

A structural sBootstrap EFT from Seiberg confinement: charge–mass duality, Yukawa inversion, and the structural core of the quark mass waterfall

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Abstract

We work under the philosophy of the *sBootstrap* [?]: a modern realisation of bootstrap and nuclear-democracy ideas in the spirit of Chew [?], in which the observed fermions are identified with confined composite fermions (mesinos and, in a quark–diquark complementarity sense, diquarkinos) rather than being endowed with arbitrary fundamental Yukawa matrices. We formulate an exploratory supersymmetric EFT that organises two related observations about Standard Model flavour data: (i) the Koide ratio $K = 2/3$ in the charged-lepton sector and the approximate inverse Koide relation in the down-quark sector, and (ii) the “waterfall” pattern whereby consecutive quark-mass triples approximately satisfy $K = 2/3$ with nontrivial sign assignments. The common structural input is a canonically normalised $U(3)$ family generator $Q_F = z_0 I + T$ whose eigenvalues q_i define a positive operator $H_Q = Q_F^\dagger Q_F = \text{diag}(q_1^2, q_2^2, q_3^2)$ with $\text{Tr}(H_Q) = 1$. For a quadratic charge law $m_i \propto q_i^2$, the Koide ratio becomes a trace identity of H_Q and is independent of the Cartan direction; we show this direction-independence is unique to the quadratic power. We then review $\mathcal{N}=1$ $SU(3)$ SQCD with $N_f = N_c = 3$: the quantum-modified constraint, implemented holomorphically with a Lagrange-multiplier superfield and deformed by diagonal mass terms $m_i Q^i \tilde{Q}_i$, yields meson VEVs $\langle M_i \rangle \propto 1/m_i$ and a Λ -independent composite spectrum $m(M^i_j) = m_i m_j / P^{1/3}$ (with $P = m_1 m_2 m_3$). On the mesonic branch, $\det M = \Lambda^6$ implies a holomorphic involution $M \leftrightarrow \Lambda^{-2} \text{adj}(M)$ (equivalently $M \rightarrow \Lambda^4 M^{-1}$), which at the supersymmetric vacuum exchanges the dependence on the deformation spurion m and on m^{-1} . This makes precise how Seiberg mass parameters can generate *inverse* Yukawa hierarchies for fields that couple to meson VEVs. Finally, we show how the composite doublet $(M_i, \text{adj}(M)_i)$ has the mathematical structure of an $\mathcal{N}=2$ hypermultiplet, emergent from confinement plus isospin, and how an effective quartic coupling among mesons mixes the direct and inverse branches. In the decoupling limit we prove an interleaving theorem: two monotone branches $m = \mu q^2$ and $\hat{m} = \nu/q^2$ necessarily form an alternating six-rung ladder in a calculable window, providing the structural core—rather than a quantitative fit—of the quark mass waterfall and of the existence of multiple charge decompositions (and sign choices) for the same observed masses. The mesino/lepton map, the quarkino completion, and the SUSY-breaking/Kähler dressing are treated throughout as EFT assumptions or working ansätze, not as consequences of SQCD alone; correspondingly, no quantitative fit to CKM/PMNS data or proof of full phenomenological viability is claimed in this paper.

1 Introduction

The sBootstrap programme [?] is an attempt to transfer the bootstrap ethos of strong-interaction physics [?] to the flavour problem: rather than postulating fundamentally free Yukawa matrices, one seeks a self-consistent effective description in which the observed spectrum arises from a small set of symmetry and dynamics inputs. In the version considered

here, the dynamical core is an $\mathcal{N}=1$ confining gauge theory with a quantum-modified moduli space whose exact moduli-space relations generate both *direct* and *inverse* hierarchies from a single set of deformation parameters. The paper should be read as a controlled *exploratory EFT proposal*: its goal is to isolate the exact holomorphic algebra, make the gauge-invariant embedding assumptions explicit, and state clearly which phenomenological steps are still ansätze.

As diagnostics (and as targets for a prospective embedding), we keep track of three empirical regularities. First, the Koide relation for charged leptons,

$$K(\sqrt{m_e}, \sqrt{m_\mu}, \sqrt{m_\tau}) \equiv \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3}, \quad (1)$$

is satisfied to the 10^{-5} level with pole masses [?, ?]. An analogous statement holds approximately for down quarks after inversion of masses (with a conventional choice of running scheme/scale) [?]. Independently, “waterfall” relations in which consecutive quark-mass triples are close to $2/3$ have been observed, but require nontrivial sign assignments inside the Koide diagnostic [?].

This paper develops a single supersymmetric framework that organises these facts through three ingredients:

1. Canonical $U(3)$ family normalisation fixes a universal offset $z_0 = 1/\sqrt{6}$ in the Cartan eigenvalues, producing three charges $q_i = z_0 + z_i$ with $\sum_i q_i^2 = 1$ and $\sum_i q_i = \sqrt{3}/2$.
2. In $\mathcal{N}=1$ $SU(3)$ SQCD with $N_f = N_c = 3$ (quantum-modified moduli space), diagonal holomorphic mass deformations $m_i Q^i \tilde{Q}_i$ generate meson VEVs $\langle M_i \rangle \propto 1/m_i$ and a composite spectrum in which the leading holomorphic mass scales are simple algebraic functions of the *Seiberg masses* m_i .
3. The composite doublet $(M_i, \text{adj}(M)_i)$ exhibits $\mathcal{N}=2$ -like structure, emergent from confinement plus isospin. An effective quartic coupling among mesons mixes the direct and inverse branches, producing an interleaved mass ladder.

The goal is not to claim a complete UV-to-SM derivation of flavour. Rather, we isolate which statements are algebraic identities, which are exact consequences of holomorphy and the quantum-modified constraint, and which require an assumed embedding of Standard Model fields into the confined sector (notably through the identification of mesinos with leptons and of diquarkino-like constituents with quarks, and through non-holomorphic Kähler dressing). Accordingly, the manuscript should be read as a structural note plus a gauge-invariant EFT ansatz, not as a completed quantitative flavour model. More specifically:

- Sections ??, ??, and Theorem ?? contain the hard algebraic/holomorphic results.
- Sections ?? and ?? formulate the gauge-invariant sBootstrap EFT ansatz and its operator map.
- Sections ?? and ?? state the viability conditions and open problems rather than claiming that they are solved.

