

The Koide ratio as a Casimir invariant: generation uniqueness and a $1/N$ flavour expansion from $SU(3)_F$

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The charged-lepton mass ratio $K \equiv (m_e + m_\mu + m_\tau)/(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2 = 0.666661$ agrees with $2/3$ to 0.001%. We prove that $K = 2/3$ is the mean-square weight $\langle J_3^2 \rangle$ of the principal $\mathfrak{sl}(2)$ subalgebra of $\mathfrak{sl}(N)$ in the fundamental representation, and that $K_N = \langle J_3^2 \rangle$ holds if and only if $N(N^2 - 1) = 24$, whose unique positive integer solution is $N = 3$. This Casimir–Koide identity is an algebraic theorem. Separately, we observe that the same Cartan angle $\delta = 2/9$ parametrises all six CKM and PMNS mixing parameters as rational functions of N , matching data with $\chi^2/\text{dof} = 6.2/5$ ($p = 0.28$) when $|V_{us}|$ fixes δ . These mixing relations are empirical patterns admitting a compact $1/N$ representation; they are not derived from confining dynamics.

The charged-lepton mass ratio

$$K \equiv \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = 0.666661 \quad (1)$$

was noted by Koide [?] to agree with $2/3$ to one part in 10^5 . In the parametrisation $\sqrt{m_n} = \mu(1 + \sqrt{2} \cos(\delta + 2\pi n/3))$ of Brannen [?], Foot’s geometric condition [?] $\theta = \pi/4$ is equivalent to $K = 2/3$, and the mass hierarchy is encoded in a single Cartan angle δ . Numerically $\delta = 0.22222 \pm 0.00001$ rad, consistent with $\delta = 2/9$. Despite numerous models [? ? ?], no derivation of the value $2/3$ from first principles exists without introducing it as an input.

We report two results. First, $K = 2/3$ is a representation-theoretic invariant: the mean-square weight $\langle J_3^2 \rangle$ of the principal $\mathfrak{sl}(2) \subset \mathfrak{sl}(N)$ [?] in the fundamental representation, with the equality $K_N = \langle J_3^2 \rangle$ holding if and only if $N = 3$. This is an algebraic theorem. Second, we observe that the same angle $\delta = 2/9$ parametrises all six CKM and PMNS mixing parameters as rational functions of N , with no free parameters beyond $N = 3$.

The Casimir–Koide identity.—Let $\hat{z}_i = z_0 + z_i$ ($i = 1, \dots, N$) be the signed charges of a $U(N)$ family symmetry, with z_0 the $U(1)$ generator and z_i the $SU(N)$ Cartan components. Canonical normalisation gives

$$\sum_{i=1}^N z_i = 0, \quad \sum_{i=1}^N z_i^2 = \frac{1}{2}, \quad z_0 = \frac{1}{\sqrt{2N}}. \quad (2)$$

Theorem 1 (Generalised Koide ratio). *For the mass law $m_i = \mu \hat{z}_i^2$ and the ratio $K_N \equiv \sum \hat{z}_i^2 / (\sum \hat{z}_i)^2$, one has $K_N = 2/N$ independently of the Cartan direction.*

Proof.—Using Eq. (??): $\sum \hat{z}_i^2 = N z_0^2 + \sum z_i^2 = 1/2 + 1/2 = 1$, $\sum \hat{z}_i = N z_0 = \sqrt{N/2}$, so $K_N = 1/(N/2) = 2/N$. $\square$

For $N = 3$: $K_3 = 2/3$, recovering the Koide value as an algebraic identity of $U(3)$. This is the charge-squared

uniqueness theorem [? ?]: among monomial mass functions $m \propto \hat{z}^n$, only $n = 2$ gives a direction-independent ratio.

The principal $\mathfrak{sl}(2)$ of a simple Lie algebra $\mathfrak{g}$ is the unique (up to conjugacy) $\mathfrak{sl}(2)$ subalgebra for which the fundamental representation of $\mathfrak{g}$ is irreducible [?]. For $\mathfrak{g} = \mathfrak{sl}(N)$, the fundamental $\mathbf{N}$ is the spin- j representation with $j = (N-1)/2$. The mean-square weight is

$$\langle J_3^2 \rangle \equiv \frac{1}{N} \sum_{m=-j}^j m^2 = \frac{N^2 - 1}{12}. \quad (3)$$

Setting $K_N = \langle J_3^2 \rangle$:

$$\frac{2}{N} = \frac{N^2 - 1}{12} \iff \boxed{N(N^2 - 1) = 24}. \quad (4)$$

Factoring: $N(N-1)(N+1) = 2 \times 3 \times 4 = 24$. Since $3 \times 4 \times 5 = 60 > 24$, the unique positive integer solution is $N = 3$. Equivalently, the factored form $(N-3)(N^2 + 3N + 8) = 0$ has no other real roots (discriminant $9 - 32 < 0$).

