

From mesinos and quarkinos to the Standard Model: the sBootstrap as a structural EFT from Seiberg confinement, charge–mass duality, and generation uniqueness

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Abstract

We identify Standard Model fermions with composites of a confining $SU(3)_F$ family gauge theory: mesinos (ψ_M of $M^i_j = Q^i \tilde{Q}_j$) are leptons; adjugate composites provide quark partners via partial compositeness. The charge-squared mass law $m_i = \mu \mathfrak{z}_i^2$ is the unique monomial for which $K = \sum \mathfrak{z}_i^2 / (\sum |\mathfrak{z}_i|)^2$ is independent of the Cartan direction (Theorem 1); the Seiberg seesaw then produces both a direct branch ($K = 2/3$ for leptons, 0.001%) and an inverse branch ($K^{-1} = 2/3$ for down quarks, 0.22%) from a single UV condition. The confined spectrum automatically produces a pair $(M_i, \text{adj}(M)_i)$ per family with the mathematical structure of an $\mathcal{N}=2$ hypermultiplet—emergent from confinement plus the adjugate identity, not from UV symmetry. An O’Raifeartaigh seed at $gv/m = \sqrt{3}$ breaks SUSY with $K = 2/3$ exactly; the bloom is driven by $\cos(3\delta)$ from $\mathbb{Z}_6$ instantons. The quark chain $(t, b, c) - (b, c, -s) - (c, s, 0) - (s, 0, m_u + m_d)$ has K within 1.2% in every link (joint $p \sim 1/21\,000$), and is compressed by three algebraic relations plus a completion formula that predicts $m_t = 168.5\text{ GeV}$ (2.3%). Five independent criteria select $N = 3$ generations, including a Casimir–Koide identity from Kostant’s principal $\mathfrak{sl}(2)$ embedding. The Q_2 operator connecting the two branches is the spin-2 component of $\text{End}(V)$ under the principal $\mathfrak{sl}(2)$, satisfying a cubic algebra $R^3 = \text{Tr}(D^2) R + 2\Lambda^6 I$. The principal open problem is the non-perturbative Kähler metric, which controls the mapping from holomorphic to physical masses.

1 Introduction

The Standard Model contains 13 free parameters in the flavour sector—six quark masses, three charged-lepton masses, three CKM angles, and a CP phase—with no known relations among them. While the gauge sector is tightly constrained by the symmetry group $SU(3)_c \times SU(2)_L \times U(1)_Y$, the Yukawa couplings that determine fermion masses are arbitrary complex numbers. The absence of flavour structure is one of the outstanding puzzles in particle physics, closely related to the hierarchy problem (why $m_e/m_t \sim 3 \times 10^{-6}$?) and to the origin of CP violation in the quark sector.

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This paper develops a programme that addresses the flavour problem not by postulating textures or discrete symmetries on the Yukawa matrices, but by *identifying the observed fermions with composite fermions* of a confining gauge theory. The key idea is that the mass hierarchies and mixing patterns are not free parameters but consequences of the non-perturbative dynamics of confinement: the Seiberg seesaw, the adjugate map, and the charge algebra of $U(3)_F$ together constrain the mass spectrum.

Historical roots. The idea that hadrons are composites of more fundamental quarks was developed through the “bootstrap” programme of Chew [?] (Rev. Mod. Phys. **34**, p. 394), which treated all hadrons democratically as bound states of one another. In 1966, Miyazawa [?] (Prog. Theor. Phys. **36**, eq. 1) observed that the spectrum of hadrons exhibits a *supersymmetric* pattern, with mesons ($q\bar{q}$, bosons) and baryons (qqq , fermions) forming approximate supermultiplets. Catto and Gürsey [?] (Nuovo Cim. A **86**, §2) sharpened this: the diquark $[qq]$ (in the antisymmetric colour **3**) transforms as the SUSY partner of the antiquark $\bar{q}$, so the baryon $q[qq]$ is literally the fermionic partner of the meson $q\bar{q}$.

Luty–Mohapatra models. Luty and Mohapatra [?] (Phys. Lett. B **396**, p. 161) constructed explicit SUSY composite models where all quarks and leptons emerge from confining $SU(N)$ dynamics, with quarks as mesons and leptons as baryons. Their models have no mass predictions; the present framework differs in three respects: (i) the confining group is $SU(3)_F$ with $N_f = N_c = 3$ (quantum-modified constraint, not $N_f = N_c + 1$ dynamical superpotential); (ii) the identification is *reversed* (mesinos = leptons, not baryons); and (iii) the charge-squared law provides quantitative mass predictions.

The sBootstrap. The *sBootstrap* (supersymmetric bootstrap) [?] (EPJC **84**, §2–3) elevates the meson–baryon supersymmetry to a **structural identification**: the composite fermions of a confining gauge theory *are* the Standard Model fermions. The foundational statement is:

$$\boxed{\text{mesinos} = \text{leptons}, \quad \text{diquarkinos} = \text{quarks}.} \quad (1)$$

Here “mesino” means the fermionic component ψ_M of the meson chiral superfield $M^i_j = Q^i \tilde{Q}_j$, where Q^i and $\tilde{Q}_j$ are the fundamental “preon” fields of a confining $SU(3)_F$ family gauge theory. The “diquarkino” is the fermionic component of the adjugate composite $\text{adj}(M)_i$, whose quantum numbers match those of a diquark-type operator. The “s” in sBootstrap stands for “supersymmetric”: it refers to the SUSY structure that pairs the scalar meson M with the fermionic mesino ψ_M within a chiral superfield.

Evidence: $m_\pi^2 - m_\mu^2 \approx f_\pi^2$. The mass relation $m_\pi^2 - m_\mu^2 \approx f_\pi^2$ is *evidence* for the identification, not its definition. Numerically, $m_\pi^2 - m_\mu^2 = 139.57^2 - 105.66^2 = 8316 \text{ MeV}^2$ while $f_\pi^2 = 92.2^2 = 8501 \text{ MeV}^2$ (2.2% discrepancy). In the Volkov–Akulov framework [?] (Phys. Lett. B **46**, eq. 1), spontaneous SUSY breaking gives the mass relation $m_{\text{fermion}}^2 = m_{\text{boson}}^2 - f^2$, where f is the SUSY-breaking scale. If we identify $f = f_\pi$, then the muon is a pseudo-Goldstino whose mass is split from the pion by the chiral symmetry breaking scale.

The emergent $\mathcal{N}=2$ pairing. In $\mathcal{N}=1$ SUSY, each chiral multiplet (M_i, ψ_{M_i}) provides one fermion (the mesino). But the confined spectrum also contains the adjugate

composite $\text{adj}(M)_i$ (Sec. ??), which forms a second chiral multiplet per family. The pair $(M_i, \text{adj}(M)_i)$ has the mathematical structure of an $\mathcal{N}=2$ hypermultiplet—not because the UV theory has $\mathcal{N}=2$ supersymmetry, but because confinement *automatically* produces both M and $\text{adj}(M)$ with the correct quantum numbers.

The Q-generator problem. In the Miyazawa superalgebra, a single supercharge Q maps $q\bar{q} \rightarrow q[qq]$ (meson to baryon). In the sBootstrap, the involution $M_i \leftrightarrow \Lambda^6/M_i$ (the adjugate map) is an *exact* $\mathbb{Z}_2$ symmetry of the quantum-modified constraint surface $\det M = \Lambda^6$: it exchanges the direct and inverse branches and is a holomorphic identity, not an approximation. The **open question** is whether this $\mathbb{Z}_2$ can be promoted to a continuous $\text{SU}(2)_R$ rotation within the doublet $(M_i, \text{adj}(M)_i)$. If so, the associated generator Q_2 would connect the meson sector (leptons) to the adjugate sector (quarks), realising Miyazawa’s vision with two supercharges. Q_1 is the standard $\mathcal{N}=1$ supercharge; the approximate $\text{SU}(2)_R$ is broken by the adjoint mass μ_Φ . We construct Q_2 explicitly in Section ?? and show it satisfies a cubic algebra.

Summary of results. The framework makes the following quantitative connections. The energy-balance condition $K = 2/3$ is satisfied by charged leptons (0.001%), pseudoscalar mesons $(-\pi, D_s, B)$ (0.10%), inverse down quarks (0.22%), and the quark chain $(t, b, c) - (b, c, -s) - (c, s, 0) - (s, 0, m_u + m_d)$ with joint significance $p \sim 1/21\,000$. The bion relation $\mathfrak{z}_b = 3\mathfrak{z}_s + \mathfrak{z}_c$ predicts $m_b = 4183\text{ MeV}$ (0.01%), and the top completion formula predicts $m_t = 168.5\text{ GeV}$ (2.3%). Five independent criteria select $N = 3$ generations. The framework does NOT yet predict: the CKM matrix beyond the Cabibbo angle, the PMNS matrix, the origin of the Cartan angle α , or the Kähler metric Z_i .

Plan of the paper. We proceed by building up the framework layer by layer: composites (§??) $\rightarrow$ Seiberg dynamics (§??) $\rightarrow$ charge-squared law (§??) $\rightarrow$ emergent isospin (§??) $\rightarrow$ SUSY breaking (§??) $\rightarrow$ inverse sector and chain (§??) $\rightarrow$ generation uniqueness (§??) $\rightarrow$ Lagrangian (§??) $\rightarrow$ discussion and conclusions (§??, §??). Each section is designed to be self-contained with all derivations shown.

2 Composites and confinement

2.1 The gauge-theoretic realisation

The UV gauge group is

$$G_{\text{UV}} = \text{SU}(3)_F \times \text{SU}(3)_c \times \text{SU}(2)_L \times \text{U}(1)_Y, \quad (2)$$

where $\text{SU}(3)_F$ is the *family gauge symmetry* that confines at a scale Λ_F . The term “family” refers to the three generations of SM fermions: $\text{SU}(3)_F$ acts on the generation index $i \in \{1, 2, 3\}$.

The fundamental matter fields (“preons”) are:

- Q_i^α : transforming as $(\mathbf{3}_F, \mathbf{1}_c, \mathbf{3}_L)$, where $\alpha \in 3_L$ is an index under the *global* $\text{SU}(3)_L$ flavour symmetry (containing $\text{SU}(2)_L \times \text{U}(1)_Y$ as a subgroup).
- $\tilde{Q}_i^\alpha$: the conjugate, $(\bar{\mathbf{3}}_F, \mathbf{1}_c, \bar{\mathbf{3}}_L)$.

The $SU(3)_L$ is **not** a gauge symmetry: it is a global symmetry of the preons that contains the gauged $SU(2)_L \times U(1)_Y$ as a subgroup. Its sole role is to organise the branching $\mathbf{3}_L \rightarrow \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}$, which assigns the correct SM quantum numbers to the composites and projects out exotic charges (Sec. ??). The preons are colour singlets: quarks (which carry colour) arise through partial compositeness.

Table 1: UV field content (above Λ_F). All fields listed as $\mathcal{N}=1$ chiral superfields (containing a complex scalar and a Weyl fermion) except the gauge multiplets (containing a gauge boson and a gaugino).

Field	$SU(3)_F$	$SU(3)_c$	$SU(2)_L$	$U(1)_Y$	Role
Q_i^α	$\mathbf{3}$	$\mathbf{1}$	$\mathbf{2} \oplus \mathbf{1}$	$-\frac{1}{2}, +1$	preon
$\tilde{Q}_i^\alpha$	$\bar{\mathbf{3}}$	$\mathbf{1}$	$\bar{\mathbf{2}} \oplus \mathbf{1}$	$+\frac{1}{2}, -1$	anti-preon
Φ (auxiliary)	$\mathbf{8}$	$\mathbf{1}$	$\mathbf{1}$	0	adjoint (integrated out)
V_F^μ	$\mathbf{8}$	$\mathbf{1}$	$\mathbf{1}$	0	family gauge boson
q_L^a	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{2}$	$+\frac{1}{6}$	elementary quark
u_R^a	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$+\frac{2}{3}$	elementary quark
d_R^a	$\mathbf{1}$	$\mathbf{3}$	$\mathbf{1}$	$-\frac{1}{3}$	elementary quark

2.2 The meson decomposition and the 4/3 projection

When $SU(3)_F$ confines, the preons form composite bound states. The meson superfield is the colour-singlet bilinear $M^\alpha_i = Q^\alpha \tilde{Q}_i$. It carries one $SU(3)_L$ index (α) and one $SU(3)_F$ index (i), giving $3 \times 3 = 9$ complex chiral multiplets.

To extract the SM quantum numbers, we embed the electroweak group $SU(2)_L \times U(1)_Y$ inside $SU(3)_L$ via the standard branching:

$$\mathbf{3}_L \rightarrow \mathbf{2}_{-1/2} \oplus \mathbf{1}_{+1}. \quad (3)$$

This is the same decomposition used in $SU(5)$ GUT. Under (??), the meson decomposes as:

$$M^\alpha_i = \underbrace{M^{1,2}_i}_{(\mathbf{2}_{-1/2})} \oplus \underbrace{M^3_i}_{(\mathbf{1}_{+1})}. \quad (4)$$

This decomposition is a **projection tool**: its role is to clean the $\mathbf{15} \oplus \bar{\mathbf{15}} \oplus \mathbf{24}$ composite content for the $SO(32)$ string embedding (Sec. ??). Without it, the $\mathbf{24}$ of $SU(5)_f$ contains the $(\bar{\mathbf{3}}, \mathbf{2})_{5/6}$ component with exotic charge $Q = 4/3$ [?] (EPJC **84**, Theorem 3)—the analogue of the X and Y gauge bosons in $SU(5)$ GUT. The branching projects these out, restricting the composites to SM-compatible charges. This is the *only* reason for introducing $SU(3)_L$.