Signed Koide diagnostic. For the waterfall discussion it is convenient to define Koide’s ratio as a function of three *real* inputs x_i ,

$$K(x_1, x_2, x_3) \equiv \frac{x_1^2 + x_2^2 + x_3^2}{(x_1 + x_2 + x_3)^2}, \quad (2)$$

so that $K(\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3})$ reproduces the usual Koide parameter while relative sign flips correspond to $x_i \rightarrow -x_i$. We also define the *inverse Koide* diagnostic by applying K to inverse square roots,

$$K^{-1}(m_1, m_2, m_3) \equiv K\left(\frac{1}{\sqrt{m_1}}, \frac{1}{\sqrt{m_2}}, \frac{1}{\sqrt{m_3}}\right) = \frac{\sum_i m_i^{-1}}{(\sum_i m_i^{-1/2})^2}. \quad (3)$$

Throughout, the underlying Seiberg masses depend only on q_i^2 . Whenever we discuss relative sign flips in Koide triples, we therefore write them as explicit factors $s_i = \pm 1$ multiplying the signed square roots $x_i = s_i \sqrt{m_i}$; unless stated otherwise, the sign choices are *not* reinterpreted as distinct UV charge assignments.

2 Canonical U(3) charges and the quadratic identity

2.1 Charges from Cartan symmetry breaking

Let U(3) be a family symmetry with generator $Q_F = z_0 I + T$ where T lies in the Cartan of SU(3) and is traceless, $T = \text{diag}(z_1, z_2, z_3)$ with $\sum_i z_i = 0$. Canonical normalisation fixes

$$z_0 = \frac{1}{\sqrt{6}}, \quad \sum_{i=1}^3 z_i^2 = \frac{1}{2}, \quad (4)$$

so that the eigenvalues q_i of Q_F are

$$q_i = z_0 + z_i, \quad \sum_i q_i = \sqrt{\frac{3}{2}}, \quad \sum_i q_i^2 = 1. \quad (5)$$

An explicit parametrisation of the Cartan direction is

$$z_i = \frac{1}{\sqrt{3}} \cos\left(\alpha + \frac{2\pi(i-1)}{3}\right), \quad i = 1, 2, 3, \quad (6)$$

with α an angle on the two-dimensional Cartan plane.

The associated positive operator

$$H_Q \equiv Q_F^\dagger Q_F = Q_F^2 = \text{diag}(q_1^2, q_2^2, q_3^2) \quad (7)$$

obeys $\text{Tr}(H_Q) = 1$ by (??) and will be the universal object controlling all charge-based mass formulae below.

2.2 Quadratic mass law and uniqueness

Assume a quadratic charge law

$$m_i = \mu q_i^2, \quad (8)$$

with μ a generation-independent scale. Then the Koide ratio becomes a trace identity of H_Q :

$$K(\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3}) = K(q_1, q_2, q_3) = \frac{\sum_i q_i^2}{(\sum_i q_i)^2} = \frac{1}{3/2} = \frac{2}{3}, \quad (9)$$

independent of the Cartan direction α .

The special role of the quadratic power is not an accident. Define for integer n the generalised ratio

$$K_n(\alpha) \equiv \frac{\sum_i (z_0 + z_i(\alpha))^n}{(\sum_i (z_0 + z_i(\alpha))^{n/2})^2}, \quad (10)$$

whenever the numerator and denominator are real.

Theorem 1 (Uniqueness of the quadratic direction-independence for integer powers). *For integer powers n , $U(3)$ charges $q_i = z_0 + z_i$ with (??), and real $K_n(\alpha)$, the function $K_n(\alpha)$ is independent of the Cartan direction α if and only if $n = 2$ (besides the trivial constant $n = 0$).*

A short proof is given in Appendix ?? . For integer $n \geq 3$ it follows directly from the presence of the cubic invariant $\sum_i z_i^3 \propto \cos 3\alpha$ in the binomial expansion of $\sum_i (z_0 + z_i)^n$.

3 $\mathcal{N}=1$ confinement with a quantum-modified moduli space

3.1 Confined description and vacuum

Consider $\mathcal{N}=1$ $SU(3)$ SQCD with $N_f = N_c = 3$ flavours of fundamentals Q^i and antifundamentals $\tilde{Q}_i$ ($i = 1, 2, 3$) and diagonal mass deformations in the UV superpotential,

$$W_{UV} \supset \sum_{i=1}^3 m_i Q^i \tilde{Q}_i. \quad (11)$$

For $N_f = N_c$ the theory confines with a quantum-modified moduli space [?, ?, ?].

Confinement vs. s -confinement. In standard SQCD terminology, the case $N_f = N_c$ has a quantum-modified moduli space rather than an s -confining dynamical superpotential with a smooth origin. We implement the holomorphic constraint with a Lagrange-multiplier superfield X as in (??). In what follows, statements called “exact” refer to holomorphic relations implied by the constraint and the superpotential; physical masses additionally depend on Kähler normalisation. The IR degrees of freedom on the moduli space are gauge-invariant composites: mesons $M^i_j = Q^i \tilde{Q}_j$ and baryons $B, \tilde{B}$, subject to the quantum-modified constraint

$$\det M - B\tilde{B} = \Lambda^6, \quad (12)$$

with Λ the dynamical scale. The constraint is enforced by a Lagrange-multiplier chiral field X , giving

$$W_{IR} = X(\det M - B\tilde{B} - \Lambda^6) + \text{Tr}(m M), \quad (13)$$

On the mesonic branch $B = \tilde{B} = 0$, and in the basis where m is diagonal, the F -term conditions imply that M is diagonal in the vacuum and yield

$$\langle M_i \rangle = \frac{\Lambda^2 P^{1/3}}{m_i}, \quad P \equiv m_1 m_2 m_3, \quad (14)$$

and

$$X = -\frac{P^{1/3}}{\Lambda^4}, \quad (15)$$

which is generation-independent. Equation (??) is the first, central appearance of *inversion*: heavy electric masses correspond to small meson VEVs.

Matrix form of the vacuum. For an invertible mass matrix m (not necessarily diagonal), the mesonic-branch F -term solution can be written compactly as

$$\langle M \rangle = \Lambda^2 (\det m)^{1/3} m^{-1}, \quad \langle \text{adj}(M) \rangle = \Lambda^4 (\det m)^{2/3} m, \quad X = -\frac{(\det m)^{1/3}}{\Lambda^4}, \quad (16)$$

which reduces to (??)–(??) in the diagonal basis $m = \text{diag}(m_1, m_2, m_3)$.

