Composite fermions from $\mathrm{SU}(3)_F$ confinement: the sBootstrap as a structural effective field theory

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We identify Standard Model fermions with composites of a confining $\mathrm{SU}(3)_F$ family gauge theory: mesinos (ψ_M of $M^i{}_j = Q^i\tilde{Q}_j$) are leptons; adjugate composites provide quark partners via partial compositeness. The charge-squared mass law $m_i = \mu\,\mathfrak{z}_i^2$ is the unique monomial for which $K = \sum \mathfrak{z}_i^2/(\sum|\mathfrak{z}_i|)^2$ is independent of the Cartan direction (Theorem 1); the Seiberg seesaw then produces both a direct branch ($K = 2/3$ for leptons, 0.001%) and an inverse branch ($K^{-1} = 2/3$ for down quarks, 0.22%) from a single UV condition. An O'Raifeartaigh seed at $gv/m = \sqrt{3}$ breaks SUSY with $K = 2/3$ exactly. Five criteria—two structural, three suggestive—select $N = 3$ generations. The principal open problem is the non-perturbative Kähler metric, which controls the mapping from holomorphic to physical masses.

I. INTRODUCTION

The Standard Model contains 13 free parameters in the flavour sector—six quark masses, three charged-lepton masses, three CKM angles, and a CP phase—with no known relations among them. While the gauge sector is tightly constrained by the symmetry group $\mathrm{SU}(3)_c\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$, the Yukawa couplings that determine fermion masses are arbitrary complex numbers. The absence of flavour structure is one of the outstanding puzzles in particle physics, closely related to the hierarchy problem (why $m_e/m_t \sim 3\times10^{-6}$?) and to the origin of CP violation in the quark sector.

This paper develops a programme that addresses the flavour problem not by postulating textures or discrete symmetries on the Yukawa matrices, but by *identifying the observed fermions with composite fermions* of a confining gauge theory. The key idea is that the mass hierarchies and mixing patterns are not free parameters but consequences of the non-perturbative dynamics of confinement: the Seiberg seesaw, the adjugate map, and the charge algebra of $\mathrm{U}(3)_F$ together constrain the mass spectrum.

The idea that hadrons are composites of more fundamental quarks was developed through the "bootstrap" programme of Chew [**?**], which treated all hadrons democratically as bound states of one another. In 1966, Miyazawa [**?**] observed that the spectrum of hadrons exhibits a *supersymmetric* pattern, with mesons ($q\bar{q}$, bosons) and baryons (qqq, fermions) forming approximate supermultiplets. Catto and Gürsey [**?**] sharpened this: the diquark $[qq]$ (in the antisymmetric colour $\overline{\mathbf{3}}$) transforms as the SUSY partner of the antiquark $\bar{q}$, so the baryon $q[qq]$ is literally the fermionic partner of the meson $q\bar{q}$.

Masiero and Veneziano [**?**] constructed the first prototype model for one leptonic family from SU(3) SQCD with $N_f = N_c = 3$, showing that soft SUSY breaking splits composite supermultiplets to favour the fermions; Karasik [**?**] revisited composite Higgs from confining SQCD with modern tools. Luty and Mohapatra [**?**] constructed explicit SUSY composite models where all quarks and leptons emerge from confining $\mathrm{SU}(N)$ dynamics, with quarks as mesons and leptons as baryons; Luty and Terning [**?**] analysed the Kähler metric problem for composite masses in $N_f = N_c$ theories. Neither model has mass predictions; the present framework differs in three respects: (i) the confining group is $\mathrm{SU}(3)_F$ with $N_f = N_c = 3$ (quantum-modified constraint, not $N_f = N_c + 1$ dynamical superpotential); (ii) the identification is *reversed* (mesinos = leptons, not baryons); and (iii) the charge-squared law provides quantitative mass predictions.

The *sBootstrap* (supersymmetric bootstrap) [**?**] takes this one step further. Rather than treating the meson–baryon supersymmetry as an approximate hadronic symmetry, it elevates it to a **structural identification**: the composite fermions of a confining gauge theory *are* the Standard Model fermions. The foundational statement is:

$$\boxed{\text{mesinos} = \text{leptons}\,,\qquad \text{diquarkinos} = \text{quarks}\,.} \tag{1}$$

Here "mesino" means the fermionic component ψ_M of the meson chiral superfield $M^i{}_j = Q^i\tilde{Q}_j$, where Q^i and $\tilde{Q}_j$ are the fundamental "preon" fields of a confining $\mathrm{SU}(3)_F$ family gauge theory. The "diquarkino" is the fermionic component of the adjugate composite $\mathrm{adj}(M)_i$, whose quantum numbers match those of a diquark-type operator.

a. Evidence: $m_\pi^2 - m_\mu^2 \approx f_\pi^2$. The mass relation $m_\pi^2 - m_\mu^2 \approx f_\pi^2$ is *evidence* for the identification, not its definition. Numerically, $m_\pi^2 - m_\mu^2 = 139.57^2 - 105.66^2 = 8316\,\mathrm{MeV}^2$ while $f_\pi^2 = 92.2^2 = 8501\,\mathrm{MeV}^2$ (2.2% discrepancy). In the Volkov–Akulov framework [**?**], spontaneous SUSY breaking gives the mass relation $m^2_{\text{fermion}} = m^2_{\text{boson}} - f^2$, where f is the SUSY-breaking scale. If we identify $f = f_\pi$, then the muon is a pseudo-Goldstino

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whose mass is split from the pion by the chiral symmetry breaking scale.

b. The emergent $\mathcal{N}$=2 pairing. In $\mathcal{N}$=1 SUSY, each chiral multiplet (M_i, ψ_{M_i}) provides one fermion (the mesino). But the confined spectrum also contains the adjugate composite $\mathrm{adj}(M)_i$ (Sec. **??**), which forms a second chiral multiplet per family. The pair $(M_i, \mathrm{adj}(M)_i)$ has the mathematical structure of an $\mathcal{N}$=2 hypermultiplet—not because the UV theory has $\mathcal{N}$=2 supersymmetry, but because confinement *automatically* produces both M and $\mathrm{adj}(M)$ with the correct quantum numbers.

c. The Q-generator problem. In the Miyazawa superalgebra, a single supercharge Q maps $q\bar{q} \to q[qq]$ (meson to baryon). In the sBootstrap, the involution $M_i \leftrightarrow \Lambda^6/M_i$ (the adjugate map) is an *exact* $\mathbb{Z}_2$ symmetry of the quantum-modified constraint surface $\det M = \Lambda^6$: it exchanges the direct and inverse branches and is a holomorphic identity, not an approximation. The **open question** is whether this $\mathbb{Z}_2$ can be promoted to a continuous $\mathrm{SU}(2)_R$ rotation within the doublet $(M_i, \mathrm{adj}(M)_i)$. If so, the associated generator Q_2 would connect the meson sector (leptons) to the adjugate sector (quarks), realising Miyazawa's vision with two supercharges. Q_1 is the standard $\mathcal{N}$=1 supercharge; the approximate $\mathrm{SU}(2)_R$ is broken by the adjoint mass μ_Φ. Constructing Q_1 and Q_2 explicitly in the confined theory is an open problem.

d. Summary of results. The framework makes the following quantitative connections. The energy-balance condition $K = 2/3$ is satisfied by charged leptons (0.001%), pseudoscalar mesons $(-\pi, D_s, B)$ (0.10%), inverse down quarks (0.22%), and the quark chain (t, b, c)–$(b, c, -s)$–$(c, s, 0)$–$(s, 0, m_u{+}m_d)$ with joint significance $p \sim 1/21\,000$. The bion relation $\mathfrak{z}_b = 3\mathfrak{z}_s + \mathfrak{z}_c$ predicts $m_b = 4183\,\mathrm{MeV}$ (0.01%), and the top completion formula predicts $m_t = 168.5\,\mathrm{GeV}$ (2.3%). Five criteria (two structural, three suggestive) select $N = 3$ generations. The framework does NOT yet predict: the CKM matrix beyond the Cabibbo angle, the PMNS matrix, the origin of the Cartan angle α, or the Kähler metric Z_i.

e. Plan of the paper. We proceed by building up the framework layer by layer: composites (§**??**) → Seiberg dynamics (§**??**) → charge-squared law (§**??**) → emergent isospin (§**??**) → SUSY breaking (§**??**) → inverse sector and chain (§**??**) → generation uniqueness (§**??**) → Lagrangian (§**??**) → discussion and conclusions (§**??**, §**??**).

II. COMPOSITES AND CONFINEMENT

A. The gauge-theoretic realisation

The UV gauge group is

$$G_{\mathrm{UV}} = \mathrm{SU}(3)_F \times \mathrm{SU}(3)_c \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y\,, \qquad (2)$$

where $\mathrm{SU}(3)_F$ is the *family gauge symmetry* that confines at a scale Λ_F. The term "family" refers to the three generations of SM fermions: $\mathrm{SU}(3)_F$ acts on the generation index $i \in \{1, 2, 3\}$.

TABLE I. UV field content (above Λ_F). All fields listed as $\mathcal{N}$=1 chiral superfields except the gauge multiplets.

Field	$\mathrm{SU}(3)_F$	$\mathrm{SU}(3)_c$	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$	Role
Q_i^α	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{2} \oplus \mathbf{1}$	$-\frac{1}{2}, +1$	preon
$\tilde{Q}_i^\alpha$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\bar{\mathbf{2}} \oplus \mathbf{1}$	$+\frac{1}{2}, -1$	anti-preon
Φ	$\mathbf{8}$	$\mathbf{1}$	$\mathbf{1}$	0	adjoint (aux.)
V_F^μ	$\mathbf{8}$	$\mathbf{1}$	$\mathbf{1}$	0	gauge boson
q_L^a	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{2}$	$+\frac{1}{6}$	elem. quark
u_R^a	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$+\frac{2}{3}$	elem. quark
d_R^a	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$-\frac{1}{3}$	elem. quark

The fundamental matter fields ("preons") are:

- Q_i^α: transforming as $(\mathbf{3}_F, \mathbf{1}_c, \mathbf{3}_L)$, where $\alpha \in 3_L$ is an index under the *global* $\mathrm{SU}(3)_L$ flavour symmetry (containing $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ as a subgroup).
- $\tilde{Q}_i^\alpha$: the conjugate, $(\bar{\mathbf{3}}_F, \mathbf{1}_c, \bar{\mathbf{3}}_L)$.

The $\mathrm{SU}(3)_L$ is **not** a gauge symmetry: it is a global symmetry whose sole role is to organise the branching $\mathbf{3}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}$, which assigns the correct SM quantum numbers to the composites and projects out exotic charges (Sec. **??**). The preons are colour singlets: quarks (which carry colour) arise through partial compositeness.

B. The meson decomposition

When $\mathrm{SU}(3)_F$ confines, the preons form composite bound states. The meson superfield $M^\alpha{}_i = Q^\alpha \tilde{Q}_i$ carries one $\mathrm{SU}(3)_L$ index and one $\mathrm{SU}(3)_F$ index, giving $3 \times 3 = 9$ chiral multiplets. Under the electroweak branching $\mathbf{3}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}$:

$$M^\alpha{}_i \;=\; \underbrace{M^{1,2}{}_i}_{(\mathbf{2}_{-1/2})} \;\oplus\; \underbrace{M^3{}_i}_{(\mathbf{1}_{+1})}\,. \qquad (3)$$

This decomposition is a **projection tool**: its role is to clean the $\mathbf{15} \oplus \overline{\mathbf{15}} \oplus \mathbf{24}$ composite content for the SO(32) string embedding (Sec. **??**). Without it, the $\mathbf{24}$ of $\mathrm{SU}(5)_f$ contains $(\bar{\mathbf{3}}, \mathbf{2})_{5/6}$ with exotic charge $Q = 4/3$ [**?**]—the analogue of X and Y bosons in SU(5) GUT. The branching projects these out, restricting composites to SM-compatible charges. This is the *only* reason for introducing $\mathrm{SU}(3)_L$. The mesino identification follows:

$$\boxed{\psi_{M^{1,2}{}_i} \equiv L_i \;=\; \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L\,, \qquad \psi_{M^3{}_i} \equiv e_i^c\,.} \qquad (4)$$

The 9 mesino fermions furnish the complete SM lepton sector: $3L_i + 3e_i^c = 3$ lepton doublets + 3 charged-lepton singlets. Anomaly matching between UV preons and IR composites is satisfied for $N_f = N_c$ [**?**].

TABLE II. IR composites (below Λ_F, $N_f = N_c = 3$). Independent DOF: $9\,(M) + 1\,(B) + 1\,(\tilde{B}) - 1\,(X) = 10 = 18 - 8$. adj$(M)$ is *derived*—not independent DOF.

Composite	Expr.	$\mathrm{SU}(2)_L$	$\mathrm{U}(1)_Y$	Ch.	SM id.
${M^{1,2}}_i$	$Q^{1,2}\tilde{Q}_i$	$\mathbf{2}$	$-\frac{1}{2}$	6	H_d, L_i
${M^3}_i$	$Q^3\tilde{Q}_i$	$\mathbf{1}$	$+1$	3	e^c_i
Derived operators:					
${\mathrm{adj}(M)^{1,2}}_i$	$\varepsilon\varepsilon MM$	$\mathbf{2}$	$+\frac{1}{2}$	(6)	H_u
${\mathrm{adj}(M)^3}_i$	$\varepsilon\varepsilon MM$	$\mathbf{1}$	-1	(3)	$\tilde{\nu}$-type
B	εQQQ	$\mathbf{1}$	0	1	sterile
$\tilde{B}$	$\varepsilon\tilde{Q}^3$	$\mathbf{1}$	0	1	sterile
X	constraint	$\mathbf{1}$	0	-1	Lagr. mult.

C. Quarks via partial compositeness

Quarks carry colour ($\mathrm{SU}(3)_c$) and cannot be mesinos (which are colour singlets). In supersymmetric gauge theories, the smooth connection between confinement and Higgs phases—the SUSY analogue of Fradkin–Shenker complementarity [?]—is established by Seiberg duality [?] and s-confinement: for SU(3) with $N_f = 3$, the confined composites ($M, B, \tilde{B}$, subject to the quantum constraint) describe the same 10 light DOF as the Higgs phase where preons develop VEVs and eat 8 Goldstone bosons ($2 \times 9 - 8 = 10$).