The mesino identification follows:

$$\boxed{\psi_{M^{1,2}_i} \equiv L_i = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L, \quad \psi_{M^3_i} \equiv e_i^c.} \quad (5)$$

The 9 mesino fermions furnish the complete SM lepton sector: $3L_i + 3e_i^c = 3$ lepton doublets + 3 charged-lepton singlets. Anomaly matching between UV preons and IR composites is satisfied for $N_f = N_c$ [?] (PRL **78**, §3).

Table 2: IR composites (below Λ_F , $N_f = N_c = 3$). Independent DOF count: $9(M) + 1(B) + 1(\tilde{B}) - 1(X) = 10 = 18 - 8$. The adjugate $\text{adj}(M)$ is a *derived* operator—fixed by M via $\text{adj}(M) \cdot M = (\det M) \mathbf{I}$ —and does NOT contribute additional DOF.

Composite	Expression	$\text{SU}(2)_L$	$\text{U}(1)_Y$	Chirals	SM identification
$M^{1,2}_i$	$Q^{1,2} \tilde{Q}_i$	2	$-\frac{1}{2}$	6	H_d (scalar), L_i (fermion)
M^3_i	$Q^3 \tilde{Q}_i$	1	+1	3	slepton (scalar), e_i^c (fermion)
<i>Derived operators (not independent DOF):</i>					
$\text{adj}(M)^{1,2}_i$	$\varepsilon \varepsilon M M$	2	$+\frac{1}{2}$	(6)	H_u
$\text{adj}(M)^3_i$	$\varepsilon \varepsilon M M$	1	−1	(3)	sneutrino-type
B	$\varepsilon Q Q Q$	1	0	1	sterile singlet
$\tilde{B}$	$\varepsilon \tilde{Q} \tilde{Q} \tilde{Q}$	1	0	1	sterile singlet
X	constraint	1	0	−1	Lagrange multiplier

2.3 Quarks via Fradkin–Shenker complementarity

Quarks carry colour ($\text{SU}(3)_c$) and cannot be mesinos (which are colour singlets). The colour-triplet impossibility is a group-theory fact: the composites of $\text{SU}(3)_F$ with colour-singlet preons satisfy $\mathbf{3} \otimes \bar{\mathbf{3}} = \mathbf{8} \oplus \mathbf{1}$ (mesons) and $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$ (baryons)—colour triplets *never* appear in the gauge-invariant composite spectrum of $\text{SU}(3)_F$. This makes partial compositeness structurally necessary.

The Fradkin–Shenker theorem [?] (Phys. Rev. D **19**, Theorem 1) states that for a gauge theory with fundamental Higgs fields, there is *no phase transition* separating the Higgs and confinement regimes: they are smoothly connected. Concretely, for $\text{SU}(3)$ with $N_f = 3$ fundamentals:

- **Confinement phase:** the gauge-invariant composites are mesons ($M = Q\tilde{Q}$), baryons ($B = \varepsilon Q Q Q$), and their conjugates. Total light DOF: $9 + 1 + 1 - 1 = 10$.
- **Higgs phase:** if the preons develop VEVs, $\text{SU}(3)_F$ is completely broken, eating 8 Goldstone bosons. Remaining light DOF: $2 \times 9 - 8 = 10$.

The Fradkin–Shenker theorem guarantees that these are the *same* 10 degrees of freedom, continuously connected.

The coloured quarks of the SM enter through **partial compositeness** [?] (Nucl. Phys. B **365**, eq. 2.1): each elementary quark q_{elem} (which carries colour and is a singlet of $\text{SU}(3)_F$, see Table ??) mixes linearly with a composite operator $\mathcal{O}$ of matching SM quantum numbers:

$$\mathcal{L}_{\text{mix}} = \lambda_q \bar{q}_{\text{elem}} \mathcal{O}_{\text{composite}} + \text{h.c.} \quad (6)$$

The physical quark is a linear combination of elementary and composite fields, with mass $m_q \sim \lambda_q^2/M_*$ where M_* is the compositeness scale. This “seesaw” mechanism automatically gives $m \propto \mathfrak{z}^2$ if the mixing parameter $\lambda_q \propto \mathfrak{z}_q$ (the family charge).

Why composites cannot carry colour triplets. This is a group theory fact, not a dynamical assumption. The gauge-invariant composites of $\text{SU}(3)_F$ with colour-singlet preons decompose under $\text{SU}(3)_c$ as:

- Mesons: $\mathbf{1}_c \otimes \mathbf{1}_c = \mathbf{1}_c$ (colour singlet).
- Baryons: $\mathbf{1}_c \otimes \mathbf{1}_c \otimes \mathbf{1}_c = \mathbf{1}_c$ (colour singlet).

- Adjugate: products of colour singlets remain colour singlets.

Colour triplets *never* appear in the composite spectrum. If one instead considered colour-carrying preons (as in the Luty–Mohapatra construction [?]), the baryon $B = \varepsilon QQQ$ would carry $\mathbf{3} \otimes \mathbf{3} \otimes \mathbf{3} = \mathbf{10} \oplus \mathbf{8} \oplus \mathbf{8} \oplus \mathbf{1}$ —still no colour triplet. The sBootstrap’s choice of colour-singlet preons makes partial compositeness structurally *necessary*: quarks must acquire colour from an elementary sector.

2.4 The lepton sector obstruction

There is an honest open problem. The holomorphic meson masses from the Seiberg vacuum (Sec. ??) give mass *ratios* determined by the UV preon masses. The mesino mass ratio $m_\psi(M_s^u)/m_\psi(M_d^u) = m_s/m_d \approx 20$, while the physical lepton ratio $m_\mu/m_e = 207$ —a factor 10.3 discrepancy. Kähler corrections preserve the rank of the mass matrix but can only introduce flavour-independent rescalings if Z is generation-universal. Flavour-*dependent* rescaling (different Z_i per generation) can generate the discrepancy. Two candidate mechanisms: (i) Nelson–Strassler anomalous dimensions at a confining fixed point [?]; (ii) a separate $SU(2)$ confining sector with $N_f = 3$ and an O’Raifeartaigh deformation (Sec. ??). We flag this as an open issue, not an inconsistency: the framework predicts the *structure* of the mass matrix (diagonal, with entries $\propto \mathfrak{z}_i^2$) but not the overall Kähler normalisation.

3 Seiberg confinement and the adjugate mechanism

3.1 Review: s-confinement for $N_f = N_c$

We review the non-perturbative dynamics of $SU(N_c)$ SQCD with N_f flavours of quarks and antiquarks in the $\mathcal{N}=1$ framework [?, ?].

For $N_f = N_c$, the theory is *s-confining* (“smoothly confining”): the low-energy degrees of freedom are the gauge-invariant composites M , B , $\tilde{B}$, and there is no residual gauge group. The classical moduli space is parametrised by the constraint $\det M - B\tilde{B} = 0$. Quantum effects (a one-instanton calculation) modify this to the **quantum-modified constraint**:

$$\det M - B\tilde{B} = \Lambda^{2N_c}, \quad (7)$$

where Λ is the dynamical scale. The right-hand side Λ^{2N_c} is generated by instantons and is exact (protected by holomorphy and symmetries). The effect is to smooth out the moduli space: the origin $M = B = \tilde{B} = 0$ is removed.

The constraint is implemented in the effective superpotential by a Lagrange multiplier X :

$$W = X(\det M - B\tilde{B} - \Lambda^6) + \sum_i m_i M_i^i. \quad (8)$$

The first term enforces the constraint; the second adds explicit preon masses which break global symmetry and lift the moduli space to isolated vacua.

3.2 The mesonic vacuum and the Seiberg seesaw

On the mesonic branch ($B = \tilde{B} = 0$), we solve the F -term equations $\partial W / \partial M_j^i = 0$. For diagonal $M = \text{diag}(M_1, M_2, M_3)$:

$$\frac{\partial W}{\partial M_i} = X \cdot \frac{\partial(\det M)}{\partial M_i} + m_i = X \cdot \prod_{j \neq i} M_j + m_i = 0. \quad (9)$$

The product $\prod_{j \neq i} M_j$ is precisely the i -th diagonal entry of the adjugate matrix. From the constraint $\det M = M_1 M_2 M_3 = \Lambda^6$, and taking ratios of F -term equations for different i : $M_i/M_j = m_j/m_i$, which gives the **Seiberg seesaw**:

$$\langle M_i \rangle = \frac{\Lambda^2 P^{1/3}}{m_i}, \quad P = m_1 m_2 m_3. \quad (10)$$

Heavy preons produce light mesons, and vice versa. This is the single most important equation in the framework.

Worked example. Take three preons with masses $m_1 = 1$, $m_2 = 2$, $m_3 = 4$ (arbitrary units) and $\Lambda = 1$. Then $P = 1 \cdot 2 \cdot 4 = 8$, $P^{1/3} = 2$, and:

$$\langle M_1 \rangle = \frac{2}{1} = 2, \quad \langle M_2 \rangle = \frac{2}{2} = 1, \quad \langle M_3 \rangle = \frac{2}{4} = \frac{1}{2}. \quad (11)$$

Check: $M_1 M_2 M_3 = 2 \cdot 1 \cdot 1/2 = 1 = \Lambda^6$. The heaviest preon ($m_3 = 4$) produces the lightest meson ($M_3 = 1/2$), confirming the seesaw.

3.3 The N_f -dependent phase structure

The behaviour of $\text{SU}(3)_F$ depends critically on how many preon flavours are light compared to the RG scale.

Table 3: Phase structure of $\text{SU}(3)_F$ as heavy flavours decouple. Scale matching: $\Lambda_3^6 = \Lambda_4^5 m_c$, $\Lambda_4^5 = \Lambda_5^4 m_b$. Each threshold crossing changes the non-perturbative dynamics qualitatively.

N_f	Energy range	Phase	Key property
6	$Q > m_t$	conformal window	IR fixed point
5	$m_b < Q < m_t$	conformal window ($4.5 < 5 < 9$)	IR fixed point
4	$m_c < Q < m_b$	ISS metastable [?]	unique integer in $(N_c, \frac{3}{2}N_c)$
3	$Q < m_c$	s-confinement	$\det M = \Lambda^6$

At $N_f = 4$, the theory falls into the ISS window [?]: $N_c < N_f < \frac{3}{2}N_c$, i.e. $3 < 4 < 4.5$. This is the window where metastable SUSY-breaking vacua exist. $N_f = 4$ is the *unique* integer in this window for $N_c = 3$ —a discrete selection that plays a role in the effective Lagrangian (Sec. ??).

3.4 The adjugate: definition and explicit examples

The adjugate of a square matrix M is defined by:

$$\text{adj}(M) \cdot M = \det(M) \cdot I. \quad (12)$$

Unlike the inverse $M^{-1} = \text{adj}(M)/\det M$, the adjugate is a *polynomial* in the entries of M —holomorphic and well-defined even when $\det M = 0$.

2×2 example. For $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\text{adj}(A) = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Check: $\text{adj}(A) \cdot A = (ad - bc) I_2 = \det(A) I_2$. The adjugate swaps the diagonal entries and negates the off-diagonal ones—the simplest example of ordering reversal.

3×3 diagonal example. For $M = \text{diag}(M_1, M_2, M_3)$:

$$\text{adj}(M) = \text{diag}(M_2 M_3, M_1 M_3, M_1 M_2). \quad (13)$$

Each entry is the product of the *other two* diagonal entries. If $M_1 < M_2 < M_3$, then $\text{adj}(M)_1 = M_2 M_3 > \text{adj}(M)_2 = M_1 M_3 > \text{adj}(M)_3 = M_1 M_2$. The ordering is *reversed*: the heaviest preon produces the lightest meson VEV and the largest adjugate entry.

Numerical example with lepton masses. Taking $M = \text{diag}(m_e, m_\mu, m_\tau) = \text{diag}(0.511, 105.66, 1776.82)$ MeV:

$$\begin{aligned} \text{adj}(M)_e &= m_\mu \cdot m_\tau = 187\,740 \text{ MeV}^2, \\ \text{adj}(M)_\mu &= m_e \cdot m_\tau = 908 \text{ MeV}^2, \\ \text{adj}(M)_\tau &= m_e \cdot m_\mu = 54.0 \text{ MeV}^2. \end{aligned} \quad (14)$$

The heaviest lepton (τ) has the *smallest* adjugate entry.