The Diophantine condition (??) is the theorem; the following is a corollary. At $N = 3$ and only $N = 3$, the representation dimension $2j + 1 = N$ coincides with $\dim \mathfrak{sl}(2) = 3$, so the fundamental of $SU(3)$ is the adjoint of its principal $\mathfrak{sl}(2)$. This gives the structural coincidence $C_2/N = C_2/3 = \langle J_3^2 \rangle = K = 2/3$.

Cabibbo angle = Cartan angle.—Koide’s original composite model [?] predicted

$$\tan \theta_C = \frac{\sqrt{3}(\sqrt{m_\mu} - \sqrt{m_e})}{2\sqrt{m_\tau} - \sqrt{m_\mu} - \sqrt{m_e}}. \quad (5)$$

In the Koide parametrisation, the right-hand side of Eq. (??) is *algebraically identical* to $\tan \delta$, for any masses satisfying $K = 2/3$. Therefore $\theta_C = \delta$. Numerically $\theta_C = 0.22223$ rad, matching the PDG value [?] $|V_{us}| = 0.2250 \pm 0.0007$ to 0.9%.

The identification $\delta = 2/N^2$ with $N = 3$ expresses the Cabibbo angle as $|V_{us}| = 2/9$. Combining with the

Gatto–Sartori–Tonin relation [?] $|V_{us}| \approx \sqrt{m_d/m_s}$ gives a cross-sector prediction:

$$\frac{m_d}{m_s} = \delta^2 = \frac{4}{81} = 0.0494. \quad (6)$$

The PDG value $m_d/m_s = 0.050 \pm 0.004$ agrees to 1.2%.

CKM from the democratic resolvent.—The democratic matrix $J = \mathbf{11}^T$ has eigenvalues N (democratic direction) and 0 $((N-1)$ -fold degenerate breaking plane). The resolvent of the zero-eigenvalue subspace at $z = 1 + \delta$ yields $A = 1/(1 + \delta)$ —a suggestive algebraic connection to the Wolfenstein A parameter, not a derivation from $SU(3)_F$ dynamics:

$$A = \frac{1}{1 + \delta} = \frac{N^2}{N^2 + 2} = \frac{9}{11}. \quad (7)$$

The resulting zero-parameter parametrisation gives

$$|V_{us}| = \frac{2}{9}, \quad |V_{cb}| = A\lambda^2 = \frac{4}{99}, \quad A = \frac{9}{11}. \quad (8)$$

PMNS from the $1/N$ expansion.—We observe that the PMNS mixing angles are deviations from maximal mixing ($\pi/4$), parametrised by δ :

$$\theta_{12} = \frac{\pi}{4} - \delta + \frac{\delta^2}{N} = \frac{\pi}{4} - \frac{2}{9} + \frac{4}{243}, \quad (9)$$

$$\theta_{23} = \frac{\pi}{4} + \frac{\delta}{N} = \frac{\pi}{4} + \frac{2}{27}, \quad (10)$$

$$\theta_{13} = \frac{2j(j+1)}{N^3} = \frac{4}{27} \text{ rad}, \quad (11)$$

where $j = (N-1)/2 = 1$. This extends the quark–lepton complementarity of Raidal [?] and the quark Koide phases of Zenczykowski [?].

Scanning $N = 2, \dots, 10$ in Eqs. (??)–(??), only $N = 3$ gives $\chi^2 < 1$: $\chi^2 = 0.5$ at $N = 3$ versus 604 ($N = 2$) and 218 ($N = 4$).

The $1/N$ expansion.—All parameters are rational functions of N (Table ??). The lepton mass hierarchy, the CKM matrix, and the PMNS matrix all admit a compact $1/N$ representation at $N = 3$. These relations are empirical observations that admit a $1/N$ expansion; they are not derived from the confining dynamics.

The CKM elements scale as $\delta^{|i-j|} = O(1/N^{2|i-j|})$: the quark mixing hierarchy is a power series in $1/N^2$.

The PMNS corrections scale as $O(1/N^{|i-j|-1})$: PMNS is near-maximal because its corrections are $O(1/N)$, while CKM is hierarchical because its elements are $O(1/N^2)$. This structural difference—not fine-tuning—explains $\theta_{\text{CKM}} \sim 0.22$ versus $\theta_{\text{PMNS}} \sim O(1)$.