*a. Colour-**3** impossibility.* Since the preons Q^i and $\tilde{Q}_i$ are colour singlets (Table ??), every polynomial composite $Q^{i_1} \cdots Q^{i_n} \tilde{Q}_{j_1} \cdots \tilde{Q}_{j_m}$ is a product of colour singlets and hence a colour singlet. In particular, $\mathrm{SU}(3)_F$ composites *never* carry colour **3**—only **1**, **6**, **8**, etc. appear. Quarks, which carry colour, therefore *cannot* be composites of this theory.

b. Quarks via partial compositeness. The coloured quarks of the SM enter through **partial compositeness** [?]: each elementary quark q_{elem} (which carries colour and is a singlet of $\mathrm{SU}(3)_F$, see Table ??) mixes linearly with a composite operator $\mathcal{O}$ of matching SM quantum numbers through both left- and right-handed couplings:

$$\mathcal{L}_{\mathrm{mix}} = \lambda_{q_L}\, \bar{q}_{L,\mathrm{elem}}\, \mathcal{O}_L + \lambda_{q_R}\, \bar{q}_{R,\mathrm{elem}}\, \mathcal{O}_R + \mathrm{h.c.} \tag{5}$$

The physical quark mass is [?]

$$m_q \sim \frac{\lambda_{q_L}\, \lambda_{q_R}}{M_*}\, \langle \mathcal{O} \rangle\,, \tag{6}$$

where M_* is the compositeness scale. If both mixing parameters scale with the family charge, $\lambda_{q_L} \propto \lambda_{q_R} \propto \mathfrak{z}_q$, then $m_q \propto \mathfrak{z}_q^2$—the charge-squared law. The simplest realisation has $\lambda_{q_L} = \lambda_{q_R}$ (parity-symmetric mixing), but even with unequal mixings the $\mathfrak{z}^2$ scaling holds provided both are proportional to the same family charge. By Seiberg duality [? ?], the confinement and Higgs phases of $\mathrm{SU}(3)_F$ are smoothly connected (complementarity), so the physical quark in the confined description IS the dressed preon in the Higgs description.

c. The lepton sector obstruction. There is an honest open problem. The holomorphic meson masses from the Seiberg vacuum (Sec. ??) give mass *ratios* determined by the UV preon masses. The mesino mass ratio $m_\psi({M^u}_s)/m_\psi({M^u}_d) = m_s/m_d \approx 20$, while the physical lepton ratio $m_\mu/m_e = 207$—a factor 10.3 discrepancy. Kähler corrections preserve the rank of the mass matrix but can only introduce flavour-independent rescalings if Z is generation-universal. Flavour-*dependent* rescaling (different Z_i per generation) can generate the discrepancy. Two candidate mechanisms: (i) Nelson–Strassler anomalous dimensions at a confining fixed point [?]; (ii) a separate SU(2) confining sector with $N_f = 3$ and an O'Raifeartaigh deformation (Sec. ??). We flag this as an open issue, not an inconsistency: the framework predicts the *structure* of the mass matrix (diagonal, with entries $\propto \mathfrak{z}_i^2$) but not the overall Kähler normalisation.

III. SEIBERG CONFINEMENT AND THE ADJUGATE MECHANISM

A. Review: s-confinement for $N_f = N_c$

We review the non-perturbative dynamics of $\mathrm{SU}(N_c)$ SQCD with N_f flavours of quarks and antiquarks in the $\mathcal{N}=1$ framework [? ?].

For $N_f = N_c$, the theory is *s-confining* ("smoothly confining"): the low-energy degrees of freedom are the gauge-invariant composites M, B, $\tilde{B}$, and there is no residual gauge group. The classical moduli space is parametrised by the constraint $\det M - B\tilde{B} = 0$. Quantum effects (a one-instanton calculation) modify this to the **quantum-modified constraint**:

$$\det M - B\tilde{B} = \Lambda^{2N_c}\,, \tag{7}$$

where Λ is the dynamical scale. The right-hand side Λ^{2N_c} is generated by instantons and is exact (protected by holomorphy and symmetries). The effect is to smooth out the moduli space: the origin $M = B = \tilde{B} = 0$ is removed.

The constraint is implemented in the effective superpotential by a Lagrange multiplier X:

$$W = X\big(\det M - B\tilde{B} - \Lambda^6\big) + \sum_i m_i\, {M^i}_i\,. \tag{8}$$

The first term enforces the constraint; the second adds explicit preon masses which break global symmetry and lift the moduli space to isolated vacua.

B. The mesonic vacuum and the Seiberg seesaw

On the mesonic branch ($B = \tilde{B} = 0$), we solve the F-term equations $\partial W / \partial {M^i}_j = 0$. For diagonal $M =$

TABLE III. Phase structure of $\mathrm{SU}(3)_F$ as heavy flavours decouple. Scale matching: $\Lambda_3^6 = \Lambda_4^5 m_c$, $\Lambda_4^5 = \Lambda_5^4 m_b$. Each threshold crossing changes the non-perturbative dynamics qualitatively.

N_f	Energy range	Phase	Key property
6	$Q > m_t$	conformal window	IR fixed point
5	$m_b < Q < m_t$	conformal window $(4.5 < 5 < 9)$	IR fixed point
4	$m_c < Q < m_b$	ISS metastable [?]	unique integer in $(N_c, \frac{3}{2}N_c)$
3	$Q < m_c$	s-confinement	$\det M = \Lambda^6$

$\mathrm{diag}(M_1, M_2, M_3)$:

$$\frac{\partial W}{\partial M_i} = X \cdot \frac{\partial(\det M)}{\partial M_i} + m_i = X \cdot \prod_{j \neq i} M_j + m_i = 0\,. \quad (9)$$

The product $\prod_{j\neq i} M_j$ is precisely the i-th diagonal entry of the adjugate matrix. From the constraint $\det M = M_1 M_2 M_3 = \Lambda^6$, and taking ratios of F-term equations for different i: $M_i/M_j = m_j/m_i$, which gives the **Seiberg seesaw**:

$$\langle M_i \rangle = \frac{\Lambda^2 P^{1/3}}{m_i}\,, \qquad P = m_1 m_2 m_3\,. \quad (10)$$

Heavy preons produce light mesons, and vice versa. This is the single most important equation in the framework.

C. The N_f-dependent phase structure

The behaviour of $\mathrm{SU}(3)_F$ depends critically on how many preon flavours are light compared to the RG scale. As the energy decreases, heavy quarks are sequentially integrated out, changing the effective number of flavours and therefore the phase of the gauge theory.

At $N_f = 5$, the theory enters the conformal window ($\frac{3}{2}N_c < N_f < 3N_c$, i.e. $4.5 < 5 < 9$): it flows to an interacting IR fixed point, the Seiberg conformal fixed point. At $N_f = 4$, it saturates the lower bound of the ISS window [?]: $N_c + 1 \leq N_f < \frac{3}{2}N_c$, i.e. $4 \leq N_f < 4.5$. This is the window where metastable SUSY-breaking vacua exist. $N_f = 4$ is the *unique* integer in this window for $N_c = 3$—a discrete selection that plays a role in the effective Lagrangian (Sec. ??). The metastable vacuum has bounce action $S_{\mathrm{bounce}} \sim 10^2$–$10^3$, giving a lifetime vastly exceeding the age of the universe. At $N_f = 3$, the theory s-confines with the quantum-modified constraint described above.

D. The adjugate: definition and properties

The adjugate of a square matrix M is defined by:

$$\mathrm{adj}(M) \cdot M = \det(M) \cdot I\,. \quad (11)$$

Unlike the inverse $M^{-1} = \mathrm{adj}(M)/\det M$, the adjugate is a *polynomial* in the entries of M—holomorphic and well-defined even when $\det M = 0$. For diagonal $M = \mathrm{diag}(M_1, M_2, M_3)$:

$$\mathrm{adj}(M) = \mathrm{diag}(M_2 M_3,\ M_1 M_3,\ M_1 M_2)\,. \quad (12)$$

Each entry is the product of the *other two* diagonal entries. Verification: $\mathrm{adj}(M)_1 \cdot M_1 = M_1 M_2 M_3 = \det M$. The ordering is *reversed*: if $M_1 < M_2 < M_3$, then $\mathrm{adj}(M)_1 = M_2 M_3 > \mathrm{adj}(M)_2 = M_1 M_3 > \mathrm{adj}(M)_3 = M_1 M_2$. This ordering reversal is the physical origin of the direct/inverse mass duality: the heaviest preon produces the lightest meson VEV and the largest adjugate entry. For general matrices, $\mathrm{adj}(M)_{ij} = (-1)^{i+j} C_{ji}$ (transpose of the cofactor matrix). The key mathematical property is that $\mathrm{adj}(M)$ is holomorphic: it is well-defined even at $\det M = 0$, unlike M^{-1}.

E. The two Yukawa operators: direct vs. inverse

From the F-term equations (??):

$$\mathrm{adj}(M)_i = -\frac{m_i}{X} = -\Lambda^3\, m_i\,, \quad (13)$$

using $X = -P^{1/3}/\Lambda^4$ from the constraint (the simpler form $X = -1/\Lambda^3$ holds in the convention $P^{1/3} = \Lambda$, i.e. equal UV masses). Together with $\langle M_i \rangle = \Lambda^2 P^{1/3}/m_i$:

$$M_i \propto \frac{1}{m_i} \quad \text{(inverse: heavy UV} \to \text{light IR)}\,, \quad (14)$$

$$\mathrm{adj}(M)_i \propto m_i \quad \text{(direct: heavy UV} \to \text{heavy IR)}\,. \quad (15)$$

These are the two natural Yukawa operators of the confined theory. The Seiberg dynamics provides both from the single composite field M, related by the holomorphic identity $\mathrm{adj}(M) = \Lambda^6 M^{-1}$ on the constraint surface.

The physical assignment [? ?]:

- **Leptons** are mesinos ψ_M. Their mass comes from the F-term $|F_{M_i}| \propto m_i$ (direct, proportional to UV mass). Hence: $K(m_e, m_\mu, m_\tau) = 0.666661$ (standard energy-balance, 0.001%).
- **Down-type quarks** couple to the meson VEV $\langle M \rangle \propto 1/m_i$ through partial compositeness. Hence: $K^{-1}(m_d, m_s, m_b) = 0.6652$ (inverse energy-balance, 0.22%).

A *single UV condition* $K(m^{\mathrm{UV}}) = 2/3$ produces both the standard lepton Koide and the inverse quark Koide, unified by the involution $M_i \leftrightarrow \Lambda^4/M_i$.

a. Automatic pairing of direct and inverse. If the UV masses satisfy $m_i = \mu \mathfrak{z}_i^2$ with $\mathrm{U}(3)_F$ charges, then on the constraint surface $a_i = \alpha \mathfrak{z}_i^2$ and $d_i = \beta/\mathfrak{z}_i^2$ with $\alpha\beta = \Lambda^6$. Both $K_{\mathrm{dir}} = K[\mathfrak{z}_1, \mathfrak{z}_2, \mathfrak{z}_3]$ and $K_{\mathrm{inv}} = K[\mathfrak{z}_1^{-1}, \mathfrak{z}_2^{-1}, \mathfrak{z}_3^{-1}]$ evaluate to $2/3$ by the same algebraic identity (Theorem ??): the $n = 2$ uniqueness holds for *both* $\mathfrak{z}_i$ and $1/\mathfrak{z}_i$. The inverse energy-balance is a holomorphic consequence of the quantum-modified vacuum, not an independent phenomenological input.

TABLE IV. Direct and inverse branch assignment. Product rule: $a_i \cdot d_i = \mu\nu = \text{const}$ ($\det M = \Lambda^6$).

Branch	Operator	Formula	Order	Sector
Direct	$\mathrm{adj}(M)_i$	$a_i = \mu\mathfrak{z}_i^2$	$a_1 < a_2 < a_3$	Leptons
Inverse	M_i	$d_i = \nu/\mathfrak{z}_i^2$	$d_1 > d_2 > d_3$	Down quarks

F. Off-diagonal masses and transparency

The off-diagonal meson masses are [?]:

$$m(M^i{}_j) = \frac{m_i m_j}{P^{1/3}}, \qquad i \neq j\,. \tag{16}$$

These depend only on ratios of the input masses—they are Λ-independent. The diagonal mass matrix has $(W_S)_{kk} = 0$ for all k (by multilinearity of the determinant: expanding $\det M$ along row k, the derivative $\partial(\det M)/\partial M_k^k$ contains no factor of M_k^k, so it cancels in the full F-term). This means the Seiberg dynamics is **transparent**: it preserves $K = 2/3$ (if present in the UV masses) but cannot generate it. The origin of $K = 2/3$ must come from the charge-squared law (Sec. **??**).

IV. THE CHARGE-SQUARED MASS LAW

We derive the mass ansatz $m_i = \mu\,\mathfrak{z}_i^2$, where $\mathfrak{z}_i = z_0 + z_i$ is the signed family charge. Three independent arguments converge on the same result.

A. The $\mathrm{U}(3)_F$ charge algebra

a. Sign conventions in the energy-balance formula. The Koide ratio $K = \sum m_i/(\sum\sqrt{m_i})^2$ is defined for positive masses. Throughout this paper, the "signed charge" $\mathfrak{z}_i$ can be negative when the corresponding particle is a pseudo-Goldstone boson approaching $m = 0$ from the chiral limit (e.g. the pion), or when a mass triple contains both direct and inverse-branch members (e.g. the quark chain link $(b, c, -s)$). In these cases, the energy-balance formula is evaluated with $m_i = \mu\,\mathfrak{z}_i^2$ and $\sqrt{m_i} = |\mathfrak{z}_i| \cdot \mathrm{sign}(\mathfrak{z}_i)$, so that *signed* square roots enter the denominator. The sign assignment is *not* freely chosen to fit: it is determined by the branch (direct vs. inverse, Sec. **??**) or by the chiral limit structure (GOR relation, $m_\pi^2 \propto m_u + m_d \to 0$).

The family symmetry $\mathrm{U}(3)_F = \mathrm{SU}(3)_F \times \mathrm{U}(1)$ has a Cartan subalgebra spanned by two diagonal generators of $\mathrm{SU}(3)_F$ plus the $\mathrm{U}(1)$ generator. With canonical normalisation:

$$\sum_{i=1}^{3} z_i = 0\,, \qquad \sum_{i=1}^{3} z_i^2 = \frac{1}{2}\,, \qquad z_0 = \frac{1}{\sqrt{6}}\,. \tag{17}$$

The signed charge is $\mathfrak{z}_i = z_0 + z_i$, with $z_i = \cos(\alpha + 2\pi(i-1)/3)/\sqrt{3}$ parametrised by the Cartan direction α (the "sector angle").

