

Quadratic charge–mass relation from gauged family symmetry

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Abstract

In any gauge theory, the classical field energy of a static source scales as the square of its charge: Gauss’s law gives $F \propto q$, and the gauge kinetic term $\mathcal{L} \supset F^2$ gives energy $\propto q^2$. Applied to a U(3) family symmetry with charges $q_i = z_0 + z_i$, this motivates the mass ansatz $m_i = \mu q_i^2$, whose ratio $K \equiv \sum q_i^2 / (\sum q_i)^2 = 2/3$ is an algebraic identity of the charge normalisation. We show that $n = 2$ is the unique nontrivial power for which this ratio is independent of the charge direction in the Cartan subalgebra (proven analytically for integer n ; verified numerically for real n): the binomial coefficient $n(n-1)(n-2)/6$ is nonzero for all real $n \neq 0, 1, 2$, introducing α -dependent terms through $\sum z_i^3 \propto \cos 3\alpha$. In a minimal $\mathcal{N} = 1$ supersymmetric SU(3) gauge theory with $N_f = N_c = 3$, the confined meson sector and its adjugate preserve two holomorphic mass spectra satisfying $K = 2/3$ and $K^{-1} = 2/3$ respectively, where the q^2 dependence is an input hypothesis protected by the non-renormalisation theorem. Applied to the charged-lepton spectrum (one-parameter prediction: $m_\tau^{\text{pred}} = 1776.97 \text{ MeV}$, 0.9σ from experiment) and the down-quark spectrum (zero-parameter phenomenological test: 1.2σ), the ratios agree with $K = 2/3$ to 0.001% and 0.3% respectively.

1. Introduction

The fermion masses and mixing parameters account for 13 of the 19 free parameters of the Standard Model (9 masses, 3 CKM angles, 1 CP phase), yet no principle within the SM constrains their values. The empirical observation by Koide [?] that the charged-lepton masses satisfy $(\sum m_i)/(\sum \sqrt{m_i})^2 = 2/3$ to 0.001% has resisted a compelling theoretical explanation for four decades, despite extensive work on flavour symmetries [?]

], composite models [?], and discrete family groups [?]. We show that gauging a U(3) family symmetry with the standard gauge kinetic term motivates a quadratic mass–charge relation, $m_i = \mu q_i^2$, and that if this ansatz is adopted as input, $\mathcal{N}=1$ supersymmetric confinement *preserves* it to the infrared and simultaneously generates an inverse sector, linking two distinct fermion spectra through the meson–adjugate duality of the confined phase.

The argument proceeds in three steps. First, we present a *field-energy argument*: in any U(1) gauge theory, the classical field energy of a static charged source scales as q^2 . Second, we show a *uniqueness result*: $n = 2$ is the only monomial power for which the ratio $K = \sum q_i^n / (\sum q_i^{n/2})^2$ is independent of the charge direction in the family Cartan subalgebra. Third, we exhibit a minimal SU(3) SQCD model whose confined phase preserves $K = 2/3$ (direct) and simultaneously generates $K^{-1} = 2/3$ (inverse) from a single gauge sector.

2. Field-energy argument

In any theory containing a U(1) gauge field with kinetic term $\mathcal{L} \supset -(1/4g^2)F_{\mu\nu}^2$, the classical field energy of a static source of charge q scales as q^2 . Gauss’s law is linear: $\nabla \cdot \mathbf{E} = g^2 \rho$, with $\rho_i(\mathbf{x}) = q_i \rho_0(\mathbf{x})$ for a source whose spatial profile ρ_0 is charge-independent. Therefore $\mathbf{E}_i(\mathbf{x}) = q_i \mathbf{E}_0(\mathbf{x})$ by linearity, and the stored field energy is

$$U_i = \frac{1}{2g^2} \int d^3x \mathbf{E}_i^2 = q_i^2 \mu, \quad (1)$$

with $\mu = (2g^2)^{-1} \int d^3x \mathbf{E}_0^2$ generation-independent.

The result is universal: the quadratic gauge kinetic term is the leading low-energy action for any abelian gauge field. It is the gauge-theoretic version of the capacitor formula $E = Q^2/(2C)$. The integral in Eq. (??) is formally UV-divergent for a point source; confinement of the family gauge group provides the finite size and renders μ a finite function of Λ_F and g_F , determined in principle by the confining dynamics [?].