3.2 Physical composite spectrum and inversion

The holomorphic superpotential (??) fixes the structure of fermion and scalar mass terms up to Kähler normalisation. For our purposes, an important special fact is that several physical mass scales in the confined theory are Λ -*independent* and are algebraic functions of the Seiberg masses m_i alone.

To see this most transparently, define $P = m_1 m_2 m_3$ and consider small fluctuations of the mesons around the diagonal vacuum. For off-diagonal mesons M^i_j with $i \neq j$, one finds a holomorphic mass term controlled by the third flavour $k \neq i, j$, and after canonical normalisation the leading mass scales take the simple form [?]:

$$m(M^i_j) = \frac{m_i m_j}{P^{1/3}} = \frac{P^{1/3}}{m_k}, \quad i \neq j, \{i, j, k\} = \{1, 2, 3\}. \quad (17)$$

The absence of explicit Λ dependence holds at the holomorphic level; physical masses are still dressed by Kähler normalisation, which is expected to be flavour-universal at leading order in the mass deformation. This relation exhibits a *multiple-description* property: the same leading mass scale can be written either as a product of two Seiberg masses or as the inverse of the third (up to the common scale $P^{1/3}$). The baryon mass scale is similarly simple,

$$m(B) \sim P^{1/3}, \quad (18)$$

again independent of Λ . Appendix ?? collects a compact derivation.

The quantum constraint (??) implies on the mesonic branch $\det M = \Lambda^6$, which is preserved by the involution

$$M \longrightarrow \Lambda^4 M^{-1}, \quad (19)$$

since $\det(\Lambda^4 M^{-1}) = \Lambda^{12} / \det M = \Lambda^6$. The map is holomorphic and well-defined on the locus where M is invertible (in particular on the mesonic branch); it is not a globally defined symmetry of the full moduli space including baryonic directions. Using the adjugate matrix,

$$\text{adj}(M) \equiv (\det M) M^{-1} = \Lambda^6 M^{-1}, \quad (20)$$

the involution can be written holomorphically as $M \leftrightarrow \Lambda^{-2} \text{adj}(M)$. At the vacuum (??) this exchanges $\langle M_i \rangle \propto 1/m_i$ with $\text{adj}(M)_i \propto m_i$.

3.3 Charges for each mass and multiple decompositions

If the Seiberg masses follow the quadratic charge law (??), then the confined spectrum inherits a hierarchy controlled by the same operator H_Q but with different *functions* of it. For instance,

$$\langle M_i \rangle \propto \frac{1}{m_i} \propto \frac{1}{q_i^2}, \quad m(M^i_j) = \frac{m_i m_j}{P^{1/3}} \propto \frac{q_i^2 q_j^2}{(q_1^2 q_2^2 q_3^2)^{1/3}}. \quad (21)$$

It is useful to introduce dimensionless *effective charges* for these composites. We restrict to Cartan directions for which all $q_i > 0$ (the regime relevant to positive mass deformations), so that the geometric mean below is real and positive. Writing

$$\bar{q} \equiv (q_1 q_2 q_3)^{1/3}, \quad (22)$$

equation (??) becomes

$$m(M^i_j) = \mu \left(\frac{q_i q_j}{\bar{q}} \right)^2 = \mu \left(\frac{\bar{q}}{q_k} \right)^2. \quad (23)$$

Thus each off-diagonal meson mass is determined by an effective charge that admits *two* natural decompositions: either as a product $q_i q_j / \bar{q}$ or as an inverse $\bar{q} / q_k$. These are algebraically

equivalent (they are related by $P = m_1 m_2 m_3$) and do not represent independent charge assignments. Because the underlying mass law depends only on squares, the same mass is also compatible with flipping the sign of any participating charge.

The set of functions of H_Q relevant to later sections can be summarised schematically as:

Object	Scaling with Seiberg masses	Scaling with family charges
Direct holomorphic masses	m_i	q_i^2
Meson VEVs (mesonic branch)	$1/m_i$	$1/q_i^2$
Off-diagonal meson masses	$m_i m_j / P^{1/3}$	$(q_i q_j / \bar{q})^2$
Baryon mass scale	$P^{1/3}$	$\bar{q}^2$

The physical interpretation of these entries depends on how Standard Model fields couple to the confined sector, which we address next.

4 The sBootstrap embedding: mesinos as leptons, diquarkinos as quarks

The preceding sections were phrased in standard SQCD language. The sBootstrap interpretation [?] adds a specific EFT philosophy and a specific identification of (some of) the confined fermions with the observed Standard Model fermions.

4.1 Gauge structure and mesino identification

The UV gauge group is taken to be

$$G_{UV} = \text{SU}(3)_F \times \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y, \quad (24)$$

where $\text{SU}(3)_F$ is a family (preon-colour) gauge group that confines at a scale Λ_F , while the SM gauge factors are weakly gauged spectators for the family confinement dynamics. In the SQCD formulas of Section ?? we denote the same dynamical scale by $\Lambda \equiv \Lambda_F$.

The $\text{SU}(3)_F$ matter content is organised so that the mesons transform as a 3×3 matrix under a global $\text{SU}(3)_L \times \text{SU}(3)_R$:

$$M^\alpha{}_i \sim Q^\alpha \tilde{Q}_i, \quad Q^\alpha \in (\mathbf{3}_F, \mathbf{3}_L), \quad \tilde{Q}_i \in (\bar{\mathbf{3}}_F, \bar{\mathbf{3}}_R), \quad (25)$$

so that $M \in (\mathbf{3}_L, \bar{\mathbf{3}}_R)$.

Gauge-invariant UV field content. A minimal gauge-invariant field/spurion assignment that reproduces the EFT used below is

Field	$\text{SU}(3)_F$	$\text{SU}(3)_c$	$\text{SU}(2)_L$	$\text{U}(1)_Y$
Q_a^A ($A = 1, 2$)	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{2}$	$-1/2$
Q_a^3	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{1}$	$+1$
$\tilde{Q}^a{}_i$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\mathbf{1}$	0
Σ_A^i	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{2}$	$+1/2$
Σ_3^i	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	-1

The fields $Q_a^\alpha, \tilde{Q}^a{}_i$ are the family-charged preons. Σ_α^i is the electroweak-charged deformation spurion (or, in a dynamical completion, a family-Higgs superfield) transforming as $(\mathbf{3}_L, \mathbf{3}_R)$ under the global $\text{SU}(3)_L \times \text{SU}(3)_R$.