B. Argument 1: Algebraic uniqueness of $n = 2$

Consider a general monomial mass law $m_i = c\,(z_0 + z_i)^n$ and define:

$$K_n(\alpha) \equiv \frac{\sum_i (z_0+z_i)^n}{\left(\sum_i (z_0+z_i)^{n/2}\right)^2}\,. \tag{18}$$

Theorem 1 (Uniqueness of $n = 2$). *For integer powers n and $\mathrm{U}(3)$ charges $\mathfrak{z}_i = z_0 + z_i$, the ratio $K_n(\alpha)$ is independent of the Cartan direction α if and only if $n = 2$ (besides $n = 0$). At $n = 2$: $K_2 = 2/3$.*

Proof. We expand $\sum(z_0 + z_i)^n$ using the binomial theorem. The power sums $S_k = \sum z_i^k$ satisfy: $S_0 = 3$; $S_1 = 0$ (tracelessness of SU(3)); $S_2 = 1/2$ (normalisation, α-independent); and S_k for $k \geq 3$: all depend on α through $\cos 3\alpha$ or higher harmonics.

For $n = 2$: $\sum(z_0 + z_i)^2 = 3z_0^2 + 2z_0 \cdot 0 + 1/2 = 1$. Also $\sum(z_0 + z_i) = 3z_0 = \sqrt{3/2}$. So $K_2 = 1/(3/2) = 2/3$, independent of α.

For $n \geq 3$: the binomial expansion of the numerator contains the term $\binom{n}{3} z_0^{n-3} S_3$ with coefficient $n(n-1)(n-2)/6 \neq 0$. Since $S_3 = \sum z_i^3 \propto \cos 3\alpha$ (using $z_i = \cos(\alpha + 2\pi(i-1)/3)/\sqrt{3}$), the numerator acquires α-dependence. The denominator $(\sum \mathfrak{z}_i^{n/2})^2$ generically also depends on α for $n \geq 3$, and the α-dependence does not cancel in the ratio. For $n = 1$: $K_1 = \sum \mathfrak{z}_i/(\sum \mathfrak{z}_i^{1/2})^2$ involves square roots of the charges, which are not polynomial in α; the ratio still varies.

Computer algebra (Sympy) verification confirms K_n varies for all tested real $n \in [-5, 10]$ at spacing 0.001, except $n = 2$ and the trivial $n = 0$. □

The physical meaning: $n = 2$ is the *unique* monomial mass law for which the energy-balance ratio K is a group-theoretic identity ($K = 2/3$), independent of which sector we are looking at. For any other exponent, K would hold in some sectors but not others—destroying the universality observed across leptons, mesons, and quarks.

C. Argument 2: Composite self-energy

Koide's original derivation [?] provides a physical mechanism. The Coulomb self-energy of a composite with total family charge $\mathfrak{z}_i$ is:

$$\delta_i = \frac{1}{2}\big[Q_F(A) + Q_F(B)\big]^2 \cdot v_F \propto \mathfrak{z}_i^2\,, \tag{19}$$

the gauge-theoretic version of the capacitor energy $E = Q^2/(2C)$: Gauss's law gives $|\mathbf{E}| \propto \mathfrak{z}_i$, and the energy $U \propto \int |\mathbf{E}|^2 \propto \mathfrak{z}_i^2$. In a confining theory, the dominant mass contribution is the string tension $\sigma \cdot L$, which is generation-independent (it depends on the colour representation, not the Cartan eigenvalue). The $\mathfrak{z}_i^2$ scaling applies to the generation-dependent Coulomb self-energy at short distances. The generation-independent piece is absorbed into the overall scale μ. In SUSY, the fermion mass is an F-term (holomorphic), protected against non-perturbative confining corrections.

D. Argument 3: Bilinear structure

The meson $M^i{}_i = Q^i\tilde{Q}_i$ is a bilinear. If each preon contributes $\propto \mathfrak{z}_i$, the composite VEV scales as $\mathfrak{z}_i^2$ [?]. This is the same structure as a seesaw: any mass from two VEV insertions gives $m \propto v^2$.

a. Status summary. Argument 1 is a rigorous algebraic theorem. Argument 2 is structural motivation (Coulomb scaling assumed to survive confinement). Argument 3 is a consistency check. The charge-squared law is motivated by three structural inputs: (i) **Holomorphy**: the superpotential $W(M)$ is holomorphic, so the mass is $\propto \mathfrak{z}^2$ (holomorphic square), not $|\mathfrak{z}|^2$ (Hermitian norm); (ii) **Bilinear structure**: $M = Q\tilde{Q}$ involves two preon fields each contributing $\propto \mathfrak{z}$; (iii) **Canonical normalisation**: $z_0 = 1/\sqrt{6}$ from the $\mathrm{U}(3)_F$ kinetic term; the $n = 2$ uniqueness (Theorem **??**) shows this is the only monomial giving $K = 2/3$ universally.

The charge-squared law $m_i = \mu\,\mathfrak{z}_i^2$ is the framework's central *postulate*: uniquely motivated but not derived from confining dynamics. The uniqueness theorem is a *necessary* condition: if masses are monomials, then $n = 2$ is selected. It does not exclude non-monomial $m = f(\mathfrak{z})$. Seven SUSY-breaking derivation attempts fail because gauge mediation gives $\mathfrak{z}^2$ scaling for soft (Kähler) masses, not for holomorphic (superpotential) masses needed for exact $K = 2/3$.

E. The energy-balance condition

For three charges $\mathfrak{z}_k = z_0 + z_k$ with $\sum z_k = 0$:

$$K = \frac{\sum \mathfrak{z}_k^2}{(\sum \mathfrak{z}_k)^2} = \frac{1}{3} + \frac{\langle z_k^2\rangle}{3z_0^2}\,. \tag{20}$$

Therefore:

$$\boxed{K = \frac{2}{3} \quad \Longleftrightarrow \quad \langle z_k^2\rangle = z_0^2\,.} \tag{21}$$

At $K = 2/3$, the variance of the SU(3) charges equals the square of the U(1) charge: the splitting scale matches the base scale. This is the *equipartition point* between the perturbative limit ($K \to 1/3$, all masses equal) and the dominant limit ($K \to 1$, one mass dominates).

In the charge plane, the variable $\langle z^2\rangle/z_0^2 = 3K-1$ measures deviation from $K = 2/3$. The zero-mass algebraic fixed points of the charge plane occur at $\alpha = 15°$ and $45°$ (SU(3) convention), with ratio $r_0 = (2+\sqrt{3})^2 \approx 13.93$. All five physical Koide tuples lie within $5.5°$ of the $15°$ fixed point—their clustering reflects proximity to a single algebraic attractor.

$K = 2/3$ is **not** an RG attractor: one-loop soft-mass renormalisation drives $K \to 1/3$, not $K \to 2/3$. The energy-balance condition is a *boundary condition* at the SUSY-breaking scale, not an RG fixed point.

a. Seed triples. For a triple $(0, \mathfrak{z}_a, \mathfrak{z}_b)$ with one massless member, the energy-balance condition becomes: $K = (\mathfrak{z}_a^2 + \mathfrak{z}_b^2)/(\mathfrak{z}_a + \mathfrak{z}_b)^2$. Setting $r = \mathfrak{z}_a/\mathfrak{z}_b$ and solving $K = 2/3$: $r^2 + 1 = 2(r+1)^2/3$, giving $r = 2-\sqrt{3} \approx 0.268$. This is a universal ratio: *any* seed triple satisfying $K = 2/3$ must have $\mathfrak{z}_{\rm small}/\mathfrak{z}_{\rm large} = 2-\sqrt{3}$.

- **Meson seed** $(0, \pi, D_s)$: $\mathfrak{z}_\pi/\mathfrak{z}_{D_s} = 0.266$ (0.6% off), $K = 0.668$ (0.18%).
- **Quark seed** $(0, s, c)$: $\mathfrak{z}_s/\mathfrak{z}_c = 0.271$ (1.2% off), $K = 0.664$ (0.35%).
- **Leptons** (e, μ, τ): $K = 0.666661$ (0.001%)—an "almost-seed" ($\mathfrak{z}_e = 0.72\,\mathrm{MeV}^{1/2}$ is small but nonzero).

b. The meson triple $(-\pi, D_s, B)$. With signed charges $\mathfrak{z}_\pi < 0$, $\mathfrak{z}_{D_s} > 0$, $\mathfrak{z}_B > 0$:

$$K(-\pi, D_s, B) = \frac{\mathfrak{z}_\pi^2 + \mathfrak{z}_{D_s}^2 + \mathfrak{z}_B^2}{(-\mathfrak{z}_\pi + \mathfrak{z}_{D_s} + \mathfrak{z}_B)^2} = 0.6674\,, \tag{22}$$

0.10% from 2/3—the second most precise triple known. Among 660 meson combinations (11 pseudoscalars, 4 sign patterns), only $(-\pi, D_s, B)$ and $(-\pi, D_s, B_s)$ lie within 0.2%.

The pseudoscalar triple $K(\pi, K, \eta) = 0.36$ is far from 2/3—and this is *expected*, not anomalous. The GOR relation [?] $M_\pi^2 = B_0(m_u + m_d)$ gives boson mass *squared* linear in quark mass; composing with $m_q \propto \mathfrak{z}^2$ gives $M_\pi^2 \propto (\mathfrak{z}_u^2 + \mathfrak{z}_d^2)$, a sum of two squares—not a single $\mathfrak{z}^2$. Heavy mesons have $m_{\rm meson} \approx m_Q + \Lambda_{\rm QCD}$ (HQET), so $m_{\rm meson} \approx m_Q \propto \mathfrak{z}_Q^2$: the charge-squared law for the heavy quark propagates to the heavy meson mass. The pion enters with a *negative* sign because it approaches $m = 0$ from below in the chiral limit (a pseudo-Goldstone boson).

c. Sumino protection. The charged-lepton $K = 0.666661$ (0.001%) is too precise to survive QED corrections without protection. Sumino [?] showed that a gauged $\mathrm{U}(3)_F$ with $\alpha_{\rm em} = \alpha_F/4$ cancels the dangerous $\alpha\, m_i \log m_i^2$ corrections, placing family gauge bosons at 10^2–10^3 TeV.

V. THE EMERGENT $\mathcal{N}$=2 STRUCTURE

A. What "emergent $\mathcal{N}$=2" means

The detailed analysis of the emergent $\mathcal{N}$=2 structure, including the Seiberg–Witten curve at the Koide point and the Cayley–Hamilton cubic, is developed in a companion paper [?]; here we summarise the key features.

In genuine $\mathcal{N}$=2 theories [?], each matter field sits in a hypermultiplet: a pair of $\mathcal{N}$=1 chirals $(\Phi, \tilde{\Phi})$ related by $\mathrm{SU}(2)_R$. In our $\mathcal{N}$=1 theory, the confined spectrum automatically produces two chiral multiplets per family:

$$(M_i,\ \mathrm{adj}(M)_i) \qquad \text{per family } i = 1, 2, 3\,, \tag{23}$$

with complementary quantum numbers: $M^{1,2} \sim (\mathbf{2}, -1/2)$ while $\mathrm{adj}(M)^{1,2} \sim (\mathbf{2}, +1/2)$. This has the *mathematical structure* of a hypermultiplet—emergent from confinement plus the adjugate identity, not from UV symmetry.

B. Breaking the doublet: the quartic

An auxiliary adjoint Φ (transforming as $\mathbf{8}$ of $\mathrm{SU}(3)_F$) with mass μ_Φ and Yukawa coupling $W \supset \sqrt{2}\, g\, \tilde{Q}\, \Phi\, Q$ [?] is integrated out via its F-term $F_\Phi = \sqrt{2}\, g\, \tilde{Q}Q + \mu_\Phi \Phi = 0$, giving $\Phi = -\sqrt{2}\, g\, \tilde{Q}Q/\mu_\Phi$. Substituting back:

$$W_{\text{quartic}} = -\frac{g^2}{\mu_\Phi}\left[\mathrm{Tr}M^2 - \frac{1}{3}(\mathrm{Tr}M)^2\right]. \tag{24}$$

The adjoint Φ is NOT a physical particle in the sBootstrap: it is a UV field with mass μ_Φ that is integrated out below that scale, leaving only the quartic coupling (**??**) as its low-energy footprint. This is standard EFT decoupling—no different from the W boson producing the Fermi four-fermion interaction below M_W—and requires no special assumption beyond μ_Φ being large compared to Λ_F. As $\mu_\Phi \to \infty$, the quartic vanishes and pure $\mathcal{N}$=1 SQCD is recovered. At finite μ_Φ, the quartic couples the diagonal meson entries to each other, mixing the direct and inverse branches.

a. The Yukawa coupling. In genuine $\mathcal{N}$=2 SUSY [? ?], the Yukawa coupling equals the gauge coupling: $y = \sqrt{2}\, g$. The UV theory here is $\mathcal{N}$=1, where y is a priori free. We *adopt* $y = \sqrt{2}\, g$ as the simplest choice consistent with the emergent doublet structure—it is the value that would hold if the $(M_i, \mathrm{adj}(M)_i)$ pairing were an exact $\mathcal{N}$=2 hypermultiplet. This is an *assumption*, not a derivation from the $\mathcal{N}$=1 dynamics. If it holds even approximately, the Yukawa hierarchy is not a hierarchy of couplings but of VEVs (via the Seiberg seesaw).

C. Both Higgs doublets from one meson

The structural consequence of the $\mathcal{N}$=2 $\to$ $\mathcal{N}$=1 breaking is that the doublet $(M_i, \mathrm{adj}(M)_i)$ splits into components with *different* electroweak quantum numbers:

$$M^{1,2} \sim (\mathbf{2}, -1/2) = H_d \qquad \text{(down-type Higgs)}\,, \tag{25}$$

$$\mathrm{adj}(M)^{1,2} \sim (\mathbf{2}, +1/2) = H_u \qquad \text{(up-type Higgs)}\,. \tag{26}$$

Both MSSM Higgs doublets emerge from a single meson matrix. This is not a postulate but a consequence of the adjugate structure: M and $\mathrm{adj}(M)$ automatically carry conjugate quantum numbers.

There is no independent $\tan\beta$ parameter. In the MSSM, $\tan\beta = \langle H_u\rangle/\langle H_d\rangle$ is free; here $H_d = M^{1,2}$ and $H_u = \mathrm{adj}(M)^{1,2}$ are both determined by the same meson matrix and constrained by $\det M = \Lambda^6$: their ratio is fixed by Seiberg dynamics, not adjustable. The "superpartners" are the mesons themselves, split from mesinos by f_π; there is no second SUSY at the TeV scale. For mixed up/down charge triples, $K(m_c, m_b, m_t) = 2/3$ involves both Higgs couplings symmetrically, consistent with equal VEVs ($\mathrm{adj}(M) \cdot M = \det M$ on the constraint surface).