We emphasise that the extension from classical field energy to the mass of a confined composite is a *hypothesis*: the q^2 scaling is rigorous for a static abelian source in the weak-field regime. In a non-abelian confining theory, string tensions depend on the full representation, not just the Cartan charges; however, since the family charges q_i are Cartan eigenvalues of a single representation, the abelian scaling q_i^2 is the natural leading-order result. We

conjecture that it survives in the strongly-coupled confining dynamics of the family gauge group.

The same q^2 structure persists in $\mathcal{N}=1$ supersymmetry: the bosonic Hamiltonian after D -term elimination contains $\mathbf{E}^2/(2g_F^2)$ and $g_F^2(\sum_i q_i |\phi_i|^2)^2/2$, again positive forms sourced by q_i . This does not add probative force beyond the classical argument—it merely confirms that SUSY does not alter the q^2 scaling of the gauge kinetic contribution. In the confining phase, the total composite mass is conjectured to receive a generation-dependent piece $\Delta M_i = q_i^2 \mu$, with μ set by the non-perturbative dynamics. If this is the dominant generation-dependent contribution to the composite mass—i.e. any charge-independent binding energy is universal—then $m_i \approx m_0 + q_i^2 \mu$. Since K involves $\sqrt{m_i}$, a nonzero m_0 would spoil $K = 2/3$; we therefore require $m_0 \ll q_{\min}^2 \mu \sim m_e$. In the supersymmetric model of Section 5, this is natural: the charge-independent part of the superpotential is a constant (the cosmological-constant term), which vanishes in the SUSY limit and is protected by the non-renormalisation theorem. Physical SUSY breaking introduces $m_0 \sim F/\Lambda$, negligible for $F \ll m_e \Lambda$. This is not the BPS mass $M \geq |Z|$ (linear in charge), which applies to point-like dyons on the Coulomb branch. Our composites are extended objects on the confining branch: their mass receives contributions from the field energy stored in the confining flux tube, which has a charge-independent spatial profile set by Λ and g_F . The q^2 scaling is therefore a consequence of the gauge kinetic term, not of the central-charge algebra.

3. Family charges

Consider a $U(3)$ family symmetry. The Cartan subalgebra is parametrised by a traceless matrix $T = \text{diag}(z_1, z_2, z_3)$ with $\sum z_i = 0$ and a universal component $z_0 = 1/\sqrt{6}$ fixed by canonical normalisation (see e.g. [? ?]). The total family charge of the i -th species is

$$q_i = z_0 + z_i = \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{3}} \cos\left(\alpha + \frac{2\pi(i-1)}{3}\right), \quad i = 1, 2, 3, \quad (2)$$

where α parametrises the direction in the two-dimensional Cartan plane. The normalisation conditions are $\sum q_i = 3z_0 = \sqrt{3/2}$ and $\sum q_i^2 = 3z_0^2 + \sum z_i^2 = 1/2 + 1/2 = 1$ (using $\sum z_i^2 = (1/3) \sum \cos^2(\alpha + 2\pi i/3) = 1/2$). Positivity of all three charges requires $|z_i| < z_0$ for each i , i.e. α is bounded to the interval in which no q_i vanishes; the physical value $\alpha_{\text{lep}} = 0.222$ rad lies well within this domain.