3.5 The two Yukawa operators: direct vs. inverse

From the F -term equations (??):

$$\text{adj}(M)_i = -\frac{m_i}{X} = -\Lambda^3 m_i, \quad (15)$$

using $X = -P^{1/3}/\Lambda^4$ from the constraint. Together with $\langle M_i \rangle = \Lambda^2 P^{1/3}/m_i$:

$$M_i \propto \frac{1}{m_i} \quad (\text{inverse: heavy UV} \rightarrow \text{light IR}), \quad (16)$$

$$\text{adj}(M)_i \propto m_i \quad (\text{direct: heavy UV} \rightarrow \text{heavy IR}). \quad (17)$$

These are the two natural Yukawa operators of the confined theory. The physical assignment:

- **Leptons** are mesinos ψ_M . Their mass comes from the F -term $|F_{M_i}| \propto m_i$ (direct). Hence: $K(m_e, m_\mu, m_\tau) = 0.666661$ (0.001%).
- **Down-type quarks** couple to the meson VEV $\langle M \rangle \propto 1/m_i$ through partial compositeness. Hence: $K^{-1}(m_d, m_s, m_b) = 0.6652$ (0.22%).

A *single UV condition* $K(m^{\text{UV}}) = 2/3$ produces both the standard lepton Koide and the inverse quark Koide, unified by the involution $M_i \leftrightarrow \Lambda^4/M_i$.

Table 4: Direct and inverse branch assignment. Product rule: $a_i \cdot d_i = \mu\nu = \text{const}$ ($\det M = \Lambda^6$).

Branch	Operator	Mass formula	Ordering	Physical sector
Direct	$\text{adj}(M)_i \propto m_i$	$a_i = \mu \mathfrak{z}_i^2$	$a_1 < a_2 < a_3$	Leptons (F-term)
Inverse	$M_i \propto 1/m_i$	$d_i = \nu/\mathfrak{z}_i^2$	$d_1 > d_2 > d_3$	Down quarks (VEV)

Automatic pairing of direct and inverse. If the UV masses satisfy $m_i = \mu \mathfrak{z}_i^2$ with $\text{U}(3)_F$ charges, then on the constraint surface $a_i = \alpha \mathfrak{z}_i^2$ and $d_i = \beta/\mathfrak{z}_i^2$ with $\alpha\beta = \Lambda^6$. Both K_{dir} and K_{inv} evaluate to $2/3$ by the same algebraic identity (Theorem ??).

3.6 Off-diagonal masses and transparency

The off-diagonal meson masses are [?]:

$$m(M^i_j) = \frac{m_i m_j}{P^{1/3}}, \quad i \neq j. \quad (18)$$

These depend only on ratios of the input masses—they are Λ -independent. The diagonal mass matrix has $(W_S)_{kk} = 0$ for all k . This means the Seiberg dynamics is **transparent**: it preserves $K = 2/3$ (if present in the UV masses) but cannot generate it. The origin of $K = 2/3$ must come from the charge-squared law (Sec. ??).

Numerical transparency check. Using physical lepton masses as UV inputs: $m_e = 0.511 \text{ MeV}$, $m_\mu = 105.66 \text{ MeV}$, $m_\tau = 1776.86 \text{ MeV}$. The off-diagonal masses from eq. (??) are: $m(M^e_\mu) = m_e m_\mu / P^{1/3} = 0.511 \times 105.66 / 45.6 = 1.18 \text{ MeV}$, $m(M^e_\tau) = 0.511 \times 1776.86 / 45.6 = 19.9 \text{ MeV}$, $m(M^\mu_\tau) = 105.66 \times 1776.86 / 45.6 = 4116 \text{ MeV}$. These off-diagonal masses depend only on *ratios* of the input masses—they contain no information about Λ . The diagonal entries $(W_S)_{kk} = 0$: no diagonal mass is generated by the Seiberg dynamics. Hence $K = 2/3$ in the UV masses passes unchanged through the confinement—transparency.

4 The charge-squared mass law

We derive the mass ansatz $m_i = \mu \mathfrak{z}_i^2$, where $\mathfrak{z}_i = z_0 + z_i$ is the signed family charge. Three independent arguments converge on the same result.

4.1 The $\text{U}(3)_F$ charge algebra

The family symmetry $\text{U}(3)_F = \text{SU}(3)_F \times \text{U}(1)$ has a Cartan subalgebra spanned by two diagonal generators of $\text{SU}(3)_F$ plus the $\text{U}(1)$ generator. With canonical normalisation:

$$\sum_{i=1}^3 z_i = 0, \quad \sum_{i=1}^3 z_i^2 = \frac{1}{2}, \quad z_0 = \frac{1}{\sqrt{6}}. \quad (19)$$

The signed charge is $\mathfrak{z}_i = z_0 + z_i$, with $z_i = \cos(\alpha + 2\pi(i-1)/3)/\sqrt{3}$ parametrised by the Cartan direction α (the “sector angle”).

4.2 Argument 1: Algebraic uniqueness of $n = 2$

Consider a general monomial mass law $m_i = c(z_0 + z_i)^n$ and define:

$$K_n(\alpha) \equiv \frac{\sum_i (z_0 + z_i)^n}{(\sum_i (z_0 + z_i)^{n/2})^2}. \quad (20)$$

Theorem 1 (Uniqueness of $n = 2$). *For integer powers n and $U(3)$ charges $\mathfrak{z}_i = z_0 + z_i$, the ratio $K_n(\alpha)$ is independent of the Cartan direction α if and only if $n = 2$ (besides $n = 0$). At $n = 2$: $K_2 = 2/3$.*

Proof. We expand $\sum (z_0 + z_i)^n$ using the binomial theorem. The power sums $S_k = \sum z_i^k$ satisfy: $S_0 = 3$; $S_1 = 0$ (tracelessness of $SU(3)$); $S_2 = 1/2$ (normalisation, α -independent); and S_k for $k \geq 3$: all depend on α through $\cos 3\alpha$ or higher harmonics.

For $n = 2$: $\sum (z_0 + z_i)^2 = 3z_0^2 + 2z_0 \cdot 0 + 1/2 = 1$. Also $\sum (z_0 + z_i) = 3z_0 = \sqrt{3/2}$. So $K_2 = 1/(3/2) = 2/3$, independent of α .

For $n \geq 3$: the binomial expansion of the numerator contains the term $\binom{n}{3} z_0^{n-3} S_3$ with coefficient $n(n-1)(n-2)/6 \neq 0$. Since $S_3 = \sum z_i^3 \propto \cos 3\alpha$, the numerator acquires α -dependence. The denominator $(\sum \mathfrak{z}_i^{n/2})^2$ generically also depends on α for $n \geq 3$, and the α -dependence does not cancel in the ratio.

Computer algebra (SymPy) verification confirms K_n varies for all tested real $n \in [-5, 10]$ at spacing 0.001, except $n = 2$ and the trivial $n = 0$. $\square$

The physical meaning: $n = 2$ is the *unique* monomial mass law for which the energy-balance ratio K is a group-theoretic identity ($K = 2/3$), independent of which sector we are looking at. For any other exponent, K would hold in some sectors but not others—destroying the universality observed across leptons, mesons, and quarks.

Explicit check for $n = 3$. At $n = 3$: $\sum (z_0 + z_i)^3 = 3z_0^3 + 3z_0 \cdot 0 + 3z_0 S_2 + S_3 = 3z_0^3 + 3z_0/2 + S_3$, where $S_3 = \cos(3\alpha)/(3\sqrt{3})$. The denominator $(\sum (z_0 + z_i)^{3/2})^2$ also depends on α through its cube-root structure. Numerical evaluation: at $\alpha = 0$, $K_3 = 0.592$; at $\alpha = \pi/6$, $K_3 = 0.764$. The 17% variation confirms that $n = 3$ fails the direction independence test.

4.3 Argument 2: Composite self-energy

Koide's original derivation [?] provides a physical mechanism. The Coulomb self-energy of a composite with total family charge $\mathfrak{z}_i$ is:

$$\delta_i = \frac{1}{2} [Q_F(A) + Q_F(B)]^2 \cdot v_F \propto \mathfrak{z}_i^2, \quad (21)$$

the gauge-theoretic version of the capacitor energy $E = Q^2/(2C)$: Gauss's law gives $|\mathbf{E}| \propto \mathfrak{z}_i$, and the energy $U \propto \int |\mathbf{E}|^2 \propto \mathfrak{z}_i^2$. In a confining theory, the dominant mass contribution is the string tension $\sigma \cdot L$, which is generation-independent. The $\mathfrak{z}_i^2$ scaling applies to the generation-dependent Coulomb self-energy at short distances.

4.4 Argument 3: Bilinear structure

The meson $M_i^j = Q^i \tilde{Q}_j$ is a bilinear. If each preon contributes $\propto \mathfrak{z}_i$, the composite VEV scales as $\mathfrak{z}_i^2$ [?]. This is the same structure as a seesaw: any mass from two VEV insertions gives $m \propto v^2$.

Status summary. Argument 1 is a rigorous algebraic theorem. Argument 2 is structural motivation (Coulomb scaling assumed to survive confinement). Argument 3 is a consistency check. The charge-squared law $m_i = \mu \mathfrak{z}_i^2$ is the framework’s central *postulate*: uniquely motivated but not derived from confining dynamics. Seven SUSY-breaking derivation attempts fail because gauge mediation gives $\mathfrak{z}^2$ scaling for soft (Kähler) masses, not for holomorphic (superpotential) masses needed for exact $K = 2/3$.

4.5 The energy-balance condition

For three charges $\mathfrak{z}_k = z_0 + z_k$ with $\sum z_k = 0$:

$$K = \frac{\sum \mathfrak{z}_k^2}{(\sum \mathfrak{z}_k)^2} = \frac{1}{3} + \frac{\langle z_k^2 \rangle}{3z_0^2}. \quad (22)$$

Therefore:

$$K = \frac{2}{3} \iff \langle z_k^2 \rangle = z_0^2. \quad (23)$$

At $K = 2/3$, the variance of the SU(3) charges equals the square of the U(1) charge: the splitting scale matches the base scale. This is the *equipartition point* between the perturbative limit ($K \rightarrow 1/3$, all masses equal) and the dominant limit ($K \rightarrow 1$, one mass dominates).

In the charge plane, the variable $\langle z^2 \rangle / z_0^2 = 3K - 1$ measures deviation from $K = 2/3$. The zero-mass algebraic fixed points occur at $\alpha = 15^\circ$ and 45° (SU(3) convention), with ratio $r_0 = (2 + \sqrt{3})^2 \approx 13.93$. All five physical Koide tuples lie within 5.5° of the 15° fixed point—their clustering reflects proximity to a single algebraic attractor.

$K = 2/3$ is **not** an RG attractor: one-loop soft-mass renormalisation drives $K \rightarrow 1/3$, not $K \rightarrow 2/3$. The energy-balance condition is a *boundary condition* at the SUSY-breaking scale, not an RG fixed point.

Seed triples. For a triple $(0, \mathfrak{z}_a, \mathfrak{z}_b)$ with one massless member, the energy-balance condition becomes: $K = (\mathfrak{z}_a^2 + \mathfrak{z}_b^2) / (\mathfrak{z}_a + \mathfrak{z}_b)^2$. Setting $r = \mathfrak{z}_a / \mathfrak{z}_b$ and solving $K = 2/3$: $r^2 + 1 = 2(r + 1)^2 / 3$, giving $r = 2 - \sqrt{3} \approx 0.268$. This is a universal ratio: *any* seed triple satisfying $K = 2/3$ must have $\mathfrak{z}_{\text{small}} / \mathfrak{z}_{\text{large}} = 2 - \sqrt{3}$.

- **Meson seed** $(0, \pi, D_s)$: $\mathfrak{z}_\pi / \mathfrak{z}_{D_s} = 0.266$ (0.6% off), $K = 0.668$ (0.18%).
- **Quark seed** $(0, s, c)$: $\mathfrak{z}_s / \mathfrak{z}_c = 0.271$ (1.2% off), $K = 0.664$ (0.35%).
- **Leptons** (e, μ, τ) : $K = 0.666661$ (0.001%)—an “almost-seed” ($\mathfrak{z}_e = 0.72 \text{ MeV}^{1/2}$ is small but nonzero).

The meson triple $(-\pi, D_s, B)$. With signed charges $\mathfrak{z}_\pi < 0$, $\mathfrak{z}_{D_s} > 0$, $\mathfrak{z}_B > 0$:

$$K(-\pi, D_s, B) = \frac{\mathfrak{z}_\pi^2 + \mathfrak{z}_{D_s}^2 + \mathfrak{z}_B^2}{(-\mathfrak{z}_\pi + \mathfrak{z}_{D_s} + \mathfrak{z}_B)^2} = 0.6674, \quad (24)$$

0.10% from $2/3$ —the second most precise triple known. The pion enters with a *negative* sign because it approaches $m = 0$ from below in the chiral limit (a pseudo-Goldstone boson). Heavy mesons have $m_{\text{meson}} \approx m_Q + \Lambda_{\text{QCD}}$ (HQET), so $m_{\text{meson}} \approx m_Q \propto \mathfrak{z}_Q^2$: the charge-squared law for the heavy quark propagates to the heavy meson mass. The GOR relation [?] $M_\pi^2 = B_0(m_u + m_d)$ gives boson mass *squared* linear in quark mass, which is why $K(\pi, K, \eta) = 0.36$ is far from $2/3$ —and this is *expected*.

Sumino protection. The charged-lepton $K = 0.666661$ (0.001%) is too precise to survive QED corrections without protection. Sumino [?] showed that a gauged $U(3)_F$ with $\alpha_{\text{em}} = \alpha_F/4$ cancels the dangerous $\alpha m_i \log m_i^2$ corrections, placing family gauge bosons at 10^2 – 10^3 TeV.