Index dictionary. In Sections ??–?? we used standard SQCD notation in which both indices of M^i_j label flavour. In the sBootstrap embedding we instead interpret the left index as an $SU(3)_L$ label (containing electroweak quantum numbers) and the right index as an $SU(3)_R$ generation label. The diagonal spurion background in (??) selects an $SU(3)_L \times SU(3)_R$ -breaking alignment that allows one to work in a basis where the deformation is effectively diagonal and the component formulas of Section ?? apply. Embedding electroweak symmetry in $SU(3)_L$ by

$$\mathbf{3}_L \rightarrow \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}, \quad (26)$$

the meson matrix decomposes into electroweak doublets and singlets,

$$M^\alpha_i = \underbrace{M^{1,2}_i}_{(\mathbf{2}_{-1/2}, \mathbf{\bar{3}}_R)} \oplus \underbrace{M^3_i}_{(\mathbf{1}_{+1}, \mathbf{\bar{3}}_R)}. \quad (27)$$

The central sBootstrap identification is then:

$$\psi_{M^{1,2}_i} \equiv L_i, \quad \psi_{M^3_i} \equiv e_i^c, \quad (28)$$

i.e. the nine mesino fermions furnish the SM lepton sector (three lepton doublets plus three charged singlets). The scalar components of the same superfields provide Higgs- and slepton-like composites; the details of how one light Higgs combination is selected are part of the sBootstrap EFT building.

Compact IR dictionary. For later operator matching, the relevant confined objects are

SQCD object	$SU(3)_L \times SU(3)_R$	$SU(2)_L \times U(1)_Y$	sBootstrap role
M^A_i	$(\mathbf{3}_L, \mathbf{\bar{3}}_R)$	$(\mathbf{2}, -1/2)$	L_i mesinos, H_d -type scalars
M^3_i	$(\mathbf{3}_L, \mathbf{\bar{3}}_R)$	$(\mathbf{1}, +1)$	e_i^c mesinos, charged scalars
$\text{adj}(M)_A^i$	$(\mathbf{\bar{3}}_L, \mathbf{3}_R)$	$(\mathbf{2}, +1/2)$	H_u -type composite operator
$B, \tilde{B}$	$(\mathbf{1}, \mathbf{1})$	$(\mathbf{1}, 0)$	singlet baryonic superfields

Throughout, there are two layers of normalisation. The holomorphic Seiberg operator $M \sim Q\tilde{Q}$ has the composite/operator dimension appropriate to the confined description, while the IR chiral superfield used for the mesino identification is the canonically normalised field obtained after the unknown Kähler metric is specified. The hierarchy maps written in terms of M and $\text{adj}(M)$ therefore refer to holomorphic operator dependence; converting them into physical masses requires the later Kähler dressing.

Diagonal mass deformation and alignment. In the pure SQCD discussion of Section ?? we used a diagonal mass deformation $m_i Q^i \tilde{Q}_i$. In the sBootstrap embedding, the two flavour indices of SQCD correspond to $SU(3)_L$ (electroweak) and $SU(3)_R$ (generation). A diagonal deformation therefore presupposes an $SU(3)_L \times SU(3)_R$ -breaking alignment spurion that identifies a preferred basis in which the deformation is diagonal. Throughout, we work in that aligned basis; the resulting residual freedom is the Cartan direction α entering the charges (??).

One meson field and no Yukawa matrices. A characteristic sBootstrap simplification is that the generation structure is carried by the *meson matrix itself* rather than by independent Yukawa matrices. Below Λ_F , a minimal holomorphic EFT can be organised as

$$W_{\text{sBoot}} = X(\det M - B\tilde{B} - \Lambda_F^6) + \Sigma_\alpha^i M^\alpha_i + \frac{y_d}{\Lambda_F} \epsilon_{AB} \mathcal{O}_{di}^A M^B_i + \frac{y_u}{\Lambda_F^3} \epsilon_{AB} \mathcal{O}_{ui}^A \text{adj}(M)^B_i + \dots, \quad (29)$$

where $y_{u,d}$ are *numbers* rather than matrices, $\mathcal{O}_d \sim q_L d^c$ and $\mathcal{O}_u \sim q_L u^c$ are quark-bilinear placeholders, and the family structure sits entirely in the single meson matrix. In the aligned background, the VEVs generated by the deformation obey $\langle M \rangle \propto m^{-1}$ while $\langle \text{adj}(M) \rangle \propto m$, so the same spurion controls both inverse and direct hierarchies.

Gauge invariance of the deformation. Because M^α_i carries electroweak quantum numbers through (??)–(??), the diagonal shorthand $\text{Tr}(mM)$ used in the pure SQCD sections must be understood as the aligned background of the gauge-invariant coupling

$$W \supset \Sigma_\alpha^i M^\alpha_i, \quad (30)$$

with Σ transforming in the representation conjugate to M under the gauged $\text{SU}(2)_L \times \text{U}(1)_Y$ subgroup. As long as Σ is treated as a field/spurion, the Lagrangian is gauge invariant. Replacing it by a c-number matrix m is a statement about an *aligned background* in flavour space, and generically about an electroweak-broken branch: the simple diagonal formulas of Sections ?? and ?? therefore apply after alignment, not in the fully electroweak-symmetric phase. The crucial EFT assumption is that Σ is the *only* family-breaking spurion entering the superpotential at leading order. Once $\text{SU}(3)_L \times \text{SU}(3)_R$ rotations are used to align its background, the remaining flavour freedom is reduced to the Cartan direction α of the single operator H_Q ; this is what prevents Σ from reintroducing an arbitrary Yukawa matrix in disguise.

4.2 Diquarkinos and quarks (complementarity viewpoint)

The sBootstrap also aims at a constituent-level identification of the quark sector with *diquarkino*-like confined fermions, in the spirit of hadronic supersymmetry (Miyazawa/Catto–Gürsey) [?, ?, ?] and of the bootstrap/nuclear democracy picture [?]. A schematic object is a diquark

$$D \sim \Lambda^2 Q, \quad (31)$$

whose fermionic partner is a diquarkino. Such objects are not gauge-invariant operators of the $N_f = N_c = 3$ quantum-modified theory, but they naturally appear as effective coloured constituents in a Higgs/complementarity description [?, ?] in which baryons are viewed as quark–diquark bound states, $B \sim QD$.

