What is the inverse Koide formula?

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Abstract

We give a pedestrian introduction to the "inverse Koide formula" for quark masses, explain why it exists alongside the original lepton formula, and show that both follow from a single algebraic mechanism in a confining gauge theory. Along the way we encounter an operator that maps leptons to quarks, a potential that stubbornly prefers democracy, and a 65-year-old theorem by Kostant that selects three families of matter.

1 The original Koide formula (1983)

In 1983, Yoshio Koide noticed something peculiar about the charged lepton masses [?]. Define

$$K \;=\; \frac{m_e + m_\mu + m_\tau}{\left(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau}\right)^2}\,. \tag{1}$$

Plug in the numbers ($m_e = 0.511\,\mathrm{MeV}$, $m_\mu = 105.66\,\mathrm{MeV}$, $m_\tau = 1776.86\,\mathrm{MeV}$) and you get

$$K = 0.666661\ldots \tag{2}$$

which is $2/3$ to 0.001%.

This is strange. The ratio K can range from $1/3$ (all masses equal) to 1 (one mass dominates). There is no obvious reason for it to be $2/3$.

Koide's formula has survived 40 years of improved measurements. The τ mass has been remeasured many times since 1983; each time, K has moved *closer* to $2/3$, not farther. The current precision makes accidental coincidence implausible ($p < 10^{-5}$ from Monte Carlo).

2 What are "signed charges"?

The key to understanding $K = 2/3$ is to change variables. Define $q_i = \sqrt{m_i}$ (the "signed charge"). Then

$$K = \frac{q_1^2 + q_2^2 + q_3^2}{(q_1 + q_2 + q_3)^2}\,. \tag{3}$$

This is just a ratio of power sums in the q_i. The numerator is the sum of squares; the denominator is the square of the sum.

Now the Cauchy–Schwarz inequality gives $1/3 \le K \le 1$, with $K = 1/3$ when all q_i are equal and $K = 1$ when one dominates. The value $K = 2/3$ is the midpoint — the "equipartition" between uniformity and hierarchy.

3 What is the *inverse* Koide formula?

Take the same ratio (**??**) but with $q_i \to 1/q_i$:

$$K^{-1} \equiv \frac{1/m_d + 1/m_s + 1/m_b}{\left(1/\sqrt{m_d} + 1/\sqrt{m_s} + 1/\sqrt{m_b}\right)^2} . \tag{4}$$

Plug in the down-type quark masses ($m_d = 4.67\,\mathrm{MeV}$, $m_s = 93.4\,\mathrm{MeV}$, $m_b = 4183\,\mathrm{MeV}$) and you get

$$K^{-1}(d, s, b) = 0.6652 , \tag{5}$$

which is 2/3 to 0.22%. Not as precise as the leptons, but far better than random ($p \sim 0.3\%$ for a single triple).

Under renormalisation-group running (5-loop Standard Model, using the Martin–Robertson SMDR code [**?**]), the inverse down-type ratio improves monotonically toward the UV:

Scale Q	$K^{-1}(d, s, b)$	Deviation from 2/3
M_Z	0.66750	+0.13%
1 TeV	0.66719	+0.08%
100 TeV	0.66675	+0.01%
280 TeV	**0.66667**	**0.00**%
10^6 GeV	0.66657	−0.01%
M_{Planck}	0.66539	−0.19%

The inverse ratio crosses 2/3 exactly at $Q = 280\,\mathrm{TeV}$. This is not a fit — it is a consequence of how down-type Yukawa couplings run in the Standard Model.

There is also a second inverse triple: $K^{-1}(s, c, -t)$ crosses 2/3 at $Q = 82\,\mathrm{TeV}$ (the $-$ sign on t means the top charge enters with opposite sign, which is needed for the formula to work).

4 Why should there be a direct *and* an inverse formula?

In supersymmetric QCD with three colours and three flavours (SU(3) with $N_f = 3$), the theory s-confines [**?**]. The low-energy degrees of freedom are "mesons" $M^i{}_j = Q^i \tilde{Q}_j$, subject to the quantum constraint

$$\det M = \Lambda^6 . \tag{6}$$

The vacuum expectation values of the mesons satisfy a seesaw:

$$\langle M_i \rangle = \frac{\Lambda^2 P^{1/3}}{m_i} , \qquad P = m_1 m_2 m_3 . \tag{7}$$

Heavy preons give light mesons, and vice versa. This is the "Seiberg seesaw."

But the *same* theory also contains the adjugate:

$$\mathrm{adj}(M)_i = \frac{\Lambda^4}{P^{1/3}} m_i , \tag{8}$$

which goes the other way: heavy preons give heavy adjugates.

So one theory produces **two** operators:

- The meson VEV: $\langle M_i \rangle \propto 1/m_i$ ("inverse")
- The adjugate: $\mathrm{adj}(M)_i \propto m_i$ ("direct")

If the ultraviolet masses satisfy $m_i = \mu\, q_i^2$ and $K(q_1, q_2, q_3) = 2/3$ (which is automatic for any U(3) charges), then *both* the direct and the inverse ratios equal 2/3. The inverse Koide is not a separate coincidence — it is the other face of the same coin.

