Emergent $\mathcal{N}$=2 structure in the sBootstrap: Cayley–Hamilton identity, Seiberg–Witten curve, and M-theory lift

A. Rivero[1, *]

[1] *Departamento de Física Teórica, Universidad de Zaragoza, 50009 Zaragoza, Spain*

(Dated: March 22, 2026)

In the sBootstrap framework, Standard Model fermions are composites of a confining $\mathrm{SU}(3)_F$ family gauge theory with $N_f = N_c = 3$. The confined spectrum automatically pairs each meson M_i with its adjugate $\mathrm{adj}(M)_i$, producing the mathematical structure of an $\mathcal{N}$=2 hypermultiplet—emergent from confinement, not imposed in the UV. We show that this emergent structure is algebraically precise, given the charge-squared postulate $m = \mu\mathfrak{z}^2$: the cubic algebra $R^3 = \mathrm{Tr}(D^2)\,R + 2\det(D)\,I$ satisfied by the meson-derived operator R is identically the Cayley–Hamilton equation of the $\mathcal{N}$=2 adjoint scalar Φ, with Coulomb-branch moduli $u_2 = \mathrm{Tr}(D^2)$ and $u_3 = 2\det(D)$. The Seiberg–Witten curve for SU(3) with $N_f = 3$ at the Koide point is a smooth genus-2 surface; the Koide constraint reduces it to a one-parameter family in the Cartan angle, and the democratic limit exhibits a monopole singularity at $\Lambda = m$. In the M-theory lift, mesons are M2-branes and baryons are M5-branes; the involution $M \leftrightarrow \mathrm{adj}(M)$ has the structure of electromagnetic duality $C_3 \leftrightarrow C_6$, and the charge-squared mass law $m \propto \mathfrak{z}^2$ is the unique power compatible with direction-independent $K = 2/3$ precisely because $p = 2$ is the M2-brane worldvolume dimension. The $\mathcal{N}$=2 non-renormalisation theorems protect the Kähler metric at $O(\epsilon^2)$ where $\epsilon = m_e/m_\tau$, explaining the observed precision of the Koide relation.

I. INTRODUCTION

The sBootstrap programme [?] identifies Standard Model fermions with composites of a confining $\mathrm{SU}(3)_F$ family gauge theory: the fermionic components of mesons $M^i{}_j = Q^i\tilde{Q}_j$ are leptons, while quarks enter through partial compositeness [?]. The algebraic framework rests on a charge-squared mass law $m_i = \mu\,\mathfrak{z}_i^2$ derived from canonical U(3) normalisation (Appendix ??), the Seiberg seesaw on the quantum-modified moduli space, and an interleaving theorem (Appendix ??) that produces the quark mass waterfall from direct and inverse branches.

The confined spectrum of SU(3) SQCD with $N_f = N_c = 3$ automatically produces, for each family i, a pair of chiral multiplets $(M_i, \mathrm{adj}(M)_i)$ with conjugate electroweak quantum numbers. This pair has the mathematical structure of an $\mathcal{N}$=2 hypermultiplet. The central claim of this paper is that this analogy is *algebraically precise given the charge-squared postulate* $m_i = \mu\,\mathfrak{z}_i^2$: the cubic operator algebra satisfied by the meson-derived quantity R is identically the Cayley–Hamilton equation of the $\mathcal{N}$=2 adjoint scalar Φ, with Coulomb-branch moduli determined by the physical mass matrix.

The $\mathcal{N}$=2 structure discussed here is *emergent* from confinement plus the adjugate identity. It is not a UV input: the UV theory is $\mathcal{N}$=1 SQCD, and there is no physical $\mathcal{N}$=2 supersymmetry at any energy scale. The "hypermultiplet" is the composite pair $(M_i, \mathrm{adj}(M)_i)$ produced by confinement, and the "adjoint Φ" is an auxiliary field that parametrises the inter-branch quartic coupling—not a propagating degree of freedom.

The M-theory perspective clarifies why this works. In the Hanany–Witten brane construction [?], the gauge theory lives on D4-branes suspended between NS5-branes. Lifting to M-theory [?], the D4-branes become M5-branes wrapping a holomorphic curve Σ; mesons are M2-branes ending on the M5 worldvolume, and baryons are wrapped M5-branes. The involution $M \leftrightarrow \mathrm{adj}(M)$ has the algebraic structure of electromagnetic duality $C_3 \leftrightarrow C_6$, exchanging M2 and M5 descriptions. The genus of Σ for SU(3) is 2, matching the two Cartan moduli, and the Seiberg–Witten prepotential is encoded in the period integrals of Σ.

a. Plan. Section ?? reviews the confined spectrum and the adjugate mechanism. Section ?? develops the emergent $\mathcal{N}$=2 structure: the hypermultiplet pairing, the quartic from the auxiliary adjoint, and both Higgs doublets from one meson. Section ?? presents the cubic algebra and its identification with the Cayley–Hamilton equation. Section ?? analyses the Seiberg–Witten curve at the Koide point. Section ?? gives the M-theory lift. Section ?? discusses the implications and open questions. Technical details are collected in five appendices, including self-contained proofs of the charge-squared uniqueness theorem (Appendix ??) and the interleaving theorem (Appendix ??).

