a factor of 6 in m_b and 4 in m_t (Fig. **??**); given the PDG range $m_d = 4.7{\pm}0.4\,\text{MeV}$, the predicted m_b spans ~ 3000–$7000\,\text{MeV}$. Setting $m_d = 4.60\,\text{MeV}$ (within 0.24σ of PDG central) simultaneously gives $m_b = 4180\,\text{MeV}$ and $m_t = 170\,\text{GeV}$ (-1.7%); the quark predictions are consistent with experiment at every m_d in the PDG range.

In contrast, $m_c = m_s\,(2{+}\sqrt{3})^2$ has unit elasticity to m_s and is the most predictive quark relation. At PDG central $m_s = 93.4\,\text{MeV}$, the chain gives $m_c = 1301\,\text{MeV}$ (1.5σ above PDG); at the Rivero value $m_s = 95\,\text{MeV}$, it gives $m_c = 1323\,\text{MeV}$ (2.7σ). This is the largest tension in the framework, and it sharpens considerably once scale dependence is examined carefully.

Holomorphic masses versus physical masses. The $\overline{\text{MS}}$ anomalous dimension γ_m is flavour-independent at all known loop orders (verified through five loops [**?**]), so the figures/sensitivity_plot.pdf

Figure 2: Inverse Koide predictions for m_b (top) and the chain prediction for m_t (bottom) as functions of the input m_d. The red bands show PDG central values $\pm 1\sigma$; the grey band is the PDG range for m_d. The red dot marks $m_d = 4.60\,\mathrm{MeV}$, where m_b and m_t simultaneously match PDG.

ratio $m_c(\mu)/m_s(\mu)$ *does not run* in QCD. The most precise lattice determination gives $m_c/m_s = 11.783 \pm 0.025$ [?], which is 18% below $(2+\sqrt{3})^2 = 13.93$. The frequently quoted "agreement" at the level of 13.6 arises only from comparing m_s at $2\,\mathrm{GeV}$ with m_c at m_c—an inconsistent mixed-scale comparison that accidentally compensates for the discrepancy.

This tension has a natural interpretation in the present framework. The mass deformations m_i entering the superpotential $W_{\text{tree}} = \mathrm{Tr}(m\,M)$ are *holomorphic parameters*: they are protected by the non-renormalisation theorem and do not run at any scale [?, ?]. Physical (pole or $\overline{\mathrm{MS}}$) masses, by contrast, are related to the holomorphic ones through the Kähler metric:

$$m_i^{\text{phys}} = \frac{m_i^{\text{hol}}}{\sqrt{Z_i}}, \tag{36}$$

where Z_i is the wavefunction normalisation of the i-th composite. In a strongly coupled s-confining theory, the Z_i are generation-dependent—different mass deformations break $\mathrm{SU}(3)_F$ differently—and non-perturbatively unknown.

If the chain relation $(2+\sqrt{3})^2$ is a statement about holomorphic mass ratios, then the 18% discrepancy with the physical m_c/m_s requires $Z_c/Z_s \approx 1.37$: a 37% difference in wavefunction normalisation between charm and strange composites. For strongly coupled scalars this is not unreasonable, and it connects the chain tension directly to the central open problem of computing the Kähler metric (Section **??**).

By contrast, the *lepton* Koide relation compares pole masses of a single generation-universal composite type (mesinos), where gauge corrections to Z_i are generation-blind at leading order and the holomorphic prediction transfers to physical masses with only $\mathcal{O}(\alpha/4\pi)$ corrections—consistent with the observed 0.04% accuracy.

(The 2024 PDG uncertainties on m_b and m_s are roughly 10 times smaller than the 2022 values, making the inverse Koide deviation of 0.25% a $\sim 1\sigma$ tension at the conventional scales. This tension is relieved at M_Z, where the deviation falls to 0.13%.) Figure **??** summarises the pulls of all CKM predictions and the lepton deviations.

`figures/pull_plot.pdf`

Figure 3: Left: pulls (prediction $-$ PDG central)$/\sigma_{\text{exp}}$ for the CKM observables. Green bars are parameter-free predictions (from R only); orange bars depend on the quark mass inputs m_d, m_s. Right: the predicted m_τ ($+0.9\sigma$), with m_e and m_μ used as inputs to fix k and α.

8.4 The pion decay constant and the confinement scale

The family confinement scale is identified as $\Lambda = f_\pi\sqrt{3}/2 \approx 80\,\text{MeV}$, where $f_\pi = 92.07\,\text{MeV}$. The quantum constraint $\det M = \Lambda^6$ implies $(m_e m_\mu m_\tau)^{1/3} = y_1\Lambda$ with $y_1 = 1/\sqrt{3}$; combined with the confinement-scale identification, this fixes

$$f_\pi = 2(m_e m_\mu m_\tau)^{1/3} \tag{37}$$

to 0.56%—a relation between the pion decay constant and the lepton mass product that is a consequence of the framework, not an additional input.

8.5 Lepton–quark angle shift

The lepton sector angle $\alpha_{\text{lep}} = 0.22222(1)$ rad $= 12.73^\circ$ and the inverse down-quark sector angle $\alpha_{\text{inv}} = 10.59^\circ$ (from $K^{-1}(d, s, b) = 2/3$ at the physical masses) differ by $\Delta\alpha = \alpha_{\text{lep}} - \alpha_{\text{inv}} = 2.14^\circ$. Empirically,

$$\tan(\Delta\alpha) = \frac{\pi}{84} = \frac{\pi}{4\,D} \tag{38}$$

to 1.6%, where $D = 21$ is the discriminant. If exact, this formula determines the quark mass ratio $m_d/m_s = 0.0492$ (PDG: 0.050 ± 0.003), reducing the parameter count from four to three: only (k, α, m_s) are needed. No derivation of (**??**) is known.

8.6 Charge-plane geometry

The Koide parametrisation $\sqrt{m_i} = \sqrt{k}\,(z_0 + z_i)$ with $\sum z_i = 0$ places the charges z_i in a two-dimensional plane, whose orientation is specified by the sector angle α. The charge-plane angles (mod 120°) of the SM Koide triples cluster within $\sim 5.5^\circ$ of the zero-mass $K = 2/3$ fixed point at $\alpha = 15^\circ$ (corresponding to the mass ratio $r_0 = (2+\sqrt{3})^2 = 13.93$). This clustering is a consequence of all SM Koide tuples having their dominant mass ratio close to r_0.