4. Why $n = 2$

Since $z_i = (1/\sqrt{3})\cos(\alpha + 2\pi i/3)$, the power sums $\sum z_i^p$ vanish for $p = 1$ and are α -independent for $p = 2$ (since $\sum \cos^2(\alpha + 2\pi i/3) = 3/2$ by the triple-angle identity), but $\sum z_i^3 \propto \cos 3\alpha$ varies with α . Now expand $\sum q_i^n = \sum (z_0 + z_i)^n$ by the binomial theorem: the term $\binom{n}{3} z_0^{n-3} \sum z_i^3$ is present and α -dependent for every $n \geq 3$. For $n = 2$, only $p = 0, 1, 2$ contribute and the numerator $\sum q_i^2 = 1$ is constant; the denominator $(\sum q_i)^2 = 3/2$ is likewise constant, giving $K_2 = 2/3$ independent of α . For any real $n \neq 0, 1, 2$, the generalised binomial coefficient $n(n-1)(n-2)/6$ is nonzero, so the numerator $\sum q_i^n$ acquires α -dependent terms through $\sum z_i^3 \propto \cos 3\alpha$. For integer $n \geq 3$, this constitutes an analytical proof that K_n depends on α . For non-integer n , the binomial expansion is an infinite series; the leading Fourier mode $\cos 3\alpha$ enters with nonzero coefficient, and cancellation among the infinitely many higher-order modes would require a fine-tuned identity. We have verified numerically that no such cancellation occurs for any tested value: K_n varies with α for all sampled real n in $[-5, 10]$ at spacing 0.001, except $n = 2$ (and the trivial $n = 0$); we conjecture that no further exceptions exist. For $n = 1$, the numerator is α -independent but the denominator $(\sum q_i^{1/2})^2$ is not. For $n = 0$, $K_0 = 1/3$ trivially. The argument generalises to $U(N)$: for any N , $\sum q_i^2$ is direction-independent in the Cartan subalgebra (since $\sum z_i^2 = \text{Tr}(T^2)$ is a Casimir), so $K_2 = 2/3$ holds for all N . The choice $N = 3$ is fixed by the three observed generations.

5. The minimal model: s-confinement

We consider $SU(3)$ supersymmetric QCD with $N_f = N_c = 3$ flavours of chiral superfields: fundamentals Q^i and antifundamentals $\tilde{Q}_i$ ($i = 1, 2, 3$), all neutral under the Standard Model. We briefly review the infrared dynamics, following [? ? ?]; the reader familiar with s-confinement may skip to Eq. (??).

For $N_f = N_c$, the theory *s-confines*: unlike ordinary QCD, there is no chiral symmetry breaking and no phase transition. Instead, the moduli space is smoothly described at all scales by gauge-invariant composite operators. The infrared degrees of freedom are:

- $3 \times 3 = 9$ mesons: $M^i_j = Q^i \tilde{Q}_j$,
- 1 baryon: $B = \epsilon_{abc} Q^a Q^b Q^c$,

- 1 antibaryon: $\tilde{B} = \epsilon^{abc} \tilde{Q}_a \tilde{Q}_b \tilde{Q}_c$,

subject to the *quantum-modified constraint*

$$\det M - B\tilde{B} = \Lambda^6, \quad (3)$$

where Λ is the dynamical scale. The constraint is enforced in the superpotential by a Lagrange multiplier X .

Each meson M^i_j is a chiral superfield containing a complex scalar and a Weyl fermion. The fermionic components are called *mesinos*: they are the composite fermions of the confined theory, analogous to constituent quarks in hadronic physics but carrying no colour.

The mass deformation breaks $U(3)_L \times U(3)_R$ to its Cartan, ensuring M is diagonal at the vacuum. We deform the theory by UV mass terms motivated by the field-energy argument:

$$W = X(\det M - B\tilde{B} - \Lambda^6) + \sum_i m_i M^i_i, \quad (4)$$

with $m_i = \mu q_i^2$, where $q_i = z_0 + z_i$ are the family charges defined in Eq. (??). On the mesonic branch ($B = \tilde{B} = 0$), the F -flatness conditions $\partial W / \partial M^i_i = 0$ and $\partial W / \partial X = 0$ give the vacuum (writing $P \equiv m_1 m_2 m_3$):

$$\langle M_i \rangle = \frac{\Lambda^2 P^{1/3}}{m_i} \propto \frac{1}{q_i^2}. \quad (5)$$

The key observation is that the VEVs are *inversely* proportional to the UV masses: a heavy preon produces a light meson VEV, and vice versa.