The two operators are linked by a family-independent product rule:

$$\langle M_i \rangle \cdot \mathrm{adj}(M)_i = \Lambda^6 \qquad \text{(same for all } i\text{)}\,. \tag{9}$$

This is the constraint $\det M = \Lambda^6$ in disguise.

5 The operator that maps leptons to quarks

Define the matrix R by

$$R_{ij} = \langle M_k \rangle \quad (k \neq i, j;\ i \neq j)\,, \qquad R_{ii} = 0\,. \tag{10}$$

This 3×3 symmetric, traceless matrix satisfies an exact identity:

$$R \cdot \mathbf{M} = 2\,\mathrm{adj}(\mathbf{M})\,. \tag{11}$$

In words: R acting on the meson VEV vector gives twice the adjugate vector. If you identify mesinos (meson fermions) with leptons and adjugate fermions with quarks, then R is the operator that *rotates leptons into quarks.*

What algebra does R satisfy? The Cayley–Hamilton theorem on the constraint surface gives:

$$R^3 = \left(\sum M_i^2\right) R + 2\Lambda^6\, I\,. \tag{12}$$

This is a *cubic*, not the quadratic $J^2 = -1$ of standard $\mathrm{SU}(2)_R$ in $\mathcal{N} = 2$ supersymmetry. The "emergent $\mathcal{N} = 2$" of the sBootstrap is therefore not standard $\mathcal{N} = 2$ — it is a cubic deformation controlled by the mass hierarchy.

A particularly beautiful property: conjugating by the diagonal meson matrix $D = \mathrm{diag}(M_1, M_2, M_3)$ gives

$$D\,R\,D = \Lambda^6\,(J - I)\,, \tag{13}$$

where J is the all-ones matrix. The right-hand side is the **democratic matrix** — completely independent of the individual masses. All hierarchy information is absorbed by the D conjugation; the R structure itself is universal.

6 Why three families?

Here is a curious coincidence (or not). For U(N) with canonical normalisation, the Koide ratio generalises to $K_N = 2/N$. Meanwhile, the principal $\mathfrak{sl}(2)$ subalgebra [?] of $\mathfrak{sl}(N)$ has mean-square weight

$$\langle J_3^2 \rangle = \frac{N^2 - 1}{12} \tag{14}$$

in the fundamental representation.

Setting $K_N = \langle J_3^2 \rangle$:

$$\frac{2}{N} = \frac{N^2 - 1}{12} \qquad \Longleftrightarrow \qquad N(N^2 - 1) = 24\,. \tag{15}$$

This factors as $(N - 3)(N^2 + 3N + 8) = 0$. The quadratic has no real roots ($\Delta = 9 - 32 < 0$).

The unique positive integer solution is $N = 3$.

In other words: the Koide value 2/3 is the Casimir eigenvalue of the principal $\mathfrak{sl}(2)$ in $\mathfrak{sl}(N)$, and this identification works *only* for three families. Kostant proved in 1959 that this principal $\mathfrak{sl}(2)$ is the unique canonical subalgebra of any simple Lie algebra [?]. He did not know about quarks, leptons, or the Koide formula. The connection was noticed 65 years later.

7 What we don't know

Lest the reader think the problem is solved, here is an honest list of what remains open:

1. **The angle.** The charge parametrisation has one free parameter (the "Cartan angle" $\alpha \approx 12.7$) that determines all mass ratios within a sector. Nobody knows where it comes from. A one-loop Coleman–Weinberg analysis shows that every perturbative contribution to the potential prefers $\alpha = 0$ (all masses equal). The physical value is a flat direction of the perturbative theory.

2. **The CKM matrix.** The off-diagonal meson masses in the confined theory block-decouple from the diagonal ones at tree level. Quark mixing cannot arise from tree-level meson mixing; it requires loop effects or Kähler corrections.

3. **Neutrinos.** The tree-level prediction for the PMNS matrix is diagonal (no mixing), which is wrong.

4. **The cubic algebra.** The R operator satisfies a cubic, not a quadratic. Does this define a new kind of extended supersymmetry?

5. **Why $m \propto q^2$?** The charge-squared mass law is uniquely selected by an algebraic theorem (among monomials, only $n = 2$ gives a direction-independent K). But no one has derived it from the dynamics of the confining theory. It is the framework's central postulate.

Five open problems. Always five.

Summary

Quantity	Value	Precision
$K(e, \mu, \tau)$	0.666661	0.001%
$K^{-1}(d, s, b)$ at 280 TeV	0.666667	exact
$K^{-1}(s, c, -t)$ at 82 TeV	0.666667	exact
m_b from bion relation	4183 MeV	0.01%
m_t from chain completion	168.5 GeV	2.3%
$\lvert V_{us}\rvert$ from $\sqrt{m_\mu/m_\tau}$	0.244	8.7%
$N = 3$ from $K = \langle J_3^2 \rangle$	unique	exact

The inverse Koide formula is not a separate observation. It is the shadow of the same algebraic structure that produces the original Koide formula, viewed through the Seiberg seesaw of a confining gauge theory. Both the direct and inverse relations are automatic for U(3) family charges. The value 2/3 is Kostant's Casimir, and it selects three families from a Diophantine equation that Bertram Kostant never had reason to write down.

References

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