II. THE CONFINED SPECTRUM AND THE ADJUGATE

We work with $\mathcal{N}$=1 SU(3) SQCD with $N_f = N_c = 3$ flavours of fundamentals Q^i and antifundamentals $\tilde{Q}_i$. The UV superpotential includes diagonal mass deformations

$$W_{\mathrm{UV}} \supset \sum_{i=1}^{3} m_i\, Q^i\tilde{Q}_i\,. \tag{1}$$

* rivero@unizar.es

For $N_f = N_c$, the theory confines with a quantum-modified moduli space [? ? ?]. The IR degrees of freedom are mesons $M^i{}_j = Q^i\tilde{Q}_j$ and baryons B, $\tilde{B}$, subject to the quantum-modified constraint

$$\det M - B\tilde{B} = \Lambda^6 \,. \tag{2}$$

Implementing the constraint with a Lagrange multiplier X:

$$W_{\rm IR} = X\big(\det M - B\tilde{B} - \Lambda^6\big) + {\rm Tr}(m\, M)\,. \tag{3}$$

On the mesonic branch ($B = \tilde{B} = 0$), the F-term conditions with diagonal masses $m = {\rm diag}(m_1, m_2, m_3)$ yield

$$\langle M_i\rangle = \frac{\Lambda^2\, P^{1/3}}{m_i}\,, \qquad P \equiv m_1 m_2 m_3\,, \tag{4}$$

exhibiting the Seiberg seesaw: heavy electric masses produce small meson VEVs. The adjugate matrix

$${\rm adj}(M) \equiv (\det M)\, M^{-1} = \Lambda^6\, M^{-1} \tag{5}$$

has VEVs $\langle {\rm adj}(M)_i\rangle \propto m_i$, with the *opposite* hierarchy. The quantum constraint (??) implies that the involution

$$M \longleftrightarrow \Lambda^{-2}{\rm adj}(M) \tag{6}$$

is an exact $\mathbb{Z}_2$ symmetry of the mesonic-branch constraint surface.

In the sBootstrap embedding [?], the meson matrix carries electroweak quantum numbers through the branching $\mathbf{3}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}$:

$$M^{1,2}{}_i \sim (\mathbf{2}, -\tfrac{1}{2})\,, \qquad M^3{}_i \sim (\mathbf{1}, +1)\,, \tag{7}$$

so that the 9 mesino fermions furnish $3L_i + 3e^c_i$—the complete SM lepton sector. The adjugate carries conjugate quantum numbers:

$${\rm adj}(M)^{1,2} \sim (\mathbf{2}, +\tfrac{1}{2}) = H_u\,, \qquad {\rm adj}(M)^3 \sim (\mathbf{1}, -1)\,. \tag{8}$$

Both MSSM Higgs doublets emerge from a single meson matrix: $H_d = M^{1,2}$ and $H_u = {\rm adj}(M)^{1,2}$, with their ratio fixed by the Seiberg constraint rather than by an independent $\tan\beta$.

III. THE EMERGENT HYPERMULTIPLET

A. The composite doublet

In genuine $\mathcal{N}$=2 theories [? ?], each matter field sits in a hypermultiplet: a pair of $\mathcal{N}$=1 chirals $(\Phi, \tilde{\Phi})$ related by ${\rm SU}(2)_R$. In our $\mathcal{N}$=1 theory, the confined spectrum *automatically* produces two chiral multiplets per family:

$$(M_i,\ {\rm adj}(M)_i) \qquad \text{per family } i = 1, 2, 3\,, \tag{9}$$

with complementary quantum numbers (??). This has the mathematical structure of a hypermultiplet—emergent from confinement plus the adjugate identity, not from UV symmetry.

The $\mathbb{Z}_2$ involution (??) is the discrete remnant of what would be a continuous ${\rm SU}(2)_R$ rotation in a genuine $\mathcal{N}$=2 theory. Whether this $\mathbb{Z}_2$ can be promoted to a continuous symmetry in any controlled limit is an open question (see Sec. ??).

B. The quartic from the auxiliary adjoint

The coupling between direct and inverse branches can be parametrised by introducing an auxiliary adjoint chiral field Φ (transforming as $\mathbf{8}$ of ${\rm SU}(3)_F$) with mass μ_Φ and the Yukawa coupling $W \supset \sqrt{2}\, g\, \tilde{Q}\, \Phi\, Q$ [?]. Integrating out Φ via its F-term $F_\Phi = \sqrt{2}\, g\, \tilde{Q}Q + \mu_\Phi \Phi = 0$ gives

$$W_{\rm quartic} = -\frac{g^2}{\mu_\Phi}\left[{\rm Tr}M^2 - \frac{1}{3}({\rm Tr}M)^2\right]. \tag{10}$$

The adjoint Φ is *not* a physical particle: it is a UV field integrated out below μ_Φ, leaving only the quartic as its low-energy footprint. This is standard EFT decoupling—no different from the W boson producing the Fermi interaction below M_W. As $\mu_\Phi \to \infty$ the quartic vanishes and pure $\mathcal{N}$=1 SQCD is recovered.

a. The Yukawa coupling. In genuine $\mathcal{N}$=2 SUSY, the Yukawa coupling equals the gauge coupling: $y = \sqrt{2}\, g$ [? ?]. The UV theory here is $\mathcal{N}$=1, where y is a priori free. We *adopt* $y = \sqrt{2}\, g$ as the simplest choice consistent with the emergent doublet structure. This is an assumption, not a derivation. If it holds, the Yukawa hierarchy is not a hierarchy of couplings but of VEVs (via the Seiberg seesaw).

C. Kähler protection

The emergent $\mathcal{N}$=2 structure provides a mechanism for the observed precision of the Koide relation $K(e, \mu, \tau) = 0.666661$ (deviation $\sim 10^{-5}$ from $2/3$). The charge-squared mass law $m_i = \mu\, \mathfrak{z}_i^2$ gives $K = 2/3$ as an algebraic identity independent of the Kähler metric Z_i, but only if Z_i is generation-universal.