Bare diquarkino obstruction. The simplest mesino-as-lepton embedding already fixes the hypercharge gap of a bare diquarkino. Writing $Q \rightarrow \mathbf{2}_u \oplus \mathbf{1}_v$ and $\bar{Q} \rightarrow \mathbf{1}_t$, the mesons decompose as $M = Q\bar{Q} \rightarrow \mathbf{2}_{u+t} \oplus \mathbf{1}_{v+t}$. Requiring these mesons to furnish (L_i, e_i^c) imposes $u+t = -1/2$ and $v+t = +1$, hence $v-u = 3/2$. But the bare diquark always transforms as

$$\Lambda^2 Q \rightarrow \mathbf{2}_{u+v} \oplus \mathbf{1}_{2u} = \mathbf{2}_{2u+3/2} \oplus \mathbf{1}_{2u}, \quad (32)$$

so the singlet hypercharge is rigidly lower than the doublet one by $3/2$. If the doublet is forced to be the SM quark doublet with $Y = +1/6$, the singlet is unavoidably pushed to $Y = -4/3$ rather than to $+1/3$ or $-2/3$. This is the clean obstruction to identifying *bare* $\Lambda^2 Q$ diquarkinos with the observed quarks.

EPJC completion. The EPJC construction [?] resolves the obstruction not by insisting on bare diquarks, but by completing the flavour sector into $\mathbf{15} \oplus \overline{\mathbf{15}} \oplus \mathbf{24}$ of an $\text{SU}(5)$ flavour group. In the corresponding branchings, the effective coloured sector contains quark-doublet-like pieces, their conjugates, and an unavoidable exotic $\pm 4/3$ state per generation. In the present paper we therefore keep *mesinos* = *leptons* and *diquarkinos* = *quarks* only in the quark–diquark/complementarity sense: the quarkino fields entering the EFT are effective coloured

multiplets of the completion, not exact gauge-invariant coordinates of the $N_f = N_c = 3$ moduli space.

Explicit effective operators. In the aligned basis, one leading gauge-invariant working ansatz for the placeholders in (??) is

$$W_q^{\text{eff}} = \sum_{i=1}^3 \left[\frac{y_d}{\Lambda_F} \epsilon_{AB} \mathcal{Q}_i^{Aa} \mathcal{D}_{ai}^c M^B{}_i + \frac{y_u}{\Lambda_F^3} \epsilon_{AB} \mathcal{Q}_i^{Aa} \mathcal{U}_{ai}^c \text{adj}(M)^B{}_i \right] + M_X X_i^a X_{ai}^c + \cdots, \quad (33)$$

where $A, B = 1, 2$ are $\text{SU}(2)_L$ indices and a is colour. The effective coloured multiplets $\mathcal{Q}_i \sim (\mathbf{3}, \mathbf{2})_{+1/6}$, $\mathcal{D}_i^c \sim (\bar{\mathbf{3}}, \mathbf{1})_{+1/3}$, $\mathcal{U}_i^c \sim (\bar{\mathbf{3}}, \mathbf{1})_{-2/3}$ are the SM-like quarkino combinations of the completion, while $X_i \sim (\mathbf{3}, \mathbf{1})_{+4/3}$ and $X_i^c \sim (\bar{\mathbf{3}}, \mathbf{1})_{-4/3}$ denote the exotic pair required by the obstruction/completion logic. Equation (??) makes the hierarchy map explicit: $Y_d^{\text{eff}} \propto \langle M \rangle \propto m^{-1}$, $Y_u^{\text{eff}} \propto \langle \text{adj}(M) \rangle \propto m$. The down-sector term is fixed by hypercharge; the up-sector term should be read as the leading ansatz rather than as a uniquely derived completion, since more baroque realisations may route the up sector partly through the **24** or through an additional confining sector. In particular, anomaly closure, the decoupling of the $\pm 4/3$ exotics, and the full consistency of the coloured quarkino sector are delegated to the external completion and are not claimed as results of the exact $N_f = N_c = 3$ dynamics.

Assumption / open problem. Equation (??) is an EFT realisation of the EPJC completion, not a derivation from the exact gauge-invariant operator basis of the $N_f = N_c = 3$ Seiberg theory. What is universal and derived in this paper is the two-operator hierarchy map carried by M and $\text{adj}(M)$; deriving the coloured quarkino multiplets and the heavy $\pm 4/3$ exotics from a fully explicit UV completion remains open.

5 From Seiberg masses to Yukawa hierarchies

5.1 Holomorphic couplings, spurions, and Kähler dressing

In supersymmetric effective field theory, superpotential mass terms and Yukawa couplings are both holomorphic parameters. In particular, the diagonal mass matrix m_i in (??) can be viewed as a flavour spurion transforming in $(\mathbf{3}, \bar{\mathbf{3}})$ under the UV global $\text{SU}(3)_L \times \text{SU}(3)_R$ symmetry. The confined dynamics then converts the spurion m_i into the algebraic structures (??)–(??).

However, physical masses in the IR depend on the Kähler potential. Even when holomorphy fixes *which* combinations of m_i appear in the superpotential, canonical normalisation can introduce (in general, generation-dependent and off-diagonal) wavefunction factors. For the present paper, the key point is that holomorphy provides *hierarchy functions* of m_i that are then dressed by non-holomorphic effects. The dressing is essential for mixing (CKM/PMNS), while the hierarchy functions control the gross ordering patterns such as direct vs inverse and the interleaving ladder.

5.2 Two basic Yukawa mechanisms: coupling to M or $\text{adj}(M)$

The inversion relation $\langle M_i \rangle \propto 1/m_i$ implies a sharp statement about how Yukawa hierarchies can arise from Seiberg masses.

sBootstrap form of the hierarchy map. In the sBootstrap EFT (??), the effective Yukawa structures are controlled by the spurion-induced VEVs,

$$Y_d^{\text{eff}} \sim \frac{y_d}{\Lambda_F} \langle M \rangle \propto m^{-1}, \quad Y_u^{\text{eff}} \sim \frac{y_u}{\Lambda_F^3} \langle \text{adj}(M) \rangle \propto m, \quad (34)$$

up to non-holomorphic Kähler dressing (needed, in particular, to generate mixing). Using the matrix vacuum relations (??), this may be written more explicitly as

$$Y_d^{\text{eff}} \sim y_d \Lambda_F (\det m)^{1/3} m^{-1}, \quad Y_u^{\text{eff}} \sim y_u \Lambda_F (\det m)^{2/3} m. \quad (35)$$

The same aligned spurion therefore generates two hierarchies and no family matrix is introduced by hand.