D. The critical scale and self-completion

The energy-balance vacuum exists only above

$$\mu_{\Phi,\text{crit}} \approx 4.5\,\text{TeV}\,. \tag{27}$$

Under 5-loop SMDR running [? ?], $K^{-1}(d, s, b)$ passes through $2/3$ at $Q \approx 280\,\text{TeV}$; the quartic must be large enough to sustain the mixed vacuum, requiring $\mu_\Phi \lesssim 4.5\,\text{TeV}$.

The model self-completes: above Λ_F, the theory is $\mathrm{SU}(3)_F \times \mathrm{SM}$ with preons, asymptotically free to the Planck scale. The only additional scale is the family gauge boson mass at 10^2–10^3 TeV [?]. There are no stops, sbottoms, gluinos, or conventional SUSY partners: the "SUSY" refers to composite-sector structure, not a TeV-scale extension.

VI. SUSY BREAKING: SEEDS, BLOOM, AND BIONS

A. The O'Raifeartaigh model and the seed

The O'Raifeartaigh superpotential [?] with three chiral superfields:

$$W_{\text{OR}} = f\,\Phi_0 + m\,\Phi_1\Phi_2 + g\,\Phi_0\Phi_1^2\,, \tag{28}$$

breaks SUSY because the F-terms

$$F_0 = f + g\,\Phi_1^2\,, \quad F_1 = m\,\Phi_2 + 2g\,\Phi_0\,\Phi_1\,, \quad F_2 = m\,\Phi_1 \tag{29}$$

cannot simultaneously vanish ($F_0 = 0$ requires $\Phi_1^2 = -f/g$, but then $F_2 = m\Phi_1 \neq 0$). The fermionic mass matrix at $\langle\Phi_0\rangle = v$, $\langle\Phi_1\rangle = \langle\Phi_2\rangle = 0$:

$$\mathcal{M}_F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2gv & m \\ 0 & m & 0 \end{pmatrix}, \tag{30}$$

has eigenvalues $m_\pm = \sqrt{g^2v^2 + m^2} \pm gv$ and $m_0 = 0$ (the goldstino). At the seed condition $gv/m = \sqrt{3}$: $m_- = (2 - \sqrt{3})\,m$ and $m_+ = (2 + \sqrt{3})\,m$. Computing K:

$$K(0, m_-, m_+) = \frac{m_- + m_+}{(\sqrt{m_-} + \sqrt{m_+})^2} = \frac{4m}{6m} = \frac{2}{3}\,, \tag{31}$$

exactly. Kähler stabilisation at the pseudo-modulus pole with $\Lambda_K = m$ requires:

$$c = -\frac{1}{12}\,. \tag{32}$$

a. Why this is "mild" SUSY breaking. The fermion masses are $\mathcal{O}(m)$, while the SUSY-breaking scale is f. In the ISS dictionary [?], $g \to h$, $m \to h\mu_{\rm ISS}$, $v \to X$, so the seed condition becomes $\langle X\rangle/\mu_{\rm ISS} = \sqrt{3}$. The "mildness" is the statement that $f \ll m^2$: in the sBootstrap, $f = f_\pi^2 \approx (92\,\text{MeV})^2$ while $m = \mu/4 \approx 471\,\text{MeV}$, giving $f/m^2 \approx 0.038$. This is why mesons and leptons appear at the same energy scale: for the electron family $m_{e,\rm scalar} = \sqrt{m_e^2 + f_\pi^2} \approx f_\pi = 92\,\text{MeV}$, while for the tau the splitting is negligible ($m_{\tau,\rm scalar} \approx m_\tau$).

B. The bloom: from seed to physical spectrum

The seed is $(0, m_-, m_+)$ with one vanishing mass. The physical lepton spectrum (m_e, m_μ, m_τ) has $K = 0.666661$ but no vanishing mass. The **bloom** is the nonperturbative process that transforms the seed into the physical spectrum.

In the charge parametrisation $\mathfrak{z}_k = z_0\big(1 + \sqrt{2}\cos(\delta + 2\pi k/3)\big)$, the seed corresponds to $\delta = 3\pi/4$ (one charge vanishes). The bloom is a rotation in δ at fixed scale. But the seed is an *unstable fixed point*: any R-symmetry-breaking perturbation pushes K away from 2/3. Shih [?] showed that in O'Raifeartaigh models with fields of R-charge $\neq 0, 2$, the one-loop Coleman–Weinberg potential gives $m_X^2 < 0$, destabilising the pseudo-modulus away from the origin—precisely the mechanism needed to push the seed toward the physical spectrum. The bloom must therefore be a nonperturbative rearrangement, not a smooth deformation—it is driven by the instanton potential described below.

C. The instanton potential and $\mathbb{Z}_6$

The anomalous $\mathrm{U}(1)_A$ of $\mathrm{SU}(N_c)$ SQCD with N_f flavours has anomaly coefficient $2N_f$ [? ?]. Instantons break $\mathrm{U}(1)_A$ to $\mathbb{Z}_{2N_f}$; for $N_f = 3$ the surviving discrete symmetry is $\mathbb{Z}_6$. (The anomaly-free $\mathrm{U}(1)_R$ with $R[Q] = (N_f - N_c)/N_f = 0$ is exact and continuous for $N_c = N_f$.)

In the charge parametrisation $\mathfrak{z}_k = z_0(1 + \sqrt{2}\cos(\delta + 2\pi k/3))$, every physical observable has the $\mathbb{Z}_3$ permutation symmetry $\delta \to \delta - 2\pi/3$. The effective potential is therefore a Fourier series in $\cos(3n\delta)$; the leading instanton term is

$$V_{\rm inst} \propto -\cos(3\delta)\,, \tag{33}$$

with three minima at $\delta = 0, 2\pi/3, 4\pi/3$. Higher harmonics $\cos(6\delta), \cos(9\delta), \ldots$ are present but suppressed by $(\Lambda/\mu)^{b_0}$ per instanton. Note that δ parametrises mass *ratios* (the Cartan direction) and is independent of the $\mathrm{U}(1)_A$ phase of $\det M$; the $\mathbb{Z}_3$ permutation symmetry constraining $V(\delta)$ is a geometric property of the charge lattice, while $\mathbb{Z}_6$ acts on the axial phase. On the constraint surface $\det M = \Lambda^6$, holomorphic invariants like $(\det M)^3$ are constant; the δ-dependent potential arises from Kähler corrections and off-shell fluctuations around the constraint. The physical lepton phase $\delta_{\rm phys}$ lies near one of these minima. At strong coupling, $V_0^{1/4} \sim 65$–$85\,\text{MeV} \sim f_\pi$.

D. The $\mathfrak{z}_b$ relation

The following empirical relation holds to remarkable precision:

$$\boxed{\mathfrak{z}_b = 3\mathfrak{z}_s + \mathfrak{z}_c\,,} \tag{34}$$

with $3\mathfrak{z}_s + \mathfrak{z}_c = 64.67$ vs. $\mathfrak{z}_b = 64.68$ (0.01%).

A *possible* dynamical origin comes from bion physics. On $\mathbb{R}^3 \times S^1$, $\mathrm{SU}(N_c)$ instantons fractionalise into N_c monopole-instantons [?]. Monopole–anti-monopole (bion) cross-terms contribute to the Kähler potential:

$$K_{\rm bion} = \frac{\zeta^2}{\Lambda^2}\Big|\sum_k s_k \mathfrak{z}_k\Big|^2\,, \tag{35}$$

where $s_k = \pm 1$ are topological signs. Organising into seed ($S_{\rm seed} = \mathfrak{z}_s + \mathfrak{z}_c$) and bloom ($S_{\rm bloom} = -\mathfrak{z}_s + \mathfrak{z}_c + \mathfrak{z}_b$) channels, the second term is a perfect square vanishing when $S_{\rm bloom} = 2\,S_{\rm seed}$, reproducing eq. (??).

a. Caveat. The Ünsal bion analysis [?] was developed for QCD(adj) (adjoint fermions on $\mathbb{R}^3 \times S^1$), not for SQCD with fundamental matter. Extending it to the present theory ($N_c = N_f = 3$ with fundamentals) is plausible but unproven. Equation (??) should therefore be regarded as an *empirical relation* (0.01%) with a suggestive but not yet rigorous bion motivation. The predicted $m_b = \mathfrak{z}_b^2 = 4183\,\text{MeV}$ (PDG: $4183 \pm 30\,\text{MeV}$).

b. Bloom dynamics: frozen-sign vs. adiabatic. The adiabatic bion potential $V_{\rm ad} = |\sum s_k \mathfrak{z}_k|^2$ is exactly flat on the energy-balance manifold (an algebraic identity: $\sum s_k|\mathfrak{z}_k| = 3v_0$ is constant). It cannot select the phase δ. When topological signs are *frozen* at the seed configuration (where one amplitude vanishes), $V_{\rm frozen} = (\sum|\mathfrak{z}_k|)^2$ provides a restoring force opposing the bloom. The equilibrium of $\cos(3\delta)$ (driving) and $V_{\rm frozen}$ (restoring) determines the physical phase.

c. Ünsal continuity caveat. Connecting the $\mathbb{R}^3 \times S^1$ analysis (where monopole-instantons are under semiclassical control) to $\mathbb{R}^4$ (where confinement is fully nonperturbative) requires the Ünsal continuity conjecture [?], supported by lattice evidence but unproven for theories with matter. The bion relation (??) should be regarded as a prediction of the semiclassical regime whose validity at $\Lambda \sim 300\,\text{MeV}$ is assumed.

d. Numerical bloom analysis. The physical lepton phase $\delta_{\rm phys} = 132.73^\circ$ (from PDG masses) lies 12.73° above the nearest instanton minimum at 120° and 2.27° below the seed at 135°. The shift $\varepsilon = \delta_{\rm seed} - \delta_{\rm phys} = 0.040\,\mathrm{rad}$ generates the electron mass through $m_e \approx \mu\varepsilon^2/6 = 0.49\,\mathrm{MeV}$ (96% of the observed value; exact parametrisation gives $0.511\,\mathrm{MeV}$).

A one-loop Coleman–Weinberg computation of $V_{\rm CW}(\delta)$ on the constraint surface—including the quartic from the adjoint mass, the off-diagonal meson loops, and the SUSY-breaking F-terms—finds that *all* perturbative contributions minimise at $\delta = 120^\circ$ (the "democratic" point where $\mathfrak{z}_1 = \mathfrak{z}_2$), not at $\delta_{\rm phys}$. The physical deviation $\alpha_{\rm lep} = \delta_{\rm phys} - 120^\circ = 12.73^\circ$ is a *flat direction* of the perturbative effective potential: the tree-level quartic gives $V_{\rm quartic} \propto \mathrm{Var}(1/\mathfrak{z}_i^2)$ (minimised at the democratic point), and the CW correction has $\mathrm{STr}\,\mathcal{M}^4 \propto V_{\rm quartic}$ (same angular profile, no shift). The angle $\alpha_{\rm lep}$ therefore requires non-perturbative Kähler dynamics, confirming that the Cartan angle and the bloom dynamics are facets of the scalar Kähler metric problem.

E. F-terms are lepton masses

When electroweak symmetry breaks, the F-term system of the confined theory becomes overconstrained: 18 real equations for 12 variables. SUSY necessarily breaks, with:

$$|F_{2j}| = m_j\,, \qquad j = e, \mu, \tau\,. \tag{36}$$

The goldstino is predominantly the τ-mesino (largest F-term). The gravitino mass $m_{3/2} = |F|/(\sqrt{3}\,M_{\rm Pl}) \approx 4 \times 10^{-16}\,\mathrm{GeV}$ is negligibly small.

a. The scalar–fermion splitting. The universal Volkov–Akulov splitting is $m^2_{\rm scalar} = m^2_{\rm fermion} + f_\pi^2$. For the pion–muon pair: $m_\pi^2 = m_\mu^2 + f_\pi^2 = 11164 + 8501 = 19665\,\mathrm{MeV}^2$, giving $m_\pi = 140.2\,\mathrm{MeV}$ (PDG: 139.6 MeV, 0.5%). This is the foundational Volkov–Akulov check: the pion (scalar meson) and the muon (fermionic mesino) are partners in a chiral multiplet, split by f_π.

F. Connection to the Seiberg dynamics

The O'Raifeartaigh model is not an independent postulate but the SUSY-breaking *deformation* of the Seiberg vacuum. The three constraining mechanisms work in concert:

1. **Seiberg constraint** $\det M = \Lambda^6$ (Sec. **??**): fixes the product $a_i \cdot d_i = \mu\nu$ (family-independent).
2. **O'Raifeartaigh seed** $gv/m = \sqrt{3}$ (this section): fixes the charge ratio within a seed triple to the algebraic value $r = 2 - \sqrt{3}$.
3. **Instanton + bion bloom**: the $\cos(3\delta)$ potential redistributes charge from the seed to the physical spectrum, while the bion restoring force constrains one mass combination ($\mathfrak{z}_b = 3\mathfrak{z}_s + \mathfrak{z}_c$).

The quartic from the adjoint mass (Sec. **??**) couples the direct and inverse branches, but whether this coupling is sufficient to produce the full waterfall structure remains open (Sec. **??**). The overall picture is a hierarchy of constraints: the Seiberg constraint gives a one-dimensional family of vacua (parametrised by X), the O'Raifeartaigh seed selects a point on this family (the $\sqrt{3}$ condition), and the bloom dynamics perturbs it to the physical spectrum.

VII. THE INVERSE SECTOR AND CHAIN

A. The dual energy-balance on (d, s, b)

Evaluating on inverse masses ($\mathfrak{z}_j \to 1/\mathfrak{z}_j$):

$$K^{-1}(d,s,b) = \frac{1/\mathfrak{z}_d^2 + 1/\mathfrak{z}_s^2 + 1/\mathfrak{z}_b^2}{(1/\mathfrak{z}_d + 1/\mathfrak{z}_s + 1/\mathfrak{z}_b)^2} = 0.6652\,, \tag{37}$$

0.22% from 2/3, using PDG 2024 $\overline{\rm MS}$ masses [?]. Under RG running, both triples evolve. Using 5-loop SMDR running with the Martin–Robertson code [?], we scan from 1 GeV to $M_{\rm Planck}$ with 391 points, computing $\overline{\rm MS}$ quark masses at each scale with the frozen-mass prescription (virtual particles below their mass threshold contribute their pole mass, not a running mass). Error propagation uses PDG fractional uncertainties, numerical $\partial K/\partial m_i$, added in quadrature.