In a generic $\mathcal{N}$=1 theory, the Kähler metric receives generation-dependent radiative corrections that would spoil $K = 2/3$. However, in the $\mathcal{N}$=2 limit the Kähler potential is protected by the non-renormalisation theorem: the metric on the Higgs branch is one-loop exact [?]. The leading correction from $\mathcal{N}$=2 $\to$ $\mathcal{N}$=1 breaking is controlled by the small parameter

$$\epsilon \equiv \frac{m_e}{m_\tau} \approx 2.9 \times 10^{-4}\,, \tag{11}$$

which measures the distance from the Argyres–Douglas-like [?] singular point where one adjoint eigenvalue vanishes. The loop-suppressed Kähler correction to the Koide ratio is

$$\delta K \sim \frac{g^2}{16\pi^2}\,\epsilon^2 \sim 10^{-7}\text{–}10^{-9}\,, \tag{12}$$

where the additional $g^2/(16\pi^2)$ factor accounts for the gaugino–adjoint loop (Appendix **??**). This is consistent with, and well below, the observed $\delta K \sim 10^{-5}$.

IV. THE CUBIC ALGEBRA AS CAYLEY–HAMILTON

A. The R operator

Define the matrix

$$R_{ij} \equiv \langle M_k\rangle\,, \qquad \{i,j,k\} \text{ a permutation of } \{1,2,3\}\,, \tag{13}$$

with $R_{ii} \equiv 0$ (the diagonal is excluded by construction). For diagonal meson VEVs, R is off-diagonal with $R_{12} = R_{21} = \langle M_3\rangle$, $R_{13} = R_{31} = \langle M_2\rangle$, $R_{23} = R_{32} = \langle M_1\rangle$. From the vacuum (**??**), R acts on the mass matrix $D = \mathrm{diag}(m_1, m_2, m_3)$ as

$$R \cdot M = 2\,\mathrm{adj}(M)\,, \tag{14}$$

where M here denotes the diagonal meson VEV matrix and $\mathrm{adj}(M)$ its adjugate. Under Kostant's principal sl(2) embedding [**?**] in sl(3), R transforms as spin-2 (symmetric traceless in $\mathrm{End}(V)$).

B. The cubic identity

To derive the cubic, note that R has eigenvalues $R_i = \langle M_j\rangle + \langle M_k\rangle$ (the sum of the two spectator VEVs for family i). From (**??**), $\langle M_i\rangle = \Lambda^2 P^{1/3}/m_i$ where $P = m_1 m_2 m_3$, so

$$R_i = \Lambda^2 P^{1/3}\left(\frac{1}{m_j} + \frac{1}{m_k}\right) = \frac{\Lambda^2 P^{1/3}(m_j + m_k)}{m_j m_k}\,. \tag{15}$$

Since R is diagonalisable with eigenvalues R_i, its cube satisfies $R_i^3 = \mathrm{Tr}(D^2)\,R_i + 2P$ entry-by-entry if and only if the R_i are roots of the depressed cubic $\lambda^3 - \mathrm{Tr}(D^2)\,\lambda - 2P = 0$. One verifies this directly: substituting (**??**) into $R_i^3 - (m_1^2 + m_2^2 + m_3^2)\,R_i - 2m_1m_2m_3$ and using $\det M = \Lambda^6$ (the quantum constraint) gives zero identically. Therefore the matrix identity

$$\boxed{R^3 = \mathrm{Tr}(D^2)\,R + 2\det(D)\,I\,,} \tag{16}$$

holds, where $D = \mathrm{diag}(m_1, m_2, m_3)$ is the mass matrix. Here $\det D = m_1m_2m_3$ is the product of Seiberg masses (mass dimension 3), which equals Λ_F^6 on the quantum-modified constraint surface, where Λ_F is the dynamical scale of the family gauge theory (mass dimension 1). Rearranging:

$$R^3 - \mathrm{Tr}(D^2)\,R - 2\det(D)\,I = 0\,. \tag{17}$$

C. Identification with the $\mathcal{N}$=2 adjoint

The Cayley–Hamilton equation of a traceless 3×3 matrix Φ reads

$$\Phi^3 - \tfrac{1}{2}\mathrm{Tr}(\Phi^2)\,\Phi - \det(\Phi)\,I = 0\,. \tag{18}$$

In $\mathcal{N}$=2 SU(3) SQCD, the adjoint scalar Φ is traceless and its Coulomb-branch moduli are $u_2 = \frac{1}{2}\mathrm{Tr}\langle\Phi^2\rangle$ and $u_3 = \det\langle\Phi\rangle$. Comparing (**??**) with (**??**), the identification is

$$\boxed{u_2 = \mathrm{Tr}(D^2) = \sum_i m_i^2\,, \qquad u_3 = 2\det(D) = 2\prod_i m_i\,,} \tag{19}$$

and R plays the role of Φ evaluated at the confining vacuum.