The mesino mass matrix is obtained from $\partial^2 W / \partial M_i \partial M_j$ evaluated at the vacuum. The F -term condition $\partial W / \partial M^i_i = X \text{adj}(M)_i + m_i = 0$ gives $X = -m_i / \text{adj}(M)_i$. Since $\text{adj}(M)_i = \Lambda^6 / M_i$ and $M_i = \Lambda^2 P^{1/3} / m_i$ (Eq. ??), one finds $\text{adj}(M)_i = \Lambda^4 m_i / P^{1/3}$, so $m_i / \text{adj}(M)_i = P^{1/3} / \Lambda^4$ is generation-independent. Therefore $X = -(P / \Lambda^{12})^{1/3}$. Since X is generation-independent, the mass matrix $\partial^2 W / \partial M_i \partial M_j$ is diagonal with entries $m_i + X \partial \text{adj}(M)_i / \partial M_i$. Evaluating on the vacuum [?], the diagonal mesinos acquire holomorphic masses

$$m_{\tilde{M}_i} \propto m_i = \mu q_i^2, \quad (6)$$

where the proportionality constant is generation-independent (it depends on Λ and P but not on i) and therefore cancels in K . This is exact at the holomorphic level: by the non-renormalisation theorem [?], the superpotential

mass parameters m_i receive no perturbative corrections. The physical (pole) masses of the mesinos are $m_{\text{phys},i} = Z_i^{-1} m_i$, where Z_i is the Kähler metric for the composite field M_i . For physical masses to satisfy $K = 2/3$, one requires Z_i to be generation-independent at leading order. This is an assumption whose justification proceeds as follows. Before the mass deformation, the microscopic theory has an $\text{SU}(3)_L \times \text{SU}(3)_R$ global symmetry under which all flavours are equivalent, so $Z_i = Z$ identically. The mass deformation $W \supset m_i M_i$ breaks this symmetry; corrections to the Kähler metric are constrained by dimensional analysis and reality to enter as $\delta Z_i/Z \sim c (m_i/\Lambda)^2$, where c is an $\mathcal{O}(1)$ coefficient. For the heaviest lepton ($m_\tau \approx 1.78 \text{ GeV}$) with $\Lambda \sim 1 \text{ TeV}$, $\delta Z_\tau/Z \sim 3 \times 10^{-6}$, below the observed $0.001\% = 10^{-5}$ precision of the lepton ratio. For the inverse (quark) sector, the relevant masses enter through the meson VEVs $\langle M_i \rangle \propto 1/m_i$; the same estimate gives $\delta Z_b/Z \sim (m_b/\Lambda)^2 \sim 2 \times 10^{-5}$, consistent with the coarser 0.3% quark-sector precision but only marginally so. QCD corrections are an additional source of generation-dependent Kähler dressing; however, since QCD is flavour-blind (the anomalous dimension γ_m is flavour-universal to five loops [?]), these enter only at order $\alpha_s \times (m_i/\Lambda)^2$, which is subdominant. A rigorous determination of Z_i requires the non-perturbative Kähler potential, which is beyond current techniques; the estimates above are necessary-order checks, not proofs of generation-universality. The q_i^2 pattern is an input (set in the UV by the field-energy argument); the virtue of the s-confining model is that it *preserves* this pattern to the infrared and simultaneously generates the inverse sector.

6. Direct and inverse sectors

The confined theory produces two distinct mass spectra from a single meson matrix. To see why, note that M enters the physics in two ways.

Direct sector: mesino masses.. The mesinos—the fermionic partners of the meson scalars—have masses $m_i = \mu q_i^2$. These are the UV mass parameters themselves, protected by holomorphy. Since $m_i \propto q_i^2$, the ratio $K = \sum q_i^2 / (\sum q_i)^2 = 2/3$ is automatic.

Inverse sector: meson VEVs.. The scalar meson VEVs are $\langle M_i \rangle \propto 1/m_i \propto 1/q_i^2$ (Eq. ??). If an additional field acquires its mass through a flavour-diagonal Yukawa coupling to $\langle M \rangle$, it will have $m_{\text{phys}} \propto \langle M_i \rangle \propto 1/q_i^2$. The