The inverse down-type triple improves monotonically toward the UV:

Q	$K^{-1}(d,s,b)$	deviation
M_Z	0.66750	+0.126%
1 TeV	0.66719	+0.078%
10 TeV	0.66695	+0.042%
100 TeV	0.66675	+0.012%
280 TeV	**0.66667**	**0.000**%
10^6 GeV	0.66657	−0.014%
$M_{\rm Planck}$	0.66539	−0.191%

The crossing $K^{-1}(d,s,b) = 2/3$ occurs at $Q = 280\,\mathrm{TeV}$ (5-loop SMDR, interpolated). A second inverse triple $K^{-1}(s,c,-t)$ crosses 2/3 at $Q = 82\,\mathrm{TeV}$. Both share the strange quark as pivot, forming an **inverse chain**: (d,s,b)–$(s,c,-t)$.

Key σ values at the $\overline{\rm MS}$ scale: $K(e,\mu,\tau)$ has $\sigma = 0.001\%$ (153σ from exact 2/3); $K^{-1}(s,c,t)$ has $\sigma = 0.13\%$ (1.0σ from 2/3); $K^{-1}(d,s,b)$ has $\sigma = 0.26\%$ (0.5σ from 2/3). The lepton Koide is measured; the quark inverse Koide values are consistent with 2/3 but with larger uncertainties from the quark mass determinations.

An exhaustive scan of all 672 configurations of six quarks ($\binom{6}{3} = 20$ mass triples $\times\ 2^3 - 1 = 7$ non-trivial sign patterns × direct and inverse = 280; plus $\binom{5}{3} \times 7 \times 2 = 140$ five-quark subsets; plus mixed-sector triples: total 672) finds 31 sub-0.5% tuples and 27 zero crossings. Cross-sector tuples (e.g. u, μ, τ) are transient zero-crossings under RG running and are discarded as unphysical.

B. The interleaving theorem

Given a single set of three charges $\mathfrak{z}_1 < \mathfrak{z}_2 < \mathfrak{z}_3$, the Seiberg dynamics produces the **direct branch** $a_i = \mu \mathfrak{z}_i^2$ and the **inverse branch** $d_i = \nu/\mathfrak{z}_i^2$, with opposite orderings.

Theorem 2 (Interleaving)**.** *The six masses form the alternating ladder*

$$d_1 > a_3 > d_2 > a_2 > d_3 > a_1 \tag{38}$$

if and only if $\max(\mathfrak{z}_1^2\mathfrak{z}_3^2, \mathfrak{z}_2^4) < \nu/\mu < \mathfrak{z}_2^2\mathfrak{z}_3^2$.

Proof. Each inequality constrains ν/μ. $d_1 > a_3$ requires $\nu/\mathfrak{z}_1^2 > \mu\mathfrak{z}_3^2$, i.e. $\nu/\mu > \mathfrak{z}_1^2\mathfrak{z}_3^2$. $a_3 > d_2$ requires $\mu\mathfrak{z}_3^2 > \nu/\mathfrak{z}_2^2$, i.e. $\nu/\mu < \mathfrak{z}_2^2\mathfrak{z}_3^2$. The remaining four inequalities are weaker or equivalent. The tightest lower bound is $\max(\mathfrak{z}_1^2\mathfrak{z}_3^2, \mathfrak{z}_2^4)$; the upper is $\mathfrak{z}_2^2\mathfrak{z}_3^2$. □

The product $a_i \cdot d_i = \mu\nu$ is family-independent (from $\det M = \Lambda^6$)—the geometric mean constraint:

$$a_i \cdot d_i = \mu\nu = \text{const}\,, \qquad \forall\, i\,. \tag{39}$$

a. Quantitative status. The interleaving theorem is *proven* as abstract mathematics on any set of three charges. However, the identification of the six physical quark masses with the six rungs faces a quantitative obstruction. If we take the down-type charges $(\mathfrak{z}_d, \mathfrak{z}_s, \mathfrak{z}_b)$ as the direct branch and compute inverse masses from $d_i = \nu/\mathfrak{z}_i^2$ with $\nu = \mathfrak{z}_d^2 \cdot m_t$ (requiring $d_1 = m_t$), then $d_2 = \nu/\mathfrak{z}_s^2 = 8638\,\text{MeV}$ versus $m_c = 1273\,\text{MeV}$—a factor 6.8 discrepancy. This is the same factor appearing in the holomorphic m_c/m_s ratio (Sec. **??**): the Kähler Z-factors $Z_c/Z_s \approx 1.37$ ($1.37^2 \approx 1.9$) reduce but do not eliminate the gap. **The waterfall cannot reproduce the up-type spectrum from down-type charges alone without Kähler corrections.** This is tied to the master obstruction (Problem 1 in Sec. **??**).

C. The chain structure

The quark mass spectrum exhibits overlapping triples:

$$\underbrace{(t,b,c)}_{\text{Fermi}} - \underbrace{(b,c,-s)}_{\text{transition}} - \underbrace{(c,s,0)}_{\text{seed}} - \underbrace{(s,0,m_u{+}m_d)}_{\text{lightest}}\,. \tag{40}$$

Link	Triple	K	Status
1	$(+t,+b,+c)$	0.6693 (0.40%)	Fermi side
2	$(+b,+c,-s)$	0.6747 (1.21%)	sign flip on s
3	$(+c,+s,0)$	0.6646 (0.31%)	seed triple
4	$(+s,0,\mathfrak{z}_{u+d})$	0.6649 (0.28%)	lightest seed

a. Connection to the interleaving. The chain and the waterfall are logically **independent** observations. The interleaving theorem predicts an alternating pattern of six masses; the chain is a sequence of overlapping triples. Whether the chain *arises from* the waterfall has not been proven. The quantitative discrepancy (factor 6.8 in the up-type spectrum) suggests that additional physics beyond the pure interleaving is needed.

b. Alternative origins for the chain. Several mechanisms could produce the overlapping-triple pattern independently of the waterfall:

1. **SUSY breaking:** the O'Raifeartaigh seed generates $(0, m_-, m_+)$ with $K = 2/3$. Successive blooms at different scales could produce a sequence of seed triples whose overlap creates the chain.
2. **Charge additivity:** the bion relation $\mathfrak{z}_b = 3\mathfrak{z}_s + \mathfrak{z}_c$ shows that $\mathfrak{z}_b$ *decomposes* into $\mathfrak{z}_c$ plus three $\mathfrak{z}_s$. If charges are additive (as expected for $\mathrm{U}(1)_F$), overlapping decompositions naturally produce triples sharing members.
3. **Kinematic constraint:** with six quarks ordered by mass, there are exactly four windows of three consecutive quarks. The chain is simply *all* windows; what requires explanation is not the chain itself but why $K \approx 2/3$ in each window.
4. **Multiple** $\mathrm{U}(1)$ **charges:** the Cartan of $\mathrm{SU}(3)_F$ has *two* $\mathrm{U}(1)$ generators; projecting onto different linear combinations could produce sign flips and different effective charges.

At present we cannot distinguish between these origins. The chain is an *observed pattern* with quantitative support; its dynamical explanation remains open.

c. The repeated seed map. The chiral tail is two iterations of the seed map $\mathfrak{z}_{k-1} = r\,\mathfrak{z}_k$ with $r = 2 - \sqrt{3}$: $\mathfrak{z}_s/\mathfrak{z}_c = 0.271$ and $\mathfrak{z}_{u+d}/\mathfrak{z}_s = 0.271$, both within 1.1% of $r \approx 0.268$.

d. Threshold decoupling and the chiral tail. The "chiral" appearance of the last two links is not accidental. Below the charm threshold, heavy flavours have been integrated out in the sequence $6 \to 5 \to 4 \to 3$ active quarks (Table **??**). In this regime the natural variables are those of chiral perturbation theory: the GOR combination $m_u + m_d$ (proportional to M_π^2) replaces the individual up and down masses, and the vanishing charge $\mathfrak{z} = 0$ represents the chiral limit. The chain's tail is an EFT reorganisation of variables after threshold decoupling, not a statement about six simultaneous current masses at a common scale. This also gives a structural reading of the coefficient 3 in the bion relation $\mathfrak{z}_b = \mathfrak{z}_c + 3\mathfrak{z}_s$: three threshold crossings ($t \to b \to c \to s$) separate the b-quark from the strange-pivoted chiral sector.

e. Top completion. The top link can be *derived* rather than fitted. Solving $K[\mathfrak{z}_t, \mathfrak{z}_b, \mathfrak{z}_c] = 2/3$ for positive $\mathfrak{z}_t$ is a quadratic equation whose positive root gives the exact completion formula:

$$\mathfrak{z}_t^{\text{pred}} = 2(\mathfrak{z}_b + \mathfrak{z}_c) + \sqrt{3}\sqrt{\mathfrak{z}_b^2 + 4\,\mathfrak{z}_b\mathfrak{z}_c + \mathfrak{z}_c^2}\,. \tag{41}$$

This is the unique positive real root; the negative root would give a "negative mass" solution which is unphysical. Numerically: $\mathfrak{z}_t^{\text{pred}} = 410.5\,\text{MeV}^{1/2}$, predicting $m_t^{\text{pred}} = \mathfrak{z}_t^2 = 168.5\,\text{GeV}$, compared with $m_t^{\text{pole}} = 172.4\,\text{GeV}$ (2.3% discrepancy). This is a genuine *prediction*: the top mass is determined by the bottom and charm masses through the energy-balance condition, with no free parameters.

f. Chain compression. The chain is thus *compressed by three relations*:

$$\begin{aligned}
\mathfrak{z}_s &\approx (2-\sqrt{3})\,\mathfrak{z}_c && \text{(seed map)}\,,\\
\mathfrak{z}_{u+d} &\approx (2-\sqrt{3})\,\mathfrak{z}_s && \text{(iterated seed)}\,,\\
\mathfrak{z}_b &\approx \mathfrak{z}_c + 3\mathfrak{z}_s && \text{(bion / threshold)}\,,
\end{aligned} \tag{42}$$

plus the positive completion (**??**) for $\mathfrak{z}_t$. Given $\mathfrak{z}_c$ as the single input, the remaining five charges are determined to within $\sim 2\%$. The seed ratio $r = 2-\sqrt{3}$ is an algebraic constant emerging from the energy-balance condition for a zero-mass triple; the bion coefficient 3 counts threshold crossings; and the completion formula is the unique positive root of a quadratic. The entire six-quark mass spectrum is thus parametrised by a single number ($\mathfrak{z}_c = 35.68\,\text{MeV}^{1/2}$) plus four algebraic relations.

g. The cross-sector product rule and the Cabibbo angle. If leptons and down quarks share the same three family charges on opposite branches, the product rule $a_i \cdot d_i = \mu\nu$ (eq. **??**) gives a *cross-sector* prediction at the holomorphic level:

$$m_e \cdot m_b^{\text{hol}} = m_\mu \cdot m_s^{\text{hol}} = m_\tau \cdot m_d^{\text{hol}}\,. \tag{43}$$

This implies $m_d^{\text{hol}}/m_s^{\text{hol}} = m_\mu/m_\tau$, and hence a holomorphic prediction for the Cabibbo angle via the Gatto–Sartori–Tonin relation [**?**]:

$$|V_{us}|_{\text{hol}} = \sqrt{m_\mu/m_\tau} = 0.244\,, \tag{44}$$

compared with the observed $|V_{us}| = 0.224$ (8.7% off). The Kähler correction needed is $Z_d/Z_s = 1.18$.

However, the full product rule (**??**) is **badly violated**: $m_e \cdot m_b = 2138\,\text{MeV}^2$ vs. $m_\mu \cdot m_s = 9869\,\text{MeV}^2$ (factor 4.6). This means the simple "same charges, opposite branches" identification requires large, generation-dependent Kähler corrections across the lepton–quark boundary—far beyond the mild ($< 1\%$) corrections needed within the lepton sector alone. The cross-sector product rule is therefore an *order of magnitude* test of the framework, not a precision prediction; its failure at the holomorphic level is the most concrete manifestation of the Kähler metric obstruction.

D. The CKM matrix: an open problem

The holomorphic cross-sector prediction $|V_{us}| = \sqrt{m_\mu/m_\tau} = 0.244$ is 8.7% from the observed value—not a precision result, but in the right ballpark, suggesting the charge identification is qualitatively correct with Kähler corrections of order $Z_d/Z_s \approx 1.2$.

The remaining CKM elements ($|V_{cb}|$, $|V_{ub}|$, δ_{CP}) are **not predicted**. Empirical patterns have been noted: the charge ratio $R = \mathfrak{z}_b^2/\mathfrak{z}_c^2 = (5+\sqrt{21})/2$ satisfies the self-dual equation $R^2 - 5R + 1 = 0$, and $R^3 = 55 + 12\sqrt{21}$ is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{21})$. But these are *observations about the data*, not derivations from the theory. Until the off-diagonal meson couplings and the Kähler metric are computed, the CKM mixing matrix remains the most significant gap in the framework.

The CP phase satisfies a suggestive $N = 3$ consistency condition: $\cos^2(2\delta) = N/(N+4) = N/(2N+1)$ holds only for $N = 3$ (since $N+4 = 2N+1$ gives $N = 3$). Whether this is structural or coincidental is not resolved.

VIII. GENERATION UNIQUENESS AND SO(32)

A. Why five quarks in the mesons

The SM has six quarks but the s-confining $\text{SU}(3)_F$ requires $N_f = N_c = 3$. The top quark ($m_t = 173\,\text{GeV} \gg \Lambda_F$) is integrated out: it is too heavy to participate in the confinement. The remaining five quarks (u,d,s,c,b) decouple sequentially as the RG scale decreases (Table **??**). The Diophantine system counts light quarks of each type (up-type and down-type) that participate in confinement, selecting $(r,s) = (3,2)$: three down-type and two up-type light quarks, with $N = rs/2 = 3$ generations.

B. The Diophantine system

The sBootstrap counting [**?**] imposes three simultaneous conditions on the numbers of up-type (s) and down-type (r) light quarks participating in confinement:

$$\begin{aligned}
rs &= 2N && \text{(total flavour pairs)}\,,\\
\frac{r(r+1)}{2} &= 2N && \text{(symmetric composites)}\,,\\
r^2 + s^2 - 1 &= 4N && \text{(adjoint content)}\,.
\end{aligned} \tag{45}$$

The first equation counts the number of meson flavour pairs. The second requires the symmetric representation **15** of $\text{SU}(5)_f$ to appear with the correct multiplicity. The third is a consistency condition on the total DOF. Together they have the unique solution $(N,r,s) = (3,3,2)$: three generations, with three down-type and two up-type light quarks. The composites fill $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$ of $\text{SU}(5)_f$.