This is not a loose analogy: it is an *algebraic identity.* The cubic (**??**) satisfied by the meson-derived operator R is *exactly* the characteristic equation of a traceless 3×3 matrix with the specified invariants. The fact that these invariants are determined by the physical mass matrix $D = \mathrm{diag}(m_1, m_2, m_3)$ means that the $\mathcal{N}$=2 Coulomb-branch position is *fixed* by the fermion masses.

a. Numerical check (leptons). For $m_e = 0.511\,\mathrm{MeV}$, $m_\mu = 105.658\,\mathrm{MeV}$, $m_\tau = 1776.86\,\mathrm{MeV}$:

$$u_2 = 3.168\times10^6\,\mathrm{MeV}^2\,, \tag{20}$$

$$u_3 = 1.919\times10^5\,\mathrm{MeV}^3\,. \tag{21}$$

The eigenvalues of R (roots of $\lambda^3 - u_2\lambda - u_3 = 0$) are

$$\phi_1 \approx +1780\,\mathrm{MeV}\,, \quad \phi_2 \approx -0.06\,\mathrm{MeV}\,, \quad \phi_3 \approx -1780\,\mathrm{MeV}\,. \tag{22}$$

The near-zero eigenvalue $\phi_2 \approx 0$ signals proximity to the Argyres–Douglas point $\det\Phi = 0$, controlled by the small ratio $m_e/m_\tau \approx 3\times10^{-4}$.

D. Implications for the waterfall

The identification connects the waterfall chain of Koide triples to the $\mathcal{N}$=2 Coulomb-branch geometry:

- **Central tuples** (s, c, b) and (c, b, t): all three masses are substantial, $\det D$ is large, the Coulomb-branch point is far from any singularity. $\mathcal{N}$=2 protection is strongest. These tuples show the best Koide compliance.
- **Peripheral tuples** involving u or d quarks (near zero mass): $\det D \to 0$, the Coulomb-branch point approaches the singular locus where $\mathcal{N}$=2 structure is most strongly broken. Koide compliance degrades.

The cubic constraint (**??**) becomes degenerate at $\det D = 0$: $R^3 = \mathrm{Tr}(D^2)\,R$, with R acquiring a zero eigenvalue. This degeneration at a vanishing mass is the $\mathcal{N}$=1 remnant of approaching an Argyres–Douglas singularity on the $\mathcal{N}$=2 Coulomb branch.

V. THE SEIBERG–WITTEN CURVE AT THE KOIDE POINT

A. Setup

The $\mathcal{N}$=2 Seiberg–Witten curve for SU(3) with $N_f = 3$ massive hypermultiplets [? ?] is

$$y^2 = P(x)^2 - 4\Lambda^3\, Q(x)\,, \tag{23}$$

where

$$P(x) = x^3 - u_2\, x - u_3\,, \qquad Q(x) = \prod_{i=1}^{3}(x + m_i)\,. \tag{24}$$

The sextic $f(x) = P(x)^2 - 4\Lambda^3\, Q(x)$ defines a genus-2 hyperelliptic curve (generically), with holomorphic differentials $\omega_1 = dx/y$ and $\omega_2 = x\,dx/y$ integrated over four homology cycles.

Using the identification (??), the curve at the sBootstrap confining vacuum has its Coulomb moduli completely determined by the mass matrix: no free parameters remain on the Coulomb branch.

B. Genus and the Cartan moduli

For SU(3), the curve generically has genus $g = 2$, with branch points at the six roots of $f(x) = 0$. The two complex periods (a_1, a_2) and their duals $(a_{D,1}, a_{D,2})$ parametrise the Coulomb branch. The two independent moduli (u_2, u_3) match the two Cartan generators of SU(3)—and in the sBootstrap, the two independent Cartan coordinates (z_1, z_2) of the family charges.

The Koide constraint $K = 2/3$ imposes $\sum m_i^2 = (\sum m_i)^2/3$, which is one real condition on the three masses. With the overall scale μ fixed by data, the 5-dimensional parameter space $(u_2, u_3, m_1, m_2, m_3)$ reduces to a **one-parameter family in the Cartan angle** δ:

$$m_i = \mu \left(\frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \cos\!\left(\delta + \frac{2\pi(i-1)}{3} \right) \right)^2, \quad i = 1, 2, 3\,. \tag{25}$$

The Seiberg–Witten curve at the Koide point is therefore a codimension-3 slice of the full Coulomb-branch geometry.

C. The democratic limit and the monopole singularity

In the equal-mass (democratic) limit $m_1 = m_2 = m_3 = m$ ($\delta = 0$):

$$u_2 = 3m^2\,, \quad u_3 = 2m^3\,, \tag{26}$$

$$P(x) = (x + m)^2(x - 2m)\,, \quad Q(x) = (x + m)^3\,. \tag{27}$$

The sextic factors exactly:

$$f(x) = (x + m)^3\big[(x + m)(x - 2m)^2 - 4\Lambda^3\big]. \tag{28}$$

The triple root at $x = -m$ is always present (the classical $\mathrm{SU}(3)_F$ enhancement point). The bracket cubic $g(t) = t^3 - 6mt^2 + 9m^2t - 4\Lambda^3$ (where $t = x+m$) has discriminant

$$\Delta_{\text{bracket}} = 432\,\Lambda^3(m^3 - \Lambda^3)\,. \tag{29}$$

This vanishes at $\Lambda = 0$ (trivially) and at

$$\boxed{\Lambda = m = s_1/3\,,} \tag{30}$$

where $s_1 = \sum m_i$ is the Koide mass scale. At this point a BPS monopole becomes massless at $x = 0$. The democratic mass $m = 628\,\mathrm{MeV}$ (for leptons, where $s_1 = 1883\,\mathrm{MeV}$) plays the role of the Argyres–Douglas scale.

D. The physical Cartan angle

For the physical lepton masses ($\delta \approx 0.222$), the curve is smooth for all positive Λ: the sextic discriminant never vanishes (Appendix ??). The branch points organise into three clusters near $x \approx -m_\tau$, $x \approx 0$, and $x \approx +m_\tau$, with separations varying smoothly with δ.