8.7 Scale dependence

Using the running Yukawa couplings of Antusch, Hinze, and Saad [**?**] (SM, $\overline{\text{MS}}$ scheme, 2024 PDG inputs), the Koide ratio for the charged leptons is $K_{\text{lep}} = 0.6678$ at all scales from M_Z to 10^7 GeV (0.17% from 2/3). The larger deviation compared to the pole-mass value (0.001%) reflects QED running corrections $\delta m \propto \alpha\,\log(\mu^2/m^2_{\text{pole}})$, which affect each lepton differently. The approximate scale invariance of K under RG flow supports Sumino's observation [**?**] that gauge interactions approximately preserve the Koide relation.

For the inverse Koide ratio of the down quarks, the running reveals an interesting structure: $K(1/m_d,\, 1/m_s,\, 1/m_b)$ passes through 2/3 at $\mu \approx 1$ TeV, with the central value $K - 2/3 = +2 \times 10^{-6}$ at 10^3 GeV. The deviation grows to +0.13% at M_Z and −0.2% at 10^7 GeV. Although the current experimental uncertainties on m_d and m_s imply a $\sim 0.4\%$ band on K, the trend is robust: the inverse Koide for down quarks is best satisfied near the TeV scale, potentially connected to the effective adjoint mass scale μ_Φ of the emergent doublet structure (§**??**). Figure **??** shows both Koide ratios as a function of scale, using the AHS running Yukawa couplings in the SM $\overline{\text{MS}}$ scheme.

9 Discussion

9.1 What is derived

The charge ratio $K = 2/3$ follows from U(3) normalization. The standard–inverse duality follows from the F-term structure of the s-confined theory: the adjugate $\text{adj}(M) \propto m$ and the meson $M \propto 1/m$ are the two natural Yukawa operators at the confinement scale, leading to standard and inverse Koide for the respective SM species. The duality is identified with the $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$ exchange of SU(5) and the Galois conjugation of $\mathbb{Q}(\sqrt{21})$. Three figures/running_koide.pdf

Figure 4: Koide ratios vs. renormalization scale in the SM. Blue: $K(e,\mu,\tau)$; red: $K^{-1}(d,s,b)$; dashed: $K=2/3$. The shaded band shows the uncertainty from m_d. Data from Antusch, Hinze, and Saad [?].

generations is uniquely selected by discriminant matching. These are exact results contingent on the charge ansatz (??)—itself uniquely selected by the direction-independence of K (Section ??)—and the family symmetry structure.

9.2 The emergent doublet structure and the critical scale

The confined spectrum contains two composite operators per family—M_i and $\mathrm{adj}(M)_i$—whose mass eigenvalues have opposite charge dependence (direct and inverse branches). This pairing has the mathematical form of an $\mathcal{N}=2$ hypermultiplet [?, ?], but is emergent from confinement plus isospin, not a UV input. The coupling between the two branches can be parametrised by an effective quartic meson interaction with scale μ_Φ:

$$W_{\text{eff}} \supset -\frac{1}{2\mu_\Phi}\mathrm{Tr}(M^2)\,, \qquad M = Q\tilde{Q}\,. \tag{39}$$

The full superpotential in the s-confined description becomes

$$W = \frac{\det M - B\tilde{B}}{\Lambda^3} + m_i M^i{}_i - \frac{1}{2\mu_\Phi}\mathrm{Tr}(M^2)\,, \tag{40}$$

with modified F-term equation

$$\frac{\mathrm{adj}(M)_i}{\Lambda^3} + m_i - \frac{M_i}{\mu_\Phi} = 0\,. \tag{41}$$

The limit $\mu_\Phi \to \infty$ recovers the pure $\mathcal{N} = 1$ theory. At finite μ_Φ, the Koide vacuum exists only above a critical scale

$$\mu_{\Phi,\text{crit}} = \frac{4\,\Lambda^2\,P_\ell^{1/3}}{m_e^2} \approx 4.5\,\text{TeV}\,. \tag{42}$$

Below this scale, the $\text{Tr}(M^2)$ correction overwhelms the quantum constraint for the electron meson VEV. The true phase boundary lies at $\mu_{\Phi,\text{bif}} = \mu_{\Phi,\text{crit}}/2^{1/3} \approx 3.5\,\text{TeV}$. Above $\mu_{\Phi,\text{bif}}$, the Koide ratio converges as $K - 2/3 \approx A/\mu_\Phi$ where $A/\mu_{\Phi,\text{crit}} = (z_0^2 - z_e^2)/6$; the observed accuracy $K = 2/3$ to 0.001% requires $\mu_\Phi \gtrsim 37\,\text{TeV}$. This constrains the effective inter-branch coupling scale in the family sector to be at least $\sim 40\,\text{TeV}$. Independently, the inverse down-quark Koide ratio $K^{-1}(d, s, b)$ crosses $2/3$ at $\mu \approx 1$–$1.6\,\text{TeV}$ under standard $\overline{\text{MS}}$ running [?], the same order of magnitude as μ_{crit}.

9.3 The Fermi scale and Kähler normalization

The meson condensate $\langle M_i \rangle$ has mass dimension 2. The canonical Higgs field is $h_i = Z^{1/2}\,M_i$ where $Z = K_{M\bar{M}}$ is the Kähler wave-function renormalization:

$$v_{\text{EW}}^2 = Z\,\langle M_\tau \rangle^2 = Z\,\frac{\Lambda^4\,m_\tau^2}{P_\ell^{2/3}}\,. \tag{43}$$

The three doublet VEVs are hierarchical: $v_i/v_\tau = m_{\ell,i}/m_\tau$, so $v_\tau/\sqrt{\sum v_i^2} = 0.998$—the electroweak VEV is dominated by the heaviest lepton. Matching $v_{\text{EW}} = 246.2\,\text{GeV}$ determines Z:

$$Z^{1/2} = \frac{v_{\text{EW}}\,P_\ell^{1/3}}{\Lambda^2\,m_\tau}\,. \tag{44}$$

If $\Lambda = f_\pi\sqrt{3}/2 \approx 79.7\,\text{MeV}$, the canonical Kähler normalization $Z = 1/\Lambda^2$ gives $v_{\text{EW}} = 3.09\,\text{GeV}$—two orders of magnitude below the observed value. The discrepancy requires a large anomalous Kähler correction $Z \gg 1/\Lambda^2$, characteristic of the strongly-coupled confined phase. By naïve dimensional analysis (NDA), the Kähler metric for a composite of dimension d scales as $Z \sim (4\pi)^{2(d-1)}/\Lambda^{2(d-1)}$ [?]; for $d = 2$ this gives $Z \sim (4\pi)^2/\Lambda^2$, an enhancement of ~ 160 over canonical. Both estimates are parametrically consistent with the required $Z/Z_{\text{canon}} \approx 6{,}300$, but off by an $O(1)$ coefficient that cannot be fixed without a non-perturbative computation—either from the Seiberg–Witten prepotential on the Coulomb branch (which is known for SU(3) with 3 flavours [?]) or from lattice SQCD. Computing the Kähler potential on the quantum-modified moduli space remains the most important quantitative open problem in the framework.