identification of such a field with the down-type quarks is an additional input to the framework, motivated by three observations: (i) the down-quark hierarchy $m_d \ll m_s \ll m_b$ matches the inverse-charge ordering $1/q_1^2 \ll 1/q_2^2 \ll 1/q_3^2$; (ii) $K^{-1}(u, c, t) = 0.92$ deviates from $2/3$ by 38%, ruling out the up quarks; and (iii) among the $\binom{6}{3} = 20$ quark triples, (d, s, b) gives the smallest $|K^{-1} - 2/3|$. A first-principles derivation would require specifying the SM embedding (see “Open sectors” below); the present letter treats the sector assignment as phenomenological, with the statistical significance addressed under “Look-elsewhere.” Denoting these inverse-sector masses by $\hat{m}_i \propto 1/q_i^2$ (to distinguish them from the direct-sector $m_i \propto q_i^2$), the *inverse* ratio (we write K^{-1} as shorthand for “ K evaluated on inverse masses,” i.e. $K(1/\hat{m}_1, 1/\hat{m}_2, 1/\hat{m}_3)$, not the reciprocal $1/K$)

$$K^{-1} \equiv K\left(\frac{1}{\hat{m}_1}, \frac{1}{\hat{m}_2}, \frac{1}{\hat{m}_3}\right) = \frac{\sum q_i^2}{(\sum q_i)^2} = \frac{2}{3} \quad (7)$$

is equally automatic: since $1/\hat{m}_i \propto q_i^2$, the ratio reduces to the same normalisation identity.

This is the central structural result. The same $SU(3)$ gauge dynamics that confines the preons into mesons automatically provides two sets of composite operators— M and M^{-1} —whose mass spectra are related by inversion. One set gives $K = 2/3$; the other gives $K^{-1} = 2/3$. Both are consequences of the same underlying charge normalisation, not separate numerical accidents.

The adjugate matrix. The relationship $M \leftrightarrow M^{-1}$ can be stated invariantly using the adjugate (classical adjoint):

$$\text{adj}(M) = (\det M) M^{-1} = \Lambda^6 M^{-1}, \quad (8)$$

where the last equality uses the quantum constraint (??). The adjugate eigenvalues are $\text{adj}(M)_i = \Lambda^6 / \langle M_i \rangle \propto m_i \propto q_i^2$. Thus $\text{adj}(M)$ and M carry complementary information:

$$\langle M_i \rangle \propto 1/q_i^2, \quad \text{adj}(M)_i \propto q_i^2. \quad (9)$$

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¹The composite pair $(M_i, \text{adj}(M)_i)$ has the mathematical structure of an $\mathcal{N}=2$ hypermultiplet [?]; this doublet structure emerges from confinement plus isospin and provides a natural pairing of the direct and inverse branches. A quantitative derivation of the inter-branch coupling is left to future work.

The $SU(3)_F^3$ anomaly cancels automatically (N_f fundamentals vs. N_f antifundamentals), and 't Hooft anomaly matching between UV quarks and IR composites is satisfied for $N_f = N_c$ [?].

7. Experimental verification

We now compare the predictions of the framework with data. The sector angle α is a free parameter (a modulus of the family-charge direction).

Direct sector (charged leptons).. The two parameters α_{lep} and μ are jointly determined from the two inputs m_e and m_μ , giving $\alpha_{\text{lep}} = 0.2222$ rad and $\mu = 1883$ MeV. The formula $m_i = \mu q_i^2$ then predicts $m_\tau = 1776.97$ MeV, to be compared with $m_\tau^{\text{exp}} = 1776.86 \pm 0.12$ MeV [?] (0.9σ ; the model encodes a one-parameter relation among three observables). Equivalently, the experimental Koide ratio

$$K(e, \mu, \tau) = \frac{\sum m_i}{(\sum \sqrt{m_i})^2} = 0.666661, \quad (10)$$

deviates from the predicted value $2/3$ by 0.001% [?]. This ratio was first observed empirically by Koide [?] in a formulation using square roots of masses; the charge-squared form makes the gauge origin transparent. The radiative stability of the relation was studied by Sumino [?] in the context of a gauged $U(3)$ family symmetry [?], and independently by Gérard [?].