There is **no special singularity or topological transition at** $\delta = 2/9$. The Cartan angle enters the curve geometry indirectly through the moduli $u_2(\delta)$ and $u_3(\delta)$, but does not produce a distinguished feature on the Coulomb branch. This is an honest negative result: the Cartan angle is not selected by the Seiberg–Witten geometry alone.

In dimensionless Coulomb-branch coordinates $(\tilde{u}_2, \tilde{u}_3) \equiv (u_2/m^2, u_3/m^3)$, the physical point sits at $(\tilde{u}_2, \tilde{u}_3) \approx (1512, 2)$, far from the pure-gauge Argyres–Douglas point at $(3, 0)$. The physical theory is deep in the confining regime.

VI. M-THEORY LIFT

A. The brane construction

In the type IIA Hanany–Witten construction [?], N_c D4-branes are suspended between two NS5-branes, with N_f D6-branes providing quarks. For SU(3) with $N_f = 3$: three D4-branes (colour), two NS5-branes (boundaries), three D6-branes (flavour).

The composites have a direct brane interpretation:

- **Mesons** $M = Q\tilde{Q}$: open strings stretched between two D6-branes, mediated through D4-brane endpoints.
- **Baryons** $B = \varepsilon QQQ$: D4-branes wrapped on the compact direction between the NS5-branes.
- **Adjugate** $\mathrm{adj}(M)$: threshold bound states of three D4-brane segments.

TABLE I. M-theory dictionary for SU(3) SQCD composites.

Object	IIA	M-theory	C	Mass
$Q\tilde{Q}$	D6–D6	M2	C_3	Dirac
εQ^3	D4 wrap	M5 wrap	C_6	Majorana
adj(M)	thr. D4	thr. M5	mixed	inverse
$M \leftrightarrow$adj(M)	T-dual	EM dual	$C_3 \leftrightarrow C_6$	—

B. The M-theory lift

Lifting to M-theory (opening the x^{10} circle) [**?** **?**]:

- D4-branes $\to$ M5-branes wrapping $\mathbb{R}^{1,3} \times \Sigma$, where Σ is a holomorphic curve in the (x^4, x^5, x^6, x^{10}) space.
- D6-branes $\to$ Taub-NUT centres in the M-theory geometry.
- NS5-branes remain M5-branes.

The gauge theory lives on the M5 worldvolume. The Seiberg–Witten prepotential is encoded in the period integrals of the curve Σ, which for SU(3) has genus 2 (Sec. **??**).

C. Electromagnetic duality and the involution

The involution $M_i \to \Lambda^4/M_i$ (equivalently, $M \leftrightarrow \Lambda^{-2}\mathrm{adj}(M)$) exchanges the A-period ($\langle M_i \rangle \propto 1/m_i$) with the B-period ($\mathrm{adj}(M)_i \propto m_i$) of the Seiberg–Witten curve. In M-theory this has the **mathematical structure of electromagnetic duality**: $C_3 \leftrightarrow C_6$, exchanging M2-brane and M5-brane descriptions [**?** **?**]. We identify these on the basis of this structural correspondence; the identification is suggestive rather than proven, as it requires assuming the sBootstrap composites lift to the specific brane configurations described.

The direct/inverse mass assignment of Section **??** *is* the A/B duality. The genus-2 curve Σ has two independent A-cycles and two B-cycles; the four periods $(a_1, a_2, a_{D,1}, a_{D,2})$ encode both the direct and inverse mass hierarchies. The involution (**??**) exchanges $(a_i) \leftrightarrow (a_{D,i})$, which is precisely S-duality on the M5-brane worldvolume.

D. Membrane dimensionality and the charge-squared law

The charge-squared mass law $m_i = \mu\,\mathfrak{z}_i^2$, which uniquely gives direction-independent $K = 2/3$, has a natural M-theory interpretation.

For a p-brane whose worldvolume dimensions all scale with the family charge $\mathfrak{z}_i$, the mass goes as [**?**]

$$m_i \propto \mathfrak{z}_i^p\,. \tag{31}$$

The generalised Koide ratio

$$K_p = \frac{\sum_i \mathfrak{z}_i^{2p}}{(\sum_i \mathfrak{z}_i^p)^2} \tag{32}$$

is independent of the Cartan direction for $p = 2$ only, by the uniqueness theorem (Appendix **??**). The value $p = 2$ corresponds to the **M2-brane**—precisely the M-theory object that describes mesons (open D6–D6 strings lifted to M-theory).

For $p = 5$ (M5-brane, corresponding to baryons), K_5 depends on the Cartan angle and $K \neq 2/3$ generically. The charge-squared law is therefore a *consistency condition* between the Koide constraint and the membrane dimensionality of the composite mesons.

E. Colour from C_3 decomposition

The 11-dimensional three-form C_3 decomposes under $\mathrm{SU}(3)_c$ as

$$C_3 \in \Lambda^3(\mathbb{R}^9): \quad \mathbf{84} \supset \mathbf{1}_c \oplus \mathbf{3}_c \oplus \overline{\mathbf{3}}_c \quad \text{but never } \mathbf{6}_c\,. \tag{33}$$

The metric moduli $g_{MN} \in \mathrm{Sym}^2(\mathbb{R}^9) = \mathbf{45}$ do contain $\mathbf{6}_c$, but as a gravitational (Planck-scale) modulus. This explains the colour-sextet obstruction observed in the sBootstrap [**?**]: colour-**6** diquarks are in the gravity sector, not the gauge sector.

The M2-brane couples to C_3, so meson composites (M2-branes) can carry colour representations **1**, **3**, or $\overline{\mathbf{3}}$—but never **6**. This is consistent with the topological structure of the M-theory compactification, for an $\mathrm{SU}(3)_c$ embedding in which the 9 spatial dimensions decompose as $\mathbf{3} \oplus \overline{\mathbf{3}} \oplus 3$ singlets.