9.4 SUSY breaking from electroweak symmetry breaking

The SUSY-preserving vacuum is diagonal: $\langle M^\alpha{}_i \rangle = v_\alpha\,\delta_{\alpha i}$, which does not break $\text{SU}(2)_L$ and has $V = 0$. When $\text{SU}(2)_L$ is broken by giving the Higgs doublet components VEVs in all three generations, the F-term system becomes overconstrained: 18 real equations for 12 real variables. SUSY necessarily breaks, with the breaking scale set by the lepton masses:

$$|F_{21}| = m_e\,, \qquad |F_{22}| = 0\,, \qquad |F_{23}| = m_\tau\,, \tag{45}$$

and $|F| = \sqrt{\sum |F_{2j}|^2} \approx m_\tau \approx 1.8\,\text{GeV}$. The goldstino is predominantly the τ-mesino (since $m_\tau/m_e \approx 3500$), and $m_{3/2} = |F|/(\sqrt{3}\,M_{\text{Pl}}) \approx 4 \times 10^{-16}\,\text{GeV}$.

The three Higgs doublets $H_i = M^{1,2}{}_i$ acquire hierarchical VEVs $v_i \propto m_{\ell,i}$ from the adjugate mechanism: v_τ dominates ($v_\tau/\sqrt{\sum v_i^2} = 0.998$), so the SM Higgs is predominantly the τ-generation doublet.

9.5 The scalar spectrum and the mild SUSY-breaking problem

Standard SUSY-breaking constructions (gravity or gauge mediation) generate soft scalar masses $m_{\rm soft} \sim F/M_{\rm mess}$ that send superpartners to the TeV scale—orders of magnitude above their Standard Model counterparts. In the present framework, the situation is structurally different. The SUSY-breaking F-terms *are* the lepton masses ($|F| \approx m_\tau \approx 1.8\,\mathrm{GeV}$, not $(100\,\mathrm{TeV})^2$), and the "superpartners" of the leptons are the composite Higgs doublets that live in the same chiral superfield M. The scalar–fermion splitting is controlled by F and the family charges, not by a distant hidden sector.

Charge-aligned Kähler deformation. The holomorphic mass formula $m_i = k(z_0+z_i)^2$ is protected by the non-renormalization theorem: any deformation that preserves the superpotential preserves it exactly. The leading SUSY-breaking correction therefore enters through the Kähler potential. The simplest charge-aligned operator is

$$\Delta K \;=\; \frac{c}{M_*^2}\sum_i (z_0+z_i)^2\,|F_X|^2\,|\Phi_i|^2\,, \tag{46}$$

where X is a SUSY-breaking spurion with $\langle F_X\rangle \neq 0$ and M_* is the mediation scale. Because the weight is $(z_0+z_i)^2$—the same charge combination that determines m_i—the resulting soft scalar mass

$$m^2_{\rm soft,\,i} \;=\; \varepsilon\, m_i\,, \qquad \varepsilon \equiv \frac{c\,|F_X|^2}{k\,M_*^2} \tag{47}$$

is automatically aligned with the mass hierarchy. The scalar-to-fermion mass ratio is $\sqrt{1+\varepsilon/m_i}$: for $\varepsilon \ll m_e$, *all* partners remain nearby. The fermion masses (and therefore the Koide formula) are unaffected at this order.

Self-mediated estimate. In the most economical scenario, the spurion is the meson sector itself ($|F_X| \approx m_\tau$, $M_* \approx \Lambda$):

$$\varepsilon \;\sim\; \frac{c\,m_\tau^2}{k\,\Lambda^2} \;\approx\; c\times 0.26\,\mathrm{MeV}\,. \tag{48}$$

For a loop-suppressed coefficient $c \sim \alpha/4\pi \sim 10^{-2}$, $\varepsilon \sim 3\,\mathrm{keV}$, giving sub-percent splittings for all three generations—consistent with the observed precision of K. For $c \sim O(1)$, $\varepsilon \sim m_e/2$: the τ and μ partners are split by $< 0.2\%$, while the electron partner receives an $O(1)$ relative correction. In either case the splitting is *mild*: no partner is pushed to the TeV scale.

Wavefunction mechanism. An equivalent viewpoint replaces the spurion with a family-dependent wave-function renormalization $Z_i = Z\big[1+\varepsilon_1(z_0+z_i)^2\big]$, giving $m_i^{\rm phys} = m_i^{\rm hol}/\sqrt{Z_i} \approx \tilde m_i\big[1-(\varepsilon_1/2k)\,\tilde m_i\big]$ at first order. The correction is proportional to m_i^2 (quartic in charge) and preserves $K = 2/3$ to $O(\varepsilon_1^2)$. This is precisely the partial-compositeness mechanism of the next subsection: the anomalous dimension γ_* controls ε_1, with $\gamma_* = 0$ (universal Z) for leptons and $\gamma_* = 1$ for down quarks.

The supertrace constraint. For F-term breaking with canonical Kähler, $\mathrm{STr}(\mathcal{M}^2) = \sum m^2_{\text{boson}} - \sum m^2_{\text{fermion}} = 0$. This means the sum of scalar mass-squared corrections vanishes: $\sum_i m^2_{\text{soft, i}} = \varepsilon \sum_i m_i = 0$ cannot hold (since $m_i > 0$ and $\varepsilon > 0$), signalling that the canonical Kähler is modified at $O(\varepsilon)$. The leading non-canonical correction is the family-dependent Z_i above, which redistributes the supertrace across generations without violating the sum rule.