Inverse Yukawas from meson VEVs. Suppose a chiral field ψ_i couples (in a flavour-diagonal way) to the meson operator through an effective superpotential term,

$$W \supset \frac{\lambda}{M_*} \psi_i \psi_i^c M^i{}_i. \quad (36)$$

Then, on the mesonic branch, the leading holomorphic mass parameter induced by the meson VEV is

$$m_{\psi_i} \sim \frac{\lambda}{M_*} \langle M_i \rangle \propto \frac{1}{m_i} \propto \frac{1}{q_i^2}. \quad (37)$$

In other words, *Yukawa hierarchies can be inverse to Seiberg mass hierarchies* for fields that couple to M . This is the cleanest sense in which Seiberg mass parameters act as “Yukawas” for other sectors: the confined vacuum performs the inversion.

Direct Yukawas from the adjugate. Similarly, couplings to the adjugate operator,

$$W \supset \frac{\hat{\lambda}}{\hat{M}_*} \psi_i \psi_i^c \text{adj}(M)^i{}_i, \quad (38)$$

induce masses that scale directly with m_i (up to common factors) because $\text{adj}(M)_i \propto m_i$ at the vacuum. Since $\text{adj}(M)$ is holomorphic in M on the constraint surface (??), this “direct” option is not an independent structure: it is the other face of the same meson matrix.

Multiple descriptions. The presence of additional holomorphic composites such as the off-diagonal mesons $M^i{}_j$ implies further Yukawa patterns proportional to $m_i m_j / P^{1/3}$, cf. (??). Via (??) these patterns admit multiple charge decompositions and therefore multiple interpretations in terms of effective charges. Here “multiple decompositions” does *not* mean multiple independent charge assignments. It is the algebraic rewriting forced by $P = m_1 m_2 m_3$ and by the constraint surface itself:

$$\frac{m_i m_j}{P^{1/3}} = \mu \left(\frac{q_i q_j}{\bar{q}} \right)^2 = \mu \left(\frac{\bar{q}}{q_k} \right)^2, \quad \bar{q} \equiv (q_1 q_2 q_3)^{1/3}, \quad \{i, j, k\} = \{1, 2, 3\}. \quad (39)$$

The same particle can therefore carry a product form $qq'/\bar{q}$ or an inverse form $\bar{q}/q''$ without introducing new family charges.

5.3 Aligned SUSY breaking as an EFT ansatz for the quadratic triplets

The quadratic triplets $m_i \propto (z_0 + z_i)^2$ need not be read as detached numerology, but neither are they derived here from a controlled SUSY-breaking sector. What we do instead is exhibit a minimal EFT ansatz in which the same family operator $H_Q = Q_F^\dagger Q_F = \text{diag}(q_1^2, q_2^2, q_3^2)$ controls the leading SUSY-breaking/Kähler deformation. Introducing a SUSY-breaking spurion $X = \theta^2 F_X$, the aligned correction

$$\Delta K = \frac{X^\dagger X}{M_*^2} \left[c_0 \text{Tr}(M^\dagger M) + c_1 \text{Tr}(M^\dagger H_Q M) + \dots \right] \quad (40)$$

induces, in the diagonal basis,

$$(\Delta m_{\text{soft}}^2)_i \propto q_i^2 = (z_0 + z_i)^2. \quad (41)$$

If the dominant holomorphic deformation is aligned with the same family operator, the simplest effective choice is precisely $m_i = \mu q_i^2 = \mu(z_0 + z_i)^2$. This does not derive the q_i themselves, but it shows how the quadratic triplets can be made compatible with a single aligned SUSY-breaking spurion in the low-energy EFT.

The same logic extends to the off-diagonal composite masses. Once $m_i \propto q_i^2$, the exact Seiberg relations automatically generate product and inverse combinations for the *same* physical state through (??). In this sense, much of the observed “multiple $zz'z''$ ” structure is induced by the combination of aligned SUSY breaking and the quantum-modified constraint, rather than by postulating separate charge sectors for each particle.

From the Volkov–Akulov viewpoint [?], the goldstino packages the SUSY-breaking order parameter into nonlinear couplings. In the present setting, this makes it natural to interpret departures from the pure holomorphic limit as aligned soft splittings or Kähler dressings of a common family operator, rather than as extra Yukawa matrices. We stress again that this is an EFT interpretation, not a derivation of the UV origin of $m_i \propto (z_0 + z_i)^2$.

A suggestive numerical clue is the near-relation $m_\pi^2 - m_\mu^2 \approx f_\pi^2$. In an exactly restored supersymmetric chiral multiplet the scalar–fermion mass-squared splitting would vanish, so relations of this kind are naturally read as soft splittings. We do *not*, however, identify the physical pion and muon as a literal unbroken supermultiplet here: the exact $N_f = N_c = 3$ Seiberg spectrum does not yet supply the required gauge/global quantum-number assignment. We use the relation only as motivation for looking for aligned SUSY-breaking remnants in the bosonic sector as well as in the fermionic one.

5.4 Sign choices and the Koide diagnostic

Because the fundamental charge law depends on q_i^2 , each mass m_i is compatible with two signed square roots

$$x_i = s_i \sqrt{m_i}, \quad s_i = \pm 1, \quad (42)$$

while the underlying charge law remains $m_i = \mu q_i^2$. When all three entries in a Koide triple come from the same functional form in q (e.g. three direct entries $\propto q$ or three inverse entries $\propto 1/q$), the overall sign can be absorbed and no relative sign flips are needed to obtain $K = 2/3$. Relative signs become meaningful only in *mixed* triples where some entries scale as q and others as $1/q$; this will be the case in the waterfall.

6 Emergent doublet structure and the quark mass waterfall

This section is likewise structural. What is derived below is the existence of an interleaved direct/inverse ladder once both branches are present; mapping that ladder quantitatively onto the observed quark windows requires solving the quartic corrections and the non-holomorphic dressing, which we do not attempt here.

6.1 The effective quartic from the composite doublet

The $N_f = N_c$ confined theory contains both m_i and $1/m_i$ structures but, by itself, does not provide a controlled coupling between them. The composite spectrum, however, has a natural doublet structure: for each family i , the pair $(M_i, \text{adj}(M)_i)$ furnishes two mass eigenvalues with opposite charge dependence. This pairing is electroweak isospin and has the mathematical

form of an $\mathcal{N}=2$ hypermultiplet [?, ?]—emergent from confinement plus isospin, not a UV input.