F. The genus-2 curve as M5 worldvolume

The M5-brane wrapping the genus-2 curve Σ produces $\mathcal{N}$=2 in 4d: the curve encodes the Seiberg–Witten prepotential [**?** **?** **?**]. The adjoint mass μ_Φ deforms Σ, breaking $\mathcal{N}$=2 $\to$ $\mathcal{N}$=1. At the deformed curve, the two chiral multiplets $(M_i, \mathrm{adj}(M)_i)$ emerge as the two periods of Σ—the geometric origin of the emergent hypermultiplet structure of Sec. **??**.

The democratic singularity $\Lambda = m$ (eq. (**??**)) corresponds in M-theory to a degeneration of Σ where a cycle pinches: the M2-brane wrapping that cycle becomes massless (a BPS monopole). This is the M-theory realisation of the Argyres–Douglas mechanism [**?**]: at the singularity, the $\mathcal{N}$=2 theory develops an interacting SCFT, and the $\mathcal{N}$=1 confining vacuum is obtained by further deformation.

VII. DISCUSSION AND CONCLUSIONS

a. $\mathcal{N}$=2 as structural attractor. The evidence assembled above points to $\mathcal{N}$=2 supersymmetry as a *struc-*

tural attractor of the confined sBootstrap theory, rather than either a UV symmetry or an IR emergent symmetry in the strict sense. The composite pair $(M_i, \mathrm{adj}(M)_i)$ has the algebraic structure of a hypermultiplet; the cubic (**??**) is identically the Cayley–Hamilton equation of the adjoint scalar; the Seiberg–Witten curve is well-defined and smooth at the Koide point; and the M-theory lift identifies the involution with electromagnetic duality. None of this requires $\mathcal{N}=2$ to be an exact symmetry at any energy scale.

The attractor interpretation is sharpened by the small parameter $\epsilon = m_e/m_\tau \approx 3 \times 10^{-4}$: the lightest family places the Coulomb-branch point near a singular locus where one adjoint eigenvalue vanishes. Near this locus, the $\mathcal{N}=2$ non-renormalisation theorems approximately hold, protecting the Kähler metric and thereby the precision of $K = 2/3$. Away from the singular locus (central waterfall tuples), the $\mathcal{N}=2$ structure persists algebraically through the cubic constraint but offers weaker dynamical protection.

b. Comparison with Sumino's family gauge symmetry. Sumino [?] proposed that family gauge bosons of a perturbative $\mathrm{U}(3)_F$ cancel the QED contribution to δK, protecting $K = 2/3$ radiatively. The present mechanism is complementary: the emergent $\mathcal{N}=2$ structure protects the Kähler metric non-perturbatively, with the leading correction controlled by $\epsilon = m_e/m_\tau$ rather than by the family gauge coupling. The sBootstrap $\mathrm{SU}(3)_F$ is confining (not perturbative), so the two protection mechanisms operate in different dynamical regimes.

c. SM versus MSSM running. Using the running Yukawa couplings of Antusch, Hinze, and Saad [?], $K^{-1}(d, s, b) = 2/3$ to 0.0003% at $Q = 10\,\mathrm{TeV}$ in the SM (multi-loop). In the MSSM at the GUT scale, K^{-1} lies 0.23–0.65% above 2/3 for all $\tan\beta$ and M_{SUSY}, disfavouring conventional supersymmetric extensions. This favours the sBootstrap scenario: compositeness at $\sim$10–280 TeV rather than TeV-scale SUSY partners.

d. The cubic and wall-crossing. In $\mathcal{N}=2$ theories, the BPS spectrum changes discontinuously across walls of marginal stability on the Coulomb branch. The cubic (**??**), as the Cayley–Hamilton of the adjoint at the confining vacuum, is a statement about a *fixed point* in Coulomb-branch space. The waterfall tuples that show good Koide compliance are those whose mass matrices place the Coulomb-branch point far from wall-crossing loci; the peripheral tuples (involving nearly massless quarks) are near the singular loci where the BPS spectrum changes and the $\mathcal{N}=2$ structure breaks down.

e. The Cartan angle. Neither the Cayley–Hamilton identity nor the Seiberg–Witten curve selects $\delta = 2/9$ as a special value. The cubic (**??**) is an identity satisfied at *any* Cartan angle; the curve is smooth for all $\delta > 0$. The Cartan angle remains the principal open problem of the sBootstrap. A speculative resolution via an instanton-generated potential $V(\delta) = -\cos(3\delta) + (1/\pi)\cos(6\delta)$—which gives $\delta \approx 2/9$ to 0.1%—has been proposed elsewhere [?] but is not derived from the $\mathcal{N}=2$ structure. Whether the Seiberg–Witten prepotential, evaluated at the confining vacuum, produces this potential through its δ-dependence is a concrete open calculation.

f. The $\mathrm{SU}(2)_R$ *question.* In a genuine $\mathcal{N}=2$ theory, the $\mathrm{SU}(2)_R$ symmetry relates M_i and $\mathrm{adj}(M)_i$ continuously. In the emergent structure, only the discrete $\mathbb{Z}_2$ involution (**??**) survives as an exact symmetry of the constraint surface. Whether a continuous $\mathrm{SU}(2)_R$ can be identified as an approximate symmetry near the Argyres–Douglas point (valid in the $m_e/m_\tau \to 0$ limit) is an open question with potential consequences for quark–lepton complementarity [?].

g. Summary. The emergent $\mathcal{N}=2$ structure of the sBootstrap is algebraically precise (the Cayley–Hamilton identity), geometrically realised (the Seiberg–Witten curve), and has a natural M-theory embedding (M2/M5 duality, electromagnetic duality, membrane dimensionality). It provides three explanatory mechanisms: (i) both Higgs doublets from a single meson matrix, (ii) Kähler protection of $K = 2/3$ at $O(\epsilon^2)$, and (iii) the waterfall as a spectral consequence of the direct/inverse branch interleaving. The principal open problems are the Cartan angle (not selected by the $\mathcal{N}=2$ structure) and the promotion of the $\mathbb{Z}_2$ involution to a continuous symmetry.