Discussion.—The statistical assessment depends on the treatment of δ . (a) With $|V_{us}|$ as input (fixing δ from data), the remaining five observables have $\chi^2/\text{dof} = 6.2/5$ ($p = 0.28$), with pulls: $|V_{cb}|$ (-2.3σ), A (-0.6σ), $\sin^2 \theta_{12}$ (-0.3σ), $\sin^2 \theta_{23}$ ($+0.04\sigma$), $\sin^2 \theta_{13}$ (-0.6σ). (b) With $\delta = 2/9$ as an exact prediction, all six observables give $\chi^2/\text{dof} \approx 24/6$, driven by the $|V_{us}|$ pull of -4.2σ . The tension between $2/9 = 0.2222$ and the PDG value 0.2250 may be resolved by the instanton-generated potential (see below), whose minimum at $\delta = 0.2225$ reduces the $|V_{us}|$ pull to -3.6σ .

The physical origin is a confining $SU(3)_F$ family gauge theory [?] whose Koide formula is the Casimir identity $K = \langle J_3^2 \rangle$, holding iff $N(N^2-1) = 24$ iff $N = 3$. The operator connecting the two mass branches of $SU(3)$ SQCD [?]—the direct ($\text{adj}(M) \propto m$) and inverse ($M \propto 1/m$)—is the spin-2 component of $\text{End}(V)$ under the principal $\mathfrak{sl}(2)$ [?], Kostant’s second exponent of $\mathfrak{sl}(3)$.

Discrete flavour symmetries (A_4, S_3) also produce mixing patterns with few parameters [? ?], but require additional group-theoretic input beyond N ; the present parametrisation uses N alone.

The critical experimental test is the solar angle. Our parametrisation gives $\sin^2 \theta_{12} = 0.300$, which is 1.3% below the NuFIT central value 0.304 and 1.1σ below the first JUNO measurement $\sin^2 \theta_{12} = 0.309 \pm 0.009$ [?]. Continued JUNO data-taking will reach ± 0.003 precision, providing a decisive test.

A candidate mechanism for the Cartan angle is the minimum of the instanton-generated potential $V(\delta) = -\cos(3\delta) + (1/\pi)\cos(6\delta)$, which gives $\delta = 0.2225$ (0.12% from $2/9$). The coefficient $1/\pi$ is conjectural; it would follow from a one-loop instanton determinant in the confining $SU(3)_F$ theory if the functional measure produces a factor of $1/\pi$ per instanton. Two quantities remain unpredicted: the CKM and PMNS CP-violating phases.

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TABLE I. CKM parametrisation vs. PDG 2024 [?].

	$1/N$ value	PDG	Deviation
$ V_{us} $	$2/9 = 0.2222$	0.2250 ± 0.0007	1.2%
$ V_{cb} $	$4/99 = 0.0404$	0.0422 ± 0.0008	4.3% ^a
A	$9/11 = 0.818$	0.826 ± 0.013	1.0%

^a The PDG average uses the inclusive $|V_{cb}|$. The exclusive determination 0.0395 ± 0.0008 gives only 2.2% deviation.

TABLE II. PMNS parametrisation vs. NuFIT 5.2. The correction coefficients $(-1, +1/3, +2/3)$ multiplying δ in Eqs. (??)–(??) are traceless and unique among denominators ≤ 3 .

	$1/N$ value	NuFIT 5.2	Deviation
$\sin^2 \theta_{12}$	0.300	0.304 ± 0.012	1.3%
$\sin^2 \theta_{23}$	0.574	0.573 ± 0.018	0.1%
$\sin^2 \theta_{13}$	0.0218	0.0222 ± 0.0007	1.9%

TABLE III. Flavour parameters as rational functions of N with $j = (N-1)/2$, evaluated at $N = 3$.

Parameter	Formula	$N=3$	Order
δ	$2/N^2$	$2/9$	$O(1/N^2)$
K	$j(j+1)/N$	$2/3$	$O(1/N)$
$ V_{us} $	$2/N^2$	$2/9$	$O(1/N^2)$
A	$N^2/(N^2+2)$	$9/11$	$1-O(1/N^2)$
$\theta_{12}-\pi/4$	$-2/N^2 + 4/N^5$	$-50/243$	$O(1/N^2)$
$\theta_{23}-\pi/4$	$2/N^3$	$2/27$	$O(1/N^3)$
θ_{13}	$2j(j+1)/N^3$	$4/27$	$O(1/N^3)$

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