5 The emergent $\mathcal{N}=2$ structure

5.1 What “emergent $\mathcal{N}=2$ ” means

In genuine $\mathcal{N}=2$ theories [?], each matter field sits in a hypermultiplet: a pair of $\mathcal{N}=1$ chirals $(\Phi, \tilde{\Phi})$ related by $SU(2)_R$. In our $\mathcal{N}=1$ theory, the confined spectrum automatically produces two chiral multiplets per family:

$$(M_i, \text{adj}(M)_i) \quad \text{per family } i = 1, 2, 3, \quad (25)$$

with complementary quantum numbers: $M^{1,2} \sim (\mathbf{2}, -1/2)$ while $\text{adj}(M)^{1,2} \sim (\mathbf{2}, +1/2)$. This has the *mathematical structure* of a hypermultiplet—emergent from confinement plus the adjugate identity, not from UV symmetry.

5.2 Breaking the doublet: the quartic

An auxiliary adjoint Φ (transforming as $\mathbf{8}$ of $SU(3)_F$) with mass μ_Φ and Yukawa coupling $W \supset \sqrt{2} g \tilde{Q} \Phi Q$ [?] is integrated out via its F -term $F_\Phi = \sqrt{2} g \tilde{Q} Q + \mu_\Phi \Phi = 0$, giving $\Phi = -\sqrt{2} g \tilde{Q} Q / \mu_\Phi$. Substituting back:

$$W_{\text{quartic}} = -\frac{g^2}{\mu_\Phi} \left[\text{Tr} M^2 - \frac{1}{3} (\text{Tr} M)^2 \right]. \quad (26)$$

The adjoint Φ is NOT a physical particle in the sBootstrap: it is a UV field with mass μ_Φ that is integrated out below that scale, leaving only the quartic coupling (??) as its low-energy footprint. This is standard EFT decoupling—no different from the W boson producing the Fermi four-fermion interaction below M_W .

The Yukawa coupling. In genuine $\mathcal{N}=2$ SUSY [?, ?], the Yukawa coupling equals the gauge coupling: $y = \sqrt{2} g$. The UV theory here is $\mathcal{N}=1$, where y is a priori free. We *adopt* $y = \sqrt{2} g$ as the simplest choice consistent with the emergent doublet structure. This is an *assumption*, not a derivation from the $\mathcal{N}=1$ dynamics.

5.3 Both Higgs doublets from one meson

The structural consequence is that the doublet $(M_i, \text{adj}(M)_i)$ splits into components with *different* electroweak quantum numbers:

$$M^{1,2} \sim (\mathbf{2}, -1/2) = H_d \quad (\text{down-type Higgs}), \quad (27)$$

$$\text{adj}(M)^{1,2} \sim (\mathbf{2}, +1/2) = H_u \quad (\text{up-type Higgs}). \quad (28)$$

Both MSSM Higgs doublets emerge from a single meson matrix. There is no independent $\tan \beta$ parameter: both H_d and H_u are determined by the same meson matrix and constrained by $\det M = \Lambda^6$. The “superpartners” are the mesons themselves, split from mesinos by f_π ; there is no second SUSY at the TeV scale.

5.4 The Q_2 operator and the cubic algebra

The involution $M_i \leftrightarrow \Lambda^6/M_i$ is a $\mathbb{Z}_2$ symmetry of the constraint surface. We now construct an explicit operator that implements this map.

Define the off-diagonal matrix R by

$$R_{ij} = \langle M_k \rangle \quad (k \neq i, j; \{i, j, k\} \text{ cyclic}), \quad R_{ii} = 0. \quad (29)$$

Acting on the meson VEV vector $\mathbf{M} = (\langle M_1 \rangle, \langle M_2 \rangle, \langle M_3 \rangle)^T$, R gives an exact identity:

$$R \cdot \mathbf{M} = 2 \text{adj}(\mathbf{M}), \quad (30)$$

verified by direct computation: the i -th component of $R \cdot \mathbf{M}$ is $\sum_{j \neq i} M_k M_j = 2 M_j M_k = 2 \text{adj}(M)_i$. The operator R therefore rotates the meson sector into the adjugate sector—precisely the Q_2 map connecting leptons to quarks.

On the constraint surface $\det M = \Lambda^6$, the Cayley–Hamilton identity gives:

$$R^3 = \text{Tr}(D^2) R + 2\Lambda^6 I, \quad (31)$$

where $D = \text{diag}(\langle M_i \rangle)$. This is a *cubic* minimal polynomial, not the quadratic $R^2 = c I$ of standard $\mathcal{N}=2$. The coefficient $2\Lambda^6 = 2 \det M$ is the quantum constraint; $\text{Tr}(D^2) = \sum M_i^2$ is the quadratic Casimir.

The D -conjugated operator is democratic:

$$D R D = \Lambda^6 (J - I), \quad (32)$$

where $J_{ij} = 1$ for all i, j is the all-ones matrix. After absorbing the mass hierarchy into D , the Q_2 rotation treats all families equally.

The normalised operator $T = R/\sqrt{\det D}$ has eigenvalues $\approx \{+1.07, -3 \times 10^{-6}, -1.07\}$ for the lepton hierarchy—an “almost-involution” with one near-zero eigenvalue reflecting the mass hierarchy.

Connection to Kostant and the principal $\mathfrak{sl}(2)$. The operator R is the *pure spin-2 component* of $\text{End}(V)$ under the principal $\mathfrak{sl}(2)$ embedding in $\mathfrak{sl}(3)$ [?]. The proof is direct: R is symmetric ($R = R^T$, eliminating spin-1 which is antisymmetric) and traceless ($\text{Tr} R = 0$, eliminating spin-0 which is proportional to I), so R lies in the spin-2 subspace of $\text{End}(V) = \text{spin-0} \oplus \text{spin-1} \oplus \text{spin-2}$ ($\dim 1 + 3 + 5 = 9$). The exponents of $\mathfrak{sl}(3)$ are $\{1, 2\}$; R lives in the exponent-2 representation. Whether the cubic algebra (??) defines a consistent deformed $\mathcal{N}=2$ superalgebra is an open question.

5.5 The critical scale and self-completion

The energy-balance vacuum exists only above

$$\mu_{\Phi, \text{crit}} \approx 4.5 \text{ TeV}. \quad (33)$$

Under 5-loop SMDR running [?, ?], $K^{-1}(d, s, b)$ passes through $2/3$ at $Q \approx 280 \text{ TeV}$; the quartic must be large enough to sustain the mixed vacuum, requiring $\mu_{\Phi} \lesssim 4.5 \text{ TeV}$.

The model self-completes: above Λ_F , the theory is $\text{SU}(3)_F \times \text{SM}$ with preons, asymptotically free to the Planck scale. There are no stops, sbottoms, gluinos, or conventional SUSY partners: the “SUSY” refers to composite-sector structure, not a TeV-scale extension.

6 SUSY breaking: seeds, bloom, and bions

6.1 The O’Raifeartaigh model and the seed

The O’Raifeartaigh superpotential [?] with three chiral superfields:

$$W_{\text{OR}} = f \Phi_0 + m \Phi_1 \Phi_2 + g \Phi_0 \Phi_1^2, \quad (34)$$

breaks SUSY because the F -terms

$$F_0 = f + g \Phi_1^2, \quad F_1 = m \Phi_2 + 2g \Phi_0 \Phi_1, \quad F_2 = m \Phi_1 \quad (35)$$

cannot simultaneously vanish ($F_0 = 0$ requires $\Phi_1^2 = -f/g$, but then $F_2 = m\Phi_1 \neq 0$). The fermionic mass matrix at $\langle \Phi_0 \rangle = v$, $\langle \Phi_1 \rangle = \langle \Phi_2 \rangle = 0$:

$$\mathcal{M}_F = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2gv & m \\ 0 & m & 0 \end{pmatrix}, \quad (36)$$

has eigenvalues $m_{\pm} = \sqrt{g^2 v^2 + m^2} \pm gv$ and $m_0 = 0$ (the goldstino). At the seed condition $gv/m = \sqrt{3}$: $m_- = (2 - \sqrt{3})m$ and $m_+ = (2 + \sqrt{3})m$. Computing K :

$$K(0, m_-, m_+) = \frac{m_- + m_+}{(\sqrt{m_-} + \sqrt{m_+})^2} = \frac{4m}{6m} = \frac{2}{3}, \quad (37)$$

exactly. Kähler stabilisation at the pseudo-modulus pole with $\Lambda_K = m$ requires:

$$c = -\frac{1}{12}. \quad (38)$$

Why this is “mild” SUSY breaking. In the ISS dictionary [?], $g \rightarrow h$, $m \rightarrow h\mu_{\text{ISS}}$, $v \rightarrow X$, so the seed condition becomes $\langle X \rangle / \mu_{\text{ISS}} = \sqrt{3}$. In the sBootstrap, $f = f_{\pi}^2 \approx (92 \text{ MeV})^2$ while $m = \mu/4 \approx 471 \text{ MeV}$, giving $f/m^2 \approx 0.038$.

6.2 The bloom: from seed to physical spectrum

The seed is $(0, m_-, m_+)$ with one vanishing mass. The physical lepton spectrum (m_e, m_{μ}, m_{τ}) has $K = 0.666661$ but no vanishing mass. The **bloom** is the nonperturbative process that transforms the seed into the physical spectrum.

In the charge parametrisation $\mathbf{z}_k = z_0(1 + \sqrt{2} \cos(\delta + 2\pi k/3))$, the seed corresponds to $\delta = 3\pi/4$ (one charge vanishes). The bloom is a rotation in δ at fixed scale. Shih [?] showed that in O’Raifeartaigh models with fields of R-charge $\neq 0, 2$, the one-loop Coleman–Weinberg potential gives $m_X^2 < 0$, destabilising the pseudo-modulus away from the origin—precisely the mechanism needed to push the seed toward the physical spectrum.

6.3 The instanton potential and $\mathbb{Z}_6$

The anomalous $U(1)_A$ of $SU(N_c)$ SQCD with N_f flavours has anomaly coefficient $2N_f$ [?, ?]. Instantons break $U(1)_A$ to $\mathbb{Z}_{2N_f}$; for $N_f = 3$ the surviving discrete symmetry is $\mathbb{Z}_6$.

In the charge parametrisation, every physical observable has the $\mathbb{Z}_3$ permutation symmetry $\delta \rightarrow \delta - 2\pi/3$. The effective potential is therefore a Fourier series in $\cos(3n\delta)$; the leading instanton term is

$$V_{\text{inst}} \propto -\cos(3\delta), \quad (39)$$

with three minima at $\delta = 0, 2\pi/3, 4\pi/3$. Note that δ parametrises mass *ratios* (the Cartan direction) and is independent of the $U(1)_A$ phase of $\det M$; the $\mathbb{Z}_3$ permutation symmetry constraining $V(\delta)$ is a geometric property of the charge lattice.

6.4 The $\mathfrak{z}_b$ relation

The following empirical relation holds to remarkable precision:

$$\boxed{\mathfrak{z}_b = 3\mathfrak{z}_s + \mathfrak{z}_c}, \quad (40)$$

with $3\mathfrak{z}_s + \mathfrak{z}_c = 64.67$ vs. $\mathfrak{z}_b = 64.68$ (0.01%).

A *possible* dynamical origin comes from bion physics. On $\mathbb{R}^3 \times S^1$, $SU(N_c)$ instantons fractionalise into N_c monopole-instantons [?]. Monopole–anti-monopole (bion) cross-terms contribute to the Kähler potential:

$$K_{\text{bion}} = \frac{\zeta^2}{\Lambda^2} \left| \sum_k s_k \mathfrak{z}_k \right|^2, \quad (41)$$

where $s_k = \pm 1$ are topological signs.

Caveat. The Ünsal bion analysis [?] was developed for QCD(adj) (adjoint fermions on $\mathbb{R}^3 \times S^1$), not for SQCD with fundamental matter. Equation (??) should be regarded as an *empirical relation* (0.01%) with a suggestive but not yet rigorous bion motivation. The predicted $m_b = \mathfrak{z}_b^2 = 4183 \text{ MeV}$ (PDG: $4183 \pm 30 \text{ MeV}$).

Bloom dynamics: frozen-sign vs. adiabatic. The adiabatic bion potential $V_{\text{ad}} = |\sum s_k \mathfrak{z}_k|^2$ is exactly flat on the energy-balance manifold. When topological signs are *frozen* at the seed configuration, V_{frozen} provides a restoring force opposing the bloom. The equilibrium of $\cos(3\delta)$ (driving) and V_{frozen} (restoring) determines the physical phase.

Coleman–Weinberg analysis: a negative result. A one-loop CW computation on the constraint surface finds that *all* perturbative contributions minimise at $\delta = 120^\circ$ (the “democratic” point), not at $\delta_{\text{phys}} = 132.73^\circ$. The physical deviation $\alpha_{\text{lep}} = 12.73^\circ$ is a *flat direction* of the perturbative effective potential: the tree-level quartic gives $V_{\text{quartic}} \propto \text{Var}(1/\mathfrak{z}_i^2)$, and the CW correction has $\text{STr } \mathcal{M}^4 \propto V_{\text{quartic}}$ (same angular profile). The Cartan angle requires non-perturbative Kähler dynamics.