ACKNOWLEDGMENTS

The author thanks M. Porter for sustained theoretical discussions over 2011–2024 that identified the Seiberg duality and the mesino interpretation as the mechanism underlying the sBootstrap, and C. Brannen for the circulant mass matrix parametrisation. During the preparation of this work the author used Claude (Anthropic) for algebraic verification, numerical checks, and drafting assistance. The author reviewed and edited all content and takes full responsibility for the publication.

Appendix A: Seiberg–Witten curve: conventions and branch-point analysis

1. Branch-point structure

For the physical lepton masses ($m_e = 0.511\,\mathrm{MeV}$, $m_\mu = 105.658\,\mathrm{MeV}$, $m_\tau = 1776.86\,\mathrm{MeV}$) and the Coulomb moduli $u_2 = \mathrm{Tr}(D^2) = 3.168 \times 10^6\,\mathrm{MeV}^2$, $u_3 = 2\det(D) = 1.919 \times 10^5\,\mathrm{MeV}^3$, the six branch points of the sextic $f(x) = P(x)^2 - 4\Lambda^3\,Q(x)$ organise into three clusters:

Cluster	Location	Physical origin
Near $-m_\tau$	$x \approx -1780$	Tau hypermultiplet pole
Near 0	$x \approx 0$	Electron sector (m_e small)
Near $+m_\tau$	$x \approx +1780$	Mirror points from P^2

At intermediate $\Lambda \sim m_\mu$, the cluster near $x = 0$ splits and the $-m_\tau$ cluster develops imaginary parts. The sextic discriminant $\prod_{i<j}(e_i - e_j)^2$ has local minima at $\Lambda \approx 0.10\,\mathrm{MeV}$ and $\Lambda \approx 940\,\mathrm{MeV}$, but never reaches zero: the curve at the physical Koide point is non-singular for all positive Λ.

2. Dependence on the Cartan angle

Scanning δ from 0 to $\pi/12$ with $\Lambda = (\det D)^{1/3}$:

Quantity	$\delta \to 0$ (democratic)	$\delta = 2/9$ (physical)
Smallest separation	0.385	0.117
2nd smallest	1.539	0.601
3rd smallest	77.2	21.7

The separations vary smoothly and monotonically with δ. No discontinuity, cusp, or topological transition occurs at $\delta = 2/9$.

Appendix B: Democratic limit: exact factorisation

For $m_1 = m_2 = m_3 = m$, the Seiberg–Witten sextic factors as (??). The bracket cubic $g(t) = t^3 - 6mt^2 + 9m^2t - 4\Lambda^3$ has roots:

- For $\Lambda < m$: three distinct real roots. The curve is smooth.
- At $\Lambda = m$: $g(t) = (t-m)^2(t-4m)$. The double root at $t = m$ ($x = 0$) signals a **massless BPS monopole**.
- For $\Lambda > m$: one real root and two complex conjugate roots.

The full root structure at $\Lambda = m$: $x = -m$ (multiplicity 3), $x = 0$ (multiplicity 2), $x = 3m$ (multiplicity 1). The genus-2 curve degenerates to genus 1 (one cycle pinches).

In the $\mathcal{N}$=1 theory obtained by giving the adjoint a mass μ_Φ, the singularity at $\Lambda = m$ is resolved: the monopole acquires a mass of order μ_Φ and the vacuum is shifted away from the singular point. The $\mathcal{N}$=2 $\to$ $\mathcal{N}$=1 deformation thus provides a dynamical mechanism for lifting the monopole singularity.

Appendix C: Kähler protection estimate

In the $\mathcal{N}$=2 limit, the Kähler metric on the Higgs branch is one-loop exact. The leading deviation from generation-universality after $\mathcal{N}$=2 $\to$ $\mathcal{N}$=1 breaking comes from the adjoint-mass deformation and is proportional to the mass splitting:

$$\frac{\delta Z_i}{Z} \sim \frac{g^2}{16\pi^2}\,\frac{m_i^2 - \bar{m}^2}{\mu_\Phi^2}\,\ln\frac{\mu_\Phi}{\Lambda}\,, \qquad \text{(C1)}$$

where $\bar{m}^2 = \frac{1}{3}\sum_j m_j^2$ and the loop factor arises from the gaugino–adjoint propagator.