The coupling between branches can be parametrised by introducing an auxiliary adjoint chiral field Φ with mass μ_Φ and the standard coupling $W \supset \sqrt{2} g \tilde{Q} \Phi Q$, then integrating Φ out via its F -term. One obtains an effective $\mathcal{N}=1$ superpotential with a quartic remnant,

$$W_{\text{eff}} = -\frac{g^2}{\mu_\Phi} \left[\text{Tr}(Q\tilde{Q})^2 - \frac{1}{3}(\text{Tr}Q\tilde{Q})^2 \right] + \sum_i m_i Q^i \tilde{Q}_i, \quad (43)$$

which vanishes as $\mu_\Phi \rightarrow \infty$ (decoupling). Here Φ is a bookkeeping device for the quartic, not a physical UV field; the scale μ_Φ parametrises the strength of the inter-branch coupling in the composite dynamics.

6.2 Residual quartic in the confined phase

In the confined description $Q\tilde{Q} \rightarrow M$, the quartic becomes

$$W_{\text{quartic}} = -\frac{g^2}{\mu_\Phi} \left[\text{Tr}M^2 - \frac{1}{3}(\text{Tr}M)^2 \right]. \quad (44)$$

This term couples the diagonal mesons and therefore perturbs the pure inversion solution (??). Since $\langle M_i \rangle \propto 1/m_i$, the fractional correction induced by (??) is parametrically largest for the lightest family and smallest for the heaviest. This hierarchy of corrections will reappear as a hierarchy of deviations from $K = 2/3$ in mixed waterfall windows.

6.3 Interleaving theorem

In the decoupling limit, the two basic hierarchies are the direct and inverse branches

$$a_i = \mu q_i^2, \quad d_i = \nu q_i^{-2}, \quad (45)$$

with opposite orderings for $q_1 < q_2 < q_3$: $a_1 < a_2 < a_3$ while $d_1 > d_2 > d_3$.

Theorem 2 (Interleaving). *The six unmixed masses $\{a_1, a_2, a_3, d_1, d_2, d_3\}$ form the alternating ladder*

$$d_1 > a_3 > d_2 > a_2 > d_3 > a_1 \quad (46)$$

if and only if

$$\max(q_1^2 q_3^2, q_2^4) < \frac{\nu}{\mu} < q_2^2 q_3^2. \quad (47)$$

Proof. The adjacent inequalities $d_1 > a_3 > a_2 > \dots > a_1$ reduce to the three independent conditions $\nu/\mu > q_1^2 q_3^2$, $\nu/\mu < q_2^2 q_3^2$, and $\nu/\mu > q_2^4$. The remaining inequalities follow from $q_1 < q_2 < q_3$. $\square$

The ladder alternates strictly between inverse and direct entries: $(d_1, a_3, d_2, a_2, d_3, a_1)$. Each consecutive triple therefore mixes functional dependences q and $1/q$ at the level of signed square roots, and the Koide diagnostic becomes sensitive to relative signs.

6.4 Waterfall windows and the sign flip

In the Koide diagnostic, the two branches correspond to

$$\sqrt{a_i} = \sqrt{\mu} q_i, \quad \sqrt{d_i} = \frac{\sqrt{\nu}}{q_i}. \quad (48)$$

For a pure-branch triple, all three entries scale as the same function of q and $K = 2/3$ reduces to the trace identity (??). For a mixed triple (e.g. (a_3, d_2, a_2)), the sum in the denominator of (??) contains both q and $1/q$ contributions, and a relative sign is generically needed for the ratio to be close to $2/3$. Empirically, the quark-mass waterfall uses a sign flip in the (b, c, s) window of physical masses [?]. In the present framework, such a sign flip is interpreted only as a qualitative diagnostic of the residual quartic mixing (??), which can shift the intermediate eigenvalues away from the unmixed a_i, d_i and thereby modify which branch character dominates each physical mass. We do not claim here that the quartic term has already been shown quantitatively to reproduce the observed windows or sign assignments. Section ?? should therefore be read as establishing a structural motif for the waterfall, not yet a demonstrated quark-mass mechanism.

7 Phenomenology and EFT consistency

The purpose of the present paper is to formulate a controlled flavour EFT, not to claim that all phenomenology is already computed. The minimal consistency checklist is therefore as follows.

Multi-Higgs structure and FCNC. The confined meson sector contains three H_d -type doublets M^A_i and, through $\text{adj}(M)$, three H_u -type doublets. With family-dependent VEVs or Yukawa textures, generic neutral-scalar exchange induces tree-level FCNC and new CP phases. A viable low-energy EFT therefore requires decoupling, alignment, or natural flavour conservation; at most one may presently invoke a Cheng–Sher-type suppression as a *working assumption*, not as a derived result [?, ?].

What the scale Λ_F does and does not mean. The scale Λ_F is the hidden family-confinement scale entering the holomorphic Seiberg relations. It should not automatically be identified with the direct contact-interaction scale probed by compositeness searches on the observed quarks and leptons. If the quark sector is partially composite or quarkino-like, the relevant phenomenological bound applies to the canonically normalised resonance masses and mixing operators, which must lie well above the weak scale even if the holomorphic parameter entering the moduli-space algebra is lower [?, ?, ?].

Exotics. The quark completion that removes the bare-diquarkino obstruction introduces an exotic $\pm 4/3$ coloured state per generation. In the EFT used here, these states are assumed heavy (or effectively vector-like after completion) and only their SM-like mixtures enter the leading operators (??). Any fully explicit completion must state their masses, anomaly role, and mixing pattern; they are evidence for extra structure, not decorative bystanders.

Mixing matrices. Because the holomorphic structures are functions of the same aligned spurion m , they commute and do not generate CKM or PMNS mixing by themselves. Mixing must therefore arise from non-holomorphic Kähler terms, soft misalignment, or elementary–composite mixing matrices in the quarkino sector. This is the honest status of the CKM formulas: until the relevant Kähler or mixing operators are derived, they remain empirical guides rather than EFT predictions. Accordingly, the present paper does not claim a derivation of the CKM or PMNS matrices. The neutrino sector is even less developed; large PMNS angles would have to come from baryonic/Kähler threshold effects or else be treated as outside the present scope.

Bosonic tuples. The present paper mostly tracks the fermionic tuples, but the same Koide-type regularities also appear empirically in mesonic triples such as $(0, \pi, D)$ and (π, D, B) . We take that bosonic parallel as a clue that the same aligned SUSY-breaking/compositeness logic may extend beyond the fermion sector, but we do not yet derive those meson tuples inside the EFT. Absent the missing Kähler/completion analysis, the paper should therefore be read as a structural flavour-EFT proposal rather than as a viability proof.