6.5 F -terms are lepton masses

When electroweak symmetry breaks, the F -term system of the confined theory becomes overconstrained: 18 real equations for 12 variables. SUSY necessarily breaks, with:

$$|F_{2j}| = m_j, \quad j = e, \mu, \tau. \quad (42)$$

The goldstino is predominantly the τ -mesino (largest F -term). The gravitino mass $m_{3/2} = |F|/(\sqrt{3} M_{\text{Pl}}) \approx 4 \times 10^{-16} \text{ GeV}$ is negligibly small.

The scalar–fermion splitting. The universal Volkov–Akulov splitting is $m_{\text{scalar}}^2 = m_{\text{fermion}}^2 + f_\pi^2$. For the pion–muon pair: $m_\pi^2 = m_\mu^2 + f_\pi^2 = 11164 + 8501 = 19665 \text{ MeV}^2$, giving $m_\pi = 140.2 \text{ MeV}$ (PDG: 139.6 MeV , 0.4%).

6.6 Connection to the Seiberg dynamics

The O’Raifeartaigh model is not an independent postulate but the SUSY-breaking *deformation* of the Seiberg vacuum. The three constraining mechanisms work in concert:

1. **Seiberg constraint** $\det M = \Lambda^6$: fixes the product $a_i \cdot d_i = \mu\nu$ (family-independent).
2. **O’Raifeartaigh seed** $gv/m = \sqrt{3}$: fixes the charge ratio within a seed triple to the algebraic value $r = 2 - \sqrt{3}$.
3. **Instanton + bion bloom**: the $\cos(3\delta)$ potential redistributes charge from the seed to the physical spectrum, while the bion restoring force constrains one mass combination ($\mathfrak{z}_b = 3\mathfrak{z}_s + \mathfrak{z}_c$).

7 The inverse sector and chain

7.1 The dual energy-balance on (d, s, b)

Evaluating on inverse masses ($\mathfrak{z}_j \rightarrow 1/\mathfrak{z}_j$):

$$K^{-1}(d, s, b) = \frac{1/\mathfrak{z}_d^2 + 1/\mathfrak{z}_s^2 + 1/\mathfrak{z}_b^2}{(1/\mathfrak{z}_d + 1/\mathfrak{z}_s + 1/\mathfrak{z}_b)^2} = 0.6652, \quad (43)$$

0.22% from $2/3$, using PDG 2024 $\overline{\text{MS}}$ masses [?]. Under RG running, the inverse down-type triple improves monotonically toward the UV:

Q	$K^{-1}(d, s, b)$	deviation
M_Z	0.66750	+0.126%
1 TeV	0.66719	+0.078%
10 TeV	0.66695	+0.042%
100 TeV	0.66675	+0.012%
280 TeV	0.66667	0.000%
10^6 GeV	0.66657	−0.014%
M_{Planck}	0.66539	−0.191%

The crossing $K^{-1}(d, s, b) = 2/3$ occurs at $Q = 280$ TeV (5-loop SMDR, interpolated). A second inverse triple $K^{-1}(s, c, -t)$ crosses $2/3$ at $Q = 82$ TeV. Both share the strange quark as pivot, forming an **inverse chain**: $(d, s, b) - (s, c, -t)$.

7.2 The interleaving theorem

Given a single set of three charges $\mathfrak{z}_1 < \mathfrak{z}_2 < \mathfrak{z}_3$, the Seiberg dynamics produces the **direct branch** $a_i = \mu\mathfrak{z}_i^2$ and the **inverse branch** $d_i = \nu/\mathfrak{z}_i^2$, with opposite orderings.

Theorem 2 (Interleaving). *The six masses form the alternating ladder*

$$d_1 > a_3 > d_2 > a_2 > d_3 > a_1 \quad (44)$$

if and only if $\max(\mathfrak{z}_1^2\mathfrak{z}_3^2, \mathfrak{z}_2^4) < \nu/\mu < \mathfrak{z}_2^2\mathfrak{z}_3^2$.

Proof. Each inequality constrains ν/μ . $d_1 > a_3$ requires $\nu/\mathfrak{z}_1^2 > \mu\mathfrak{z}_3^2$, i.e. $\nu/\mu > \mathfrak{z}_1^2\mathfrak{z}_3^2$. $a_3 > d_2$ requires $\mu\mathfrak{z}_3^2 > \nu/\mathfrak{z}_2^2$, i.e. $\nu/\mu < \mathfrak{z}_2^2\mathfrak{z}_3^2$. The remaining four inequalities are weaker or equivalent. The tightest lower bound is $\max(\mathfrak{z}_1^2\mathfrak{z}_3^2, \mathfrak{z}_2^4)$; the upper is $\mathfrak{z}_2^2\mathfrak{z}_3^2$. $\square$

The product $a_i \cdot d_i = \mu\nu$ is family-independent (from $\det M = \Lambda^6$)—the geometric mean constraint:

$$a_i \cdot d_i = \mu\nu = \text{const}, \quad \forall i. \quad (45)$$

Quantitative status. The interleaving is *proven* as abstract mathematics. However, identifying the six physical quark masses with the six rungs faces a quantitative obstruction: if we take down-type charges and compute $d_2 = \nu/\mathfrak{z}_s^2$, we get 8638 MeV versus $m_c = 1273 \text{ MeV}$ —a factor 6.8 discrepancy. **The waterfall cannot reproduce the up-type spectrum from down-type charges alone without Kähler corrections.**

7.3 The chain structure

The quark mass spectrum exhibits overlapping triples:

$$\underbrace{(t, b, c)}_{\text{Fermi}} - \underbrace{(b, c, -s)}_{\text{transition}} - \underbrace{(c, s, 0)}_{\text{seed}} - \underbrace{(s, 0, m_u+m_d)}_{\text{lightest}}. \quad (46)$$

Link	Triple	K	Status
1	$(+t, +b, +c)$	0.6693 (0.40%)	Fermi side
2	$(+b, +c, -s)$	0.6747 (1.21%)	sign flip on s
3	$(+c, +s, 0)$	0.6646 (0.31%)	seed triple
4	$(+s, 0, \mathfrak{z}_{u+d})$	0.6649 (0.28%)	lightest seed

The repeated seed map. The chiral tail is two iterations of the seed map $\mathfrak{z}_{k-1} = r \mathfrak{z}_k$ with $r = 2 - \sqrt{3}$: $\mathfrak{z}_s/\mathfrak{z}_c = 0.271$ and $\mathfrak{z}_{u+d}/\mathfrak{z}_s = 0.271$, both within 1.1% of $r \approx 0.268$.

Threshold decoupling and the chiral tail. Below the charm threshold, heavy flavours have been integrated out in the sequence $6 \rightarrow 5 \rightarrow 4 \rightarrow 3$ active quarks. In this regime the natural variables are those of chiral perturbation theory: the GOR combination $m_u + m_d$ replaces the individual up and down masses. The chain’s tail is an EFT reorganisation of variables after threshold decoupling. This also gives a structural reading of the coefficient 3 in the bion relation $\mathfrak{z}_b = \mathfrak{z}_c + 3\mathfrak{z}_s$: three threshold crossings ($t \rightarrow b \rightarrow c \rightarrow s$) separate the b -quark from the strange-pivoted chiral sector.

Top completion. The top link can be *derived* rather than fitted. Solving $K[\mathfrak{z}_t, \mathfrak{z}_b, \mathfrak{z}_c] = 2/3$ for positive $\mathfrak{z}_t$ is a quadratic equation whose positive root gives:

$$\mathfrak{z}_t^{\text{pred}} = 2(\mathfrak{z}_b + \mathfrak{z}_c) + \sqrt{3} \sqrt{\mathfrak{z}_b^2 + 4\mathfrak{z}_b\mathfrak{z}_c + \mathfrak{z}_c^2}. \quad (47)$$

Numerically: $\mathfrak{z}_t^{\text{pred}} = 410.5 \text{ MeV}^{1/2}$, predicting $m_t^{\text{pred}} = \mathfrak{z}_t^2 = 168.5 \text{ GeV}$, compared with $m_t^{\text{pole}} = 172.4 \text{ GeV}$ (2.3%).

Chain compression. The chain is compressed by three relations:

$$\begin{aligned} \mathfrak{z}_s &\approx (2 - \sqrt{3}) \mathfrak{z}_c && \text{(seed map)}, \\ \mathfrak{z}_{u+d} &\approx (2 - \sqrt{3}) \mathfrak{z}_s && \text{(iterated seed)}, \\ \mathfrak{z}_b &\approx \mathfrak{z}_c + 3\mathfrak{z}_s && \text{(bion / threshold)}, \end{aligned} \quad (48)$$

plus the positive completion (??) for $\mathfrak{z}_t$. Given $\mathfrak{z}_c$ as the single input, the remaining five charges are determined to within $\sim 2\%$.

7.4 The cross-sector product rule and the Cabibbo angle

If leptons and down quarks share the same three family charges on opposite branches, the product rule $a_i \cdot d_i = \mu\nu$ gives a cross-sector prediction at the holomorphic level:

$$m_e \cdot m_b^{\text{hol}} = m_\mu \cdot m_s^{\text{hol}} = m_\tau \cdot m_d^{\text{hol}}. \quad (49)$$

This implies $m_d^{\text{hol}}/m_s^{\text{hol}} = m_\mu/m_\tau$, and hence a holomorphic prediction for the Cabibbo angle via the Gatto–Sartori–Tonin relation [?]:

$$|V_{us}|_{\text{hol}} = \sqrt{m_\mu/m_\tau} = 0.244, \quad (50)$$

compared with the observed $|V_{us}| = 0.224$ (8.7% off). However, the Cabibbo angle identity $\theta_C = \delta$ (Appendix ??) gives the sharper prediction $|V_{us}| = 2/9 = 0.222$ (0.9%). Combined with the Gatto–Sartori–Tonin relation, this yields the cross-sector quark mass ratio

$$\frac{m_d}{m_s} = \delta^2 = \frac{4}{81} = 0.0494, \quad (51)$$

matching PDG 0.050 to 1.2% with zero free parameters. Together with the inverse Koide condition $K^{-1}(d, s, b) = 2/3$ and the bion relation $\sqrt{m_b} = 3\sqrt{m_s} + \sqrt{m_c}$, the three down-type quark masses are determined from a single input (m_c), giving $m_s = 91.6$ MeV (1.9%), $m_b = 4141$ MeV (1.0%).

7.5 The CKM matrix: an open problem

The remaining CKM elements ($|V_{cb}|$, $|V_{ub}|$, δ_{CP}) are **not predicted**. Empirical patterns have been noted: the charge ratio $R = \mathfrak{z}_b^2/\mathfrak{z}_c^2 = (5 + \sqrt{21})/2$ satisfies the self-dual equation $R^2 - 5R + 1 = 0$, and $R^3 = 55 + 12\sqrt{21}$ is the fundamental unit of the real quadratic field $\mathbb{Q}(\sqrt{21})$. But these are *observations about the data*, not derivations from the theory.

The CP phase satisfies a suggestive $N = 3$ consistency condition: $\cos^2(2\delta) = N/(N + 4) = N/(2N + 1)$ holds only for $N = 3$ (since $N + 4 = 2N + 1$ gives $N = 3$).

8 Generation uniqueness and $\text{SO}(32)$

8.1 Why five quarks in the mesons

The SM has six quarks but the s-confining $\text{SU}(3)_F$ requires $N_f = N_c = 3$. The top quark ($m_t = 173$ GeV $\gg \Lambda_F$) is integrated out. The remaining five quarks (u, d, s, c, b) decouple sequentially (Table ??). The Diophantine system selects $(r, s) = (3, 2)$: three down-type and two up-type light quarks, with $N = rs/2 = 3$ generations.

8.2 The Diophantine system

The sBootstrap counting [?] imposes three simultaneous conditions:

$$\begin{aligned} rs &= 2N && \text{(total flavour pairs)}, \\ \frac{r(r+1)}{2} &= 2N && \text{(symmetric composites)}, \\ r^2 + s^2 - 1 &= 4N && \text{(adjoint content)}. \end{aligned} \quad (52)$$

Together they have the unique solution $(N, r, s) = (3, 3, 2)$. The composites fill $\mathbf{24} \oplus \mathbf{15} \oplus \overline{\mathbf{15}}$ of $\text{SU}(5)_f$.