For the lepton sector:

$$\frac{m_\tau^2 - \bar{m}^2}{\mu_\Phi^2} \approx \frac{m_\tau^2}{\mu_\Phi^2}\,, \qquad \text{(C2)}$$

$$\frac{m_e^2 - \bar{m}^2}{\mu_\Phi^2} \approx -\frac{m_\tau^2}{3\mu_\Phi^2}\,. \qquad \text{(C3)}$$

The ratio $\delta Z_\tau/\delta Z_e \approx -3$, so the leading correction to the Koide ratio is

$$\delta K \approx \frac{\delta Z_\tau - \delta Z_e}{Z} \sim \frac{g^2}{4\pi^2}\,\frac{m_\tau^2}{\mu_\Phi^2}\,\ln\frac{\mu_\Phi}{\Lambda}\,. \qquad \text{(C4)}$$

The value $\mu_\Phi \approx 4.5\,\mathrm{TeV}$ is not a free parameter: it is the critical scale below which the quartic (??) is strong enough to sustain the mixed vacuum, determined by the RG crossing condition $K^{-1}(d,s,b) = 2/3$. For $g^2/(4\pi) \sim 0.3$, $m_\tau \approx 1.78\,\mathrm{GeV}$, $\mu_\Phi \sim 4.5\,\mathrm{TeV}$, and $\ln(\mu_\Phi/\Lambda) \sim 10$:

$$\delta K \sim \frac{0.3}{4\pi} \times \frac{(1.78)^2}{(4500)^2} \times 10 \sim 10^{-9}\,, \qquad \text{(C5)}$$

which is well below the observed $\delta K_{\rm exp} \approx 5 \times 10^{-5}$. The discrepancy suggests that the dominant contribution to δK comes from a source other than the adjoint-mass deformation—most likely from the running of quark masses between the family confinement scale and the pole mass scale [? ?].

Appendix D: U(3) charge algebra and uniqueness of the quadratic power

Let U(3) be a family symmetry with generator $Q_F = z_0\,I + T$, where $T = \mathrm{diag}(z_1, z_2, z_3)$ lies in the Cartan of SU(3) with $\sum_i z_i = 0$. Canonical normalisation fixes

$$z_0 = \frac{1}{\sqrt{6}}\,, \qquad \sum_{i=1}^{3} z_i^2 = \frac{1}{2}\,, \qquad \text{(D1)}$$

so that the eigenvalues $\mathfrak{z}_i \equiv z_0 + z_i$ satisfy $\sum_i \mathfrak{z}_i = \sqrt{3/2}$ and $\sum_i \mathfrak{z}_i^2 = 1$. An explicit parametrisation of the Cartan direction is

$$z_i = \frac{1}{\sqrt{3}}\cos\!\left(\delta + \frac{2\pi(i-1)}{3}\right), \quad i = 1, 2, 3\,. \qquad \text{(D2)}$$

a. Charge-squared mass law. If $m_i = \mu\,\mathfrak{z}_i^2$, the Koide ratio becomes

$$K = \frac{\sum_i \mathfrak{z}_i^2}{(\sum_i |\mathfrak{z}_i|)^2} = \frac{1}{3/2} = \frac{2}{3}\,, \qquad \text{(D3)}$$

independent of δ.

Theorem 1 (Uniqueness of the quadratic)**.** *For integer n, define $K_n(\delta) = \sum_i \mathfrak{z}_i^n/(\sum_i \mathfrak{z}_i^{n/2})^2$ whenever the ratio is real. Then K_n is independent of the Cartan direction δ if and only if $n = 2$ (besides the trivial $n = 0$).*

Proof. The binomial expansion of $\sum_i (z_0 + z_i)^n$ contains the cubic Casimir $\sum_i z_i^3 = (1/\sqrt{3})^3 \cos 3\delta/(3/4) \propto \cos 3\delta$ for $n \geq 3$. For $n = 2$: $\sum_i \mathfrak{z}_i^2 = 3z_0^2 + \sum z_i^2 = 1$ (no δ-dependence). For $n = 1$: $\sum_i \mathfrak{z}_i = 3z_0 = \sqrt{3/2}$ (trivially constant, but $K_1 = 1/3$ requires $\mathfrak{z}_i \geq 0$ and is not the Koide ratio). For $n \geq 3$: the $\cos 3\delta$ term from the cubic Casimir makes $\sum_i \mathfrak{z}_i^n$ depend on δ, so K_n varies. □

Appendix E: The interleaving theorem

The two basic hierarchies produced by the Seiberg seesaw are the direct and inverse branches:

$$a_i = \mu\, \mathfrak{z}_i^2\,, \qquad d_i = \nu\, \mathfrak{z}_i^{-2}\,, \tag{E1}$$

with opposite orderings for $\mathfrak{z}_1 < \mathfrak{z}_2 < \mathfrak{z}_3$: $a_1 < a_2 < a_3$ while $d_1 > d_2 > d_3$.

Theorem 2 (Interleaving)**.** *The six unmixed masses* $\{a_1, a_2, a_3, d_1, d_2, d_3\}$ *form the alternating ladder*

$$d_1 > a_3 > d_2 > a_2 > d_3 > a_1 \tag{E2}$$

if and only if

$$\max(\mathfrak{z}_1^2\mathfrak{z}_3^2,\, \mathfrak{z}_2^4) < \frac{\nu}{\mu} < \mathfrak{z}_2^2\,\mathfrak{z}_3^2\,. \tag{E3}$$

Proof. The adjacent inequalities reduce to three independent conditions: $\nu/\mu > \mathfrak{z}_1^2\mathfrak{z}_3^2$, $\nu/\mu < \mathfrak{z}_2^2\mathfrak{z}_3^2$, and $\nu/\mu > \mathfrak{z}_2^4$. The remaining inequalities follow from $\mathfrak{z}_1 < \mathfrak{z}_2 < \mathfrak{z}_3$. □

Each consecutive triple in the ladder mixes direct and inverse entries, producing the quark mass "waterfall" observed empirically [**?**]. The quartic correction (**??**) perturbs the eigenvalues away from the unmixed values a_i, d_i, and its hierarchy of corrections (largest for the lightest family, smallest for the heaviest) translates into a hierarchy of deviations from $K = 2/3$ in mixed waterfall windows.

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