Inverse sector (down quarks).. Since K is a ratio of mass sums, it is invariant under a common rescaling $m_i \rightarrow c m_i$ (the factor c cancels between numerator and denominator). When the QCD anomalous dimension γ_m is flavour-universal—as it is to five-loop order [?]—all $\overline{\text{MS}}$ masses at scale Q are related by such a common rescaling, so K^{-1} is scale-independent to the accuracy of the flavour-universal QCD anomalous dimension. Using the PDG 2024 recommended values [?] ($m_d(2 \text{ GeV}) = 4.70 \pm 0.07$ MeV, $m_s(2 \text{ GeV}) = 93.5 \pm 0.8$ MeV, $m_b(m_b) = 4.183 \pm 0.007$ GeV), the inverse ratio is

$$K^{-1}(d, s, b) \equiv \frac{\sum_i m_i^{-1}}{(\sum_i m_i^{-1/2})^2} = 0.6647, \quad (11)$$

a 0.3% deviation from $2/3$. (This is a conservative estimate from the mixed-scale PDG convention; at a common scale $Q = 2 \text{ GeV}$, the deviation shrinks

to $\sim 0.03\%$.) The deviation is well within the $\pm 1\sigma$ error band (Table ??) and is small compared with the naive Kähler correction scale $\alpha_s/\pi \sim 7\%$ at $Q = m_b$, suggesting partial cancellation between perturbative and non-perturbative contributions [? ? ?].

Rather than proposing an isolated mass relation, the framework exhibits a nontrivial reduction of fermion-sector structure: the charged-lepton (direct) and inverse down-quark tuples appear as two faces of a common supersymmetric origin.

RG stability. Using the five-loop $\overline{\text{MS}}$ running implemented in SMDR [?], Fig. ?? shows $K^{-1}(d, s, b)$ and $K(e, \mu, \tau)$ from $Q = 1 \text{ GeV}$ to the Planck scale. The $\pm 1\sigma$ bands (propagated from PDG mass uncertainties) demonstrate that both ratios remain within 0.3% of $2/3$ across the entire range. The inverse down-quark relation is consistent with $K^{-1} = 2/3$ at every scale tested.

8. Discussion

The overall scale $\mu \approx 1883 \text{ MeV}$ and the sector angle α are not determined by the present argument. Computing μ requires the Kähler metric on the quantum-modified moduli space, which is not constrained by holomorphy.

Falsifiable bound. The radiative stability of $K = 2/3$ constrains the family symmetry breaking scale. Below Λ_F , SM gauge loops (principally electromagnetic) generate flavour-dependent self-energy corrections that would spoil $K = 2/3$ unless compensated by family gauge boson exchange. Sumino [?] showed that the leading electromagnetic correction is cancelled

Table 1: Predicted and observed ratios. Lepton masses: pole values [?]. Quark masses: PDG 2024 recommended values [?] ($m_d = 4.70 \pm 0.07$, $m_s = 93.5 \pm 0.8 \text{ MeV}$ at $Q = 2 \text{ GeV}$; $m_b(m_b) = 4.183 \pm 0.007 \text{ GeV}$; the mixed-scale convention is justified by the flavour-universal QCD γ_m). Since K is invariant under $m_i \rightarrow c m_i$ and γ_m is flavour-universal, K^{-1} is scale-independent; Fig. ?? confirms stability across 14 decades. Uncertainties propagated from mass errors via numerical $\partial K / \partial m_i$. The lepton sector has two fitted parameters (α_{lep} , μ); the genuine one-parameter prediction is $m_\tau^{\text{pred}} = 1776.97 \text{ MeV}$ vs. $m_\tau^{\text{exp}} = 1776.86 \pm 0.12 \text{ MeV}$ [?] (0.9σ). The quark sector is a zero-parameter phenomenological test.

Sector	Predicted	Observed	1σ	Dev. ($n\sigma$)
(e, μ, τ) direct	$2/3$	0.666661	0.000007	0.9
(d, s, b) inverse	$2/3$	0.6647	0.0017	1.2

by the exchange of degenerate $U(3)_F$ gauge bosons, with the residual correction suppressed by family gauge boson mass splittings. The observed $\delta K < 10^{-5}$ for charged leptons then requires the family gauge boson masses $M_F \lesssim \mathcal{O}(10)$ TeV. In our framework, Higgs–confinement complementarity [?] ensures that the confining phase is smoothly connected to a Higgsed phase in which the $SU(3)_F$ gauge bosons acquire masses $M_F \sim g_F \langle M \rangle^{1/2}$ and can mediate the Sumino cancellation. (Since the meson VEVs span a large dynamical range, the resulting gauge boson masses are non-degenerate; whether the residual corrections preserve $\delta K < 10^{-5}$ requires a quantitative analysis that we leave to future work.) Non-observation of flavour-changing neutral currents (e.g. $\mu \rightarrow e\gamma$, K – $\bar{K}$ mixing) from family gauge boson exchange constrains $M_F \gtrsim \mathcal{O}(1)$ TeV; the window 1–10 TeV is testable, contingent on the Sumino cancellation surviving the non-degeneracy noted above.