Contrast with the MSSM. In the MSSM, the μ-problem and the hierarchy problem arise because soft masses must be tuned against $|\mu|^2$ to get $v_{\text{EW}} \ll M_{\text{Pl}}$. Here, both v_{EW} and the soft masses are determined by the *same* family charges (eq. **??**): the electroweak VEV is $v_{\text{EW}} = \Lambda^2 m_\tau / P_\ell^{1/3}$ (eq. **??**), the soft masses are $\varepsilon \sim m_\tau^2/(k\Lambda^2)$, and the Higgs mass arises from Coleman–Weinberg corrections at $M_S \approx \mu_{\text{crit}}$ (eq. **??**). There is no separate hidden sector; all scales are tied to the lepton mass product P_ℓ and the confinement scale Λ.

What remains open. Computing the coefficient c in (**??**) from first principles requires the Kähler metric on the quantum-modified moduli space of the s-confined theory. For the genus-0 Seiberg–Witten curve relevant to $N_f = N_c = 3$, this metric is not constrained by holomorphy and receives non-perturbative corrections from M2-brane and M5-brane instantons (Appendix **??**). The full scalar spectrum—three Higgs doublets, three charged sleptons, and the baryonic modulus—depends additionally on the $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ D-terms, which enforce vacuum alignment but whose quantitative contribution to the mass splittings is model-dependent. Identifying a mildly broken SUSY sector that keeps superpartners nearby—rather than sending them to the TeV scale as in standard constructions—is the most pressing open problem of the framework.

9.6 Partial compositeness and quark masses

Quarks are elementary fields, external to the confining sector. They acquire mass through *partial compositeness* [**?**]: each elementary quark mixes linearly with a composite operator of the same quantum numbers: $\mathcal{L} \supset \lambda_L \bar{q}_L \mathcal{O}_R + \lambda_R \bar{q}_R \mathcal{O}_L + \text{h.c.}$ The mixing coefficients λ_i are generation-dependent, determined by anomalous dimensions at the confining fixed point [**?**, **?**]. The superpotential is holomorphic in M and $\mathrm{adj}(M)$, so the most general Yukawa operator is $W \supset q\,\bar{q}\,M^a\,\mathrm{adj}(M)^b/\Lambda^n$ with $a, b \in \mathbb{Z}_{\geq 0}$. On the constraint surface $\mathrm{adj}(M) = \Lambda^6 M^{-1}$, the generation-dependent mass scaling is $m_q \propto m_{\text{UV}}^{a-b}$: the exponent $p = a - b$ is automatically an integer. EFT power counting selects the two leading operators: $(a, b) = (1, 0)$ (renormalisable, p=+1) and $(a, b) = (0, 1)$ (p=−1), related by the $\mathbf{15} \leftrightarrow \overline{\mathbf{15}}$ exchange. The *assignment* (why down quarks couple to M rather than $\mathrm{adj}(M)$) is fixed by hypercharge ($Y(M^{1,2}) = -\frac{1}{2}$, $Y(\mathrm{adj}(M)^{1,2}) = +\frac{1}{2}$).

9.7 What remains open

1. **The dynamical origin of α_{lep}.** All SU(3)-invariant potentials for the traceless diagonal field depend on α only through $\cos 3\alpha$, whose extrema lie at $\alpha = 0$ and $\pi/3$. The one-loop Coleman–Weinberg effective potential remains $\mathbb{Z}_3$-invariant and monotonic in α at every renormalization scale. Fixing α_{lep} therefore requires non-perturbative Kähler effects, cross-sector constraints, or input from a UV completion.

2. $|V_{ub}|$ **and the exclusive/inclusive tension.** $|V_{ub}| = 2\sqrt{2}/((2N+1)R^3)$ gives 0.003674, matching the *exclusive* determination $|V_{ub}|_{\rm excl} = 0.003682 \pm 0.000110$ (0.08σ) but deviating from the inclusive value $|V_{ub}|_{\rm incl} = 0.00425 \pm 0.00010$ (5.8σ). If the exclusive/inclusive tension is resolved in favour of the inclusive value, the framework is falsified.

3. **PMNS mixing.** The baryonino Majorana matrix $\mathcal{M}_R$ is diagonal on the mesonic branch, and the mesino–baryonino Dirac coupling vanishes at tree level. The most promising path is non-diagonal Kähler corrections to the baryonic sector: the composite operator $B = \epsilon\, QQQ$ involves all three flavours symmetrically, so its Kähler metric need not inherit the meson hierarchy. A non-diagonal Z_B rotates the diagonal $\mathcal{M}_R$ into a near-democratic matrix whose eigenvectors can reproduce large PMNS angles. This naturally explains the CKM/PMNS split: mesons (hierarchical VEVs $\to$ small CKM) vs. baryoninos (flavour-symmetric Kähler $\to$ large PMNS).

4. **The Higgs mass.** At tree level the F-term self-quartic vanishes, so $m_h \leq m_Z$ as in the MSSM; lifting m_h to 125 GeV requires composite-loop (Coleman–Weinberg) corrections. Using the MSSM one-loop formula as a parametric estimate:

$$\Delta m_h^2 \;=\; \frac{3m_t^4}{4\pi^2 v^2}\left[\log\frac{M_S^2}{m_t^2} + \frac{X_t^2}{M_S^2}\left(1 - \frac{X_t^2}{12M_S^2}\right)\right], \tag{49}$$

   for small mixing ($X_t \approx 0$), $m_h = 125\,\mathrm{GeV}$ requires $M_S \approx 4.6\,\mathrm{TeV} \approx \mu_{\rm crit}$—the same parameter that controls the Koide accuracy.

5. **Flavour-changing neutral currents.** Three Higgs doublets with generation-dependent VEVs generically mediate tree-level FCNC. The VEV hierarchy provides a natural Cheng–Sher suppression [**?**]: off-diagonal Yukawa couplings scale as $\sqrt{m_i m_j}/v$, consistent with K^0–$\bar{K}^0$ mixing for $M_H \gtrsim O(\,\mathrm{TeV})$.

6. **The composite/elementary asymmetry.** Leptons are composites (mesinos), quarks are elementary. This identification is *forced by colour*: mesons of $\mathrm{SU}(3)_F$ are colour singlets. The Fradkin–Shenker complementarity of Appendix **??** renders the distinction frame-dependent: in the Higgs phase, "dressed quarks" are smoothly connected to mesinos, and K is invariant.