C. Additional $N = 3$ uniqueness criteria

Beyond the Diophantine system, four additional criteria each select $N = 3$. We rank them by the strength of the physical argument. The five criteria arise from different mathematical structures (number theory, algebraic geometry, representation theory, CP-violation counting, and Casimir invariants). They are algebraically independent in the sense that no one implies another, though they all constrain the same physical quantity N and share the common feature of being rooted in SU(N) representation theory.

1. **Discriminant matching** [*suggestive*]: the discriminant of the characteristic polynomial of SU(N) gives $D_{\rm alg} = (N + 2)^2 - 4$, while the dimension of the s-confining constraint surface is $D_{\rm dyn} = N(2N + 1) = \dim(\mathrm{Sp}(2N))$. Requiring $D_{\rm alg} = D_{\rm dyn}$ gives $(N + 2)^2 - 4 = N(2N + 1)$, which has the unique solution $N = 3$ ($21 = 21$). The physical principle connecting the algebraic discriminant to the constraint-surface dimension is not established; this criterion is a numerical coincidence that may indicate a deeper structure.

2. **Binomial identity** [*suggestive*]: $\binom{3N}{N} = (N + 1) \cdot N(2N + 1)$ holds only at $N = 3$: $84 = 4 \times 21$. The number 84 counts the C_3 polarisations of 11d SUGRA under SO(9) ($\Lambda^3(\mathbb{R}^9) = \mathbf{84}$), and 21 is dim(SU(3)$_F$ composites at $N_f = N_c = 3$). While suggestive of a connection between 11d geometry and 4d confinement, this remains an arithmetic coincidence without a derivation from dynamics.

3. **CP-phase consistency** [*conditional*]: requiring $\cos^2(2\delta) = N/(N + 4)$ and $\cos^2(2\delta) = N/(2N + 1)$ simultaneously gives $N + 4 = 2N + 1$, which requires $N = 3$. This uses the observed CP phase $\delta_{\rm CP}$, which is not currently predicted by the framework; it is a consistency check rather than a selection criterion.

4. **Casimir–Koide identity** [*structural*]: for U(N) with canonical normalisation, the energy-balance ratio $K_N = 2/N$ (Theorem **??** generalised). The mean-square weight $\langle J_3^2\rangle = N(N^2 - 1)/(12N) = (N^2 - 1)/12$ of the principal $\mathfrak{sl}(2)$ embedding in $\mathfrak{sl}(N)$ [**?**] (where J_3 is the weight operator in the fundamental representation). The condition $K_N = \langle J_3^2\rangle$ gives $N(N^2 - 1) = 24$, whose unique positive integer solution is $N = 3$. Equivalently, at $N = 3$ and only $N = 3$, the representation dimension $2j + 1 = N$ coincides with the algebra dimension $\dim \mathfrak{sl}(2) = 3$, giving $C_2/N = C_2/3 = \langle J_3^2\rangle = K = 2/3$.

TABLE V. SU(5)$_f$ composite content and sBootstrap scorecard.

SU(5)$_f$ rep	SM decomposition	Role	Status
24	$(\mathbf{8}, \mathbf{1})_0 + \ldots$	gauge + Higgs	
	$+(\mathbf{3}, \mathbf{2})_{-5/6} + \text{c.c.}$	Q=4/3 proj.	
15	$(\mathbf{1}, \mathbf{3})_{-1}$	sleptons	✓
	$(\mathbf{3}, \mathbf{2})_{1/6}$	leptoquarks	≈
	$(\mathbf{6}, \mathbf{1})_{-2/3}$	diquarks	×
84 DOF: exact 16 (19%), approx. 44 (52%), obstr. 24 (29%).			

D. SO(32) selection

The ten-dimensional heterotic string has two anomaly-free gauge groups: SO(32) and $E_8 \times E_8$. The sBootstrap composite content $\mathbf{24}\oplus\mathbf{15}\oplus\overline{\mathbf{15}}$ of SU(5)$_f$ selects between them.

For SO(32): $\mathrm{SO}(32) \supset \mathrm{SU}(16) \times \mathrm{U}(1) \supset \mathrm{SU}(5) \times \mathrm{SU}(3) \times \mathrm{U}(1)^2$. The adjoint **496** contains **24** and $\mathbf{15} + \overline{\mathbf{15}}$—precisely the sBootstrap content. The $Q = 4/3$ states in the **24** are projected out by the SU(3)$_L$ decomposition of Sec. **??**, eq. (**??**): this is the cleaning that makes the **15** SM-compatible.

For $E_8 \times E_8$: every decomposition of E_8 under SU(5) produces only the antisymmetric **10**, never the symmetric **15** [**?**]. The sBootstrap therefore selects SO(32) over $E_8 \times E_8$.

In 11d SUGRA, $\mathbf{6}_C$ belongs to $\mathrm{Sym}^2(9) = \mathbf{45} \subset g_{MN}$ (gravity sector), not $\Lambda^3(9) = \mathbf{84} \subset C_3$ (gauge sector), explaining the colour-sextet obstruction.

IX. THE EFFECTIVE LAGRANGIAN

A. Scenario (a): sequential integration

a. The UV theory. The starting point is SU(3)$_F$ SQCD with $N_f = 5$ massive flavours (u, d, s, c, b). As the RG scale decreases below m_b and m_c, heavy flavours are integrated out sequentially (Table **??**). At each threshold the dynamical scale matches: $\Lambda_3^6 = \Lambda_4^5 m_c$, $\Lambda_4^5 = \Lambda_5^4 m_b$.

b. The $N_f = 4$ ISS window. Between m_c and m_b, the theory has $N_f = 4$ and sits at the lower boundary of the ISS window [**?**] ($N_c{+}1 \le N_f < \frac{3}{2}N_c$, i.e. $4 \le N_f < 4.5$). The IR-free magnetic dual has gauge group SU($N_f - N_c$) = SU(1) = trivial (for $N_f = 4$, $N_c = 3$); $N_f = 4$ is a boundary case where the magnetic gauge dynamics is absent but the rank-condition SUSY breaking persists [**?**]. The macroscopic superpotential is:

$$W_{\rm ISS} = h \,\mathrm{Tr}\, \varphi\, \Phi\, \tilde{\varphi} - h\mu^2\, \mathrm{Tr}\Phi\,, \tag{46}$$

where Φ_{ij} is the meson of the electric theory (now classical in the magnetic description), φ and $\tilde{\varphi}$ are magnetic quarks, h is the magnetic Yukawa, and $\mu^2 \sim m\Lambda$ is the SUSY-breaking parameter. The ISS variables map to sBootstrap composites: $\Phi_{ij} \to M^i{}_j/\Lambda$ (normalised meson), $h\mu^2 \to m_i\Lambda$ (preon mass × scale).

SUSY is broken by the rank condition: $F_\Phi \sim \tilde{\varphi}\varphi - \mu^2\,\mathbf{1}$ has rank $N_f - N_c = 1$, but $\mu^2\,\mathbf{1}$ has rank $N_f = 4$, so not all F-terms can vanish. The metastability parameter $\epsilon = \mu/\Lambda_m \sim \sqrt{m/\Lambda} \ll 1$ ensures parametrically long lifetime. At $N_f = 3$ the magnetic gauge group is trivial (SU(0)), so the magnetic quarks disappear and eq. (**??**) reduces to the s-confined superpotential (**??**)—the ISS window serves only as a *transient* SUSY-breaking phase during the RG descent from $N_f = 4$ to $N_f = 3$.

c. The $N_f = 3$ confined theory. Below m_c, the theory s-confines with the effective superpotential:

$$W_{\rm conf} = X\big(\det M^{(3)} - B\bar{B} - \Lambda_3^6\big) + \mathrm{Tr}(\hat{m}\,M^{(3)})\,, \quad (47)$$

where $M^{(3)}$ is the 3×3 light-flavour meson matrix, X is the Lagrange multiplier enforcing the quantum constraint, and $\hat{m} = \mathrm{diag}(m_u, m_d, m_s)$ are the light preon masses.

d. The full superpotential. Combining all sectors:

$$\begin{aligned} W_{\rm (a)} = X\big(&\det M^{(3)} - B\bar{B} - \Lambda_3^6\big) + \mathrm{Tr}(\hat{m}\,M^{(3)}) \\ &+ c_3\,\frac{(\det M^{(3)})^3}{\Lambda_3^{18}} \quad \text{(three-instanton, §??)} \\ &+ y_c\,H_u\,M^c{}_c + y_b\,H_d\,M^b{}_b \quad \text{(heavy-flavour Yukawas)} \\ &+ y_t\,H_u\,Q^t\bar{Q}_t \quad \text{(top: partial compositeness)}\,. \end{aligned} \quad (48)$$

The first line is the s-confined sector. The second is the $\cos(3\delta)$ instanton potential (Sec. **??**). The third couples the charm and bottom mesons to the Higgs doublets $H_u = \mathrm{adj}(M)^{1,2}$ and $H_d = M^{1,2}$. The fourth is the top quark Yukawa: the top is too heavy to confine ($m_t \gg \Lambda_F$) and enters through partial compositeness [**?**], coupling directly to the elementary preon fields. Note that the charm and bottom enter differently from the top: they participate in the confinement at the $N_f = 4, 5$ levels but are integrated out before s-confinement at $N_f = 3$, so they contribute through the threshold matching $\Lambda_3^6 = \Lambda_4^5 m_c$ rather than through the confined superpotential directly.

e. The Kähler potential.

$$K_{\rm (a)} = \mathrm{Tr}(M^\dagger M) + |X|^2 + |B|^2 + |\bar{B}|^2 - \frac{|X|^4}{12\mu^2} + \frac{\zeta^2}{\Lambda^2}\Big|\sum_k s_k \mathfrak{z}_k\Big|^2\,. \quad (49)$$

The first four terms are canonical. The $-|X|^4/(12\mu^2)$ term is the negative quartic Kähler correction that stabilises the pseudo-modulus at $gv/m = \sqrt{3}$ (Sec. **??**, eq. (**??**)). The last term is the bion contribution to the Kähler potential [**?**], with topological signs $s_k = \pm1$. The soft potential $V_{\rm soft} = f_\pi^2\,\mathrm{Tr}(M^\dagger M)$ gives universal splitting $m^2_{\rm scalar} - m^2_{\rm fermion} = f_\pi^2$.

f. The scalar–fermion splitting. The soft SUSY-breaking potential is $V_{\rm soft} = f_\pi^2\,\mathrm{Tr}(M^\dagger M)$, giving a universal splitting $m^2_{\rm scalar} - m^2_{\rm fermion} = f_\pi^2$ for every chiral multiplet. The supertrace $\mathrm{STr}[\mathcal{M}^2] = 18\,f_\pi^2$ (from 9 meson multiplets × 2 real scalars each).

TABLE VI. Framework inputs and their roles.

Input	Sets	Status
δ (Cartan)	mass ratios	fitted (2/9)
μ (scale)	abs. masses	fitted (1883 MeV)
Λ_F	composites	$\alpha_s(M_Z)$
$y{=}\sqrt{2}g$	Yukawa–gauge	assumed ($\mathcal{N}{=}2$)
μ_Φ	$\mathcal{N}{=}2{\to}1$	$\lesssim 4.5$ TeV
$m{\propto}\mathfrak{z}^2$	mass law	postulate
Z_i (Kähler)	hol. → phys.	**unknown**

Continuous: δ, μ (2 fitted). *Discrete:* $m \propto \mathfrak{z}^2$, $y = \sqrt{2}g$, $\mathrm{SU}(3)_F$ with $N_f{=}N_c{=}3$.

g. Parameter counting. Table **??** lists all inputs to the framework. In the holomorphic sector, the lepton mass *ratios* are set by $\delta_{\rm lep}$ (the Cartan angle); quark chain ratios by $\delta_{\rm quark}$ (a different Cartan angle, giving the seed ratio $\mathfrak{z}_s/\mathfrak{z}_c \approx 2-\sqrt{3}$). The absolute scale μ converts charge ratios to masses. Λ_F is fixed by matching to $\alpha_s(M_Z)$; μ_Φ is empirical ($\lesssim 4.5$ TeV); and the Z_i are the master unknowns (Problem 1 in Sec. **??**). The mapping to physical masses also requires Z_i, μ_Φ, and the bloom dynamics—none of which are derived from δ and μ alone.

h. Strengths and limitations. Scenario (a) uses standard, well-established Seiberg dynamics. The ISS metastable vacuum provides a natural SUSY-breaking mechanism with calculable lifetime (bounce action $S \sim 10^2$–10^3). The limitation: the top Yukawa is an external coupling (partial compositeness), not generated by confinement, breaking the "everything from one meson matrix" paradigm.

B. Scenario (b): speculative Pati–Salam extension

Scenario (b) requires a Pati–Salam gauge group ($N_c = 4$, $N_f = 6$) that is NOT part of the core sBootstrap framework. In this scenario:

1. The $\mathcal{N}{=}2$ SU(N_c) theory with N_f hypermultiplets has superpotential [**? ?**] $W = \sqrt{2}\,g\,Q^i\Phi\tilde{Q}_i + \sqrt{2}\,m_i^j Q^i\tilde{Q}_j$, where the Yukawa $\sqrt{2}\,g$ equals the gauge coupling by $\mathcal{N}{=}2$ SUSY—it is not free.

2. Adding an adjoint mass $\Delta W = \mu_\Phi\,\mathrm{Tr}\Phi^2$ breaks $\mathcal{N}{=}2 \to \mathcal{N}{=}1$ and generates the quartic (**??**).

3. At the baryonic root [**?**], the magnetic adjoint VEV breaks the gauge group to $\mathrm{SU}(N_f - N_c) = \mathrm{SU}(2)$, which IS $\mathrm{SU}(2)_L$.

The strength is that the Yukawa hierarchy is a hierarchy of VEVs (Seiberg seesaw), not couplings. The limitation: the Pati–Salam embedding requires gauge structure beyond $\mathrm{SU}(3)_F$, the KK monopole identification for the top is speculative, and the Kähler metric is still needed. The results of Secs. **??**–**??** are independent of

TABLE VII. M-theory dictionary for SU(3) SQCD composites.

Field theory	IIA	M-theory	C	Mass
Meson $Q\tilde{Q}$	D6–D6	M2	C_3	Dirac
Baryon εQ^3	D4	M5	C_6	Majorana
adj(M)	thr. D4	thr. M5	mixed	branch
Involution	T-dual	EM dual	$C_3 \leftrightarrow C_6$	—

Scenario (b). By Fradkin–Shenker complementarity, the "UV $\mathcal{N}$=2" picture (Higgs phase) and the "emergent $\mathcal{N}$=2" picture (confined phase) may describe the same physics, smoothly connected.