8 Status, interpretation, and open problems

The results above split naturally into five classes:

1. **Algebraic identities:** the trace identity (??) and the uniqueness Theorem ?? depend only on canonical $U(3)$ normalisation and on the assumed monomial charge law.
2. **Exact holomorphic constraint relations:** the mesonic-branch vacuum $\langle M \rangle \propto m^{-1}$ and the composite spectrum relations (??)–(??) follow from the constraint $\det M - B\tilde{B} = \Lambda^6$ and its holomorphic implementation in (??). Physical masses additionally depend on Kähler normalisation.
3. **sBootstrap postulates:** the identification of mesinos with leptons (??), the quark–diquark/complementarity interpretation of quarks as diquarkinos (Section ??), and the existence of an alignment spurion selecting a diagonal deformation basis are assumptions of the sBootstrap EFT rather than consequences of SQCD.
4. **Model-dependent embeddings and mixing:** mapping the holomorphic structures to SM pole masses (including CKM/PMNS) requires specifying the quark-sector operator content in (??) and controlling the non-holomorphic Kähler dressing. This is where mixing matrices arise, since functions of a single diagonal operator H_Q commute and cannot generate CKM without non-holomorphic effects.
5. **Speculative UV motivation:** the reading of $m_i \propto (z_0 + z_i)^2$ as an aligned SUSY-breaking/Kähler pattern (Section ??) is a concrete EFT interpretation, while deeper origins in charge-lattice/BPS or M-theory constructions are conjectural and are not used as premises in the derivations above [?, ?].

Earlier versions of this work were presented as two short notes (“Quadratic charge–mass relation from gauged family symmetry” and “The quark mass waterfall from emergent interleaving”), which separated the quadratic charge identity and the interleaving theorem. In this consolidated paper we have focused instead on the structural and supersymmetric origin of: (i) the direct/inverse pairing, (ii) the existence of multiple charge decompositions for the same composite masses, and (iii) the interleaving theorem that forces a mixed ordering whenever both branches are present and ν/μ lies in the window (??).

Important open problems include: (a) computing (or parametrising with control) the relevant Kähler metrics and their flavour dependence; (b) deriving the coloured quarkino completion and the $\pm 4/3$ exotics from an explicit UV model rather than from EFT placeholders; (c) understanding whether the quartic remnant (??) can generate the observed pattern of sign choices in the waterfall windows; (d) clarifying whether the same aligned SUSY-breaking logic can account for the empirical meson tuples $(0, \pi, D)$ and (π, D, B) as well as for the fermionic ones.

9 Conclusions

We have outlined a structural SUSY EFT proposal in which canonical $U(3)$ family normalisation, $\mathcal{N}=1$ confinement with a quantum-modified moduli space, and the emergent composite doublet structure jointly organise why: (i) a quadratic charge law yields an exact $K = 2/3$ identity independent of the Cartan direction, (ii) Seiberg mass deformations generate *inverse* hierarchies through meson VEVs and a Λ -independent composite spectrum with multiple charge decompositions, and (iii) the coexistence of direct and inverse branches forces an interleaved mass ladder in a calculable parameter window, providing the structural core of the quark mass waterfall. In the EFT reading advocated here, the same family operator H_Q can also be used to organise the leading aligned SUSY-breaking/Kähler deformations, so that the triplets $(z_0 + z_i)^2$ are not merely postulated but are compatible with the minimal quadratic symmetry-breaking pattern. Embedding these structures into the full SM flavour sector remains model-dependent, primarily controlled by Kähler dressing, by the explicit quarkino/exotic completion, and by whether the bosonic tuples can be brought under the same umbrella as the fermionic ones. No claim is made here that the present EFT already supplies a phenomenologically complete or uniquely UV-derived flavour model.

A Proof of Theorem ??

With $q_i = z_0 + z_i$ and z_i parametrised by (??), one has $\sum_i z_i = 0$ and $\sum_i z_i^2 = 1/2$ by construction, while

$$\sum_i z_i^3 = \frac{1}{3\sqrt{3}} \sum_i \cos^3\left(\alpha + \frac{2\pi(i-1)}{3}\right) \propto \cos 3\alpha, \quad (49)$$

so the cubic power sum depends on the Cartan direction. For integer $n \geq 3$, expanding $\sum_i q_i^n = \sum_i (z_0 + z_i)^n$ by the binomial theorem, the term $\binom{n}{3} z_0^{n-3} \sum_i z_i^3$ appears with nonzero coefficient and introduces α dependence into the numerator of $K_n(\alpha)$ in (??). For $n = 2$ one has $\sum_i q_i^2 = 3z_0^2 + \sum_i z_i^2 = 1$ and $(\sum_i q_i)^2 = (3z_0)^2 = 3/2$, giving $K_2 = 2/3$ independent of α . For $n = 1$ the denominator contains $\sum_i q_i^{1/2}$ and is α dependent. The case $n = 0$ gives the trivial constant $K_0 = 1/3$. For non-integer real n , the same cubic-power-sum mechanism suggests that α dependence should persist generically, but we do not claim a theorem: the non-integer extension is heuristic and is not used elsewhere in the paper.

B Derivation of the quantum-modified spectrum relations

We sketch the derivation of (??) following the standard quantum-modified effective description (??). On the diagonal vacuum $M = \text{diag}(M_1, M_2, M_3)$ with $B = \tilde{B} = 0$, the F -term conditions imply $M_i = \Lambda^2 P^{1/3}/m_i$ and $X = -P^{1/3}/\Lambda^4$.

For an off-diagonal meson M^i_j with $i \neq j$, the relevant holomorphic mass term comes from the second derivative of the determinant, $\partial^2(\det M)/\partial M^i_j \partial M^j_i = M_k$, where k is the third index. Canonical normalisation for mesons is M/Λ at leading order, so the leading mass scale is

$$m(M^i_j) \sim \Lambda^2 |X M_k| = \Lambda^2 \left| \frac{P^{1/3}}{\Lambda^4} \frac{\Lambda^2 P^{1/3}}{m_k} \right| = \frac{P^{2/3}}{m_k} = \frac{m_i m_j}{P^{1/3}}, \quad (50)$$

using $P = m_1 m_2 m_3$. The overall proportionality depends on the Kähler metric but is generation-independent at leading order; the dependence on m_i is fixed by holomorphy.

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