7. **Proton stability.** SM baryon number $B_{\rm SM}$ and family baryon number B_F are independent symmetries. All family-sector composites carry $B_{\rm SM} = 0$; the dangerous dimension-6 operator $QQQL/M^2$ cannot be generated because there is no $\mathrm{SU}(3)_F$-invariant coupling. The $N_f = 4$ Pati–Salam structure is safe because $\mathrm{SU}(4)_R$ is a *global* symmetry. Proton stability is an *exact structural prediction.*

9.8 The sector angle and the critical angle

The self-dual angle (**??**) is $\alpha_{\rm sd} \approx 0.162$ rad. The physical lepton angle $\alpha_{\rm lep} = 0.22222(1)$ rad, fitted from m_τ/m_μ, predicts m_τ to 0.9σ.

At the critical angle $\alpha_{\rm crit} = \pi/12$, the lightest Koide mass vanishes: $z_0 + z_1 = 0$. Expanding around this zero,

$$m_e \;\approx\; k\,\frac{\varepsilon^2}{6}\,, \qquad \varepsilon \equiv \frac{\pi}{12} - \alpha_{\rm lep} \approx 0.040\,, \tag{50}$$

which reproduces m_e/k to 3.8% (0.05% at $O(\varepsilon^2)$). The lepton mass hierarchy $m_e/m_\tau \sim 3\times10^{-4}$ is not a fine-tuning problem: it is a geometric consequence of $\alpha_{\rm lep}/\alpha_{\rm crit} \approx 0.85$. This admits a 't Hooft naturalness interpretation: at $\alpha = \alpha_{\rm crit}$, the first-generation mass vanishes exactly and the symmetry is enhanced.

9.9 Logical status of $K = 2/3$

$K(m_i) = 2/3$ is a *theorem* on the charge lattice (Section **??**), never used as input. $K(1/m_i) = 2/3$ is a *constraint* that selects the self-dual angle and determines R. All predictions in Table **??** follow from the charge ansatz, the involution, and the discriminant condition.

Status of key claims. Table **??** classifies the main results by their logical status. "Proven" means derived algebraically from the stated assumptions. "Conditional" means derived contingent on a specific additional hypothesis. "Empirical" means a numerical relation that holds to stated accuracy but lacks a derivation. "Motivated" means supported by a structural argument but not fully derived from first principles.

Claim	Status	Comment
$K = 2/3$ from U(3)	Proven	derived as theorem; never used as input
Seiberg transparency $(W_S)_{kk} = 0$	Proven	det multilinearity
Involution $M_i \leftrightarrow \Lambda^4/M_i$	Proven	moduli space
$\tan\beta$ elimination	Proven*	adj(M) holomorphic
$N = 3$ uniqueness	Conditional	equal quartic couplings
$\delta_{CP} = \arctan\sqrt{R}$	Conditional	equivalent to N=3
$\lvert V_{us}\rvert = \sqrt{m_d/m_s}$	Empirical	Gatto–Sartori–Tonin [?]
$\lvert V_{cb}\rvert = 1/(5R)$	Empirical	no derivation yet
Inverse Koide for down quarks	Empirical	assumes $\gamma_* = 1$
$K^{-1}(d,s,b) = 2/3$ at $\mu\approx 1$ TeV	Empirical	AHS running [?]
$\mathcal{M}_R^{ii} = m_i$ (baryonino)	Conditional‡	N_f=4 F-terms
$\Lambda = f_\pi\sqrt{3}/2$	Empirical	0.2%
$M_S(X_t{=}0) \approx \mu_{\rm crit}$	Empirical	3%, gives m_h=125 GeV
$m_e \approx k\varepsilon^2/6$, $\varepsilon{=}\pi/12{-}\alpha_{\rm lep}$	Exact	Taylor expansion of (**??**), 3.8%
$K(b,c,s)^{\pm} = 2/3$ (bridge tuple)	Tentative	1.2% at PDG, exact at $m_s \approx 84$ MeV
$\gamma \in \{0,1\}$ (holomorphic quantisation)	Proven	W polynomial in M, adj(M)
Proton stability	Proven	$B_{\rm SM}$ exact; no dim-6 $QQQL$
Nelson–Barr strong CP	Conditional	involution = CP
$m_i = k(z_0{+}z_i)^2$	Motivated	field energy $\propto q_i^2$; k non-pert.

Table 4: Logical status of the main results. *At the superpotential level; the Kähler potential can in principle reintroduce an independent ratio v_u/v_d. ‡Requires canonical Kähler normalization of composite baryonic operators.

9.10 TeV-scale convergence

Three TeV-range scales emerge from the framework with no TeV-scale input: the $\mathcal{N} = 2$ adjoint mass $\mu_{\rm crit} \approx 4.5$ TeV, the composite stop mass $M_S(X_t{=}0) \approx 4.6$ TeV required for $m_h = 125$ GeV, and the inverse Koide crossing scale $\mu_K \approx 1$ TeV. These three scales form a geometric progression $v_{\rm EW}$, $\sqrt{v\,\mu_{\rm crit}} \approx 1\,{\rm TeV}$, $\mu_{\rm crit}$, with common ratio determined entirely by the lepton mass product. The full scale hierarchy is

$$\mu_{\mathcal{N}=2} \gtrsim 37\,{\rm TeV} > \Lambda_{\rm CST} > v_{\rm EW} = 246\,{\rm GeV} > \Lambda \approx 80\,{\rm MeV}\,. \tag{51}$$

10 Conclusion

The flavour sector of the Standard Model can be organised by two structural ingredients: the moduli space of SU(3) SQCD, whose quantum constraint provides an involution connecting the charged-lepton and down-quark sectors; and the canonical normalization of a U(3) family symmetry, which fixes $z_0 = 1/\sqrt{6}$ and guarantees $K = 2/3$. The charge ansatz $m_i = k(z_0{+}z_i)^2$ is uniquely selected: it is the only monomial for which the Koide ratio is direction-independent in the Cartan subalgebra, and the q_i^2 pattern follows from gauge theory (field energy $\propto q_i^2$ by Gauss's law and $\mathcal{L} \supset F_{\mu\nu}^2$). The discriminant $D = 21$ appears simultaneously in the self-dual charge spectrum and the family Higgs potential, uniquely at $N = 3$ generations. Extension to $N_f = 4$ recovers the Pati–Salam decomposition from the baryon sector, with the top quark as the decoupling fourth flavour.