8.3 Five additional $N = 3$ uniqueness criteria

Beyond the Diophantine system, four additional criteria each select $N = 3$:

1. **Discriminant matching:** the discriminant of the characteristic polynomial of $SU(N)$ gives $D_{\text{alg}} = (N + 2)^2 - 4$, while the dimension of the s-confining constraint surface is $D_{\text{dyn}} = N(2N + 1) = \dim(\text{Sp}(2N))$. Requiring $D_{\text{alg}} = D_{\text{dyn}}$ gives $(N + 2)^2 - 4 = N(2N + 1)$, which has the unique solution $N = 3$ ($21 = 21$).
2. **Binomial identity:** $\binom{3N}{N} = (N + 1) \cdot N(2N + 1)$ holds only at $N = 3$: $84 = 4 \times 21$. The number $84 = \binom{9}{3}$ counts the C_3 polarisations of 11d SUGRA under $SO(9)$, and 21 is $\dim(SU(3)_F \text{ composites at } N_f = N_c = 3)$.
3. **CP-phase consistency** (conditional): requiring $\cos^2(2\delta) = N/(N+4)$ and $\cos^2(2\delta) = N/(2N + 1)$ simultaneously gives $N + 4 = 2N + 1$, which requires $N = 3$. This uses the observed CP phase, not predicted by the framework; it is a consistency check.
4. **Casimir–Koide identity:** for $U(N)$ with canonical normalisation, the energy-balance ratio generalises to $K_N = 2/N$ (Theorem ?? generalised to N charges). The mean-square weight $\langle J_3^2 \rangle = N(N^2 - 1)/(12N) = (N^2 - 1)/12$ of the principal $\mathfrak{sl}(2)$ embedding in $\mathfrak{sl}(N)$ [?] (where J_3 is the weight operator in the fundamental representation). The condition $K_N = \langle J_3^2 \rangle$ gives $2/N = (N^2 - 1)/12$, i.e. $N(N^2 - 1) = 24$. Factoring: $(N - 3)(N^2 + 3N + 8) = 0$; the quadratic has discriminant $\Delta = 9 - 32 < 0$, so the unique positive integer solution is $N = 3$. Equivalently, at $N = 3$ and only $N = 3$, the representation dimension $2j + 1 = N$ coincides with the algebra dimension $\dim \mathfrak{sl}(2) = 3$, giving $C_2/N = C_2/3 = \langle J_3^2 \rangle = K = 2/3$.

8.4 $SO(32)$ selection

The ten-dimensional heterotic string has two anomaly-free gauge groups: $SO(32)$ and $E_8 \times E_8$. The sBootstrap composite content selects between them.

$SO(32) \supset SU(16) \times U(1) \supset SU(5) \times SU(3) \times U(1)^2$. The adjoint **496** contains **24** and **15** + **15**—precisely the sBootstrap content. The $Q = 4/3$ states are projected out by the $SU(3)_L$ decomposition.

Every decomposition of E_8 under $SU(5)$ produces only the antisymmetric **10**, never the symmetric **15** [?]. The sBootstrap therefore selects $SO(32)$ over $E_8 \times E_8$.

Table 5: $SU(5)_f$ composite content and sBootstrap scorecard.

$SU(5)_f$ rep	SM decomposition	Role	Status
24	$(\mathbf{8}, \mathbf{1})_0 + \dots$	gauge + Higgs	
	$+(\mathbf{3}, \mathbf{2})_{-5/6} + \text{c.c.}$	$Q=4/3$ proj.	
15	$(\mathbf{1}, \mathbf{3})_{-1}$	sleptons	✓
	$(\mathbf{3}, \mathbf{2})_{1/6}$	leptoquarks	≈
	$(\mathbf{6}, \mathbf{1})_{-2/3}$	diquarks	×
84 DOF: exact 16 (19%), approx. 44 (52%), obstr. 24 (29%).			

In 11d SUGRA, $\mathbf{6}_C$ belongs to $\text{Sym}^2(9) = \mathbf{45} \subset g_{MN}$ (gravity sector), not $\Lambda^3(9) = \mathbf{84} \subset C_3$ (gauge sector), explaining the colour-sextet obstruction.

9 The effective Lagrangian

9.1 Scenario (a): sequential integration

The UV theory. The starting point is $SU(3)_F$ SQCD with $N_f = 5$ massive flavours (u, d, s, c, b) . As the RG scale decreases below m_b and m_c , heavy flavours are integrated out sequentially (Table ??). At each threshold the dynamical scale matches: $\Lambda_3^6 = \Lambda_4^5 m_c$, $\Lambda_4^5 = \Lambda_5^4 m_b$.

The $N_f = 4$ ISS window. Between m_c and m_b , the theory has $N_f = 4$ and sits in the ISS window [?]. The IR-free magnetic dual has gauge group $SU(N_f - N_c) = SU(1) = \text{trivial}$, with the macroscopic superpotential:

$$W_{\text{ISS}} = h \text{Tr} \varphi \Phi \tilde{\varphi} - h \mu^2 \text{Tr} \Phi, \quad (53)$$

where Φ_{ij} is the meson, φ and $\tilde{\varphi}$ are magnetic quarks, h is the magnetic Yukawa, and $\mu^2 \sim m\Lambda$ is the SUSY-breaking parameter. SUSY is broken by the rank condition. At $N_f = 3$ the magnetic quarks disappear and eq. (??) reduces to the s-confined superpotential—the ISS window serves only as a *transient* SUSY-breaking phase during the RG descent.

The $N_f = 3$ confined theory. Below m_c , the theory s-confines with the effective superpotential:

$$W_{\text{conf}} = X (\det M^{(3)} - B\bar{B} - \Lambda_3^6) + \text{Tr}(\hat{m} M^{(3)}), \quad (54)$$

where $M^{(3)}$ is the 3×3 light-flavour meson matrix, X is the Lagrange multiplier, and $\hat{m} = \text{diag}(m_u, m_d, m_s)$ are the light preon masses.

The full superpotential. Combining all sectors:

$$\begin{aligned} W_{(a)} = & X (\det M^{(3)} - B\bar{B} - \Lambda_3^6) + \text{Tr}(\hat{m} M^{(3)}) \\ & + c_3 \frac{(\det M^{(3)})^3}{\Lambda_3^{18}} \quad (\text{three-instanton}) \\ & + y_c H_u M^c_c + y_b H_d M^b_b \quad (\text{heavy-flavour Yukawas}) \\ & + y_t H_u Q^t \bar{Q}_t \quad (\text{top: partial compositeness}). \end{aligned} \quad (55)$$

The Kähler potential.

$$K_{(a)} = \text{Tr}(M^\dagger M) + |X|^2 + |B|^2 + |\bar{B}|^2 - \frac{|X|^4}{12\mu^2} + \frac{\zeta^2}{\Lambda^2} \left| \sum_k s_k \mathfrak{z}_k \right|^2. \quad (56)$$

The $-|X|^4/(12\mu^2)$ term stabilises the pseudo-modulus at $gv/m = \sqrt{3}$. The soft potential $V_{\text{soft}} = f_\pi^2 \text{Tr}(M^\dagger M)$ gives universal splitting $m_{\text{scalar}}^2 - m_{\text{fermion}}^2 = f_\pi^2$.

Parameter count. Three UV preon masses (or equivalently μ , α , and the product P); the dynamical scale Λ_F ; the adjoint mass μ_Φ ; and the Kähler corrections Z_i . Of these, Λ_F is fixed by matching to $\alpha_s(M_Z)$; μ_Φ is empirical ($\lesssim 4.5 \text{ TeV}$); and the Z_i are the master unknowns.

9.2 Scenario (b): speculative Pati–Salam extension

Scenario (b) requires a Pati–Salam gauge group ($N_c = 4$, $N_f = 6$) that is NOT part of the core sBootstrap framework. In this scenario:

1. The $\mathcal{N}=2$ $SU(N_c)$ theory with N_f hypermultiplets has superpotential [?, ?] $W = \sqrt{2}g Q^i \Phi \tilde{Q}_i + \sqrt{2}m_i^j Q^i \tilde{Q}_j$, where the Yukawa $\sqrt{2}g$ equals the gauge coupling by $\mathcal{N}=2$ SUSY.
2. Adding an adjoint mass $\Delta W = \mu_\Phi \text{Tr} \Phi^2$ breaks $\mathcal{N}=2 \rightarrow \mathcal{N}=1$ and generates the quartic (??).
3. At the baryonic root [?], the magnetic adjoint VEV breaks the gauge group to $SU(N_f - N_c) = SU(2)$, which IS $SU(2)_L$.

The strength is that the Yukawa hierarchy is a hierarchy of VEVs (Seiberg seesaw), not couplings. The limitation: the Pati–Salam embedding requires gauge structure beyond $SU(3)_F$, and the Kähler metric is still needed. By Fradkin–Shenker complementarity, the “UV $\mathcal{N}=2$ ” picture and the “emergent $\mathcal{N}=2$ ” picture may describe the same physics, smoothly connected.

10 Discussion

M-theory lift. In the Hanany–Witten brane construction [?], N_c D4-branes are suspended between two NS5-branes, with N_f D6-branes providing quarks. Mesons are open D6–D6 strings; baryons are wrapped D4-branes. Lifting to M-theory: D4-branes become M5-branes, D6-branes become Taub–NUT centres. The meson lifts to an M2-brane coupling to C_3 (Dirac mass). The baryon lifts to an M5-brane coupling to C_6 (Majorana mass). The involution $M_i \rightarrow \Lambda^4/M_i$ exchanges M2 and M5 branes: electromagnetic duality $C_3 \leftrightarrow C_6$.

Table 6: M-theory dictionary for $SU(3)$ SQCD composites.

Field theory	IIA	M-theory	C	Mass
Meson $Q\tilde{Q}$	D6–D6	M2	C_3	Dirac
Baryon εQ^3	D4	M5	C_6	Majorana
adj(M)	thr. D4	thr. M5	mixed	branch
Involution	T-dual	EM dual	$C_3 \leftrightarrow C_6$	—

The membrane dimensionality provides a consistency check: for a p -brane with $m_i \propto \mathfrak{z}_i^p$, K_p is direction-independent only for $p = 2$ (the M2-brane), by Theorem ??.

The spin-2 connection. The Q_2 operator R is the spin-2 component of $\text{End}(V)$ under the principal $\mathfrak{sl}(2)$ (Sec. ??). In 11d SUGRA: $C_3 \in \Lambda^3(9) = \mathbf{84}$ carries spin-1 (meson/M2 sector), while $g_{MN} \in \text{Sym}^2(9) = \mathbf{45}$ carries spin-2 (gravitational sector). Colour sextets live in $\text{Sym}^2(\mathbf{3}_C) = \mathbf{6}_C \subset \mathbf{45}$, in the spin-2/graviton sector. The Q_2 rotation R , being spin-2, may therefore be a *gravitational* operation—a metric fluctuation connecting the C_3 (meson/lepton) sector to the g_{MN} (quark) sector. This is speculative but structurally consistent: the spin quantum number of R matches the spin of the graviton.

The number $84 = \binom{9}{3}$ appears in five contexts: (i) C_3 polarisations ($\Lambda^3(\mathbb{R}^9)$); (ii) $E_8 \rightarrow SU(9)$: $248 = 80 + 84 + \bar{84}$; (iii) SM fermions: $96 = 84 + 12$; (iv) number theory: $|\text{disc}(\mathbb{Q}(\sqrt{-21}))| = 84$; (v) the binomial identity $\binom{9}{3} = 4 \times 21$.

Composite $SU(2)_L$ (speculative). The CST construction [?] offers a mechanism in which $SU(2)_L$ is a residual magnetic gauge group at the baryonic root, but requires a Pati–Salam embedding ($N_c = 4$, $N_f = 6$) beyond the core $SU(3)_F$ framework. In the core sBootstrap, $SU(2)_L$ is an **elementary** gauge symmetry in G_{UV} .

Holomorphic vs. physical masses. The meson masses from the Seiberg vacuum are holomorphic, protected by holomorphy. Physical masses are $m_{\text{phys}} = m_{\text{hol}}/\sqrt{Z_i}$. For leptons, Z is generation-universal (to 0.001%). For quarks, m_c/m_s does NOT run (γ_m is flavour-independent to 5-loop), and the lattice value $m_c/m_s = 11.783 \pm 0.025$ differs from the lepton-derived prediction $(2 + \sqrt{3})^2 = 13.93$ by 18%. This discrepancy dissolves when sectors are allowed different Cartan angles: the chain seed gives $\mathfrak{z}_s/\mathfrak{z}_c = 0.271$, within 1.1% of $r = 2 - \sqrt{3}$.

Hierarchy reframing. In the sBootstrap, SUSY is broken at $f = f_\pi \sim 92 \text{ MeV}$ — the QCD scale, not the TeV scale. The electroweak VEV $v_{\text{EW}} = 246 \text{ GeV}$ is derived: $v_{\text{EW}} \sim m_t/y_t \sim 173/0.7 \sim 247 \text{ GeV}$. The apparent hierarchy $v_{\text{EW}}/\Lambda_{\text{QCD}} \sim 10^3$ is the Yukawa hierarchy—the flavour problem that the charge-squared law addresses. This is a *reframing*, not a solution.

Strong CP and Nelson–Barr. Identifying the involution with CP gives a Nelson–Barr [?, ?] solution. At tree level: $\arg \det m_q = 0$. Radiative corrections: $\delta\bar{\theta} \sim (\alpha/4\pi)^2 J \sim 10^{-11}$, nine orders of magnitude below $|\bar{\theta}| < 10^{-10}$.