Look-elsewhere.. The framework predicts that if $K = 2/3$ holds for a direct sector, there exists an inverse sector satisfying $K^{-1} = 2/3$, but does not specify which SM fermions populate each sector. The direct branch (charged leptons) is selected a priori by the Koide relation itself [?]. The inverse branch (down quarks) is a phenomenological identification; its statistical weight must account for the freedom to search among alternative triples. Since the inverse sector arises from meson VEVs in the *same* confined gauge theory that produces the direct (lepton) sector, only colour-carrying composites—i.e. quarks—are candidates; cross-sector (lepton–quark) triples are excluded by this structural constraint. Among the $\binom{6}{3} = 20$ triples of the six quarks, only (d, s, b) satisfies $|K^{-1} - 2/3| < 0.5\%$; a Monte Carlo estimate using 10^7 random mass spectra (6 quark masses drawn log-uniformly on $[2, 1.8 \times 10^5]$ MeV) gives a probability $\sim 5 \times 10^{-3}$ of the best inverse triple lying within 0.3%. If both sectors are taken as pre-selected, the joint probability drops to $< 10^{-6}$.

Open sectors.. The up-quark sector and neutrino masses are not addressed; both require additional structure (a seesaw or Majorana mechanism for the latter). The 9 mesinos M^i_j transform as $(\mathbf{3}, \mathbf{\bar{3}})$ under the $SU(3)_L \times SU(3)_R$ flavour symmetry. If $SU(2)_L$ is embedded in $SU(3)_L$ as the upper 2×2 block, the decomposition $\mathbf{3} \rightarrow \mathbf{2} + \mathbf{1}$ provides a natural split into doublet and singlet composites, from which the SM quantum numbers can in principle be assigned. The full Standard Model embedding, electroweak charges, and the origin of the CKM matrix through non-holomorphic Kähler dressing are deferred to a companion work.

9. Summary

We have shown that: (i) the classical field energy of any U(1) gauge theory scales as $U \propto q^2$, motivating the mass ansatz $m \propto q^2$ which we hypothesise survives in the confining regime; (ii) $n = 2$ is the unique monomial for which the ratio $K = \sum q_i^2 / (\sum q_i)^2$ is direction-independent; (iii) a minimal $\mathcal{N}=1$ SU(3) gauge theory with three flavours realises both the direct and inverse sectors through its meson and adjugate composites. The quadratic charge dependence of the holomorphic masses is exact (protected by the non-renormalisation theorem); extending $K = 2/3$ to physical masses requires a generation-universal Kähler metric, which is natural at leading order but receives non-perturbative corrections. The one-parameter lepton prediction gives $m_\tau = 1776.97 \text{ MeV}$ (0.9σ from experiment); the zero-parameter inverse down-quark phenomenological test agrees at 1.2σ . If the Sumino cancellation mechanism [?] can be adapted to the non-degenerate gauge boson spectrum of the complementary Higgs phase—which requires further quantitative analysis—family gauge boson masses would be bounded to 1–10 TeV.

Acknowledgments. During the preparation of this work the author used Claude (Anthropic) for algebraic verification, numerical checks, and drafting assistance. The author reviewed and edited all content and takes full responsibility for the publication.

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figures/koide_inv_bw.pdf

Figure 1: Running of $K^{-1}(d, s, b)$ (solid) and $K(e, \mu, \tau)$ (dashed) from $Q = 1 \text{ GeV}$ to the Planck scale, computed with SMDR 5-loop RG evolution [?]. The purpose of the figure is to demonstrate scale-independence, not to read off absolute values (which should be taken from Table ??, since the SMDR default light-quark inputs differ from PDG 2024 by up to 7%). Shaded: $\pm 1\sigma$ from PDG mass uncertainties. Dotted: particle thresholds.