The effective Lagrangian involves a single meson field M: both electroweak Higgs doublets ($H_d = M^{1,2}$ and $H_u = {\rm adj}(M)^{1,2}$) are holomorphic in M alone, constraining $\tan\beta$ at the superpotential level (subject to Kähler corrections). The mass parameter m_i in the UV superpotential serves simultaneously as mass deformation, Yukawa coupling, and meson-VEV determinant—there are no independent Yukawa parameters. The family confinement scale $\Lambda \approx f_\pi\sqrt{3}/2$ relates the meson condensate to the electroweak VEV through the Kähler normalization (eq. **??**).

The framework has four continuous inputs (k, α, m_d, m_s), with m_e and m_μ fixing k and α. The predicted m_τ agrees to 0.91σ. The CKM elements $|V_{cb}| = 1/(5R)$, $|V_{ub}| = 2\sqrt{2}/(7R^3)$, and $\delta_{CP} = \arctan\sqrt{R}$ depend only on the self-dual ratio R and are independent of all continuous inputs, though derivations from the confined operator basis remain open. The overall CKM fit has $\chi^2/{\rm dof} < 1$ (Fig. **??**).

The EWSB vacuum necessarily breaks SUSY (the F-term system is overconstrained), with $|F_{2j}| = m_j$: SUSY breaking and the lepton mass hierarchy are structurally unified.

The quark chain splits into a "Fermi side" (t, b, c)—connected through the signed bridge tuple (b, c, s)—to a "chiral side" $(c, s, 0)$, $(s, 0, m_u{+}m_d)$ where zero-charge triples connect to meson physics through the isospin reparametrization $u \to 0$, $d \to m_u{+}m_d$. This split reflects the transition from fermionic to scalar degrees of freedom across the SUSY-breaking scale.

Three TeV-range scales emerge with no TeV-scale input: $\mu_{\rm crit} \approx 4.5\,{\rm TeV}$, $M_S(X_t{=}0) \approx 4.6\,{\rm TeV}$ (for $m_h = 125\,{\rm GeV}$), and the inverse Koide crossing $\mu_K \approx 1\,{\rm TeV}$. If composite stops exist, this framework predicts them at 1–5 TeV.

The M-theory lift (Appendix **??**) provides a geometric interpretation: mesons are M2-branes, baryons are M5-branes, the involution is electromagnetic duality of the C-field, and $K = 2/3$ is singled out by the membrane dimensionality ($p = 2$). The top quark is an asymptotic Kaluza–Klein monopole whose 12 degrees of freedom reside in the SU(9) adjoint $\mathbf{80}$ rather than the $\Lambda^3(\mathbf{9}) = \mathbf{84}$ that houses the family-structured fermions. Identifying the moduli-space involution with CP yields a Nelson–Barr solution to the strong CP problem, with $\delta_{CP} = \arctan\sqrt{R}$ arising from spontaneous CP violation.

The most pressing open problems are: (i) computing the Kähler metric on the quantum-modified moduli space (needed to fix $v_{\rm EW}$, $\tan\beta$, the overall mass scale $k = \mu$, and the scalar spectrum); (ii) identifying the mild SUSY-breaking mechanism that keeps scalar partners nearby rather than at the TeV scale (Section ??); (iii) identifying a source of large PMNS mixing; (iv) deriving the CKM relations involving R from the confined operator basis.

Acknowledgments

A Alternative: $\mathrm{SU}(2)_L$ as a magnetic dual gauge group

This appendix records an alternative construction considered during the investigation; it does not affect the numerical results of the paper.

The embedding $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y \subset \mathrm{SU}(3)_L$ used in the main text (Section ??) is the minimal route to the Higgs identification $H_d = M^{1,2}$, $H_u = \mathrm{adj}(M)^{1,2}$. An alternative, due to Csaki, Shirman, and Terning [?], derives $\mathrm{SU}(2)_L$ dynamically as the magnetic dual gauge group of a confining theory at a scale $\Lambda_{\mathrm{CST}} > v_{\mathrm{EW}}$.

The mechanism. Consider an electric theory with gauge group $\mathrm{SU}(N_e)$ and N_f flavours. Seiberg duality [?] gives a magnetic description with gauge group $\mathrm{SU}(N_f - N_e)$. Setting $N_f - N_e = 2$, the magnetic gauge group is SU(2), identified with $\mathrm{SU}(2)_L$. The magnetic description contains:

1. Magnetic quarks q_i, $\tilde{q}^j$ $(i = 1, \ldots, N_f)$, transforming as doublets under $\mathrm{SU}(2)_{\mathrm{mag}}$.
2. The electric meson $\mathcal{M}^i{}_j = Q^{i,\mathrm{el}}\tilde{Q}^{\mathrm{el}}_j/\Lambda_{\mathrm{CST}}$, a gauge singlet in the magnetic description.
3. The magnetic superpotential $W_{\mathrm{mag}} = q\,\mathcal{M}\,\tilde{q}/\mu$, which generates Yukawa couplings between the magnetic quarks and the Higgs.

Scale hierarchy. The construction requires three scales:

$$\Lambda_{\mathrm{CST}} \;>\; v_{\mathrm{EW}} \;>\; \Lambda_F\,. \tag{52}$$

Above Λ_{CST}, the electric theory confines and is replaced by its magnetic dual. Between v_{EW} and Λ_{CST}, the magnetic quarks are $\mathrm{SU}(2)_L$ doublets, and $\mathrm{SU}(3)_F$ sees $2N_f = 6$ effective flavours—inside the conformal window, so $\mathrm{SU}(3)_F$ does not confine. Below v_{EW}, the doublet partners decouple, leaving $N_f = 3$ and $\mathrm{SU}(3)_F$ confines at $\Lambda_F \approx f_\pi\sqrt{3}/2$.

Connection to the Pati–Salam extension. For $N_f = 4$ (Section ??), the electric theory is $\mathrm{SU}(N_e) = \mathrm{SU}(2)$ with 4 flavours. Since $N_f = 2N_c$, this is a self-dual point: the electric and magnetic theories have the same gauge group.

Comparison with the main text. The CST construction provides a dynamical origin for $\mathrm{SU}(2)_L$ but costs at least two additional parameters (Λ_{CST} and the electric gauge coupling) that do not enter any prediction. The branching-rule embedding of the main text ($\mathbf{3}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}$) achieves the same Higgs identification with zero additional parameters. The two approaches are compatible; the choice between them does not affect any numerical result.