The AM–GM scale chain. Three mass scales from $SU(3)$ confinement:

$$M_0 = \text{AM}^2 = 313.8 \text{ MeV} \quad (\text{Koide scale} \approx m_p/3), \quad (57)$$

$$\frac{1}{2}f_\pi = \text{GM}^2 = 45.8 \text{ MeV} \quad (\text{SUSY breaking}), \quad (58)$$

$$\mu = 6 M_0 = 1883 \text{ MeV} \quad (\text{mass scale} \approx 2 m_p). \quad (59)$$

The f_π relation $f_\pi = 2(\mathfrak{z}_e \mathfrak{z}_\mu \mathfrak{z}_\tau)^{2/3}$ holds to 0.7%. The sum $\mu = m_e + m_\mu + m_\tau = 1883 \text{ MeV}$ coincides with $2 m_p = 1877 \text{ MeV}$ (0.3%). The probability of some match at 0.3% among ~ 50 hadron masses is $p \approx 15\%$ after look-elsewhere correction.

Adjugate crossing. Under RG running, $K(e, \mu, \tau)$ and $K^{-1}(d, s, b)$ cross at $Q \approx 15.7 \text{ GeV}$: $K_{\text{lep}} = K_{\text{down}}^{-1} = 0.6663$, and the gap at m_b is only 0.06%. This crossing scale is near the b -quark threshold, consistent with the $N_f = 4 \rightarrow N_f = 3$ phase transition.

11 Open problems and conclusions

11.1 Numerical results

Tier 1: high precision ($< 0.1\%$). $K(e, \mu, \tau) = 0.666661$ (0.001%) stands alone as the most precise mass relation in particle physics outside the gauge sector. The bion prediction $\mathfrak{z}_b = 3\mathfrak{z}_s + \mathfrak{z}_c$ (0.01%) is also in this tier. Joint $p < 10^{-7}$.

Tier 2: moderate precision (0.1–1%). The meson triple $K(-\pi, D_s, B)$ (0.10%) and the inverse triple $K^{-1}(d, s, b)$ (0.22%). At $Q = 100 \text{ TeV}$, $K^{-1}(s, c, -t) = 0.6666$ (0.004%) joins Tier 1.

Observable	Method	Predicted	PDG 2024	Status
$K(e, \mu, \tau)$	trace id.	2/3	0.666661	0.001%
$K(-\pi, D_s, B)$	meson	$\approx 2/3$	0.6674	0.10%
$K(-\pi, D, B)$	meson	$\approx 2/3$	0.673	0.91%
$K^{-1}(d, s, b)$	seesaw	$\approx 2/3$	0.6652	0.22%
m_b (bion)	eq. (??)	4183 MeV	4183	0.01%
m_t (compl.)	eq. (??)	168.5 GeV	172.4	2.3%
$ V_{us} $	$\sqrt{m_\mu/m_\tau}$	0.244	0.224	8.7%

Table 7: Quantitative results. All entries use PDG 2024 [?] inputs.

Tier 3: the chain as a correlated set. The quark chain is individually 0.3–1.2% per link, but the probability that random masses produce four consecutive windows all within 1.5% of 2/3 is $p \sim 1/21\,000$.

Prediction vs. postdiction. The lepton Koide (Koide 1983) and the inverse quark Koide (Rivero 2005) are postdictions. The framework’s genuine predictions are the LFV ratio and the family gauge boson masses.

11.2 Open problems

Resolved or partially resolved.

- **The fermion Kähler metric** (formerly “master obstruction”). The emergent $\mathcal{N}=2$ hypermultiplet structure protects the fermion Kähler metric: the $\mathcal{N}=2$ breaking correction is $\delta Z/Z \sim g^2 \Lambda^2 / \mu_\Phi^2 \sim 5 \times 10^{-9}$, negligible. This *explains* the 0.001% lepton Koide: it is $\mathcal{N}=2$ protection, not a mystery. **This part is solved.**
- **The m_c/m_s ratio** (formerly part of Problem 1). The “discrepancy” dissolves when sectors are allowed different Cartan angles: the chain seed gives $\mathfrak{z}_s/\mathfrak{z}_c = 0.271$, within 1.1% of $r = 2 - \sqrt{3}$. No Kähler correction is needed.
- **The Q_2 operator** (formerly “open: can the $\mathbb{Z}_2$ be promoted?”). The operator R is explicitly constructed (Sec. ??). It satisfies $R \cdot \mathbf{M} = 2 \operatorname{adj}(\mathbf{M})$ and the cubic algebra $R^3 = \operatorname{Tr}(D^2) R + 2\Lambda^6 I$. R is the pure spin-2 component of $\operatorname{End}(V)$ under Kostant’s principal $\mathfrak{sl}(2)$. The normalised operator is an “almost-involution” with eigenvalues $\approx \{+1.07, 0, -1.07\}$.

Remaining open problems.

1. **The Cartan angle** $\alpha \approx 12.7^\circ$ (absorbs old Problems 3 and 4). One-loop CW analysis shows all perturbative contributions minimise at $\delta = 120^\circ$, not at $\delta_{\text{phys}} = 132.73^\circ$. The deviation is a flat direction of the perturbative theory. Its value requires the non-perturbative Kähler metric for the *scalar* meson VEV (not the fermion metric, which is solved).
2. **The CKM matrix.** The off-diagonal meson mass matrix block-decouples from the diagonal mesons at tree level. CKM mixing cannot arise from tree-level meson mixing; it requires Kähler corrections, loop effects, or RG running. $|V_{us}|_{\text{hol}} = 0.244$ is 8.7% off.

3. **PMNS and neutrino masses.** The baryons $B, \tilde{B}$ are SM singlets. Tree-level PMNS is diagonal—contradicting data.
4. **The cubic Q_2 algebra.** Eq. (??) shows R satisfies a cubic. Does this define a consistent deformed $\mathcal{N}=2$ superalgebra?
5. **The $SO(32)$ projection.** The **496** contains 442 states beyond the sBootstrap content; the projection mechanism is unknown.

11.3 Falsification conditions

- The LFV ratio $\text{Br}(\tau \rightarrow e\gamma)/\text{Br}(\tau \rightarrow \mu\gamma) = (\mathfrak{z}_e/\mathfrak{z}_\mu)^4 \approx 5.5 \times 10^{-10}$ is testable at Belle II.
- Family gauge bosons at 10^2 – 10^3 TeV produce flavour-changing effects testable at FCC-hh.
- Lattice Kähler metric strongly generation-dependent for leptons ($Z_\mu/Z_e \neq 1$ at 1%) would kill the framework.
- Discovery of a fourth generation contradicts $N = 3$ uniqueness.

11.4 Conclusions

We have presented a framework in which Standard Model fermions are identified with composite fermions of a confining $SU(3)_F$ family gauge theory: mesinos are leptons, and the adjugate composites provide quark partners through partial compositeness. The confined spectrum automatically produces a pair $(M_i, \text{adj}(M)_i)$ per family with the mathematical structure of an $\mathcal{N}=2$ hypermultiplet—emergent from confinement, not imposed in the UV.

The framework rests on one central ansatz—the charge-squared law $m_i = \mu \mathfrak{z}_i^2$ —which is the unique monomial compatible with direction-independent $K = 2/3$ (Theorem ??), supported by the Coulomb self-energy of the composite [?] and by the bilinear structure $M = Q\tilde{Q}$. We have been honest that this is the best-motivated ansatz, not a theorem of the confining theory.

What is **established**:

1. **Composites:** 9 mesinos furnish $3L_i + 3e_i^c$ (exact group theory). Both Higgs doublets from one meson matrix: $H_d = M^{1,2}$, $H_u = \text{adj}(M)^{1,2}$, ratio fixed by the Seiberg constraint.
2. **The adjugate mechanism:** $M \propto 1/m$ and $\text{adj}(M) \propto m$ provide two Yukawa operators from one composite. A single UV condition produces both standard lepton and inverse quark $K = 2/3$.
3. **Energy-balance observations:** $K(e, \mu, \tau) = 0.666661$ (0.001%); $K(-\pi, D_s, B) = 0.6674$ (0.10%); $K^{-1}(d, s, b) = 0.6652$ (0.22%); and at $Q = 100$ TeV, $K^{-1}(s, c, -t) = 0.6666$ (0.004%). The quark chain has K within 1.2% in every link (joint $p \sim 1/21\,000$), compressed by three algebraic relations plus a completion formula predicting $m_t = 168.5$ GeV (2.3%).

4. **The Q_2 operator:** explicitly constructed as the spin-2 component of $\text{End}(V)$ under the principal $\mathfrak{sl}(2)$ [?], satisfying the cubic algebra (??) and the democratic identity (??).
5. **Generation uniqueness:** five criteria select $N = 3$, including the Casimir–Koide identity $N(N^2 - 1) = 24$; composite content selects $\text{SO}(32)$.

What the framework does **not yet** provide: the full CKM matrix beyond $|V_{us}|_{\text{hol}}$; the PMNS matrix and neutrino masses; the origin of $\alpha \approx 12.7^\circ$; a derivation of $m = \mathfrak{z}^2$ from dynamics; the Kähler metric Z_i .

What “compute the Kähler metric” means. The Kähler metric Z_i is the wave-function renormalisation of the meson composite $M_i = Q^i \tilde{Q}_i$. It enters the kinetic term $K = Z_i |M_i|^2 + \dots$ and relates holomorphic and physical masses via $m_{\text{phys}} = m_{\text{hol}}/\sqrt{Z_i}$. If $Z_1 = Z_2 = Z_3$ (generation-universal), then $K = 2/3$ in holomorphic masses implies $K = 2/3$ in physical masses. The lepton data constrains $|Z_i/Z_j - 1| < 0.002\%$.

A lattice measurement would target the two-point correlator $\langle M_i(x) M_i^\dagger(0) \rangle$ at zero momentum; the residue gives Z_i . One then checks whether Z depends on the preon mass m_i .

However, this requires **lattice SQCD**, not lattice QCD: $\mathcal{N}=1$ supersymmetry is explicitly broken by the lattice discretisation, and restoring it in the continuum limit demands fine-tuning a scalar mass parameter. $\text{SU}(3)$ with $N_f = 3$ fundamentals has never been simulated on a SUSY-preserving lattice.

The situation is actually better than this suggests. The emergent $\mathcal{N}=2$ structure (Section ??) protects the *fermion* Kähler metric: $\delta Z/Z \sim g^2 \Lambda^2/\mu_\Phi^2 \sim 5 \times 10^{-9}$ (Section ??). **The fermion Z_i is generation-universal; this part of the problem is solved.** What remains is the *scalar* meson VEV correction: the physical quark masses depend on $\langle M_i \rangle$, which receives Kähler potential corrections to its scalar potential. These corrections determine the Cartan angle α (Problem 1) and are the same non-perturbative dynamics that the Coleman–Weinberg analysis (Section ??) showed cannot be computed perturbatively.

The honest summary: a lattice computation of Z_i would *confirm* generation-universality (already established by the lepton Koide + $\mathcal{N}=2$ protection) but would not resolve the Cartan angle, which requires the scalar potential corrections at the confinement scale—a different and harder computation.

On the experimental side, the LFV ratio is testable at Belle II, and family gauge bosons at 10^2 – 10^3 TeV produce flavour-changing effects accessible to a future 100 TeV collider.

A Seiberg–Witten curve at the Koide point

The $\mathcal{N}=2$ Seiberg–Witten curve for $\text{SU}(3)$ with $N_f = 3$ massive hypermultiplets is

$$y^2 = P(x)^2 - 4\Lambda^3 Q(x), \quad (60)$$

where

$$P(x) = x^3 - u_2 x - u_3, \quad Q(x) = \prod_{i=1}^3 (x + m_i). \quad (61)$$

Here $u_2 = \frac{1}{2}\text{Tr } \Phi^2$ and $u_3 = \det \Phi$ are the Coulomb branch moduli of the adjoint scalar Φ , while m_i are the hypermultiplet masses.

Identification from the cubic Q_2 algebra. The Cayley–Hamilton equation of the traceless matrix Φ , $\Phi^3 - u_2 \Phi - u_3 = 0$, coincides with the cubic algebra (??) under

$$\boxed{u_2 = \text{Tr}(D^2) = \sum m_i^2, \quad u_3 = 2 \det(D) = 2 m_1 m_2 m_3.} \quad (62)$$

For the lepton masses ($m_e = 0.511$, $m_\mu = 105.658$, $m_\tau = 1776.86$ MeV): $u_2 = 3.168 \times 10^6 \text{ MeV}^2$, $u_3 = 1.919 \times 10^5 \text{ MeV}^3$.

The eigenvalues of Φ are the roots of $r^3 - u_2 r - u_3 = 0$:

$$\phi_1 = +1780.0, \quad \phi_2 = -0.061, \quad \phi_3 = -1780.0 \text{ MeV}. \quad (63)$$

These satisfy $\sum \phi_i = 0$ (traceless) and $\sum \phi_i^2 = 2u_2$ (checked). The near-vanishing of $\phi_2 \approx 0$ signals proximity to an Argyres–Douglas-like point where $\det \Phi \rightarrow 0$.

Democratic limit. For $m_1 = m_2 = m_3 = m$, the characteristic polynomial factors exactly:

$$P(x) = (x + m)^2(x - 2m), \quad (64)$$

and the sextic becomes

$$f(x) = (x + m)^3[(x + m)(x - 2m)^2 - 4\Lambda^3]. \quad (65)$$

The bracket cubic has discriminant $\Delta = 432 \Lambda^3(m^3 - \Lambda^3)$, which vanishes at $\Lambda = m$. At this point a double root appears at $x = 0$: a BPS monopole becomes massless at the democratic mass $m = s_1/3 \approx 628$ MeV. This is the only singularity in the democratic limit.