X. DISCUSSION

a. M-theory lift. In the Hanany–Witten brane construction [**?**], N_c D4-branes are suspended between two NS5-branes, with N_f D6-branes providing quarks. Mesons $M = Q\tilde{Q}$ are open D6–D6 strings; baryons $B = \varepsilon QQQ$ are wrapped D4-branes. Lifting to M-theory (the x^{10} circle opens up): D4-branes become M5-branes, D6-branes become Taub-NUT centres, NS5-branes remain M5-branes. The gauge theory lives on the M5 worldvolume wrapping a holomorphic curve Σ in the (x^4, x^5, x^6, x^{10}) space. The meson (open D6–D6 string) lifts to an M2-brane ending on the M5 worldvolume, coupling to C_3 (Dirac mass type). The baryon (wrapped D4) lifts to an M5-brane wrapping S^4 with three string endpoints, coupling to C_6 (Majorana mass type).

The involution $M_i \to \Lambda^4/M_i$ exchanges the A-period ($\langle M_i \rangle \propto 1/m_i$) with the B-period ($\mathrm{adj}(M)_i \propto m_i$) of the SW curve. In M-theory this is electromagnetic duality: $C_3 \leftrightarrow C_6$, exchanging M2 and M5 branes. The direct/inverse mass assignment of Sec. **??** IS the A/B duality. The genus of the M5 worldvolume curve Σ for SU(3) is 2, matching the two Cartan moduli (z_1, z_2).

The M5-brane wrapping Σ gives $\mathcal{N}$=2 in 4d: the curve encodes the Seiberg–Witten prepotential. The adjoint mass μ_Φ deforms Σ, breaking $\mathcal{N}$=2 $\to$ $\mathcal{N}$=1. At the deformed curve, the two chiral multiplets $(M_i, \mathrm{adj}(M)_i)$ emerge as the two periods of Σ—the geometric origin of the emergent hypermultiplet structure of Sec. **??**.

The membrane dimensionality provides a consistency check: for a p-brane whose worldvolume dimensions all scale with $\mathfrak{z}_i$, $m_i \propto \mathfrak{z}_i^p$. The ratio $K_p = \sum \mathfrak{z}_i^{2p} / (\sum \mathfrak{z}_i^p)^2$ is direction-independent only for $p = 2$ (the M2-brane), by Theorem **??**. For $p = 5$ (M5-brane), K_5 depends on the Cartan angle. This confirms the composite mesons have the right dimensionality for M2-branes—not an independent derivation, since assuming both worldvolume directions scale with $\mathfrak{z}_i$ is equivalent to the bilinear structure. Under SU(3)$_c$, the 11d bosonic fields decompose as: $C_3 \in \Lambda^3(9) = \mathbf{84}$ contains $\mathbf{1}_C$, $\mathbf{3}_C$, $\overline{\mathbf{3}}_C$ but **never** $\mathbf{6}_C$; while $g_{MN} \in \mathrm{Sym}^2(9) = \mathbf{45}$ contains $\mathbf{6}_C$ as a gravitational modulus. This explains the colour-sextet obstruction: $\mathbf{6}_C$ diquarks are in the gravity sector (Planck scale), not the gauge sector (confinement scale).

The number $84 = \binom{9}{3}$ appears in five contexts: (i) C_3 polarisations ($\Lambda^3(\mathbb{R}^9)$); (ii) $E_8 \to \mathrm{SU}(9)$: $\mathbf{248} = \mathbf{80} + \mathbf{84} + \overline{\mathbf{84}}$; (iii) SM fermions: $96 = 84 + 12$; (iv) number theory: $|\mathrm{disc}(\mathbb{Q}(\sqrt{-21}))| = 84$; (v) the binomial identity $\binom{9}{3} = 4 \times 21$.

b. Composite SU(2)$_L$ *(speculative).* The CST construction [**?**] offers a mechanism in which SU(2)$_L$ is a residual magnetic gauge group at the baryonic root, but requires a Pati–Salam embedding ($N_c = 4$, $N_f = 6$) beyond the core SU(3)$_F$ framework. An honest caveat: the Seiberg dual of SU(3) at $N_f = 3$ has *trivial* gauge group—there is no residual SU(2) to identify with SU(2)$_L$. The Pati–Salam embedding is a separate gauge sector that must be grafted onto the framework; it does not arise from SU(3)$_F$ alone. In the core sBootstrap framework, SU(2)$_L$ is an **elementary** gauge symmetry already present in G_{UV} (eq. (**??**)), embedded via the branching $\mathbf{3}_L \to \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}$. The CST construction offers a speculative *alternative* that is NOT required by the results of this paper.

c. Yukawa emergence. If the $\mathcal{N}$=2 relation $y = \sqrt{2}\, g$ is adopted (Section **??**), the same gauge coupling produces both branches: the F-term gives mesino masses $\propto \mathfrak{z}_i^2$ (direct), while the meson VEV Yukawa gives masses $\propto 1/\mathfrak{z}_i^2$ (inverse). In the holomorphic sector, flavour parameters are controlled by g (overall scale) and α (mass ratios); the Yukawa hierarchy is not a hierarchy of couplings but of VEVs (Seiberg seesaw). The mapping to physical observables also requires the Kähler metric Z_i, the adjoint mass μ_Φ, and the overall scale $\mu \approx 1883\,\mathrm{MeV}$ (fitted, not derived).

d. Holomorphic vs. physical masses. The meson masses from the Seiberg vacuum are holomorphic: they arise from the superpotential $W(M)$ and are protected by holomorphy. The physical fermion masses are $m_{\mathrm{phys}} = m_{\mathrm{hol}}/\sqrt{Z_i}$ where Z_i is the Kähler metric. For leptons, Z is generation-universal (to 0.001%), so the holomorphic masses directly determine the mass *ratios*. For quarks, m_c/m_s does NOT run (γ_m is flavour-independent to 5-loop), and the lattice value $m_c/m_s = 11.783 \pm 0.025$ differs from the lepton-derived prediction $(2+\sqrt{3})^2 = 13.93$ by 18%. This discrepancy dissolves when sectors are allowed different Cartan angles: the chain seed gives $\mathfrak{z}_s/\mathfrak{z}_c = 0.271$, within 1.1% of the algebraic ratio $r = 2 - \sqrt{3}$, without invoking Kähler corrections (see "Resolved" in Section **??**).

e. Kähler sensitivity of quark-sector comparisons. All quark-sector numerical comparisons in Table **??** operate at the holomorphic level. The physical masses are $m_{\mathrm{phys}} = m_{\mathrm{hol}}/\sqrt{Z_i}$, where Z_i is the unknown Kähler metric. For leptons, Z_i is generation-universal to 0.001%—the precision of $K(e, \mu, \tau) = 2/3$—protected by the emergent $\mathcal{N}$=2 structure (Sec. **??**). For quarks, which enter through partial compositeness [**?**] (eq. **??**), the physical mass is $m_q \sim \lambda_{q_L} \lambda_{q_R} \langle \mathcal{O} \rangle / M_*$, where $\lambda_{q_L,R}$ are elementary–composite mixing parameters. The $\mathcal{N}$=2 protection does not obviously extend to partial compositeness masses, so Z_i may receive $\mathcal{O}(1\text{–}10\%)$ generation-dependent corrections. These could shift $K^{-1}(d,s,b)$ by up to $\sim 1\%$, the bion mass m_b by up to $\sim 5\%$, and m_t by up to $\sim 10\%$. The cross-sector product rule (**??**)—already violated by a factor 4.6 at the holomorphic level—demonstrates that generation-dependent Kähler corrections across the lepton–quark boundary are large. *Until the Kähler metric is computed or bounded, the quark-sector holomorphic agreements should be regarded as suggestive, not as precision tests of the framework.*

f. Why $m_i = \mathfrak{z}_i^2$: honest status. The charge-squared law is motivated by three structural inputs: holomorphy ($W(M)$ holomorphic $\Rightarrow$ mass $\propto \mathfrak{z}^2$), bilinear structure ($M = Q\tilde{Q}$ $\Rightarrow$ two preon factors), and canonical normalisation (Theorem **??**). However, seven attempts to derive $m = \mathfrak{z}^2$ from SUSY-breaking dynamics all fail for the following structural reason: gauge mediation gives $\mathfrak{z}^2$ scaling for *soft* (Kähler) masses, but we need it for *holomorphic* (superpotential) masses—the protected masses that determine exact $K = 2/3$. The diagonal mesons M_i^i are $\mathrm{U}(1)_F$-neutral, which kills D-term approaches; and soft masses enter the scalar potential, not the fermion mass matrix. The charge-squared law remains the best-motivated ansatz, not a theorem of the confining theory. The Coulomb self-energy argument [**?**] (a capacitor energy $E = Q^2/(2C)$) is the physically most transparent route: it is rigorous for weakly-coupled Abelian fields but must be assumed to survive in the strongly-coupled confining regime.

g. Hierarchy reframing. In the sBootstrap, SUSY is broken at $f = f_\pi \sim 92\,\mathrm{MeV}$ —the QCD scale, not the TeV scale. The electroweak VEV $v_{\mathrm{EW}} = 246\,\mathrm{GeV}$ is a *derived* quantity: $v_{\mathrm{EW}} \sim m_t/y_t \sim 173/0.7 \sim 247\,\mathrm{GeV}$. The apparent hierarchy $v_{\mathrm{EW}}/\Lambda_{\mathrm{QCD}} \sim 10^3$ is the Yukawa hierarchy—which is the flavour problem that the charge-squared law addresses. This is a *reframing*, not a solution: the hierarchy is moved from $v_{\mathrm{EW}}/M_{\mathrm{Pl}}$ to y_t/y_e, and the latter still requires explanation (the Cartan angle α, which is not derived).

h. Strong CP and the Nelson–Barr mechanism. Identifying the involution $M_i \to \Lambda^4/M_i$ with CP gives a Nelson–Barr [**? ?**] solution to the strong CP problem. CP is an exact symmetry of the UV theory (the involution is a symmetry of the quantum constraint); it is spontaneously broken by the meson VEVs $\langle M_i \rangle$ (which are real but unequal). At tree level: $\arg\det m_q = 0$ (the superpotential is real under the involution). Radiative corrections generate $\delta\bar{\theta} \sim (\alpha/4\pi)^2 J \sim (10^{-3})^2 \times 3 \times 10^{-5} \sim 10^{-11}$, nine orders of magnitude below the experimental bound $|\bar{\theta}| < 10^{-10}$.

i. The AM-GM scale chain. The framework produces three mass scales from SU(3) confinement and the bilinear structure $M = Q\tilde{Q}$. Define the arithmetic and geometric means of the lepton charges: $\mathrm{AM} = \frac{1}{3}\sum_i \mathfrak{z}_i$ and $\mathrm{GM} = (\prod_i \mathfrak{z}_i)^{1/3}$. Then:

$$M_0 = \mathrm{AM}^2 = 313.8\,\mathrm{MeV} \quad (\text{Koide scale} \approx m_p/3)\,, \tag{50}$$

$$\tfrac{1}{2} f_\pi = \mathrm{GM}^2 = 45.8\,\mathrm{MeV} \quad (\text{SUSY breaking})\,, \tag{51}$$

$$\mu = 6\,M_0 = 1883\,\mathrm{MeV} \quad (\text{mass scale} \approx 2\,m_p)\,. \tag{52}$$

The f_π relation $f_\pi = 2(\mathfrak{z}_e\,\mathfrak{z}_\mu\,\mathfrak{z}_\tau)^{2/3}$ holds to 0.7%. The ratio $M_0/f_\pi = \frac{1}{2}(\mathrm{AM}/\mathrm{GM})^2$ measures the hierarchy: $\mathrm{AM}/\mathrm{GM} = 2.618$ encodes the τ/e mass ratio. The sum $\mu = m_e + m_\mu + m_\tau = 1883\,\mathrm{MeV}$ coincides with $2\,m_p = 1877\,\mathrm{MeV}$ (0.3%). Both arise from SU(3) confinement at similar scales; the probability of some match at 0.3% among $\sim$50 hadron masses is $p \approx 15\%$ after look-elsewhere correction. We note the coincidence without claiming it is derived.

XI. OPEN PROBLEMS AND CONCLUSIONS

A. Numerical results

a. Tier 1: high precision ($<$ 0.1%). $K(e,\mu,\tau) = 0.666661$ (0.001%) stands alone as the most precise mass relation in particle physics outside the gauge sector. The bion prediction $\mathfrak{z}_b = 3\mathfrak{z}_s + \mathfrak{z}_c$ (0.01%) is also in this tier. These are the two results for which accidental coincidence is least plausible: a joint Monte Carlo with 10^7 random mass triples gives $p < 10^{-7}$.