B Higgs–confinement complementarity

If the mesino (the fermionic component of the meson superfield M) *is* a Standard Model lepton, a conceptual question arises: the composites are built from preons Q, $\tilde{Q}$, but those preons carry the same quantum numbers as the composites themselves. This appendix shows that the question is resolved by *Higgs–confinement complementarity* [?, ?]: the two descriptions—elementary preons with a VEV (Higgs phase) and confined composites (confinement phase)—are smoothly connected, with no phase boundary separating them.

The Fradkin–Shenker theorem. For a gauge theory with matter in the fundamental representation, Fradkin and Shenker [?] proved that the Higgs and confinement regimes are analytically connected: no local order parameter distinguishes them. The theorem applies whenever the fundamental matter can screen all centre charges. For SU(3) with $N_f = 3$ fundamentals, the centre $\mathbb{Z}_3$ is fully screened, and the theory is s-confining—precisely the regime in which complementarity holds [?].

Higgs-phase spectrum. Giving the squarks a diagonal VEV, $\langle Q^i_a \rangle = v_i\,\delta^i_a$, $\langle \tilde{Q}^a_j \rangle = \tilde{v}_j\,\delta^a_j$, all 8 generators of SU(3) are broken and the 8 gauge bosons acquire masses. After Higgsing, 8 massive vector multiplets absorb one chiral multiplet each, leaving $18-8=10$ light chiral multiplets. In the confined description, 9 mesons + 2 baryons − 1 quantum constraint = 10: an exact match.

Operator dictionary. The Higgs-phase and confinement-phase variables are related by

$$M^i{}_j = Q^i_a\,\tilde{Q}^a_j\,, \qquad B = \epsilon_{abc}\,Q^{a\,1}Q^{b\,2}Q^{c\,3}\,, \qquad \tilde{B} = \epsilon^{abc}\,\tilde{Q}^1_a\tilde{Q}^2_b\tilde{Q}^3_c\,. \tag{53}$$

The quantum constraint $\det M - B\tilde{B} = \Lambda^6$ is automatically satisfied. The "mesino" ψ_M in the confined description is the *same fermion* as the "dressed quark" in the Higgs phase, $\psi^{i,j}_M \sim \tilde{v}_j\,\psi^i_Q + v_i\,\psi^j_{\tilde{Q}}$.

Anomaly matching. Under $\mathrm{SU}(3)_F \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$, all anomaly coefficients involving the electroweak quantum numbers match exactly between UV preons and IR composites (verified numerically in the ancillary script `anomaly_matching_uv_ir.py`).

Why the Koide structure survives. Since K is a *ratio*, multiplicative factors from the Kähler metric cancel, and $K(m^{\mathrm{dressed}}) = K(m^{\mathrm{hol}}) = 2/3$. Complementarity guarantees the two computations agree.

Dirac masses from mesons, Majorana masses from baryons. The meson bilinear $M^i{}_j = Q^i\tilde{Q}_j$ pairs *different* fields—a **Dirac-type** structure producing charged-lepton masses. The baryon sector involves $B = \epsilon\,Q^3$ and $\tilde{B} = \epsilon\,\tilde{Q}^3$—trilinears of the *same* field type, producing **Majorana-type** masses. On the quantum-modified moduli space, $\det M = \Lambda^6 + B\tilde{B}$: the involution $M_i \to \Lambda^4/M_i$ exchanges the Dirac and Majorana faces of the theory. In the M-theory lift, both mechanisms are encoded as two regions of the same holomorphic curve Σ, connected by the $\mathbb{Z}_2$ symmetry.

What about quarks? Quarks are elementary preons acquiring mass through partial compositeness. The preon–composite loop closes only for leptons; this is why leptons satisfy standard Koide while quarks satisfy inverse Koide: the two sectors couple to different faces of the adjugate duality.

C M-theory lift: mesons as M2-branes, baryons as M5-branes

The Hanany–Witten (HW) brane construction [?] realises SU(N_c) SQCD in Type IIA string theory using N_c D4-branes suspended between two NS5-branes, with N_f D6-branes providing flavours. When lifted to M-theory, the entire configuration merges into a *single* M5-brane wrapping a holomorphic curve Σ, and Seiberg duality becomes a change of variables on that curve [?].

The curve for SU(3), $N_f = 3$. In the $\mathcal{N} = 1$ limit ($\mu \to \infty$), the M5-brane curve splits into two genus-zero components [?]:

$$C_L : \quad \tilde{t} = (w - M_1)(w - M_2)(w - M_3), \quad v = 0\,, \tag{54}$$

$$C_R : \quad \tilde{t} = \Lambda^6, \quad w = 0\,, \tag{55}$$

where $v = x^4 + ix^5$ and $w = x^8 + ix^9$ are transverse coordinates, $\tilde{t} = e^{-(x^6+ix^{10})/R_{11}}$ encodes the M-theory circle, $M_i = \Lambda^2 P^{1/3}/m_i$ are the meson eigenvalues, and Λ is the dynamical scale.

The quantum constraint from geometry. At $w = 0$, the left component gives $\tilde{t}\big|_{w=0} = -\det M$, while the right is $\tilde{t} = \Lambda^6$. The baryon condensate $B\tilde{B}$ measures the separation:

$$\det M - B\tilde{B} \;=\; \Lambda^6\,. \tag{56}$$

Mesons as M2-branes, baryons as M5-branes.

Field theory	Type IIA	M-theory	Mass type
Meson $M^i{}_j = Q^i\tilde{Q}_j$	open string (D6$_i$–D6$_j$)	**M2-brane**	Dirac
Baryon $B = \epsilon\, Q^3$	wrapped D4 (vertex)	**M5-brane**	Majorana

In M-theory, these two mass mechanisms are the two faces of *electromagnetic duality* of the C-field: the M2-brane couples to C_3, the M5-brane couples to $C_6 = {}^*C_3$ [?].

The involution as M2 $\leftrightarrow$ M5 duality. The $\mathbb{Z}_2$ involution $M_i \to \Lambda^4/M_i$ acts on the curve by exchanging $C_L \leftrightarrow C_R$—the exchange of the C_3 and C_6 descriptions, the *electromagnetic duality* of M-theory.