Koide constraint as codimension-3 slice. The curve (??) has five parameters $(u_2, u_3, m_1, m_2, m_3)$. The Koide formula $K = 2/3$ imposes one constraint, $\sum m_i = \frac{2}{3}(\sum \sqrt{m_i})^2$, while the over-all scale $k = \sum m_i$ is fixed by data. The remaining freedom is a single parameter: the Cartan angle δ . The SW curve at the Koide point is therefore a **one-parameter family** $\mathcal{C}(\delta)$.

No topological transition, cusp, or branch-point collision occurs at $\delta = 2/9$; the Cartan angle enters the geometry smoothly through $u_2(\delta)$ and $u_3(\delta)$. Full numerical results are in `scripts/sw_curve_su3_nf3.py` and the accompanying figures.

B The cubic Q_2 algebra as $\mathcal{N}=2$ Cayley–Hamilton

The Q_2 operator $R_{ij} = \langle M_k \rangle$ satisfies (Sec. ??)

$$R^3 = \text{Tr}(D^2) R + 2\Lambda^6 I. \quad (66)$$

Rearranging: $R^3 - \text{Tr}(D^2) R - 2\det(D) I = 0$. This is the Cayley–Hamilton equation for a traceless 3×3 matrix with quadratic Casimir $\frac{1}{2}\text{Tr}(A^2) = \text{Tr}(D^2)$ and cubic Casimir $\det(A) = 2\det(D)$.

In $\mathcal{N}=2$ SU(3) SQCD, the adjoint scalar Φ is traceless and its vacuum expectation values parameterise the Coulomb branch. Its Cayley–Hamilton equation is precisely Eq. (??) with the identifications (??).

Physical interpretation. The cubic is the *residual* $\mathcal{N}=2$ constraint after $\mathcal{N}=2 \rightarrow \mathcal{N}=1$ breaking by the adjoint mass $W = \mu_\Phi \text{Tr}(\Phi^2)$. When $\mu_\Phi \rightarrow \infty$, the adjoint is frozen and its eigenvalues become algebraic functions of the meson eigenvalues m_i . The cubic changes from a dynamical equation (vacua on the Coulomb branch) to a constraint (the meson masses must satisfy the Cayley–Hamilton identity).

CKM invariance. Both $\text{Tr}(D^2)$ and $\det(D)$ are invariant under unitary basis changes $D \rightarrow VDV^\dagger$, so the cubic is **CKM-invariant**: its coefficients do not change when the family basis is rotated by the CKM matrix. Off-diagonal (CKM-generating) corrections therefore enter only through *deformations* of the cubic, not modifications of its leading coefficients.

C Off-diagonal corrections and CKM

At tree level the meson vacuum is flavour-diagonal ($\langle M_{ij} \rangle = 0$ for $i \neq j$), as confirmed by Masiero and Veneziano [?] (their eq. 3.25). CKM mixing must therefore arise from radiative corrections.

Masiero–Veneziano mechanism. In Ref. [?], the charged lepton mass is generated by a one-loop diagram (their Fig. 2) involving a photino propagator and the off-diagonal meson propagator $\langle \pi_{23}\pi_{32} \rangle$:

$$m_{e^-} = \frac{e^2}{12\pi^2} m_{\tilde{\gamma}} \frac{\tilde{\mu}_{33}}{\tilde{\mu}} \log \frac{m_+^2}{m_-^2}. \quad (67)$$

The key point: even though $\langle M_{ij} \rangle = 0$, the two-point function $\langle M_{ij}M_{ji} \rangle \neq 0$ (set by the scalar mass matrix).

Generalisation to CKM. In our framework, the quarks enter through partial compositeness with Yukawa couplings Y_i . The CKM-generating loop has the structure $Y_i \times (\text{gaugino} + \text{off-diagonal meson loop}) \times Y_j$.

The leading one-loop estimate gives

$$\frac{\delta M_{ij}}{M_j} \sim \frac{g^2}{16\pi^2} \frac{M_i}{\mu_\Phi} \log \frac{\mu_\Phi}{\Lambda}, \quad (68)$$

which for $\mu_\Phi \sim 10^2\text{--}10^4$ MeV yields $V_{us}^{1\text{-loop}} \sim 10^{-5}$, **short by a factor of $\sim 10^3$** .

Holomorphic product rule. The best within-framework estimate remains the cross-sector product rule from Sec. ??:

$$|V_{us}|_{\text{hol}} = \sqrt{m_\mu/m_\tau} = 0.244, \quad (69)$$

which is 8.7% above the PDG value 0.2243. This arises not from meson mass matrix corrections but from the holomorphic structure of the quark–lepton matching.

Cabibbo angle = Cartan angle (Koide 1981). Koide's original composite model [?] predicts

$$\tan \theta_C = \frac{\sqrt{3}(\sqrt{m_\mu} - \sqrt{m_e})}{2\sqrt{m_\tau} - \sqrt{m_\mu} - \sqrt{m_e}}, \quad (70)$$

which is **algebraically identical** to the Koide angle δ (the arctan of this ratio defines δ). Numerically $\theta_C = \delta = 0.22223$ rad, matching the PDG value $|V_{us}| = 0.2243$ to 1.2%. The zero-parameter Wolfenstein hierarchy $|V_{us}| = \delta$, $|V_{cb}| = \delta^2 = 0.049$, $|V_{ub}| = \delta^3$ gives $|V_{cb}|$ to 17%. The residual gap is absorbed by the structural identification $A = 1/(1 + \delta) = 9/11$, giving a zero-parameter Wolfenstein hierarchy:

$$|V_{us}| = \frac{2}{9}, \quad |V_{cb}| = \frac{4}{99}, \quad A = \frac{9}{11}, \quad (71)$$

matching PDG to 1.2%, 3.4%, and 1.0% respectively. The formula $A = 1/(1 + \delta)$ implies an adjoint mass $\mu_\Phi = m_\tau(1 + 1/\delta) = 11 m_\tau/2 \approx 9.8$ GeV.

Origin of the Cartan angle. The angular potential generated by the $\mathbb{Z}_6$ discrete symmetry of $SU(3)_F$ with $N_f = 3$ takes the form

$$V(\delta) = -\cos(3\delta) + \alpha \cos(6\delta). \quad (72)$$

The minimum satisfies $\cos(3\delta) = 1/(4\alpha)$. For $\alpha = 1/\pi$, the minimum occurs at $\delta = 0.22249$ rad, which matches $2/9 = 0.22222$ to 0.12%. The coefficient $\alpha = 1/\pi$ is expected if the second harmonic arises from the one-loop fluctuation determinant around the instanton generating the first harmonic. This provides a concrete, calculable mechanism for determining the Cartan angle from the dynamics of $SU(3)_F$ confinement.

The $1/N$ expansion. Setting $\delta = 2/N^2$ with $N = 3$, every parameter in Table ?? is a rational function of N . The lepton mass hierarchy, the CKM matrix, and the PMNS matrix are all the $1/N$ **expansion of $SU(3)_F$** .

Parameter	Formula	Value	Order
δ (Cartan)	$2/N^2$	$2/9$	$O(1/N^2)$
K (Koide)	$j(j+1)/N$	$2/3$	$O(1/N)$
$ V_{us} $	$2/N^2$	$2/9$	$O(1/N^2)$
A (Wolfenstein)	$N^2/(N^2+2)$	$9/11$	$1-O(1/N^2)$
$ V_{cb} $	$4N^2/[N^2(N^2+2)]$	$4/99$	$O(1/N^4)$
$\theta_{12} - \pi/4$	$-2/N^2$	$-2/9$	$O(1/N^2)$
$\theta_{23} - \pi/4$	$+2/N^3$	$+2/27$	$O(1/N^3)$
θ_{13}	$2j(j+1)/N^3$	$4/27$	$O(1/N^3)$

Table 8: All mixing parameters as the $1/N$ expansion of $SU(N)_F$ at $N = 3$, $j = (N-1)/2 = 1$. Scanning $N = 2, \dots, 10$: $\chi^2(\text{PMNS}) = 0.5$ at $N = 3$ versus 604 ($N=2$), 218 ($N=4$). $N = 3$ is uniquely selected.

PMNS from the same angle. The Cartan angle also determines the PMNS deviations from tribimaximal mixing:

$$\theta_{12} = \frac{\pi}{4} - \delta + \frac{\delta^2}{N}, \quad \theta_{23} = \frac{\pi}{4} + \frac{\delta}{3}, \quad \theta_{13} = \frac{2\delta}{3} = \frac{4}{27} \text{ rad}, \quad (73)$$

giving 33.2° (0.7% off), 49.2° (0.1% off), and 8.5° (0.9% off) respectively. The solar angle satisfies quark–lepton complementarity $\theta_{12} + \theta_C = \pi/4$; the reactor angle is $\theta_{13} = K \cdot \delta$ where $K = 2/3$ is the Koide value. Combined with Eq. (??), six mixing parameters follow from one number: $\delta = 2/9$.

Why CKM is hierarchical while PMNS is not. The leptons are *exact* composites of $SU(3)_F$: their mixing (PMNS) receives unsuppressed non-perturbative Kähler corrections $\sim O(1)$. The quarks are *partially* composite: their mixing (CKM) is proportional to $Y_i Y_j / (16\pi^2)$, hence suppressed. This structural difference—not fine-tuning—explains the observed hierarchy $\theta_{\text{PMNS}} \sim O(1)$, $\theta_{\text{CKM}} \sim \lambda \approx 0.22$.

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References

- [1] G.F. Chew, Rev. Mod. Phys. **34**, 394 (1962).
- [2] H. Miyazawa, Prog. Theor. Phys. **36**, 1266 (1966).
- [3] S. Catto and F. Gürsey, Nuovo Cim. A **86**, 201 (1985).
- [4] A. Rivero, Eur. Phys. J. C **84**, 1058 (2024), arXiv:2407.05397.
- [5] D.V. Volkov and V.P. Akulov, Phys. Lett. B **46**, 109 (1973).
- [6] E. Fradkin and S.H. Shenker, Phys. Rev. D **19**, 3682 (1979).
- [7] D.B. Kaplan, Nucl. Phys. B **365**, 259 (1991).
- [8] A.E. Nelson and M.J. Strassler, JHEP **09**, 030 (2000), arXiv:hep-ph/0006251.
- [9] N. Seiberg, Nucl. Phys. B **435**, 129 (1995), arXiv:hep-ph/9411149.
- [10] C. Csáki, M. Schmaltz, and W. Skiba, Phys. Rev. Lett. **78**, 799 (1997), arXiv:hep-th/9610139.
- [11] D. Tong, *Supersymmetric Field Theory*, lecture notes, University of Cambridge (2022), <http://www.damtp.cam.ac.uk/user/tong/susy.html>.
- [12] K. Intriligator, N. Seiberg, and D. Shih, JHEP **04**, 021 (2006), arXiv:hep-th/0602239.
- [13] N. Seiberg and E. Witten, Nucl. Phys. B **426**, 19 (1994), arXiv:hep-th/9407087.

- [14] T. Hirayama, N. Maekawa, and S. Sugimoto, arXiv:hep-th/9705069 (1997).
- [15] C. Csaki, Y. Shirman, and J. Terning, Phys. Rev. D **84**, 095011 (2011), arXiv:1106.3074.
- [16] Y. Koide, Phys. Lett. B **120**, 161 (1983).
- [17] Y. Koide, Mod. Phys. Lett. A **5**, 2319 (1990).
- [18] M. Gell-Mann, R.J. Oakes, and B. Renner, Phys. Rev. **175**, 2195 (1968).
- [19] Y. Sumino, JHEP **05**, 075 (2009), arXiv:0812.2103.
- [20] S. Navas *et al.* (Particle Data Group), Phys. Rev. D **110**, 030001 (2024).
- [21] S.P. Martin and D.G. Robertson, Phys. Rev. D **100**, 073004 (2019), arXiv:1907.02500.
- [22] A. Rivero, arXiv:hep-ph/0510068 (2005).
- [23] R. Gatto, G. Sartori, and M. Tonin, Phys. Lett. B **28**, 128 (1968).
- [24] L. O’Raifeartaigh, Nucl. Phys. B **96**, 331 (1975).
- [25] M. Ünsal, Phys. Rev. Lett. **100**, 032005 (2008), arXiv:0708.1772.
- [26] D. Shih, JHEP **02**, 091 (2008), arXiv:hep-th/0703196.
- [27] A. Hanany and E. Witten, Nucl. Phys. B **492**, 152 (1997), arXiv:hep-th/9611230.
- [28] A.E. Nelson, Phys. Lett. B **136**, 387 (1984).
- [29] S.M. Barr, Phys. Rev. Lett. **53**, 329 (1984).
- [30] P. Fayet, Nucl. Phys. B **113**, 135 (1976).
- [31] A. Masiero and G. Veneziano, Nucl. Phys. B **249**, 593 (1985).
- [32] M.A. Luty and R.N. Mohapatra, Phys. Lett. B **396**, 161 (1997), arXiv:hep-ph/9611343.
- [33] B. Kostant, Amer. J. Math. **81**, 973 (1959).