*b. Tier 2: moderate precision (*0.1–1%*).* The meson triple $K(-\pi, D_s, B) = 0.6674$ (0.10%) and the inverse triple $K^{-1}(d,s,b) = 0.6652$ (0.22%). The cross-sector product rule predicts $|V_{us}|_{\mathrm{hol}} = \sqrt{m_\mu/m_\tau} = 0.244$ (8.7% off), which is an order-of-magnitude test rather than a precision result. At $Q = 100\,\mathrm{TeV}$, the inverse triple $K^{-1}(s,c,-t) = 0.6666$ (0.004%) joins Tier 1 under RG running.

c. Tier 3: the chain as a correlated set. The quark chain (t,b,c)–$(b,c,-s)$–$(c,s,0)$–$(s,0,u{+}d)$ is individually less precise than the lepton Koide (0.3–1.2% per link). But the chain is not four independent observations: it is a **correlated set of overlapping triples** sharing members. The four links use only 6 quarks and contain $\binom{6}{3} = 20$ possible triples, of which exactly 4 (the consecutive windows) have $K \approx 2/3$ simultaneously. The probability that a random mass spectrum produces four consecutive windows all within 1.2% of 2/3 is $p \sim 1/21\,000$, estimated as follows. We draw six quark masses independently from a log-uniform distribution over $[m_u, m_t] = [2.2\,\mathrm{MeV}, 172.4\,\mathrm{GeV}]$, assign random signs ($\pm 1$, uniform) to each mass in the Koide formula, and evaluate K on the four consecutive windows of three. A "hit" requires all four windows to satisfy $|K - 2/3|/(2/3) < 0.012$ simultaneously. Over 10^6 trials, we find 47 ± 7 hits, giving $p = (4.7 \pm 0.7) \times 10^{-5} \approx 1/21\,000$. This is *before* any look-elsewhere correction; the chain structure (consecutive windows with a fixed ordering) is specified a priori,

Observable	Method	Predicted	PDG 2024	Deviation	Kähler sensitivity
$K(e, \mu, \tau)$	trace id.	2/3	0.666661	0.001%	protected ($\delta Z/Z \sim 10^{-9}$)
$K(-\pi, D_s, B)$	meson	$\approx 2/3$	0.6674	0.10%	protected (HQET: Z cancels in ratio)
$K(-\pi, D, B)$	meson	$\approx 2/3$	0.673	0.91%	protected (HQET)
$K^{-1}(d, s, b)$	seesaw	$\approx 2/3$	0.6652	0.22%	**uncontrolled** ($\delta K \sim$ 1–10% from δZ_i)
m_b (bion)	eq. (**??**)	4183 MeV	4183	0.01%	**uncontrolled** ($\delta m_b/m_b \sim \delta\sqrt{Z_b}$)
m_t (compl.)	eq. (**??**)	168.5 GeV	172.4	2.3%	**uncontrolled** ($\delta m_t \sim$ 5–10% possible)
$\lvert V_{us}\rvert$	$\sqrt{m_\mu/m_\tau}$	0.244	0.224	8.7%	cross-sector: needs $Z_d/Z_s \approx 1.18$

TABLE VIII. Quantitative results with Kähler sensitivity. All entries use PDG 2024 [**?**] inputs. "Protected" means the comparison involves ratios within a sector where $\mathcal{N}$=2 Kähler protection applies (Sec. **??**). "Uncontrolled" means the holomorphic-to-physical mapping depends on unknown generation-dependent Kähler corrections from partial compositeness; the stated deviation is holomorphic and may shift by $\mathcal{O}(Z_i)$.

not selected from the data. The chain is a single coherent observation, and its significance should be assessed jointly, not link by link.

d. The inverse chain at high scale. Under 5-loop SMDR running, two inverse triples converge on $K^{-1} = 2/3$ at 100 TeV: $K^{-1}(d, s, b) = 0.6667$ (0.012%) and $K^{-1}(s, c, -t) = 0.6666$ (0.004%). Both share the strange quark as pivot, forming an inverse chain (d, s, b)–$(s, c, -t)$ shorter than the direct chain but more precise at high scales.

e. Prediction vs. postdiction. The lepton Koide (Koide 1983 [**?**]) and the inverse quark Koide (Rivero 2005 [**?**]) were observed before the present framework; they are postdictions. The bion relation $\mathfrak{z}_b = 3\mathfrak{z}_s + \mathfrak{z}_c$ is also a postdiction (identified after seeing data). The framework's genuine predictions—testable going forward—are the LFV ratio and the family gauge boson masses. The CKM matrix beyond the Cabibbo angle is not currently predicted.

B. Open problems

a. Resolved or partially resolved.

- **The fermion Kähler metric** (formerly "master obstruction"). The emergent $\mathcal{N}$=2 hypermultiplet structure protects the fermion Kähler metric: the $\mathcal{N}$=2 breaking correction is $\delta Z/Z \sim g^2\Lambda^2/\mu_\Phi^2 \sim 5\times 10^{-9}$, negligible. This *explains* the 0.001% lepton Koide: it is $\mathcal{N}$=2 protection, not a mystery. The fermion Z_i is generation-universal. **This part of Problem 1 is solved.**
- **The m_c/m_s ratio** (formerly part of Problem 1). The holomorphic ratio $m_c/m_s = (2+\sqrt{3})^2 = 13.93$ vs. lattice 11.78 (18%) was attributed to $Z_c/Z_s \approx 1.37$. However, m_c/m_s does NOT run (γ_m flavour-independent to 5-loop; $<1\%$ drift from 2 GeV to 282 TeV). The "discrepancy" dissolves when sectors are allowed different Cartan angles: the chain seed gives $\mathfrak{z}_s/\mathfrak{z}_c = 0.271$, within 1.1% of the algebraic ratio $r = 2 - \sqrt{3}$. No Kähler correction is needed.
- **The Q_2 operator** (formerly "open: can the $\mathbb{Z}_2$ be promoted?"). The operator $R_{ij} = \langle M_k\rangle$ ($k \neq i, j$; $R_{ii} = 0$) is explicitly constructed. It satisfies the *exact* identity

$$R \cdot \mathbf{M} = 2\,\mathrm{adj}(\mathbf{M}) \tag{53}$$

(verified symbolically): R acting on the meson VEV vector gives twice the adjugate vector. Its Cayley–Hamilton identity on the constraint surface is

$$R^3 = \mathrm{Tr}(D^2)\,R + 2\Lambda^6\,I\,, \tag{54}$$

where $D = \mathrm{diag}(\langle M_i\rangle)$. This is a *cubic* minimal polynomial, not the quadratic $R^2 = c\,I$ of standard $\mathcal{N}$=2. The normalised operator $T = R/\sqrt{\det D}$ has eigenvalues $\approx \{+1.07, -3\times 10^{-6}, -1.07\}$ for the lepton hierarchy—an "almost-involution" with one near-zero eigenvalue. Whether the cubic algebra (**??**) defines a *deformed* $\mathcal{N}$=2 structure (controlled by the mass hierarchy $\mathrm{Tr}D^2$) is an open question.

b. Remaining open problems.

1. **The Cartan angle** $\alpha \approx 12.7°$ (absorbs old Problems 3 and 4). One-loop CW analysis shows all perturbative contributions to $V(\delta)$ minimise at $\delta = 120°$ (the democratic point), not at $\delta_{\rm phys} = 132.73°$: the tree-level quartic gives $V \propto \mathrm{Var}(1/\mathfrak{z}_i^2)$, and the CW correction has the same angular profile ($\mathrm{STr}\,\mathcal{M}^4 \propto V_{\rm quartic}$). The deviation $\alpha = 12.73°$ from the democratic point is a flat direction of the perturbative theory. Its value requires either the non-perturbative Kähler metric for the *scalar* meson VEV (not the fermion metric, which is solved) or an as-yet-unknown selection mechanism.
2. **The CKM matrix.** The off-diagonal meson mass matrix block-decouples from the diagonal mesons at tree level (the 10×10 fermion mass matrix decomposes into a 4×4 diagonal block and three 2×2 off-diagonal blocks). CKM mixing therefore *cannot* arise from tree-level meson mixing; it

requires Kähler corrections, loop effects, or RG running. The holomorphic cross-sector prediction $|V_{us}| = \sqrt{m_\mu/m_\tau} = 0.244$ is 8.7% off, suggesting Kähler corrections $Z_d/Z_s \approx 1.2$.

3. **PMNS and neutrino masses.** The baryons $B, \tilde{B}$ are SM singlets (sterile fermions, Table **??**). A tree-level seesaw gives $M_\nu \propto m_\ell^2/M_B$, but the tree-level PMNS matrix is diagonal—contradicting the observed large mixing.

4. **The cubic Q_2 algebra.** Eq. (**??**) shows R satisfies a cubic, not a quadratic, minimal polynomial. The coefficient $2\Lambda^6 = 2\det M$ is the quantum constraint; $\mathrm{Tr}(D^2) = \sum M_i^2$ is the quadratic Casimir. Does this cubic define a consistent deformed $\mathcal{N}$=2 superalgebra? If so, the "emergent $\mathcal{N}$=2" label becomes precise: a cubic deformation of the hypermultiplet algebra controlled by the family hierarchy.

5. **The SO(32) projection.** The **496** contains 442 states beyond the sBootstrap content; the projection mechanism is unknown.

C. Falsification conditions

- The LFV ratio $\mathrm{Br}(\tau \to e\gamma)/\mathrm{Br}(\tau \to \mu\gamma) = (\mathfrak{z}_e/\mathfrak{z}_\mu)^4 \approx 5.5 \times 10^{-10}$ is a parameter-free prediction testable at Belle II.
- Family gauge bosons at 10^2–10^3 TeV produce flavour-changing effects testable at FCC-hh.
- Lattice Kähler metric strongly generation-dependent for leptons ($Z_\mu/Z_e \neq 1$ at 1%) would kill the framework.
- LFV ratio deviating from $(\mathfrak{z}_e/\mathfrak{z}_\mu)^4$ by more than a factor 2 falsifies the $\mathrm{U}(3)_F$ charge assignment.
- Discovery of a fourth generation contradicts $N = 3$ uniqueness.

D. Conclusions

We have presented a framework in which Standard Model fermions are identified with composite fermions of a confining $\mathrm{SU}(3)_F$ family gauge theory: mesinos are leptons, and the adjugate composites provide quark partners through partial compositeness. The confined spectrum automatically produces a pair $(M_i, \mathrm{adj}(M)_i)$ per family with the mathematical structure of an $\mathcal{N}$=2 hypermultiplet—emergent from confinement, not imposed in the UV.

The framework rests on one central ansatz—the charge-squared law $m_i = \mu\,\mathfrak{z}_i^2$—which is the unique monomial compatible with direction-independent $K = 2/3$ (Theorem **??**), supported by the Coulomb self-energy of the composite [**?**] and by the bilinear structure $M = Q\tilde{Q}$. We have been honest that this is the best-motivated ansatz, not a theorem of the confining theory: seven attempts to derive it from SUSY-breaking dynamics all fail for holomorphic masses.

What is **established**:

1. **Composites:** 9 mesinos furnish $3L_i + 3e_i^c$ (exact group theory). Both Higgs doublets from one meson matrix: $H_d = M^{1,2}$, $H_u = \mathrm{adj}(M)^{1,2}$, ratio fixed by the Seiberg constraint. The "superpartners" are the mesons, split from mesinos by f_π.

2. **The adjugate mechanism:** $M \propto 1/m$ and $\mathrm{adj}(M) \propto m$ provide two Yukawa operators from one composite. A single UV condition produces both standard lepton and inverse quark $K = 2/3$.

3. **Energy-balance observations:** $K(e, \mu, \tau) = 0.666661$ (0.001%); $K(-\pi, D_s, B) = 0.6674$ (0.10%); $K^{-1}(d, s, b) = 0.6652$ (0.22%); and at $Q = 100\,\mathrm{TeV}$, $K^{-1}(s, c, -t) = 0.6666$ (0.004%). The quark chain (t, b, c)–$(b, c, -s)$–$(c, s, 0)$–$(s, 0, u{+}d)$ has K within 1.2% in every link (joint $p \sim 1/21\,000$), and is compressed by three algebraic relations (seed map, iterated seed, bion/threshold) plus a positive-completion formula that predicts $m_t = 168.5\,\mathrm{GeV}$ (2.3%).

4. **Three mass scales from one confinement:** $M_0 = \mathrm{AM}^2 = 313.8\,\mathrm{MeV} \approx m_p/3$; $f_\pi/2 = \mathrm{GM}^2$; $\mu = 6\,M_0 = 1883\,\mathrm{MeV} \approx 2\,m_p$. The Fermi scale and the QCD scale are *not* independently set—both arise from SU(3) confinement.

5. **Generation uniqueness:** five criteria (two structural—Diophantine and Casimir–Koide—and three suggestive) select $N = 3$; composite content selects SO(32).

The seven open problems of Section **??** constitute what the framework does **not yet** provide; the most pressing are the Cartan angle (Problem 1), the CKM matrix (Problem 2), and the cubic Q_2 algebra (Problem 4).

On the experimental side, the LFV ratio $\mathrm{Br}(\tau \to e\gamma)/\mathrm{Br}(\tau \to \mu\gamma) = (\mathfrak{z}_e/\mathfrak{z}_\mu)^4$ is a parameter-free prediction testable at Belle II, and family gauge bosons at 10^2–10^3 TeV produce flavour-changing effects accessible to a future 100 TeV collider.

a. Adjugate crossing. Under RG running, the charged-lepton energy-balance $K(e, \mu, \tau)$ and the inverse down-quark energy-balance $K^{-1}(d, s, b)$ both remain close to 2/3 but at slightly different values. Remarkably, they cross at $Q \approx 15.7\,\mathrm{GeV}$: at this scale, $K_{\mathrm{lep}} = K^{-1}_{\mathrm{down}} = 0.6663$, and the gap at m_b is only 0.06%. This crossing is a quantitative consequence of the adjugate mechanism: both K and K^{-1} are determined by the same charges, and their coincidence at a particular RG scale reflects the interplay of direct and inverse operators under running. The crossing scale $Q \approx 15.7\,\mathrm{GeV}$

is near the b-quark threshold, consistent with the phase transition from $N_f = 4$ to $N_f = 3$ in Table **??**.

The framework represents a structural EFT, not a UV completion: it organises the flavour sector using composites of a confining gauge theory, with exact results (group theory, holomorphy, anomaly matching) supplemented by a well-motivated ansatz (the charge-squared law) and empirically verified patterns (the chain, the bion relation, the seed structure). The gap between what is established and what is not is clearly delineated by the seven open problems listed above.

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Note added. *(i) Cayley–Hamilton.* The cubic algebra $R^3 = \mathrm{Tr}(D^2)\, R + 2\Lambda^6\, I$ of Sec. **??** coincides with the Cayley–Hamilton equation of the $\mathcal{N}$=2 adjoint scalar Φ under $u_2 = \mathrm{Tr}(D^2)$, $u_3 = 2\det(D)$. *(ii) Seiberg–Witten curve.* The SW curve for SU(3) with $N_f = 3$ at the Koide point is a smooth genus-2 surface; in the democratic limit it develops a monopole singularity at $\Lambda = m = s_1/3$. The Koide constraint reduces the curve to a one-parameter family in the Cartan angle δ. *(iii) Cabibbo angle.* Koide's original Cabibbo prediction [**?**] $\tan\theta_C = \sqrt{3}(\sqrt{m_\mu} - \sqrt{m_e})/(2\sqrt{m_\tau} - \sqrt{m_\mu} - \sqrt{m_e})$ is algebraically identical to the Cartan angle δ ($\delta = 0.222$, 1.2% from PDG). This extends the quark–lepton complementarity of Raidal [**?**] and the quark Koide phases of Zenczykowski [**?**]. The full CKM and PMNS parametrisations from δ—including $|V_{us}| = 2/9$, $|V_{cb}| = 4/99$, and PMNS deviations from tribimaximal mixing—are phenomenological observations (not yet derived from the $\mathrm{SU}(3)_F$ dynamics) and are presented in a companion letter [**?**]. *(iv) RG comparison.* Using multi-loop SM running [**? ?**], $K^{-1}(d, s, b) = 2/3$ at $Q \approx 10\,\mathrm{TeV}$; in the MSSM, K^{-1} at the GUT scale is 0.23–0.65% above 2/3 for all $\tan\beta$ and M_{SUSY}, disfavouring conventional supersymmetric extensions.

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