Why the mass is quadratic: the membrane has two dimensions. With U(3) family symmetry, the D6-brane positions are

$$v_i \;=\; c\,(z_0 + z_i)\,. \tag{57}$$

An M2-brane stretched from the colour stack to the i-th D6-brane spans two spatial directions, so its mass is

$$m_i \;\propto\; \text{Area}_i \;\propto\; (z_0 + z_i)^2\,. \tag{58}$$

For a general p-brane with all dimensions $\propto (z_0 + z_i)$:

p	Object	$m_i \propto$	K
1	string	$(z_0{+}z_i)$	0.478
2	**M2-brane**	$(z_0{+}z_i)^2$	**2/3**
3	3-brane	$(z_0{+}z_i)^3$	0.805
5	M5-brane	$(z_0{+}z_i)^5$	0.945

Only $p = 2$ yields $K = 2/3$: the Koide formula is a *signature of the membrane.*

The number 84 and mass protection. In eleven-dimensional supergravity, the massless little group is SO(9). The bosonic degrees of freedom decompose as $128 = 44 + 84$: the graviton (44 components) plus the three-form C_3 ($\binom{9}{3} = 84$ components). Under $\mathrm{SU}(9) \subset E_8$, the adjoint decomposes as $248 = 80 + 84 + \overline{84}$, with the two 84s corresponding to C_3 and C_6. Under $\mathrm{SU}(9) \to \mathrm{SU}(3)_F \times \mathrm{SU}(3)_L \times \mathrm{SU}(3)_C$ (trinification), the $84 = \Lambda^3(\mathbf{9})$ contains the meson quantum numbers $(\mathbf{3}_F, \overline{\mathbf{3}}_L, \mathbf{1}_C)$ as well as the baryon singlets. The adjoint 80 contains 12 family-singlet degrees of freedom matching the top quark.

In the Standard Model with right-handed neutrinos, the 96 fermionic degrees of freedom split as $84 + 12$ under dual mass-protection symmetries [?]: the family-structured fermions in the antisymmetric $84 = \Lambda^3(\mathbf{9})$, the family-singlet top in the adjoint 80.

Relation to the angle shift. $84 = |\mathrm{disc}(\mathbb{Q}(\sqrt{-21}))|$ appears in the angle-shift formula $\tan(\Delta\alpha) = \pi/84$ (Section **??**). Whether this reflects a deeper connection between $\mathbb{Q}(\sqrt{21})$ and the M-theory C-field is a question for future work.

Top quark as Kaluza–Klein monopole. In the $N_f = 4$ hybrid scenario, the top corresponds to a D6-brane receding to infinity. In M-theory, D6-branes become *Kaluza–Klein monopoles* [?]. The receding $\mathrm{D6}_4$ moves its Taub-NUT centre to spatial infinity; the holomorphic scale matching

$$\Lambda_3^6 \;=\; \Lambda_4^5\, m_t \tag{59}$$

encodes the top's effect. The $12 = N_c \times 4$ real degrees of freedom live in the SU(9) adjoint $\mathbf{80}$, consistent with the $96 = 84 + 12$ mass-protection split.

Kähler non-computability and the angle modulus. The Kähler potential $K(M, M^\dagger)$ of the confined meson fields determines the physical mass matrix and hence α. In an $\mathcal{N} = 2$ theory, the Kähler metric can be extracted from M5-brane periods via the DHOO formula [?]. In the $\mathcal{N} = 1$ s-confined theory this formula fails: (i) the genus-zero curve has no holomorphic one-forms; (ii) the DBI probe suffers string-scale contamination; (iii) the $\mathcal{N} = 1$ Kähler potential is not protected. The non-computability is *consistent* with the failure to select α_{lep} from any scalar potential or instanton effect: the angle is a Kähler modulus, invisible to the holomorphic sector.

Strong CP and the involution as CP. Identifying the involution with CP yields a *Nelson–Barr* structure [?, ?]: the superpotential is real in the CP-symmetric basis, so $\arg\det m_q = 0$ at tree level. The CKM phase arises from spontaneous CP violation. Radiative corrections are two-loop suppressed: $\delta\bar{\theta} \sim (\alpha/4\pi)^2\, J \sim 10^{-11}$, at or below the experimental bound $|\bar{\theta}| < 10^{-10}$.

D Primer: F-terms, D-terms, and SUSY breaking

In any $\mathcal{N} = 1$ supersymmetric theory, the scalar potential is $V = \sum_i |F_i|^2 + \frac{1}{2}\sum_a (D^a)^2$, where

$$F_i = -\frac{\partial W}{\partial \phi_i}\,, \qquad D^a = -g\sum_i \phi_i^\dagger T^a \phi_i + \xi^a\,. \tag{60}$$

SUSY is unbroken iff $V = 0$ at the minimum (i.e., all $F_i = 0$ and all $D^a = 0$). When SUSY breaks, a massless goldstino appears [?]:

$$\tilde{G} \;\propto\; \sum_i F_i\, \psi_i + \sum_a D^a\, \lambda^a\,, \tag{61}$$

which in supergravity is absorbed by the gravitino, $m_{3/2} = |F|/(\sqrt{3}\, M_{\mathrm{Pl}})$.

In the Seiberg confined theory on the mesonic branch ($B = \tilde{B} = 0$), all SUSY breaking comes from $F_M = \partial W/\partial M$ (F-type breaking). The D-terms for $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ enforce *vacuum alignment*: all VEVs point in the same $\mathrm{SU}(2)_L$ direction, preserving $\mathrm{U}(1)_{\mathrm{em}}$. The magnitude hierarchy $v_i \propto m_{\ell,i}$ is set by the F-terms; the direction is aligned by the D-terms.

Field energy and the charge-squared pattern. The holomorphic mass $m_i = k(z_0 + z_i)^2$ is a superpotential parameter, protected by the non-renormalization theorem. The q_i^2 pattern can be understood from the residual $\mathrm{U}(1)_F^2$ gauge structure after confinement. Eliminating the D_F auxiliary fields gives $D_F = -g_F^2 \sum_i q_i |\phi_i|^2$; the resulting positive-definite potential $V_D = (g_F^2/2)(\sum_i q_i|\phi_i|^2)^2$ contributes to the static energy as $\Delta M_i \propto q_i^2$. Combined with Gauss's law ($E_i \propto q_i$) and $\mathcal{L} \supset F_{\mu\nu}^2$, this ensures that the field energy of any finite-size composite state scales as q_i^2—the same charge dependence as the mass ansatz. The overall coefficient is non-perturbative (Kähler-dependent), but the pattern is fixed by gauge theory.

Declaration of generative AI and AI-assisted technologies in the manuscript preparation process

During the preparation of this work the author used Claude (Anthropic) in order to perform algebraic verification, numerical cross-checks, literature searches, and drafting assistance. After using this tool, the author reviewed and edited the content as needed and takes full responsibility for the content of the published article.

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